Mathematical analysis 2
Chapter 1: Multivariable and vectorial functions
Part 1: Elements of topology

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Metric

Definition

Let E be a non empty vectorial space on $\mathbb{R}$.

1. *A **metric** (distance) on E is a mapping $d : E \times E \to \mathbb{R}^+$ satisfying*
   - $\forall x, y \in E$: $d(x,y) = 0 \Longleftrightarrow x = y$.
   - $\forall x, y \in E$: $d(x,y) = d(y,x)$. *(symmetrically)*
   - $\forall x, y, z \in E$: $d(x,y) \leq d(x,z) + d(z,y)$. *(triangle inequality).*
2. *A **metric space** is a pair (E,d), where d is a metric on E.*

Metric example

Example.

We can define the following metrics on $\mathbb{R}^n$

1. *The* ***taxicab or Manhattan*** *distance defined by*
$$d_1(X,Y) = \sum_{i=1}^{n} |X_i - Y_i|$$
2. *The* ***euclidian distance*** *defined by*
$$d_2(X,Y) = \sqrt{\sum_{i=1}^{n} |X_i - Y_i|^2}$$
3. *The* ***Minikowski*** *distance defined by*
$$d_p(X,Y) = \left(\sum_{i=1}^{n} |X_i - Y_i|^p\right)^{1/p}$$
4. *The* ***uniform or Chebyshev*** *distance defined by*
$$d_\infty(X,Y) = \max_{1\leq i\leq n} |X_i - Y_i|$$

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Norm

Definition

Let E be a non empty vercorial space on $\mathbb{R}$.

1. *A **norm** on E is a mapping $||.|| : E \to \mathbb{R}^+$ satisfing*
   - $\forall X \in E$: $||X|| = 0 \Longleftrightarrow X = 0$ *(definiteness).*
   - $\forall X \in E$, $\forall \lambda \in \mathbb{R}$: $||\lambda X|| = |\lambda| ||X||$. *(homogeneity).*
   - $\forall X, Y \in E$: $||X + Y|| \leq ||X|| + ||Y||$. *(triangle inequality).*
2. *A **normed space** is a pair $(E, ||.||)$, where $||.||$ is a norm on E.*

Norm example

Example.

We can define the following metrics on $\mathbb{R}^n$

1. *The **Manhattan** norm defined by*
$$||X||_1 = \sum_{i=1}^{n} |X_i|.$$
2. *The **euclidian** norm defined by*
$$||X||_2 = \sqrt{\sum_{i=1}^{n} |X_i|^2}.$$
3. *The **Minikowski** norm defined by*
$$||X||_p = \left(\sum_{i=1}^{n} |X_i|^p \right)^{1/p}.$$
4. *The **uniform or Chebyshev** norm defined by*
$$||X||_\infty = \max_{1 \leq i \leq n} |X_i|.$$

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Equivalence of norms

Definition

Tow norms $||.||$, $||.||'$ on a vectorial space E said to be equivalent if there exists tow positive constant C_1, C_2 such that

$$\forall X \in E : \quad C_1||X|| \leq ||X||' \leq C_2||X||.$$

Proposition

All norms on a finite dimensional space E are equivalent.

Proposition

The usual norms on $\mathbb{R}^n$ are equivalent and for all X in $\mathbb{R}^n$ and we have

$$||X||_\infty \leq ||X||_1 \leq n||X||_\infty.$$

$$||X||_\infty \leq ||X||_2 \leq \sqrt{n}||X||_\infty.$$

$$\frac{1}{\sqrt{n}}||X||_2 \leq ||X||_1 \leq n||X||_2.$$

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Link between metric and norm

Proposition

Let $||.||$ be a norm on $\mathbb{R}^n$. The mapping d defined on $\mathbb{R}^n \times \mathbb{R}^n$ by $(X,Y) \longmapsto d(X,Y) = ||X-Y||$ is a distance on $\mathbb{R}^n$.

Proof.

It remains to show that the mapping d satisfies the conditions of norms.

1. $d(X,Y) = 0 \Leftrightarrow ||X-Y|| = 0 \Leftrightarrow X - Y = 0 \Leftrightarrow X = Y.$
2. $d(X,Y) = ||X-Y|| = ||-1(Y-X)|| = |-1|||Y-X|| = ||Y-X|| = d(Y,X).$
3. $d(X,Y) = ||X-Y|| = ||X-Z+Z-Y|| \leq ||X-Z|| + ||Z-Y|| = d(X,Z) + d(Z,Y).$

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Open balls, closed balls, spheres

Definition

Let $||.||$ *be a norm on* $\mathbb{R}^n$, $a \in \mathbb{R}^n$ *and* $r \in \mathbb{R}_+^*$.

- $B(a,r) = \{X \in \mathbb{R}^n : ||X - a|| < r\}$ *is called the open ball of radius* r *and center* a.
- $\bar{B}(a,r) = \{X \in \mathbb{R}^n : ||X - a|| \leq r\}$ *is called the closed ball of radius* r *and center* a.
- $S(a,r) = \{X \in \mathbb{R}^n : ||X - a|| = r\}$ *is called the sphere ball of radius* r *and center* a.

Open balls, closed balls, spheres

Example.

Let us determine $B(a,1)$ *of* $\mathbb{R}^2$ *for the usual norms.*

- *With the norm* $\|\cdot\|_1$*:*

$$B_1\left(0_{\mathbb{R}^2},1\right) = \left\{(x,y)\in\mathbb{R}^2 / \|(x,y)-(0,0)\|_1 < 1\right\}$$
$$= \left\{(x,y)\in\mathbb{R}^2 / |x|+|y| < 1\right\}.$$

  - *If* $x \geq 0, y \geq 0 : |x|+|y| < 1 \iff x+y < 1 \iff y < 1-x.$
  - *If* $x \leq 0, y \geq 0 : |x|+|y| < 1 \iff -x+y < 1 \iff y < 1+x.$
  - *If* $x \geq 0, y \leq 0 : |x|+|y| < 1 \iff x-y < 1 \iff y > x-1.$
  - *If* $x \leq 0, y \leq 0 : |x|+|y| < 1 \iff -x-y < 1 \iff y > -1-x.$

$B_1\left(0_{\mathbb{R}^2},1\right)$ *is the interior of diamond centered at the origin.*

Open balls, closed balls, spheres

Example.

- *With the norm* $\|\cdot\|_2$ *:*

$$B_2\left(0_{\mathbb{R}^2},1\right)=\left\{(x,y)\in\mathbb{R}^2/\,\|(x,y)-(0,0)\|_2<1\right\}=\left\{(x,y)\in\mathbb{R}^2/\,\sqrt{x^2+y^2}<1\right\}$$
$$=\left\{(x,y)\in\mathbb{R}^2/\,x^2+y^2<1\right\}.$$

$B_2\left(0_{\mathbb{R}^2},1\right)$ *is the interior of circle centred at the origin and radius 1 .*

- *With the norm* $\|\cdot\|_\infty$ *:*

$$B_\infty\left(0_{\mathbb{R}^2},1\right)=\left\{(x,y)\in\mathbb{R}^2/\,\|(x,y)-(0,0)\|_\infty<1\right\}$$
$$=\left\{(x,y)\in\mathbb{R}^2/\,\max\{|x|,|y|\}<1\right\}$$
$$B_\infty\left(0_{\mathbb{R}^2},1\right)=\left\{(x,y)\in\mathbb{R}^2/\,|x|<1\ et\ |y|<1\right\}=\left]-1,1\right[\times\left]-1,1\right[.$$

$B_\infty\left(0_{\mathbb{R}^2},1\right)$ *is the interior of square centered at the origin and side 2.*

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Bounded sets, open and closed sets

Definition

*Let $||.||$ be a norm on $\mathbb{R}^n$ and E a non empty subset of $\mathbb{R}^n$. We say that E is **bounded set** if*

$$\exists a \in \mathbb{R}^n, \exists r > 0 : E \subset B(a,r) \quad (or\ E \subset \bar{B}(a,r)).$$

Definition

Let $||.||$ be a norm on $\mathbb{R}^n$

- *A non empty subset E of $\mathbb{R}^n$ is said to be an **open of** $\mathbb{R}^n$ if*
$$\forall X \in E, \exists r > 0 : B(X,r) \subseteq E.$$
- *A non empty subset E of $\mathbb{R}^n$ is said to be a **closed of** $\mathbb{R}^n$ if E^c is open.*

Bounded sets, open and closed sets

Example.

- *A singleton* $\{X\}$, $X \in \mathbb{R}^n$ *is closed.*
- *An open ball in* $\mathbb{R}^n$ *is open.*
- *A closed ball in* $\mathbb{R}^n$ *is closed.*
- $\emptyset$ *and* $\mathbb{R}^n$ *are opens and closed in the same time.*
- *The set* $\{(X,Y) \in \mathbb{R}^2 : 0 \leq X < 1\}$ *is neither open nor closed.*

Proposition

We have

- *An* ***arbitrary union*** *of open sets is open.*
- *A* ***finite intersection*** *of open sets is open.*
- *An* ***arbitrary intersection*** *of closed sets is closed.*
- *A* ***finite union*** *of closed sets is closed.*

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Neighborhood, interior, exterior, boundry

Definition

Let $||.||$ *be a morm on* $\mathbb{R}^n$, E *a subset of* $\mathbb{R}^n$ *and* $X_0 \in \mathbb{R}^n$.

- E *is* ***neighborhood*** *of* X_0 *if:* $\exists r > 0, B(X_0, r) \subseteq E.$
- X_0 *is an* ***interior point*** *of* E *if:* $\exists r > 0, B(X_0, r) \subseteq E.$
- *The set of all interior points of is denoted by* $Int\, E$
- *The* ***exterior*** *of* E *is the interior of* E^c *and is denoted by* $Ext(E)$.
- X_0 *is said to be* ***boundary*** *point or frontier point of* E *if each open set containing* X_0 *intersects both* E *and* E^c.
- *The set of all boundary points of* E *is denoted by* ∂E.

Neighborhood, interior, exterior, boundry

Example.

Let us consider the subset of $\mathbb{R}^n$ *defined by*

$$E = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1\}$$

Then for the euclidean norm

- *The points* $(-1/2, -1)$ *and* $(1/3, 5)$ *are interior points. but the point* $(2, 1)$ *is an exterior point.*
- $Int\, E = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1\}$.
- $Ext\, E = \{(x, y) \in \mathbb{R}^2 : x < -1 \ or\ x > 1\}$.
- $\partial E = \{(x, y) \in \mathbb{R}^2 : x = -1 \ or\ x = 1\}$.

Neighborhood, interior, exterior, boundry

Proposition

Let E be a non empty subset of $\mathbb{R}^n$.

- *$Int\, E$ is the larget open subset of E.*
- *E is open if and only if $Int\, E = E$.*
- *If E is closed then $E = Int\, E \cup \partial E$.*

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Clusure, accumulation point

Definition

Let $||.||$ be a norm on $\mathbb{R}^n$ and E a subset of $\mathbb{R}^n$.

- *A point $X \in E$ is an **adherent point** for E if every open set containing X contains at least one point of E:*

$$\forall r > 0 : B(X,r) \cap E \neq \emptyset.$$

- *The set of all adherent points of E is called **closure** of E and is denoted by $\overline{E}$.*
- *A point X is an **accumulation point** if*

$$\forall r > 0 : B(X,r) \backslash \{X\} \cap E \neq \emptyset.$$