Mathematical analysis 3
Chapter 0 : Improper Integrals

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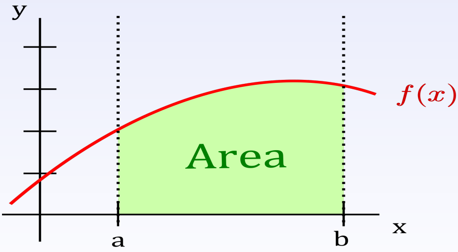
2023/2024

Recall that the Fundamental Theorem of Calculus says that if f is a **continuous** function on the **closed interval** $[a,b]$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a),$$

where F is any antiderivative of f.
Both the continuity condition and closed interval must hold to use the Fundamental Theorem of Calculus, and in this case, $\int_a^b f(x)\,dx$ represents the net **area** under $f(x)$ from a to b:



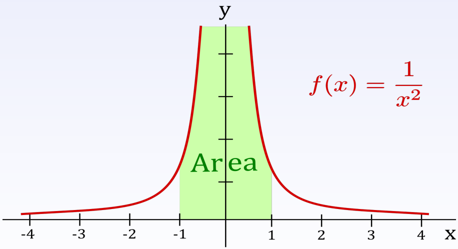
We begin with an example where blindly applying the Fundamental Theorem of Calculus can give an incorrect result.

Example.

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{1} x^{-2}\,dx = -\frac{1}{x}\Big|_{-1}^{1} = -2$$

However, the above answer is WRONG! Since $f(x) = \frac{1}{x^2}$ is not continuous on $[-1, 1]$, we cannot directly apply the Fundamental Theorem of Calculus.

Intuitively, we can see why -2 is not the correct answer by looking at the graph of $f(x) = \frac{1}{x^2}$ on $[-1, 1]$. The shaded area appears to grow without bound as seen in the figure below.



- There are two ways to extend the Fundamental Theorem of Calculus. One is to use an infinite interval, i.e., $[a,\infty)$, $(-\infty,b]$, or $(-\infty,\infty)$. The second is to allow the interval $[a,b]$ to contain an infinite discontinuity of $f(x)$. In either case, the integral is called an improper integral.
- One of the most important applications of this concept is probability distributions because determining quantities like the cumulative distribution or expected value typically require integrals on infinite intervals.

Definition

- *If* f *is continuous on* $[a,+\infty)$, *then the improper integral of* f *over* $[a,+\infty)$ *is*
$$\int_a^{+\infty} f(x)\,dx = \lim_{R\to+\infty}\int_a^R f(x)\,dx.$$
- *If* f *is continuous on* $(-\infty,b]$, *then the improper integral of* f *over* $(-\infty,b]$ *is*
$$\int_{-\infty}^b f(x)\,dx = \lim_{R\to-\infty}\int_R^b f(x)\,dx.$$
- *If the limit exists and is a finite number, we say the improper integral* ***converges***. *If the limit is* $\pm\infty$ *or does not exist, we say the improper integral* ***diverges***.
- *If both* $\displaystyle\int_a^{-\infty} f(x)\,dx$ *and* $\displaystyle\int_{\infty}^a f(x)\,dx$ *are convergent, then the improper integral of* f *over* $(-\infty,\infty)$ *is*

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^{+\infty} f(x)\,dx.$$

Example. Determine whether $\int_1^{+\infty} \frac{1}{x}\,dx$ is convergent or divergent.
Using the definition for improper integrals, we write this as:

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{R\to+\infty} \int_1^{R} \frac{1}{x}\,dx = \lim_{R\to+\infty} \ln|x|\Big|_1^{R} = \lim_{R\to+\infty} \ln|R| - \ln|1|$$

$$= \lim_{R\to+\infty} \ln|R| = +\infty.$$

Therefore, the integral is divergent.

Example. Determine whether $\int_{-\infty}^{\infty} x\sin(x^2)\,dx$ is convergent or divergent. We compute both $\int_{-\infty}^{0} x\sin(x^2)\,dx$ and $\int_{0}^{+\infty} x\sin(x^2)\,dx$. Note that we don't have to split the integral up at 0, any finite value a will work. First, we compute the indefinite integral. Let $u = x^2$, then $du = 2x\,dx$, and hence,

$$\int x\sin(x^2)\,dx = \frac{1}{2}\int \sin(u)\,du = -\frac{1}{2}\cos(x^2) + C.$$

Using the definition of the improper integral gives:

$$\int_0^{+\infty} x\sin(x^2)\,dx = \lim_{R\to+\infty}\int_0^R x\sin(x^2)\,dx = \lim_{R\to+\infty}\left[-\frac{1}{2}\cos(x^2)\right]\Big|_0^R$$

$$= -\frac{1}{2}\lim_{R\to+\infty}\cos(R^2) + \frac{1}{2}.$$

This limit does not exist since $\cos x$ oscillates between -1 and $+1$. In particular, $\cos x$ does not approach any particular value as x gets larger and larger. Thus, $\int_0^{+\infty} x\sin(x^2)\,dx$ diverges, and hence, the integral $\int_{-\infty}^{\infty} x\sin(x^2)\,dx$ diverges.

Definition

- *If* f *is continuous on* $(a,b]$, *then the improper integral of* f *over* $(a,b]$ *is*

$$\int_a^b f(x)\,dx = \lim_{R\to a^+} \int_R^b f(x)\,dx.$$

- *If* f *is continuous on* $[a,b)$, *then the improper integral of* f *over* $[a,b)$ *is*

$$\int_a^b f(x)\,dx = \lim_{R\to b^-} \int_a^R f(x)\,dx.$$

Discontinuities Within Integration Interval

Definition

If f *has a discontinuity at* $x = c$ *where* $c \in [a,b]$, *and both* $\int_a^c f(x)\,dx$ *and* $\int_c^b f(x)\,dx$ *are convergent, then the improper integral of* f *over* $[a,b]$ *is*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example. Determine if $\int_{-1}^{1} \frac{1}{x^2}\,dx$ is convergent or divergent.
The function $f(x) = \frac{1}{x^2}$ has a discontinuity at $x = 0$, which lies in $[-1, 1]$. We compute $\displaystyle\int_{-1}^{0} \frac{1}{x^2}\,dx$ and $\displaystyle\int_{0}^{1} \frac{1}{x^2}\,dx$.

Let's start with $\displaystyle\int_{0}^{1} \frac{1}{x^2}\,dx$:

$$\int_{0}^{1} \frac{1}{x^2}\,dx = \lim_{R\to 0^+} \int_{R}^{1} \frac{1}{x^2}\,dx = \lim_{R\to 0^+} \left(-\frac{1}{x}\Big|_{1}^{R} \right) = -1 + \lim_{R\to 0^+} \frac{1}{R},$$

which diverges to $+\infty$.
Therefore, $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$ is divergent since one of $\displaystyle\int_{-1}^{0} \frac{1}{x^2}\,dx$ and $\displaystyle\int_{0}^{1} \frac{1}{x^2}\,dx$ is divergent.

Example. Determine if $\int_0^1 \ln x\, dx$ is convergent or divergent.

Note that $f(x) = \ln x$ is discontinuous at the endpoint $x = 0$. We first use Integration by Parts to compute $\int \ln x\, dx$. We let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}dx$, $v = x$, giving:

$$\int \ln x\, dx = x\ln x - \int x \cdot \frac{1}{x}\, dx = x\ln x - \int 1\, dx = x\ln x - x + C$$

Now using the definition of the improper integral for $\int_0^1 \ln x\,dx$:

$$\int_0^1 \ln x\,dx = \lim_{R\to 0^+}\int_R^1 \ln x\,dx = \lim_{R\to 0^+}(x\ln x - x)\Big|_R^1 = -1 - \lim_{R\to 0^+}(R\ln R) + \lim_{R\to 0^+} R$$

Note that $\lim_{R\to 0^+} R = 0$. Next, compute $\lim_{R\to 0^+}(R\ln R)$. Rewrite the expression as:

$$\lim_{R\to 0^+}(R\ln R) = \lim_{R\to 0^+}\frac{\ln R}{1/R}$$

Now apply l'Hopital's Rule:

$$\lim_{R\to 0^+}(R\ln R) = \lim_{R\to 0^+}\frac{\ln R}{1/R} = \lim_{R\to 0^+}\frac{1/R}{-1/R^2} = \lim_{R\to 0^+}(-R) = 0$$

Thus, $\lim_{R\to 0^+}(R\ln R) = 0$. Therefore,

$$\int_0^1 \ln x\,dx = -1,$$

and the integral is convergent to -1.

Example. Determine if $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$ is convergent or divergent.

Note that $\frac{1}{\sqrt{4-x}}$ is discontinuous at the endpoint $x = 4$. Let $u = 4 - x$, then $du = -dx$, giving:

$$\int \frac{dx}{\sqrt{4-x}} = \int -\frac{du}{\sqrt{u}} = \int -u^{-1/2}\,du = -2\sqrt{u} + C = -2\sqrt{4-x} + C$$

Now using the definition of improper integrals for $\int_0^4 \frac{dx}{\sqrt{4-x}}$:

$$\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{R\to 4^-} \left(-2\sqrt{4-x}\right)\Big|_0^R = \lim_{R\to 4^-} \left(-2\sqrt{4-R} + 2\sqrt{4}\right) = 4$$

Therefore, $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$ is convergent to 4.

Example. Determine if $\int_1^2 \frac{dx}{(x-1)^{1/3}}$ is convergent or divergent.
Note that $f(x) = \frac{1}{(x-1)^{1/3}}$ is discontinuous at the endpoint $x = 1$. We first use substitution to find $\int \frac{dx}{(x-1)^{1/3}}$. Let $u = x - 1$. Then $du = dx$, giving:

$$\int \frac{dx}{(x-1)^{1/3}} = \int \frac{du}{u^{1/3}} = \int u^{-1/3}\, du = \frac{3}{2}u^{2/3} + C = \frac{3}{2}(x-1)^{2/3} + C.$$

Now using the definition of the improper integral for $\int_1^2 \frac{dx}{(x-1)^{1/3}}$:

$$\int_1^2 \frac{dx}{(x-1)^{1/3}} = \lim_{R\to 1^+} \left(\frac{3}{2}(x-1)^{2/3}\right)\Big|_R^2 = \lim_{R\to 1^+} \left(\frac{3}{2}(2-1)^{2/3} - \frac{3}{2}(R-1)^{2/3}\right) = \frac{3}{2},$$

and the integral is convergent to $\frac{3}{2}$. Graphically, one might interpret this to mean that the net area under $\frac{1}{(x-1)^{1/3}}$ on $[1,2]$ is $\frac{3}{2}$.

Theorem

For $a > 0$:

- *If* $p > 1$, *then* $\displaystyle\int_a^{\infty} \frac{1}{x^p}\,dx$ *converges.*
- *If* $p \leq 1$, *then* $\displaystyle\int_a^{\infty} \frac{1}{x^p}\,dx$ *diverges.*
- *If* $p < 1$, *then* $\displaystyle\int_0^{a} \frac{1}{x^p}\,dx$ *converges.*
- *If* $p \geq 1$, *then* $\displaystyle\int_0^{a} \frac{1}{x^p}\,dx$ *diverges.*

Example. Determine if the following integrals are convergent or divergent.

$$\int_1^{\infty} \frac{1}{x^3}\,dx, \qquad \int_0^{5} \frac{1}{x^4}\,dx$$

Theorem

Assume that $f(x) \geq g(x) \geq 0$ for $x \geq a$.

- *If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ also converges.*
- *If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ also diverges.*

Example. Show that $\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx$ converges.
We use the Comparison Test to show that it converges. Note that $0 \leq \cos^2(x) \leq 1$ and hence

$$0 \leq \frac{\cos^2(x)}{x^2} \leq \frac{1}{x^2}.$$

Thus, taking $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{\cos^2(x)}{x^2}$, we have $f(x) \geq g(x) \geq 0$. One can easily see that $\int_2^\infty \frac{1}{x^2}\,dx$ converges. Therefore, $\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx$ also converges.