Chapter 3
Quantifier-free logic

Terms

Chapter 3, Section 1

Terms

A *term* is an expression that can be used for naming something. (Well, *any* expression can be used for naming your cat. But we are talking about names within the usual conventions of mathematics.)

The most common examples of terms are constant symbols, variables, definite descriptions and complex mathematical terms.

Constant symbols

(a) Constant symbols : These are single mathematical symbols that are used as fixed names of particular numbers, functions or sets. For example,

$$0, 1, 42, \pi, \infty, \mathbb{R}, \emptyset, \cos .$$

Variables

(b) Variables : These are letters that are used to stand for any numbers or any elements of some particular set. For example,

x, y, z, etc. ranging over real numbers,

m, n, p, etc. ranging over integers,

f, g, F, G, etc. ranging over functions of some kind.

Definite descriptions

(c) Definite descriptions: These are singular expressions of the form ‘the . . . ’ or ‘this . . . ’ or ‘our . . . ’, etc. For example,

the number,

the number $\sqrt{\pi}$,

the last nonzero remainder in the above steps,

this graph G,

our original assumption,

its top node,

Proposition 2.6.(as short for ‘the proposition numbered 2.6’.)

Complex mathematical terms

(d) Complex mathematical terms: These are mathematical expressions that do the same job as definite descriptions, but in purely mathematical notation. For example,

$$\sqrt[3]{x+y}$$

is a complex mathematical term; it means the same as 'the cube root of the result of adding *x* to *y*'.

Also,

$$\int_{-\pi}^{\pi} z^2 dz$$

is a complex mathematical term meaning the same as 'the integral from $-\pi$ to π of the function $(z \to z^2)$'.

Free and bound variables

Consider the integral

(5.1) $$\int_x^y z^2 dz$$

There are three variables in (5.1): x, y, z. The variable z occurs twice. There is an important difference between x and y on the one hand, and z on the other. *We can meaningfully read x and y as naming particular numbers.* For example, the integral still makes sense if we have already put $x = 1$ and $y = 2$. We can check this by replacing x and y :

$$\int_1^2 z^2 dz$$

We cannot do the same with z. For example, replacing z by π is nonsense.

Free and bound variables

Definition 5.1.1 An occurrence of a variable in a piece of mathematical text is said to be *free* if the text allows us to read the variable as a name. If not free, the occurrence is *bound*. When an occurrence of a variable in the text is bound, there must always be something in the text that prevents us taking the occurrence as a name. For example, the integral sign in (5.1) causes both occurrences of z to be bound; we say that the integral *binds* the occurrences.

The talk of occurrences in Definition 5.1.1 is necessary, because it can happen that the same variable has free occurrences and bound occurrences in the same text. For example:

$$3z + \int_1^2 z^2 dz$$

Here the first z is free but the other two occurrences of z are bound by the integral.

Free and bound variables

It is also necessary to mention the surrounding text. An occurrence of a variable can be free in text T_1 but bound in text T_2 that contains T_1. For example, the integral (5.1) contains the expression

(5.2) $$z^2$$

In (5.2), z is free, although it is bound in (5.1). This is a very common situation: probably most occurrences of variables become bound if we take a large enough piece of surrounding text.

Free and bound variables

The rewriting test must be used with care. For example, it makes sense to write

$$\int_x^y \pi^2 dz$$

which suggests that the first occurrence of z in (5.1) was really free. But this is wrong. There is no way that we can read the first z in the integral (5.1) as standing for π and the second z as the variable of integration. As mathematicians say, the two occurrences are ‘the same z’. In order to make Definition 5.1.1 precise, we would need to restrict it to a precisely defined language…

Free and bound variables

Integrals are not the only expressions that bind variables. For example,

(5.3) $$\{c : |c| > 2\}$$

Here both occurrences of *c* are bound by the set expression.

Free and bound variables

(5.4) For every integer n there is an integer $m > n$

In this example the variable n is bound by the expression ‘For every integer’; it does not make sense to say ‘For every integer 13.5’, for example. Also the variable m is bound by the expression ‘there is an integer’.

Free and bound variables

(5.5) For every $\varepsilon > 0$ there is a $\delta > 0$ such that for every real x with $|x| < \delta$ we have $f(x) < \varepsilon$

Here the variable ε is bound by ‘For every’ and the variable δ is bound by ‘there is’. The symbol f is free in the sentence - for example, it could stand for the sine function.

Quantifiers

The language LR(σ) will have symbols to stand as variables. It will also have a symbol $\forall$ to stand for the expression 'for every', and a symbol $\exists$ for the expression 'there is'. (You may have noticed that $\forall$ and $\exists$ come from the capitalized letters in 'for All' and 'there Exists', rotated through angle π.) The symbol $\forall$ is known as the *universal quantifier symbol*, and the symbol $\exists$ as the *existential quantifier symbol* . These two quantifier symbols will be part of the lexicon of the language LR, and in fact they will be the only symbols in LR that bind variables.

Free and bound variables

In Chapter 7 we will see how to interpret the quantifier symbols in the semantics of LR, and we will give a precise definition of ‘free’ and ‘bound’ occurrences of variables in formulas of LR. But for later sections of the present chapter we will confine ourselves to formulas of LR in which no quantifier symbols occur, that is, to *quantifier-free* (qf) formulas. In these formulas there are no bound variables.