Logical Equivalence

Chapter 2, Section 6

Definition

Lemma 3.6.1

Let σ be a signature and ϕ, ψ formulas of LP(σ). Then the following are equivalent :

(i) For every σ-structure A, A is a model of ϕ if and only if it is a model of ψ.

(ii) For every σ-structure A, $A^*(\phi) = A^*(\psi)$.

(iii) $\vDash (\phi \leftrightarrow \psi)$.

Definition 3.6.2

Let σ be a signature and ϕ, ψ *formulas of LP(σ)*. We say that ϕ *and* ψ *are logically equivalent*, in symbols

$$\phi \quad \text{eq} \quad \psi$$

if any of the equivalent conditions (i)–(iii) of Lemma 3.6.1 hold.

Example

Example 3.6.3

Clause (ii) in Lemma 3.6.1 says that ϕ and ψ have the same head column in their truth tables. We can use this fact to check logical equivalence. For example, the following truth table shows that

$$(p_1 \lor (p_2 \lor p_3)) \quad \mathrm{eq} \quad ((p_1 \lor p_2) \lor p_3)$$

p_1	p_2	p_3	$(p_1$	$\lor$	$(p_2$	$\lor$	$p_3))$	$((p_1$	$\lor$	$p_2)$	$\lor$	$p_3)$
T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	F	T	T	T	T	F
T	F	T	T	T	F	T	T	T	T	F	T	T
T	F	F	T	T	F	F	F	T	T	F	T	F
F	T	T	F	T	T	T	T	F	T	T	T	T
F	T	F	F	T	T	T	F	F	T	T	T	F
F	F	T	F	T	F	T	T	F	F	F	T	T
F	F	F	F	F	F	F	F	F	F	F	F	F
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Equivalence Relation

Theorem 3.6.4

Let σ be a signature. Then eq *is an equivalence relation on the set of all formulas of LP(σ). In other words it is*

- *Reflexive : For every formula ϕ, ϕ* eq *ϕ.*
- *Symmetric : If ϕ and ψ are formulas and ϕ* eq *ψ, then ψ* eq *ϕ.*
- *Transitive : If ϕ, ψ and χ are formulas and ϕ* eq *ψ and ψ* eq *χ, then ϕ* eq *χ.*

Some Logical Equivalences

Example 3.6.5
Here follow some commonly used logical equivalences.

$(p_1 \vee (p_2 \vee p_3))$ eq $((p_1 \vee p_2) \vee p_3)$
$(p_1 \wedge (p_2 \wedge p_3))$ eq $((p_1 \wedge p_2) \wedge p_3)$

Associative Laws

$(p_1 \vee (p_2 \wedge p_3))$ eq $((p_1 \vee p_2) \wedge (p_1 \vee p_3))$
$(p_1 \wedge (p_2 \vee p_3))$ eq $((p_1 \wedge p_2) \vee (p_1 \wedge p_3))$

Distributive Laws

$(p_1 \vee p_2)$ eq $(p_2 \vee p_1)$
$(p_1 \wedge p_2)$ eq $(p_2 \wedge p_1)$

Commutative Laws

Some Logical Equivalences

$(\neg(p_1 \vee p_2))$ eq $((\neg p_1) \wedge (\neg p_2))$

$(\neg(p_1 \wedge p_2))$ eq $((\neg p_1) \vee (\neg p_2))$

De Morgan Laws

$(p_1 \vee p_1)$ eq p_1

$(p_1 \wedge p_1)$ eq p_1

Idempotent Laws

$(\neg(\neg p_1))$ eq p_1

Double negation Law