Mathematical analysis 2
Chapter 4 : Sequences and Series of functions
Part 2: Series of functions

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Course outline

1. Generalities
2. Types of convergence for functions series
   - Pointwise convergence
   - Uniform convergence
   - Normal convergence
3. Properties of the sum of series of functions
   - Continuity of the sum
   - Integrability of the sum
   - Differentiability of the sum

Generalities

Definition

Let E be a non-empty subset of $\mathbb{R}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on E.

1. *We call a series of functions with general term f_n the expression:*
$$\sum_{n\geq 0} f_n = f_0 + f_1 + f_2 + \cdots + f_n + \dots$$
2. *The sequence of functions $(S_n)_{n\in\mathbb{N}}$ where*
$$S_n(x) = \sum_{k=0}^{n} f_k = f_0 + f_1 + f_2 + \cdots + f_n$$
is called the sequence of partial sums associated with f_n.

Definition

We call the domain of convergence of the series of functions $\sum f_n$ the set denoted by D, given by:
$$D = \{x \in E \mid \sum f_n(x) \text{ converges}\}.$$

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Pointwise and absolute convergence

Definition (Pointwise Convergence)

Let (f_n) *be a sequence of functions defined on* E.

1. *We say that the series of functions* $\sum f_n$ *is pointwise convergent at the point* x_0 *in* E *if the sequence* $(S_n(x_0))$ *is convergent.*
2. *We say that the series of functions* $\sum f_n$ *is pointwise convergent on* $I \subseteq E$ *if its sequence of partial sums* (S_n) *is pointwise convergent on* I.

In other words:

1. *The series of functions* $\sum f_n$ *is pointwise convergent at the point* x_0 *in* E *if the numerical series* $\sum f_n(x_0)$ *is convergent.*
2. *The series of functions* $\sum f_n$ *is pointwise convergent on* $I \subseteq E$ *if for every fixed* $x \in I$, *the numerical series* $\sum f_n(x)$ *is convergent. In this case, we denote:*

$$\forall x \in I, \quad F(x) = \sum_{n=0}^{\infty} f_n(x).$$

Pointwise and absolute convergence

Example.

Study the pointwise convergence of $\sum f_n$ *where for all* $x \in [0,1]$

$$f_n(x) = \frac{nx}{1+n^3x^3}.$$

- *If* $x = 0$, *then* $\sum f_n(0) = 0$ *converges.*
- *If* $x \in]0,1]$, *then* $\dfrac{nx}{1+n^3x^3} \underset{\infty}{\sim} \dfrac{1}{n^2x^2} > 0$.

  However, $\sum \dfrac{1}{n^2x^2}$ *is a convergent Riemann series.*
  Therefore, by the comparison test, $\sum f_n$ *converges pointwise on* $]0,1]$.

Conclusion: $\sum f_n$ *converges pointwise on* $[0,1]$.

Definition (Absolute Convergence)

We say that the series of functions $\sum f_n$ *converges absolutely on* I *if the series of functions* $\sum |f_n|$ *converges pointwise on* I.

Pointwise and absolute convergence

Example.

Study the pointwise convergence of $\sum f_n$ *where* $f_n(x) = \dfrac{(-1)^n x^n}{n}$, $\forall x \in [0,1]$.

- *If* $x = 0$, *then* $\sum f_n(0) = 0$ *converges.*
- *For* $x \in]0,1]$, *let's study the absolute convergence. That is to say the convergence of the series* $\sum |f_n(x)|$. *Apply the ratio test (D'Alembert's test):*

  $$\lim_{n\to+\infty} \left| \frac{f_{n+1}(x)}{f_n(x)} \right| = \lim_{n\to+\infty} \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} = x.$$

  Then for $x \in]0,1[$, $\sum |f_n|$ *converges.*
  For $x = 1$, *the ratio test is inconclusive.*
- *When* $x = 1$, $f_n(1) = (-1)^n \frac{1}{n}$. *Now,* $\sum (-1)^n \frac{1}{n}$ *is an alternating series convergent via Leibniz's theorem.*

Conclusion: $\sum f_n$ *converges pointwise on* $[0,1]$.

Pointwise and absolute convergence

Definition

*The series $\sum f_n$ is said to be **conditionally convergent** on I if it is pointwise convergent without being absolutely convergent on I.*

Proposition

1. *If $\sum f_n$ is absolutely convergent on I, then it is pointwise convergent on I.*
2. *If $\sum f_n$ is pointwise convergent on I, then the sequence of functions (f_n) converges pointwise to 0 on I.*

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Uniform convergence

Definition

Let (f_n) be a sequence of functions defined on E. We say that the series of functions $\sum f_n$ converges uniformly on $I \subseteq E$ if and only if its sequence of partial sums (S_n) converges uniformly on I.

Example.

Consider $\sum f_n$ where $f_n(x) = e^{-nx}$ for all $x \in \mathbb{R}_+^*$.

1. *Show that this series of functions converges pointwise on $\mathbb{R}_+^*$.*
2. *Does the series $\sum f_n$ converge uniformly on $\mathbb{R}_+^*$?*
3. *Provide the domains where there is uniform convergence.*

1. **Pointwise Convergence:** Let (S_n) be the sequence of partial sums related to $\sum f_n$,
$$S_n(x) = \sum_{k=0}^{n} f_k = \sum_{k=0}^{n} e^{-xk} = \frac{1-e^{-x(n+1)}}{1-e^{-x}}.$$
The limit as $n \to +\infty$ of $S_n(x)$ equals $\frac{1}{1-e^{-x}}$. Therefore, $S_n \xrightarrow{\text{Pointwise}} S$ on $\mathbb{R}_+^*$, where $S(x) = \frac{1}{1-e^{-x}}$. Hence, $\sum f_n$ converges pointwise on $\mathbb{R}_+^*$.

Uniform convergence

2. **Uniform Convergence:** To verify if $\lim_{n\to+\infty}\|S_n - S\| = 0$:
   Let $g_n(x) = |S_n(x) - S(x)| = \left|\frac{e^{-x(n+1)}}{1-e^{-x}}\right| = \frac{e^{-x(n+1)}}{1-e^{-x}}$. Notice that the function $g_n(x)$ is not bounded on $\mathbb{R}_+^*$. Indeed,
   $$\sup_{x\in\mathbb{R}_+^*}|S_n(x) - S(x)| \geq \lim_{x\to 0^+}\frac{e^{-x(n+1)}}{1-e^{-x}} = +\infty \Rightarrow \lim_{n\to+\infty}\|S_n - S\| \neq 0.$$
   Hence, $S_n \nrightarrow S$ uniformly on $\mathbb{R}_+^*$.
3. We observe that there is no uniform convergence in any interval containing the neighborhood of 0. Let's consider intervals of the form $[\alpha, +\infty[$, $\alpha > 0$:
   For all $x \in [\alpha, +\infty[$, $g_n(x) = \dfrac{e^{-x(n+1)}}{1-e^{-x}} \leq \dfrac{e^{-\alpha(n+1)}}{1-e^{-\alpha}}$ and
   $\lim_{n\to+\infty}\dfrac{e^{-\alpha(n+1)}}{1-e^{-\alpha}} = 0$. Therefore, $S_n \xrightarrow{\text{Unif}} S$ over any interval of the form $[\alpha, +\infty[$, $\alpha > 0$.

Uniform convergence

Remark

- *The previous example demonstrates that uniform convergence is highly dependent on the considered interval. It's essential to specify on which interval the uniform convergence occurs.*
- *If $\sum f_n$ converges uniformly on I, then $\sum f_n$ converges uniformly on any subset interval $A \subseteq I$.*

Uniform convergence

Definition

We call the nth remainder of a series of functions $\sum f_n$ pointwise convergent on $I \subseteq E$, the sequence of functions R_n defined as:

$$\forall x \in I, \forall n \in \mathbb{N}, \quad R_n(x) = \sum_{k=n+1}^{\infty} f_k.$$

Proposition (Necessary and sufficient condition)

Let (f_n) be a sequence of functions defined on E. Let (R_n) be the nth remainder of the series of functions $\sum f_n$. We say that the series of functions $\sum f_n$ converges uniformly on $I \subseteq E$ if and only if $R_n \xrightarrow{Unif} 0$ on I.

Proposition (Necessary condition)

Let (f_n) be a sequence of functions defined on E. If $\sum f_n$ converges uniformly on $I \subseteq E$, then $f_n \xrightarrow{Unif} 0$ on I.

Sufficient conditions for uniform convergence

Theorem (Uniform Leibniz test)

Let $\sum f_n$ *be a series of functions where the general term is of the form* $f_n(x) = (-1)^n u_n(x)$*, where* (u_n) *is a sequence of functions defined on* I*.*
If:

1. *For each* $x \in I$*, the numerical sequence* $(u_n(x))$ *is positive and decreasing sequence.*
2. *The sequence of functions* (u_n) *converges uniformly on* I *to* 0*, i.e.,* $\lim_{n\to\infty} ||u_n|| = 0$*,*

then the series $\sum f_n$ *converges uniformly on* I*.*

Sufficient conditions for uniform convergence

Example.

Study the uniform convergence of $\sum f_n$ *where* $f_n(x) = (-1)^n \frac{1}{x^2+n}$ *for all* $x \in \mathbb{R}$.

Solution: Let's use the Uniform Leibniz test. Set $u_n(x) = \frac{1}{x^2+n}$.

1. For all $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$ $u_n(x) = \frac{1}{x^2+n} > 0$.
2. For all $x \in \mathbb{R}$, $\frac{u_{n+1}}{u_n} = \frac{x^2+n}{x^2+n+1} < 1$
   thus the sequence (u_n) is decreasing.
3. Is $\lim\limits_{n\to\infty} ||u_n|| = 0$?

   For all $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$, $|u_n(x)| = \frac{1}{x^2+n} \leq \frac{1}{n}$ and $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

   Therefore, $u_n \xrightarrow{\text{Unif}} 0$ on $\mathbb{R}$. Thus, $\sum f_n$ converges uniformly on $\mathbb{R}$.

Sufficient conditions for uniform convergence

Theorem (Uniform Abel test)

Let $\sum f_n$ be a series of functions where the general term is of the form $f_n(x) = u_n(x)v_n(x)$, where (u_n) and (v_n) are two sequences of functions defined on I. If:

1. *For each $x \in I$, the numerical sequence $(u_n(x))$ is decreasing,*
2. *The sequence of functions (u_n) converges uniformly on I to 0, i.e., $\lim_{n\to\infty} ||u_n|| = 0$,*
3. *There exists a real number $M > 0$ (independent of n and x) such that $|S_n(x)| = \left|\sum_{k=0}^{n} v_k\right| \leq M$,*

then the series $\sum f_n$ converges uniformly on I.

Sufficient conditions for uniform convergence

Example.

Show that the series $\sum \frac{\cos(nx)}{n}$ *converges uniformly on every interval of the form* $[\alpha, 2\pi - \alpha]$, *where* $0 < \alpha < \pi$.

Solution: Let's apply the Uniform Abel criterion: set $u_n(x) = \frac{1}{n}$ and $v_n(x) = \cos(nx)$.

- The sequence (u_n) is decreasing, and $\lim_{n\to\infty} u_n = 0$.
- $|S_n(x)| = \left|\sum_{k=0}^{n} \cos(kx)\right| \leq \left|\frac{1}{\sin\left(\frac{\alpha}{2}\right)}\right|$ for $\alpha \leq x \leq 2\pi - \alpha$.
  Thus, $\frac{1}{\sin\left(\frac{\alpha}{2}\right)}$ is independent of n and x, denoted as M.
  Therefore, for all $x \in [\alpha, 2\pi - \alpha]$, where $0 < \alpha < \pi$, $|S_n(x)| \leq M$, independent of n and x.

Hence, $\sum \frac{\cos(nx)}{n}$ converges uniformly on every interval of the form $[\alpha, 2\pi - \alpha]$, where $0 < \alpha < \pi$.

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Normal convergence

Definition (Normal Convergence)

$\sum f_n$ is said to converge normally on $I \subseteq E$ if and only if

1. $\sup\limits_{x\in I} |f_n(x)|$ exists,
2. and the numerical series $\sum \sup\limits_{x\in I} |f_n(x)|$ converges.

Theorem (Dominated Convergence or Weierstrass test)

$\sum f_n$ converges normally on $I \subseteq E$ if there exists a numerical sequence (u_n) such that:

1. *$\forall n \geq n_0$ and $\forall x \in I$, $|f_n(x)| \leq u_n$,*
2. *and $\sum u_n$ converges.*

Theorem

If the series of functions $\sum f_n$ converges normally on $I \subseteq E$, then it converges uniformly on I.

Normal convergence

Example.

Show that $\sum f_n$ *with general term* $f_n(x) = ne^{-nx}$ *converges normally on every* $[\alpha, +\infty[$, *where* $\alpha > 0$, *but does not converge normally on* $[0, +\infty[$.

Solution:

1. For any $n \in \mathbb{N}$ and $x \in [\alpha, +\infty[$ where $\alpha > 0$:

$$|f_n(x)| = ne^{-nx} \leq ne^{-n\alpha}$$

Since $\sum ne^{-n\alpha}$ converges (reference), we deduce from the dominated convergence criterion that $\sum f_n$ converges normally, and hence uniformly, on $[\alpha, +\infty[$, where $\alpha > 0$.

2. On $[0, +\infty[$:

$$\sup_{x \in [0,+\infty[} |f_n(x)| = \sup_{x \in [0,+\infty[} ne^{-nx} = n$$

As $\sum n$ diverges, $\sum f_n$ does not converge normally on $[0, +\infty[$.

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Continuity of the sum

Theorem

Let $\sum f_n$ be a series of functions and I an interval in $\mathbb{R}$. If:

1. *All functions f_n are continuous at $a \in I$,*
2. *The series of functions $\sum f_n$ converges uniformly on I,*

then the sum function F of the series, defined as $F(x) = \sum_{n=0}^{\infty} f_n(x)$, is continuous at a.

In other words,

$$\lim_{x \to a} F(x) = \lim_{x \to a} \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \lim_{x \to a} f_n(x) = \sum_{n=0}^{\infty} f_n(a) = F(a).$$

Corollary

Let $\sum f_n$ be a series of functions and I an interval in $\mathbb{R}$. If:

1. *All functions f_n are continuous on I,*
2. *The series of functions $\sum f_n$ converges uniformly on I,*

then the sum function F of the series $F(x) = \sum_{n=0}^{\infty} f_n(x)$ is continuous on I.

Continuity of the sum

Remark

As continuity is a pointwise property, the above result remains valid if we replace the uniform convergence on I by uniform convergence on any segment $[\alpha, \beta] \subseteq I$.

Example.

Consider the series of functions $\sum f_n$ where $f_n(x) = ne^{-nx}$, $x \in [1, +\infty[$. Show that the sum function F of the series $\sum f_n$ is continuous on $[1, +\infty[$.

Solution: We have:

1. All f_n functions are continuous on $[1, +\infty[$ as they are compositions of two continuous functions.
2. $\sum f_n$ converges uniformly on $[1, +\infty[$ (already proven).

Hence, the sum function F is continuous on $[1, +\infty[$.

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Integrability of the sum

Theorem

Let $\sum f_n$ be a series of functions and $[a,b]$ an interval in $\mathbb{R}$. If:

1. *All f_n functions are integrable on $[a,b]$,*
2. *The series of functions $\sum f_n$ is uniformly convergent on $[a,b]$,*

then the sum function F of the series, defined as $F(x) = \sum_{n=0}^{\infty} f_n(x)$, is integrable on $[a,b]$, and we have

$$\int_a^b F(x)\,dx = \int_a^b \sum_{n=0}^{\infty} f_n(x)\,dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)\,dx.$$

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Differentiability of the sum

Theorem

Let $\sum f_n$ be a series of functions and I an interval not reduced to a point. If:

1. *All f_n functions are of class C^1 on I.*
2. *There exists $x_0 \in I$ such that the numerical series $\sum f_n(x_0)$ converges.*
3. *The series of derivative functions $\sum f_n'$ converges uniformly on I.*

Then:

1. *The series of functions $\sum f_n$ converges uniformly on I.*
2. *The sum function F of the series, defined as $F(x) = \sum_{n=0}^{\infty} f_n(x)$, is of class C^1 on I, and we have $F'(x) = \left(\sum_{n=0}^{\infty} f_n(x)\right)' = \sum_{n=0}^{\infty} f_n'(x)$.*

Differentiability of the sum

Example.

Let $\sum f_n$ *series of functions with* $f_n(x) = \frac{1}{n^2}e^{-nx}$ *for* $x \in \mathbb{R}$.

1. *Determine the domain of convergence* D *of this series.*
2. *Show that the sum function* F *of the series* $\sum f_n$ *is continuous on* D.
3. *Show that* F *is differentiable on* $[a, +\infty)$ *with* $a > 0$.

Solution:

1. **The domain of convergence** D of $\sum f_n$ is given by:
   $D = \{x \in \mathbb{R} \mid \sum f_n(x) \text{ converges}\}$.
   - If $x < 0$, $\lim\limits_{n\to+\infty} f_n(x) = +\infty$. Hence, $\sum f_n(x)$ diverges.
   - If $x \geq 0$, for all $x \in \mathbb{R}^+$ and $n \in \mathbb{N}^*$, $|f_n(x)| = \frac{1}{n^2}e^{-nx} \leq \frac{1}{n^2}$.
     As $\sum \frac{1}{n^2}$ converges, by the dominated convergence, $\sum f_n$ converges normally, and thus uniformly and pointwise on $\mathbb{R}^+$.

   Therefore, $D = \mathbb{R}^+$.

Differentiability of the sum

2. **Continuity of the sum**: we have
   - All f_n functions are continuous on $\mathbb{R}^+$ as compositions of two continuous functions.
   - $\sum f_n$ converges uniformly on $\mathbb{R}^+$ (already demonstrated).

   Hence, the sum function F is continuous on $\mathbb{R}^+$.

Differentiability of the sum

3. **The differentiability of the sum**
   - All f_n functions are of class C^1 on $\mathbb{R}^+$.
   - $\sum f_n$ converges pointwise on $\mathbb{R}^+$, so $\exists x_0 \in \mathbb{R}^+$ such that the numerical series $\sum f_n(x_0)$ converges.
   - Study of the uniform convergence of $\sum f_n'$:
     Consider intervals of the form $[a, +\infty[$, $a > 0$:
     For all $x \in [a, +\infty[$ and $n \in \mathbb{N}^*$,
     $|f_n'(x)| = \left|\frac{1}{n}e^{-nx}\right| \leq \frac{1}{n}e^{-n\alpha}$.
     As $\sum \frac{1}{n}e^{-n\alpha}$ converges, $\sum f_n'$ converges normally, and thus uniformly, on $[a, +\infty[$, $a > 0$.

Therefore, the sum function F of the series $\sum f_n$ is C^1 on $[a, +\infty[$, $a > 0$. Thus, F is C^1 on $]0, +\infty[$ by continuity.