Mathematical analysis 2
Chapter 1: Multivariable and vectorial functions
Part 3: Differentiability

R. KECHKAR

The National School of
Artificial Intelligence
المدرسة الوطنية للذكاء الاصطناعي

2023/2024

Plan

Plan

Diractionnal derivation

Definition

Let f be a function defined on $D \subseteq \mathbb{R}^n \to \mathbb{R}$ and $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ a non null vector. We say f is derivable at $a = (a_1, \ldots, a_n) \in D$ in the diriction of v if

$$\lim_{t \to 0} \frac{f(a+tv) - f(a)}{t} \text{ exist and is finite.}$$

This limit is denoted by $D_v f(a)$.

Diractionnal derivation

Example.

Find the derivatives of f in the direction of $v_1 = (2,1)$ et $v_2 = (1,0)$ at point $a = (0,0)$ if there exist , where :$f(x,y) = x^2 - 2xy + |\sin(y)|$.

- *In the direction of v_1:*

$$\lim_{t\to 0} \frac{f((0,0)+t(2,1)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(2t,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{|\sin(t)|}{t} \nexists.$$

Then f is not derivable at the point $(0,0)$ in the direction of v_1.

- *In the direction of v_2:*

$$\lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{t^2}{t} = 0.$$

Then f is derivable at $(0,0)$ in the direction of v_2 and we have

$$D_{v_2} f(0,0) = 0$$

Plan

Partial derivatives

Definition

Let $(e_1,\ldots,e_i,\ldots,e_n)$ *the canonical base of* $\mathbb{R}^n$ *and* $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$.

- *We call first partial derivative of* f *at the point* $a = (a_1,\ldots,a_n) \in U$ *with respect to* x_i *variable and we denote* $\frac{\partial f}{\partial x_i}(a)$, *the derivative of* f *at* a *in the direction of the vector* e_i, *that is to say*

$$\begin{aligned}\frac{\partial f}{\partial x_i}(a) = D_{e_i}f(a) &= \lim_{t\to 0}\frac{f(a+te_i)-f(a)}{t}\\ &= \lim_{t\to 0}\frac{f(a_1,\ldots,a_i+t,\ldots,a_n)-f(a_1,\ldots,a_i,\ldots,a_n)}{t}\\ &= \lim_{x_i\to a_i}\frac{f(a_1,\ldots,x_i,\ldots,a_n)-f(a_1,\ldots,a_i,\ldots,a_n)}{x_i-a_i}.\end{aligned}$$

- *In particular, for* $n = 2$ *the first partial derivatives of* f *at the point* $(x_0,y_0) \in U$ *are written as follow*

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}, \quad \frac{\partial f}{\partial y}(x_0,y_0) = \lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}.$$

Gradient

Definition

Let $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $a \in D$. Suppose that all first partial derivatives of f exist. The vector denoted by $\nabla f(a) \in \mathbb{R}$ defined by

$$\nabla f(a) = \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(a) \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(a) \end{pmatrix}$$

*is called **gradient** of f at a.*

Gradient: Example

Example.

Find the first partial derivatives of f at all points of f $\mathbb{R}^2$. if there exist, where f the function defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{si } (x,\ y) \neq (0,0)\ , \\ 0 & \text{si } (x,\ y) = (0,0)\ . \end{cases}$$

1. On $\mathbb{R}^2_*$ the partial derivatives of first order exist since f is the quotient of tow polynomial and we have $\forall (x,\ y) \in \mathbb{R}^2_*$,

$$\frac{\partial f}{\partial x}(x,\ y) = \frac{y(x^2+y^2) - xy(2x)}{(x^2+y^2)^2} = y\frac{y^2-x^2}{(x^2+y^2)^2}.$$

$$\frac{\partial f}{\partial y}(x,\ y) = \frac{x(x^2+y^2) - xy(2y)}{(x^2+y^2)^2} = x\frac{x^2-y^2}{(x^2+y^2)^2}.$$

Gradient: Example

Then on $\mathbb{R}^2_*$ the gradient of f is

$$\nabla f(x,y) = \begin{pmatrix} y\dfrac{y^2-x^2}{(x^2+y^2)^2} \\ x\dfrac{x^2-y^2}{(x^2+y^2)^2} \end{pmatrix}$$

- The first partial derivatives at $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0}\frac{0}{x^3} = 0 \in \mathbb{R}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0}\frac{0}{y^3} = 0 \in \mathbb{R}.$$

Then at $(0,0)$ the gradient of f is

$$\nabla f(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Gradient: Important remark

Remark

- *Note that for the pervious example, limit of* f *at* $(0,0)$ *does not exist then* f *is not continuous at* $(0,0)$ *however it is derivable.*
- *Generally, the existence of partial derivatives at a point* a *does not imply continuity at* a.

Tangent plane

Definition

Suppose f has continuous partial derivatives. An equation of the ***tangent plane*** *to the surface $z = f(x,y)$ at the point $(x_0, y_0, f(x_0,y_0))$ is*

$$z = \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0) + f(x_0,y_0)$$

Plan

1. Differentiability
   - First partial derivatives
     - Diractionnal derivation
     - Partial derivatives of numerical function
     - Partial derivatives of vectorial function
     - Functions of class C^1
   - Higher order partial derivatives
     - Second order partial derivatives
     - Functions of class C^k
   - Differentiability
     - Differentiability of numerical functions
     - Differentiability of vectorial functions
     - Differentiation of composite functions (The chain rule)
     - Taylor's formula
     - Implicit derivation

Partial derivatives of vectorial function

Definition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ $(m \geq 2)$ and $a = (a_1, \cdots a_n) \in D$. Then the derivative with respect to the variable x_i at the point a of f exist ***if and only if*** *the derivatives with respect to the variable x_i at the point a of all $f_j : D \subseteq \mathbb{R}^n \to \mathbb{R}$ for $j = 1, \cdots, m$ exist. In this case we have*

$$\frac{\partial f}{\partial x_i}(a) = \left(\frac{\partial f_1}{\partial x_i}(a), \cdots, \frac{\partial f_m}{\partial x_i}(a)\right)$$

Jacobian Matrix

Definition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ $(m \geq 2)$ *that all partial derivatives at the point* a *exist. The matrix*

$$Jf(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \cdots & \frac{\partial f_2}{\partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}$$

of size (m,n) *is called* ***Jacobian matrix*** *of* f *at* a.

Plan

Function of class C^1

Definition

- *Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. We say that f is of calss C^1 on D **if and only if** all first partial derivatives $\dfrac{\partial f}{\partial x_i}$ exist and are continuous on D.*
- *Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ $(m \geq 2)$ be a vectorial function we have*

$$f \textit{ is of class } C^1 \textit{ on } D \Longleftrightarrow \forall i = 1, \cdots, m \ f_i \textit{ is of class } C^1 \textit{ on } D$$

Plan

Plan

1. Differentiability
   - First partial derivatives
     - Diractionnal derivation
     - Partial derivatives of numerical function
     - Partial derivatives of vectorial function
     - Functions of class C^1
   - Higher order partial derivatives
     - Second order partial derivatives
     - Functions of class C^k
   - Differentiability
     - Differentiability of numerical functions
     - Differentiability of vectorial functions
     - Differentiation of composite functions (The chain rule)
     - Taylor's formula
     - Implicit derivation

Second order partial derivatives

Definition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. If the first partial derivative of f at the point a with respect to the variable x_i exist and if the derivative of $\frac{\partial f}{\partial x_i}$ at the point a with respect to the variable x_j exist. Then we denote by

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(a) = \begin{cases} \dfrac{\partial^2 f}{\partial x_j \partial x_i} & \text{if } j \neq i \\ \dfrac{\partial^2 f}{\partial x_2^2} & \text{if } j = i \end{cases}$$

the second order partial derivative of f with respect to the variables x_i and x_j.

Schwarz theorem

Theorem

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ *be a function. If the partial derivatives* $\frac{\partial^2 f}{\partial x_i \partial x_j}$ *and* $\frac{\partial^2 f}{\partial x_j \partial x_i}$ ***exist*** *on an open containing* a *and are* ***continuous*** *at* a *then*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a)$$

Plan

Function of class C^k

Proposition

- *Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. We say that f is of calss C^k on D* ***if and only if*** *all partial derivatives up to order k exist and are continuous on D.*
- *Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ $(m \geq 2)$ be a vectorial function we have*

$$f \text{ is of class } C^k \text{ on } D \Longleftrightarrow \forall i = 1, \cdots, m \ f_i \text{ is of class } C^k \text{ on } D$$

Remark

The function f is said to be C^∞ if f is of class C^k for all $k \in \mathbb{N}$.

Hessian matrix

Definition

*Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ such that $f \in C^2$ on D. We call **Hessian matrix** of f at the point a the matrix defined by*

$$Jf(a) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Second order derivatives: Example

Example.

Let us consider the function defined on $\mathbb{R}^2$ *by*

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \textit{if } (x,y) \neq (0,0), \\ 0 & \textit{if } (x,y) = (0,0). \end{cases}$$

Find $\dfrac{\partial^2 f}{\partial y \partial x}$ *and* $\dfrac{\partial^2 f}{\partial x \partial y}$ *at the point* $(0,0)$. *What can we conclude.*

Solution

- Partial derivatives of f at $(0,0)$:

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0}{x}=0\in\mathbb{R}$$
$$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to 0}\frac{0}{y}=0\in\mathbb{R}.$$

- On $\mathbb{R}^2_*$, f is of class C^1 sice f is quotient of tow polynomials, we have

$$\forall (x,y)\in\mathbb{R}^2_*,\quad \frac{\partial f}{\partial x}(x,y)=y\frac{\left(3x^2-y^2\right)\left(x^2+y^2\right)-2x\left(x^3-xy^2\right)}{\left(x^2+y^2\right)^2}.$$
$$\forall (x,y)\in\mathbb{R}^2_*,\quad \frac{\partial f}{\partial y}(x,y)=-x\frac{\left(3y^2-x^2\right)\left(x^2+y^2\right)-2y\left(y^3-yx^2\right)}{\left(x^2+y^2\right)^2}.$$

Solution

- Now, let's find $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$ at $(0,0)$:

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y \to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y-0} = \lim_{y \to 0} \frac{-\frac{y^5}{y^4} - 0}{y} = -1 \in \mathbb{R}.$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x \to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x-0} = \lim_{x \to 0} \frac{\frac{x^5}{x^4} - 0}{x} = 1 \in \mathbb{R}.$$

- Since $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) \neq \dfrac{\partial^2 f}{\partial x \partial y}(0,0)$, we conclude using Schwartz theorem that at least $\dfrac{\partial^2 f}{\partial y \partial x}$ or $\dfrac{\partial^2 f}{\partial x \partial y}$ is not continuous at $(0,0)$. Then f is not C^2 on $\mathbb{R}^2$.

Plan

Plan

Differentiability definition

Definition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$.

- *We say that* f *is* ***differentiable*** *at the point* $a \in \mathbb{R}^n$ *if there exist a linear mapping* $L : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - L(h)}{||h||} = 0$$

- *The linear mapping* L *is called* ***differential*** *of* f *at the point* a*, denoted by* $df(a)$ *or* df_a.
- *The function* f *is said to be differentiable on* D *if it is differentiable at every point of* D.

Proposition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ *be a differentiable function then* L *is unique.*

Differentiability results

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function at point a. Then:

- *The function f has partial derivatives at point a with respect to all its variables.*
- $df_a(h) = h_1 \dfrac{\partial f}{\partial x_1}(a) + h_2 \dfrac{\partial f}{\partial x_2}(a) + \ldots + h_n \dfrac{\partial f}{\partial x_n}(a).$

Example

Example.

Let f a function defined by

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

We have

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=0$$

Also,

$$\lim_{(h,k)\to(0,0)}\frac{f(0+h,0+k)-f(0,0)-\frac{\partial f}{\partial x}(0,0)\cdot h-\frac{\partial f}{\partial y}(0,0)\cdot k}{\sqrt{h^2+k^2}}=\lim_{(h,k)\to(0,0)}\frac{hk}{h^2+k^2}$$

does not exist. Thus, f has partial derivatives with respect to x and y at the point $(0,0)$, but f is not differentiable at $(0,0)$.

Differentiability

Theorem

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$. Let f is differentiable at point a if and only if the following two conditions are satisfied:

- *f has partial derivatives at a with respect to all its variables.*
- *and*

$$\lim_{h \to 0} \frac{f(a_1+h_1,\dots,a_n+h_n) - f(a) - h_1 \dfrac{\partial f}{\partial x_1}(a) - \dots - h_n \dfrac{\partial f}{\partial x_n}(a)}{\sqrt{h_1^2+h_2^2+\dots+h_n^2}} = 0.$$

Plan

1. Differentiability
   - First partial derivatives
     - Diractionnal derivation
     - Partial derivatives of numerical function
     - Partial derivatives of vectorial function
     - Functions of class C^1
   - Higher order partial derivatives
     - Second order partial derivatives
     - Functions of class C^k
   - Differentiability
     - Differentiability of numerical functions
     - Differentiability of vectorial functions
     - Differentiation of composite functions (The chain rule)
     - Taylor's formula
     - Implicit derivation

Differential

Definition

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$. $(m \geq 2)$ *and* $a = (a_1, \ldots, a_n) \in D$. *Then,* f *is differentiable at* a *if and only if each of the* m *component functions* $f_j : U \subset \mathbb{R}^n \to \mathbb{R}$ *is differentiable at* a *for* $j = 1, \ldots, m$. *In this case, the* ***differential*** *of* f *at* a *is given by:*

$$df_a = (df_1(a), df_2(a), \ldots, df_m(a))$$

Proposition

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$. $(m \geq 1)$. *If* f *is differentiable at* $a \in U$, *then* f *is continuous at* a.

Relation between function of class C^1 and and function differentiable

Proposition

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$. ($m \geq 1$). If all the partial derivatives of f exist in a neighborhood of $a \in U$ and are continuous at a, then f is differentiable at a, i.e.

$$f \text{ is } C^1 \text{ on } U \implies f \text{ is differentiable on } U.$$

Example

Example.

Study the differentiability of f *in* $\mathbb{R}^2$ *where*

$$f(x,y) = \begin{cases} \dfrac{x^3 + xy^3}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution

• On $\mathbb{R}^2_*$, f is of class C^1 as it is a quotient and composition of two functions of class C^1.

• Study of the differentiability of f at $(0,0)$: If f is differentiable at $(0,0)$, then the differential of f is expressed using partial derivatives at $(0,0)$. Let's calculate the partial derivatives of f at $(0,0)$:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0}\frac{x^3}{x|x|} = 0 \in \mathbb{R}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0}\frac{0}{y} = 0 \in \mathbb{R}$$

Solution

If f is differentiable at $(0,0)$, then its differential at $(0,0)$ is given by:

$$d_{(0,0)}f(h_1,h_2) = h_1\frac{\partial f}{\partial x}(0,0) + h_2\frac{\partial f}{\partial y}(0,0) = 0.$$

According to the definition, f is differentiable at $(0,0)$ if there exists a linear transformation $d_{(0,0)}f$ such that:

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(0+h_1,0+h_2) - f(0,0) - d_{(0,0)}f(h_1,h_2)}{\|(h_1,h_2)\|} = 0.$$

Let's choose the Euclidean norm:

$$\begin{aligned}
&\lim_{(h_1,h_2)\to(0,0)} \frac{f(h_1,h_2) - f(0,0) - d_{(0,0)}f(h_1,h_2)}{\|(h_1,h_2)\|^2} \\
&= \lim_{(h_1,h_2)\to(0,0)} \frac{h_1^3 + h_1h_2^3}{h_1^2 + h_2^2} \\
&= \lim_{(h_1,h_2)\to(0,0)} \left(h_1\frac{h_1^2}{h_1^2+h_2^2} + h_1h_2\frac{h_2^2}{h_1^2+h_2^2}\right) = 0.
\end{aligned}$$

Solution

We can conclude that f is differentiable at $(0,0)$ with a zero differential.

Conclusion: f is differentiable over $\mathbb{R}^2$.

Plan

1. Differentiability
   - First partial derivatives
     - Diractionnal derivation
     - Partial derivatives of numerical function
     - Partial derivatives of vectorial function
     - Functions of class C^1
   - Higher order partial derivatives
     - Second order partial derivatives
     - Functions of class C^k
   - Differentiability
     - Differentiability of numerical functions
     - Differentiability of vectorial functions
     - Differentiation of composite functions (The chain rule)
     - Taylor's formula
     - Implicit derivation

Chain rule (case 1)

Proposition

Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function on D, where $x = x(t)$ and $y = y(t)$ are both differentiable functions with respect to t. Then f is a differentiable function with respect to t

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Chain rule: Example

Example.

Let consider the function f *defined by*

$$\begin{aligned} f : D = \mathbb{R}^+ \times \mathbb{R}_+^* &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto f(x, y) = \frac{x}{y}. \end{aligned}$$

We set $x = t^2 \quad y = \ln t$*. Find* $\dfrac{df}{dt}$ *using two different methods.*

Solution

Note that $f \in C^1$ on D

- **Using direct method**: we substitute the value of x and y in the expression of f:

$$f(x,y) = f(t^2, \ln t) = \frac{t^2}{\ln t} \Longrightarrow \frac{df}{dt} = \frac{2t\ln t - \frac{t^2}{t}}{\ln^2 t} = \frac{2t\ln t - t}{\ln^2 t}$$

Then

$$\frac{df}{dt} = \frac{2t\ln t - t}{\ln^2 t}$$

Solution

- **Using chain rule**: We have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

$$\begin{cases} x = t^2 \\ y = \ln t \end{cases} \Longrightarrow \begin{cases} \dfrac{dx}{dt} = 2t \\ \dfrac{dy}{dt} = \dfrac{1}{t} \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial f}{dx} = \dfrac{1}{y} = \dfrac{1}{\ln t} \\ \dfrac{\partial f}{dy} = -\dfrac{x}{y^2} = -\dfrac{t^2}{\ln^2 t}. \end{cases}$$

Then

$$\frac{df}{dt} = \frac{1}{\ln t}\cdot 2t - \frac{t^2}{\ln^2 t}\cdot\frac{1}{t} = \frac{2t\ln t - t}{\ln^2 t}$$

Consequently

$$\frac{df}{dt} = \frac{2t\ln t - t}{\ln^2 t}$$

Chain rule (case 2)

Proposition

Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function on D x and y, where $x = x(s,t)$ and $y = y(s,t)$ are differentiable functions with respect to s and t. Then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Chain rule: Example

Example.

Given $f:\mathbb{R}^2 \to \mathbb{R}$ of class C^2. Let $u = x - y$ and $v = x + y$. Express $\dfrac{\partial f}{\partial x}$, and $\dfrac{\partial f}{\partial y}$ in terms of the partial derivatives of f with respect to u and v.

Solution.

Considering that $x = \dfrac{u+v}{2}$ and $y = \dfrac{u-v}{2}$, we have:

$$\begin{cases} \dfrac{\partial f}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u} \\[2ex] \dfrac{\partial f}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v} \end{cases} \Longrightarrow \begin{cases} \dfrac{\partial f}{\partial u} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\right) \\[2ex] \dfrac{\partial f}{\partial v} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial y}\right) \end{cases} \Longrightarrow \begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u} + \dfrac{\partial f}{\partial v} \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial u} - \dfrac{\partial f}{\partial v} \end{cases}$$

Plan

Taylor's formula of first order

Proposition

- *Let* $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ *be a function of class* C^1 *on* D. *For* $p = (\alpha_1, \ldots, \alpha_n) \in D$ *fixed, there exists a function* ε *defined on* D *with* $\lim\limits_{u \to p} \varepsilon(u) = 0$ *such that for all* $u = (u_1, \ldots, u_n) \in D$, *we have*

$$f(u) = f(p) + (u_1 - \alpha_1)\frac{\partial f}{\partial x_1}(p) + \ldots + (u_m - \alpha_m)\frac{\partial f}{\partial x_n}(p) + ||u - p||\varepsilon(u).$$

- *In particular for* $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ *we have*

$$f(u) = f(p) + (u_1 - \alpha_1)\frac{\partial f}{\partial x_1}(p) + (u_2 - \alpha_2)\frac{\partial f}{\partial x_2}(p) + ||u - p||\varepsilon(u, v).$$

Taylor's formula of second order

Proposition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ *be a function of class* C^2 *on* D*. For* $p \in D$*, there exists a function* ε *defined on* D *with* $\lim_{u \to p} \varepsilon(u) = 0$*. Then, for all* $u = (u_1, \ldots, u_n)$*, we have*

$$f(u) = f(p) + \sum_{1 \le i \le n} (u_i - p_i) \frac{\partial f}{\partial x_i}(p) + \frac{1}{2} \sum_{1 \le i,j \le n} (u_i - p_i)(u_j - p_j) \frac{\partial^2 f}{\partial x_i \partial x_j}(p) + \|u - p\|^2 \varepsilon(u - p).$$

Plan

Implicit Function Theorem (2D version)

Theorem

Let $F : D \subset \mathbb{R}^n \to \mathbb{R}$ be a C^k function on D with $k \geq 1$. Consider $(a,b) \in \mathbb{R}^2$ such that

$$F(a,b) = 0 \quad and \quad \frac{\partial F}{\partial y}(a,b) \neq 0.$$

Then, there exist neighborhoods V and W of a and b and a C^k function $\varphi : V \to W$ such that $V \times W \subset D$ and

$$\forall x \in V,\ \forall y \in W, \quad F(x,y) = 0 \Longleftrightarrow y = \varphi(x).$$

Furthermore,we have for all $x \in V$, the derivative $\varphi'(x)$ is given by

$$\varphi'(x) = -\frac{\dfrac{\partial F}{\partial x}(x,\varphi(x))}{\dfrac{\partial F}{\partial y}(x,\varphi(x))}.$$

Important remark

Remark

- *If* $\frac{\partial F}{\partial y}(a,b) \neq 0$ *then, there exist also neighborhoods* V *and* W *of* a *and* b *and a* C^k *function* $\psi : W \to V$ *such that* $V \times W \subset D$

$$\forall x \in V,\ \forall y \in W, \quad F(x,y) = 0 \Longleftrightarrow x = \psi(y).$$

Furthermore,we have for all $y \in W$*, the derivative* $\psi'(y)$ *is given by*

$$\psi'(y) = -\frac{\frac{\partial F}{\partial y}(\psi(y),y}{\frac{\partial F}{\partial x}(\psi(y),y}.$$

Important remark

Remark

- *The result of the previous proposition can be generalized to a function with n variables . If* $F : D \subset \mathbb{R}^n \to \mathbb{R}$ *be a* C^k *function on* D *with* $k \geq 1$. *and* $p = (\alpha_1, \ldots, \alpha_m) \in D \subset \mathbb{R}^m$ *such that*
$$f(p) = 0, \quad \textit{and} \quad \frac{\partial f}{\partial x_m}(p) \neq 0.$$
Then there exists a neighborhood V *of* $(\alpha_1, \ldots, \alpha_{m-1})$ *and an interval* J *centered at* α_m *such that* $V \times J \subset D$, *and a function* $\varphi : V \to J$ *satisfying*
$$\forall (x_1, \ldots, x_m) \in V \times J : f(x_1, \ldots, x_m) = 0 \Longleftrightarrow x_m = \varphi(x_1, \ldots, x_{m-1}),$$
Furthermore $\forall (x_1, \ldots, x_m) \in V \times J$, $\forall j \in \{1, \ldots, m-1\}$
$$\frac{\partial \varphi}{\partial x_j}(x_1, \ldots, x_{m-1}) = -\frac{\dfrac{\partial f}{\partial x_j}(x_1, \ldots, x_{m-1}, \varphi(x_1, \ldots, x_{m-1}))}{\dfrac{\partial f}{\partial x_m}(x_1, \ldots, x_{m-1}, \varphi(x_1, \ldots, x_{m-1}))}$$

.

Implicit derivation: example

Example.

Let consider the function f *defined by*

$$\begin{array}{rcl} f:\mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & f(x,y) = x^4 + x^3y^2 - y + y^2 + y^3 - 1. \end{array}$$

Apply the implicit function theorem at point $p = (-1, 1)$.

Solution

We have $f \in C^{\infty}(\mathbb{R}^2)$ $f(-1,1) = 0$,

$$\frac{\partial f}{\partial y}(x,y) = 2x^3y - 1 + 2y + 3y^2,$$

$$\frac{\partial f}{\partial y}(-1,1) = 2 \neq 0.$$

Then there exist neighborhoods V of -1 and W of 1 and a function $\varphi : V \to W$ of class C^1 such that $V \times W \subset \mathbb{R}^2$ and

$$\forall x \in V,\ \forall y \in W, \quad F(x,y) = 0 \Longleftrightarrow y = \varphi(x).$$

We have

$$g'(-1) = -\frac{\dfrac{\partial f}{\partial x}(-1,1)}{\dfrac{\partial f}{\partial y}(-1,1)}.$$

but

$$\frac{\partial f}{\partial x}(x,y) = 4x^3 + 3x^2y^2 \Longrightarrow \frac{\partial f}{\partial x}(-1,1) = -1$$

Then

$$\varphi'(-1) = \frac{1}{2}.$$