Mathematical analysis 3
Chapter 2 : Integral transformations
Part 2: Laplace transform

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Course outline

Definition of Laplace transform

Definition

Let $t \mapsto f(t)$ be a function from $\mathbb{R}^+$ to $\mathbb{R}$. The Laplace transform of f, denoted by $L(f)$ or $\mathscr{L}(f)$, is defined by:

$$L(f)(x) = \int_0^{+\infty} f(t)e^{-xt}\,dt.$$

Remark

1. *We also use the notation F to denote $L(f)$. F is called the image of f and f is called the original of F.*
2. *The Laplace transform is defined in the same way for complex x.*

Definition of Laplace transform

Example.

If $U(t) = 1$ *for* $t \geq 0$ *and is zero for* $t < 0$ *(called the unit step function or Heaviside function), then the Laplace transform of* U *is defined if* $x > 0$ *and we have:*

$$L(U)(x) = \int_0^{+\infty} e^{-xt}\,dt = \frac{1}{x}.$$

Existence

The function F only exists if the integral defining it is convergent.

Definition

A function f is said to be of exponential order at infinity if for sufficiently large t we have:

$$|f(t)| \leq Me^{\rho t},$$

where $\rho \in \mathbb{R}$ and $M > 0$ are constants.

Practical remark

1. f is of exponential order at infinity if and only if there exists $k \in \mathbb{R}$ such that $\lim\limits_{t \to +\infty} e^{-kt} f(t) = 0$.
2. If the constants M and ρ exist, they are not unique.

Existence

Example:
Bounded functions (such as sine and cosine) are of exponential order at infinity. Indeed, if f is bounded by a constant C, then

$$|f(t)e^{-t}| \leq Ce^{-t} \to 0, \text{ as } t \to +\infty.$$

Existence

Theorem

If f satisfies the following two conditions:

- f *is piecewise continuous on every interval* $[0,L] \subset [0,+\infty[$,
- f *is of exponential order* ρ *at infinity,*

then the Laplace transform of f *exists for all* $x > \rho$ *and*
$\lim\limits_{x \to +\infty} L(f)(x) = 0$.

Abscissa of convergence

Theorem

Suppose there exists $x \in \mathbb{R}$ *such that* $\int_0^{+\infty} f(t)e^{-xt}\,dt$ *converges.*

Then, for any $x_0 > x$, $\int_0^{+\infty} f(t)e^{-x_0 t}\,dt$ *also converges.*

Theorem (Abscissa of convergence)

Let f *be a function from* $\mathbb{R}^+$ *to* $\mathbb{R}$ *that is continuous. Then, there exists a unique* $\rho \in \mathbb{R}$ *(called Abscissa of convergence) such that*

$$x > \rho \Rightarrow \int_0^{+\infty} f(t)e^{-xt}\,dt \text{ converges } (L(f) \text{ exists}).$$

$$x < \rho \Rightarrow \int_0^{+\infty} f(t)e^{-xt}\,dt \text{ diverges } (L(f) \text{ does not exist}).$$

Examples

For $f(t) = 0$ for all $t < 0$

L'origine f en t	L'image $F = \mathscr{L}(f)$ en x	Conditions d'existence
1	$\frac{1}{x}$	$x > 0$
t^n	$\frac{n!}{x^{n+1}}$	$x > 0$
$\sqrt{t}$	$\frac{1}{2}\sqrt{\frac{\pi}{x^3}}$	$x > 0$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{x}}$	$x > 0$
e^{at}	$\frac{1}{x-a}$	$x > a$
te^{at}	$\frac{1}{(x-a)^2}$	$x > a$
$\cos(at)$	$\frac{x}{x^2+a^2}$	$x > 0$
$\sin(at)$	$\frac{a}{x^2+a^2}$	$x > 0$
$\mathrm{ch}(at)$	$\frac{x}{x^2-a^2}$	$x > \|a\|$
$\mathrm{sh}(at)$	$\frac{a}{x^2-a^2}$	$x > \|a\|$

Properties of Laplace Transform

Proposition (Linearity)

Let f and g be two causal functions admitting Laplace transforms $L(f)$ and $L(g)$ respectively, such that

- $L(f)$ *exists for* $x > \rho_1$,
- $L(g)$ *exists for* $x > \rho_2$.

Then, for any $\alpha, \beta \in \mathbb{R}$, *we have*

$$L(\alpha f + \beta g)(x) = \alpha L(f)(x) + \beta L(g)(x), \forall x > \max(\rho_1, \rho_2)$$

Example: Calculate the Laplace transform of the function

$$f(t) = 3\cos(t) - 2t.$$

We have

$$L(f)(x) = 3L(\cos(t))(x) - 2L(t)(x) = \frac{3x}{x^2+1} - \frac{2}{x^2}, \quad \forall x > 0.$$

Properties of Laplace Transform

Proposition (Laplace Transform of Translation)

For any $\alpha > 0$, we have $L(f(t-\alpha))(x) = e^{-\alpha x}F(x)$, for all $x > \rho$.

Example: Since $L(\sin(t))(x) = \dfrac{1}{x^2+1}$ for all $x > 0$, then

$$L(\sin(t-3))(x) = \frac{e^{-3x}}{x^2+1}, \text{ for all } x > 0$$

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Proposition (Translated Laplace Transform)

For any $\alpha \in \mathbb{R}$, we have $L(e^{\alpha t}f(t))(x) = F(x-\alpha)$, for all $x > \rho + \alpha$.

Example: Since $L(\cos(t))(x) = \dfrac{x}{x^2+1}$ for all $x > 0$, then

$$L(e^{2t}\cos(t))(x) = \frac{x-2}{(x-2)^2+1}, \quad \forall x > 2$$

Properties of Laplace Transform

Proposition

For any $\alpha > 0$,

$$L(f(\alpha t))(x) = \frac{1}{\alpha} F\left(\frac{x}{\alpha}\right), \forall x > \alpha\rho$$

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Example: Since $L(\sin(t))(x) = \dfrac{1}{x^2+1}$ for all $x > 0$, then

$$L(\sin(2t))(x) = \frac{2}{x^2+4}, \forall x > 0$$

. Since $L(\cos(t))(x) = \dfrac{x}{x^2+1}$ for all $x > 0$, then

$$L(\cos(3t))(x) = \frac{x}{x^2+9}, \forall x > 0$$

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Properties of Laplace Transform

Proposition (Laplace Transform of the Derivative)

Let f *be a function from* $\mathbb{R}^+$ *to* $\mathbb{R}$ *such that*

- f *is piecewise continuous on each interval* $[0,N] \subset \mathbb{R}^+$.
- f *is of exponential order at the infinity.*
- f *differentiable on* $\mathbb{R}^+$.
- f' *piecewise continuous on each interval* $[0,N] \subset \mathbb{R}^+$

and if $L(f)(x) = F(x)$ *for all* $x > \rho$. *Then*

$$L(f'(t))(x) = xF(x) - f(0^+), \forall x > \rho$$

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Properties of Laplace Transform

Proposition (Generalization)

Let f be a function from $\mathbb{R}^+$ to $\mathbb{R}$.

- *For all $k \in \{0, \ldots, n\}$, $f^{(k)}$ exists and is piecewise continuous on $\mathbb{R}^+$.*
- *For all $k \in \{0, \ldots, n-1\}$, there exist M and $\rho \in \mathbb{R}$ such that $|f^{(k)}(t)| \leq Me^{\rho t}$ for t large enough.*

Then, for all $x > \rho$, we have

$$L(f^{(n)}(t))(x) = x^n F(x) - x^{n-1} f(0^+) - \ldots - x f^{(n-2)}(0^+) - f^{(n-1)}(0^+).$$

Properties of Laplace Transform

Proposition (Derivative of the Laplace Transform)

If $f:\mathbb{R}^+ \to \mathbb{R}$ such that $L(f)(x) = F(x)$ for $x > \rho$, then F is C^∞ on $]\rho, +\infty[$ and for all $n \in \mathbb{N}$ we have

$$F^{(n)}(x) = (-1)^n L(t^n f(t))(x), \quad \forall x > \rho.$$

Example. Calculate $L(t^2 \sin(t))$.

We set : $f(t) = \sin(t)$, then we have $F(x) = L(f(t)) = \frac{1}{x^2+1}$ for $x > 0$. So

$L[t^2 \sin(t)] = (-1)^2 F''(x) = \left(\frac{1}{(x^2+1)}\right)''$ for $x > 0$.

$L[t^2 \sin(t)] = \dfrac{6x^2 - 2}{(x^2+1)^3}$ for $x > 0$.

Properties of Laplace Transform

Proposition (Integration of the Laplace Transform)

If $f:\mathbb{R}^+ \to \mathbb{R}$ *such that* $L(f)(x) = F(x)$ *for* $x > \rho$ *and* $\lim_{t\to 0^+} \frac{f(t)}{t}$ *is finite, then*

$$\int_x^{\infty} F(s)\,ds = L\left(\frac{f(t)}{t}\right)(x), \forall x > \rho.$$

Proposition (Transform of the Primitive)

Let f *be a function from* $\mathbb{R}^+$ *to* $\mathbb{R}$ *such that* $L(f) = F$ *for* $x > \rho$. *Then,*

$$L\left(\int_0^t f(s)\,ds\right)(x) = \frac{F(x)}{x}, \forall x > \max(0,\rho).$$

Properties of Laplace Transform

Proposition

If f is a T-periodic function, defined, continuous, and bounded on $\mathbb{R}^+$, then

$$L(f)(x) = \frac{\int_0^T f(t)e^{-xt}}{1-e^{-xT}}\,dt, \forall x > \max(0,\rho).$$

Initial and final value

Proposition

Suppose f and f' are piecewise continuous on the interval $[0, L]$ and of exponential order.
1) If f has a Laplace transform $L(f) = F$ for $x > \rho$, then

$$\lim_{x \to +\infty} xL(f)(x) = \lim_{t \to 0} f(t).$$

2) If f has a Laplace transform $L(f) = F$ for $x > \rho$ with $\rho < 0$, and $\lim_{t \to +\infty} f(t)$ exists, then

$$\lim_{x \to 0} xL(f)(x) = \lim_{t \to +\infty} f(t).$$

Laplace transform of the convolution product

Definition

The convolution product of two functions f and g is defined by:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(u)g(t-u)\,du$$

Proposition (Laplace Transform of the Convolution Product)

Let f and g be two functions from $\mathbb{R}^+$ to $\mathbb{R}$ such that $L(f) = F$ and $L(g) = G$ respectively for $x > \rho_1$ and $x > \rho_2$ respectively. Then,

$$L(f * g)(x) = F(x)G(x), \forall x > \max(\rho_1, \rho_2)$$