Corrigé test 2 (Section 2) 23/24.

$\forall k \in \mathbb{N}, \; \mathbb{P}(X=k) = \dfrac{\theta^k}{(1+\theta)^{k+1}} \quad ; \theta > 0$

soit $S_n = X_1 + X_2 + \dots + X_n$.

1) Du fait que $X_1, X_2, \dots, X_n$ sont indépendants on a :

$$G_{S_n}(t) = \mathbb{E}[t^{S_n}] = \mathbb{E}[t^{X_1 + \dots + X_n}] = \prod_{i=1}^{n} \mathbb{E}[t^{X_i}] = (G_{X_1}(t))^n \quad 0,5$$

On a : $G_{X_1}(t) = \mathbb{E}[t^{X_1}] = \sum_{k=0}^{\infty} t^k \, \mathbb{P}(X=k) = \dfrac{1}{1+\theta} \sum_{k=0}^{\infty} \left(\dfrac{\theta t}{1+\theta}\right)^k$ $\quad 0,5$

Comme $\dfrac{\theta}{1+\theta} < 1$ et $t \leq 1. \Rightarrow \dfrac{\theta t}{1+\theta} < 1$.

d'où $G_{X_1}(t) = \dfrac{1}{1+\theta} \cdot \dfrac{1}{1 - \frac{\theta t}{1+\theta}} \Rightarrow \boxed{G_{X_1}(t) = \dfrac{1}{1+\theta-\theta t}}$ $\quad 0,5$

On en déduit que :

$$\boxed{G_{S_n}(t) = (G_{X_1}(t))^n = \left(\dfrac{1}{1+\theta-\theta t}\right)^n} \quad 0,5$$

* $\mathbb{E}[S_n] = G'_{S_n}(1)$ on a : $G'_{S_n}(t) = n \cdot \dfrac{\theta}{(1+\theta-\theta t)^2} \cdot \left(\dfrac{1}{1+\theta-\theta t}\right)^{n-1}$ $\quad 1$

$\Rightarrow G'_{S_n}(1) = \boxed{\mathbb{E}[S_n] = n\theta}$

* $\mathbb{E}[S_n^2] = G''_{S_n}(1) + G'_{S_n}(1)$. on a : $G''_{S_n}(t) = n \dfrac{2\theta^2}{(1+\theta-\theta t)^3} \left(\dfrac{1}{1+\theta-\theta t}\right)^{n-1}$

$+ n(n-1) \dfrac{\theta^2}{(1+\theta-\theta t)^4} \left(\dfrac{1}{1+\theta-\theta t}\right)^{n-2}$

$\Rightarrow G''_{S_n}(1) + G'_{S_n}(1) = n \cdot 2\theta^2 + n(n-1)\theta^2 + n\theta = n^2\theta^2 + n\theta^2 + n\theta$

$\Rightarrow \mathbb{V}ar(S_n) = \mathbb{E}[S_n^2] - (\mathbb{E}[S_n])^2 = n^2\theta^2 + n\theta^2 + n\theta - n^2\theta^2$

$\Rightarrow \boxed{\mathbb{V}ar(S_n) = n\theta(\theta+1)}$ $\quad 1$

2) $G_{S_n}(t) = \left(\frac{1}{(1+\theta-\theta t)}\right)^n = \frac{1}{(1+\theta)^n} \cdot \frac{1}{\left(1-\frac{\theta}{1+\theta}t\right)^n}$ 1

$= \frac{1}{(1+\theta)^n} \cdot \sum_{k=0}^{\infty} C_{n+k-1}^{k} \frac{t^k \theta^k}{(1+\theta)^k} = \sum_{k=0}^{\infty} t^k \cdot C_{n+k-1}^{k} \frac{\theta^k}{(1+\theta)^{k+n}}$

$\Rightarrow \forall k \in \mathbb{N} \, ; \quad \mathbb{P}(S_n = k) = C_{n+k-1}^{k} \frac{\theta^k}{(1+\theta)^{k+n}}$.

3) On a : $\quad \mathbb{P}(0 \leq S_n \leq n) = \sum_{k=0}^{n} \mathbb{P}(S_n = k) = \sum_{k=0}^{n} C_{n+k-1}^{k} \frac{\theta^k}{(1+\theta)^{n+k}}$

Donc $\lim_{n\to\infty} \sum_{k=0}^{\infty} C_{n+k-1}^{k} \frac{\theta^k}{(1+\theta)^{k+n}} = \lim_{n\to\infty} \mathbb{P}(0 \leq S_n \leq n)$. 0,5

Du Théorème central limit on a :

$Z_n = \frac{S_n - \mathbb{E}[S_n]}{\sqrt{Var(S_n)}} = \frac{S_n - n\theta}{\sqrt{n\theta(1+\theta)}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1)$.

Alors $\mathbb{P}(0 \leq S_n \leq n) = \mathbb{P}\left(\frac{-n\theta}{\sqrt{n\theta(1+\theta)}} \leq Z_n \leq \frac{n(1-\theta)}{\sqrt{n\theta(1+\theta)}}\right)$. 0,5

$= \mathbb{P}\left(-\sqrt{n} \cdot \sqrt{\frac{\theta}{1+\theta}} \leq Z_n \leq \sqrt{n} \cdot \frac{1-\theta}{\sqrt{\theta(1+\theta)}}\right)$.

$= \Phi\left(\sqrt{n}\frac{1-\theta}{\sqrt{\theta(1+\theta)}}\right) - \Phi\left(-\sqrt{n}\sqrt{\frac{\theta}{1+\theta}}\right)$.

D'où
$$\boxed{\lim_{n\to\infty} \sum_{k=0}^{n} C_{n+k-1}^{k} \frac{\theta^k}{(1+\theta)^{n+k}} = \begin{cases} 1 & \text{si } 0 < \theta < 1 \\ 1/2 & \text{si } \theta = 1 \\ 0 & \text{si } \theta > 1. \end{cases}}$$
1