Mathematical analysis 2
Chapter 2: Multiple Integrals
Part : Double Integrals

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Course outline

Generalities

- Let f be a function of two variables x, y defined on $\mathbb{R}^2$,

$$\begin{array}{rccc} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & f(x,y)=z \end{array}$$

- The graph of a function of two variables in the space referred to Cartesian coordinate system $(O,\vec{i},\vec{j},\vec{k})$ is a surface S of equation

$$z=f(x,y)$$

.

- The integral of f on a domain D is a double integral denoted by

$$I=\iint_D f(x,y)dxdy$$

Generalities

- The integral represents the volume between the plane (xOy) of equation

$$z = 0$$

delimited by the domain D and the surface S of equation

$$z = f(x, y)$$

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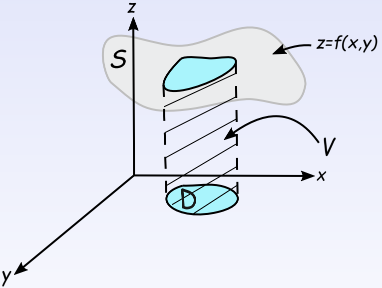

Course outline

Properties of double integrals

Theorem

Let D a closed bounded subset of $\mathbb{R}^2$. If $f : D \to \mathbb{R}$ is continuous, then f is integrable on D.

Properties of double integrals

Theorem

Let f and g be tow integrable functions over a domain D then

- *The sum $f+g$ is integrable and $\forall \alpha, \beta \in \mathbb{R}$*
$$\iint_D (\alpha f(x,y) + \beta g(x,y))dxdy = \alpha \iint_D f(x,y)dxdy + \beta \iint_D g(x,y)dxdy$$

Properties of double integrals

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$$\iint_D (\alpha f(x,y) + \beta g(x,y))dxdy = \alpha \iint_D f(x,y)dxdy + \beta \iint_D g(x,y)dxdy$$
- *If $D = D_1 \cup D_2$ with $D_1 \cap D_2 = \emptyset$ then*
$$\iint_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy$$

Properties of double integrals

Theorem

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- *If $D = D_1 \cup D_2$ with $D_1 \cap D_2 = \emptyset$ then*
$$\iint_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy$$
- *If $\forall (x,y) \in \mathbb{R}^2$ $f(x,y) \leq g(x,y)$ then*
$$\iint_D f(x,y)dxdy \leq \iint_D g(x,y)dxdy$$

Properties of double integrals

Theorem

Let f and g be tow integrable functions over a domain D then

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- *If $D = D_1 \cup D_2$ with $D_1 \cap D_2 = \emptyset$ then*
$$\iint_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy$$
- *If $\forall (x,y) \in \mathbb{R}^2\ f(x,y) \leq g(x,y)$ then*
$$\iint_D f(x,y)dxdy \leq \iint_D g(x,y)dxdy$$
- *we have*
$$\left|\iint_D f(x,y)dxdy\right| \leq \iint_D |f(x,y)|\, dxdy$$

Course outline

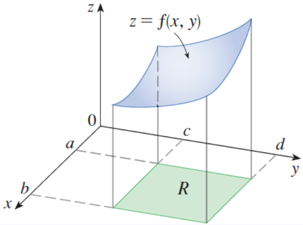
Integrals over rectangular domains

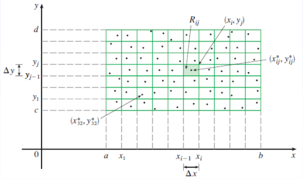
- Let f be a function of tow variables

$$\begin{array}{rccc} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & f(x,y) = z \end{array}$$

- Suppose that f is continuous on the rectangle $R = [a,b] \times [c,d]$ of $\mathbb{R}^2$.

- We subdivide $[a,b]$ into n subintervals and $[c,d]$ into m subintervals then R is subdivided to $n \times m$ sub rectangles $r_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i]$.

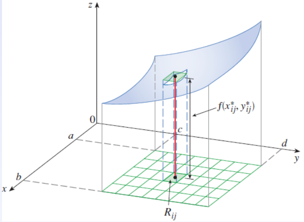




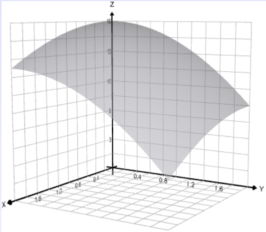
Integrals over rectangular domains

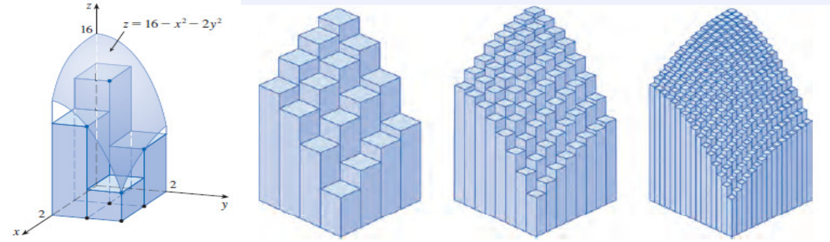
- The integral of f over R is defined by

$$\iint_D f(x,y)dxdy = \lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i,y_j)\Delta x_i \Delta y_j$$



Integrals over rectangular domains





Integrals over rectangular domains

Theorem (Fubini)

Let f be an integrable function over $R = [a,b] \times [c,d]$. then

$$\iint_D f(x,y)dxdy = \int_a^b \left[\int_c^d f(x,y)dy \right] dx = \int_c^d \left[\int_a^b f(x,y)dx \right] dy$$

The notation $\int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$ means that we integrate $f(x,y)$ with respect to y while holding x constant. Similarly, the notation $\int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$ means that we integrate $f(x,y)$ with respect to x while holding y constant.

Integrals over rectangular domains

Example.

Find $\iint_D xy\,dx\,dy$ *on* $D = [0,1] \times [2,3]$

The domain D can be written as follows $D = \left\{(x,y) \in \mathbb{R}^2 :\ 0 \le x \le 1,\ 2 \le y \le 3\right\}$

$$\begin{aligned}
\iint_D xydxdy &= \int_0^1 \left(\int_2^3 xydy\right) dx \\
&= \int_0^1 x \left[\frac{y^2}{2}\right]_2^3 dx \\
&= \frac{5}{2}\int_0^1 xdx \\
&= \frac{5}{2}\left[\frac{x^2}{2}\right]_0^1 = \frac{5}{4}
\end{aligned}$$

$$\begin{aligned}
\iint_D xydxdy &= \int_2^3 \left(\int_0^1 xydx\right) dy \\
&= \int_2^3 y \left[\frac{x^2}{2}\right]_0^1 dy \\
&= \frac{1}{2}\int_2^3 ydy \\
&= \frac{1}{2}\left[\frac{y^2}{2}\right]_2^3 = \frac{5}{4}
\end{aligned}$$

Integrals over rectangular domains

Proposition (Particular case)

If $g : [a,b] \to \mathbb{R}$ *and* $h : [c,d] \to \mathbb{R}$ *are two integrable functions then*

$$\iint_{[a,b]\times[c,d]} g(x)h(y)\,dxdy = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$$

Integrals over rectangular domains

Example.

Calculer l'intégrale $\iint_D e^{x-y}\,dxdy$ *sur* $D = [0,1] \times [1,2]$

$$\begin{aligned}
\iint_D e^{x-y}dxdy &= \iint_D e^{x} \cdot e^{-y} dxdy \\
&= \left(\int_0^1 e^x dx\right)\left(\int_1^2 e^{-y} dy\right) \\
&= [e^x]_0^1 \times [-e^{-y}]_1^2 \\
&= (e^1 - 1)(-e^{-2} + e^{-1})
\end{aligned}$$

Course outline

Integrals over non-rectangular domains

Theorem

Let f be an integrable function over a domain D of $\mathbb{R}^2$.

- *If* $D = \left\{(x,y) \in \mathbb{R}^2 / f_1(x) \leq y \leq f_2(x), a \leq x \leq b\right\}$ *then*

$$\iint_D f(x,y)dxdy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)dy\right] dx$$

(Domain between the graph of two functions and tow vertical lines).

- *If* $D = \left\{(x,y) \in \mathbb{R}^2 / g_1(y) \leq x \leq g_2(y), c \leq y \leq d\right\}$ *then*

$$\iint_D f(x,y)dxdy = \int_c^d \left[\int_{g_1(y)}^{g_2(y)} f(x,y)dx\right] dy.$$

(Domain between the graph of two functions and two horizontal lines).

Integrals over non-rectangular domains

Example.

Find $\iint_D x^2 y dxdy$ *where* D *is the triangle with vertices* $(0,0)$, (1.0), $(0,1)$.

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ et } 0 \le y \le 1-x\}$$

Integrals over non-rectangular domains

We have:

$$\begin{aligned}\int_0^1 \left(\int_0^{1-x} x^2 y\, dy\right) dx &= \int_0^1 x^2 \left[\frac{y^2}{2}\right]_0^{1-x} dx \\ &= \int_0^1 x^2 \frac{(1-x)^2}{2} dx = \int_0^1 \frac{x^4 - 2x^3 + x^2}{2} dx \\ &= \frac{1}{2}\left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]_0^1 = \frac{1}{60}\end{aligned}$$

Remark

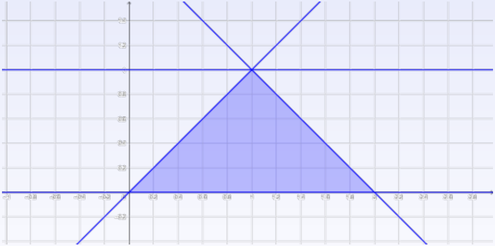
we can interchange the order of integration, and then we have

$$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1 \; et \; 0 \leq x \leq 1 - y\}$$

Integrals over non-rectangular domains

Example.

Find the double integral $I = \iint_D (x+y)dxdy$ *where* D *is the triangle with vertices* $(0,0)$, $(2,0)$, $(1,1)$.



$$D = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 1 \text{ et } y \leq x \leq 2-y\}$$

Integrals over non-rectangular domains

$$\begin{aligned} I &= \int_0^1 \left(\int_y^{2-y} (x+y)dx \right) dy = \int_0^1 \left[\frac{x^2}{2} + yx \right]_y^{2-y} dy \\ &= \int_0^1 (-2y^2+2)dy = \left[2y - \frac{2y^3}{3} \right]_0^1 = \frac{4}{3}. \end{aligned}$$

Remark

We can interpret D as the union of two domains D_1 et D_2 where

$$D_1 = \{(x,y) \in \mathbb{R}^2 : \ 0 \le x \le 1 \ et \ 0 \le y \le x\}$$

$$D_2 = \{(x,y) \in \mathbb{R}^2 : \ 1 \le x \le 2 \ et \ 0 \le y \le 2-x\}$$

Integrals over non-rectangular domains

Example.

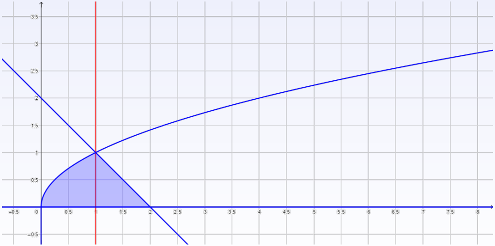
1. *Find the integral* $I = \iint_D y\,dxdy$,
   $D = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 1 \ et \ y^2 \leq x \leq 2 - y\}$.
2. *Represent the domain* D *then interchange the order of integration*

Integrals over non-rectangular domains

1.Calculation of I

$$\begin{aligned} I &= \int_0^1 \int_{y^2}^{2-y} y\,dxdy = \int_0^1 y \int_{y^2}^{2-y} dxdy \\ &= \int_0^1 (2y - y^2 - y^3)dy = \left[y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{5}{12} \end{aligned}$$

2.Representation of the domain D



Integrals over non-rectangular domains

Permutation de l'ordre d'intégration

We have

$$\begin{cases} 0 \le y \le 1 \\ y^2 \le x \le 2-y \end{cases} \implies \begin{cases} x \le 2-y \\ x \ge y^2 \end{cases} \implies \begin{cases} 0 \le y \le 2-x \\ 0 \le y \le \sqrt{x} \end{cases}$$

Then

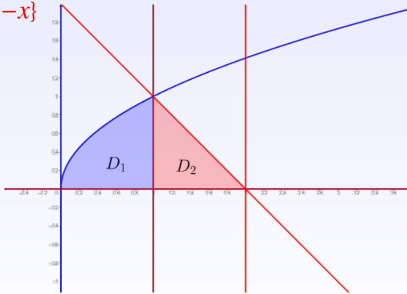
$D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2,\ 0 \le y \le \sqrt{x},\ 0 \le y \le 2-x\}$

We can write the domain D as a union of two domains D_1 and D_2 where

$D_1 = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le \sqrt{x}\}$

$D_2 = \{(x,y) \in \mathbb{R}^2 : 1 \le x \le 2,\ 0 \le y \le 2-x\}$

$D_1 \cap D_2 = \{\text{a line of equation } x = 1\}$



Integrals over non-rectangular domains

We have then

$$\begin{aligned}
\iint_D y\,dxdy &= \iint_{D_1} y\,dxdy + \iint_{D_2} y\,dxdy \\
&= \int_0^1 \int_0^{\sqrt{x}} y\;dydx + \int_0^1 \int_0^{2-x} y\;dydx. \\
&= \frac{5}{12}
\end{aligned}$$

Course outline

Change of variable

Definition

Let $\Omega \subset \mathbb{R}^2$ *and* φ *amapping of class* $C^1(\Omega)$ *such that*

$$\begin{array}{rccl} \varphi: & \Omega & \longrightarrow & \mathbb{R}^2 \\ & (u,v) & \longmapsto & \varphi(u,v) = (x(u,v), y(u,v)) \end{array}$$

• *We call Jacobian matrix of the mapping* φ *the square matrix denoted by* J_φ *and defined by*

$$J_\varphi = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

• *We call Jacobien of the mapping* φ *the determinant of Jacobian matrix and we have*

$$det\ J_\varphi = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{\partial x}{\partial u} \times \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \times \frac{\partial y}{\partial u}$$

Change of variable

Theorem

Let φ be a bijection of class $\mathbb{C}^1$ of Ω to D. Where

$$\Omega = \{(u,v) \in \mathbb{R}^2,\ a \leq u \leq b,\ c \leq v \leq d\} \quad et \quad D = \varphi(\Omega)$$

then

$$\iint_D f(x,y)dxdy = \iint_\Omega f \circ \varphi(u,v)\left|det\ J_\varphi(u,v)\right| dudv$$
$$= \iint_\Omega f(x(u,v),y(u,v))\left|det\ J_\varphi(u,v)\right|\ dudv$$

Remark

A bijective mapping is invertible. That is to say $det\ J_\varphi \neq 0$.

Change of variable

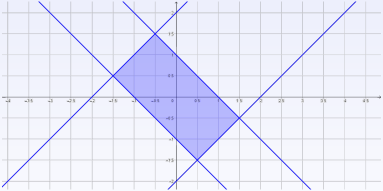
Example.

Find $I = \iint_D (x-1)^2 dxdy$ *on* $D = \{(x,y) : -1 \leq x+y \leq 1, -2 \leq x-y \leq 2\}$

1.**Representation of** D

$$-1 \leq x+y \leq 1 \implies \begin{cases} x+y \leq 1 \implies y \leq 1-x \\ x+y \geq -1 \implies y \geq -1-x. \end{cases}$$

$$-2 \leq x-y \leq 2 \implies \begin{cases} x-y \leq 2 \implies y \geq x-2 \\ x-y \geq -2 \implies y \leq x+2. \end{cases}$$



Change of variable

2- **Change of variable** On pose

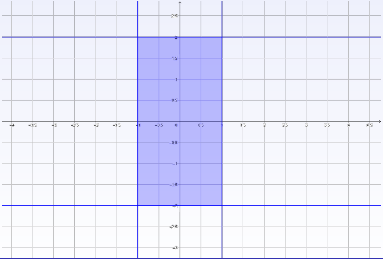
$$\begin{cases} u = x + y \\ v = x - y \end{cases} \Longrightarrow \begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{u-v}{2} \end{cases}$$

We set

$$\begin{array}{rccl} \varphi : & \Omega & \longrightarrow & \mathbb{R}^2 \\ & (u,v) & \longmapsto & \varphi(u,v) = \left(x = \dfrac{u+v}{2}, y = \dfrac{u-v}{2}\right) \end{array}$$

The image of the domain D under the change of variables is

$$\Omega = \{(u,v) \in \mathbb{R}^2 : \; -1 \leq u \leq 1, \; -2 \leq v \leq 2\}.$$



Change of variable

3- **The Jacobian matrix**

$$J_\varphi = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

4- **The Jacobian**

$$det\ J_\varphi = \frac{-1}{2} \Rightarrow |det\ J_\varphi| = \frac{1}{2}$$

5- **Calculation of the integral**

$$\begin{aligned} \iint_D f(x,y)dxdy &= \iint_\Omega f(x(u,v),y(u,v))\left|det\ J_\varphi(u,v)\right|\ du\ dv \\ &= \iint_\Omega \left(\frac{u+v}{2}-1\right)^2\left(\frac{1}{2}\right)dudv \\ &= \frac{1}{8}\int_{-2}^{2}\left[\int_{-1}^{1}(u+v-2)^2du\right]dv = \frac{136}{3} \end{aligned}$$

Course outline

Change of variable in polar coordinates

Let φ be a mapping of class C^1 on $\mathbb{R}^2$ defined by

$$\begin{array}{rccl} \varphi: & \mathbb{R}_+^* \times [0,2\pi] & \longrightarrow & \mathbb{R}^2 \\ & (r,\theta) & \longmapsto & \varphi(r,\theta) = (x(r,\theta), y(r,\theta)) \end{array}$$

with

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

The Jacobian matrix

$$J_\varphi = \begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Change of variable in polar coordinates

The Jacobian

$$det\ J_{\varphi}(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta) = r$$

and we have then

$$I = \iint_D f(x,y)dxdy = \iint_\Omega f(r\cos\theta, r\sin\theta)\, r\, drd\theta,\ \Omega = I_1 \times I_2;\ \ r \in I_1;\ \ \theta \in I_2.$$

Remark

Generally if the domain of integration D is circular then we use the change of variable in polar coordinates.

Change of variable in polar coordinates

Example.

Find the double integral $I = \iint_D \frac{1}{x^2+y^2}dxdy$

Where

$$D = \left\{(x,y) : x \geq 0, y \geq 0, 1 \leq x^2+y^2 \leq 4\right\}$$

Change of variable in polar coordinates

1) Representation of the domain D

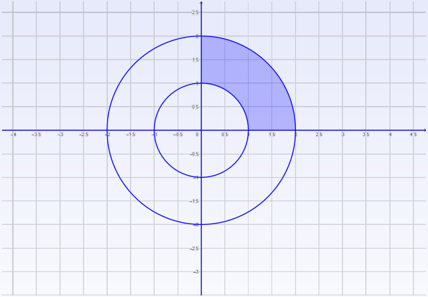
- $x \geq 0$, $y \geq 0$: First quadrant of the plane.
- $1 \leq x^2 + y^2 \leq 4$: Annulus $C(1,2) = D_1 \cap D_2$, with:

D_1 is the disk centered at $(0,0)$ and of radius $r_1 = 2$

$$D_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$$

and D_2 is the exterior of the disk centered at $(0,0)$ and of radius $r_2 = 1$

$$D_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$$



Change of variable in polar coordinates

2- Change of variable

We set

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

We have then

$$1 \le x^2 + y^2 \le 4 \implies 1 \le r^2 \le 4 \implies 1 \le r \le 2.$$

and

$$\begin{cases} x = r\cos\theta \ge 0 \\ y = r\sin\theta \ge 0 \end{cases} \implies 0 \le \theta \le \frac{\pi}{2}.$$

Consequently

$$\Omega = \{(r,\theta) \in \mathbb{R}^2 : 1 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2}\}.$$

Change of variable in polar coordinates

3- **Calculation of the integral** I

$$I = \iint_D \frac{1}{x^2+y^2}dxdy = \iint_\Omega \frac{1}{r^2}\, r\, drd\theta = \int_0^{\frac{\pi}{2}} \int_1^2 \frac{1}{r^2}\, r\, drd\theta = \frac{\pi}{2}\ln 2$$

Applications

a) **Calculation of volume**

The volume between the surface S of equation $z = f(x, y)$ and the domain D situated on plan (XOY) is given by

$$\iint_D f(x, y)dxdy.$$

b) **Calculation of domain area**

When $f(x, y) = 1 \ \forall (x, y) \in D$, this volume measurement corresponds to the area of the domain D and we have then

$$Area(D) = \iint_D dxdy$$

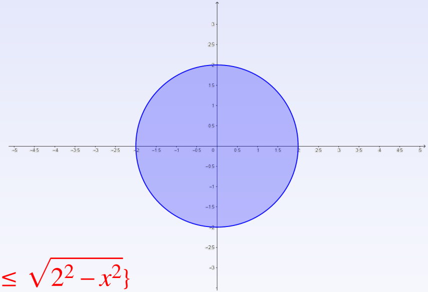
Applications

Example.

Find the aria delimited by the circle of equation $x^2 + y^2 = 4 = 2^2$.

D is the disk of inequality

$$x^2 + y^2 \leq 2^2$$



We know that the area of a circle is given by

$$A = 2^2\pi$$

The domain D is given by

$$D = \{(x, y) \in \mathbb{R}^2 : -2 \leq x \leq 2, -\sqrt{2^2 - x^2} \leq y \leq \sqrt{2^2 - x^2}\}$$

since

$$x^2 + y^2 \leq 2^2 \implies y^2 \leq 2^2 - x^2 \implies |y| \leq \sqrt{2^2 - x^2} \implies -\sqrt{2^2 - x^2} \leq y \leq \sqrt{2^2 - x^2}$$

Applications

By the change of variable in polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies 0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

Consequently

$$\Omega = \{(r,\theta) \in \mathbb{R}^2 : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

then

$$Aire(D) = \iint_{\Omega} r\,drd\theta = \int_0^{2\pi}\int_0^2 r\,drd\theta = 2^2\pi$$