ENSIA — Academic year 2023-2024

Mathematical Logic — Third Academic Semester

Worksheet 2 : Propositional Logic

Exercise 1 : For each of the following formulas, draw a parsing tree that has the formula as its associated formula. (Trial and error should suffice, but later we will give an algorithm for this.)

(a) $(p \wedge q)$.

(b) p.

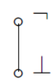
(c) $(p \to (q \to (r \to s)))$.

(d) $((\neg(p_2 \to (p_1 \leftrightarrow p_0))) \wedge (p_2 \to \bot))$.

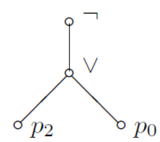
(e) $(((\neg(p_3 \wedge p_7)) \vee (\neg(p_1 \vee p_2))) \to (p_2 \vee (\neg p_5)))$.

Exercise 2 : Find the associated formula of each of the following parsing trees.

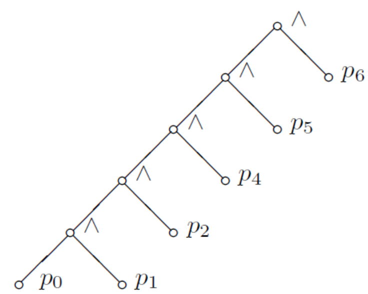
(a)



(b)



(c)



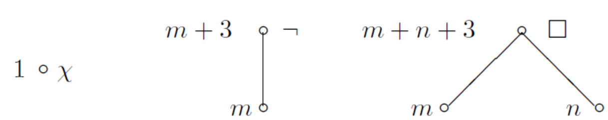
Exercise 3 : For each of the formulas in Exercise 1, find a smallest possible signature σ such that the formula is in the language $\mathbf{LP}(\sigma)$.

Exercise 4 : List all the subformulas of the following formula. (You found its parsing tree in Exercise 1(d).)

$$((\neg(p_2 \to (p_1 \leftrightarrow p_0))) \wedge (p_2 \to \bot)).$$

Exercise 5 : Take σ to be the default signature $\{p_0, p_1, \dots\}$. Draw six parsing trees $\pi_1, \dots, \pi_6$ for $\mathbf{LP}(\sigma)$, so that each π_i has i nodes.

Exercise 6 : Consider the following compositional definition, which uses numbers as labels :



where χ is atomic and $\Box \in \{\wedge, \vee, \to, \leftrightarrow\}$.

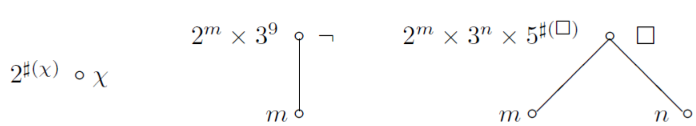
If π is any parsing tree for $\mathbf{LP}$, and δ is the definition above, what is $\delta(\pi)$? Justify your answer. [You might find it helpful to try the definition in your trees from Exercise 5.]

Exercise 7 : Construct a compositional definition δ so that for each parsing tree π, $\delta(\pi)$ is the number of parentheses '(' or ')' in the associated formula of π.

Exercise 8 : This exercise turns formulas into numbers. Let σ be the default signature $\{p_0, p_1, p_2, \dots\}$. We assign distinct odd positive integers $\sharp$ to symbols s of $\mathbf{LP}(\sigma)$ as follows :

s	$\wedge$	$\vee$	$\to$	$\leftrightarrow$	$\neg$	$\bot$	p_0	p_1	$\dots$
$\sharp(s)$	1	3	5	7	9	11	13	15	$\dots$

The following compositional definition



where χ is atomic and $\Box \in \{\wedge, \vee, \to, \leftrightarrow\}$.

assigns a number to each node of any parsing tree. The number on the root is called the *Godel number* of the associated formula of the tree.
Explain how, if you know the Godel number of a formula of **LP**(σ), you can reconstruct the formula. (Use unique prime decomposition.) Illustrate by reconstructing the formula with Godel number

$$2^{2^{2^{15}} \times 3^9} \times 3^{2^{13}} \times 5^3.$$

(For goodness sake do not try to calculate the number!)
Exercise 9 : Suppose (N_1, D_1) and (N_2, D_2) are planar trees. An isomorphism from (N_1, D_1) to (N_2, D_2) is a bijection $f : N_1 \to N_2$, such that for every node $\mu \in N_1$, if $D_1(\mu) = (\nu_1, \ldots, \nu_n)$, then $D_2(f\mu) = (f\nu_1, \ldots, f\nu_n)$. We say that two planar trees are isomorphic if there is an isomorphism from the first to the second. (Then, isomorphism is an equivalence relation.)
Prove : If f and g are two isomorphisms from (N_1, D_1) to (N_2, D_2), then $f = g$. (If (N_1, D_1) has height n, prove by induction on k that for each k, the functions f and g agree on all nodes of height $n - k$ in N_1.)
Exercise 10 : Calculate the depths of all the initial segments of the string

$$(\neg(p_{22} \leftrightarrow (\neg\bot))).$$

Exercise 11 : Calculate the heads of the following formulas of **LP** :

(a) $((((\neg p_0)) \leftrightarrow (\neg(p_1 \to \bot))) \to p_1) \to p_1)$.

(b) $(\neg((\neg p_0) \leftrightarrow ((((\neg p_1) \to \bot) \to p_1) \to p_1)))$.

Exercise 12 : For each of the following strings, either write down its parsing tree (thereby, showing that it is a formula of **LP**), or show that it is not a formula of **LP**.

(a) $((p_3 \wedge (p_3 \wedge p_2))$.
(b) $((((\neg p_1) \leftrightarrow \bot) \vee p_1) \wedge p_2)$.
(c) $(((\neg(p_0 \vee p_1)) \wedge (p_2 \to p_3))) \to (p_3 \wedge p_4))$.
(d) $(p_1 \wedge \neg\neg(p_2 \vee p_0))$.
(e) $((\neg p_1) \to (\neg p_2) \wedge p_1))$.
(f) $((p_1 \wedge (p_2 \vee p_3)) \leftrightarrow (\neg(\neg p_0)))$.
(g) $(((p_1 \wedge p_2)) \wedge (p_3 \wedge p_4))$.
(h) $((p_1 \to (\neg(p_2))) \leftrightarrow p_3)$.
(i) $(p_1 \to (p_2 \to (p_3 \wedge p_4) \to p_5)))$.

Exercise 13 :

(a) Prove the following : let S be a set of expressions such that

(1) every atomic formula of **LP**(σ) is in S ;

(2) if s and t are any expressions in S, then the expressions

$$(\neg s) \quad (s \wedge t) \quad (s \vee t) \quad (s \to t) \quad (s \leftrightarrow t)$$

are all in S.

Then every formula of **LP**(σ) is in S. [Let π be a parsing tree. Show (by induction on height of) that if (1) and (2) are true, then for every node ν of π, the formula $\bar{\nu}$ at ν is in S.]

(b) Use (a) to show that every formula of **LP**(σ) has equal numbers of left parentheses '(' and right parentheses ')'. [Put $S = \{s : s$ has equal numbers of left and right parentheses$\}$, and remember to prove that (1) and (2) are true for this S.]
Deduce that

$$(((p \leftrightarrow q) \vee (\neg p))$$

is not a formula of **LP**(σ).

Exercise 14 : In each case below, use Exercise 14(a) to show that the given expression is not a formula of **LP**(σ), where $\sigma = \{p_0, p_1, \ldots\}$, by finding a set that satisfies (1) and (2) above but does not contain the given expression. [The expressions all do have equal numbers of left and right parentheses, so you cannot just use S from (b) of the previous exercise. Avoid mentioning 'formulas' in the definition of your S.]

(a) $p_{\sqrt{2}}$.

(b) $)p_0($.

(c) $(p_1 \wedge p_2 \to p_3)$.

(d) $(\neg\neg p_1)$. [WARNING. The obvious choice $S = \{s \mid s$ does not have two $\neg$ next to each other$\}$ does not work, because it fails (2); $\neg$ is in S but $(\neg\neg)$ is not.]

(e) $(p_1 \to ((p_2 \to p_3)) \to p_2)$.

(f) $(\neg p_1)p_2$.

(g) $(\neg p_1 \to (p_1 \vee p_2))$.

Exercise 15 : Let σ be the default signature $\{p_0, p_1, \dots\}$. Neither of the following two diagrams is a σ-derivation. In them, find all the faults that you can.

(a)
$$\dfrac{(\neg(p_0 \to p_1)) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p_0 \qquad \cancel{(p_0 \to \bot)}}{\bot}\,(\to\text{E})}{p_1}\,(\neg\text{E})}{(p_1 \to p_0)}\,(\to\text{I}) \qquad \cancel{(\neg(p_1 \to p_0))}}{\bot}\,(\to\text{E})}{\bot}\,(\text{RAA})}{((\neg(p_0 \to p_1)) \to \bot)}\,(\to\text{I})$$

(b)
$$\dfrac{\dfrac{\dfrac{\cancel{(\neg((\neg p_2) \vee q))} \qquad \dfrac{\cancel{(\neg p_2)}}{((\neg p_2) \vee q)}\,(\vee\text{I})}{\bot}\,(\neg\text{E})}{p_2}\,(\neg\text{I}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\cancel{p_2} \qquad \cancel{(p_2 \to q)}}{q}\,(\to\text{E})}{((\neg p_2) \vee q)}\,(\vee\text{I}) \qquad \cancel{(\neg((\neg p_2) \vee q))}}{\bot}}{(\neg p_2)}\,(\text{RAA})}{\dfrac{\bot}{((\neg p_2) \vee q)}\,(\neg\text{I})}\,(\neg\text{I})$$

Exercise 16 : Let ρ and σ be signatures with $\rho \subseteq \sigma$. Show that :

(a) Every parsing tree for $\mathbf{LP}(\rho)$ is also a parsing tree for $\mathbf{LP}(\sigma)$.

(b) Every formula of $\mathbf{LP}(\rho)$ is also a formula of $\mathbf{LP}(\sigma)$.

(c) Every ρ-derivation is also a σ-derivation.

(d) If $(\Gamma \vdash_\rho \psi)$ is a correct sequent, then so is $(\Gamma \vdash_\sigma \psi)$.

Exercise 17 : Let ρ and σ be signatures with $\rho \subseteq \sigma$.

(a) Suppose D is a σ-derivation, and D' is got from D by writing $\bot$ in place of each symbol in D that is in σ but not in ρ. Show that D' is a ρ-derivation.

(b) Suppose Γ is a set of formulas of $\mathbf{LP}(\rho)$ and ψ is a formula of $\mathbf{LP}(\rho)$, such that the sequent $(\Gamma \vdash_\sigma \psi)$ is correct. Show that the sequent $(\Gamma \vdash_\rho \psi)$ is correct. [If D is a σ-derivation proving $(\Gamma \vdash_\sigma \psi)$, apply part (a) to D and note that this does not change the conclusion or the undischarged assumptions.]

Exercise 18 : Let σ be a signature, Γ a set of formulas of $\mathbf{LP}(\sigma)$ and ϕ a sentence of $\mathbf{LP}(\sigma)$. Show the following :

(a) The sequent $\Gamma \cup \{(\neg\phi)\} \vdash \bot$ is correct if and only if the sequent $\Gamma \vdash_\sigma \phi$ is correct.

(b) The sequent $\Gamma \cup \{\phi\} \vdash \bot$ is correct if and only if the sequent $\Gamma \vdash_\sigma (\neg\phi)$ is correct.

Exercise 19 : We have stated several rules about correct sequents, and verified them informally. With our new formal definition of sequents we can prove them mathematically. Do so using Definition 3.4.1. (The formulas mentioned are all assumed to be in $\mathbf{LP}(\sigma)$.)

(a) The Axiom Rule : If ψ is a formula in Γ, then $(\Gamma \vdash_\sigma \psi)$ is correct.

(b) Monotonicity : If $\Gamma \subseteq \Delta$ and $(\Gamma \vdash_\sigma \psi)$ is correct, then $(\Delta \vdash_\sigma \psi)$ is correct.

(c) The Transitive Rule : If $(\Delta \vdash_\sigma \psi)$ is correct and for every formula χ in Δ, $(\Gamma \vdash_\sigma \chi)$ is correct, then $(\Gamma \vdash_\sigma \psi)$ is correct.

(d) The Cut Rule : If $(\Gamma \vdash_\sigma \phi)$ is correct and $(\Gamma \cup \{\phi\} \vdash_\sigma \psi)$ is correct, then $(\Gamma \vdash_\sigma \psi)$ is correct.

[The proofs of (c) and (d) involve taking derivations and fitting them together to create a new derivation.]

Exercise 20 : Prove by truth tables that the following are tautologies.

(a) $(p \leftrightarrow p)$.
(b) $(p \to (q \to p))$.
(c) $((p_1 \to p_2) \leftrightarrow ((\neg p_2) \to (\neg p_1)))$.
(d) $((p_1 \to (\neg p_1)) \leftrightarrow (\neg p_1))$.
(e) $(p_1 \vee (\neg p_1))$.
(f) $(\bot \to p_1)$.
(g) $((p_1 \to (p_2 \to p_3)) \leftrightarrow ((p_1 \wedge p_2) \to p_3))$.

Exercise 21 : Write out truth tables for the formulas in the following list. For each of these formulas, say whether it is (a) a tautology, (b) a contradiction, (c) satisfiable. (It can be more than one of these.)

(a) $((p_0 \to \bot) \leftrightarrow (\neg p_0))$.
(b) $(p_1 \leftrightarrow (\neg p_1))$.
(c) $((p_2 \wedge p_1) \to (\neg p_1))$.
(d) $(((p_1 \leftrightarrow p_2) \wedge ((\neg p_1) \leftrightarrow p_3)) \wedge (\neg (p_2 \vee p_3)))$.
(e) $((((p \to q) \to r) \to ((r \to p) \to q)) \to ((q \to r) \to p))$.
(f) $((p_0 \to p_1) \to ((\neg (p_1 \wedge p_2)) \to (\neg (p_0 \wedge p_1))))$.
(g) $(((p \wedge q) \vee (p \wedge (\neg r))) \leftrightarrow (p \vee (r \to q)))$.
(h) $((p \wedge (\neg ((\neg q) \vee r))) \wedge (r \vee (\neg p)))$.

Exercise 22 : You forgot whether the Logic class is at 11 or 12. Your friend certainly knows which; but sometimes he tells the truth and at other times he deliberately lies, and you know that he will do one of these but you do not know which. What should you ask him? [Let p be the statement that your friend is telling the truth, and let q be the statement that the lecture is at 11. You want to ask your friend whether a certain formula ϕ is true, where ϕ is chosen so that he will answer 'Yes' if and only if the lecture is at 11. The truth table of ϕ will be that of q if p is true, and that of $(\neg q)$ if p is false. Find an appropriate ϕ which contains both p and q.]

Exercise 23 : Prove the following equivalences :

(a) $(p \wedge q)$ is logically equivalent to $(\neg ((\neg p) \vee (\neg q)))$, and to $(\neg (p \to (\neg q)))$.
(b) $(p \vee q)$ eq $(\neg ((\neg p) \wedge (\neg q)))$, and to $((p \to q) \to q)$.
(c) $(p \to q)$ eq $(\neg (p \wedge (\neg q)))$, and to $((\neg p) \vee q)$.
(d) $(p \leftrightarrow q)$ eq $((p \to q) \wedge (q \to p))$, and to $((p \wedge q) \vee ((\neg p) \wedge (\neg q)))$.
(e) $(p_1 \leftrightarrow p_2)$ eq $(p_2 \leftrightarrow p_1)$.
(f) $(p_1 \leftrightarrow (p_2 \leftrightarrow p_3))$ eq $((p_1 \leftrightarrow p_2) \leftrightarrow p_3)$.
(g) $(\neg (p_1 \leftrightarrow p_2))$ eq $((\neg p_1) \leftrightarrow p_2)$.
(h) $(p_1 \leftrightarrow (p_2 \leftrightarrow p_2))$ eq p_1.

Exercise 24 : Suppose ρ and σ are signatures with $\rho \subseteq \sigma$, and ϕ and ψ are formulas of $\mathbf{LP}(\rho)$. Show that ϕ and ψ are logically equivalent when regarded as formulas of $\mathbf{LP}(\rho)$ if and only if they are logically equivalent when regarded as formulas of $\mathbf{LP}(\sigma)$.

Exercise 25 : Show that the following are equivalent, for any formula ϕ of $\mathbf{LP}(\sigma)$:

(a) ϕ is a tautology.
(b) $(\neg \phi)$ is a contradiction.
(c) ϕ is logically equivalent to $(\neg \bot)$.
(d) ϕ is logically equivalent to some tautology.

Exercise 26 : Carry out the following substitutions :

(a) $(p \to q)[p/q]$.
(b) $(p \to q)[p/q][q/p]$.
(c) $(p \to q)[p/q, q/p]$.
(d) $(r \wedge (p \vee q))[((t \to (\neg p)) \vee q)/p, (\neg (\neg q))/q, (q \leftrightarrow p)/s]$.

Exercise 27 :

(a) Using one of the De Morgan Laws, show how the following equivalences follow from the Replacement and Substitution Theorems :

$$\begin{aligned}(\neg ((p_1 \wedge p_2) \wedge p_3)) \quad & \text{eq} \quad ((\neg (p_1 \wedge p_2)) \vee (\neg p_3)) \\ & \text{eq} \quad (((\neg p_1) \vee (\neg p_2)) \vee (\neg p_3)).\end{aligned}$$

(b) Show the following generalised De Morgan Law, by induction on n :
If $\phi_1, \ldots, \phi_n$ are any formulas, then

$$(\neg (\ldots (\phi_1 \wedge \phi_2) \wedge \ldots) \wedge \phi_n)) \quad \text{eq} \quad (\ldots ((\neg \phi_1) \vee (\neg \phi_2)) \vee \ldots) \vee (\neg \phi_n)).$$

(c) The four formulas below are logically equivalent. Justify the equivalences, using the Replacement and Substitution Theorems, the De Morgan and Double Negation Laws and (a), (b) above.

(i) $(\neg((p_1 \wedge (\neg p_2)) \vee (((\neg p_1) \wedge p_2) \wedge p_3)))$.

(ii) $((\neg(p_1 \wedge (\neg p_2))) \wedge (\neg(((\neg p_1) \wedge p_2) \wedge p_3)))$.

(iii) $(((\neg p_1) \vee (\neg(\neg p_2))) \wedge (((\neg(\neg p_1)) \vee (\neg p_2)) \vee (\neg p_3)))$.

(iv) $(((\neg p_1) \vee p_2) \wedge ((p_1 \vee (\neg p_2)) \vee (\neg p_3)))$.

Exercise 28 : We know that $(p \to q)$ is logically equivalent to $((\neg p) \vee q)$, and hence (by the Substitution Theorem) for all formulas ϕ and ψ, $\phi \to \psi$ is logically equivalent to $((\neg\phi) \vee \psi)$.

(a) Deduce that every formula of **LP** is logically equivalent to one in which $\to$ never occurs. [Using the Substitution and Replacement Theorems, show that if ϕ contains n occurrences of $\to$ with $n > 0$, then ϕ is logically equivalent to a formula containing $n-1$ occurrences of $\to$.]

(b) Illustrate this by finding a formula that is logically equivalent to

$$((((p \to q) \to r) \leftrightarrow s) \to t)$$

in which $\to$ never appears.

Exercise 29 : both $(p \wedge q)$ and $(p \vee q)$ are logically equivalent to formulas in which no truth function symbols except $\to$ and $\neg$ occur.

(a) Show that each of $(p \leftrightarrow q)$ and $\bot$ is logically equivalent to a formula in which no truth function symbols except $\to$ and $\neg$ occur.

(b) Find a formula of **LP** logically equivalent to

$$(q \vee (((\neg p) \wedge q) \leftrightarrow \bot))$$

in which no truth function symbols except $\to$ and $\neg$ occur.

Exercise 30 : For each of the following formulas ϕ, find a formula ϕ^{DNF} in DNF, and a formula ϕ^{CNF} in CNF, which are both equivalent to ϕ.

(a) $\neg(p_1 \to p_2) \vee \neg(p_2 \to p_1)$.

(b) $(p_2 \leftrightarrow (p_1 \wedge p_3))$.

(c) $\neg(p_1 \to p_2) \vee (p_0 \leftrightarrow p_2)$.

(d) $\neg(p_1 \wedge p_2) \to (p_1 \leftrightarrow p_0)$.

(e) $\neg(p \wedge q) \to (q \leftrightarrow r)$.

(f) $((p_0 \to p_1) \to p_2) \to (p_0 \wedge p_1)$.

(g) $((p \leftrightarrow q) \to r) \to q$.

(h) $(p_0 \to p_1) \to (\neg(p_1 \wedge p_2) \to \neg(p_0 \wedge p_2))$.

(i) $(p_1 \wedge p_2) \to \neg(p_3 \vee p_4)$.

(j) $p_1 \to (p_2 \to (p_3 \to p_4))$.

Exercise 31 : Consider the formula

$$(p_1 \wedge \neg p_2 \wedge p_3 \wedge \neg p_4) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4).$$

The two conjuncts are identical except that one has p_3 where as the other has $\neg p_3$. In this case we can leave out p_3 ; the whole formula is logically equivalent to $p_1 \wedge \neg p_2 \wedge \neg p_4$.

(a) Justify the statement above.

(b) Use this method to find a shorter formula in DNF that is logically equivalent to the following :

$$(p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge p_4) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4) \vee (p_1 \wedge p_2 \wedge p_3 \wedge \neg p_4).$$

Exercise 32 :

(a) For each of the following formulas in DNF, either find a model if there is one, or show that there is none (Do not write out truth tables for the formulas.).

(i) $(p1 \wedge p_2 \wedge \neg p_1) \vee (p_2 \wedge \neg p_3 \wedge p_3) \vee (\neg p_1 \wedge p_3)$.

(ii) $(p_2 \wedge \neg p_1 \wedge p_3 \neg p_5 \wedge \neg p_2 \wedge p_4) \vee (\neg p_2 \wedge p_1 \wedge \neg p_3 \wedge p_5 \wedge \neg p_8 \wedge \neg p_1)$.

(b) In your own words, write instructions for your younger sister (who is not a mathematician) so that she can answer (i), (ii) and similar questions by herself.

Exercise 33 : Let σ be a signature containing k symbols. The relation eq splits the set of formulas of $\mathbf{LP}(\sigma)$ into classes, where each class consists of a formula and all the other formulas logically equivalent to it. (These are the equivalence classes of the equivalence relation eq.) Calculate the number of equivalence classes of the relation eq.

Exercise 34 : Let σ be a non-empty signature. Show that every formula of $\mathbf{LP}(\sigma)$ is logically equivalent to a formula of $\mathbf{LP}(\sigma)$ in which no truth function symbols are used except $\to$ and $\neg$.

Exercise 35 : The following argument shows that Peirce's Formula $(((p \to q) \to p) \to p)$ can't be proved using just the Axiom Rule and the rules $(\to I)$ and $(\to E)$; your task is to fill in the details. Instead of two truth values T and F, we introduce three truth values $1, \frac{1}{2}, 0$. Intuitively 1 is truth, 0 is falsehood and $\frac{1}{2}$ is somewhere betwixt and between. If ϕ has the value i and ψ has the value j (so $i, j \in \{1, \frac{1}{2}, 0\}$), then the value of $(\phi \to \psi)$ is

$$\text{the greatest real number } r \leq 1 \text{ such that } \min\{r, i\} \leq j.$$

(a) Write out the truth table for $\to$ using the three new truth values. (For example, you can check that $(p \to q)$ has the value $\frac{1}{2}$ when p has value 1 and q has value $\frac{1}{2}$.)

(b) Find values of p and q for which the value of $(((p \to q) \to p) \to p)$ is not 1.

(c) Show that the following holds for every derivation D using at most the Axiom Rule, $(\to I)$ and $(\to E)$: If ψ is the conclusion of D and A is a structure with $A^*(\psi) < 1$, then $A^*(\phi) \leq A^*(\psi)$ for some undischarged assumption ϕ of D.