Exercise 1 (1+3)+(1+1+3)+(3+3)

1) Completeness Theorem

Let Γ be a set of formulae of $LP(\sigma)$ and ψ a formula of $LP(\sigma)$. Then

$$\Gamma \vdash_\sigma \psi \iff \Gamma \models_\sigma \psi.$$

Recall that we write $\Gamma \models_\sigma \psi$ to mean that for every σ-structure A, if A is a model of Γ then A is a model of ψ.

The implication $\Gamma \vdash_\sigma \psi \Rightarrow \Gamma \models_\sigma \psi$ means that we can't derive a formula that is false in a given σ-structure A from formulae that are true in A. This ensures that we did not make some dreadful mistake when we set up the rules of natural deduction.

The converse says that if the truth of Γ guarantees the truth of ψ, then our natural deduction rules allow us to derive ψ from Γ, that is, we do not need any more natural deduction rules besides those that we already have.

2) See Definition 5.3.1. and Example 5.3. (b), page 112.

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To show that ϕ is a formula of $LR(\sigma_{group})$, we can use Theorem 5.3.8.

First we consider terms :

- x_1 and x_2 are variables, so they are terms by (a).
- $\cdot$ is a function symbol with arity 2, then by (c), $\cdot(x_1, x_2)$ is a term.
- e is a constant symbol, so it is a term by (b).

Now we consider formulae

- $\cdot(x_1, x_2)$ and e are terms, so $(\cdot(x_1,x_2)=e)$ is a formula by (b).
- Similarly, $(\cdot(x_2, x_1)=e)$ is a formula.
- Then, by (d), $\psi := ((\cdot(x_1,x_2)=e)\wedge(\cdot(x_2,x_1)=e))$ is a formula
- Finally, by (f), we have successively :

$$\exists x_2\, \psi \quad \text{is a formula,}$$
$$\forall x_1\, \exists x_2\, \psi \quad \text{is a formula.}$$

therefore, ϕ is a formula of $LR(\sigma_{group})$.

3) (a)

$$1\ \dfrac{\dfrac{(\phi\wedge(\psi\vee\chi))}{(\psi\vee\chi)}(\wedge E)\qquad \dfrac{\dfrac{\dfrac{(\phi\wedge(\psi\vee\chi))}{\phi}(\wedge E)\quad \cancel{\psi}^{1}}{(\phi\wedge\psi)}(\wedge I)}{((\phi\wedge\psi)\vee(\phi\wedge\chi))}(\vee I)\qquad \dfrac{\dfrac{\dfrac{(\phi\wedge(\psi\vee\chi))}{\phi}(\wedge E)\quad \cancel{\chi}^{1}}{(\phi\wedge\chi)}(\wedge I)}{((\phi\wedge\psi)\vee(\phi\wedge\chi))}(\vee I)}{((\phi\wedge\psi)\vee(\phi\wedge\chi))}(\vee E)$$

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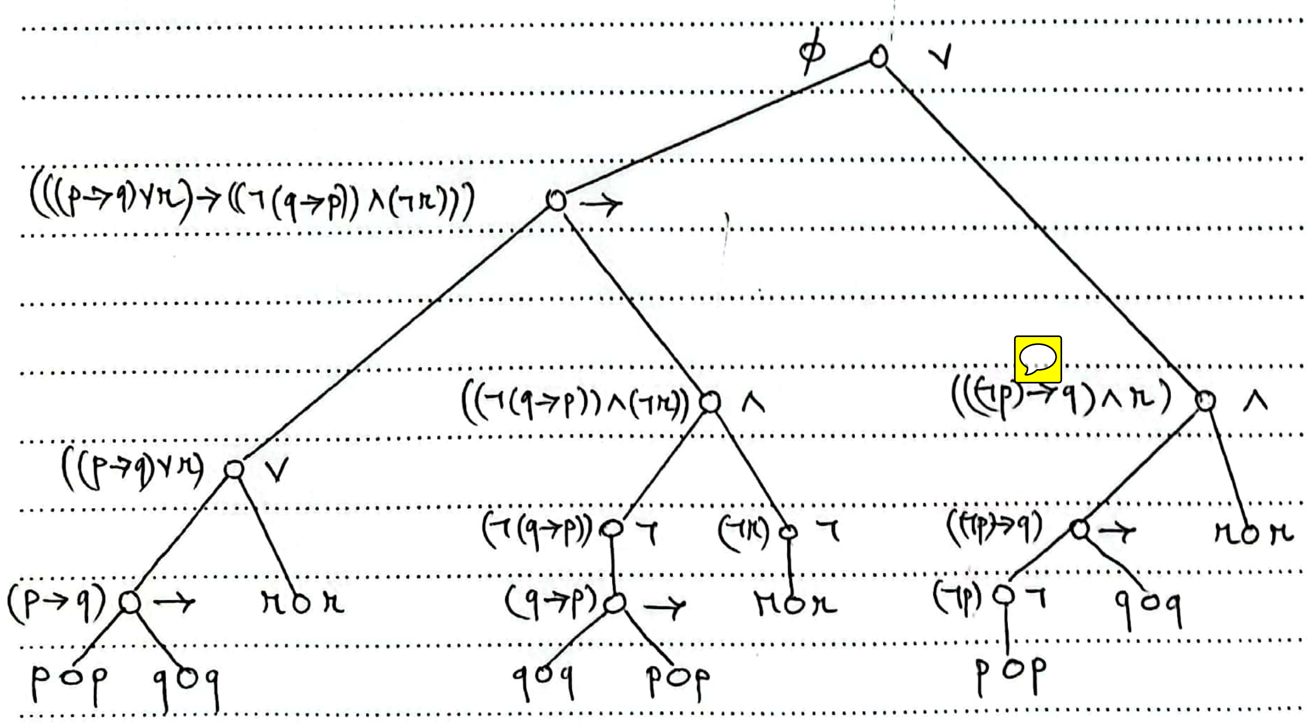
3) (b)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\cancel{\phi}^{1} \quad \cancel{(\phi \to (\psi \wedge \chi))}^{3}}{(\psi \wedge \chi)}\,(\to E)}{\chi}\,(\wedge E)}{(\phi \to \chi)}\,(\to I)\;1 \qquad \dfrac{\dfrac{\dfrac{\cancel{\phi}^{2} \quad \cancel{(\phi \to (\psi \wedge \chi))}^{3}}{(\psi \wedge \chi)}\,(\to E)}{\psi}\,(\wedge E)}{(\phi \to \psi)}\,(\to I)\;2}{((\phi \to \chi) \wedge (\phi \to \psi))}\,(\wedge I)}{((\phi \to (\psi \wedge \chi)) \to ((\phi \to \chi) \wedge (\phi \to \psi)))}\,(\to I)\;3$$

Exercise 2 $\quad 2+3+(1+2)+3+(1+1+1)+(1+2)+(1+1+1+2)$

1) Computing the depths of the initial segments with functor on the right, we find that the head of the formula ϕ is the second occurrence of the functor $\vee$.

2) The parsing tree of ϕ:



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3) The tree π has 9 leaves. The formula ϕ has complexity 5. (...)

4) The truth table of ϕ :

$((((p \to q) \vee r) \to ((\neg(q \to p)) \wedge (\neg r))) \vee ((\neg((\neg p) \to q)) \wedge r))$											
T	T	F	F	T	F	F	F	F	F	T	F
T	T	F	F	T	F	T	F	F	F	T	F
F	T	F	F	T	F	F	F	F	F	T	F
F	F	T	F	T	F	T	T	F	F	T	F
T	T	F	T	F	F	F	F	F	T	T	F
T	T	T	T	F	T	T	T	F	T	T	F
T	T	F	F	T	F	F	T	T	T	F	T
T	T	F	F	T	F	T	F	T	T	F	F

$\Uparrow$

(the σ-structures are ordered as usual.)

5) The formula ϕ is satisfiable since there exists a σ-structure A such that $A^*(\phi) = T$, for example the σ-structure in the fourth row. ϕ is neither a tautology since there exists a σ-structure A such that $A^*(\phi) = F$, nor a contradiction since it is satisfiable.

6) From the truth table we find

$\phi^{DNF} = (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$

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6) Set $\theta : (\neg\phi)^{DNF}$. Then θ eq $(\neg\phi)$, hence $(\neg\theta)$ eq $(\neg(\neg\phi))$ eq ϕ. Since $(\neg\phi)$ is true if and only if ϕ is false, then from the truth table of ϕ we get

$$(\neg\phi)^{DNF} = (p\wedge q\wedge r)\vee(p\wedge q\wedge\neg r)\vee(p\wedge\neg q\wedge r)$$
$$\vee(\neg p\wedge q\wedge r)\vee(\neg p\wedge\neg q\wedge\neg r),$$

So we deduce

$$\phi^{CNF} = (\neg p\vee\neg q\vee\neg r)\wedge(\neg p\vee\neg q\vee r)\wedge(\neg p\vee q\vee\neg r)$$
$$\wedge(p\vee\neg q\vee\neg r)\wedge(p\vee q\vee r).$$

7) Define

p : the gold is in the first box

q : the gold is in the second box

r : the gold is in the third box.

Since one box contains gold and the other two are empty, then the following formula is true:

$$(p\wedge\neg q\wedge\neg r)\vee(\neg p\wedge q\wedge\neg r)\vee(\neg p\wedge\neg q\wedge r). \quad (1)$$

Notice that this is just ϕ^{DNF}.

Since only one message is true, then the following formula is true:

$$(\neg p\wedge\neg\neg q\wedge\neg q)\vee(\neg\neg p\wedge\neg q\wedge\neg q)\vee(\neg\neg p\wedge\neg\neg q\wedge q) \quad (2)$$

This last formula is equivalent to:

$$(p\wedge\neg q)\vee(p\wedge q)$$

which is true if and only if p is true. Using the truth table of ϕ, we can see that (1) and (2) are true if and only if p is true and q and r are false, which implies that the gold is in the first box.

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Exercise 3 $(1+2+2)+3$

1) We prove the property in 1) by induction on the complexity of ϕ.

If ϕ has complexity 0, then $\phi \in \sigma$, so if $A(\phi)=T$ then $B(\phi)$ is true by assumption.

Suppose the property is true for all formulae of F_+ that have complexities $\leqslant k$. Let $\phi \in F_+$ of complexity $k+1$ such that $A^*(\phi)=T$. From the definition of F_+, ϕ has one of the forms $(\psi \vee \chi)$ or $(\psi \wedge \chi)$. Since the complexities of ψ and χ are $< k+1 \leqslant k$, then we have :

i) $A^*(\phi) = A^*(\psi \vee \chi) = T$, then $A^*(\psi)=T$ or $A^*(\chi)=T$, so by the induction hypothesis $B^*(\psi)=T$ or $B^*(\chi)=T$, thus $B^*(\psi \vee \chi) = B^*(\phi) = T$,

or

ii) $A^*(\phi) = A^*(\psi \wedge \chi) = T$, then $A^*(\psi) = A^*(\chi) = T$, so by the induction hypothesis $B^*(\psi) = B^*(\chi) = T$, hence $B^*(\psi \wedge \chi) = B^*(\phi) = T$.

2) The statement in 1) doesn't remain true if we replace F_+ by F. Here is a counter-example.

Let $\sigma = \{P_1, P_2\}$ and let A and B be the σ-structures defined by $A(P_1)=T$, $A(P_2)=F$, $B(P_1)=T$ and $B(P_2)=T$. Then we have :

$$A(P_i)=T \Rightarrow B(P_i)=T \text{ for all } P_i \in \sigma.$$

But, if we take $\phi = (\neg P_2)$, we have :

$$A^*(\phi) = A^*(\neg P_2) = T \text{ and } B^*(\phi) = B^*(\neg P_2) = F.$$

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