# Mathematical analysis 3
# Chapter 1 : Integrals Depending On a Parameter

ensia
The National School of
Artificial Intelligence
المدرسة الوطنية للذكاء الاصطناعي

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# Course outline

# Introduction and motivation

- In this chapter, we focus on the study of the **continuity** and **differentiability** of functions defined by

$$x \longmapsto F(x) = \int_a^b f(t,x)dt; \quad x \longmapsto F(x) = \int_{u(x)}^{v(x)} f(t,x)dt,$$

(where f is a real function of two variables), In both cases:

  - The integral is proper (Riemann)
  - The integral is improper

- The integrals depending on a parameter are used in various fields

Differential equations, partial differential equations, probability and statistics, physics and engineering, economics and finance, computer science, machine learning, artificial intelligence, robotics, biology,...

- The determination of **the domain of definition** of the function $F$ is not always evident, as generally one cannot provide an expression for $F$ without using the symbol $\int$.
- Suppose $x \to F(x)$ is well-defined on a certain interval $I \subset \mathbb{R}$. Many natural questions arise:
  - If $f$ is **continuous**, will $F$ be **continuous**? And for every $x_0 \in I$ and $[\alpha, \beta] \subset I$, is it true that

$$\lim_{x \to x_0} \int_a^b f(t,x)\,dt \overset{?}{=} \int_a^b \lim_{x \to x_0} f(t,x)\,dt,$$

$$\int_\alpha^\beta \int_a^b f(t,x)\,dt\,dx \overset{?}{=} \int_a^b \int_\alpha^\beta f(t,x)\,dx\,dt.$$

  - If $f$ is **differentiable**, will $F$ be **differentiable**? And is it true that

$$\frac{\partial}{\partial x} \int_a^b f(t,x)\,dt \overset{?}{=} \int_a^b \frac{\partial}{\partial x} f(t,x)\,dt.$$

**Example.** Consider the function

$$F(x) = \int_0^{+\infty} x\sin(x)e^{-tx^2}\,dt.$$

We have

- For $x = 0$, $F(0) = 0$,
- If $x \neq 0$, We have

$$F(x) = x\sin(x)\int_0^{+\infty} e^{-tx^2}\,dt = x\sin(x)\left[\frac{-e^{-tx^2}}{x^2}\right]_0^{+\infty} = \frac{\sin(x)}{x}.$$

Therefore, $D_F = \mathbb{R}$.
The function $(t,x) \mapsto f(t,x) = x\sin(x)e^{-tx^2}$ is continuous on $\mathbb{R}^2$, but the function $F$ is not continuous on $\mathbb{R}$. Indeed, the issue arises at $x = 0$:

$$\lim_{x\to 0} F(x) = \lim_{x\to 0}\int_0^{+\infty} f(t,x)\,dt = \lim_{x\to 0}\frac{\sin(x)}{x} = 1 \neq 0 = \int_0^{+\infty}\lim_{x\to 0} f(t,x)\,dt = F(0).$$

# Course outline

## Introduction

# Proper integrals depending on a parameter

Throughout this section, we adopt the following notations:

- $I$ is an interval of $\mathbb{R}$,
- $a, b \in \mathbb{R}$, and $\Delta. = [a,b] \times I$,
- $f$ is a function of two variables such that

$$f : [a,b] \times I \longrightarrow \mathbb{R} \quad (t,x) \longmapsto f(t,x)$$

- $F(x) = \displaystyle\int_a^b f(t,x)\,dt.$

# Course outline

# Continuity of Proper Integrals Depending on a parameter

## Theorem (Continuity Preservation under Integration)

*If $f$ is a continuous function on $\Delta = [a,b] \times I$, then the function $F$ defined for every $x \in I$ by*

$$F(x) = \int_a^b f(t,x)\,dt$$

*is continuous on $I$, and*

$$\forall x_0 \in I, \lim_{x \to x_0} \int_a^b f(t,x)\,dt = \int_a^b \lim_{x \to x_0} f(t,x)\,dt = \int_a^b f(t,x_0)\,dt.$$

# Continuity of Proper Integrals Depending on a parameter

**Example.**

*Let*

$$F(x) = \int_0^{\pi} \sin(x+t)e^{xt^2}\,dt.$$

*1) Show that* $F$ *is continuous on* $\mathbb{R}$.
*2) Calculate* $\lim_{x\to 0} F(x)$.

**Solution.**
1) The integral is a proper Riemann integral. Set $f(t,x) = \sin(x+t)e^{xt^2}$. Since $(t,x) \longmapsto f(t,x)$ is continuous on $\Delta = [0,\pi] \times \mathbb{R}$ (composition of continuous functions), then according to the previous theorem, the function $x \longmapsto F(x)$ is continuous on $\mathbb{R}$.

## Continuity of Proper Integrals Depending on a parameter

2) Consequently, we deduce the continuity of $F$ at $0$. Therefore,

$$\lim_{x \to 0} F(x) = F(0) = \int_0^{\pi} \sin(t)\, dt = 2.$$

In this example, we were able to calculate the limit of $F$ at $0$ without having an explicit formula.

# Generalization of the continuity theorem

The following result generalizes the theorem of continuity preservation under the integral sign $\int$:

**Theorem**

*If $f$ is a continuous function on $\Delta = [a,b] \times I$, $x \mapsto u(x)$ and $x \mapsto v(x)$ are two continuous functions on $I$ within $[a,b]$, then the function $F$ defined for every $x \in I$ by*

$$F(x) = \int_{u(x)}^{v(x)} f(t,x)\,dt$$

*is continuous on $I$.*

## Course outline

# Integration of Integrals Depending on a parameter

Is $F(x) = \int_a^b f(t,x)\,dt$ integrable over $[\alpha,\beta] \subset I$? And

$$\int_\beta^\alpha F(x)\,dx = ???$$

## Theorem (Integration under the integral sign)

*If $f$ is a continuous function on $\Delta = [a,b] \times I$, then the function $F$ defined for every $x \in I$ by*

$$F(x) = \int_a^b f(t,x)\,dt$$

*is integrable over any closed and bounded interval $[\alpha,\beta] \subset I$, and*

$$\int_\beta^\alpha F(x)\,dx = \int_\beta^\alpha \left(\int_a^b f(t,x)\,dt\right) dx = \int_a^b \left(\int_\beta^\alpha f(t,x)\,dx\right) dt$$

# Course outline

# Differentiation of Integrals Depending on a Parameter

## Theorem

*If $f$ is a continuous function on $\Delta = [a,b] \times I$, the partial derivative $(t,x) \mapsto \frac{\partial f}{\partial x}(t,x)$ exists and is continuous on $\Delta$, then the function $F$ defined for every $x \in I$ by*

$$F(x) = \int_a^b f(t,x)\,dt$$

*is of class $C^1$ on $I$, and*

$$\forall x \in I, F'(x) = (\int_a^b f(t,x)\,dt)' = \int_a^b \frac{\partial f}{\partial x}(t,x)\,dt.$$

## Corollary

*If $f$ is of class $C^n$ (with $n \in \mathbb{N} \cup \{+\infty\}$) on $\Delta$, then $F$ is of class $C^n$ on $I$, and for all $x \in I$, we have*

$$F^{(n)}(x) = \left(\int_a^b f(t,x)\,dt\right)^{(n)} = \int_a^b \frac{\partial^n f}{\partial x^n}(t,x)\,dt.$$

# Differentiation of Integrals Depending on a Parameter

## Example.

*Let's consider the function*

$$f(t,x) = \frac{1}{t^2 + x^2}$$

*and let*

$$F(x) = \int_0^1 \frac{1}{t^2 + x^2}\, dt.$$

*1) Show that* $F$ *is of class* $C^1$ *on* $]0,+\infty[$ *and calculate* $F'(x)$.
*2) Calculate* $F$ *directly and deduce the value of*

$$\int_0^1 \frac{1}{(t^2 + x^2)^2}\, dt.$$

# Differentiation of Integrals Depending on a Parameter

**Solution.** The integral is a proper Riemann integral.

1) We have:

- $f$ is a continuous function on $[0,1]\times]0,+\infty[$,
- $(t,x)\mapsto \dfrac{\partial f}{\partial x}(t,x) = \dfrac{-2x}{(t^2+x^2)^2}$ exist and is continuous on $[0,1]\times]0,+\infty[$,

Then the function $x\mapsto F(x)$ is of class $C^1$ on $]0,+\infty[$ and

$$F'(x) = \int_0^1 \frac{-2x}{(t^2+x^2)^2}\,dt.$$

## Differentiation of Integrals Depending on a Parameter

2) We can easily verify from the definition of $F$ that $F(x) = \frac{1}{x} \arctan \frac{1}{x}$, thus

$$F'(x) = -\frac{1}{x^2} \arctan \frac{1}{x} - \frac{1}{x(1+x^2)}.$$

Hence, we have:

$$\int_0^1 \frac{1}{(t^2+x^2)^2}\, dt = \frac{1}{2x^3} \arctan \frac{1}{x} + \frac{1}{2x^2}(1+x^2).$$

# Differentiation of Integrals Depending on a Parameter

The following result generalizes the theorem of derivative preservation under integration to parameter-dependent integrals with parameterized bounds.

## Theorem

*If $f$ and $\frac{\partial f}{\partial x}$ are continuous on $\Delta = [a,b] \times I$, $x \mapsto u(x)$ and $x \mapsto v(x)$ are two $C^1$ functions on $I$ with values in $[a,b]$, then the function $F$ defined on $I$ by*

$$F(x) = \int_{u(x)}^{v(x)} f(t,x)\,dt$$

*is of class $C^1$ on $I$, and*

$$F'(x) = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}(t,x)\,dt + v'(x)f(v(x),x) - u'(x)f(u(x),x).$$

# Differentiation of Integrals Depending on a Parameter

**Example.**

*Let's consider the following function:*

$$F(x) = \int_0^x \frac{1}{t^2+x^2+1}\,dt.$$

*Show that* $F$ *is differentiable on* $[0,+\infty[$ *and calculate* $F'(x)$.

Answer: Let $\beta > 0$. Apply the previous theorem for

$$f(t,x) = \frac{1}{t^2+x^2+1}, \quad [a,b] = [0,\beta], \quad I = [0,\beta].$$

We have: $f$ and $\frac{\partial f}{\partial x}$ are continuous on $\Delta = [0,\beta]\times[0,\beta]$, $x \mapsto u(x) = 0$ and $x \mapsto v(x) = x$ are two $C^1$ functions on $[0,\beta]$ with values in $[0,\beta]$.

## Differentiation of Integrals Depending on a Parameter

Then the function $F$ is of class $C^1$ on every $[0,\beta]\subset[0,+\infty[$ and

$$\forall x\in[0,\beta], F'(x)=\int_0^x \frac{-2x}{(t^2+x^2+1)^2}\,dt+\frac{1}{2x^2+1}.$$

By extension, we deduce that $F$ is of class $C^1$ on $[0,+\infty[$ and

$$\forall x\in[0,+\infty[, F'(x)=\int_0^x \frac{-2x}{(t^2+x^2+1)^2}\,dt+\frac{1}{2x^2+1}.$$

# Differentiation of Integrals Depending on a Parameter

## Example.

*Let's consider the function*

$$F(x) = \int_0^1 \frac{1}{(t^2+x^2)(t^2+1)}\,dt.$$

*1) Show that $F$ is continuous on $\mathbb{R}^*$.*

*2) Deduce the value of $\displaystyle\int_0^1 \frac{1}{(t^2+1)^2}\,dt$.*

# Differentiation of Integrals Depending on a Parameter

We set

$$f(t,x) = \frac{1}{(t^2+x^2)(t^2+1)}.$$

The function

$$F(x) = \int_0^1 \frac{1}{(t^2+x^2)(t^2+1)}dt$$

is a proper parameterized integral.

1) First, let's study the continuity of $F$ on $\mathbb{R}_+^*$. Since $f$ is continuous as a quotient of continuous functions on $[0,1]\times\mathbb{R}_+^*$ with a non-zero denominator, by using the theorem of continuity preservation under the integral sign for proper parameterized integrals, we deduce that $F$ is continuous on $\mathbb{R}_+^*$. Furthermore, since the function $F$ is even, we conclude that $F$ is continuous on all of $\mathbb{R}^*$.

# Differentiation of Integrals Depending on a Parameter

2) From the previous question, we have in particular

$$\lim_{x \to 1} F(x) = F(1) = \int_0^1 \frac{1}{(t^2+1)^2 \, dt}.$$

Furthermore,

$$f(t,x) = \frac{1}{(t^2+x^2)(t^2+1)} = \frac{1}{x^2-1}\left(\frac{1}{t^2+1} - \frac{1}{t^2+x^2}\right)$$

for $x \neq 1$ and $x \neq -1$.

# Differentiation of Integrals Depending on a Parameter

Then

$$\lim_{x\to 1} F(x) = \lim_{x\to 1} \frac{1}{x^2-1}\left(\int_0^1 \frac{1}{t^2+1}\,dt - \int_0^1 \frac{1}{t^2+x^2}\,dt\right)$$

$$= \lim_{x\to 1} \frac{1}{x^2-1}\left[\arctan(t)\Big|_0^1 - \frac{1}{x}\arctan\left(\frac{t}{x}\right)\Big|_0^1\right]$$

$$= \lim_{x\to 1} \frac{1}{x^2-1}\left(\frac{\pi}{4} - \frac{1}{x}\arctan\left(\frac{1}{x}\right)\right)$$

$$= \lim_{x\to 1} \frac{\frac{\pi}{4} - \frac{1}{x}\arctan\left(\frac{1}{x}\right)}{x^2-1}$$

# Differentiation of Integrals Depending on a Parameter

Using L'Hopital's rule, it follows that

$$\lim_{x \to 1} F(x) = \lim_{x \to 1} \frac{-\left( \frac{-1}{x^2} \arctan\left(\frac{1}{x}\right) + \frac{1}{x} \frac{\frac{-1}{x^2}}{\left(\frac{1}{x}\right)^2 + 1} \right)}{2x} = \frac{\pi}{8} + \frac{1}{4}.$$

Conclusion: From (1) and (2), we deduce that

$$\int_0^1 \frac{1}{(t^2+1)^2}\, dt = \frac{\pi}{8} + \frac{1}{4}.$$

## Course outline

# Improper integrals depending on a parameter

Throughout this section, we adopt the following notations:

- $I$ is any interval in $\mathbb{R}$,
- $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$, and $\Delta = [a,b[\times I$, where $[a,b[$ is open on the side of $b$,
- $f$ is a function of two variables such that

$$\begin{aligned} f\colon [a,b[\times I &\to \mathbb{R} \\ (t,x) &\mapsto f(t,x). \end{aligned}$$

- $F(x) = \int_a^b f(t,x)\,dt$

# Improper Integrals Depending on a Parameter

- for all $x \in I$, $t \mapsto f(t,x)$ is assumed to be locally Riemann integrable on $[a,b[$ according to $t$ ($f \in R_{loc}([a,b[)$ with respect to $t$)
- if $F(x) = \int_a^b f(t,x)dt$, then its domain of definition is given by

$$D_F = \{x \in I \mid \int_a^b f(t,x)dt \text{ converges}\}.$$

# Improper Integrals Depending on a Parameter

In this paragraph, we focus on improper integrals dependent on a parameter. In this case, for example, the continuity of $f$ cannot systematically imply that $F$ is well-defined on $I$, nor that it is continuous. To see this, let's consider the example of the function

$$F(x) = \int_0^{+\infty} x \sin(x) \exp(-tx^2)\, dt.$$

By calculation, we have

$$F(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Note: Although $f(t,x) = x \exp(-tx^2) \sin(x)$ is $C^\infty$ on $[0,+\infty[\times\mathbb{R}$, the function $F$ is not continuous at $0$.
To preserve the "analytic" properties under the integral sign, we will need to add certain hypotheses called domination hypotheses.

# Dominated convergence

## Definition

*If there exists a function* $\varphi : t \mapsto \varphi(t)$ *(independent of* $x$*) piecewise continuous on* $[a,b[$ *such that*

- $\forall (t,x) \in [a,b[\times A, \left|f(t,x)\right| \leq \varphi(t),$
- $\int_a^b \varphi(t)\,dt$ *is convergent.*

*then we say that the improper parameterized integral* $\int_a^b f(t,x)\,dt$ *satisfies the criterion of dominated convergence on* $I$*.*

## Proposition

*Dominated convergence* $\Rightarrow$ *Simple convergence*

## Dominated convergence

**Example.** Let's consider the integral $\int_1^{+\infty} \frac{\sin(xt)}{t^2}\,dt$.

We set

$$f(t,x) = \frac{\sin(xt)}{t^2}$$

then

- $\forall (t,x) \in [1,+\infty[\times\mathbb{R}: \ \left|f(t,x)\right| \leq \frac{1}{t^2},$
- $\int_1^{+\infty} \frac{1}{t^2}\,dt$ converges

Therefore,

$$\int_1^{+\infty} \frac{\sin(xt)}{t^2}\,dt.$$

satisfies the criterion of dominated convergence on $\mathbb{R}$.

# Course outline

# Continuity of an improper integral depending on one parameter

## Theorem

*If*

- $f$ *is a continuous function on* $\Delta = [a,b[\times I$,
- $F(x) = \int_a^b f(t,x)\,dt$ *satisfies the criterion of dominated convergence on* $I$,

*Then* $F$ *is* ***well-defined*** *and* ***continuous*** *on* $I$, *and* ***integrable*** *over any* $[\alpha,\beta] \subseteq I$.

# Continuity of an improper integral depending on one parameter

## Example.

*Let's Consider the function* $F(x) = \int_1^{+\infty} \sin(3xt)e^{-t^2}\,dt.$
*1) Determine the domain of definition* $D$ *of* $F$.
*2) Study the continuity of* $F$ *on* $D$.

**Answer:**
1) The parameterized integral is improper and poses a problem at $+\infty$.
Let

$$f(t,x) = \sin(3xt)e^{-t^2}.$$

We have

$$\forall (t,x) \in \Delta = [1,+\infty[\times\mathbb{R}, \left|f(t,x)\right| \leq e^{-t^2} \leq e^{-t},$$

and

$$\int_1^{+\infty} e^{-t}\,dt \ \ \textit{converges}$$

# Continuity of an improper integral depending on one parameter

Therefore, $\int_1^\infty \sin(3xt)e^{-t^2}\,dt$ satisfies the criterion of dominated convergence, which implies its simple convergence. Thus, $D=\mathbb{R}$.

2) We have

- $(t,x)\mapsto f(t,x)$ is continuous on $\Delta$,
- $F(x)=\int_1^\infty f(t,x)\,dt$ satisfies the criterion of dominated convergence on $\mathbb{R}$.

Verifying the hypotheses of the theorem of the conservation of continuity under the integral sign, the function $F$ is well-defined and continuous on $\mathbb{R}$.

## Continuity of an improper integral depending on one parameter

### Remark

*As continuity is a local property, using covering, the theorem remains valid if we replace the hypothesis of dominated convergence over $I$ by domination over any $[\alpha, \beta]$ of $I$.*

# Continuity of an improper integral depending on one parameter

## Example.

*Study the continuity of the function* $F(x) = \int_0^{+\infty} \frac{e^{-xt}}{1+t}\,dt$ *on* $]0,+\infty[$.

Let's set $f(t,x) = \frac{e^{-xt}}{1+t}$ and take $0 < \alpha < \beta$. We have:

- $f$ is continuous on $[0,+\infty[\times]0,+\infty[$,
- For all $t \in [0,+\infty[$, for all $x \in [\alpha,\beta] \subset ]0,+\infty[$,

$$|f(t,x)| \leq e^{-xt} \leq e^{-\alpha t},$$

- The integral $\int_0^{\infty} e^{-\alpha t}\,dt$ converges.

Verifying the hypotheses of the continuity preservation theorem, the function $F$ is well-defined and continuous on any $[\alpha,\beta] \subset ]0,+\infty[$. Thus, by covering, we deduce the continuity of $F$ on $]0,+\infty[$.

## Course outline

# Differentiation of an improper integral depending on one parameter

## Theorem

*If*

- $\exists x_0 \in I$ *such that* $\int_a^b f(t,x_0)dt$ is *convergent*,
- $f$ is *continuous on* $\Delta = [a,b[\times I$,
- *The partial derivative* $(t,x) \mapsto \frac{\partial f}{\partial x}(t,x)$ *exists and is continuous on* $\Delta$,
- $\int_a^b \frac{\partial f}{\partial x}(t,x)dt$ *satisfies the dominated convergence criterion on* $I$,

*then*

- $\int_a^b f(t,x)dt$ *converges*,
- *The function* $F(x) = \int_a^b f(t,x)dt$ is *well-defined and of class* $C^1$ *on* $I$, *and*
$$F'(x) = \int_a^b \frac{\partial f}{\partial x}(t,x)dt.$$

# Differentiation of an improper integral depending on one parameter

**Example.**

*Study the differentiability of* $F(x) = \int_1^{+\infty} \frac{\sin(xt)}{t^3}\, dt$ *on* $\mathbb{R}$.

# Differentiation of an improper integral depending on one parameter

**Answer.** The integral is improper and poses a problem at $+\infty$.
We have

$$f(t,x)=\frac{\sin(xt)}{t^3},\quad \frac{\partial f}{\partial x}(t,x)=\frac{\cos(xt)}{t^2},\ \forall (t,x)\in\Delta=[1,+\infty[\times\mathbb{R}$$

- There exists $x_0=0\in\mathbb{R}$ such that $\int_1^{\infty} f(t,0)\,dt=0$ is convergent,
- $f$ and $\frac{\partial f}{\partial x}$ are continuous over $\Delta$.
- $\forall (t,x)\in\Delta: \left|\frac{\partial f}{\partial x}(t,x)\right|\le\frac{1}{t^2}$, and $\int_1^{\infty}\frac{1}{t^2}\,dt$ converges. Therefore, $\int_1^{\infty}\frac{\partial f}{\partial x}(t,x)\,dt$ satisfies the dominated convergence criterion over $\mathbb{R}$.

Then the function $F$ is well-defined and of class $C^1$ over $\mathbb{R}$ and

$$F'(x)=\int_1^{\infty}\frac{\cos(xt)}{t^2}\,dt.$$

# Differentiation of an improper integral depending on one parameter

**Example.**

*Let the function $F$ be given by:*

$$F(x) = \int_1^{\infty} \frac{\cos(xt)}{t(t^2+1)}\, dt;$$

*1) Find the domain of definition $D$ of the function $F$.*
*2) Study the continuity of $F$ on $D$:*

**Answer**

- Let $f(t,x) = \dfrac{\cos(xt)}{t(t^2+1)}$ on $\Delta = [1, +\infty[ \times \mathbb{R}$,
- $f \in R_{loc}$(problem at $+\infty$ only according to the variable $t$).
- $F$ is an improper parameterized integral.

$$D_F = \left\{ x \in \mathbb{R} : \int_1^{+\infty} \frac{\cos(xt)}{t(t^2+1)}\, dt \text{ converges} \right\}.$$

# Differentiation of an improper integral depending on one parameter

- We have:

$$\frac{\cos(xt)}{t(t^2+1)} \leq \frac{1}{t(t^2+1)} \leq \frac{1}{t^3} = \varphi(t) \forall x \in \mathbb{R},\ \forall t \in [1,+\infty[,$$

moreover,

$$\int_1^{\infty} \frac{1}{t^3}\,dt \ \ converges$$

Hence $F$ verifies the criterion for dominated convergence over $\mathbb{R}$. We obtain simple convergence over $\mathbb{R}$. So $D = \mathbb{R}$.

# Differentiation of an improper integral depending on one parameter

2) Let's apply the theorem of continuity preservation for an improper parameterized integral:

- $f$ is continuous on $U = \mathbb{R}^* \times \mathbb{R}$, $\Delta \subseteq U$, as composed and ratio of continuous functions.
- $\displaystyle\int_1^{\infty} f(t;x)\,dt$ verifies the criterion for dominated convergence over $\mathbb{R}$ ( from question 1).

Then $F$ is continuous on $\mathbb{R}$.

# Differentiation of an improper integral depending on one parameter

We have

$$\frac{\partial f}{\partial x}(t;x) = \frac{-t\sin(xt)}{t(t^2+1)} = \frac{-\sin(xt)}{t^2+1} \quad \text{for } [1;+\infty[\times\mathbb{R};$$

and

$$\left|\frac{-\sin(xt)}{t^2+1}\right| \leq \frac{1}{t^2+1} \leq \frac{1}{t^2} = \psi(t) \quad \text{for } t\in\mathbb{R} \text{ and } t\in[1;+\infty[;$$

Moreover, we have

$$\int_1^{+\infty}\frac{1}{t^2}\,dt \ \ converges.$$

This implies the dominated convergence of

$$\int_1^{+\infty}\frac{\partial f}{\partial x}(x;t)\,dt \text{ over } \mathbb{R}.$$

# Differentiation of an improper integral depending on one parameter

We apply the theorem of conservation of differentiability:

- $f$ is $C^1$ on $U$ where $f$ and $\frac{\partial f}{\partial x}$ are continuous, as they are composed, multiplied, or ratios of $C^1$ functions.
- The integral
$$\int_1^{+\infty} \frac{\partial f}{\partial x}(t;x)\,dt$$
satisfies the criterion of dominated convergence on $R$.
- $F$ is defined on $\mathbb{R}$ so $\forall x_0 \in \mathbb{R} \int_1^{+\infty} f(t;x_0)\,dt$ is convergent.

Then $F$ is differentiable on $\mathbb{R}$ and

$$F'(x) = \int_1^{+\infty} \frac{\partial f}{\partial x}(t;x)\,dt = \int_1^{+\infty} \frac{\sin(xt)}{t^2+1}\,dt.$$