1.

set $A\{\vec{a}\in R^k \mid p(0)=1\} = \{\vec{a}\in R^k \mid a_0=1\}$, If $\vec{x}_1,\vec{x}_2\in A \Rightarrow \theta\vec{x}+(1-\theta)\vec{y} = \begin{bmatrix}\theta x_0+(1-\theta)y_0\\ \vdots\\ \theta x_k+(1-\theta)y_k\end{bmatrix} = \begin{bmatrix}1\\ \theta x_1+(1-\theta)y_1\\ \vdots\\ \theta x_k+(1-\theta)x_k\end{bmatrix}$ for $\theta\in R \Rightarrow$ A is affine $\Rightarrow$ A is convex

set $B\{\vec{a}\in R^k \mid |p(t)|\le 1 \text{ fo } \alpha\le t\le\beta\} = \{\vec{a}\in R^k \mid \vec{t}^T\vec{a}\le 1,\ -\vec{t}^T\vec{a}\le -1,\ \vec{t}=\begin{bmatrix}1\\ t\\ \vdots\\ t^{k-1}\end{bmatrix},\ \alpha\le t\le\beta\}$

For a given $t$, $B_t$ is a polyhedra $\Rightarrow$ $B_t$ is convex with respect to $\vec{a}$ $\Rightarrow$ B is an intersection of infinite convex set $\Rightarrow$ B is convex with respect to $\vec{a}$

$\Rightarrow \{\vec{a}\in R^k \mid p(0)=1;\ |p(t)|\le 1 \text{ for } \alpha\le t\le\beta\},\ p(t)=a_1+a_2t+\ldots a_kt^{k-1} = A\cap B$ is an intersection of 2 convex set $\Rightarrow$ convex ※

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2. $\mathrm{dom} f = \{(\vec{x},t)\in R^n\times R \mid \|\vec{x}\|<t\}$ is a norm cone $\Rightarrow$ dom f is convex

$f(x,t) = -\log(t^2-\vec{x}^T\vec{x}) = -\log t - \log(t-\frac{\vec{x}^T\vec{x}}{t})$

$f_1(x,t) = -\log t$, $t>0$ is convex and nonincreasing.

$f_2(x,t) = t-\frac{\vec{x}^T\vec{x}}{t} = t+\sum_{i=1}^{n}\frac{-x_i^2}{t}$, $t>\|\vec{x}\|$   $t$ is concave, $\frac{x_i^2}{t}$ is quadratic-over-linear function

$\Rightarrow \frac{x_i^2}{t}$ is convex $\Rightarrow -\frac{x_i^2}{t}$ is concave

$\Rightarrow f_2(x,t)$ is sum of concave functions $\Rightarrow f_2(x,t)$ is concave, and $\because t>\|\vec{x}\|$, $f_2(x,t)>0$

$\Rightarrow -\log(t-\frac{\vec{x}^T\vec{x}}{t}) = f_1(f_2(x,t))$ is the composition of a convex and nondecreasing function $f_1$ with a concave function $f_2$ $\Rightarrow -\log(t-\frac{\vec{x}^T\vec{x}}{t})$ is convex

$\Rightarrow f(x,t)$ is sum of two convex function $\Rightarrow f(x,t)$ is convex ※

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PSD inside domain ?

3. $\mathrm{dom} f = \{ x \in R^2 \mid x_2 > 0,\ x_1 - \frac{1}{x_2} > 0 \}$. If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in f \Rightarrow x_2, y_2 > 0$. $x_1 x_2 > 1$, $y_1 y_2 > 1$

① $\Rightarrow$ For $\theta \in [0,1]$. $\theta x_2 + (1-\theta) y_2 > 0$

② $(\theta x_1 + (1-\theta) y_1)(\theta x_2 + (1-\theta) y_2) \geq \theta^2 x_1 x_2 + (1-\theta)^2 y_1 y_2 > \theta^2 + (1-\theta)^2 = \underbrace{2\theta^2 - 2\theta}_{\leq 0} + 1 \geq 1$

$\Rightarrow \theta x_1 + (1-\theta) y_1 - \dfrac{1}{\theta x_2 + (1-\theta) y_2} > 0$

$\Rightarrow \theta \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (1-\theta) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in f$, $\theta \in [0,1] \Rightarrow \mathrm{dom} f$ is convex

$f(\vec{x}) = \dfrac{1}{x_1 - \frac{1}{x_2}} = g(h(x_1, x_2))$. $g(x) = \frac{1}{x}$, $x > 0$, $h(\vec{x}) = x_1 - \frac{1}{x_2}$, $x_2 > 0$, $x_1 - \frac{1}{x_2} > 0$

$\frac{1}{x}$ is convex and nonincreasing for $x > 0$, $h(\vec{x}) = x_1 + (-\frac{1}{x_2})$ is sum of two concave function

$\Rightarrow f(\vec{x})$ is a composition of a convex and nondecreasing function $g(x) = \frac{1}{x}$ and a concave function $h(\vec{x}) = x_1 + (\frac{-1}{x_2})$

$\Rightarrow f(\vec{x})$ is a convex function ※

② consider $g: R^2 \to R$, $g(\vec{x}) = \dfrac{1}{x_1 - \frac{1}{x_2}}$, $x_2 > 0$, $x_1 - \frac{1}{x_2} > 0$, $h: R^4 \to R^2$, $h(\vec{x}) = \begin{bmatrix} x_1 \\ x_2 - \frac{1}{x_3 - \frac{1}{x_4}} \end{bmatrix}$, $x_4$, $x_3 - \frac{1}{x_4}$, $x_2 - \dfrac{1}{x_3 - \frac{1}{x_4}}$, $x_1 - \dfrac{1}{x_2 - \frac{1}{x_3 - x_4}} > 0$

$g(\vec{x})$ is convex by ①. Besides, $\dfrac{\partial g(\vec{x})}{\partial x_1} = -\dfrac{1}{(x_1 - \frac{1}{x_2})^2} < 0$, $\dfrac{\partial g(\vec{x})}{\partial x_2} = -\dfrac{1}{x_2^2 (x_1 - \frac{1}{x_2})^2} < 0 \Rightarrow g(\vec{x})$ is convex and nonincreasing in each argument.

$x_1$ is concave, $x_2 - \dfrac{1}{x_3 - \frac{1}{x_4}} = x_2 + \left(-\dfrac{1}{x_3 - \frac{1}{x_4}}\right)$ is sum of two concave function $\Rightarrow$ is concave $\Rightarrow h_1, h_2$ are concave

$\Rightarrow f = g \circ h = \dfrac{1}{x_1 - \frac{1}{x_2 - \frac{1}{x_3 - \frac{1}{x_4}}}}$, $x_4$, $x_3 - \frac{1}{x_4}$, $x_2 - \dfrac{1}{x_3 - \frac{1}{x_4}}$. $x_1 - \dfrac{1}{x_2 - \frac{1}{x_3 - \frac{1}{x_4}}} > 0$ is a composition of a convex and nonincreasing in each argument function $g(\vec{x})$ and a concave fn $h_1, h_2$ function $h(\vec{x}) \Rightarrow f$ is convex ※