The National Higher School of Artificial Intelligence | **2022-2023** | **Probability and Statistics Semestre 3 Term 2**

Midterm Exam (1h30)

Exercise 1 *We have an urn $\mathcal{U}$ containing $\frac{1}{3}$ black balls and $\frac{2}{3}$ white balls and a target $\mathcal{C}$.*

We consider the following game: A player shoots a first time at the target $\mathcal{C}$: if he hits he has won the game; otherwise he draws randomly a ball in the urn $\mathcal{U}$; if he gets a black ball he has lost the game, otherwise he puts the white ball back in the urn and shoots again to the target. He repeats the same operations another time and so on...

Let $p \in [0,1]$ be the probability that the player hits the target during a shot. Let A_n, B_n, A, B and C be the following events $(n \in \mathbb{N}^*)$

A_n *"the player wins the game after the n^{th} shot".*

B_n *"the player loses the game after the n^{th} shot".*

A *"the player wins the game"*

B *"the player loses the game"*

C *"the game does not end".*

1. *Calculate the probabilities of A_n and B_n as a function of n and p.*
2. *Calculate, as a function of p, the probabilities of A, B and C.*
3. *Let X be the random variable equal to the number of moves needed to stop the game. Determine the distribution of X and calculate $\mathbb{E}[X]$.*
4. *Knowing that the player has lost the game, calculate the probability that it is after the n^{th} shot (we will assume in this question $p \neq 1$).*
5. *Study, according to the values of p, the fact that the player has more or less chance to win at this game.*

Exercise 2 *A real random variable X is said to follow a Pareto distribution of parameters (a, b), if the probability density of X is defined by*

$$f_X(x) = \frac{ba^b}{x^{b+1}} \mathbb{I}_{[a,\infty[}(x), a > 0, b > 0.$$

1. *Determine F_X the cumulative distribution function of the r.v. X.*
2. *For what value of b, are $\mathbb{E}[X]$ and $Var(X)$ finite?*
3. *Determine the density distribution of $Y = \ln \frac{X}{b}$.*