Mathematical analysis 2
Chapter 4 : Power series

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2023/2024

Course outline

Generalities

Definition

A ***power series*** *is defined as a series of the form* $\sum a_n x^n$, *where* (a_n) *is a numerical sequence and* $x \in \mathbb{R}$.

- *The sequence of partial sums* (S_n) *defined by*

$$S_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \sum_{k=0}^{n} a_k x^k$$

  is a polynomial of degree n.
- $\sum a_n (x - x_0)^n$, *where* $x_0 \in \mathbb{R}$, *represents a power series centered at* x_0.

Domain of Convergence

In the study of power series, one seeks a set D such that

$$D = \{x \in \mathbb{R} \text{ such that } \sum a_n x^n \text{ converges}\}$$

which is called the **domain of convergence**.

Example.

The series $\sum x^n$ is a convergent geometric series if and only if $x \in]-1, 1[$, hence $D =]-1, 1[$.

Radius of convergence

For any power series $\sum a_n x^n$, there exists a unique real number $R \in [0, +\infty[$ (R can be equal to $+\infty$) called the **radius of convergence** such that

- If $|x| < R$, the series $\sum a_n x^n$ converges absolutely.
- If $|x| > R$, the series $\sum a_n x^n$ diverges.
- If $|x| = R$, nothing can be said about the nature of the series $\sum a_n x^n$.
- The open interval $]-R, R[$ is called the interval of convergence.

Example.

The series $\sum x^n$ converges absolutely if and only if $|x| < 1$ and diverges if $|x| \geq 1$. Hence, the radius of convergence of the series $\sum x^n$ is $R = 1$.

Computation of radius of convergence

Proposition (D'Alembert's ratio test)

Let $\sum_{n=0}^{\infty} a_n x^n$ *be a power series,* $(a_n)_n$ *a numerical such that* $a_n > 0, \ \forall n \geq n_0$.

If

$$\lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| = l \in [0, +\infty],$$

Then the radius of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ *is* $R = \frac{1}{l}$.

Remark

1. *If* $\ell = 0$, *then* $R = +\infty$. *In this case, the series* $\sum a_n x^n$ *converges absolutely for all* $x \in \mathbb{R}$.
2. *If* $\ell = +\infty$, *then* $R = 0$. *In this case, the series* $\sum a_n x^n$ *converges absolutely only for* $x = 0$.

Computation of radius of convergence

Example.

Determine the convergence domain of the power series: $P = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution: Given $a_n = \frac{1}{n!} > 0, \forall n \in \mathbb{N}$, let's apply D'Alembert's ratio test:

$$l = \lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = \lim_{n \to +\infty} \frac{n!}{(n+1)!} = \lim_{n \to +\infty} \frac{1}{n+1} = 0,$$

Thus, we deduce that $R = \frac{1}{l} = +\infty$, and hence $D = \mathbb{R}$.

Computation of radius convergence

Example.

Determine the radius of convergence and the domain of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$.

We have $a_n = \frac{1}{n^2}$ and $\ell = \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to +\infty} \frac{n^2}{(n+1)^2} = 1.$

Therefore, the radius of convergence is given by $R = \frac{1}{\ell} = 1$.

The series $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ converges absolutely if $|x| < 1$ and diverges for $|x| > 1$.

- If $x = 1$, then $u_n = \frac{1}{n^2}$ and the series $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges (convergent Riemann series).
- If $x = -1$, then $u_n = \frac{(-1)^n}{n^2}$ and the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$ is convergent (convergent alternating series).

Therefore, the convergence domain is $D = [-1, 1]$.

Computation of radius convergence

Example.

Determine the radius of convergence and the convergence domain of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ *and* $\sum_{n=0}^{\infty} n!x^n$.

- For the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have $a_n = \frac{1}{n!}$ and
  $\ell = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \frac{n!}{(n+1)!} = 0.$
  Hence, the radius of convergence $R = +\infty$, and thus the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all $x \in \mathbb{R}$.
- For the series $\sum_{n=0}^{\infty} n!x^n$, we have $R = 0$, and thus the series converges only for $x = 0$.

Computation of radius convergence

Example.

Determine the radius of convergence and the convergence domain of the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$.

We have $a_n = \frac{1}{3^n}$ and $\ell = \lim_{n\to+\infty} \sqrt[n]{|u_n|} = \frac{1}{3}$. Hence,

$$R = \frac{1}{\ell} = 3.$$

Thus, the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$ converges for all $|x-2| < 3$, that is,

$$-3 < x-2 < 3$$
$$2-3 < x < 2+3$$
$$x \in]-1, 5[$$

and diverges for $|x-2| > 3$.

Computation of radius convergence

Let's verify the boundaries:

- If $x=-1$, then $u_n=\dfrac{(-3)^n}{3^n}=\dfrac{(-1)^n(3)^n}{3^n}=(-1)^n$ and the series $\sum_{n=0}^{\infty}(-1)^n$ diverges.
- If $x=5$, then $u_n=\dfrac{3^n}{3^n}=1$ and the series $\sum_{n=0}^{\infty}1$ diverges.

Therefore, we conclude that the convergence domain is $D=]-1,5[$.

Computation of radius of convergence

Proposition (Cauchy root test)

Let $\sum_{n=0}^{\infty} a_n x^n$ *be a power series, If*

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = l \in [0,+\infty],$$

Then the radius of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ *is*

$$R = \frac{1}{l}.$$

Computation of radius convergence

Example.

Determine the radius of convergence and the domain of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$.

We have

$$a_n = \frac{1}{2^n} \quad \text{and} \quad \ell = \lim_{n\to+\infty} \sqrt[n]{|u_n|} = \frac{1}{2}.$$

Therefore,

$$R = \frac{1}{\ell} = 2.$$

Hence, the series $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converges absolutely on $]-2,2[$, which means the series converges absolutely for $|x| < 2$ and diverges for all x such that $|x| > 2$.

Computation of radius convergence

Let's examine the case $|x| = 2$:

- If $x = 2$, then $u_n = 1$ and the series $\sum_{n=0}^{\infty} 1$ diverges because $\lim_{n\to+\infty} u_n \neq 0$.
- If $x = -2$, then $u_n = \dfrac{(-2)^n}{2^n} = (-1)^n$ and the series $\sum_{n=0}^{\infty} (-1)^n$ diverges.

Therefore, we conclude that the convergence domain is $D =]-2, 2[$.

Computation of radius convergence

Lemma (d'Hadamard)

The radius of convergence of a power series $\sum a_n x^n$ *is given by*

$$R = \frac{1}{\ell} \quad \textit{where} \quad \ell = \limsup_{n \to +\infty} \sqrt[n]{|a_n|}.$$

Computation of radius convergence

Example.

Determine the radius of convergence and the domain of convergence of the series $\sum_{n=0}^{\infty} e^{n\cos(n)} x^n$.

- We have $a_n = e^{n\cos(n)}$, and $\limsup_{n\to+\infty} \sqrt[n]{|a_n|} = \limsup_{n\to+\infty} \sqrt[n]{e^{n\cos(n)}} = e$ then $R = \frac{1}{e}$
- For $x = \frac{1}{e}$, the series $\sum_{n=0}^{\infty} \left(e^{\cos(n)-1}\right)^n$ is divergent (since $a_{2n\pi} \nrightarrow 0$)
- $x = -\frac{1}{e}$, the series $\sum_{n=0}^{\infty} \left(e^{\cos(n)-1}\right)^n (-1)^n$ is divergent (since $a_{2n\pi} \nrightarrow 0$).
- Then the domain of convergence is $]-1/e; 1/e[$.

Lacunary series $\sum a_n x^{kn+l}$

Definition

A Lacunary series is a series that has an infinite number of zero terms.

Example.

The series $\sum_{n=0}^{\infty} 3^n x^{2n}$ *is a Lacunary power series. Indeed,*

$$\begin{aligned} \sum_{n=0}^{\infty} 3^n x^{2n} &= 1 + 3x^2 + 3^2 x^4 + 3^3 x^6 + \cdots \\ &= 1 + 0x + 3x^2 + 0x^3 + 3^2 x^4 + \cdots \end{aligned}$$

⚠ To calculate the radius of convergence of a Lacunary power series, we apply the **D'Alembert and Cauchy tests** as specified in the following examples:

Examples of lacunary series

Example.

Determine the radius of convergence for the series $\sum_{n=0}^{\infty} 3^n x^{2n}$.

Let's define $f_n(x) = 3^n x^{2n}$.

We have $\lim_{n\to+\infty} \sqrt[n]{|f_n(x)|} = \lim_{n\to+\infty} \sqrt[n]{|3^n x^{2n}|} = |3x^2|$

Therefore, the series converges if $|3x^2| < 1$.

$$|3x^2| < 1 \implies |x^2| < \frac{1}{3} \implies |x| < \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Hence,

$$R = \frac{\sqrt{3}}{3}.$$

Examples of lacunary series

Example.

Find the radius of convergence for the series $\sum_{n=0}^{\infty} 2^n x^{2n+3}$.

We have

$$\lim_{n\to+\infty}\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \lim_{n\to+\infty}\left|\frac{2^{n+1}x^{2n+5}}{2^n x^{2n+3}}\right| = |2x^2|$$

The series converges if $|2x^2| < 1$.

$$|2x^2| < 1 \implies |x^2| < \frac{1}{2} \implies |x| < \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Hence,

$$R = \frac{\sqrt{2}}{2}.$$

Comparison of radius of convergence

Proposition

Let $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ *be two power series with respective radius of convergence* R_a *and* R_b. *Then*

1. *If for all* $n \geq n_0$, $|a_n| \leq |b_n|$, *then* $R_a \geq R_b$.
2. *If* $|a_n| \underset{+\infty}{\sim} |b_n|$, *then* $R_a = R_b$.

Corollaire

Let $\sum_{n=0}^{\infty} a_n x^n$ *be a divergent power series at* $x_0 \in \mathbb{R}$, *then it diverges for all* $x \in \mathbb{R}$ *such that* $|x| \geq |x_0|$. *That is to say, The power series diverges on* $]-\infty; -|x_0|] \cup [|x_0|; +\infty[$.

Comparison of radius of convergence

Proposition

Let $\sum_{n=0}^{\infty} a_n x^n$ *be a power series.*

1. *If there exist two strictly positive real numbers* m *and* M *such that for all* n, $m \le |a_n| \le M$, *then* $R = 1$.
2. *If* a_n *is a non-zero rational fraction, then* $R = 1$.

Proposition

The series $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} \alpha a_n x^n$ *have the same radius of convergent for all* $\alpha \in \mathbb{R}^*$

Comparison of radius of convergence

Proposition (Sum of power series)

Let $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ *be two power series with respective radius of convergence* R_a *and* R_b*. Then*

- *The power series* $\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$ *has a radius of convergence* $R_c \geq \inf\{R_a, R_b\}$.
- *For all* x *such that* $|x| < \inf\{R_a, R_b\}$*, we have*

$$\sum_{n=0}^{\infty} (a_n + b_n) x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n.$$

- *Moreover, if* $R_a \neq R_b$*, then* $R_c = \inf\{R_a, R_b\}$.

Course outline

Properties of Power Series

Theorem (**Continuity**)

The sum function of a power series $S(x)=\sum_{n=0}^{\infty} a_n x^n$ *with a convergence radius* R *is a continuous function on the interval* $]-R,R[$.

Theorem (**Integration**)

Let $\sum_{n\geq 0} a_n x^n$ *be a power series with a convergence radius* R. *For any compact interval* $[a,b]\subset]-R,R[$,

$$\int_a^b S(x)dx=\int_a^b \sum_{n\geq 0} a_n x^n dx=\sum_{n\geq 0} a_n \int_a^b x^n dx=\left[\sum_{n\geq 0} a_n \frac{x^{n+1}}{n+1}\right]_a^b.$$

The sum of a power series with a convergence radius R *is integrable over any segment* $[a,b]\subset]-R,R[$.

Properties of Power Series

Proposition

Any power series $\sum_{n\geq 0} a_n x^n$ *with a convergence radius* R *is integrable and has a primitive on* $]-R,R[$ *given by*

$$F(x) = cste + \sum_{n\geq 0} \frac{a_n}{n+1} x^{n+1}, \quad cste \in \mathbb{R},$$

which is a power series with the same convergence radius.

Example

Example.

Let $f(x)=\sum_{n\geq 0} x^n$, $R=1$, $x\in]-1,1[$.

We have

$$f(x)=\sum_{n\geq 0} x^n=\frac{1}{1-x} \quad \forall x\in]-1,1[.$$

The primitive series is

$$\begin{aligned} F(x) &= cste+\sum_{n\geq 0}\frac{x^{n+1}}{n+1} \quad \forall x\in]-1,1[. \\ &= \int \frac{1}{1-x}dx+cste \\ &= -\ln(1-x)+cste. \end{aligned}$$

Therefore, $$\ln(1-x)=-\sum_{n\geq 0}\frac{x^{n+1}}{n+1}, \quad R=1, \quad x\in]-1,1[.$$

Properties of Power Series

Theorem (Derivative of a Power Series)

Any power series $f(x) = \sum_{n\geq 0} a_n x^n$ *with a convergence radius* R *is differentiable on* $]-R, R[$*, and its derivative is*

$$f'(x) = \sum_{n\geq 1} n a_n x^{n-1}.$$

It is a power series with the same convergence radius R.

Corollaire

The sum function of a power series $\sum_{n\geq 0} a_n x^n$ *with a convergence radius* R *is* $\mathscr{C}^\infty$ *on* $]-R, R[$.

Example

Example.

Let $f(x) = \sum_{n\geq 0} \frac{x^{2n+1}}{2n+1}$, $R = 1$. *Determine the function* f' *on* $]-1,1[$.

We have

$$f'(x) = \sum_{n\geq 0} \frac{d}{dx}\left(\frac{x^{2n+1}}{2n+1}\right) = \sum_{n\geq 1} x^{2n} = \sum_{n\geq 1} (x^2)^n = \frac{1}{1-x^2}.$$

Hence,

$$f'(x) = \sum_{n\geq 1} x^{2n} = \frac{1}{1-x^2}.$$

Course outline

Representing Functions as Power Series

Definition

A real function defined on $I \subset \mathbb{R}$ *has a representation as power series in the neighborhood of* $x_0 \in I$ *if there exists a real numerical sequence* (a_n) *and* $R > 0$ *such that*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \forall x \in]x_0 - R, x_0 + R[.$$

Example.

The function f *defined on* $]-1, 1[$ *by* $f(x) = \dfrac{1}{1-x}$

has a representation as power series in the neighborhood of $x_0 = 0$*, and we have*

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \forall x \in]-1, 1[, \quad R = 1, \quad a_n = 1.$$

Representation functions as power series: Examples

Example.

Expand the function defined by $f(x) = \dfrac{1}{2-x}$ *into a power series around* $x_0 = 1$.

We have

$$f(x) = \frac{1}{2-x} = \frac{1}{1-(x-1)} = \sum_{n=0}^{\infty} (x-1)^n, \quad \forall x \text{ such that } |x-1| < 1,$$

With $\quad R = 1, \quad a_n = 1.$

Therefore,

$$f(x) = \frac{1}{2-x} = \sum_{n=0}^{\infty} (x-1)^n, \quad \forall x \in]0,2[, \quad R = 1.$$

Representation functions as power series:Examples

Example.

Expand the function defined by $f(x) = \dfrac{1}{1+x^2}$ *into a power series around* $x_0 = 0$.

We have

$$\begin{aligned} f(x) = \frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} \\ &= \sum_{n=0}^{\infty} (-x^2)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad \forall x \in]-1,1[,\ R = 1,\ a_n = (-1)^n \end{aligned}$$

Therefore,

$$f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad \forall x \in]-1,1[,\ R = 1.$$

Representation functions as power series: Taylor series

Definition

*Let f be a real function indefinitely differentiable on $]-R,R[$. We call **Taylor series** of the function f in the neighborhood of $x_0 = 0$ the power series :*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Remark

In the neighborhood of $x_0 = a$, Taylor series associated to the function f is written as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

Representation functions as power series: Taylor series

Example.

Expand the function defined by $f(x) = e^x$ *to a power series in the neighborhood of* $x_0 = 0$.

The function $f(x) = e^x$ is infinitely differentiable ($f \in \mathscr{C}^\infty(\mathbb{R})$) with $f^{(n)}(x) = e^x$.
Therefore, the Taylor series of the functin f around $x_0 = 0$, is

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x \in \mathbb{R}, \quad R = +\infty.$$

Representation functions as power series

Proposition

Let $f : I \to \mathbb{R}$ be a function that can be represented as a power series around 0. Then, there exists $r > 0$ such that $f \in C^{\infty}(]-r,r[)$ and f is equal to its Taylor series on $]-r,r[$,that is to say

$$\forall x \in]-r,r[, f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

where $f^{(n)}(0)$ denotes the nth derivative of f evaluated at 0.

Proposition

Let f be a function in C^{∞} on $]-r,r[$. If there exists a real number $M > 0$ such that for all $x \in]-r,r[$ and for all $n \in \mathbb{N}$, $|f^{(n)}(x)| \leq M$, then f can be expanded as a power series around 0.

Power series associated to some elementary functions

The function f defined by

- $f(x) = \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \forall x \in]-1,1[, \quad R = 1.$
- $f(x) = \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \quad \forall x \in]-1,1[,$
  $R = 1.$
- $f(x) = \ln(1+x) \Longrightarrow f'(x) = \dfrac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$

  By integrating the derivative series, we get

  $f(x) = \ln(x+1) = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n+1} x^{n+1}, \quad \forall x \in]-1,1[, \quad R = 1.$

Power series associated to some elementary functions

- $f(x) = \ln(1-x) \implies f'(x) = \dfrac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n$

  By integrating the derivative series, we get

  $$f(x) = \ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \quad \forall x \in]-1,1[, \quad R = 1.$$

- $f(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad \forall x \in]-1,1[, \quad R = 1.$

- $f(x) = \arctan(x) \implies f'(x) = \dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

  By integrating the derivative series, we get

  $$f(x) = \arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad \forall x \in]-1,1[, \quad R = 1.$$

Power series associated to some elementary functions

- **The sine function** $f(x) = \sin(x)$
  The function f is defined over $\mathbb{R}$ and $f \in \mathscr{C}^{\infty}(\mathbb{R})$.

$$\begin{aligned}
f(x) &= \sin(x) & f(0) &= 0.\\
f'(x) &= \cos(x) & f'(0) &= 1.\\
f''(x) &= -\sin(x) & f''(0) &= 0.\\
f^{(3)}(x) &= -\cos(x) & f^{(3)}(0) &= -1.\\
f^{(4)}(x) &= \sin(x) & f^{(4)}(0) &= 0.\\
f^{(5)}(x) &= \cos(x) & f^{(5)}(0) &= 1.\\
&\vdots\\
f^{(n)}(x) &= \sin(x + \frac{n\pi}{2}) & f^{(n)}(0) &= \sin(\frac{n\pi}{2})
\end{aligned}$$

By the Taylor series expansion around $x_0 = 0$, we obtain

$$\begin{aligned}
f(x) = \sin(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n\\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots
\end{aligned}$$

Power series associated to some elementary functions

We conclude that

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R}, \quad R = +\infty.$$

- **The cosine function** $f(x) = \cos(x)$
  The function f is defined over $\mathbb{R}$, and we have

$$f(x) = \cos(x) = \frac{d}{dx}\sin(x) = \frac{d}{dx}\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

  Consequently,

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Power series associated to some elementary functions

- **The hyperbolic cosine function**

$$f(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

The function f is defined and infinitely differentiable over $\mathbb{R}$.
We have

$$\begin{aligned}
\cosh(x) &= \frac{e^x + e^{-x}}{2} \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}\right) \\
&= \frac{1}{2}\sum_{n=0}^{\infty} (1 + (-1)^n) \frac{x^n}{n!} \\
&= \frac{1}{2}\left(2 + 0 + \frac{2x^2}{2!} + 0 + \frac{2x^4}{4!} + 0 + \frac{2x^6}{6!} + \cdots\right) \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}
\end{aligned}$$

Power series associated to some elementary functions

- **The hyperbolic sine function**

$$f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2}$$

The function f is defined and infinitely differentiable over $\mathbb{R}$.
We have

$$\begin{aligned}
\sinh(x) &= \frac{e^x - e^{-x}}{2} \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}\right) \\
&= \frac{1}{2}\sum_{n=0}^{\infty} (1-(-1)^n)\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

Therefore,

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R}, \quad R = +\infty.$$

Course outline

Application to differential equations

Example.

Solve the following differential equation:

$$y'(x) - 2y(x) = 0, \quad y(0) = 1. \quad (1)$$

We seek the solution in the form of a power series, therefore, we assume

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n, \quad x \in]-R, R[. \\ y(0) &= 1 \implies a_0 = 1. \end{aligned}$$

We have

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \implies y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

.

Application to differential equations

Substituting into equation (1), we obtain

$$\begin{aligned}
y' - 2y = 0 \quad &\Longrightarrow \quad \sum_{n\geq 0}(n+1)a_{n+1}x^n - 2\sum_{n\geq 0} a_n x^n = 0 \\
&\Longrightarrow \quad \sum_{n\geq 0}\left[(n+1)a_{n+1} - 2a_n\right]x^n = 0 \\
&\Longrightarrow \quad (n+1)a_{n+1} - 2a_n = 0 \\
&\Longrightarrow \quad a_{n+1} = \frac{2}{n+1}a_n, \quad n = 0, 1, \cdots
\end{aligned}$$

Application to differential equations

We have

$$
\begin{aligned}
a_0 &= 1\\
a_1 &= \frac{2}{0+1}a_0 = \frac{2}{1}\\
a_2 &= \frac{2}{1+1}a_1 = \frac{2}{2}\cdot\frac{2}{1} = \frac{2^2}{2!}\\
a_3 &= \frac{2}{2+1}a_2 = \frac{2}{3}\cdot\frac{2^2}{2!} = \frac{2^3}{3!}\\
a_4 &= \frac{2}{3+1}a_3 = \frac{2}{4}\cdot\frac{2^3}{3!} = \frac{2^4}{4!}\\
&\vdots\\
a_n &= \frac{2^n}{n!} \quad \forall n\in\mathbb{N}
\end{aligned}
$$

Therefore $$y(x) = \sum_{n=0}^{\infty}\frac{2^n}{n!}x^n = \sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = e^{2x}.$$

Application to differential equations

Example.

Solve the differential equation

$$2x(1+x)y''+(5x+3)y'+y=0$$

The previous equation can be written as

$$2xy''+2x^2y''+5xy'+3y'+y=0 \tag{1}$$

Let's assume

$$y(x)=\sum_{n=0}^{\infty}a_nx^n \quad \text{on }]-R,R[.$$

Application to differential equations

Therefore,

$$\begin{aligned}
y'(x) &= \sum_{n\geq 1} n a_n x^{n-1} = \sum_{n\geq 0} (n+1) a_{n+1} x^n.\\
y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n\geq 0} n(n+1) a_{n+1} x^{n-1}.\\
xy'(x) &= \sum_{n\geq 1} n a_n x^n = \sum_{n\geq 0} n a_n x^n.\\
xy''(x) &= \sum_{n\geq 1} n(n+1) a_{n+1} x^n = \sum_{n\geq 0} n(n+1) a_{n+1} x^n.\\
x^2y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^n = \sum_{n\geq 0} n(n-1) a_n x^n.
\end{aligned}$$

Application to differential equations

By substituting into equation (1), we obtain

$$\sum_{n\geq 0} \left(2n(n+1)a_{n+1} + 2n(n-1)a_n + 5na_n + 3(n+1)a_{n+1} + a_n\right)x^n = 0.$$

$$\Longrightarrow \quad 2n(n+1)a_{n+1} + 2n(n-1)a_n + 5na_n + 3(n+1)a_{n+1} + a_n = 0.$$

$$\Longrightarrow \quad (2n+3)(n+1)a_{n+1} + (2n(n-1) + 5n + 1)a_n = 0.$$

$$\Longrightarrow \quad (2n+3)(n+1)a_{n+1} + (2n^2 + 3n + 1)a_n = 0.$$

$$\Longrightarrow \quad a_{n+1} = \frac{-(2n^2+3n+1)}{(2n+3)(n+1)}a_n.$$

We have

$$(2n^2 + 3n + 1) = (2n+1)(n+1)$$

Therefore,

$$a_{n+1} = \frac{-(2n+1)}{(2n+3)}a_n$$

Application to differential equations

$$n=0 \qquad a_1=\frac{-1}{3}a_0.$$

$$n=1 \qquad a_2=\frac{-3}{5}a_1=\frac{-3}{5}\left(\frac{-1}{3}a_0\right)=\frac{1}{5}a_0$$

$$n=2 \qquad a_3=\frac{-5}{7}a_2=\frac{-5}{7}\left(\frac{1}{5}a_0\right)=\frac{-1}{7}a_0$$

$$\vdots \qquad \vdots$$

$$a_n=\frac{(-1)^n}{2n+1}a_0$$

Therefore,

$$y(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}a_0x^n=a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^n, \quad x\in]-1,1[.$$

Application to differential equations

Let's determine the expression of y.

- For $x \in]0,1[$: We know that

$$\operatorname{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Furthermore, we have

$$\begin{aligned} y(x) &= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2n+1}. \\ &= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n}}{2n+1}. \\ &= \frac{a_0}{\sqrt{x}} \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n+1}}{2n+1}. \end{aligned}$$

Therefore, $y(x) = \frac{a_0}{\sqrt{x}} \operatorname{Arctan}(\sqrt{x})$.

Application to differential equations

- For $x \in]-1,0[$: We know that

$$\operatorname{argth}(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}.$$

Furthermore, we have

$$\begin{aligned} y(x) &= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2n+1}. \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-x)^n}{2n+1}. \\ &= a_0 \sum_{n=0}^{\infty} \frac{(\sqrt{-x})^{2n}}{2n+1}. \\ &= \frac{a_0}{\sqrt{-x}} \sum_{n=0}^{\infty} \frac{(\sqrt{-x})^{2n+1}}{2n+1}. \end{aligned}$$

Application to differential equations

Consequently

$$y(x) = \frac{a_0}{\sqrt{-x}} argth(\sqrt{-x}).$$

- forr $x = 0$ we have

$$y(0) = a_0$$

Application to differential equations

Example.

Find the function y as a power series around 0, which is a solution to the differential equation

$$x^2y'' + 4xy' + 2y = e^x \tag{2}$$

We seek a solution to the equation in the form of a power series given by

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Thus, we have

$$y'(x) = \sum_{n\geq 1} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n\geq 2} n(n-1) a_n x^{n-2}.$$

Application to differential equations

We then have

$$\sum_n n(n-1)a_n x^n + \sum_n 4na_n x^n + \sum_n 2a_n x^n = \sum_n \frac{x^n}{n!} \quad (\text{since} \quad e^x = \sum_n \frac{x^n}{n!})$$

hence we deduce

$$\sum_n \left(n^2+3n+2\right) a_n x^n = \sum_n \frac{x^n}{n!}$$

Consequently

$$\left(n^2+3n+2\right) a_n = \frac{1}{n!}$$

Now

$$n^2+3n+2 = (n+2)(n+1).$$

which yields

$$a_n = \frac{1}{(n+2)(n+1)n!} = \frac{1}{(n+2)!}$$

Finally

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n+2)!}, \quad R = +\infty.$$

Application to differential equations

Let's now determine the expression of y:

1. For $x \neq 0$, we have

$$y(x) = \frac{1}{x^2} \sum_{n=2}^{+\infty} \frac{x^n}{(n+2)!} = \frac{1}{x^2} \sum_{n=2}^{+\infty} \frac{x^n}{n!} = \frac{1}{x^2} \left(\sum_{n=0}^{+\infty} \frac{x^n}{n!} - 1 - x \right) = \frac{e^x - 1 - x}{x^2}.$$

2. For $x = 0$, we have

$$y(0) = \frac{1}{2}.$$