Worksheet N 3
Special functions
Gamma and Beta functions

Exercise 1: We call the Gamma function the function defined by

$$\Gamma : x \mapsto \int_0^{+\infty} e^{-t}t^{x-1}dt.$$

1. Show that Γ is defined on $]0, +\infty[$.
2. Show that Γ is continuous on $]0, +\infty[$.
3. Show that Γ is of class C^∞ on $]0, +\infty[$ with $\forall x \in N^*, \forall x \in R_+^*, \Gamma(k)(x) = \displaystyle\int_0^{+\infty} (\ln t)^k e^{-t}t^{x-1}dt$.
4. Show that for all $x > 0, \Gamma(x+1) = x\Gamma(x)$. Deduce that $\forall n \in N^*, \Gamma(n) = (n-1)!$.
5. Calculate $\Gamma\left(\frac{1}{2}\right)$.

Exercise 2:

1. Show that
$$\int_0^1 (\ln(\frac{1}{y}))^{n-1}dy = \Gamma(n), \qquad \int_0^{+\infty} x^n e^{-k^2x^2}dx = \frac{1}{2k^{n+1}}\Gamma(n).$$
2. Calculate the following integrals:
$$\int_0^{+\infty} e^{-x^2}dx, \qquad \int_0^{+\infty} \sqrt{x}e^{-3\sqrt{x}}dx.$$

Exercise 3: Calculate the following integrals

$$\int_0^{+\infty} x^5e^{-x^4}dx, \qquad \int_0^{+\infty} x^3e^{-\frac{x^2}{2}}dx, \qquad \int_0^{+\infty} 2^{-3x^2}dx, \qquad \int_0^{+\infty} \sqrt{2}e^{-\sqrt{x}},$$

$$\int_1^{+\infty} \frac{(lnx)^3}{x^2}dx, \qquad \int_0^{+\infty} 2\sqrt{x}e^{-x^2}dx, \qquad \int_0^{+\infty} x^6e^{-4x^2}dx, \qquad \int_0^1 \left(ln\frac{1}{x}\right)^{a-1} dx$$

Exercise 4: Calculate the following integrals

$$\int_0^{\pi/2} \sin^2\theta d\theta, \qquad \int_0^{\pi/4} \sin^3(2x)\cos^4(2x)dx, \qquad \int_0^{\pi} \sin^5(\frac{x}{2})\cos^2(\frac{x}{2})dx, \qquad \int_0^4 \frac{x^3}{\sqrt{4-x}}dx,$$

$$\int_0^1 \frac{4}{\sqrt[4]{1-x^4}}dx, \qquad \int_0^1 x^3(1-\sqrt{x})^5dx, \qquad \int_0^1 \frac{x}{\sqrt{1-x^3}}dx, \qquad \int_0^1 (x\log x)^3dx,$$

$$\int_3^7 \sqrt[4]{(x-3)(7-x)}dx, \qquad \int_0^e \frac{x}{\sqrt{1-\ln x}}dx$$
