Completeness for Propositional Logic

Chapter 2, Section 10

Adequacy Theorem

Our target in this section is the following theorem. Throughout the section, σ is assumed to be the default signature $\{p_0, p_1, \cdots\}$. We will briefly discuss this assumption at the end of the section.

Theorem 3.10.1 (Adequacy of Natural Deduction for Propositional Logic)

Let Γ *be a set of formulas of* LP(σ) *and* ψ *a formula of* LP(σ). *If* $\Gamma \vDash_\sigma \psi$ *then* $\Gamma \vdash_\sigma \psi$.

Syntactically Consistent Set

Definition 3.10.2

We say that a set Γ of formulas of LP*(σ)* is *syntactically consistent* if $\Gamma \nvdash_{\sigma} \bot$. (This notion is independent of what signature σ we choose, so long as LP(σ) contains all of Γ.)

Lemma 3.10.3

To prove the Adequacy Theorem it is enough to show that every syntactically consistent set of formulas of LP(σ) *has a model.*

Hintikka Set

Definition 3.10.4

We say that a set Γ of formulas of (the stripped-down) LP is a *Hintikka set* (for LP) if it has the following properties :

(H1) If a formula $(\phi \wedge \psi)$ is in Γ, then ϕ is in Γ and ψ is in Γ.

(H2) If a formula $(\neg(\phi \wedge \psi))$ is in Γ, then at least one of $(\neg\phi)$ and $(\neg\psi)$ is in Γ.

(H3) If a formula $(\neg(\neg\phi))$ is in Γ, then ϕ is in Γ.

(H4) $\bot$ is not in Γ.

(H5) There is no propositional symbol p such that both p and $(\neg p)$ are in Γ.

Hintikka Set Properties

Lemma 3.10.5

Every Hintikka set has a model.

Lemma 3.10.6

If Γ *is a syntactically consistent set of formulas of* LP(σ)*, then there is a Hintikka set* Δ *of formulas of* LP(σ) *with* $\Gamma \subseteq \Delta$.

Proof of Adequacy Theorem

Proof

From Lemma 3.10.3, it suffices to show that every syntactically consistent set of formulas has a model.

Let Γ *be a syntactically consistent set of formulas. By Lemma 3.10.6, there exists a Hintikka set* Δ *of formulas of* $\mathrm{LP}(\sigma)$ *such that* $\Gamma \subseteq \Delta$.

By Lemma 3.10.5, Δ *has a model* A. *Since* $\Gamma \subseteq \Delta$, *then* A *is also a model of* Γ.

Completeness Theorem

Theorem 3.10.7 (Completeness Theorem)

Let Γ *be a set of formulas of* LP(σ) *and* ψ *a formula of* LP(σ). *Then*

$$\Gamma \vdash_\sigma \psi \quad \Leftrightarrow \quad \Gamma \vDash_\sigma \psi.$$

Proof This is the Soundness and Adequacy Theorems combined.