# Weapons, Power, and Defense

A Look at Mathematical Models of Arms Races

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# 1 Introduction

The applications of nonlinear systems are diverse. Single equations like the logistic map can be used to model a myriad of scenarios from population growth to infection spread to economic growth. Similar equations can be used to explain social dynamics and interactions between peoples. A simple two-dimensional linear system can be interpreted as a measure of the love/hate that two people may hold for one another.[1] Restricting this idea to a less romantic example, armed combat between parties can be similarly idealized and described by easily understandable equations.[2] Relatively simple nonlinear systems can behave in such a way that can be intuitively interpreted to behaviour that would be expected in real-world situations.

Expanding these ideas to a more macroscopic scale, it is possible to consider the interactions between nations in a similar sense. Specifically, rivalries between nations and how this may manifest in an arms race, converge to a stable relation, or perhaps behave in a much more complicated manner. An arms race is defined as "a pattern of competitive acquisition of military capability between two or more countries."[3] Some historical examples[4] of arms races are the construction of dreadnought-class warships by Great Britain and Germany prior to the First World War, and the well-known Cold War; a nuclear arms race between the USA and the USSR. In contemporary history, the Arab League and Israel have been in a cycle of arms races, peace treaties and open hostilities; whereas India and Pakistan have experienced an arms race for over half a century that has transcended traditional weapons into the nuclear domain.[5]

# 2 Richardson Model

The classic example of arms race modelling is given by the Richardson model[6]

$$\begin{aligned} \dot{x} &= ky - ax + g \\ \dot{y} &= k'y - a'y + g' \end{aligned} \tag{1a}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -a & k \\ -a' & k' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} g \\ g' \end{bmatrix}, \quad \dot{\vec{\mathbf{x}}} = \mathbf{A} \cdot \vec{\mathbf{x}} + \vec{\mathbf{g}} \tag{1b}$$

Where the variables $x$ and $y$ represent the militaristic strength of countries X and Y, respectively. The exact meaning of military strength is ambiguous, however in most realistic cases it suffices to restrict $x, y > 0$.[7] These variables are generally considered to measure military expenditure. The parameters $k$ and $k'$ represent

[1] Strogatz, p. 139-141.

[2] Epstein, lecture 2.

[3] Perlo-Freeman.

[4] Mahnken et al., ch. 1, 6–8.

[5] Mahnken et al., ch. 9, 10.

[6] Richardson, p. 16.

[7] Richardson, p. 14, 19.

the reactionary response of each nation to their rival's strength; the stronger their enemy, the more they ought to spend to catch up. A country cannot spend all of its disposable income on arms, there are other expenses to cover. The higher $x$ is, the less willing country X is to spend even more on its military, similarly for country Y. Parameters $a$ and $a'$ control the strength of this friction term. Finally, $g$ and $g'$ were originally referred to as "grievances" and represented anything that provided a constant source of change in military spending in a nation. These sources may be ambitions, grudges, some ever-present external factor, etc. All parameters except $g$ and $g'$ were originally defined to be positive,[8] hence the minus sign in front of $a$ and $a'$.

Note that these parameters need not be positive. Negative values can still be interpreted e.g. $k < 0$ describes a system where country X spends less on its military the more its rival spends. This would be a scenario of appeasement; country X does not wish to aggravate or compete with country Y. Another example, if $a < 0$ there is a sort of positive feedback loop, the more country X spends on its military the more it will spend in the future. This can describe a country run by a military junta where the more is spent on the military the more power it consolidates thereby creating this loop. Lifting the restrictions on the parameters is advantageous as it allows the system to describe more qualitatively different behaviours.

Nonhomogeneous linear systems written in the form of eq. (1b) can be solved exactly[9] using the general solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\mathbf{A}t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \int_0^t e^{\mathbf{A}(t-s)} \begin{bmatrix} g \\ g' \end{bmatrix} ds \tag{2}$$

Where $x_0, y_0$ are determined by the initial conditions of the system. However, eq. (2) is not very enlightening. A more productive approach would be to analyze the stability conditions on eq. (1a).

$$0 = \dot{x} = ky - ax + g$$
$$0 = \dot{y} = k'y - a'y + g'$$

yields

$$\begin{aligned} x^* &= \frac{g'k + ga'}{aa' - kk'} \\ y^* &= \frac{gk' + g'a}{aa' - kk'} \end{aligned} \tag{3}$$

There is a fixed point so long as $aa' - kk' \neq 0$, this condition is to be expected as this value is simply the determinant of $\mathbf{A}$ in eq. (1b). Linearizing the system about $x^*, y^*$ yields as eigenvalues:

$$\lambda_{1,2} = \frac{-a - a' \pm \sqrt{(a + a')^2 - 4aa' + 4kk'}}{2}$$

After the final trivial step of finding the eigenvectors, there is sufficient information to sketch the phase portrait and determine the behaviour of countries X and Y.

[8] Richardson, p. 14-15.

[9] Howell, ch. 41, p. 12.

Another advantage of not restricting the parameters' signs are that the classic Richardson model can now incorporate the behaviour of a proposed alternative that took into consideration the size difference of the rival's military strength,[10]

$$\begin{aligned} \dot{x} &= k(y-x) - ax + g \\ \dot{y} &= k'(x-y) - a'y + g' \end{aligned} \tag{4}$$

Clearly, if we allow no restrictions on $a$, $k$, $g$, or their primed cousins, this nonhomogeneous linear system is equivalent to eq. (1a) written in the form:

$$\dot{x} = ky - (a+k)x + g$$

$$\dot{y} = k'x - (a'+k')y + g'$$

A special case in the Richardson system arises when $a = a'$ in eq. (4). There is an equivalent condition for this case in eq. (1a), however proceeding with the formalism in eq. (4) is much more straightforward. Let

$$\delta = x - y$$

$$\dot{\delta} = \dot{x} - \dot{y} = k(y-x) - ax + g - k'(x-y) + a'y - g'$$

recalling that $a = a'$

$$\dot{\delta} = -\delta(k + a + k') + g - g'$$

This differential equation can be solved exactly giving

$$|(k + k' + a)\delta - g + g'| = Ce^{-(k+k'+a)t}$$

$$x - y = \delta \longrightarrow \frac{g - g'}{k + k' + a} \quad as \quad t \longrightarrow \infty$$

The rivals' military spending will approach the line $x = y + \frac{g-g'}{k+k'+a}$ in the long time limit under the conditions $a = a'$ and $k + k' + a > 0$.

## 3 Alternative Models

The Richardson model of arms races is an easily understandable simplification of a complex process. However with simplification comes inaccuracy. There are several glaring issues:

- The linear nature of eq. (1a) allows for scenarios that predict infinite growth in both nations' military strength.[11] Similarly, these variables can take on negative values. Assuming $x$ and $y$ measure military expenditure or nuclear weapons stockpile these results are nonsensical.

[10] Caspary, p. 69.
[11] Hill, p. 203.

- The original restriction of parameters $a$, $a'$, $k$, and $k'$ to the strictly positive domain limited the dynamic behaviour of eq. (1a) to stable, unstable, and saddle nodes.[12] The current treatment of this system removes this restriction but does not consider a scenario with multiple fixed points.

Several alternatives have been proposed to solve these problems while still sticking to the spirit of Richardson's classic model.[13]

### 3.1 Richardson Model with Carrying Capacity

The first major issue of unlimited growth in $x$ and $y$ can be solved by adding a carrying capacity term to eq. (1a) in a manner identical to that of the logistic growth model:[14]

$$\begin{aligned} \dot{x} &= (1 - \frac{x}{x_{max}})(ky - ax + g) \\ \dot{y} &= (1 - \frac{y}{y_{max}})(k'x - a'y + g') \end{aligned} \tag{5}$$

Where $x_{max}$ and $y_{max}$ are the maximum values that countries X and Y are willing to spend on their military readiness. This model also addresses the second issue in that there are four fixed points of eq. (5)

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} \frac{g'k+ga'}{aa'-kk'} \\ \frac{gk'+g'a}{aa'-kk'} \end{pmatrix}, \begin{pmatrix} \frac{g'k+ga'}{aa'-kk'} \\ y_{max} \end{pmatrix}, \begin{pmatrix} x_{max} \\ \frac{gk'+g'a}{aa'-kk'} \end{pmatrix}, \begin{pmatrix} x_{max} \\ y_{max} \end{pmatrix} \tag{6}$$

The stability of this last fixed point is of special interest as it will determine whether countries X and Y will continuously increase their military expenditure as much as possible; this is exactly the condition for an arms race. Computing the linear stability of $(x_{max},\ y_{max})$ using the Jacobian of eq. (5) yields

$$\begin{aligned} \lambda_1 &= a - \frac{g}{x_{max}} - \frac{ky_{max}}{x_{max}},\ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \lambda_2 &= a' - \frac{g'}{y_{max}} - \frac{k'x_{max}}{y_{max}},\ v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned} \tag{7}$$

as eigenvalues and eigenvectors.

### 3.2 Economic Constraints

While eq. (5) places a hard cap on $x$ and $y$, it is still conceivable that a nation may spend more than this model predicts it should. Nonetheless this increased military expenditure strains the economy and ought to

[12] Richardson, p. 37-41.
[13] Caspary; Lehmann, McEwen, and Lane; Hill.
[14] Lehmann, McEwen, and Lane, p. 2.

be reflected in any model. One such model uses exponential constraints to limit the growth of $x$ and $y$[15]

$$\begin{aligned} \dot{x} &= \lambda(1 - \exp(\frac{ax}{\lambda})) + ky + g \\ \dot{y} &= \lambda'(1 - \exp(\frac{a'y}{\lambda'})) + k'x + g' \end{aligned} \tag{8}$$

It is not possible for $x$ and $y$ to grow to infinity since for large enough values the exponential in eq. (8) will overtake the linear terms making $\dot{x}$ and $\dot{y}$ negative. This prevents the infinite growth found in the Richardson equations. Note that for small values of $x$ and $y$ Taylor expanding eq. (8) and keeping the leading order terms returns exactly eq. (1a).[16]

# 4 Fitting Models

As with all models, it is important to consider the assumptions made in their derivation and application.

**Assumption 4.1.** Military strength is directly proportional to military expenditure. Richardson initially described his model with the dependent variables representing defenses.[17] While this may not seem like an unreasonable assumption, the detail is in the linear relation between these two. There may be many other factors that effect a nation's military strength/defensive capabilities such as geography, war fatigue, foreign aid, etc., herein lies the assumption.

**Assumption 4.2.** Two nations in an arms race are only in an arms race with one another. The models described so far only consider the interactions between two countries X and Y. They assume that the change in military expenditure year-to-year is predominantly affected by their interactions.

**Assumption 4.3.** The proposed models operate with only three terms: reactions to foreign military expenditure, economic constraints, and grievances. Other factors such as internal elections, migration, major world events, etc. are not considered.

## 4.1 India-Pakistan

In order to fit the model to the data in fig. 1[18] it is first required to find India's and Pakistan's change in spending year-to-year from 1960 to 2020. This can be achieved by taking a rolling average of the data to reduce the amount of noise then using the secant line method. A window of five years was chosen for the rolling average as this is the length of the election cycles in both India and Pakistan. Once $\dot{x}$ and $\dot{y}$ have been derived, MATLAB is used to fit the data to the models described by eq. (1a), eq. (5), and eq. (8).

[15] Hill, p. 203.
[16] Hill, p. 203.
[17] Richardson, p. 14.
[18] "Military expenditure (current USD)".

Military expenditure can be measured in terms of absolute dollars spent or as a percentage of gross domestic product. One advantage of considering the dollar amount is that it is possible to see, after adjusting for inflation, whether or not the size of a nation's military is growing. However, it is expected that a nation's military grows in tandem with its demographic and economic growth. A nation's spending as a percentage of GDP will indicate the importance that it places on the military.

Figure 1: India's and Pakistan's military expenditure by year in absolute terms and as a percent of GDP.

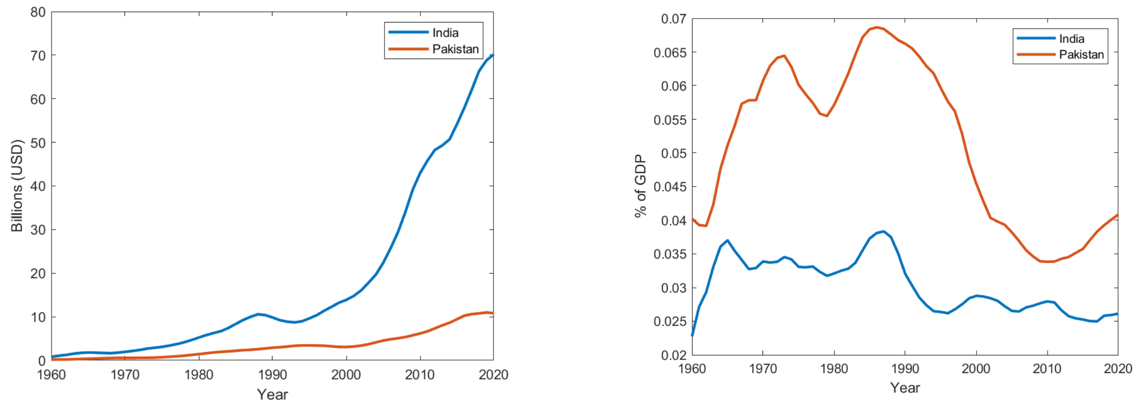
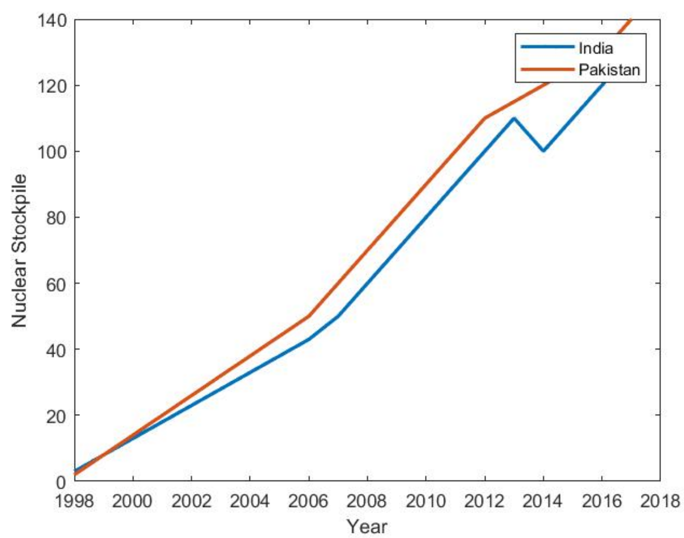



Figure 2: India's and Pakistan's nuclear stockpile by year.




## 4.2 Nuclear Weapons

It is possible to apply the exact same procedure bar the rolling average to India's and Pakistan's nuclear arms race. The variables $x$ and $y$ would now represent the amount of nuclear warheads in each countries' possession.[19] This relaxes assumption 4.1 and assumption 4.2. A countries nuclear capabilities are much more closely correlated to the number of nuclear warheads it has access to. Note that this relation is still not linear as there are always extra factors to consider such as delivery methods, bomb yield, target's missile defense capabilities, etc. Furthermore, the decision made by India and Pakistan to pursue nuclear weapons was largely as a result of competition between solely these two nations.[20]

# 5 Results

The Richardson, carrying capacity, and economic constraints models predict qualitatively different behaviour when fitted to the data on military spending in dollar values and to that on nuclear weapons. However, all three models are in agreement on the long-term behaviour of India's and Pakistan's military spending as a percent of their GDP.

## 5.1 Dollar-Value Spending Results

The Richardson model in fig. 3 exhibits the previously mentioned infinite growth behaviour. As a result, this model cannot be used to predict India's and Pakistan's future military spending. Even so, it does indicate that qualitatively these two nations are indeed in an arms race with one another. The fixed point of this system is a saddle node therefore it is possible for the spending to decrease under different initial conditions. However this would then lead to negative values, which is a breakdown in the application of the Richardson model to this data.

Table 1: Predicted behaviour of India's and Pakistan's military spending in billions USD. The timescale column gives an estimate for the number of years it would take to reach the long-term behaviour starting from 2020. An asterisk denotes predicted stable behaviour; the system will asymptotically approach this limit.

| | Long-Term Behaviour | | |
|---|---|---|---|
| Model | India (Billions USD) | Pakistan (Billions USD) | Timescale (Years) |
| Richardson | $\infty$ | $\infty$ | $t \to \infty$ |
| Carrying Capacity | 0 | 42.5 | 142 |
| Economic Constraints | 72.1 | 11 | 0* |

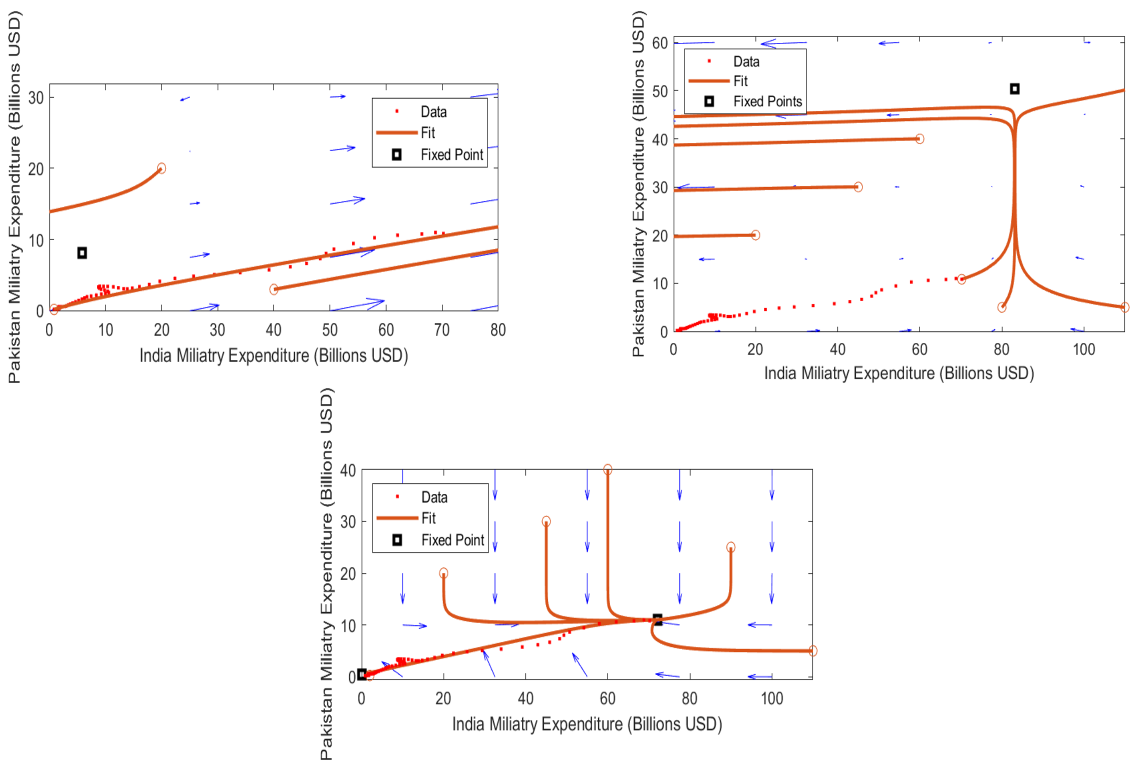
The carrying capacity model suffers from the same problem. This model predicts that the trajectory of

[19] Roser and Nagdy.

[20] Jalil.

India's and Pakistan's military spending will follow the line $x = x_{max}$, then turn at a saddle point to follow $y = y_{max}$ before finally reaching the $y$-axis. At this point the model also breaks down. Qualitatively, this system predicts that India and Pakistan will both be willing to spend as much money as possible on their military, as one would in an arms race, before lessening their expenses as the strain on their economy grows too large.

Figure 3: Phase portrait and predicted behavior of India's and Pakistan's military expenditure in billions USD. Data points are plotted in red and predicted trajectories in orange. The initial conditions of trajectories are given by orange circles. The arrows indicate the direction of flow at a given point. Models in order (left-right-down): Richardson (eq. (1a)), carrying capacity (eq. (5)), economic constraints (eq. (8)).




Finally, the economic constraints model produces entirely different results; it predicts that the system is already near a stable equilibrium. India and Pakistan will not spend any more or less money (after adjusting for inflation) on their military as the years progress.

## 5.2 %GDP Spending Results

All three models predict that India and Pakistan will tend towards a stable limit in their military spending at around 3% and 5.5% respectively (see fig. 4 and table 2). This indicates that these countries are not in an arms race; the increase in their spending can be attributed to their economic growth rather than hostile or defensive intentions.

Table 2: Predicted behaviour of India's and Pakistan's military spending as a percentage of GDP (1 indicating 100%). The timescale column gives an estimate for the number of years it would take to reach the long-term behaviour starting from 2020. An asterisk denotes predicted stable behaviour; the system will asymptotically approach this limit.

| | Long-Term Behaviour | | |
|---|---|---|---|
| Model | India (% GDP) | Pakistan (% GDP) | Timescale (Years) |
| Richardson | 0.0309 | 0.0536 | $t \to \infty^*$ |
| Carrying Capacity | 0.0314 | 0.0560 | $t \to \infty^*$ |
| Economic Constraints | 0.0336 | 0.0646 | $t \to \infty^*$ |

The carrying capacity can predict an arms race in a fashion similar to what is predicted by the carrying capacity fitted to military expenditure given in terms of absolute spending. However, this would require wildly different initial conditions on the data. It is also worth noting that the economic constraints model predicts another stable fixed point in the positive domain found at 8.7133% and 50.922%. This is an unrealistically high spending rate for Pakistan and so is not a useful prediction.

## 5.3 Nuclear Weapons Results

Finally, fitting all three models to India's and Pakistan's nuclear arsenal produces the most interesting results. All three models predict some sort of arms race before long-term stability. So while these countries are increasing their nuclear capabilities year-by-year, the momentum that drives this change is slowing down in recent years. The Richardson model demonstrates this quite clearly, both India and Pakistan will build more nuclear weapons before mutually lessening their stockpiles. The system described by the first graph in fig. 5 is actually an unstable spiral circling around a fixed point outside of the domain. However, this instability is not a factor in interpreting the results since, as previously discussed, the model breaks down at the $x$-axis.

The carrying capacity model predicts local stability given the initial data. The fixed point given by $(x_{max} \approx 152,\ y_{max} \approx 165)$ is a stable node. Both India and Pakistan are willing to spend as much resources as their economies will allow on their nuclear programs. This is exactly the requirement that the carrying capacity model sets for an arms race. The presence of a saddle means that for slightly different initial conditions the trajectories will diverge to a state of disarmament for India while Pakistan continues to add to its weapons stockpile.

Figure 4: Phase portrait and predicted behavior of India's and Pakistan's military expenditure as a percent of GDP (1 indicating 100%). Data points are plotted in red and predicted trajectories in orange. The initial conditions of trajectories are given by orange circles. The arrows indicate the direction of flow at a given point. Models in order (left-right-down): Richardson (eq. (1a)), carrying capacity (eq. (5)), economic constraints (eq. (8)).

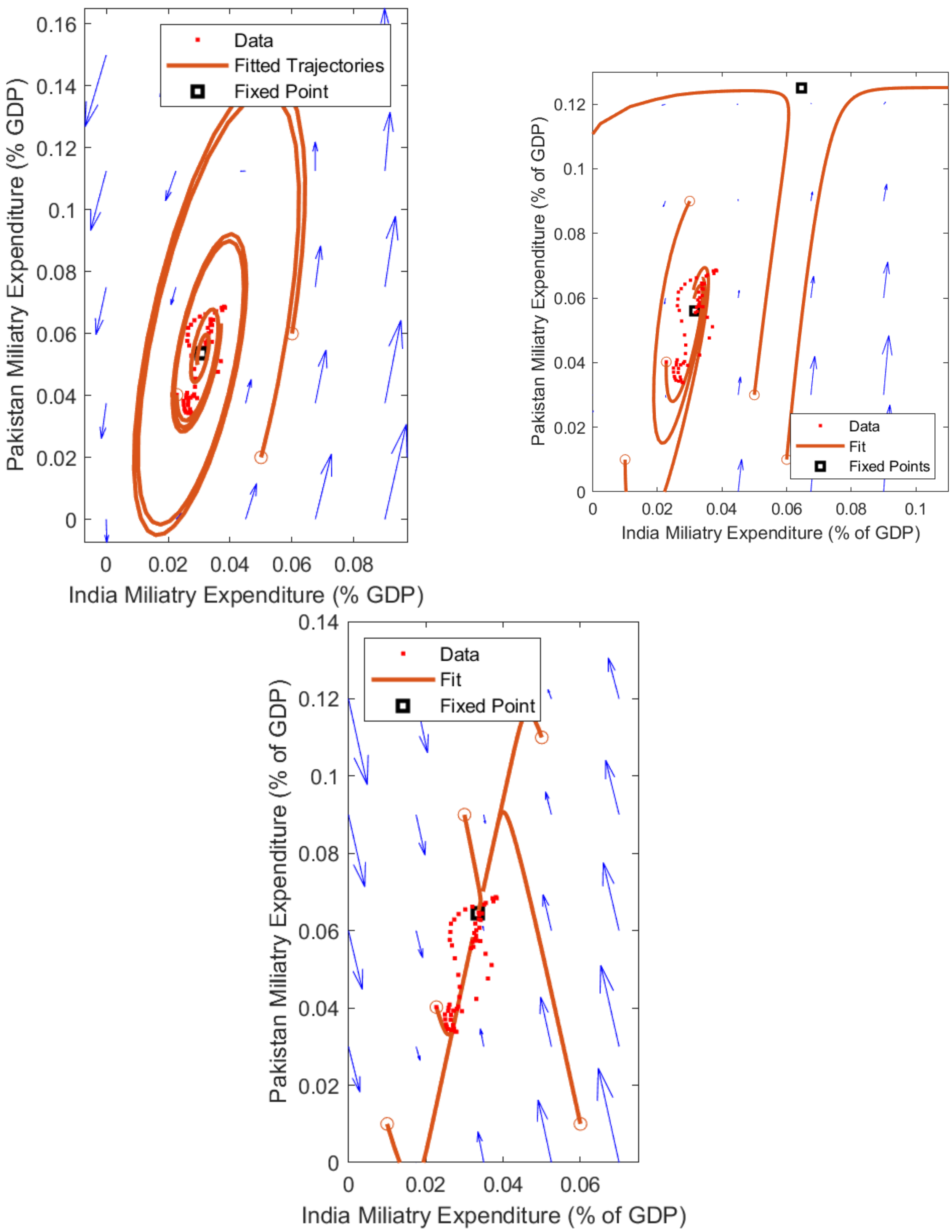


Figure 5: Phase portrait and predicted behavior of India's and Pakistan's nuclear arsenal. Data points are plotted in red and predicted trajectories in orange. The initial conditions of trajectories are given by orange circles. The arrows indicate the direction of flow at a given point. Models in order (left-right-down): Richardson (eq. (1a)), carrying capacity (eq. (5)), economic constraints (eq. (8)).

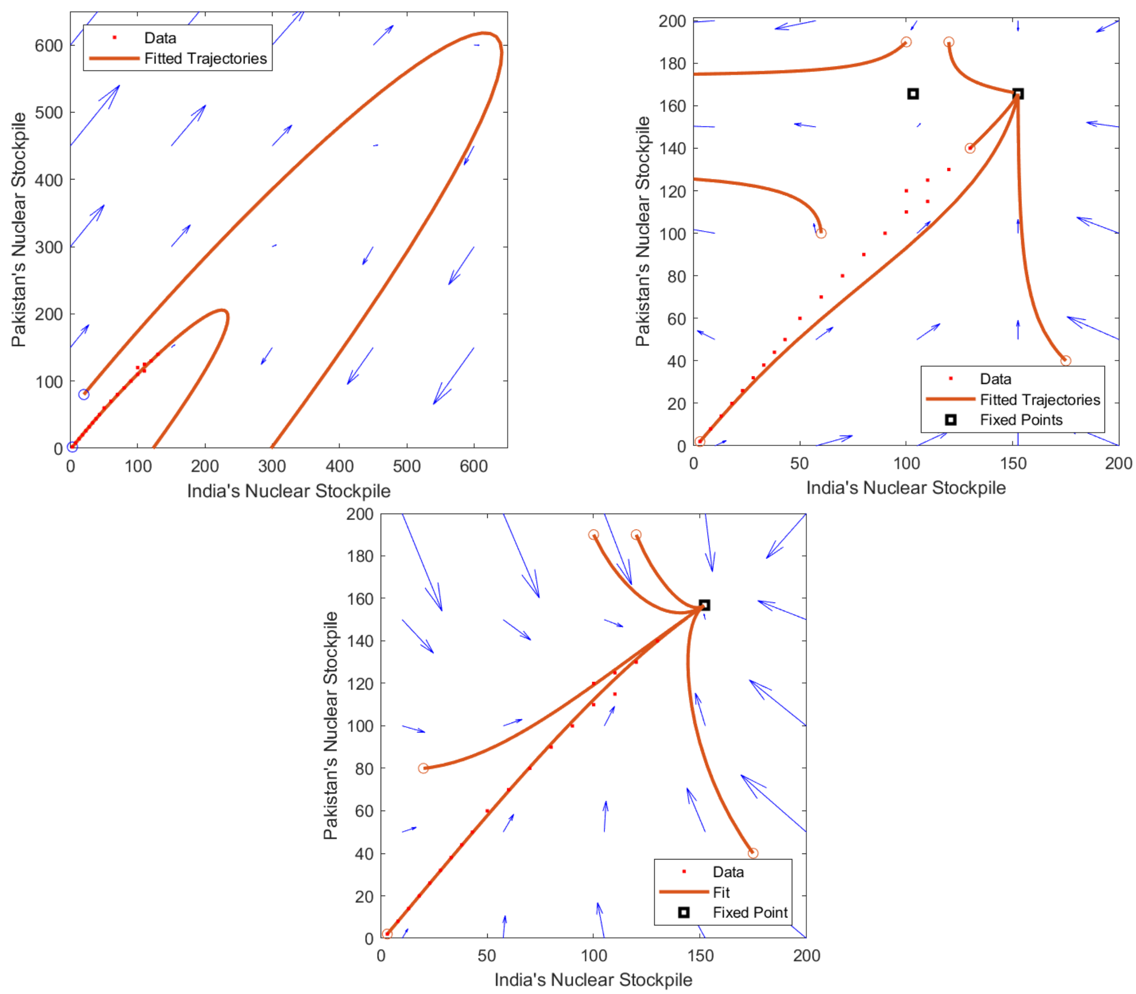


Table 3: Predicted behaviour of India's and Pakistan's nuclear arsenal. The timescale column gives an estimate for the number of years it would take to reach the long-term behaviour starting from 2020. An asterisk denotes predicted stable behaviour; the system will asymptotically approach this limit however since nuclear weapons can only be counted in whole numbers the system will reach a stable value in finite time.

| | Long-Term Behaviour | | |
|---|---|---|---|
| Model | India | Pakistan | Timescale (Years) |
| Richardson | 122 | 0 | 60 |
| Carrying Capacity | 152 | 165 | 12* |
| Economic Constraints | 152 | 156 | 13* |

The economic constraints model predicts stable behaviour across the entire positive domain. This is a scenario where the balance of power has been equalized. Both India and Pakistan are content with their nuclear capabilities relative to the other as a return on their investment. Qualitatively, this is in close agreement with the results from the carrying capacity model although their interpretations differ.

## 6 Conclusion

A major limitation in all proposed models is the restriction of the variables to the positive domain. It is at the $x$ and $y$ axes that the models and the assumptions that $x$ and $y$ represent military spending break down. Caution must be used when interpreting results whose trajectories are on or near the axes. Despite this, all proposed models qualitatively describe trends in data even if their predictions on future behaviour and interpretations differ.

Furthermore, assumption 4.2 may also be relaxed by taking a generalization of the Richardson model,

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dots \\ \dot{x_n} \end{bmatrix} = \begin{bmatrix} -a_1 & k_{1,2} & \dots & k_{1,n-1} & k_{1,n} \\ k_{2,1} & -a_2 & \dots & k_{2,n-1} & k_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ k_{n,1} & k_{n,2} & \dots & k_{n,n-1} & -a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ \dots \\ g_n \end{bmatrix} \tag{9}$$

Such a model in again linear. Finding the parameters of eq. (9) would follow from a simple application of linear regression to military spending data. The procedure for finding $\dot{x_i}$ would again be the secant line method, but choosing the window size for the rolling average would be a nontrivial question. A similar extension can be applied to the carrying capacity model and the economic constraints model.

Arms races are complex international phenomena that arise from a multitude of events. However, people are not powerless, destined to follow these fitted trajectories. The actions of nations always follow from the decisions of individuals; whether they be members of government or private citizens.

## References


Caspary, William R. "Richardson's Model of Arms Races: Description, Critique, and an Alternative Model". In: *International Studies Quarterly* 11 (1967), pp. 63–88. DOI: https://doi.org/10.2307/3013990.

Epstein, Joshua M. *Nonlinear Dynamics, Mathematical Biology, and Social Science*. CRC Press, 1997, p. 180. DOI: https://doi-org.proxy3.library.mcgill.ca/10.1201/9780429493409.

Hill, Walter H. "Several sequential augmentations of Richardson's arms race model". In: *Mathematical and Computer Modelling* 16 (1992), pp. 201–212. DOI: https://doi.org/10.1016/0895-7177(92)90096-4.

Howell, Kenneth. *Nonhomogeneous Linear Systems*. 2016, Ch. 41. URL: http://howellkb.uah.edu/DEtext/Additional_Chapters/index.html.

Jalil, Ghazala Yasmin. "Nuclear Arms Race in South Asia: Pakistan's Quest for Security". In: *Strategic Studies* 37 (2017), pp. 18–41. URL: https://www.jstor.org/stable/48535985.

Lehmann, Brian, John McEwen, and Brian Lane. *Modifying the Richardson Arms Race Model With a Carrying Capacity*. URL: http://plaza.ufl.edu/blane116/modifying_the_richardson_arms_race_model.pdf.

Mahnken, Thomas et al. *Arms Races in International Politics: From the Nineteenth to the Twenty-First Century*. 2016. DOI: 10.1093/acprof:oso/9780198735267.001.0001.

*Military expenditure (current USD)*. The World Bank. URL: https://data.worldbank.org/indicator/MS.MIL.XPND.CD.

Perlo-Freeman, Sam. *arms race*. In: Encyclopædia Britannica, 2020. URL: https://www.britannica.com/topic/arms-race.

Richardson, Lewis F. *Arms and Insecurity*. The Boxwood Press, 1960.

Roser, Max and Mohamed Nagdy. *Nuclear Weapons*. 2013. URL: https://ourworldindata.org/nuclear-weapons.

Strogatz, Steven H. *Nonlinear Dynamics and Chaos. With Applications to Physics, Biology, Chemistry, and Engineering*. 2nd ed. 2015.


## Appendix

The MATLAB code used to fit models and generate plots can be found at https://github.com/WillPugs/arms-race.