The National School of Artificial Intelligence
ENSIA

Probability
Semester 3
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Worksheet n°1

Exercise 1. Consider the random experiment consisting in throwing two dice and observing, once these dice are immobilized, the number of dots on the top face of each of them. And let us define the following random events:
A: "the number of dots on the first die is odd";
B: "the number of dots on the two dice is even";
C: "the sum of the dots on the two dice is divisible by 4".

1. Describe the proof space Ω, associated with this random experiment.
2. Represent the following events, by subsets of Ω:
   - The number of dots on each side is greater than 3.
   - The total number of dots is greater than 9.
   - The total number of dots is 8.
   - On each face, there is an even number of dots.
   - On one face, there is an even number of dots and on the other; there is more than one dot.

Exercise 2. A die is rigged in the following way: when it is thrown, the probability P that a number appears is proportional to that number. Determine P(1), ..., P(6).

Exercise 3. A queue is formed by randomly assigning sequence numbers to n people (n ≥ 2). Two friends A and B are in this queue.

1. What is the probability that the two friends are located one behind the other?
2. What is the probability that the two friends are r seats apart (i.e. separated by r - 1 people)?

Exercise 4. A letter is stored with probability p in a chest of drawers. This chest has seven drawers. Six drawers have been looked at without success.
Calculate the probability that the letter is in the seventh drawer.

Exercise 5. A runner is randomly selected from a group of athletes to undergo a doping control.
We call T the event: "The test is positive". According to statistics, we admit that P(T)=0,05.
We call D the event: "The runner is doped".
The anti-doping control is not 100% reliable, we know that:

If a runner is doped, the test is positive in 97% of the cases.
If a runner is not doped, the test is positive in 1% of the cases.

1. We note p the probability of D. Determine the p value.
2. A runner has a positive test. What is the probability that he is not doped?

Exercise 6. We consider a succession of bags that we call $S_1, S_2, \dots, S_n, \dots$
At the beginning the bag S_1 contains 2 black chips and 1 white chip; all the other bags contain 1 black chip and 1 white chip each. We randomly draw a chip from bag S_1 and place it in bag S_2. Then, we randomly draw a chip from bag S_2, which we place in bag S_3, and so on.
Let us note B_k the event: "the chip drawn from the bag S_k is white", and $p_k = P(B_k)$ its probability.

a. Show that $P(B_2/B_1) = \frac{2}{3}$ and $P(B_2/\bar{B}_1) = \frac{1}{3}$.

b. Show that for any integer $n \geq 1$: $p_{n+1} = \frac{1}{3}(p_n + 1)$.

c. For any $n \in \mathbb{N}^*$, we pose $q_n = p_n - \frac{1}{2}$. Show that the sequence $(p_n)_{n\in\mathbb{N}^*}$ is convergent and determine its limit.

Exercise 7. Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space where $\Omega = \{\omega_1, \omega_2, \ldots, \omega_6\}, \mathcal{A} = \mathcal{P}(\Omega)$ and the probability is given by

$$\mathbb{P}(\{\omega_i\}) = \frac{i}{21}, i = 1,2, \ldots, 6.$$

Let the events

$$A = \{\omega_1, \omega_3\}; B = \{\omega_2, \omega_4\} \; and \; C = \{\omega_6\}.$$

Let us consider the application $X(.)$ From Ω to $\mathbb{R}$, given by

$$X(\omega) = 3\mathbb{I}_A - 4\mathbb{I}_B + \mathbb{I}_C,$$

Where $\mathbb{I}_A$ is the indicatrice of A.

1. Show that X is a random variable on $(\Omega, \mathcal{A})$.
2. Determine the distribution of X.
3. Determine the cumulative distribution function of X.
4. Calculate the probabilities

$$\mathbb{P}(X > -4); \mathbb{P}(-4 \leq X < 3).$$

Exercise 8. Let X be the discrete random variable of probability distribution

X=k	-1	1	3
P(X=k)	1/3	1/4	α

1. Determine α.
2. Plot the cumulative distribution function of X.
3. Calculate the expectation of X, the variance and P(X<2,5 / X≥0,5).

Exercise 9. An urn contains 3 white balls and 2 black balls. The number of balls drawn is a random variable Y which depends on the number displayed by a die.

$$Y = \begin{cases} 1 \; ball \; is \; drawn \; if \; the \; die \; display \; 1 \; or \; 2 \; or \; 3 \\ 2 \; balls \; are \; drawn \; if \; the \; die \; display \; 4 \; or \; 5 \\ 3 \; balls \; are \; drawn \; if \; the \; die \; display \; 6 \end{cases}$$

1. Give the probability distribution of Y then calculate $E[Y]$ and Var(Y).
2. What is the probability of drawing at least one black ball? (The draw being without put-back).

Exercise 10. For $\theta \in]0,1[$ and $n \geq 2$, we define the sequence p_k by

$$p_k = \begin{cases} C_n(1-\theta)\min\{k, n-k\} & if \; k = 1, \ldots, n-1 \\ \theta & if \; k = n \end{cases}$$

Where C_n is a positive constant. Determine C_n such that p_k is a probability on $\{1, \ldots, n\}$.

Exercise 11. Let X be a random variable with values in $\mathbb{N}$ and having expectation $\mathbb{E}[X] = m$ where $1 \leq m \leq 4$. We suppose that the distribution of X verify

$$\forall n \in \mathbb{N}, 6\mathbb{P}(x = n+2) - 7\mathbb{P}(x = n+1) + 2\mathbb{P}(x = n) = 0.$$

1. Determine $p_n = \mathbb{P}(x = n)$. Deduce the value of p_0 and p_1.
2. Calculate $Var(X)$ when
   A. $m = 1$.
   B. $m = 2$.

Exercise 12. The cumulative distribution function of a random variable X is given by

$$F_X(x)=\begin{cases}0 & \text{if} \quad x<2 \\ (x-2)^2 & \text{if} \quad 2\le x<3. \\ 1 & \text{if} \quad x\ge 3\end{cases}$$

1. Calculate $P\left(1\le X\le\frac{5}{2}\right); P\left(X\ge\frac{5}{2}\Big/X\le\frac{7}{2}\right)$.
2. Find the density function of the random variable X.

Exercise 13.

1. Is there a constant C such that the function f defined by
$$f(x)=C(x^2-4x)1_{[-1,1]}(x)$$
is the density of a random variable X.
2. Consider X the random variable of density f: $f(x)=C(x^2-4x)1_{[0,2]}(x)$.
   a. Calculate C and determine the cumulative distribution function of X.
   b. Calculate $E[X^k], k\ge 1$, deduce $E[X]$ and Var(X).
   c. Calculate $P\left(X\ge\frac{1}{2}/X<1\right)$.
   d. Determine the density of the random variable $Y=\sqrt{X}$.

Exercise 14. Let be the random variable of density $f(x)=\frac{\alpha}{1+x^2}, x\in\boldsymbol{R}$

1. Determine α, F and $E[X]$.
2. Determine the density of the random variable $Y=|X|$.

Exercise 15. A die is rolled 10 times. Let X be the random variable representing the number of marks multiple of 3 obtained. What is the probability distribution of X? Determine E[X] and Var(X).

Exercise 16. A player tosses a coin n times. The probability of getting tails on a toss is p. The player wins if he gets tails exactly once. For what values of p is he most likely to win?

Exercise 17. Two players of equal strength are playing against each other. What is the probability of one of the players winning 5 out of 8 games? At least 5 games out of 8?

Exercise 18. A colony of snails has a well-localized habitat and a population of 20. In a first operation 8 snails were marked and put back in the colony. A few days later 5 snails are collected at random. What can be said about the distribution of X equal to the number of marked snails among the 5? Determine E[X] and Var(X).

Exercise 19. A doctor knows that one tenth of his patients have a disease A. In one morning he makes 20 visits.

1. What is the probability distribution of the variable N expressing the number of patients suffering from A.
2. Calculate P(N≥2).
3. Determine E[N] and Var(N).

Exercise 20. A die is rolled until a "6" is obtained and the experiment is stopped.

1. Give the probability distribution of the random variable X corresponding to the number of throws until a "6" appears.
2. Calculate the mathematical expectation of X and its variance.

Exercise 21. An urn contains white balls in proportion p=0.4 and black balls in proportion q=1-p. We draw a ball, note its color and put it back in the urn. We note by X the random variable equal to the number of draws at the end of which appears the first time a white ball. Determine the distribution of X. Calculate E[X] and Var(X).

Exercise 22. The proportion of defective tubes produced by a company is 2%.

a. What is the distribution of the number of defective tubes in a sample of 200 tubes? Determine its expectation and variance.
b. Are the conditions required to approximate this distribution by a Poisson distribution satisfied? Calculate the probabilities of obtaining a number of defective tubes:

- Zero
- equal to 5
- less than or equal to 6
- greater than or equal to 10

Exercise 23. A switchboard receives on average 2 calls per minute. The calls are randomly distributed in time.

1. What is the probability law governing the number of calls received in 3 minutes?
2. What is the probability that there are no calls in 3 minutes?
3. What is the probability that the number of calls in 2 minutes is greater than or equal to 5?

Exercise 24. Assume that the lifetime of a hard disk is distributed according to an exponential law. The manufacturer wants to guarantee that the hard drive has a probability of less than 0.001 of failing over one year. What is the minimum average life of the hard drive?

Exercise 25. Let X be a centered reduced normal random variable.

1. Determine the following probabilities:
    - $\mathbb{P}(X \geq 0)$ *and* $\mathbb{P}(X > 0)$;
    - $\mathbb{P}(X \leq 1), \mathbb{P}(X \geq 1), \mathbb{P}(X \leq -1), \mathbb{P}(X \geq -1), \mathbb{P}(X > -2.33)$ *and* $\mathbb{P}(X < 5.3)$;
    - $\mathbb{P}(-1 \leq X \leq 1), \mathbb{P}(-3 \leq X \leq -2)$ *and* $\mathbb{P}(|X| \geq 2.2)$.
2. Find x such that $\mathbb{P}(X \leq x) = 0.975, \mathbb{P}(X \geq x) = 0.90, \mathbb{P}(X \leq x) = 0.95$ *and* $\mathbb{P}(|X| < x) = 0.90$.

Exercise 26. Machines make sheet metal plates for stacking.

1. Let X be the Random Variable: "thickness of the plate in mm"; it is assumed that X follows the normal distribution of parameters μ=0.3 and σ=0.1.
   Calculate the probability that X is less than 0.36 mm and the probability that X is between 0.25 and 0.35?
2. The use of these plates consists in stacking n of them, numbered from 1 to N by taking them at random: let X_i be the random variable: "thickness of plate number i in mm" and Z the random variable: "thickness of n plates in mm".
   For n=20, what is the law of Z, its expectation and its variance?