Propositional natural deduction

Chapter 2, Section 4

Introduction

- Intuitively speaking, derivations in this chapter are the same thing as derivations in Chapter 2.
- But now we use formulas of LP instead of English statements.
- We need to be more precise than this, for two reasons :

1. We want to be sure that we can check unambiguously whether a given diagram is a derivation or not.
2. We need a description of derivations that will support our later mathematical analysis (e.g. the Soundness proof, or the general results about provability).

Introduction

- Our starting point will be the fact that the derivations of Chapter 1 have a tree-like feel to them.
- But they have their root at the bottom and they branch upwards instead of downwards.
- We can borrow the definitions of Section 3.2, but with the trees the other way up.
- We will think of derivations as a kind of left-and right-labelled tree, and we will define exactly which trees we have in mind.
- In practice we will continue to write derivations in the style of Chapter 2.

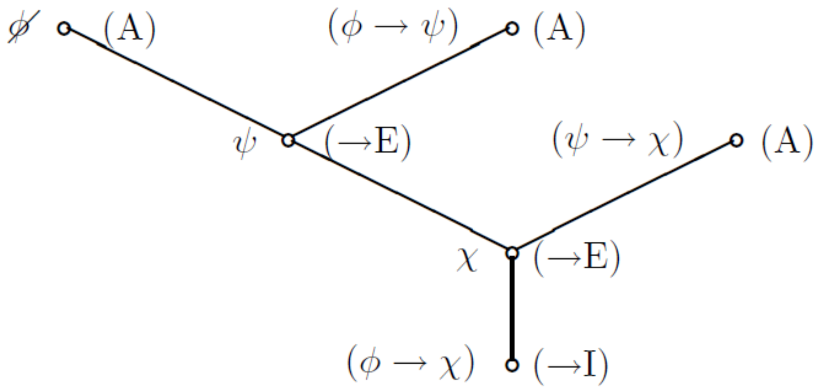
Tree of a derivation

To illustrate our approach, here is the derivation of Example 2.4.5:

$$\dfrac{\dfrac{\dfrac{\cancel{\phi}\ \textcircled{1} \qquad (\phi \to \psi)}{\psi}\,(\to\mathrm{E}) \qquad (\psi \to \chi)}{\chi}\,(\to\mathrm{E})}{\textcircled{1}\ (\phi \to \chi)}\,(\to\mathrm{I}) \tag{3.33}$$

Here is our tree version of it.

(3.34)



Tree of a derivation

- The formulas are the left labels.
- The right-hand label on a node tells us the rule that was used to bring the formula to the left of it into the derivation.
- Formulas not derived from other formulas are allowed by the Axiom Rule of Section 2.1, so we label them (A).
- Also we leave out the numbering of the discharged assumptions, which is not an essential part of the derivation.

Definition of a derivation

- We can give a mathematical definition of derivation that runs along the same lines as Definition 3.2.4 for parsing trees.
- The definition is long and repeats things we said earlier; so we have spelt out only the less obvious clauses.
- You should check that the conditions in (d)–(g) correspond exactly to the natural deduction rules as we defined them in Chapter 2.

Definition of a σ −derivation

Definition 3.4.1

Let σ be a signature. Then a *σ-derivation* or, for short, a *derivation* is a left-and-right-labelled tree (drawn branching upwards) such that the following hold:

(a) Every node has arity 0, 1, 2 or 3.

(b) Every left label is either a formula of LP(σ), or a formula of LP(σ) with a dandah.

(c) Every node of arity 0 carries the right-hand label (A).

Definition of a σ −derivation (arity 1)

(d) If ν is a node of arity 1, then one of the following holds:

(i) ν has right-hand label (→I), and for some formulas ϕ and ψ, ν has the left label $(\phi \to \psi)$ and its daughter has the left label ψ;

(ii) ν has right-hand label (¬I) (resp. (RAA)), the daughter of ν has left label ⊥, and if the right-hand label on ν is (¬I) (resp. (RAA)) then the left label on ν is of the form $(\neg\, \phi)$ (resp. ϕ).

Exercise : Complete the definition by (iii), (iv), (v) corresponding to (∧E), (∨I) and (↔E).

Definition of a σ −derivation (arity 2)

(e) If ν is a node of arity 2, then one of the following holds:

(i) ν has right-hand label (→E), and there are formulas ϕ and ψ such that ν has the left label ψ, and the left labels on the daughters of ν are (from left to right) ϕ and $(\phi \to \psi)$.

Exercise :

Complete the definition by (ii), (iii), (iv) corresponding to $(\wedge \mathrm{I})$, $(\neg$ E) and (↔I).

Definition of a σ −derivation (arity 3)

(f) If ν is a node of arity 3, then the right-hand label on ν is ($\vee$E), and there are formulas ϕ, ψ such that the leftmost daughter of ν is a node with left label $(\phi \vee \psi)$, and the other two daughters of ν carry the same left label as ν,

Definition of a σ —derivation (dandah)

(g) If a node μ has left label χ with a dandah, then μ is a leaf, and the branch to μ (Definition 3.2.2(d)) contains a node ν where one of the following happens :

(i) Case (d)(i) occurs with formulas ϕ and ψ, and ϕ is χ,

(ii) Case (d)(ii) occurs; if the right-hand label on ν is (¬I) then the left label on ν is $(\neg\chi)$, while if it is (RAA) then χ is $(\neg\,\phi)$ where ϕ is the left label on ν.

(iii) ν has label (∨E) with formulas ϕ and ψ as in Case (f), and either χ is ϕ and the path from the root to ν goes through the middle daughter of ν, or χ is ψ and the path goes through the right-hand daughter.

The *conclusion* of the derivation is the left label on its root, and its *undischarged assumptions* are all the formulas that appear without dandah as left labels on leaves. The derivation is a *derivation of* its conclusion.

Algorithm for recognizing a $\boldsymbol{\sigma}$ $-$derivation

Theorem 3.4.2

Let σ be a finite signature, or the signature $\{p_0, p_1, \cdots\}$*. There is an algorithm that, given any diagram, will determine in a finite amount of time whether or not the diagram is a σ-derivation.*

Example

Example 3.4.3

Suppose D is a σ-derivation whose conclusion is $\bot$, and ϕ is a formula of LP(σ). Let D' be the labelled tree got from D by adding one new node below the root of D, putting left label ϕ and right label (RAA) on the new node, and writing a dandah on $(\neg\, \phi)$ whenever it labels a leaf. We show that D' is a σ-derivation. The new node has arity 1, and its daughter is the root of D. Clearly (a)–(c) of Definition 3.4.1 hold for D' since they held for D. In (d)–(f) we need to only check for (d)(ii), the case for (RAA); D' satisfies this since the root of D carried $\bot$. There remains (g) with $(\neg\, \phi)$ for χ: here D' satisfies (g)(ii), so the added dandahs are allowed.

Example

You will have noticed that we wrote *D′* as

(3.35)

$$\dfrac{\begin{array}{c}\cancel{(\neg\phi)}\\ D\\ \bot\end{array}}{\phi}\ (\text{RAA})$$

in the notation of Chapter 2. We will continue to use that notation, but now we also have Definition 3.4.1 to call on when we need to prove theorems about derivations.

σ —Sequents

Definition 3.4.4 Let σ be a signature. A *σ-sequent*, or for short just a *sequent*, is an expression

(3.36) $$\Gamma \vdash_{\sigma} \psi$$

where ψ is a formula of LP(σ) (the *conclusion* of the sequent) and Γ is a set of formulas of LP(σ) (the *assumptions* of the sequent). The sequent (3.36) means

(3.37) There is a σ-derivation whose conclusion is ψ and whose undischarged assumptions are all in the set Γ.

When (3.37) is true, we say that the sequent is *correct*, and that the σ-derivation *proves* the sequent. The set Γ can be empty, in which case we write the sequent as

(3.38) $$\vdash_{\sigma} \psi$$

This sequent is correct if and only if there is a σ-derivation of ψ with no undischarged assumptions.

When the context allows, we leave out σ and write $\vdash_{\sigma}$ as $\vdash$.

(See Exercises 3.4.3 and 3.4.4)

Example

Example 3.4.3 (continued) Let Γ be a set of formulas of LP(σ) and ϕ a formula of LP(σ). We show that if the sequent $\Gamma \cup \{(\neg\phi)\} \vdash_\sigma \bot$ is correct, then so is the sequent $\Gamma \vdash_\sigma \phi$. Intuitively this should be true, but thanks to the definition (3.37) we can now prove it mathematically. By that definition, the correctness of $\Gamma \cup \{(\neg\phi)\} \vdash_\sigma \bot$ means that there is a σ-derivation *D* whose conclusion is $\bot$ and whose undischarged assumptions are all in $\Gamma \cup \{(\neg\phi)\}$. Now let *D′* be the derivation constructed from *D* earlier in this example. Then *D′* has conclusion ϕ and all its undischarged assumptions are in Γ, so it proves $\Gamma \vdash_\sigma \phi$ as required.

The Greek metavariables ϕ, ψ etc. are available to stand for any formulas of LP. For example, the derivation in Example 2.4.5 whose tree we drew is strictly not a derivation but a pattern for derivations. Taking σ as the default signature $\{p_0, p_1, \dots\}$, an example of a genuine σ-derivation that has this pattern is

$$\textcircled{1}\ \dfrac{\dfrac{\dfrac{\cancel{p_5}^{\textcircled{1}} \qquad (p_5 \to p_3)}{p_3}\,(\to\mathrm{E}) \qquad (p_3 \to \bot)}{\bot}\,(\to\mathrm{E})}{(p_5 \to \bot)}\,(\to\mathrm{I})$$

This derivation is a proof of the sequent

$$\{(p_5 \to p_3), (p_3 \to \bot)\} \vdash_\sigma (p_5 \to \bot).$$

In practice, we will continue to give derivations using Greek letters, as a way of handling infinitely many derivations all at once.

Unacceptable Sequents

A sequent $\Gamma \vdash \psi$ is unacceptable if there is a way of reading the propositional symbols in it so that Γ becomes a set of truths and ψ becomes a falsehood.

Example 3.4.6

We show that the sequent $\{(p_0 \to p_1)\} \vdash p_1$ is unacceptable. To do this we *interpret* the symbols p_0 and p_1 by making them stand for certain sentences that are known to be true or false. The following example shows a notation for doing this :

$$(3.39) \qquad \begin{array}{ll} p_0 & 2 = 3 \\ p_1 & 2 = 3 \end{array}$$

Under this interpretation p_1 is false, but $(p_0 \to p_1)$ says 'If 2 = 3 then 2 = 3', which is true. So any rule which would deduce p_1 from $(p_0 \to p_1)$ would be unacceptable.

Counterexample

Definition 3.4.7

Let $(\Gamma \vdash \psi)$ be a σ-sequent, and let *I* be an interpretation that makes each propositional symbol appearing in formulas in the sequent into a meaningful sentence that is either true or false. Using this interpretation, each formula in the sequent is either true or false. (For the present, this is informal common sense; in the next section we will give a mathematical definition that allows us to calculate which formulas are true and which are false under a given interpretation.) We say that *I* is a *counterexample* to the sequent if *I* makes all the formulas of Γ into true sentences and ψ into a false sentence.