MATRIX OF ZZ, XX AND YY

$ZZ(\beta) = e^{-i\beta\sigma_z\otimes\sigma_z}$

$$\sigma_z\otimes\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$$

diagonal matrix $\Rightarrow$

$$\boxed{ZZ(\beta) = e^{-i\beta\sigma_z\otimes\sigma_z} = \begin{pmatrix}e^{-i\beta}&0&0&0\\0&e^{i\beta}&0&0\\0&0&e^{i\beta}&0\\0&0&0&e^{-i\beta}\end{pmatrix}}$$

$YY(\beta) = e^{-i\beta\sigma_y\otimes\sigma_y}$

$$\sigma_y\otimes\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\otimes\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}0&0&0&i^2\\0&0&-i^2&0\\0&-i^2&0&0\\i^2&0&0&0\end{pmatrix} = \begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix} \rightarrow \text{NON-DIAGONAL MATRIX!}$$

$$\Downarrow$$

$$\boxed{e^A = P e^D P^{-1}}$$

- $A = \sigma_y\otimes\sigma_y$
- $D = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$
- $P = \begin{pmatrix}0&-1&0&1\\1&0&-1&0\\1&0&1&0\\0&1&0&1\end{pmatrix}$
- $P^{-1} = \frac{1}{2}\begin{pmatrix}0&1&1&0\\-1&0&0&1\\0&-1&1&0\\1&0&0&1\end{pmatrix}$

$$e^{-i\beta A} = P\begin{pmatrix}e^{-i\beta}&0&0&0\\0&e^{-i\beta}&0&0\\0&0&e^{i\beta}&0\\0&0&0&e^{i\beta}\end{pmatrix}P^{-1} =$$

$$= \begin{pmatrix}0&-1&0&1\\1&0&-1&0\\1&0&1&0\\0&1&0&1\end{pmatrix}\cdot\begin{pmatrix}e^{-i\beta}&0&0&0\\0&e^{-i\beta}&0&0\\0&0&e^{i\beta}&0\\0&0&0&e^{i\beta}\end{pmatrix}P^{-1} = \begin{pmatrix}0&-e^{-i\beta}&0&e^{i\beta}\\e^{-i\beta}&0&-e^{i\beta}&0\\e^{-i\beta}&0&e^{i\beta}&0\\0&e^{-i\beta}&0&e^{i\beta}\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}0&1&1&0\\-1&0&0&1\\0&-1&1&0\\1&0&0&1\end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix}e^{-i\beta}+e^{i\beta}&0&0&-e^{-i\beta}+e^{i\beta}\\0&e^{-i\beta}+e^{i\beta}&e^{-i\beta}-e^{i\beta}&0\\0&e^{-i\beta}-e^{i\beta}&e^{-i\beta}+e^{i\beta}&0\\-e^{-i\beta}+e^{i\beta}&0&0&e^{-i\beta}+e^{i\beta}\end{pmatrix} = \left|\begin{array}{l}e^{-i\beta}+e^{i\beta} = \cos\beta - i\sin\beta + \cos\beta + i\sin\beta = 2\cos\beta\\-e^{-i\beta}+e^{i\beta} = -\cos\beta + i\sin\beta + \cos\beta + i\sin\beta = 2i\sin\beta\\e^{-i\beta}-e^{i\beta} = \cos\beta - i\sin\beta - \cos\beta - i\sin\beta = -2i\sin\beta\end{array}\right| =$$

$$= \frac{1}{2}\begin{pmatrix}2\cos\beta&0&0&2i\sin\beta\\0&2\cos\beta&-2i\sin\beta&0\\0&-2i\sin\beta&2\cos\beta&0\\2i\sin\beta&0&0&2\cos\beta\end{pmatrix} = \underline{\underline{\begin{pmatrix}\cos\beta&0&0&i\sin\beta\\0&\cos\beta&-i\sin\beta&0\\0&-i\sin\beta&\cos\beta&0\\i\sin\beta&0&0&\cos\beta\end{pmatrix}}}$$

$XX(\beta) = e^{-i\beta\sigma_x\otimes\sigma_x}$

$$\sigma_x\otimes\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix} \rightarrow \text{NON-DIAGONAL MATRIX!}$$

$$\Downarrow$$

$$\boxed{e^A = P e^D P^{-1}}$$

- $A = \sigma_x\otimes\sigma_x$
- $D = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$
- $P = \begin{pmatrix}0&1&0&-1\\1&0&-1&0\\1&0&1&0\\0&1&0&1\end{pmatrix}$
- $P^{-1} = \frac{1}{2}\begin{pmatrix}0&1&1&0\\1&0&0&1\\0&-1&1&0\\-1&0&0&1\end{pmatrix}$

$$e^{-i\beta A} = P e^{-i\beta D} P^{-1} =$$

$$= \begin{pmatrix}0&1&0&-1\\1&0&-1&0\\1&0&1&0\\0&1&0&1\end{pmatrix}\begin{pmatrix}e^{-i\beta}&0&0&0\\0&e^{-i\beta}&0&0\\0&0&e^{i\beta}&0\\0&0&0&e^{i\beta}\end{pmatrix}P^{-1} = \begin{pmatrix}0&e^{-i\beta}&0&-e^{i\beta}\\e^{-i\beta}&0&-e^{i\beta}&0\\e^{-i\beta}&0&e^{i\beta}&0\\0&e^{-i\beta}&0&e^{i\beta}\end{pmatrix}P^{-1} =$$

$$= \begin{pmatrix}0&e^{-i\beta}&0&-e^{i\beta}\\e^{-i\beta}&0&-e^{i\beta}&0\\e^{-i\beta}&0&e^{i\beta}&0\\0&e^{-i\beta}&0&e^{i\beta}\end{pmatrix}\frac{1}{2}\begin{pmatrix}0&1&1&0\\1&0&0&1\\0&-1&1&0\\-1&0&0&1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}e^{-i\beta}+e^{i\beta}&0&0&e^{-i\beta}-e^{i\beta}\\0&e^{-i\beta}+e^{i\beta}&e^{-i\beta}-e^{i\beta}&0\\0&e^{-i\beta}-e^{i\beta}&e^{-i\beta}+e^{i\beta}&0\\e^{-i\beta}-e^{i\beta}&0&0&e^{-i\beta}+e^{i\beta}\end{pmatrix} =$$

$$= \left|\begin{array}{l}e^{-i\beta}+e^{i\beta} = \cos\beta - i\sin\beta + \cos\beta + i\sin\beta = 2\cos\beta\\-e^{-i\beta}+e^{i\beta} = -\cos\beta + i\sin\beta + \cos\beta + i\sin\beta = 2i\sin\beta\\e^{-i\beta}-e^{i\beta} = \cos\beta - i\sin\beta - \cos\beta - i\sin\beta = -2i\sin\beta\end{array}\right| = \frac{1}{2}\begin{pmatrix}2\cos\beta&0&0&-2i\sin\beta\\0&2\cos\beta&-2i\sin\beta&0\\0&-2i\sin\beta&2\cos\beta&0\\-2i\sin\beta&0&0&2\cos\beta\end{pmatrix} =$$

$$= \underline{\underline{\begin{pmatrix}\cos\beta&0&0&-i\sin\beta\\0&\cos\beta&-i\sin\beta&0\\0&-i\sin\beta&\cos\beta&0\\-i\sin\beta&0&0&\cos\beta\end{pmatrix}}}$$

$$XX(\beta)YY(\beta) = e^{-i\beta \sigma_x \otimes \sigma_x} e^{-i\beta \sigma_y \sigma_y}$$

$$XX(\beta)YY(\beta) = \begin{pmatrix} \cos\beta & 0 & 0 & -i\sin\beta \\ 0 & \cos\beta & -i\sin\beta & 0 \\ 0 & -i\sin\beta & \cos\beta & 0 \\ -i\sin\beta & 0 & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & 0 & i\sin\beta \\ 0 & \cos\beta & -i\sin\beta & 0 \\ 0 & -i\sin\beta & \cos\beta & 0 \\ i\sin\beta & 0 & 0 & \cos\beta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos^2\beta - i^2\sin^2\beta & 0 & 0 & \cos\beta\, i\sin\beta - i\sin\beta\cos\beta \\ 0 & \cos^2\beta + i^2\sin^2\beta & -\cos\beta\, i\sin\beta - i\sin\beta\cos\beta & 0 \\ 0 & -i\sin\beta\cos\beta - \cos\beta\, i\sin\beta & \cos^2\beta + i^2\sin^2\beta & 0 \\ -i\sin\beta\cos\beta + \cos\beta\, i\sin\beta & 0 & 0 & \cos^2\beta - i^2\sin^2\beta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2\beta - \sin^2\beta & -2i\cos\beta\sin\beta & 0 \\ 0 & -2i\cos\beta\sin\beta & \cos^2\beta - \sin^2\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\beta = 15^\circ$ $\Rightarrow$ $\boxed{\begin{array}{l} \cos 15^\circ = (\sqrt{6}+\sqrt{2})/4 \\ \sin 15^\circ = (\sqrt{6}-\sqrt{2})/4 \end{array}}$

- $\cos^2 15^\circ - \sin^2 15^\circ = \dfrac{(\sqrt{6}+\sqrt{2})^2}{16} - \dfrac{(\sqrt{6}-\sqrt{2})^2}{16} = \dfrac{1}{16}\left[6 + 2\sqrt{6}\sqrt{2} + 2 - (6 - 2\sqrt{6}\sqrt{2} + 2)\right] =$
  $= \dfrac{1}{16}\left(8 + 2\sqrt{6}\sqrt{2} - 8 + 2\sqrt{6}\sqrt{2}\right) = \dfrac{1}{16} 4\sqrt{6}\sqrt{2} = \dfrac{1}{4}\sqrt{12} = \dfrac{1}{4}(\pm 2\sqrt{3}) = \pm\dfrac{\sqrt{3}}{2} = \begin{cases} \dfrac{\sqrt{3}}{2} = \cos 30^\circ \\ -\dfrac{\sqrt{3}}{2} = \cos 150^\circ \end{cases}$
- $-2i\cos 15^\circ \sin 15^\circ = -2i\dfrac{1}{16}(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2}) = -\dfrac{i}{8}(6 - \sqrt{6}\sqrt{2} + \sqrt{2}\sqrt{6} - 2) = -\dfrac{i}{8}\cdot 4 = -i\dfrac{1}{2} = -i\sin 30^\circ$

$$\Rightarrow XX(15^\circ)YY(15^\circ) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 30^\circ & -i\sin 30^\circ & 0 \\ 0 & -i\sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

QISKIT:

$$R_x(2\cdot\theta) = \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix} = \left| \theta = 2\cdot\beta \right| = R_x(4\cdot\beta) = \begin{pmatrix} \cos 30^\circ & -i\sin 30^\circ \\ -i\sin 30^\circ & \cos 30^\circ \end{pmatrix}$$

IMPORTANT NOTES:

1) IN THE SOURCE CODE, RADIANS ARE USED INSTEAD OF DEGREES!

2) SIMILAR WAY WE ARE ABLE TO FIND THE SOLUTION FOR $\beta = 30^\circ$, WITH THE SAME RESULT: $R_x(4\beta)$.