Academic year (2023/2024)
Mathematical analysis 2

Midterm
Duration: 2 hours

Exercise 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as follows:

$$f(x,y) = \begin{cases} \dfrac{x^2(y+1)}{x^2+(y+1)^2} & \text{if } (x,y) \neq (0,-1), \\ 0 & \text{if } (x,y) = (0,-1). \end{cases}$$

1. Study of the function on $\mathbb{R}^2 \setminus \{(0,-1)\}$:
   a) Show that f is continuous on $\mathbb{R}^2 \setminus \{(0,-1)\}$.
   b) Calculate the gradient of f for $(x,y) \in \mathbb{R}^2 \setminus \{(0,-1)\}$.
   c) Show that f is of class C^1 on $\mathbb{R}^2 \setminus \{(0,-1)\}$.
   d) What can we conclude about the differentiability of f on $\mathbb{R}^2 \setminus \{(0,-1)\}$?
2. Study of the function at $(0,-1)$:
   a) Show that f is continuous at $(0,-1)$.
   b) Calculate the gradient of f at $(0,-1)$.
   c) Show that f is not differentiable at $(0,-1)$.
   d) Is f of class C^1 at $(0,-1)$?

Exercise 2. Let f be the function defined on $\mathbb{R}^2$ by

$$f(x,y) = x^2 + y^2 - 2(x+y) + 1$$

1. Represent the level curve at $k = 0$.
2. State the implicit function theorem for a function of two variables.
3. Show that the equation $f(x,y) = 0$ defines implicitly $y = \varphi(x)$ in the neighborhood of the point $(\frac{2+\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2})$.
4. calculate φ' in the neighborhood of the point $(\frac{2+\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2})$.
5. Find the equation of the tangent line to the graph of φ at point $(\frac{2+\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2})$.

Exercise 3. Find the area of the following domains without using change of variable

1. $D_1 = \{(x,y) \in \mathbb{R}^2 : |x-y| \leq 1, |x+y| \leq 1\}$.
2. D_2 is delimited by the curves of equations $y = 1 - x^2$ and $y = (x-1)^2$.

Dr. KECHKAR

Exercise 1. (10.5pts)

1. Study of the function on $\mathbb{R}^2 \setminus \{(0,-1)\}$:

   (a) **(1pt)** f is continuous on $\mathbb{R}^2 \setminus \{(0,-1)\}$ since it is the quotient, sum, product and composition of continuous functions on $\mathbb{R}^2 \setminus \{(0,-1)\}$.

   (b) **(2.5pts= 0.5pt existence of partial derivatives+ 1pt first partial derivative+ 1pt second partial derivative)**
   •f is derivable on $\mathbb{R}^2 \setminus \{(0,-1)\}$ since it is the quotient, sum, product and composition of derivable functions on $\mathbb{R}^2 \setminus \{(0,-1)\}$.
   •The gradient of f for $(x,y) \in \mathbb{R}^2 \setminus \{(0,-1)\}$ is the vector with components

$$\frac{\partial f}{\partial x}(x,y) = \frac{2x(y+1)^3}{(x^2+(y+1)^2)^2}, \quad \frac{\partial f}{\partial y}(x,y) = x^2\frac{x^2-(y+1)^2}{(x^2+(y+1)^2)^2}.$$

   (c) **(1pt)** f is continuous and derivable on $\mathbb{R}^2 \setminus \{(0,-1)\}$; its partial derivatives are continuous on $\mathbb{R}^2 \setminus \{(0,-1)\}$ as they are the quotients of continuous functions. Therefore, f is of class $C^1(\mathbb{R}^2 \setminus \{(0,-1)\})$.

   (d) **(1pt)** Since f is of class $C^1(\mathbb{R}^2 \setminus \{(0,-1)\})$, f is differentiable on $\mathbb{R}^2 \setminus \{(0,-1)\}$.

2. Study of the function at $(0,-1)$:

   (a) **(1pt)** The function f continuous at $(0,-1)$ if and only if $\lim\limits_{(x,y)\to(0,-1)} f(x,y) = f(0,-1)$. We switch to polar coordinates:

$$x = 0 + r\cos(\theta), \quad y = -1 + r\sin(\theta).$$

   $f(r,\theta) = f(x(r,\theta), y(r,\theta)) = r\cos^2(\theta)\sin(\theta)$. Since $|f(r,\theta)| \leq r \to 0$ as $r \to 0$, we have

$$\lim_{(x,y)\to(0,-1)} f(x,y) = 0 = f(0,-1),$$

   and therefore, f is continuous at $(0,-1)$.

   (b) **(2pts: 1pt for each partial derivative)** The gradient of f at $(0,-1)$ is the vector with components

$$\frac{\partial f}{\partial x}(0,-1) = \lim_{x\to 0}\frac{f(x,-1)-f(0,-1)}{x-0} = 0, \quad \frac{\partial f}{\partial y}(0,-1) = \lim_{y\to -1}\frac{f(0,y)-f(0,-1)}{y+1} = 0.$$

   (c) **(1pt)**To prove that f is not differentiable at $(0,-1)$, it is necessary to show that

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k-1)-f(0,-1)-h\frac{\partial f}{\partial x}(0,-1)-k\frac{\partial f}{\partial y}(0,-1)}{\sqrt{h^2+k^2}} \neq 0.$$

   Let $H(h,k) = \dfrac{f(h,k-1)-f(0,-1)-h\frac{\partial f}{\partial x}(0,-1)-k\frac{\partial f}{\partial y}(0,-1)}{\sqrt{h^2+k^2}} = \dfrac{h^2k}{(h^2+k^2)^{3/2}}$.
   Since $H(h,h) = \frac{1}{2^{3/2}} \neq 0$, then f is not differentiable at $(0,-1)$.

   (d) **(1pt)** Since f is not differentiable at $(0,-1)$, it is not of class C^1 at $(0,-1)$.

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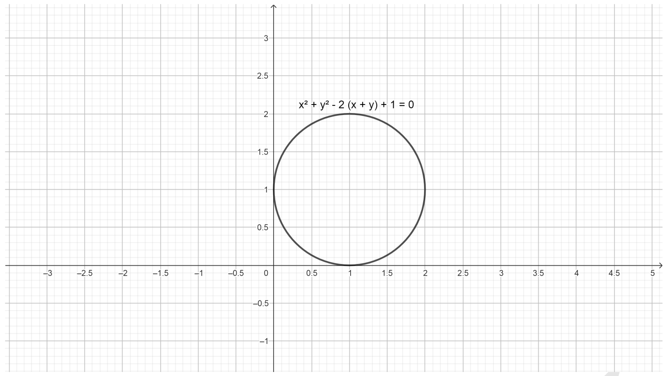
Exercise 2. (5.5pts)

1. **1pt=0.5pt identification of the curve+0.5pt drawing**
At $k=0$ $f(x,y)=0 \iff (x-1)^2+(y-1)^2=1$, it is a circle centered in $(1,1)$ with radius 1.



2. **(1pt)** Implicit function theorem 2D: Let $F: D \subset \mathbb{R}^n \to \mathbb{R}$ be a C^k function on D with $k \geq 1$. Consider $(a,b) \in \mathbb{R}^2$ such that

$$F(a,b)=0 \quad \text{and} \quad \frac{\partial F}{\partial y}(a,b) \neq 0.$$

Then, there exist neighborhoods V and W of a and b and a C^k function $\varphi: V \to W$ such that $V \times W \subset D$ and

$$\forall x \in V,\ \forall y \in W, \quad F(x,y)=0 \Longleftrightarrow y=\varphi(x).$$

Furthermore,we have for all $x \in V$, the derivative $\varphi'(x)$ is given by

$$\varphi'(x) = -\frac{\frac{\partial F}{\partial x}(x,\varphi(x))}{\frac{\partial F}{\partial y}(x,\varphi(x))}.$$

3. **(1.5pts)** We have $f(x,y)=x^2+y^2-2(x+y)+1$ and $\frac{\partial f}{\partial y}(x,y)=2y-2$.
At the point $(a,b)=(\frac{2+\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2})$ we have $f(a,b)=0$ and $\frac{\partial f}{\partial y}(a,b)=\sqrt{2} \neq 0$. Then, there exist neighborhoods V and W of a and b and a C^k function $\varphi: V \to W$ such that $V \times W \subset D$ and

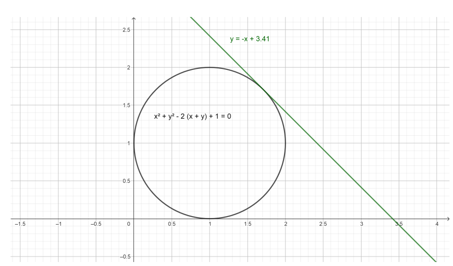
$$\forall x \in V,\ \forall y \in W, \quad F(x,y)=0 \Longleftrightarrow y=\varphi(x).$$

4. **(1pt)** We have $\forall x \in V$, $\varphi'(x) = -\dfrac{\partial F/\partial x(x,\varphi(x))}{\partial F/\partial y(x,\varphi(x))}$. and $\frac{\partial f}{\partial x}(x,y)=2x-2$. then $\varphi'(x)=\dfrac{1-x}{\varphi(x)-1}$

5. **(1pt)** The equation of tangent line to the graph of φ at the point $(a,b)=(\frac{2+\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2})$ is given by $y=\varphi'(a)x+\alpha$, where

   - $\varphi'(a) = \dfrac{1-a}{\varphi(a)-1} = -1$ (since $\varphi(a)=b$).
   - $\alpha = b - \varphi'(a)a \Longrightarrow \alpha = 2+\sqrt{2}$.

   then $\quad y = -x + 2 + \sqrt{2}$.



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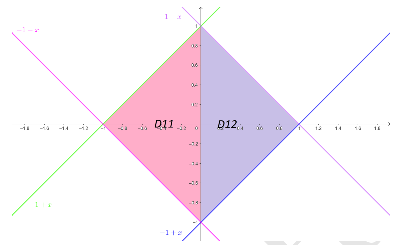
Exercise 3. (4pts)

- **2 pts= 0.5pt drawing+1pt domain+ 0.5 calculation**
  Area of $D_1 = \{(x,y) \in \mathbb{R}^2 : |x-y| \leq 1, |x+y| \leq 1\}$.:
  We divide D_1 to two sub domains as follows
  $D_{11} = \{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 0, -1-x \leq y \leq 1+x\}$.
  $D_{12} = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, x-1 \leq y \leq 1-x\}$.
  Then

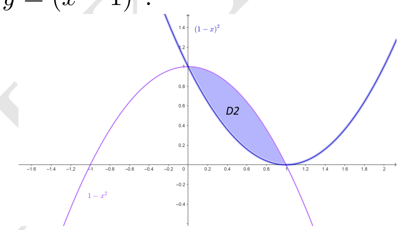


$$\text{Area}(D_1) = \text{Area}(D_{11}) + \text{Area}(D_{12})$$

$$\iint_{D_1} dxdy = \iint_{D_{11}} dxdy + \iint_{D_{12}} dxdy$$

$$\iint_{D_1} dxdy = \int_{-1}^{0} \int_{-1-x}^{1+x} dydx + \int_{0}^{1} \int_{x-1}^{1-x} dydx = 2.$$

- **2 pts= 0.5pt drawing+1pt domain+ 0.5 calculation**
  Area of D_2 is delimited by the curves of equations $y = 1-x^2$ and $y = (x-1)^2$:



$$D_2 = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, (x-1)^2 \leq y \leq 1-x^2\}.$$

$$\text{Area}D_2 = \iint_{D_2} dxdy = \int_{0}^{1} \int_{(x-1)^2}^{1-x^2} dydx = \frac{1}{3}.$$