Truth Tables

Chapter 2, Section 5

Definition

Definition 3.5.1

We take for granted henceforth that truth and falsehood are two different things. It will never matter exactly what things they are, but in some contexts it is convenient to identify truth with the number 1 and falsehood with 0. We refer to truth and falsehood as the *truth values*, and we write them as T and F, respectively. The *truth value of* a statement is T if the statement is true and F if the statement is false.

Example

Example 3.5.2

We want to calculate the truth value of a statement $(\phi \wedge \psi)$ from the truth values of ϕ and ψ. The following table shows how to do it :

(3.40)

ϕ	ψ	$(\phi \wedge \psi)$
T	T	T
T	F	F
F	T	F
F	F	F

The table has four rows corresponding to the four possible assignments of truth values to ϕ and ψ.

Example

Example 3.5.2

We can do the same for all the symbols $\vee, \to, \leftrightarrow, \neg, \bot$, using the meanings that we gave them when we introduced them in Chapter 2. The following table shows the result :

(3.41)

ϕ	ψ	$(\phi \wedge \psi)$	$(\phi \vee \psi)$	$(\phi \to \psi)$	$(\phi \leftrightarrow \psi)$	$(\neg\phi)$	$\bot$
T	T	T	T	T	T	F	F
T	F	F	T	F	F		
F	T	F	T	T	F	T	
F	F	F	F	T	T		

The table has four rows corresponding to the four possible assignments of truth values to ϕ and ψ.

The case $\to$

The only part of this table that may raise serious doubts is the listing for $\to$. In fact not all linguists are convinced that the table is correct for 'If . . . then' in ordinary English. But the following argument confirms that the table for $\to$ does correspond to normal mathematical usage.

In mathematics we accept as true that

(3.42) if p is a prime > 2 then p is odd.

In particular,

(3.43) if 3 is a prime > 2 then 3 is odd. (If T then T.)

This justifies the T in the first row. But also

(3.44) if 9 is a prime > 2 then 9 is odd. (If F then T.)

This justifies the T in the third row. Also

(3.45) if 4 is a prime > 2 then 4 is odd. (If F then F.)

This justifies the T in the fourth row. There remains the second row. But everybody agrees that if ϕ is true and ψ is false then 'Ifϕ then ψ' must be false.

$\sigma - \text{structure}$

Definition 3.5.3

Let σ be a signature. By a *σ-structure* we mean a function A with domain σ, that assigns to each symbol p in σ a truth value $A(p)$.

For practical application and for comparison with the later structures, we note that if $\sigma = \{q_1, \cdots, q_n\}$ (where the symbols are listed without repetition), then we can write the structure A as a chart

$$\begin{array}{ccc} q_1 & \cdots & q_n \\ \hline A(q_1) & \cdots & A(q_n) \end{array}$$

Every σ-structure A gives a truth value $A^*(\chi)$ to each formula χ of LP(σ) in accordance with table (3.41). It is useful to think of $A^*(\chi)$ as the truth value that χ has 'in A' .

Example

We can calculate the value $A^*(\chi)$ by climbing up the parsing tree of χ.

Example 3.5.4

Let χ be $(p_1 \ \wedge \ (\neg(p_0 \to p_2)))$. Given the $\{p_0, p_1, p_2\}$-structure:

$$A: \quad \begin{array}{ccc} p_0 & p_1 & p_2 \\ \hline F & T & T \end{array},$$

we calculate the truth value $A^*(\chi)$ of χ as follows.

Example

The following table contracts (3.48) into two lines, together with an optional bottom line showing a possible order for visiting the nodes. The truth value at each node v is written under the head of the subformula corresponding to v. The head of χ itself is indicated by $\Uparrow$. We call the column with $\Uparrow$ the *head column* of the table; this is the last column visited in the calculation, and the value shown in it is the truth value of χ.

(3.49)

p_0	p_1	p_2	$(p_1$	$\wedge$	$(\neg$	$(p_0$	$\to$	$p_2)))$
F	T	T	T	F	F	F	T	T
			1	$\Uparrow$	5	2	4	3

Example

Table (3.49) shows the truth value of χ for one particular $\{p_0, p_1, p_2\}$-structure. There are eight possible $\{p_0, p_1, p_2\}$-structures. The next table lists them on the left, and on the right it calculates the corresponding truth value for χ. These values are shown in the head column.

(3.50)

p_0	p_1	p_2	$(p_1$	$\wedge$	$(\neg$	$(p_0$	$\rightarrow$	$p_2)))$
T	T	T	T	F	F	T	T	T
T	T	F	T	T	T	T	F	F
T	F	T	F	F	F	T	T	T
T	F	F	F	F	T	T	F	F
F	T	T	T	F	F	F	T	T
F	T	F	T	F	F	F	T	F
F	F	T	F	F	F	F	T	T
F	F	F	F	F	F	F	T	F
				$\Uparrow$				

Truth Tables

Definition 3.5.5

A table like (3.50), which shows when a formula is true in terms of the possible truth values of the propositional symbols in it, is called the *truth table* of the formula. Note the arrangement: the first column, under p_0, changes slower than the second column under p_1, the second column changes slower than the third, and T comes above F. It is strongly recommended that you keep to this arrangement, otherwise other people (and very likely you yourself) will misread your tables.

Truth tables were invented by Charles Peirce in an unpublished manuscript of 1902, which may have been intended for a correspondence course in logic.

Recursive definition of A^*

Definition 3.5.6

(a) If p is a propositional symbol in σ then $A^*(p) \;=\; A(p)$.

(b) $A^*(\bot) \;=\; \mathrm{F}$.

(c) $A^*((\neg\phi)) \;=\; \mathrm{T}$ if and only if $A^*(\phi) \;=\; \mathrm{F}$.

(d) $A^*((\phi \wedge \psi))$ is T if $A^*(\phi) \;=\; A^*(\psi) \;=\mathrm{T}$, and is F otherwise.

(e) $A^*((\phi \vee \psi))$ is T if $A^*(\phi) = \mathrm{T}$ or $A^*(\psi) \;=\mathrm{T}$, and is F otherwise.

(f) $A^*((\phi \to \psi))$ is F if $A^*(\phi) = \mathrm{T}$ and $A^*(\psi) \;=\mathrm{F}$, and is T otherwise.

(g) $A^*((\phi \leftrightarrow \psi))$ is T if $A^*(\phi) = \; A^*(\psi)$, and is F otherwise.

Model

Definition 3.5.7

Let σ be a signature, A a σ-structure and ϕ a formula of LP(σ). When $A^*(\phi) = \mathrm{T}$, we say that A is a *model* of ϕ, and that ϕ is *true in A*.

We will use the notation

$$\vDash_A \phi$$

for "A is a model of ϕ".

Tautology, Contradiction and Satisfiable formula

Definition 3.5.8

Let σ be a signature and ϕ a formula of LP(σ).

(a) We say that ϕ is *valid* , and that it is a *tautology*, in symbols $\vDash_\sigma \phi$, if every σ-structure is a model of ϕ. (So ($\vDash_\sigma \phi$) says that $A^*(\phi) = \mathrm{T}$ for all σ-structures A.) When the context allows, we drop the subscript σ and write $\vDash \quad \phi$.

(b) We say that ϕ is *consistent*, and that it is *satisfiable*, if some σ-structure is a model of ϕ.

(c) We say that ϕ is a *contradiction*, and that it is *inconsistent*, if no σ-structure is a model of ϕ.

Example

Example 3.5.9

We confirm that Peirce's Formula, which we proved in Example 3.4.5, is a tautology :

(3.51)

p_1	p_2	$(((p_1$	$\to$	$p_2)$	$\to$	$p_1)$	$\to$	p_1
T	T	T	T	T	T	T	T	T
T	F	T	F	F	T	T	T	T
F	T	F	T	T	F	F	T	F
F	F	F	T	F	F	F	T	F
							$\Uparrow$	

Principle of Irrelevance

Lemma 3.5.10 (Principle of Irrelevance)

If σ is a signature, ϕ a formula of LP(σ) and A a σ-structure, then $A^*(\phi)$ does not depend on the value of A at any propositional symbol that does not occur in ϕ.