Theory of Computing:

8. Context-Free Languages + Revision

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Outline :

- Chomsky Classification of Languages
- Pumping Lemma
  - Explanation
  - Examples
- Closure Properties
- Decisionable Problems
- Revision : CFG / NFA-RegEx/ Prove CFG
- Software & Tools

Chomsky Classification of Languages

- Who is Noam Chomsky
  - American Linguist
  - Political Activist.
  - Professor Emeritus at MIT
  - Published more than 150 books on topics such as linguistics, war, politics, and mass media

Wikipedia

Chomsky Classification of Languages

- We have moved to study Context-Free Languages and Grammar

Type	Grammar	Language	Automaton
Type-3	Regular Grammar	Regular Languages	DFA/NFA
Type-2	Context-Free Grammar	Context Free Languages	PDA

Chomsky Classification of Languages

- What about Context-sensitive languages ?

Type	Grammar	Language	Automaton
Type-3	Regular Grammar	Regular Languages	DFA/NFA
Type-2	Context-Free Grammar	Context Free Languages	PDA
Type-1	Context-Sensitive Grammar	Context-Sensitive Languages	Linear-bounded automaton

Chomsky Classification of Languages

- We have started with Regular languages

Type	Grammar	Language	Automaton
Type-3	Regular Grammar	Regular Languages	DFA/NFA

Chomsky Classification of Languages

- What about Context-sensitive languages ?
  - Example of Grammars
    - $S \rightarrow abc \mid aAbc$
    - $Ab \rightarrow bA$
    - $Ac \rightarrow Bbcc$
    - $bB \rightarrow Bb$
    - $aB \rightarrow aa \mid aaA$

What language is for this grammar ?

Chomsky Classification of Languages

- What about Context-sensitive languages ?
  - Example of Grammars
    - $S \rightarrow abc \mid aAbc$
    - $Ab \rightarrow bA$
    - $Ac \rightarrow Bbcc$
    - $bB \rightarrow Bb$
    - $aB \rightarrow aa \mid aaA$

$S \rightarrow aAbc$

$\rightarrow abAc$

$\rightarrow abBbcc$

$\rightarrow aBbbcc$

$\rightarrow aaAbbcc$

$\rightarrow aabAbcc$

$\rightarrow aabbAcc$

$\rightarrow aabbBbccc$

$\rightarrow aabBbbccc$

$\rightarrow aaBbbbccc$

$\rightarrow$ **aaa**bbb**ccc**

Chomsky Classification of Languages

- Lastly, Turing Machine will be explained next week

Type	Grammar	Language	Automaton
Type-3	Regular Grammar	Regular Languages	DFA/NFA
Type-2	Context-Free Grammar	Context Free Languages	PDA
Type-1	Context-Sensitive Grammar	Context-Sensitive Languages	Linear-bounded automaton
Type-0	Unrestricted grammar	Recursively enumerable language	Turing Machine

Chomsky Classification of Languages

- Context-free languages are those that can be **generated** by

  context-free grammar.
- Example : Language L = { 0^n1^n | n >= 0 }

Pumping Lemma for Context Free Grammar

- Context-free languages are those that can be **generated** by context-free grammar.
- Example : Language L = { 0^n1^n | n >= 0 }

Pumping Lemma for Context Free Grammar

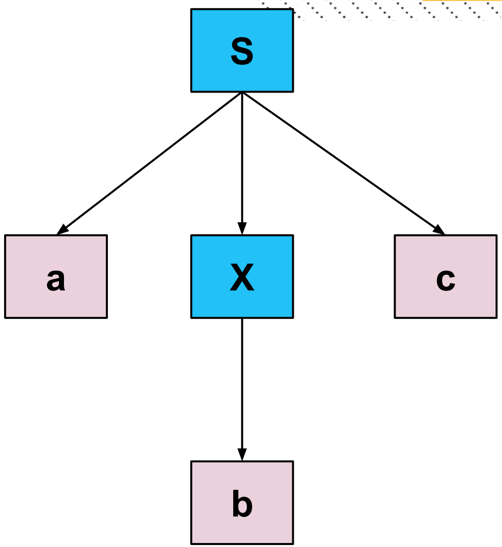
- Context-free languages are those that can be **generated** by context-free grammar.
- What about : Language L = { $a^nb^nc^n$ | n >= 0 } ?

Pumping Lemma for Context Free Grammar

- Given the following grammar:
  - $S \rightarrow aX$
  - $X \rightarrow bc$
- Let's derive the string **abc**

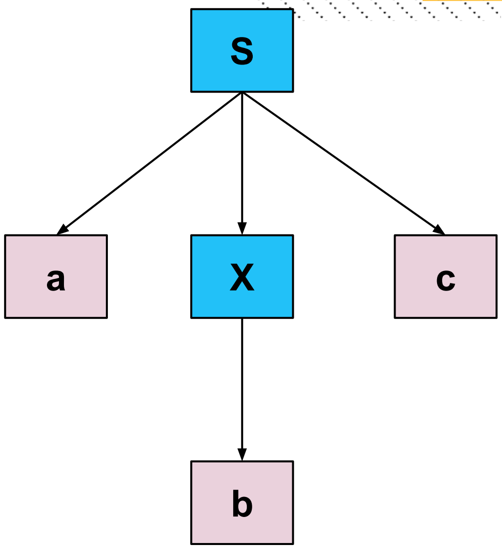
Pumping Lemma for Context Free Grammar

- Given the following grammar:
  - S → aXc
  - X → b
- Let's derive the string **abc**



Pumping Lemma for Context Free Grammar

- Given the following grammar:
  - $S \rightarrow aXc$
  - $X \rightarrow b$
- **Can we generate:**
  - More or infinite number of words ?
  - **Longer** words ?



Pumping Lemma fo
Context Free Gramm

We can only Generate a single word

- Given the following grammar:
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  - X → b
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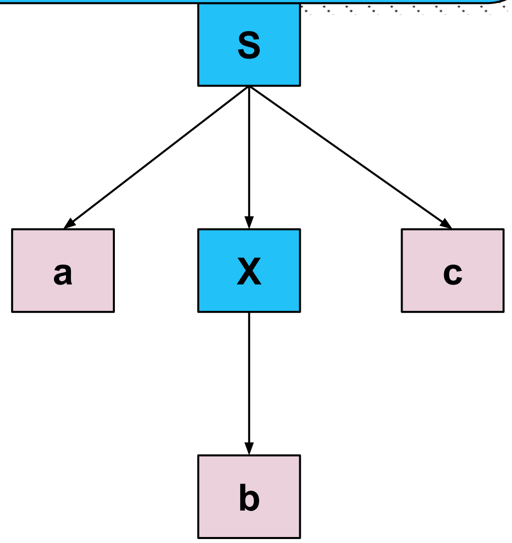
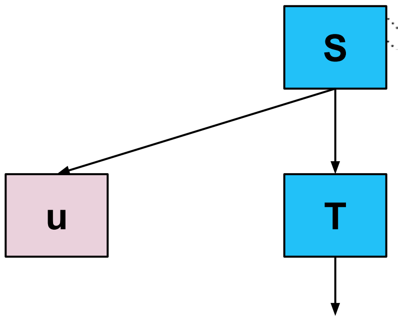
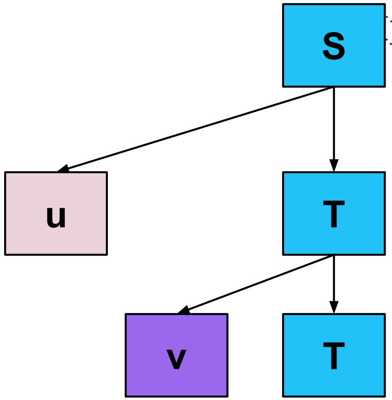


Pumping Lemma for Context Free Grammar

- Let's consider the grammar :
  - S → uT | ε
  - T → vT | x | ε
- **Can we generate:**
  - ? ??



Pumping Lemma for Context Free Grammar



- Let's consider the grammar :
  - $S \rightarrow uT \mid \varepsilon$
  - $T \rightarrow vT \mid x \mid \varepsilon$
- **Can we generate:**
  - u**vvv**x , u**vvvvvvvv**x, u**vvvvvvv...vv**x

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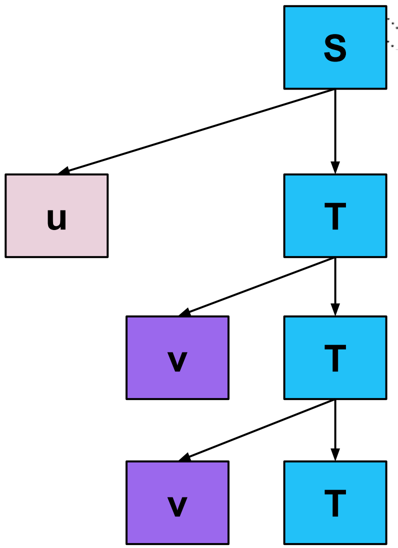

Pumping Lemma for Context Free Grammar

- Let's consider the grammar :
  - S → uT | ε
  - T → T**y** | x | ε
- **Can we generate:**
  - ux**yy** , ux**yyyyyyy**, ux**yyyyyyy…yy**

Pumping Lemma for Context Free Grammar

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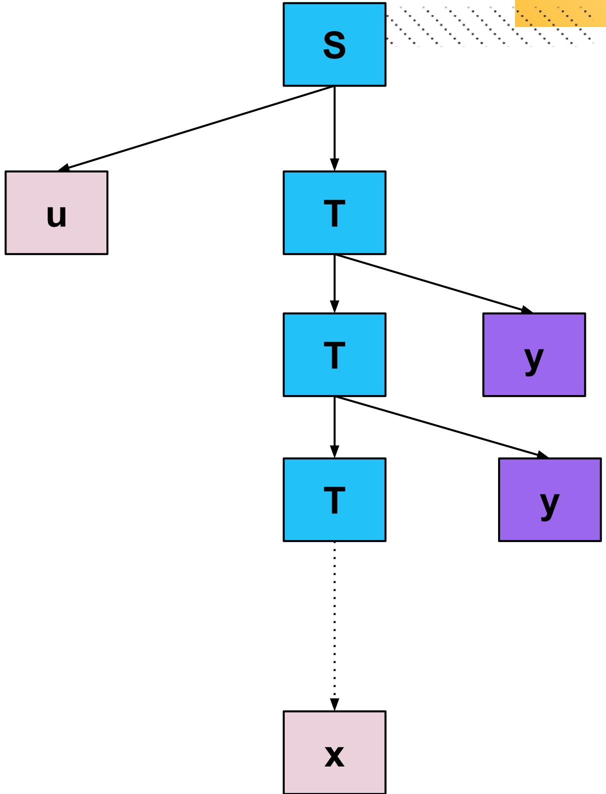
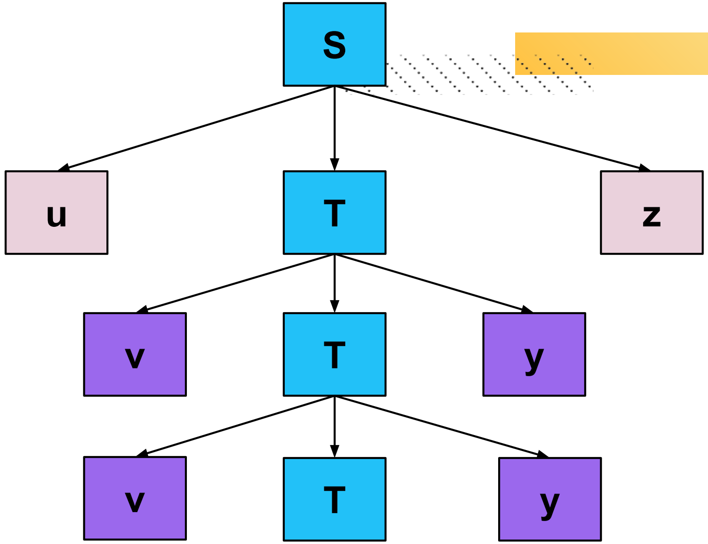


Pumping Lemma for Context Free Grammar

- Let's consider the grammar :
  - S → uTz | ε
  - T → **vTy** | x | ε
- **Can we generate:**
  - u**vvv**x**yyy**z , u**vvvvvvv**x**yyyyyyz**, u**vvvvvvv…vv**x**yyyyyyy…yy**z

Pumping Lemma for Context Free Grammar



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- **Can we generate:**
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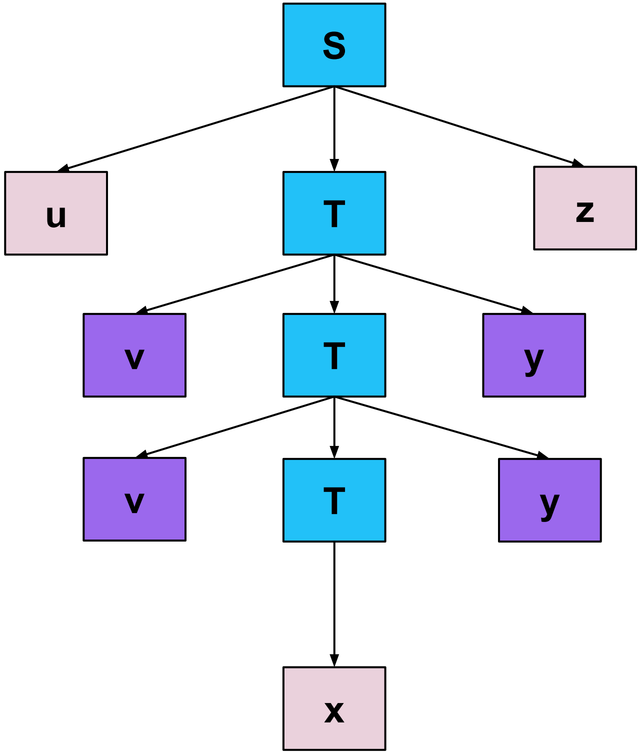
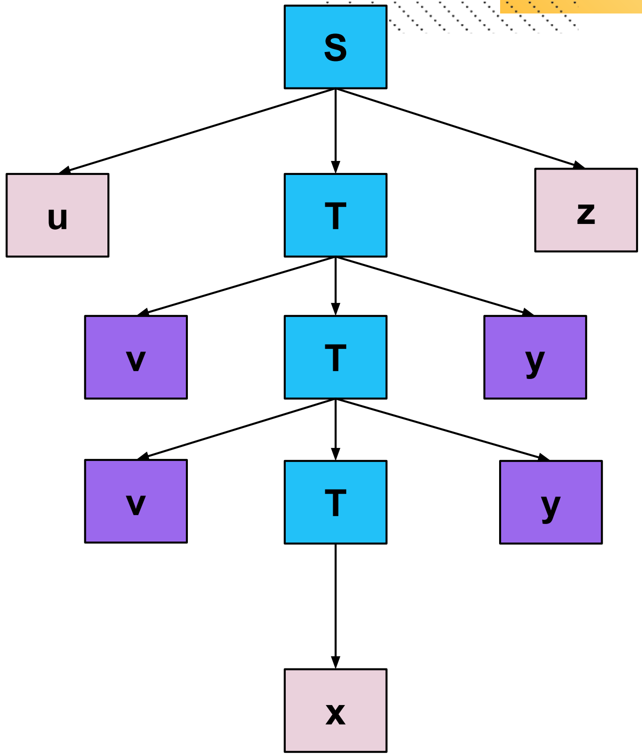
Pumping Lemma for Context Free Grammar



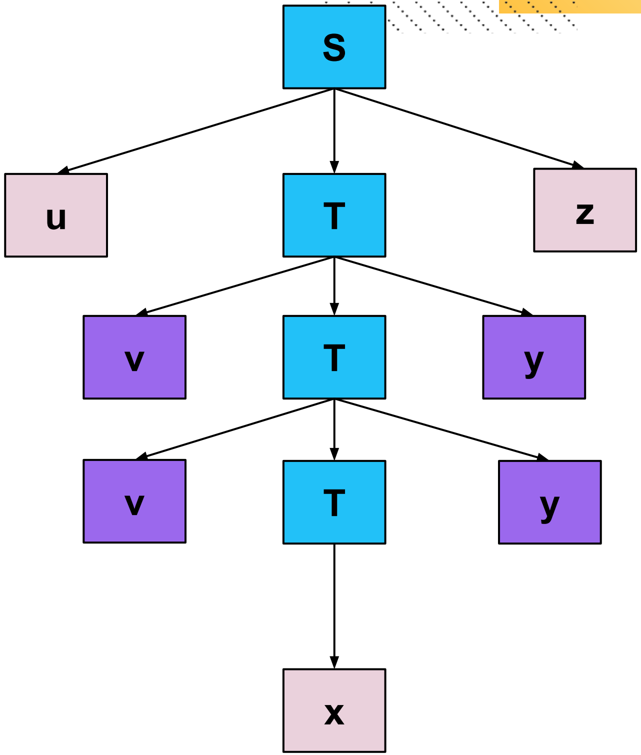
Pumping Lemma for Context Free Grammar

- For a given string belonging to the language which is infinite, it must be in the form : uvxyz
- Such that :



Pumping Lemma for Context Free Grammar

- For a given string belonging to the language which is infinite, it must be in the form : uvxyz
- Such that : **pumping can happen for v or y or both**



Pumping Lemma for Context Free Grammar

Pumping lemma for context-free languages If A is a context-free language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ satisfying the conditions

1. for each $i \geq 0$, $uv^ixy^iz \in A$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

Pumping Lemma for Context Free Grammar

- What if |vy|= 0 ?
  - $S \rightarrow uTz \mid \varepsilon$
  - $T \rightarrow \mathbf{vTy} \mid x \mid \varepsilon$
- Would be :
  - $S \rightarrow uTz \mid \varepsilon$
  - $T \rightarrow T \mid x \mid \varepsilon$

Pumping Lemma for Context Free Grammar

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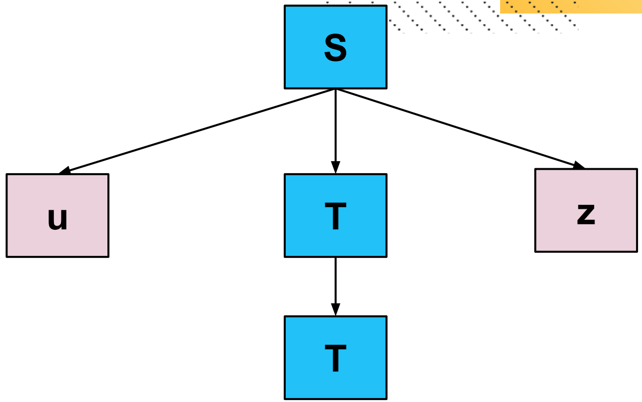

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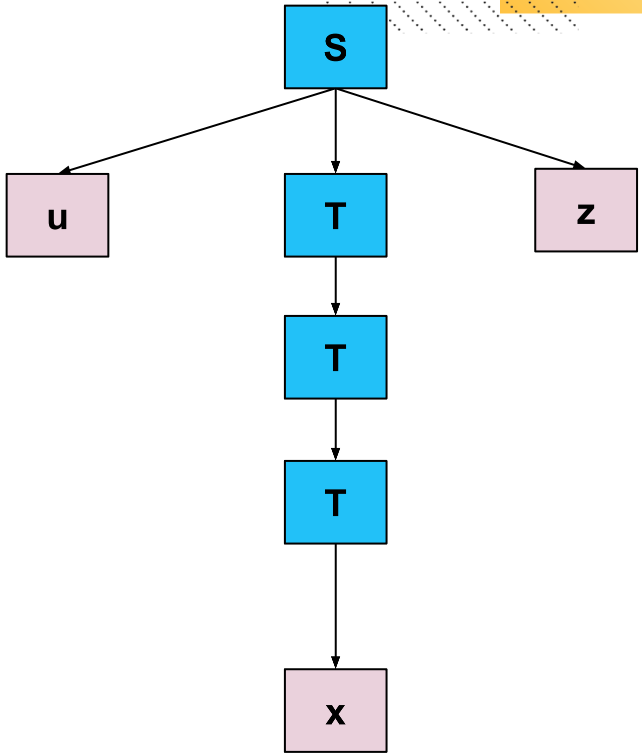
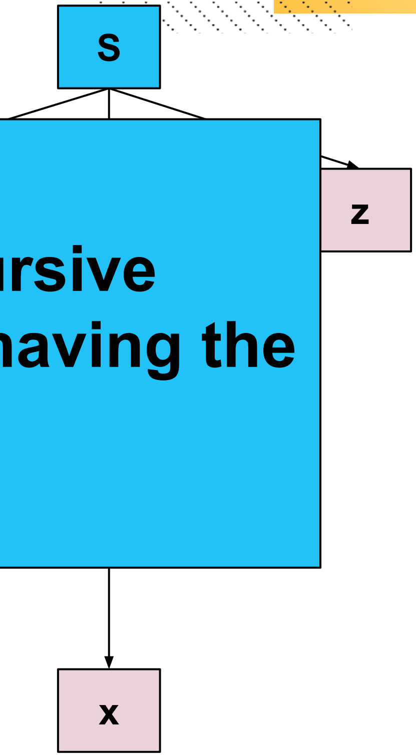


Pumping Lemma for Context Free Grammar

- W
  -
  -
- W
  - S → uTz | ε
  - T → T | x | ε

No matter you how many recursive substitutions you do, you end up having the same word : uxz



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There is another technique called Ogden's lemma

Pumping Lemma for Context Free Grammar

- If you cannot for a given language :
  - Create the Context-Free Grammar
  - Create a Pushdown automaton
- You may use the pumping lemma to prove it is not a context-free language

Pumping Lemma for Context Free Grammar

- To use the pumping lemma via Proof by Contradiction:
  - Assume the language is a context free language
  - Assume the pumping length ***P***
  - Think of a string which is part of the language , such that if you put it in the form : u**v**x**y**z ,
    - Regardless how you choose **v** and **y** from the string , pumping them, would lead to generate strings not belonging to the language

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language $B = \{a^n b^n c^n \mid n \geq 0\}$ is not context free

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language B = $\{a^n b^n c^n \mid n \geq 0\}$ is not context free
    - We assume it is context-free.
    - We assume the pumping length **P**
    - Let's try this string : $a^P b^P c^P$

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language $B = \{a^n b^n c^n \mid n \geq 0\}$ is not context free
    - String : $a^P b^P c^P$ ⇒ u**v**x**y**z such that : **|vxy| <= P**
    - Let's try all possible combinations for placing the **vxy** on our string $a^P b^P c^P$ **and see if we can pump**

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language $B = \{a^n\ b^n\ c^n \mid n \geq 0\}$ is not context free

$$\overbrace{\underbrace{a}_{u}\underbrace{aaaa \ldots aaa}_{vxy}}^{a^p}\ \underbrace{\overbrace{bbbb \ldots bbb}^{b^p}\overbrace{cccc \ldots ccc}^{c^p}}_{z}$$

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language $B = \{a^n b^n c^n \mid n \geq 0\}$ is not context free

$$\underbrace{\epsilon}_{u}\quad \underbrace{\overbrace{aaaaa \ldots aaa}^{a^p}}_{vxy}\ \underbrace{\overbrace{bbbb \ldots bbb}^{b^p}\overbrace{cccc \ldots ccc}^{c^p}}_{z}$$

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language B = $\{a^n b^n c^n \mid n \geq 0\}$ is not context free
  - String : $a^P b^P c^P$ ⇒ u**v**x**y**z such that : **|vxy| <= P**
    - At first part : a^P ⇒ **you can pump just a ? generated words would have more a than b and c ⇒ not in the language**

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language B = $\{a^n\ b^n\ c^n \mid n \geq 0\}$

  is not context free

$$\underbrace{\overbrace{aaaaa \ldots aaa}^{a^p}}_{u}\ \underbrace{\overbrace{bbbb \ldots bbb}^{b^p}}_{vxy}\underbrace{\overbrace{cccc \ldots ccc}^{c^p}}_{z}$$

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language B = $\{a^n b^n c^n \mid n \geq 0\}$

  is not context free

  - String : $a^P b^P c^P$ **⇒** u**v**x**y**z such that : **|vxy| <= P**
    - At middle part between a and b ?

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language $B = \{a^n\, b^n\, c^n \mid n \geq 0\}$ is not context free

a^p b^p c^p

aaaaa ... aaa bbbb ... bbbcccc ... ccc

u vxy z

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language B = $\{a^n b^n c^n \mid n \geq 0\}$

  is not context free

  - String : $a^P b^P c^P$ ⇒ u**v**x**y**z such that : **$|vxy| <= P$**

    - At middle part between a and b :

      You will pump only a and b but not C.

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language B = $\{a^n b^n c^n \mid n \geq 0\}$

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  - String : $a^P b^P c^P$ ⇒ u**v**x**y**z such that : **|vxy| <= P**
    - At middle part : Between b and c : Same

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language $B = \{a^n b^n c^n \mid n \geq 0\}$ is not context free
  - String : $a^P b^P c^P$ ⇒ u**v**x**y**z such that : **|vxy| <= P**
    - Between a^P and b^P ⇒ **regardless of what you take for v and y, the c will not be pumped.**

Pumping Lemma for Context Free Grammar

- Language :

WordWord

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language D={ww| w ∈ {0,1}*

  } is not context free
  - We assume it is context-free.
  - We assume the pumping length **P**
  - Let's try this string : ?

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language D={ww| w ∈ {0,1}*

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  - We assume it is context-free.
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  - Let's try this string : $0^{P}\,1\,0^{P}1$

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language D={ww| w ∈ {0,1}* } is not context free
  - String : $0^P 1 0^P 1$ ⇒ u**v**x**y**z such that : **|vxy| <= P**
  - Let's try all possible combinations for placing the uvxyz on our string $0^P 1 0^P 1$ **and see if we can pump**

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language D={ww| w ∈ {0,1}*

  } is not context free

$$\underbrace{\epsilon}_{u}\;\underbrace{\overbrace{0000 \ldots 001}^{0^p1}}_{vxy}\;\underbrace{\overbrace{\color{red}{0000 \ldots 001}}^{0^p1}}_{z}$$

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language D={ww| w ∈ {0,1}*

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$$\underbrace{\overbrace{0000 \ldots 001}^{0^p1}}_{vxy}\ \overbrace{\color{red}{0000 \ldots 001}}^{0^p1}$$

The second word would not be equal to the first word

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language D={ww| w ∈ {0,1}* } is not context free

$$\overbrace{0000 \ldots 001}^{0^p1}\ \underbrace{\overbrace{0000 \ldots 001}^{0^p1}}_{vxy}$$

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language D={ww| w ∈ {0,1}* } is not context free

0^p1 0^p1

0000 … 001 0000 … 001

vxy

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language D={ww| w ∈ {0,1}*

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$$\overbrace{0000 \ldots 001}^{0^p1}\;\overbrace{0000 \ldots 001}^{0^p1}$$

vxy

v=0
x=1
y=0

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language D={ww| w ∈ {0,1}*

  } is not context free

$$\overbrace{0000 \ldots 001}^{0^p1}\ \overbrace{0000 \ldots 001}^{0^p1}$$

vxy

v=0
x=1
y=0

- We can pump the selected string and ends up generating words in the same language

Pumping Lemma for Context Free Grammar

- U ∈ {0,1}*

} i

This does not mean that the language is context-free, it means you chose a bad string

$\underbrace{}_{\textbf{vxy}}$

x=1
y=0

- We can pump the selected string and ends up generating words in the same language

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language D={ww| w ∈ {0,1}* } is not context free
  - String : $0^P 1 0^P 1 \Rightarrow$ uvxyz such that : **|vxy| <= P**
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- Use the pumping lemma to show that the language D={ww| w ∈ {0,1}*

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$$\overbrace{\underbrace{0000 \ldots 00}_{vxy}1111..111}^{0^p1^p}\ \overbrace{\color{red}{0000 \ldots 001111...111}}^{0^p1^p}$$

More ones in the first word + if you break to get two words ? => second word starts with 1, not zero

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- Use the pumping lemma to show that the language D={ww| w ∈ {0,1}*

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$$\overbrace{0000 \ldots 00\underbrace{1111..111}_{vxy}}^{0^p1^p}\ \overbrace{0000 \ldots 001111...111}^{0^p1^p}$$

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$$\overbrace{0000\ \underbrace{\ldots\ 0011}_{vxy}11..111}^{0^p1^p}\ \overbrace{\color{red}{0000\ \ldots\ 001111\ldots111}}^{0^p1^p}$$

More numbers added ? if you break to get two word ? => second word starts with 1, not zero

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language D={ww| w ∈ {0,1}* } is not context free

0^p1^p 0^p1^p

0000 … 001111..111 0000 … 001111…111

vxy

Even if you take x as empty string …

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language $C = \{a^i b^j c^k \mid 0 \le i \le j \le k\}$ is not context free
  - We assume it is context-free.
  - We assume the pumping length **P**
  - Let's try this string : $a^{\mathbf{P}} b^{\mathbf{P}} c^{\mathbf{P}}$

Pumping Lemma for Context Free Grammar

Pumping lemma for context-free languages If A is a context-free language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ satisfying the conditions

1. for each $\boxed{i \geq 0,}$ $uv^ixy^iz \in A$,
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Pumping Lemma for Context Free Grammar

- Construct a PDA for the following Language :

  $C = \{a^i\, b^j\, c^k \mid 0 \le i \le j \le k\}$

  Example of words:

  - abbccc
  - bc
  - abcc

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language C = $\{a^i b^j c^k \mid 0 \le i \le j \le k\}$ is not context free

$$\overbrace{\text{aaaa...aaaa bbbb ....bbbb cccc}\underbrace{\text{... ccccc}}_{vxy}}^{a^pb^pc^p}$$

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language C = $\{a^i b^j c^k \mid 0 \le i \le j \le k\}$ is not context free

$$\overbrace{\text{aaaa...aaaa bbbb ....bbbb cccc... ccccc}}^{a^p b^p c^p}$$

vxy

Seems Easy ?
We just pump **up** v and y which can be c ?

Words will be Always in the language

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language C = $\{a^i b^j c^k \mid 0 \le i \le j \le k\}$ is not context free

$$\overbrace{\text{aaaa...aaaa } \textbf{bbbb ....bbbb} \text{ cc} \underbrace{\text{cc... cccc}}_{vxy}\text{c}}^{a^pb^pc^p}$$

Seems Easy ?
We just pump **up** v and y which can be c ?

Words will be Always in the language

Can we design the PDA ?

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language $C = \{a^i b^j c^k \mid 0 \le i \le j \le k\}$ is not context free

$$\overbrace{\text{aaaa...aaaa bbbb ....bbbb }\underbrace{\text{cccc... ccccc}}_{vxy}}^{a^p b^p c^p}$$

v or y contains only c

If we pump UP ?

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language $C = \{a^i b^j c^k \mid 0 \le i \le j \le k\}$ is not context free

$$\overbrace{\text{aaaa...aaaa } \color{red}{\text{bbbb ....bbbb}} \text{ cccc... } \underbrace{\text{ccccc}}_{vxy}}^{a^p b^p c^p}$$

v or y contains only c
Let's assume i=0 for the pumping lemma
" Pumping down"

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language C = {$a^i b^j c^k$ | 0 ≤ i ≤ j ≤ k} is not context free

$$\overbrace{\text{aaaa...aaaa } \textcolor{red}{\text{bbbb ....bbbb}} \text{ cccc}\underbrace{\text{... ccccc}}_{\text{vxy}}}^{a^p b^p c^p}$$

v^0xy^0 ⇒ Certainly, we will drop at least one C

Remember |vy|>1

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language $C = \{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}$ is not context free

$$\overbrace{\text{aaaa...aaaa } \color{red}{\text{bbbb ....bbbb}} \text{ } \underbrace{\text{cccc... ccccc}}_{vxy}}^{a^p b^p c^p}$$

$uv^0xy^0z \Rightarrow$ does it belong to the language ?

Pumping Lemma for Context Free Grammar

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$$\overbrace{\text{aaaa...aaaa bbbb ....bbbb cccc... ccccc}}^{a^p b^p c^p}$$

$$\text{aaaa...aaaa bbbb ....bbbb } \underbrace{\text{cccc... ccccc}}_{vxy}$$

$uv^0xy^0z \Rightarrow$ does it belong to the language ?
No because the number of C is less than P

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language $C = \{a^i b^j c^k \mid 0 \le i \le j \le k\}$ is not context free

$$\overbrace{\text{aaaa...aaaa bbbb ....}\underbrace{\text{bbbb cccc...}}_{vxy}\text{ ccccc}}^{a^pb^pc^p}$$

Same case as before, number of A should <= B or C

Pumping Lemma for Context Free Grammar

- Use the pumping lemma to show that the language $C = \{a^i b^j c^k \mid 0 \le i \le j \le k\}$ is not context free

$$\overbrace{\underbrace{aaaa\ldots aaaa}_{vxy}\ bbbb\ \ldots\ldots bbbb\ cccc\ldots\ ccccc}^{a^pb^pc^p}$$

Pump down or pump up ?

Closure Properties for Context Free Grammar

- If L_1 and L_2 are context-free languages over an alphabet Σ, then
  - $L_1 \cup L_2$
  - $L_1 L_2$
  - $L_1{*}$
- are also Context-free languages.

Closure Properties for Context Free Grammar

- **Union**
  - Given two languages:
    - $G_1 = (N_1, \Sigma, S_1, P_1)$ be CFG for L_1 .
    - $G_2 = (N_2, \Sigma, S_2, P_2)$ be CFG for L_2
  - $G_u = G_1 \cup G_2$ where $G_u = (N_u, \Sigma, S_u, P_u)$
    - $N_u = N_1 \cup N_2 \cup \{S_u\}$
    - $P_u = P_1 \cup P_2 \cup \{S_u \rightarrow S_1 \mid S_2\}$

N : non-terminal variables
P : Production rules
S : Start Variable.

Closure Properties for Context Free Grammar

- Union : Important
  - If L_1 and L_2 are context Free Languages ⇒ $L_1 \cup L_2$ is context free language.
  - If L_1 and $L_1 \cup L_2$ are context free languages ⇒ does it imply that L_2 is also context free language ?

Closure Properties for Context Free Grammar

- Union : Important
  - Consider : $L_1 = \Sigma^*$
  - Consider L_2 any non context free language ($a^n b^n c^n$ where n>0)
  - $L_1 \cup L_2 = L_1$

Closure Properties for Context Free Grammar

- **Union : Important**
  - If L_1 and L_2 are context Free Languages ⇒ $L_1 \cup L_2$ is context free language.
  - If L_1 and $L_1 \cup L_2$ are context free languages ⇒ does it imply that L_2 is also context free language ?
    - **Not necessarily**

Closure Properties for Context Free Grammar

- **Concatenation**
  - Given two languages:
    - $G_1 = (N_1, \Sigma, S_1, P_1)$ be CFG for L_1.
    - $G_2 = (N_2, \Sigma, S_2, P_2)$ be CFG for L_2
  - $G_c = G_1 G_2$ where $G_c = (N_c, \Sigma, S_c, P_c)$
    - $N_c = N_1 \cup N_2 \cup \{S_c\}$
    - $P_c = P_1 \cup P_2 \cup \{S_c \to S_1 S_2\}$

Closure Properties for Context Free Grammar

- **Star**
  - Given the following language:
    - $G_1 = (N_1, \Sigma, S_1, P_1)$ be CFG for L_1 .
  - G_s is the grammar for L^* where $G_s = (N_s, \Sigma, S_s, P_s)$
    - $N_s = N_1 \cup \{S_s\}$
    - $P_s = P_1 \cup \{S_s \rightarrow S_1 S_s \mid \varepsilon\}$

Closure Properties for Context Free Grammar

- Intersection is not closed for context-free grammar :
  - Given the two context free languages languages:
    - $L_1 = \{a^n b^n c^k \mid n \text{ and } k >= 0\}$
    - $L_2 = \{a^k b^n c^n \mid n \text{ and } k >= 0\}$
  - $L_1 \cap L_2 = ?$

Closure Properties for Context Free Grammar

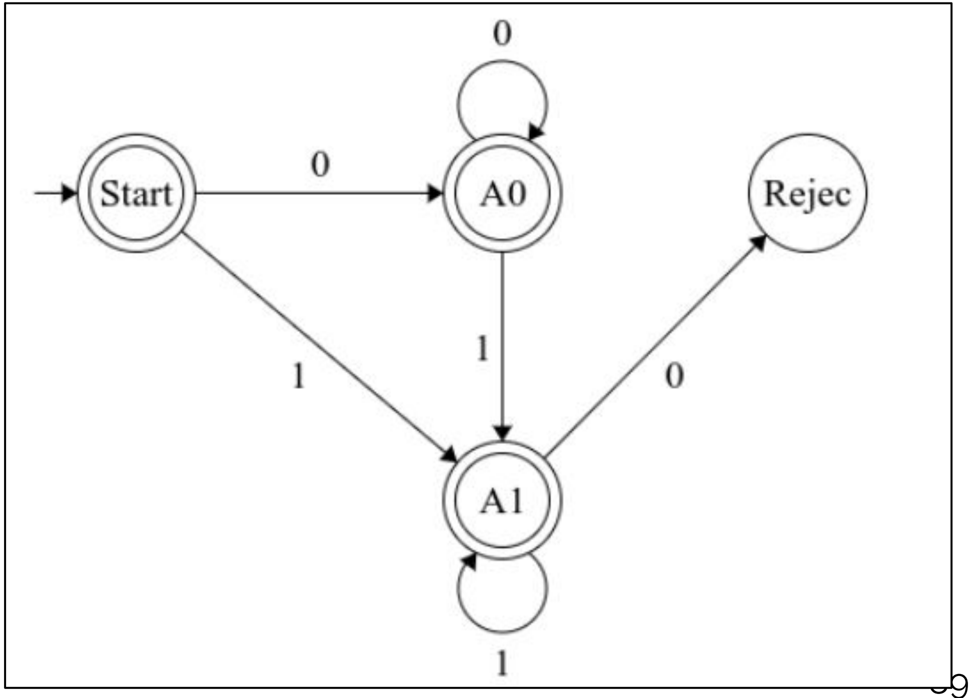
- Intersection is not closed for context-free grammar :
  - Given the two context free languages languages:
    - $L_1 = \{a^n b^n c^k \mid n \text{ and } k >= 0\}$
    - $L_2 = \{a^k b^n c^n \mid n \text{ and } k >= 0\}$
  - $L_1 \cap L_2 = \{a^n b^n c^n \mid n >= 0\}$ this is non context free grammar

Closure Properties for Context Free Grammar

- Complement is not closed under the Context-Free Languages
  - Example :
    - Language { wx such that w is not x }
    - Can we design DFA/NFA/PDA for it ? or the CFG ?
      - **YES.**

Revision : DFA

- What's this language for ?



Revision : CFG

- Construct the CFGs for the following languages :
  - Words which starts and ends with the same symbol over {0,1}
  - Words of Odd Length over {0,1}
  - L ={ $a^i b^j c^k$ such that i = j + k } over {a,b,c} (number of a=number of b+c
  - L ={ $a^i b^j c^k$ such that j = i + k } over {a,b,c} (number of b=number of a+c
  - (Number of a) +2 = number of b

Revision : CFG

Please , try to do it on your own without seeing the solution :

How to know you are correct, simulate basic words...

- Construct the CFGs for the following languages :
  - Words which starts and ends with the same symbol over {0,1}

$$S_0 \rightarrow 0S_10|1S_11|\varepsilon$$

$$S_1 \rightarrow 0S_1|1S_1|\varepsilon$$

Revision : CFG

- Construct the CFGs for the following languages :
  - Words of Odd Length over {0,1}

$$S_0 \rightarrow 0S_1|1S_1$$

$$S_1 \rightarrow 00S_1|01S_1|10S_1|11S_1|\varepsilon$$

Revision : CFG

- Construct the CFGs for the following languages :
  - L ={ $a^i b^j c^k$ such that i = j + k } over {a,b,c} (number of a=number of b+c

Revision : CFG

- Construct the CFGs for the following languages :
  - L ={ $a^i b^j c^k$ such that i = j + k } over {a,b,c} (number of a=number of b+c

$$S_0 \rightarrow \mathrm{a}\mathrm{S}_0 c \mid S_1$$

$$S_1 \rightarrow \mathrm{a}\mathrm{S}_1 b \mid \varepsilon$$

Revision : CFG

- Construct the CFGs for the following languages :
  - L ={ $a^i b^j c^k$ such that i = j + k } over {a,b,c} (number of a=number of b+c

$$S_0 \rightarrow \mathrm{a}S_0c \,|\, S_1$$
$$S_1 \rightarrow \mathrm{a}S_1b \,|\, \varepsilon$$

PDA done in the Tutorial sessions ?

Revision : CFG

- Construct the CFGs for the following languages :
    - L ={ $a^i b^j c^k$ such that j = i + k } over {a,b,c} (number of b=number of a+c

Revision : CFG

- Construct the CFGs for the following languages :
    - L ={ $a^i b^j c^k$ such that j = i + k } over {a,b,c} (number of b=number of a+c

$$S_0 \rightarrow \mathrm{a}\mathrm{S}_1\mathrm{b}\mathrm{S}_2 | S_1\mathrm{b}\mathrm{S}_2 c | \varepsilon$$

$$S_1 \rightarrow \mathrm{a}\mathrm{S}_1 b | \varepsilon$$

$$S_2 \rightarrow \mathrm{b}\mathrm{S}_2 c | \varepsilon$$

Revision : CFG

- Construct the CFGs for the following languages :
    - L ={ $a^i b^j c^k$ such that j = i + k } over {a,b,c} (number of b=number of a+c

$$S_0 \rightarrow \mathrm{a}S_1\mathrm{b}S_2 | S_1\mathrm{b}S_2c | \varepsilon$$

$$S_1 \rightarrow \mathrm{a}S_1b | \varepsilon$$

$$S_2 \rightarrow \mathrm{b}S_2c | \varepsilon$$

Can you design the PDA from scratch without converting from the CFG ?

Revision : NFA to RegEx

- Exam Question

Software & Tools

- You can download the JFLAP :
    - https://www.jflap.org/tutorial/turing/one/index.html

Software & Tools

- Online
  - https://automatonsimulator.com/
  - https://turingmachine.io/

Software & Tools

- Mobile Apps :
    - https://play.google.com/store/apps/details?id=com.TripleVGames.MFLAP
    - https://play.google.com/store/apps/details?id=com.singh.tuhina.automatasimulationcopy&hl=en&gl=US

TD6 - Solutions

In each case below, say what language is generated by the context-free grammar:

1. S → aS | bS | ε **{a,b}***
2. S → SS | bS | a **{a,b}*a**
3. S → SaS | b **bababab a starts with b, there is a between two b**
4. S → SaS | b | ε **does not contain bb**
5. S → T T **contains exactly two b**
   T → aT | T a | b
6. S → aSa | bSb | aAb | bAa **not palindromes**
   A → aAa | bAb | a | b |ε | S
7. S → aT | bT | ε **Even number of letters**
   T → aS | bS
8. S → aT | bT **odd number of letters**
   T → aS | bS | ε

TD6 - Solutions

Give the context-free grammars that generate the following languages. Alphabet Σ is {0,1}.

1. {w| w contains at least three 1s}
   S→P1P1P1P
   P → 0P | 1P | e

2. {w| w starts and ends with the same symbol}
   S → 0 P 0 | 1 P 1 | 1 | 0
   P → 0P | 1 P | e

3. {w| the length of w is odd}
   S→ 0 | 1 | 00S | 10S | 10S | 11S
   Or
   S→ 0 | 1 | 0S0 | 0S1 | 1S0 | 1S1
   Or
   See previous exercise

4. {w| the length of w is odd and its middle symbol is a 0}
   S → 0 | 0S0 | 0S1 | 1S0 | 1S1

TD6 - Solutions

Give the context-free grammars that generate the following languages. Alphabet Σ is {0,1}.

1. {w| w = w^R , that is, w is a palindrome}

   S → 0S0 | 1S1 | 1 | 0 | ε

2. {w| w is not equal to w^R , that is, w is not a palindrome}

   S → 0S0 | 1S1 | 0A1 | 1A0
   A → 0A0 | 1A1 | 0 | 1 | ε | S

3. {number of 0 is the same as 1}

   S → ε | S0S1S | S1S0S
   OR
   S → 0S1 | 1S0 | SS | ε

4. All strings with more a's than b's

   $S \rightarrow S_1aS_1$
   $S_1 \rightarrow bS_1a|aS_1b|S_1S_1|aS_1|\ \varepsilon$

   Test String : aabbaa :

   $S \rightarrow S_1aS_1 \rightarrow aS_1b\ aS_1 \rightarrow aaS_1bb\ aS_1 \rightarrow aaS_1bb\ aS_1 \rightarrow aabb\ aS_1 \rightarrow aabbaaS_1 \rightarrow aabbaa$