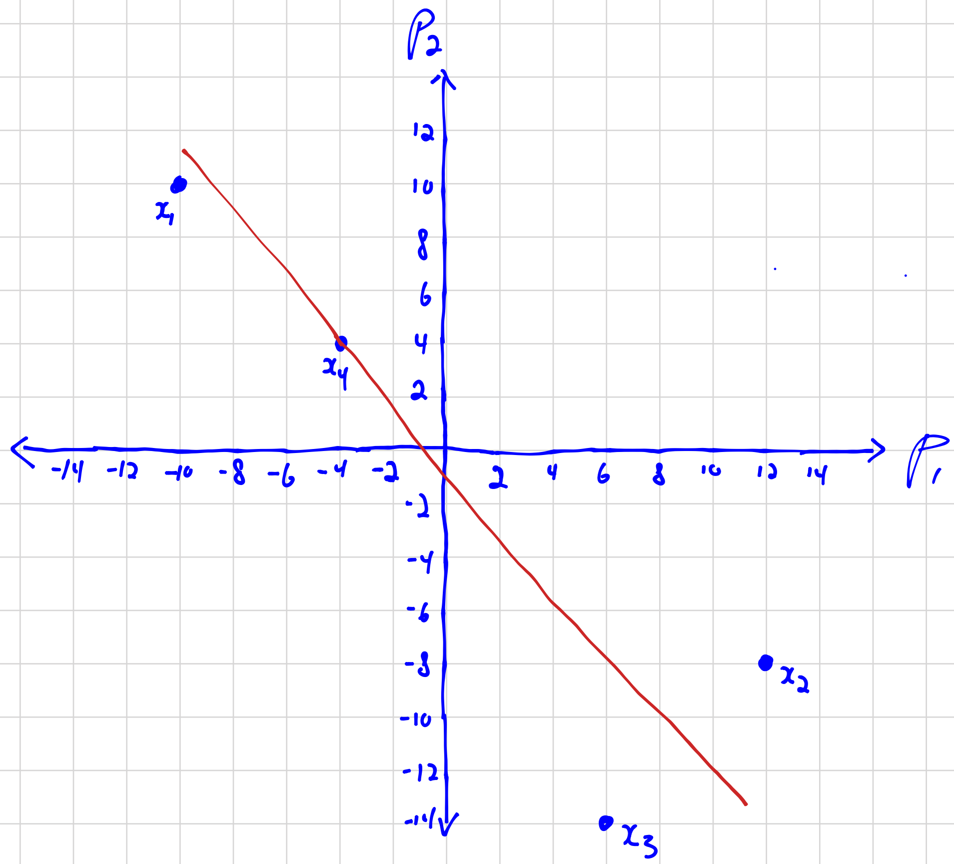
DATA 558 - HOMEWORK 4

a.) PLOT THE DATA:

$$X = \begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \end{array} & \begin{bmatrix} -10 & 12 & 6 & -4 \\ 10 & -8 & -14 & 4 \end{bmatrix} \end{array}$$



b.) DRAW AN ESTIMATE FOR THE SUBSPACE CLOSEST TO THE DATA

(SEE RED LINE)

ESTIMATE THE SLOPE AND INTERCEPT:

SLOPE $\approx -5/4$

INTERCEPT ≈ -1

c.) CENTER THE DATA

$$\bar{P}_1 = \frac{1}{4} \cdot (-10 + 12 + 6 + (-4)) = 1$$

$$\bar{P}_2 = \frac{1}{4} \cdot (10 + (-8) + (-14) + 4) = -2$$

SUBTRACT $\bar{P}_1$ AND $\bar{P}_2$ FROM THEIR RESPECTIVE ROWS OF X:

$$X^{CENTERED} = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} -11 & 11 & 5 & -5 \\ 12 & -6 & -12 & 6 \end{bmatrix} \end{array}$$

I'LL REFER TO $X^{CENTERED}$ AS SIMPLY X MOVING FORWARDS.

d.) COMPUTE THE EMPIRICAL COVARIANCE MATRIX.

$$\hat{S} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^T$$

$$= \frac{1}{4}\left(\begin{bmatrix}-11\\12\end{bmatrix}\begin{bmatrix}-11 & 12\end{bmatrix} + \begin{bmatrix}11\\-6\end{bmatrix}\begin{bmatrix}11 & -6\end{bmatrix} + \begin{bmatrix}5\\-12\end{bmatrix}\begin{bmatrix}5 & -12\end{bmatrix} + \begin{bmatrix}-5\\6\end{bmatrix}\begin{bmatrix}-5 & 6\end{bmatrix}\right)$$

$$= \frac{1}{4}\left(\begin{bmatrix}121 & -132\\-132 & 144\end{bmatrix} + \begin{bmatrix}121 & -66\\-66 & 36\end{bmatrix} + \begin{bmatrix}25 & -60\\-60 & 144\end{bmatrix} + \begin{bmatrix}25 & -30\\-30 & 36\end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix}292 & -288\\-288 & 360\end{bmatrix}$$

$$\hat{S} = \begin{bmatrix}73 & -72\\-72 & 90\end{bmatrix}$$

e) COMPUTE THE EIGENVALUES λ_1, λ_2 OF THE COVARIANCE MATRIX.

$$\det(\lambda I_2 - \hat{S}) = 0$$

$$= \det\left(\lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}73 & -72\\-72 & 90\end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix} - \begin{bmatrix}73 & -72\\-72 & 90\end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix}\lambda-73 & 72\\72 & \lambda-90\end{bmatrix}\right)$$

$$= (\lambda-73)(\lambda-90) - 5184 = \lambda^2 - 90\lambda - 73\lambda + 6570 - 5184$$

$$= \lambda^2 - 163\lambda + 1386 = 0$$

USE QUADRATIC FORMULA TO SOLVE

$$\frac{163 \pm \sqrt{26569 - 5544}}{2}$$

$$= \frac{163 \pm 145}{2} \quad \frac{163+145}{2} = 154 \quad \frac{163-145}{2} = 9$$

$$\lambda_1 = 154 \quad \lambda_2 = 9$$

f) COMPUTE THE EIGENVECTORS V_1, V_2 OF THE EMPIRICAL COVARIANCE MATRIX.

REVIEW: $A\vec{v} = \lambda\vec{v} \Rightarrow \vec{0} = \lambda\vec{v} - A\vec{v} \Rightarrow \vec{0} = \lambda I_n\vec{v} - A\vec{v} \Rightarrow \vec{0} = (\lambda I_n - A)\vec{v}$

$$E_\lambda = N(\lambda I_n - A)$$

EIGENSPACE NULL SPACE

For $\lambda = 9$:

$$E_9 = N\left(\begin{bmatrix}9 & 0\\0 & 9\end{bmatrix} - \begin{bmatrix}73 & -72\\-72 & 90\end{bmatrix}\right) = N\left(\begin{bmatrix}-64 & 72\\72 & -81\end{bmatrix}\right)$$

Find reduced echelon form:

$$\begin{bmatrix}-64 & 72\\72 & -81\end{bmatrix}\begin{matrix}\div -8\\ \div 9\end{matrix} \Rightarrow \begin{bmatrix}8 & -9\\8 & -9\end{bmatrix}\begin{matrix}\\ -p_1\end{matrix} \Rightarrow \begin{bmatrix}8 & -9\\0 & 0\end{bmatrix}\div 8 \Rightarrow \begin{bmatrix}1 & -9/8\\0 & 0\end{bmatrix}$$

So:

$$\begin{bmatrix}1 & -9/8\\0 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \qquad v_1 + (-9/8\, v_2) = 0 \qquad v_1 = 9/8\, v_2$$

$$E_9 = \left\{\begin{bmatrix}v_1\\v_2\end{bmatrix} = t\begin{bmatrix}9/8\\1\end{bmatrix},\ t \in \mathbb{R}\right\} = \operatorname{SPAN}\left(\begin{bmatrix}9/8\\1\end{bmatrix}\right)$$

For $\lambda = 154$:

$$E_{154} = N\left(\begin{bmatrix}154 & 0\\0 & 154\end{bmatrix} - \begin{bmatrix}73 & -72\\-72 & 90\end{bmatrix}\right) = N\left(\begin{bmatrix}81 & 72\\72 & 64\end{bmatrix}\right)$$

Reduced echelon form:

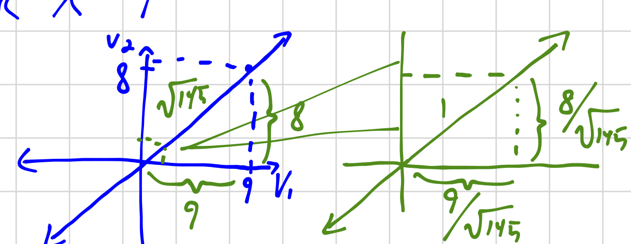
$$\begin{bmatrix}81 & 72\\72 & 64\end{bmatrix}\begin{matrix}\div 9\\ \div 8\end{matrix} \Rightarrow \begin{bmatrix}9 & 8\\9 & 8\end{bmatrix}\begin{matrix}\\ -p_1\end{matrix} \Rightarrow \begin{bmatrix}9 & 8\\0 & 0\end{bmatrix}\div 9 \Rightarrow \begin{bmatrix}1 & 8/9\\0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 8/9\\0 & 0\end{bmatrix}\vec{v} = \vec{0} \qquad v_1 + 8/9\, v_2 = 0 \qquad v_1 = -8/9\, v_2$$

$$E_{154} = \left\{\begin{bmatrix}v_1\\v_2\end{bmatrix} = t\begin{bmatrix}-8/9\\1\end{bmatrix},\ t \in \mathbb{R}\right\} = \operatorname{SPAN}\left(\begin{bmatrix}-8/9\\1\end{bmatrix}\right)$$

Convert to unit-vectors:

For $\lambda = 9$



$\sqrt{9^2+8^2} = \sqrt{145}$ UNIT VECTOR FOR E_9 IS $\begin{bmatrix}9/\sqrt{145}\\8/\sqrt{145}\end{bmatrix}$

SIMILARLY, FOR $\lambda = 154$, THE EIGENVECTOR IS $\begin{bmatrix}-8/\sqrt{145}\\9/\sqrt{145}\end{bmatrix}$

g.) COMPUTE THE PCA PROJECTIONS OF THE POINTS ONTO A LINE:

STARTING WITH $\lambda = 154$, WE HAVE $W = \begin{bmatrix}-8\\9\end{bmatrix}\cdot\frac{1}{\sqrt{145}}$

KNOWING $\operatorname{Proj}_W(x_j) = w^T x_j$, WE COMPUTE:

$$x_1' = \mathrm{Proj}_W(x_1) = \frac{1}{\sqrt{145}}\begin{bmatrix}-8 & 9\end{bmatrix}\begin{bmatrix}-10\\10\end{bmatrix} = \frac{1}{\sqrt{145}}\left(-8(-10)+9\cdot 10\right) = 14.12$$

$$x_2' = \mathrm{Proj}_W(x_2) = \frac{1}{\sqrt{145}}\begin{bmatrix}-8 & 9\end{bmatrix}\begin{bmatrix}12\\-8\end{bmatrix} = \frac{1}{\sqrt{145}}\left(-8(12)+9(-8)\right) = -13.95$$

$$x_3' = \mathrm{Proj}_W(x_3) = \frac{1}{\sqrt{145}}\begin{bmatrix}-8 & 9\end{bmatrix}\begin{bmatrix}6\\-14\end{bmatrix} = \frac{1}{\sqrt{145}}\left(-8(6)+9(-14)\right) = -14.45$$

$$x_4' = \mathrm{Proj}_W(x_4) = \frac{1}{\sqrt{145}}\begin{bmatrix}-8 & 9\end{bmatrix}\begin{bmatrix}-4\\4\end{bmatrix} = \frac{1}{\sqrt{145}}\left(-8(-4)+9\cdot 4\right) = 5.65$$

h.) TRANSFORM THE PROJECTED POINTS BACK TO THE ORIGINAL SPACE USING THE TOP PRINCIPAL COMPONENT AND MEAN VECTOR FROM PART (b.)

$$\hat{X} = W \cdot X_{proj}^T + \mu$$

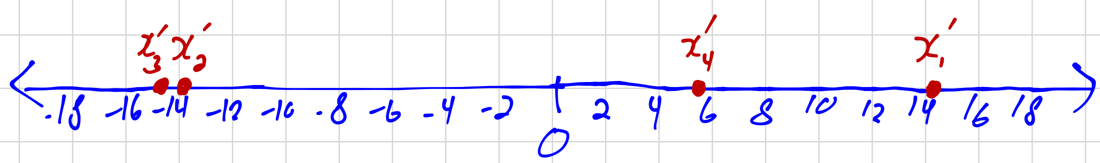
$$= \frac{1}{\sqrt{145}}\begin{bmatrix}-8\\9\end{bmatrix}\begin{bmatrix}14.12 & -13.95 & -14.45 & 5.65\end{bmatrix} + \begin{bmatrix}1\\-2\end{bmatrix}$$

$$= \frac{1}{\sqrt{145}}\begin{bmatrix}-122.96 & 111.61 & 115.60 & 45.18\\ 127.08 & -122.56 & -130.05 & 50.82\end{bmatrix} + \begin{bmatrix}1\\-2\end{bmatrix}$$

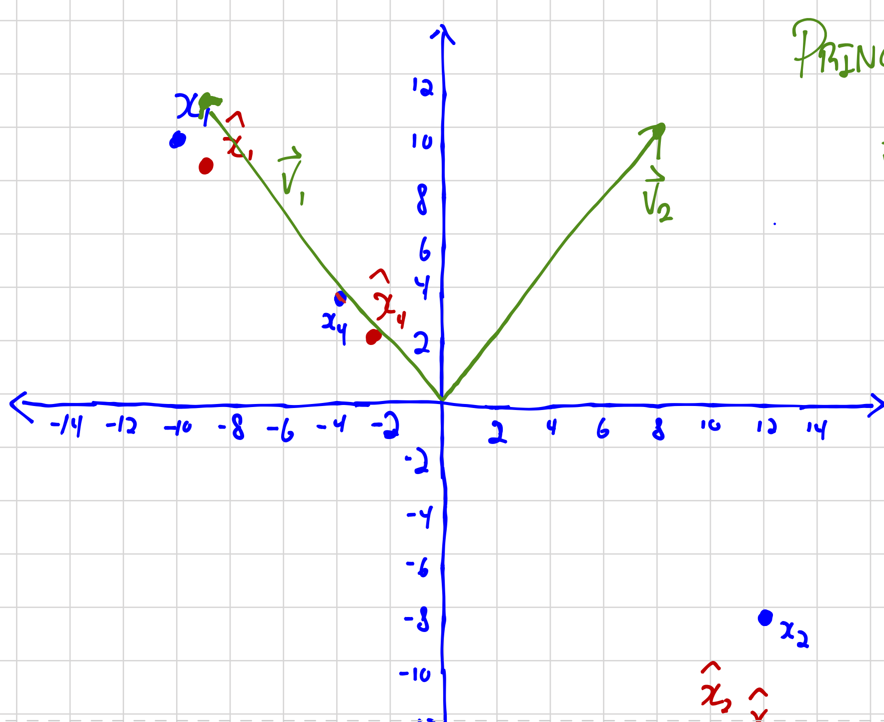
$$= \begin{bmatrix}-9.21 & 10.26 & 10.60 & -2.75\\ 8.55 & -12.42 & -12.80 & 2.22\end{bmatrix}$$

$$\hat{x}_1 = \begin{bmatrix}-9.21\\8.55\end{bmatrix} \quad \hat{x}_2 = \begin{bmatrix}10.26\\-12.42\end{bmatrix} \quad \hat{x}_3 = \begin{bmatrix}10.60\\-12.80\end{bmatrix} \quad \hat{x}_4 = \begin{bmatrix}-2.75\\2.22\end{bmatrix}$$

i.) PLOT x_1', x_2', x_3', x_4'



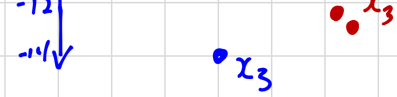
j.) PLOT THE FOLLOWING QUANTITIES: THE ORIGINAL POINTS, THE PRINCIPAL COMPONENTS V_1, V_2 SHIFTED BY THE MEAN μ OF THE ORIGINAL DATA, AND THE RECONSTRUCTED POINTS. DOES THIS LOOK REASONABLE?



PRINCIPAL COMPONENTS SHIFTED BY MEAN:

$$\vec{v}_1 - \mu = \begin{bmatrix}-8\\9\end{bmatrix} - \begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}-9\\11\end{bmatrix}$$

$$\vec{v}_2 - \mu = \begin{bmatrix}9\\8\end{bmatrix} - \begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}8\\10\end{bmatrix}$$



THIS SEEMS MOSTLY REASONABLE WITH SOME LATITUDE FOR MY LINE-DRAWING ABILITY.