Chapitre 1. Random Variables

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Definitions and notations

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The service life of a spare part can be represented by a r.r.v.

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We throw two coins and let X be the number of tails obtained. We know that $\Omega = \{(F, F) ; (F, P) ; (P, F) ; (P, P)\}$ and the values of X are $\{0, 1, 2\}$.

1. Show that X is a random variable on Ω endowed with the algebra $\mathcal{P}(\Omega)$.
2. Show that X is not a random variable on Ω endowed with the algebra $\mathcal{A}_1 = \{\Omega, \emptyset, \{(F, F)\}, \{(F, F) ; (F, P) ; (P, F)\}\}$.

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Let Ω be the space of trials associated to a Bernoulli random experiment and let $I_A(\cdot)$ be the function from Ω to $\{0, 1\}$ defined by

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4. $\mathbb{P}(I_A(\omega) = 0) = 1 - \mathbb{P}(A) = \mathbb{P}(\overline{A}), \forall A \in \mathcal{A}$.

Induced probability

Theorem

Let X be a r.r.v. defined on probabilized space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in a probabilizable space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. The application $\mathbb{P}_X$ of $\mathcal{B}_{\mathbb{R}}$ in $\mathbb{R}$ defined by $\mathbb{P}_X(B) = \mathbb{P}\left(X^{-1}(B)\right)$, is a probability on $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$.

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Remark

The definition is due to the existence of $\mathbb{P}$ on $(\Omega, \mathcal{T})$, hence the notion of induced probability.

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Moreover $\mathbb{P}$ verifies these conditions:

$$\mathbb{P}_X(\mathbb{R}) = \mathbb{P}\left(X^{-1}(\mathbb{R})\right) = \mathbb{P}(\Omega) = 1$$

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$$\begin{aligned}
\mathbb{P}_X\left(\underset{i\geq 1}{\cup} B_i\right) &= \mathbb{P}\left(X^{-1}\left(\underset{i\geq 1}{\cup} B_i\right)\right) = \mathbb{P}\left(\underset{i\geq 1}{\cup} X^{-1}(B_i)\right) \\
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\end{aligned}$$

noting that $X^{-1}(B_i)$ and $X^{-1}(B_j)$ are incompatible $\forall i \neq j$. $\square$

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Remark

$\lim_{n\to\infty} A_n$ exists if and only if the sequence $(A_n)_{n\geq 1}$ is monotonic.

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Lemma

(Property of the continuity of $\mathbb{P}$*)*
If $(A_n)_{n\geq 1}$ *is a monotonic sequence of events, then we have:*

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}\left(\lim_{n\to\infty} A_n\right).$$

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4. $\lim_{x \to \infty} F(x) = 1$ *and* $\lim_{x \to -\infty} F(x) = 0$.

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3. Let us show that for any real sequence (ε_n) decreasing and converging to 0, $\lim_{n\to\infty} F(x+\varepsilon_n) = F(x)$ .We set $A_n =]x, x+\varepsilon_n]$ .The (A_n) are decreasing and $\lim_{n\to\infty} A_n = \emptyset$, hence from the lemma $\lim_{n\to\infty} \mathbb{P}_X(A_n) = \mathbb{P}_X(\lim_{n\to\infty} A_n) = \mathbb{P}_X(\emptyset) = 0$.Since

$$\begin{aligned}
\mathbb{P}_X(A_n) &= \mathbb{P}(x < X \leq x+\varepsilon_n) = \mathbb{P}(X \leq x+\varepsilon_n) - \mathbb{P}(X \leq x) \\
&= F(x+\varepsilon_n) - F(x),
\end{aligned}$$

Cumulative distribution function of a random variable

Proof.

1. Obvious because $F(X) = \mathbb{P}(X \leq x)$ so $0 \leq F(X) \leq 1$.
2. Suppose that $x_1 \leq x_2$, hence $]-\infty, x_1] \subset]-\infty, x_2]$and $X^{-1}(]-\infty, x_1]) \subset X^{-1}(]-\infty, x_2])$ .It follows that $\mathbb{P}\left(X^{-1}(]-\infty, x_1])\right) \leq \mathbb{P}\left(X^{-1}(]-\infty, x_2])\right)$ hence $F(x_1) \leq F(x_2)$.
3. Let us show that for any real sequence (ε_n) decreasing and converging to 0, $\lim_{n\to\infty} F(x+\varepsilon_n) = F(x)$ .We set $A_n =]x, x+\varepsilon_n]$ .The (A_n) are decreasing and $\lim_{n\to\infty} A_n = \emptyset$, hence from the lemma $\lim_{n\to\infty} \mathbb{P}_X(A_n) = \mathbb{P}_X(\lim_{n\to\infty} A_n) = \mathbb{P}_X(\emptyset) = 0$.Since

$$\begin{aligned}\mathbb{P}_X(A_n) &= \mathbb{P}(x < X \leq x+\varepsilon_n) = \mathbb{P}(X \leq x+\varepsilon_n) - \mathbb{P}(X \leq x) \\ &= F(x+\varepsilon_n) - F(x),\end{aligned}$$

then $\lim_{n\to\infty} F(x+\varepsilon_n) = F(x)$.

Cumulative distribution function of a random variable

Proof.

4. Let $B_n =]-\infty, x_n]$ where the x_n is decreasing and converging to $-\infty$.

Cumulative distribution function of a random variable

Proof.

4. Let $B_n =]-\infty, x_n]$ where the x_n is decreasing and converging to $-\infty$.We deduce that (B_n) is a decreasing sequence and $\lim B_n = \emptyset$, and according to the lemma

$$\lim_{n\to\infty} \mathbb{P}_X(B_n) = \mathbb{P}_X\left(\lim_{n\to\infty} B_n\right) = \mathbb{P}_X(\emptyset) = 0$$

Cumulative distribution function of a random variable

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$$\lim_{n\to\infty} \mathbb{P}_X(B_n) = \mathbb{P}_X\left(\lim_{n\to\infty} B_n\right) = \mathbb{P}_X(\emptyset) = 0$$

or $\lim_{n\to\infty} \mathbb{P}_X(B_n) = \lim_{n\to\infty} \mathbb{P}_X(X \leq x_n) = \lim_{n\to\infty} F(x_n) = 0$.

Cumulative distribution function of a random variable

Proof.

4. Let $B_n =]-\infty, x_n]$ where the x_n is decreasing and converging to $-\infty$.We deduce that (B_n) is a decreasing sequence and $\lim B_n = \emptyset$, and according to the lemma

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   or $\lim_{n\to\infty} \mathbb{P}_X(B_n) = \lim_{n\to\infty} \mathbb{P}_X(X \leq x_n) = \lim_{n\to\infty} F(x_n) = 0$.
   Let us consider the sequence defined by $C_n =]-\infty, y_n]$ where (y_n) is an increasing real sequence such that $\lim_{n\to\infty} y_n = +\infty$.

Cumulative distribution function of a random variable

Proof.

4. Let $B_n = \left]-\infty, x_n\right]$ where the x_n is decreasing and converging to $-\infty$.We deduce that (B_n) is a decreasing sequence and $\lim B_n = \emptyset$, and according to the lemma

$$\lim_{n\to\infty} \mathbb{P}_X (B_n) = \mathbb{P}_X \left(\lim_{n\to\infty} B_n\right) = \mathbb{P}_X (\emptyset) = 0$$

or $\lim_{n\to\infty} \mathbb{P}_X (B_n) = \lim_{n\to\infty} \mathbb{P}_X (X \leq x_n) = \lim_{n\to\infty} F(x_n) = 0.$
Let us consider the sequence defined by $C_n = \left]-\infty, y_n\right]$ where (y_n) is an increasing real sequence such that $\lim_{n\to\infty} y_n = +\infty$.We deduce that the (C_n) are increasing and $\lim_{n\to\infty} C_n = \mathbb{R}$.

Cumulative distribution function of a random variable

Proof.

4. Let $B_n =]-\infty, x_n]$ where the x_n is decreasing and converging to $-\infty$.We deduce that (B_n) is a decreasing sequence and $\lim B_n = \emptyset$, and according to the lemma

$$\lim_{n\to\infty} \mathbb{P}_X(B_n) = \mathbb{P}_X\left(\lim_{n\to\infty} B_n\right) = \mathbb{P}_X(\emptyset) = 0$$

or $\lim_{n\to\infty} \mathbb{P}_X(B_n) = \lim_{n\to\infty} \mathbb{P}_X(X \leq x_n) = \lim_{n\to\infty} F(x_n) = 0.$

Let us consider the sequence defined by $C_n =]-\infty, y_n]$ where (y_n) is an increasing real sequence such that $\lim_{n\to\infty} y_n = +\infty$.We deduce that the (C_n) are increasing and $\lim_{n\to\infty} C_n = \mathbb{R}$.

We have $\lim_{n\to+\infty} F(y_n) = \lim_{n\to+\infty} \mathbb{P}(X \leq y_n) = \lim_{n\to+\infty} \mathbb{P}_X(C_n) = \mathbb{P}_X(\lim_{n\to+\infty} C_n) = \mathbb{P}_X(\mathbb{R}) = 1.$