

Worksheet 1 : Informal Natural Deduction

Exercise 1 : What are the conclusion and the assumptions of the following argument ?

Theorem. Let r be a positive real number. Then r has a square root.

Proof. Write $f(x) = x^2 - r$ for any real x . Then f is a continuous function on \mathbb{R} . If $x = 0$, then $f(x) = 0 - r < 0$ since r is positive. If x is very large, then $f(x) = x^2 - r > 0$. So, by the Intermediate Value Theorem, there must be x such that $f(x) = 0$. For this value x , we have

$$r = r + 0 = r + f(x) = r + (x^2 - r) = x^2.$$

Exercise 2 : From your understanding of mathematical arguments, which (if any) of the following possible sequent rules seem to be true? Give reasons.

Possible sequent rule A : If the sequent $(\Gamma \vdash \psi)$ is a correct sequent, and every statement in Γ is also in Δ , then the sequent $(\Delta \vdash \psi)$ is also correct.

Possible sequent rule B : If the sequent $(\{\phi\} \vdash \psi)$ is correct, then so is the sequent $(\{\psi\} \vdash \phi)$.

Possible sequent rule C : If the sequents $(\Gamma \vdash \psi)$ and $(\Delta \vdash \psi)$ are both correct, then so is the sequent $(\Gamma \cap \Delta \vdash \psi)$.

Exercise 3 : Express the following using \wedge between statements :

1. The real number r is positive but not an integer.
2. v is a nonzero vector.
3. ϕ if and only if ψ . [Here ϕ and ψ stand for statements.]

Exercise 4 : Write out derivations that prove the following sequents :

1. $\{\phi, \psi, \chi\} \vdash (\phi \wedge (\psi \wedge \chi))$.
2. $\{\phi, \psi\} \vdash (\phi \wedge \psi)$.
3. $\{\phi\} \vdash ((\phi \wedge \phi) \wedge \phi)$.
4. $\{\phi, \psi\} \vdash ((\phi \wedge \psi) \wedge (\phi \wedge \psi))$.
5. $\{(\phi \wedge \psi)\} \vdash (\phi \wedge \phi)$.
6. $\{((\phi \wedge \psi) \wedge \chi)\} \vdash (\phi \wedge (\psi \wedge \chi))$.
7. $\{\phi, (\psi \wedge \chi)\} \vdash (\chi \wedge \phi)$.
8. $\{(\phi \wedge (\psi \wedge \chi))\} \vdash ((\chi \wedge \phi) \wedge \psi)$.

Exercise 5 : Show that $\{\phi_1, \phi_2\} \vdash \psi$ if and only if $\{(\phi_1 \wedge \phi_2)\} \vdash \psi$.

Exercise 6 : In the following two derivations, the names of the rules are missing, and so are the dandahs and step numbers for the assumptions that are discharged. Write out the derivations, including these missing pieces.

1. A proof of $\vdash ((\phi \wedge \psi) \rightarrow (\psi \wedge \phi))$

$$\frac{\frac{\frac{(\phi \wedge \psi)}{\psi}}{(\psi \wedge \phi)}}{((\phi \wedge \psi) \rightarrow (\psi \wedge \phi))}$$

2. A proof of $\vdash ((\psi \rightarrow \chi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)))$

$$\frac{\frac{\frac{\frac{\frac{\phi}{\psi}}{(\psi \rightarrow \chi)}}{\chi}}{((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))}}{((\psi \rightarrow \chi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)))}$$

Exercise 7 : Each of the following derivations proves a sequent. Write out the sequent that it proves.

1.

$$\frac{\frac{\frac{\phi}{\textcircled{1}}}{(\psi \rightarrow \phi)} (\rightarrow I)}{(\phi \rightarrow (\psi \rightarrow \phi))} (\rightarrow I).$$

2.

$$\frac{\frac{\phi}{(\psi \rightarrow \phi)} (\rightarrow I)}{(\phi \rightarrow (\psi \rightarrow \phi))} (\rightarrow I).$$

3.

$$\frac{\frac{\phi}{\textcircled{1}}}{(\phi \rightarrow \phi)} (\rightarrow I).$$

Exercise 8 : Write out derivations to prove each of the following sequents.

1. $\vdash (\phi \rightarrow (\psi \rightarrow \psi)).$
2. $\vdash ((\phi \rightarrow \phi) \wedge (\psi \rightarrow \psi)).$
3. $\vdash ((\phi \rightarrow (\theta \rightarrow \psi)) \rightarrow (\theta \rightarrow (\phi \rightarrow \psi))).$
4. $\{(\phi \rightarrow \psi), (\phi \rightarrow \chi)\} \vdash (\phi \rightarrow (\psi \wedge \chi))$
5. $\{(\phi \rightarrow \psi), ((\phi \wedge \psi) \rightarrow \chi)\} \vdash (\phi \rightarrow \chi).$

Exercise 9 : Show that $\{\phi\} \vdash \psi$ if and only if $\vdash (\phi \rightarrow \psi)$. [Prove the directions \Rightarrow and \Leftarrow separately.]

Exercise 10 : Give derivations to prove the following sequents :

1. $\{\phi, (\phi \leftrightarrow \psi)\} \vdash \psi.$
2. $\vdash (\phi \leftrightarrow \phi).$
3. $\{(\phi \leftrightarrow \psi), (\psi \leftrightarrow \chi)\} \vdash (\phi \leftrightarrow \chi).$
4. $\{(\phi \leftrightarrow (\psi \leftrightarrow \psi))\} \vdash \phi.$

Exercise 11 : Let S be any set of statements, and let \sim be the relation on S defined by : for all $\phi, \psi \in S$,

$$\phi \sim \psi \text{ if and only if } \vdash (\phi \leftrightarrow \psi).$$

Show that \sim is an equivalence relation on S .

Exercise 12 : Show that if we have a derivation D of ψ with no undischarged assumptions, then we can use it to construct, for any statement ϕ , a derivation of $((\phi \leftrightarrow \psi) \leftrightarrow \phi)$ with no undischarged assumptions.

Exercise 13 : Find natural deduction proofs for the following sequents :

1. $\vdash (\neg(\phi \wedge (\neg\phi))).$
2. $\vdash ((\neg(\psi \rightarrow \psi)) \rightarrow (\neg\psi)).$
3. $\vdash ((\phi \wedge \psi) \rightarrow (\neg(\phi \rightarrow (\neg\psi)))).$
4. $\{((\neg(\phi \wedge \psi)) \wedge \phi)\} \vdash (\neg\psi).$
5. $\{((\neg\psi) \rightarrow (\neg\phi))\} \vdash (\phi \rightarrow \psi).$
6. $\vdash ((\neg(\phi \rightarrow \psi)) \rightarrow \phi).$
7. $\vdash (\phi \rightarrow ((\neg\phi) \rightarrow \psi)).$
8. $\{(\neg(\phi \leftrightarrow \psi))\} \vdash ((\neg\phi) \leftrightarrow \psi). \text{ (Hard.)}$

Exercise 14 : Give natural deduction proofs of the following sequents :

1. $\vdash (\phi \rightarrow (\phi \vee \psi)).$
2. $\{(\neg(\phi \vee \psi))\} \vdash ((\neg\phi) \wedge (\neg\psi)).$
3. $\vdash ((\phi \rightarrow \psi) \rightarrow ((\neg\phi) \vee \psi)).$
4. $\{(\phi \vee \psi)\} \vdash (\psi \vee \phi).$
5. $\{(\phi \vee \psi), (\phi \rightarrow \chi), (\psi \rightarrow \chi)\} \vdash \chi.$
6. $\{(\phi \vee \psi), (\neg\phi)\} \vdash \psi.$
7. $\{((\neg\phi) \wedge (\neg\psi))\} \vdash (\neg(\phi \vee \psi)).$