Academic year (2023/2024)
Mathematical analysis 2

Work sheet N 1
Multivariable functions

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Exercise 1. Let $u = (x, y) \in \mathbb{R}^2$. We define the following norms:

$$\|u\|_1 = |x| + |y|, \quad \|u\|_2 = \sqrt{x^2 + y^2}, \quad \|u\|_3 = \max(|x|, |y|).$$

Show that $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_3$ are equivalent norms over $\mathbb{R}^2$.

Exercise 2. Show that the union of a family of open sets is an open set. Deduce that an intersection of closed sets is closed.

Exercise 3. Show that any open subset of $\mathbb{R}^2$ is the union of open balls.

Exercise 4. Represent the definition sets of the following functions

$$f_1(x,y) = \ln(2x + y - 2), \quad f_2(x,y) = \sqrt{1 - xy},$$
$$f_3(x,y) = \frac{\ln(y - x)}{x}, \qquad f_4(x,y) = \frac{1}{\sqrt{x^2 + y^2 - 1}} + \sqrt{4 - x^2 - y^2}.$$

Exercise 5. Represent the level curves (Solutions of the equation $f(x, y) = k$)

$$f_1(x,y) = y^2, \text{ with } k = -1 \text{ and } k = 1, \quad f_2(x,y) = \frac{x^4 + y^4}{8 - x^2y^2}, \text{with } k = 2.$$

Exercise 6. Finde the limite of each function at the indecated point.

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + y^4}{x^2 + y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{2x^2 + 3xy + y^2}{x^2 + 5y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{(x + y)^2}{x^2 + y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2(y)}{x^2 + 3y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{\arctan((x + y)^2)}{x^2} \qquad \lim_{(x,y)\to(0,0)} \frac{x^2 y e^y}{x^4 + 4y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \qquad \lim_{(x,y)\to(0,0)} \frac{x^2 y^3}{2x^2 + y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{|x| + |y|}{x^2 + y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^2}{x^2 + y^2} \qquad \lim_{(x,y)\to(1,2)} \frac{y \sin(x + 1)}{x^2 - 2x + 1} \qquad \lim_{(x,y)\to(0,0)} x e^{x/y};$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \qquad \lim_{(x,y)\to(0,1)} \frac{x \ln y}{\sqrt{x^2 + (y - 1)^2}} \qquad \lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2 + y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2} \qquad \lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4} \qquad \lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2 + y^2}}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - x^3 + y^2 + y^3}{x^2 + y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{x - y}{x + y} \qquad \lim_{(x,y)\to(0,0)} (x^2 + y^2) \ln(x^2 + y^2) + 5$$

$$\lim_{(x,y)\to(0,0)} \frac{y^2 \sin^2(x)}{x^4 + y^4} \qquad \lim_{(x,y,z)\to(\pi,0,1/3)} e^{y^2} \tan(xz) \qquad \lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}.$$

Exercise 7. Examine the continuity and differentiability of functions defined by

$$f(0,0)=0 \text{ and } \quad f(x,y)=\frac{x^4+y^4}{x^2+xy+y^2},\ (x,y)\neq(0,0)$$

$$f(0,0)=0 \text{ and } \quad f(x,y)=\frac{xy^2}{x^2+y^4},\ (x,y)\neq(0,0)$$

$$f(0,0)=0 \text{ and } \quad f(x,y)=\frac{y\sin x^2-x\sin y^2}{x^2+y^2},\ (x,y)\neq(0,0)$$

$$f(0,0)=0 \text{ and } \quad f(x,y)=\frac{x^2\ln(1+|y|)}{\sqrt{x^2+y^2}},\ (x,y)\neq(0,0)$$

Exercise 8. Find the first partial derivatives of the following functions

$$\begin{array}{lll}
(x,y)\mapsto x^4+5xy^3 & (x,y)\mapsto x^2y-3y^4 & (x,y)\mapsto x^3\sin y \\
(x,t)\mapsto e^{xt} & (x,t)\mapsto \ln(x+t^2) & (u,v)\mapsto \dfrac{u}{v^2} \\
(x,y)\mapsto ye^{xy} & (x,y)\mapsto (x^2+xy)^3 & (x,y)\mapsto y(x+x^2y)^5 \\
(x,y)\mapsto \dfrac{x}{x+y^2} & (x,y)\mapsto \dfrac{ax+by}{cx+dy} & (u,v)\mapsto \dfrac{e^v}{u+v^2} \\
(u,v)\mapsto (u^2v-v^3)^5 & (r,\theta)\mapsto \sin(r\cos\theta) & (p,q)\mapsto \tan^{-1}(pq^2) \\
(x,y)\mapsto xy & (x,y)\mapsto \displaystyle\int_y^x \cos(e^t)\,dt & (\alpha,\beta)\mapsto \displaystyle\int_\alpha^\beta \sqrt{t^3+1}\,dt \\
(x,y,z)\mapsto x^3yz^2+2yz & (x,y,z)\mapsto xy^2\exp(-xz) & (x,y,z)\mapsto \ln(x+2y+3z) \\
(x,y,z)\mapsto y\tan(x+2z) & (t,u,v)\mapsto \sqrt{t^4+u^2\cos v} & (x,y,z)\mapsto \dfrac{xy}{z} \\
(x,y,z,)\mapsto x^2y\cos\left(\frac{z}{t}\right) & (x,y,z,t)\mapsto \dfrac{\alpha x+\beta y^2}{\gamma z+\delta t^2} &
\end{array}$$

Exercise 9. Find all the second partial derivatives

$$\begin{array}{lll}
f(x,y)=x^4y-2x^3y^2 & f(x,y)=\ln(ax+by) & f(x,y)=\dfrac{y}{x+3y} \\
f(r,\theta)=e^{-2r}\cos\theta & f(s,t)=\sin(s^2-t^2) & f(x,y)=\arctan\left(\dfrac{x+y}{1-xy}\right)
\end{array}$$

Exercise 10. Study the differentiability of the following functions.

$$f(x,y)=\begin{cases}\dfrac{(x+y)\sin(xy)}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases} \qquad f(x,y)=\begin{cases}(x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

Exercise 11. Assume that all the given functions are differentiable.

1. If $z=f(x,y)$, where $x=r\cos(\theta)$ and $y=r\sin(\theta)$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ and show that:

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

2. If $u=f(x,y)$, where $x=e^s\cos(t)$ and $y=e^s\sin(t)$, show that:

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]$$

Exercise 12. Assume that all the given functions have continuous second-order partial derivatives. 1. Show that any function of the form

$$z = f(x+at) + g(x-at)$$

is a solution of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

[Hint: Let $u = x + at$ and $v = x - at$.]

2. If $z = f(x,y)$, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}.$$

Exercise 13. (Homogeneous Functions) A function f is called homogeneous of degree n if it satisfies the equation

$$f(tx, ty) = t^n f(x,y)$$

for all t, where n is a positive integer and f has continuous second-order partial derivatives.

1. Verify that $f(x,y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.
2. Show that if g is homogeneous of degree n, then

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = ng(x,y).$$

[Hint: Use the Chain Rule to differentiate $g(tx, ty)$ with respect to t.]

Also, show that

$$x^2\frac{\partial^2 g}{\partial x^2} + 2xy\frac{\partial^2 g}{\partial x \partial y} + y^2\frac{\partial^2 g}{\partial y^2} = n(n-1)g(x,y).$$

3. If h is homogeneous of degree n, show that

$$\frac{\partial x h(tx,ty)}{\partial x} = t^{n-1}\frac{\partial x h(x,y)}{\partial x}.$$

Exercise 14.

1. Give the Taylor expansion up to order 2 of the function $f(x,y) = e^x \sin(x+y)$ around the point $(0,0)$.
2. Give the Taylor expansion up to order 3 of the function $g(x,y) = -x^2 + 2xy + 3y^2 - 6x - 2y + 4$ around the point $(-2,1)$.

Exercise 15. Use Implicit Differentiation to find dy/dx

1) $y\cos(x) = x^2 + y^2$ 2) $\cos(xy) = 1 + \sin(y)$ 3) $\tan^{-1}(x^2y) = x + xy^2$ 4) $e^y \sin(x) = x + xy$.

Exercise 16. Use Implicit Differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$

1) $x^2 + 2y^2 + 3z^2 = 1$ 2) $x^2 - y^2 + z^2 - 2z = 4$ 3) $e^z = xyz$ 4) $yz + x\ln(y) = z^2$.

Additional exercises

Exercise 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x,y) = \dfrac{x+y}{1+x^2+y^2}$.

1. Determine and represent its level curves.
2. Calculate its first partial derivatives.
3. Write the equation of the tangent plane to f at (0,0).

Exercise 2. Let f be the function defined on $\mathbb{R}^2$ by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2-y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

1. Show that f is continuous on $\mathbb{R}^2$.
2. Calculate $\nabla f(x,y)$.
3. Show that ∇f is continuous on $\mathbb{R}^2$.
4. Demonstrate that f has second partial derivatives at every point.
5. What can you deduce from the calculation of $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$?

Exercise 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as follows:

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

1. Is it continuous on $\mathbb{R}^2$?
2. Calculate $\nabla f(x,y)$.
3. Is the function f of class $C^1(\mathbb{R}^2)$?
4. What can be concluded about the differentiability of the function f on $\mathbb{R}^2$?

Exercise 4. Suppose that the equation $F(x,y,z) = 0$ implicitly defines each of the three variables x, y, and z as functions of the other two: $z = f(x,y)$, $y = g(x,z)$, $x = h(y,z)$. If F is differentiable and $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, and $\frac{\partial F}{\partial z}$ are all nonzero, show that

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = -1.$$
