Mathematical analysis 3
Chapter 2 : Integral transformations
Part 1: Fourier transform

The National School of
Artificial Intelligence
المدرسة الوطنية للذكاء الاصطناعي

Course outline

Introduction

- For all $y \in \mathbb{R}$, $e^{iy} = \cos(y) + i\sin(y)$.
- For any function $g : \mathbb{R} \to \mathbb{C}$, we have

$$\lim_{t\to\pm\infty} g(t) = 0 \iff \lim_{t\to\pm\infty} |g(t)| = 0.$$

- For any function $g : \mathbb{R} \to \mathbb{C}$, the integral

$$\int_{-\infty}^{+\infty} g(t)dt$$

converges if and only if the integrals

$$\int_{-\infty}^{+\infty} \mathrm{Re}(g)(t)dt \quad \text{and} \quad \int_{-\infty}^{+\infty} \mathrm{Im}(g)(t)dt$$

converge, and

$$\int_{-\infty}^{+\infty} g(t)dt = \int_{-\infty}^{\infty} \mathrm{Re}(g)(t)dt + i\int_{-\infty}^{+\infty} \mathrm{Im}(g)(t)dt.$$

The space $\mathscr{L}^1$

Definition and notation

We define

$$\mathscr{L}^1(\mathbb{R}) = \left\{ f, f : \mathbb{R} \to \mathbb{C} \text{ , piecewise continuous, } \int_{-\infty}^{\infty} |f(t)| dt < \infty \right\}.$$

$\int_{-\infty}^{+\infty} |f(t)| dt < \infty$ means that $\int_{-\infty}^{+\infty} |f(t)| dt$ converges.

The space $\mathscr{L}^1$

Example. Let $a > 0$ and

$$f(t) = \begin{cases} x^2 & \text{if } t \in [-a, a], \\ 0 & \text{elsewhere.} \end{cases}$$

Since

- f is piecewise continuous on $\mathbb{R}$,
- we have
$$\int_{-\infty}^{+\infty} |f(t)|dt = \int_{-a}^{a} x^2 dt = \frac{2}{3}a^3 < \infty.$$

Thus $f \in \mathscr{L}^1(\mathbb{R})$.

The space $\mathscr{L}^1$

Example. Let $a > 0$ and $f(t) = e^{-a|t|}$.
We have
1) f is continuous on $\mathbb{R}$.
2) We have

$$\int_{-\infty}^{+\infty} |f(t)|dt = \int_{0}^{+\infty} |f(t)|dt + \int_{-\infty}^{0} |f(t)|dt.$$

$$\begin{aligned}\int_{0}^{+\infty} |f(t)|dt &= \int_{0}^{+\infty} e^{-at} dt \\ &= \lim_{R \to +\infty} \left[-\frac{1}{a} e^{-at} \right]_0^R \\ &= \frac{1}{a}.\end{aligned}$$

The space $\mathscr{L}^1$

and

$$\int_{-\infty}^{0} |f(t)|dt = \int_{-\infty}^{0} e^{at} dt$$
$$= \lim_{R \to -\infty} \left[\frac{1}{a} e^{at} \right]_R^0$$
$$= \frac{1}{a},$$

thus

$$\int_{-\infty}^{+\infty} |f(t)|dt = \int_{0}^{+\infty} |f(t)|dt + \int_{-\infty}^{0} |f(t)|dt = \frac{2}{a} < \infty.$$

Therefore, $f \in \mathscr{L}^1(\mathbb{R})$.

Course outline

Fourier transform TF

Definition

Let $f \in \mathscr{L}^1(\mathbb{R})$. The Fourier transform of f, denoted Ff or $\hat{f}$, is given by:

$$Ff(x) = \int_{-\infty}^{+\infty} e^{-ixt} f(t)dt.$$

Remark

Depending on the application domain, we have other equivalent definitions of the Fourier transform with the same properties up to a multiplicative factors:

$$F(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)e^{-ixt}dt$$

or

$$F(f)(x) = \int_{-\infty}^{+\infty} f(t)e^{-i2\pi xt}dt.$$

Fourier transform *TF*

Proposition

Let $f \in \mathscr{L}^1(\mathbb{R})$. The Fourier transform Ff is a well-defined, bounded (i.e., its modulus is bounded), and continuous function.

Proof.
Remember that: If $f \in \mathscr{L}^1(\mathbb{R})$, then:

$$Ff(x) = \int_{-\infty}^{+\infty} e^{-ixt} f(t)dt = \int_{-\infty}^{+\infty} \cos(xt)f(t)dt - i\int_{-\infty}^{+\infty} \sin(xt)f(t)dt.$$

Proposition

Let $f \in \mathscr{L}^1(\mathbb{R})$. Then $\lim\limits_{x\to\pm\infty} F(f)(x) = 0.$

Fourier transform *TF*

Example.

Determine the Fourier transform of the function f defined on $\mathbb{R}$ by

$$f(t) = \begin{cases} 1 & \text{if } |t| < 3, \\ 0 & \text{if } |t| > 3. \end{cases}$$

Answer: First, let's show that $f \in \mathscr{L}^1(\mathbb{R})$. We have:

- f is piecewise continuous on $\mathbb{R}$.
- $\displaystyle\int_{-\infty}^{+\infty} |f(t)|dt = \int_{-3}^{3} 1dt = [t]_{-3}^{3} = 6 < \infty$, which is convergent.

Thus, $f \in \mathscr{L}^1(\mathbb{R})$ and Ff exists.

Fourier transform TF

Calculation of Fourier transform of f: We have

$$Ff(x) = \int_{-\infty}^{+\infty} e^{-ixt} f(t)dt = \int_{-3}^{3} e^{-ixt} dt.$$

Case 1: $x \neq 0$:

$$Ff(x) = \frac{-1}{ix}[e^{-ixt}]_{-3}^{3} = \frac{-1}{ix}(e^{-3ix} - e^{3ix}) = \frac{2}{x}\sin(3x).$$

Case 2: $x = 0$:

$$Ff(0) = 6$$

(also, $Ff(0) = \lim_{x\to 0} Ff(x) = \lim_{x\to 0} \frac{2}{x}\sin(3x) = 6$).

Conclusion:

$$Ff(x) = \begin{cases} \frac{2}{x}\sin(3x) & \text{if } x \neq 0, \\ 6 & \text{if } x = 0. \end{cases}$$

Fourier transform *TF*

Example.

Let $f(t) = e^{-|t|}$, $\forall t \in \mathbb{R}$. Determine the Fourier transform of f.

Answer. We have

$$\begin{aligned}
F(f)(x) &= \int_{-\infty}^{+\infty} f(t)e^{-ixt}dt \\
&= \int_{-\infty}^{0} e^{-(ix-1)t}dt + \int_{0}^{+\infty} e^{-(ix+1)t}dt \\
&= \left[\frac{-1}{ix-1}e^{-(ix-1)t}\right]_{t\to-\infty}^{0} + \left[\frac{-1}{ix+1}e^{-(ix+1)t}\right]_{0}^{t\to+\infty} \\
&= -\frac{1}{ix-1} + \frac{1}{ix+1}.
\end{aligned}$$

Therefore, $F(f)(x) = \frac{2}{x^2+1}$.

Fourier transform *TF*

Proposition

Let $f \in \mathscr{L}^1(\mathbb{R})$.

- *If* f *is even, then* Ff *is even and we have*

$$Ff(x) = 2\int_0^{+\infty} \cos(xt)f(t)dt.$$

- *If* f *is odd, then* Ff *is also odd and we have*

$$Ff(x) = -2i\int_0^{+\infty} \sin(xt)f(t)dt$$

Course outline

Fourier inversion theorem

Theorem

If $f \in \mathscr{L}^1(\mathbb{R})$ and f is differentiable from the left and right at a point t, then

$$\lim_{R\to+\infty} \frac{1}{2\pi} \int_{-R}^{R} e^{ixt} F(f)(x)dx = \frac{1}{2}(f(t^+) + f(t^-)).$$

Moreover, if $\displaystyle\int_{-\infty}^{+\infty} e^{ixt} F(f)(x)dx$ *converges, then*

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} F(f) e^{ixt} dx = \frac{1}{2}(f(t^+) + f(t^-)).$$

Fourier inversion theorem

Remark

We have

$$\int_{-R}^{R} g(x)dx \text{ converges } \Rightarrow \lim_{R\to+\infty}\int_{-R}^{R} g(x)dx = \int_{-\infty}^{\infty} g(x)dx.$$

However, the converse is false. Counter-example:

$$\lim_{R\to+\infty}\int_{-R}^{R} xdx = 0 \;\; and \;\; \int_{-\infty}^{+\infty} xdx \;\; does\, not\, exist.$$

Fourier inversion theorem

Example.

Apply the Fourier inversion theorem to the function $f(t) = e^{-|t|}$ *to find the value of the integral*

$$\int_0^{+\infty} \frac{1}{x^2+1} \cos(xt)dx.$$

We have $F(f) = \frac{2}{x^2+1}$ and $|e^{ixt}F(f)(x)| = \left|\frac{2}{x^2+1}\right| = \frac{2}{x^2+1}$, and

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1} dx$$

converges. Therefore,

$$\int_{-\infty}^{+\infty} e^{ixt} F(f)(x) dx$$

converges.

Fourier inversion theorem

Thus, by the Fourier inversion theorem,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{x^2+1}e^{ixt}dx=\frac{1}{2}(f(t^+)+f(t^-)),$$

where f is continuous on $\mathbb{R}$,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{2}{x^2+1}e^{ixt}dx=e^{-|t|}.$$

However,

$$\int_{-\infty}^{+\infty}\frac{1}{x^2+1}e^{ixt}dx=\int_{-\infty}^{+\infty}\frac{1}{x^2+1}\cos(xt)dx+i\int_{-\infty}^{+\infty}\frac{1}{x^2+1}\sin(xt)dx,$$

so (by parity),

$$\int_{0}^{+\infty}\frac{1}{x^2+1}\cos(xt)dx=\frac{\pi}{2}e^{-|t|}.$$

Fourier inversion theorem

Definition

The inverse Fourier transform of a function F is a function given for all t by

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(x)e^{ixt}dx.$$

Fourier inversion theorem

Proposition

Let f and g be two functions in $\mathscr{L}^1(\mathbb{R})$ and C^1 piecewise on $\mathbb{R}$. If $F(f)(x) = F(g)(x)$ for all $x \in \mathbb{R}$, then $f(t) = g(t)$ at every point t where f and g are continuous.

Fourier inversion theorem

Proof: Let $f, g \in \mathscr{L}^1(\mathbb{R})$ and f be differentiable from the left and right at a point t. Then

$$\lim_{R\to+\infty} \frac{1}{2\pi} \int_{-R}^{R} e^{ixt} F(f)(x)dx = \frac{1}{2}(f(t^+) + f(t^-))$$

and

$$\lim_{R\to+\infty} \frac{1}{2\pi} \int_{-R}^{R} e^{ixt} F(g)(x)dx = \frac{1}{2}(g(t^+) + g(t^-)).$$

If $F(f)(x) = F(g)(x)$ for all $x \in \mathbb{R}$, then

$$\frac{1}{2}(f(t^+) + f(t^-)) = \frac{1}{2}(g(t^+) + g(t^-)).$$

Thus, if f and g are continuous at a point $t \in \mathbb{R}$, we have $f(t) = g(t)$.

Course outline

Properties of Fourier Transform

Proposition (Linearity)

Let f, g be two functions of $\mathscr{L}^1(\mathbb{R})$. Then, for all $\alpha, \beta \in \mathbb{C}$, we have

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g).$$

Proposition (The Fourier Transform of the Translated Function)

Let f be a function in $L^1(\mathbb{R})$. Then, for any real number α, we have

$$\mathscr{F}(f(t+\alpha)) = e^{i\alpha x}\mathscr{F}(f).$$

Properties of Fourier Transform

Proposition (Translation of the Fourier Transform)

Let f be a function in $\mathscr{L}^1(\mathbb{R})$. Then, for any $\alpha \in \mathbb{R}$, we have

$$F(e^{i\alpha t}f(t))(x) = F(f)(x-\alpha).$$

Proposition (The Scaling Property)

Let f be a function in $\mathscr{L}^1(\mathbb{R})$. Then, for any $\alpha \in \mathbb{R}^*$, we have

$$F(f(\alpha t)) = \frac{1}{|\alpha|}F(f)\left(\frac{x}{\alpha}\right).$$

Properties of Fourier Transform

Proposition (Derivative of the Fourier Transform)

Let $f \in \mathscr{L}^1(\mathbb{R})$ *such that* $tf(t) \in \mathscr{L}^1(\mathbb{R})$. *Then,* $F(f)$ *is differentiable and* $(F(f)(x))' = -iF(tf(t))(x)$.

Proposition (The Fourier Transform of the Derivative)

Let f *be a real function differentiable such that* $f, f' \in \mathscr{L}^1(\mathbb{R})$. *Then,* $F(f')(x) = ixf(x)$.

Proposition (The Fourier Transform of the nth Derivative)

Let f *be a real function such that its first* $(n-1)$ *derivatives exist, are continuous, and belong to* $\mathscr{L}^1(\mathbb{R})$, *and* $f^{(n)}$ *exists and belongs to* $\mathscr{L}^1(\mathbb{R})$. *Then,* $F(f^{(n)})(x) = (ix)^n F(f)(x)$.

Parseval's Formula

Proposition

Let $f \in \mathscr{L}^1(\mathbb{R})$ *be a piecewise* C^1 *function such that*

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty.$$

Then,

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(f)(x)|^2 \, dx.$$

Parseval's Formula

Example. Let's consider the function $f(t) = e^{-|t|}$. Using Parseval's formula, calculate

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx$$

Answer. We have that f is C^1 piecewise and

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt$$

converges, because

$$\int_0^{\infty} |f(t)|^2\, dt = \int_0^{\infty} e^{-2t}\, dt = \frac{1}{2}$$

Thus,

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = 2\int_0^{\infty} |f(t)|^2\, dt = 1 < \infty$$

Parseval's Formula

We deduce from Parseval's theorem that

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(f)(x)|^2\, dx$$

But $F(f)(x) = \frac{2}{x^2+1}$, so

$$1 = \int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{(x^2+1)^2}\, dx$$

Hence

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx = \frac{\pi}{2}$$

Convolution

The convolution product of two integrable functions on $\mathbb{R}$ is another function defined as follows:

Definition

The convolution product of two functions f and g is defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(u)g(t-u)\,du.$$

In the integral, both functions are traversed in opposite directions to each other.

Convolution

- The convolution product is **commutative**, i.e., $f * g = g * f$.
- The convolution product is **associative**, i.e., $(f * g) * h = f * (g * h)$.
- The convolution product is **bilinear**, for any scalar α we have

$$f * (g + \alpha h) = (f * g) + \alpha (f * h).$$

- The convolution product is **invariant by translation**, i.e.,

$$\forall \alpha \in \mathbb{R}, (f * g)(t - \alpha) = (f * g)(t).$$

Convolution

Example. Let $f(t)$ be defined as follows:

$$f(t) = \begin{cases} 1 & \text{if } t \in [-1, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $f * f$.
Answer. We have

$$(f * f)(t) = \int_{-\infty}^{\infty} f(u) f(t-u)\, du = \int_{-1}^{1} f(t-u)\, du \overset{s=t-u}{=} \int_{t+1}^{t-1} f(s)\, ds.$$

Convolution

- **Case 1:** If $t \leq -2$ or $t \geq 2$, then $(f * f)(t) = \int_{t-1}^{t+1} f(s)\, ds = 0$ since $(t \leq -2 \Rightarrow t-1 < t+1 \leq -1)$, $(2 \leq t \Rightarrow 1 - 1 < t+1)$.
- **Case 2:** If $-2 < t \leq 0 \iff -3 < t-1 \leq -1 < t+1 < 1$, then $(f * f)(t) = \int_{-1}^{t+1} f(s)\, ds = \int_{-1}^{t+1} ds = [s]_{t+1}^{-1} = t+2.$
- **Case 3:** If $0 < t < 2 \iff -1 < t-1 < 1 < t+1 < 3$, then $(f * f)(t) = \int_{t-1}^{t+1} f(s)\, ds = \int_{t-1}^{1} f(s)\, ds = \int_{t-1}^{1} ds = [s]_{1}^{t-1} = -t+2.$

In summary:

$$(f * f)(t) = \begin{cases} -|t| + 2 & \text{if } t \in]-2, 2[, \\ 0 & \text{otherwise.} \end{cases}$$

Convolution Formula

Proposition

Let f, g *be two functions in* $\mathscr{L}^1(\mathbb{R})$. *Then,* $(f * g) \in \mathscr{L}^1(\mathbb{R})$,

$$F(f * g) = F(f) \cdot F(g).$$

Convolution Formula

Example.

Let $g(t)$ be the function defined as

$$g(t) = \begin{cases} -|t| + 2 & \text{if } t \in]-2, 2[, \\ 0 & \text{otherwise}. \end{cases}$$

Calculate $F(g)$.

Answer. We have seen in the previous example that $g = f * f$ with

$$f(t) = \begin{cases} 1 & \text{if } t \in [-1, 1], \\ 0 & \text{otherwise}. \end{cases}$$

Thus,

$$F(g) = F(f * f) = F(f) \cdot F(f).$$

Convolution Formula

But,

$$F(f)(x) = \begin{cases} \dfrac{2\sin(x)}{x} & \text{if } x \neq 0, \\ 2 & \text{if } x = 0, \end{cases}$$

which implies that

$$F(g)(x) = \begin{cases} \dfrac{4\sin^2(x)}{x^2} & \text{if } x \neq 0, \\ 4 & \text{if } x = 0. \end{cases}$$

Course outline

Integral equations (Convolution equations)

Definition

We call a convolution equation an equation of the form $f * y = g$ where f and g are given functions and y is an unknown function.

Remark

The resolution of a convolution equation is done as follows (under favorable conditions):
$f * y = g \Rightarrow F(f * y) = F(g) \Rightarrow (Ff) \cdot (Fy) = Fg \Rightarrow Fy = \frac{Fg}{Ff}$.
Then it will be necessary to apply the inverse Fourier transform theorem to obtain y.

Integral equations (Convolution equations)

Example.

1) Calculate the Fourier transform of $e^{-|t|}$.
2) Use the inverse Fourier transform to calculate the value of

$$\int_0^1 \frac{\cos(tx)}{1+x^2}\,dx.$$

3) Solve the integral equation:

$$\int_{-1}^1 \frac{y(u)}{(t-u)^2+1}\,du = \frac{1}{t^2+4}$$

where $y \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$.

Integral equations (Convolution equations)

Answer.
1) Let $f(t) = e^{-|t|}$. We have $f \in L^1(\mathbb{R})$, since f is continuous on $\mathbb{R}$ (as it is composed of continuous functions).

$$\int_{-1}^{1} |f(t)|\, dt = \int_{-1}^{1} |e^{-t}|\, dt = 2$$

since $|f|$ is even.

$$\int_{0}^{1} e^{-t}\, dt$$

which converges (exponential integral, reference). So, Ff exists and we have:

$$Ff(x) = \int_{-1}^{1} e^{-ixt} f(t)\, dt = \int_{-1}^{1} e^{-ixt} \cdot e^{-|t|}\, dt$$

since f is even.

$$= 2\int_{0}^{1} \cos(xt) \cdot e^{-t}\, dt.$$

Integral equations (Convolution equations)

Using an integration by parts we obtain

$$Ff(x) = \frac{2}{1+x^2}$$

2) Since f is differentiable on $\mathbb{R} \setminus \{0\}$ (and continuous on $\mathbb{R}$) and has a derivative from the right and from the left at 0, we can apply the IFT. Furthermore, f is even, so:

$$f(a) = e^{-|a|} = \frac{1}{\pi}\int_0^{+\infty} \cos(ax)Ff(x)\,dx = \frac{2}{\pi}\int_0^{+\infty} \frac{\cos(ax)}{1+x^2}\,dx \quad \forall a \in \mathbb{R}.$$

Setting $a \to t$, we get:

$$\int_0^1 \frac{\cos(tx)}{1+x^2}\,dx = \frac{2}{\pi}e^{-|t|} \quad \forall t \in \mathbb{R}.$$

Integral equations (Convolution equations)

3) Let's define: $g(t) = \frac{1}{t^2+1}$, $h(t) = \frac{1}{t^2+4}$, and the given integral equation is actually the convolution product: $y * g = h$.
Let's calculate Fy, we have $y \in L^1(\mathbb{R})$ by assumption.
First, let's verify that g and h are in $L^1(\mathbb{R})$ in order to calculate their Fourier transforms. Indeed, g and h are continuous on $\mathbb{R}$ (as ratios of polynomials), furthermore:

$$\int_{-1}^{+\infty} |g(t)|\,dt = \int_{0}^{+\infty} \frac{1}{t^2+1}\,dt$$

which converges by equivalence and the Riemann criterion ($\frac{1}{t^2+1} \sim \frac{1}{t^2}$ for $t \to \pm\infty$), and the same holds for

$$\int_{-1}^{+\infty} |h(t)|\,dt = \int_{0}^{+\infty} \frac{1}{t^2+4}\,dt.$$

Thus, Fg and Fh exist.

Integral equations (Convolution equations)

$$Fg(x) \overset{\text{since g is even}}{=} 2 \cdot \int_0^{+\infty} \frac{\cos(xt)}{t^2+1}\,dt = \pi e^{-|x|} \quad \forall x \in \mathbb{R}.$$

$$Fh(x) = 2 \cdot \int_0^{+\infty} \frac{\cos(xt)}{t^2+4}\,dt$$

Letting $u = t/2$, we have:

$$Fh(x) = \int_0^{1} \frac{\cos(2xu)}{u^2+1}\,du$$

Then

$$= \frac{\pi}{2} e^{-2|x|} \quad \forall x \in \mathbb{R}.$$

Integral equations (Convolution equations)

So we obtain $Fy(x) = \dfrac{Fh(x)}{Fg(x)} = \dfrac{e^{-|x|}}{2} \quad \forall x \in \mathbb{R}.$

Let's find y using the inverse Fourier transform (TIF), we have $y \in C^1(\mathbb{R})$:

$$y(a) = \frac{1}{2\pi} \lim_{A\to+\infty} \int_{-A}^{A} e^{iax} \cdot Fy(x)\, dx$$

Since Fy is even: $y(a) = \dfrac{1}{\pi} \displaystyle\int_0^{\infty} \cos(ax) \frac{e^{-x}}{2}\, dx$

$$y(a) = \frac{1}{4\pi}\left(2\int_0^{+\infty} \cos(ax) e^{-x}\, dx\right) = \frac{1}{4\pi} \cdot \frac{1}{1+a^2}$$

Conclusion:

$$y(t) = \frac{1}{4\pi} \cdot \frac{1}{1+a^2} \quad \text{for all } t \in \mathbb{R}.$$

Course outline

Application of *TF* for resolution of ODE

The problem: Let $g(t) = e^{-|t|}$. We seek a function f twice differentiable with $f, f', f'' \in L^1(\mathbb{R})$ such that

$$f'' - f = g(t), \quad \forall t \in \mathbb{R}.$$

Solution: Since $F(f'')(x) = -x^2 F(f)(x)$, applying the Fourier transform to this ODE gives

$$F(f)(x) = -\frac{1}{x^2+1} F(g)(x).$$

But as seen in a previous example, $F(g)(x) = \frac{2}{x^2+1}$, so

$$F(f)(x) = -\frac{1}{2} F(g)(x) F(g)(x) = F\left(-\frac{1}{2}(g * g)\right)(x).$$

Thus, by the Fourier inversion theorem (justify!), we obtain

$$f(t) = -\frac{1}{2}(g * g)(t) = \int_{-\infty}^{\infty} e^{-|u|-|t-u|} du.$$

Application of *TF* for resolution of ODE

The problem: We seek a function $f \in L^1(\mathbb{R})$ such that

$$\forall t \in \mathbb{R}, \int_{-\infty}^{\infty} f(u)f(t-u)du = f(t).$$

Solution: The previous equation can be written as $f * f = f$. Applying the Fourier transform to both sides, we get

$$\forall x \in \mathbb{R}, F(f)(x)F(f)(x) = F(f)(x).$$

Thus,

$$\forall x \in \mathbb{R}, F(f)(x)(F(f)(x) - 1) = 0 \Rightarrow \forall x \in \mathbb{R}, (F(f)(x) = 0) \vee (F(f)(x) = 1).$$

Application of *TF* for resolution of ODE

However, since $F(f)$ is continuous on $\mathbb{R}$, we have two possibilities:

- Either $\forall x \in \mathbb{R}, F(f)(x) = 0$.
- Or $\forall x \in \mathbb{R}, F(f)(x) = 1$, which contradicts the fact that $\lim_{x \to \pm\infty} F(f)(x) = 0$.

We deduce that $\forall x \in \mathbb{R}, F(f)(x) = 0$, so $f(t) = 0$ almost everywhere!