Mathematical analysis 2
Chapter 2: Multiple Integrals
Part : Triple Integrals

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2023/2024

- Let f be a continuous function of three variables x, y, z on a sub domain $D \subset \mathbb{R}^3$:

$$\begin{cases} f: & D \to \mathbb{R} \\ & (x; y; z) \mapsto f(x, y, z) \end{cases}$$

- The triple integral of f over D is denoted by

$$\iiint_D f(x, y, z)dx\, dy\, dz$$

Theorem

Let f and g be tow integrable functions over a domain $D \subset \mathbb{R}^3$ then

- *The sum $f+g$ is integrable and $\forall \alpha, \beta \in \mathbb{R}$*
  $$\iiint_D (\alpha f(x,y,z)+\beta g(x,y,z))dxdydz = \alpha \iiint_D f(x,y,z)dxdydz + \beta \iint_D g(x,y,z)dxdydz$$
- *If $D = D_1 \cup D_2$ with $D_1 \cap D_2 = \emptyset$ then*
  $$\iiint_D f(x,y,z)dxdydz = \iiint_{D_1} f(x,y,z)dxdydz + \iiint_{D_2} f(x,y,z)dxdydz$$
- *If $\forall (x,y,z) \in \mathbb{R}^3$ $f(x,y,z) \leq g(x,y,z)$ then*
  $$\iiint_D f(x,y,z)dxdydz \leq \iiint_D g(x,y,z)dxdydz$$
- *we have*
  $$\left|\iiint_D f(x,y,z)dxdydz\right| \leq \iiint_D |f(x,y,z)|\, dxdydz$$

Triple integrals over parallelepiped

Theorem (Fubini)

Let f *be a continuous on a parallelepiped* $D = [a,b] \times [c,d] \times [e,f]$,

$$\iiint_D f(x,y,z) = \int_a^b \int_c^d \int_e^f f(x,y,z)dzdydx$$
$$= \int_e^f \int_c^d \int_a^b f(x,y,z)dxdydz$$
$$= \int_e^f \int_a^b \int_c^d f(x,y,z)dydxdz$$

Triple integrals over parallelepiped

Example.

Find $I = \iiint_D (x+3y+z)dxdydz$ *where* $D = [0:1] \times [1:2] \times [1;3]$

$$\begin{aligned}
I = \iiint_D (x+3y+z)dxdydz &= \int_0^1 \int_1^2 \left(\int_1^3 (x+3y+z)dz \right) dydx \\
&= \int_0^1 \int_1^2 \left[xz + 3yz + \frac{z^2}{2} \right]_1^3 dydx \\
&= \int_0^1 \int_1^2 2x + 6y + 4 \; dydx \\
&= \int_0^1 \left[2xy + 3y^2 + 4y \right]_1^2 \; dx \\
&= \int_0^1 (2x+13) \, dx = 14
\end{aligned}$$

Triple integrals over general bounded domains

If the domain D is of the following type:

$$D = \left\{(x,y,z) \in \mathbb{R}^3; a \leq x \leq b; u_1(x) \leq y \leq u_2(x); v_1(x,y) \leq z \leq v_2(x,y)\right\}$$

Then

$$\iiint_D f(x,y,z)dxdydz = \int_a^b \left(\int_{u_1(x)}^{u_2(x)} \left(\int_{v_1(x,y)}^{v_2(x,y)} f(x,y,z)dz\right) dy\right) dx$$

Remark

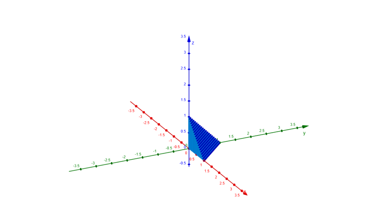
- *z is between two surfaces.*
- *y is between two curves.*
- *x is between two lines.*

Example.

Find $I = \iiint_D x\,dx\,dy\,dz$ *where*
$D = \left\{(x,y,z) \in \mathbb{R}^3; x \geq 0; y \geq 0; z \geq 0; x+y+z \leq 1\right\}$

- We have $x+y+z \leq 1 \Rightarrow z \leq 1-x-y$, but $z \geq 0 \Rightarrow 0 \leq z \leq 1-x-y$.
- On the plan (xoy), $z = 0$ then $x \geq 0$; $y \geq 0$; $x+y \leq 1 \Rightarrow 0 \leq y \leq 1-x$ therefore $0 \leq y \leq 1-x$ et $0 \leq x \leq 1$

Then the domain D is given by:



$$D = \left\{(x,y,z) \in \mathbb{R}^3; 0 \leq x \leq 1; 0 \leq y \leq 1-x; 0 \leq z \leq 1-x-y\right\}$$

$$\begin{aligned} I = \iiint_D x dx dy dz &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x dz dy dx \\ &= \int_0^1 x \int_0^{1-x} [z]_0^{1-x-y} dy dx \\ &= \int_0^1 x \int_0^{1-x} 1 - x - y \; dy dx \\ &= \int_0^1 x \left[y - yx - \frac{y^2}{2} \right]_0^{1-x} dx \\ &= \int_0^1 \frac{x}{2} - x^2 + \frac{x^3}{2} \} dx \\ &= \left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} \right]_0^1 = \frac{1}{24} \end{aligned}$$

- Let Ω be a subset of $\mathbb{R}^3$. Let φ be a bijective mapping of class C^1 such that

$$\begin{cases} \varphi: & \Omega \to \mathbb{R}^3 \\ & (u,v,w) \mapsto \varphi(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w)). \end{cases}$$

- Then we have:

$$\iiint_D f(x,y,z)dxdydz = \iiint_\Omega f \circ \varphi(u,v,w) \left| DetJ_\varphi(u,v,w) \right| dudvdw$$

$$= \iiint_\Omega f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| DetJ_\varphi(u,v,w) \right| dudvdw$$

With

$$J_\varphi = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

Change of variable in cylindrical coordinates

- Cylindrical coordinates of a point $M(x;y;z) \in \mathbb{R}^3$ are given by:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

- We define then the mapping

$$\begin{aligned} \varphi \quad &: \Omega = \mathbb{R}_+^* \times [0;2\pi] \times \mathbb{R} \to \mathbb{R} \\ &(r,\theta,z) \mapsto (x(r,\theta), y(r,\theta), z) \end{aligned}$$

- The Jacobian matrix associated to the mapping φ is given by

$$J_\varphi = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Change of variable in cylindrical coordinates

- The Jacobian is given by

$$\left|J_\varphi\right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\ drd\theta dz$$

- We have then

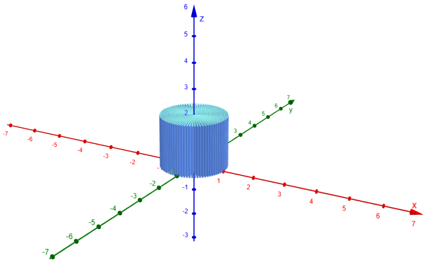
$$\iiint_D f(x,y,z)dxdydz = \iiint_\Omega f(r\cos\theta), r\sin\theta, z)\ r\ drd\theta dz.$$

Example.

Using the change of variable in cylindrical find:

$$I = \iiint_D (x^2 + y^2 + 1)\,dxdydz$$

where $D = \left\{(x;y;z) \in \mathbb{R}; x^2 + y^2 \leq 1; 0 \leq z \leq 2\right\}$



We have

$$x^2+y^2 \leq 1 \Rightarrow \quad \text{l'interieur du cercle unité} \Rightarrow 0 \leq r \leq 1 \text{ et } 0 \leq \theta \leq 2\pi$$

We set

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

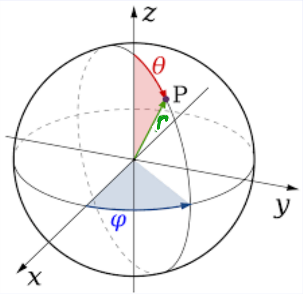
Then $\Omega = \{(r;\theta;z) \in \mathbb{R}; 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 2\}$. Therefor

$$\begin{aligned} I = \iiint_D (x^2+y^2+1)dxdydz &= \iiint_\Omega (r^2+1)r\,dr\,d\theta\,dz \\ &= \int_0^2 \int_0^{2\pi} \left(\int_0^1 (r^3+r)dr\right) d\theta\,dz \\ &= \left(\int_0^2 dz\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 (r^3+r)dr\right) \\ &= [z]_0^2\,[\theta]_0^{2\pi}\left[\frac{r^4}{4}+\frac{r^2}{2}\right]_0^1 \\ &= 3\pi \end{aligned}$$

Change of variable in spherical coordinates

- Spherical coordinates of a point $M(x,y,z) \in \mathbb{R}^3$ are given by:

$$\begin{cases} x = r\sin\theta\cos\varphi, \\ y = r\sin\theta\sin\varphi, \\ z = r\cos\theta, \end{cases}$$



- We define then the mapping

$$\begin{aligned} \psi \quad &: \Omega = \mathbb{R}_+^* \times [0;\pi] \times [0;2\pi] \to \mathbb{R} \\ &(r,\theta,\varphi) \mapsto (x(r,\theta,\varphi), y(r,\theta,\varphi), z(r,\theta,\varphi)) \end{aligned}$$

- The Jacobian matrix associated to the mapping ψ is given by:

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

- The Jacobian is given by

$$|J| = r^2\sin\theta$$

- Then the integral is given by

$$\iiint_D f(x,y,z)dx\,dy\,dz = \iiint_\Omega f\left(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta\right) r^2\sin\theta\,dr\,d\theta\,d\varphi$$

Example.

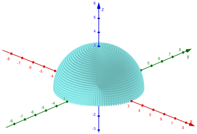
Find

$$I = \iiint_D z dx dy dz$$

where

$D = \{(x,y,z) : x^2 + y^2 + z^2 \leq R^2; z \geq 0\}$

The domain D is the upper hemisphere (centered at the origin and radius R).



Using spherical coordinates, we set then

$$\begin{cases} x = r\sin\theta\cos\varphi, \\ y = r\sin\theta\sin\varphi, \\ z = r\cos\theta, \end{cases}$$

Then $\Omega = \left\{(r,\theta,\varphi) \in \mathbb{R}; 0 \le r \le R; 0 \le \theta \le \frac{\pi}{2}; 0 \le \varphi \le 2\pi\right\}$

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{R} r\cos\theta r^2 \sin\theta \, d\varphi \, d\theta \, dr \\ &= \left(\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta \, d\theta\right)\left(\int_0^{R} r^3 \, dr\right)\left(\int_0^{2\pi} d\varphi\right) \\ &= 2\pi \times \frac{R^4}{4} \times \left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi R^4}{4} \end{aligned}$$

- **Calcul du volume** In particular for $f(x,y,z) = 1$ the domain's volume is given by

$$\iiint_D dx\, dy\, dz$$

Example.

Find the volume of the following domain

$$D = \left\{(x,y,z) \in \mathbb{R}; x \geq 0; y \geq 0; z \geq 0; x+y+z \leq 1\right\}$$

We have

$D = \left\{(x,y,z) \in \mathbb{R}; 0 \leq x \geq 1; 0 \leq y \geq 1-x; z \geq 0; 0 \leq z \leq 1-x-y1\right\}$ (Example 1)

Then

$$V(D) = \iiint_D dx\, dy\, dz = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} dz\right) dy\right) dx = \frac{1}{6}$$