**Academic year (2023/2024)** **Work sheet N 4**
**Mathematical analysis 2** **Power Series**

**Exercise 1:** Let $\sum a_n x^n$ and $\sum b_n x^n$ two power series with radius of convergence $R_a$ and $R_b$. Show the following:

1. $\forall n \geq n_0, (n_0 \in \mathbb{N}) :\in \mathbb{N} : |a_n| \leq |b_n| \Rightarrow R_a \geq R_b$.
2. $\left(\forall \alpha, \forall \beta \in \mathbb{R}_+^*, \forall n \geq n_0, (n_0 \in \mathbb{N}) : \alpha\, |b_n| \leq |a_n| \leq \beta\, |b_n|\right) \Rightarrow R_a = R_b$.
3. If $|a_n| \underset{+\infty}{\sim} |b_n| \Rightarrow R_a = R_b$.

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**Exercise 2**: Determine the radius of convergence and the domain of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2} x^n, \qquad \sum_{n=1}^{\infty} (2n)! \frac{x^n}{2^n},$$
$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}, \qquad \sum_{n=0}^{\infty} \sin^n(n),$$
$$\sum_{n>0} \frac{(-1)^n}{n^n} x^n, \qquad \sum_{n>1} \arccos\left(1 - \frac{1}{n^2}\right) x^n,$$
$$\sum_{n>1} \frac{cos^2 n}{n} x^n, \qquad \sum_{n\geq 0} \left(\frac{1}{(2n)!} + n4^n\right) x^n.$$

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**Exercise 3:** Find the convergence radius and the sum of the following power series:

$$\sum_{n=1}^{\infty} \frac{n^2+n+1}{n} x^n, \quad \sum_{n=0}^{\infty} \frac{n+1}{n+2} x^n, \quad \sum_{n=2}^{\infty} (-1)^{n+1} n x^{2n+1}, \quad \sum_{n=0}^{\infty} n^{(-1)^n} x^n, \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)} x^n.$$

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**Exercise 4:** Develop the following functions into power series around 0:

$$f(x) = \frac{x}{9-x^2}, \quad f(x) = \ln\left(\frac{1+x}{1-x}\right), \quad f(x) = \frac{x}{(x-1)(x-2)}, \quad f(x) = \int_0^x e^{t^2}\, dt.$$

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**Exercise 5**: Consider the power series defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

1) Determine the radius $R$ and the domain of convergence $D$, then find its sum $S(x)$.

2) Calculate $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$.