Data Structures and Algorithms 2

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Chapter 2

Algorithm Complexity

Chapter Outline

- Algorithms efficiency
- Algorithm analysis:
  - Machine-dependent vs Machine-independent
- Function ordering
  - Weak Order; Big-Oh (asymptotically $\leq$); Big-Ω (asymptotically $\geq$); Big-θ (asymptotically =); Little-oh (asymptotically <).
- Algorithm complexity analysis
  - Rules for complexity analysis
  - Analysis of 4 algorithms for the max subsequence sum
- Master Theorem

Design challenges

- Designing an algorithmic solution is (almost) good:
  - The algorithm should not take “ages”
  - It should not consume “too much” memory.
- In this chapter, we will discuss the following:
  - How to estimate the time required for a program.
  - How to reduce the running time of a program from days or years to fractions of a second.
  - The results of careless use of recursion.
  - Very efficient algorithms to raise a number to a power and to compute the greatest common divisor of two numbers.

Algorithms Efficiency

- A city has *n* view points
- Buses move from one view point to another
- A bus driver wishes to follow the shortest path (travel time wise).
- Every view point is connected to another by a road.
- However, some roads are less congested than others.
- Also, roads are one-way, i.e., the road from view point 1 to 2, is different from that from view point 2 to 1.

- How to find the shortest path between any two pairs?
- Naïve approach:
  - List all the paths between a given pair of view points
  - Compute the travel time for each.
  - Choose the shortest one.
- How many paths are there between any two view points (without revisits)?

$$\mathbf{n! \cong (n/e)^n}$$

It will be impossible to run the algorithm for n = 30 (roughly 11^{30} paths)

What is efficiency of an algorithm?

- Run time in the computer is machine-dependent
- Example: Multiply two positive integers a and b
- Subroutine 1: Multiply a and b
- Subroutine 2:

```
v = a,    w =  b
While   w > 1
      { v  = v + a;
        w = w - 1}
Output v
```

Machine-Dependent Analysis

- First subroutine has 1 multiplication.
- Second subroutine has *b* additions and *b* subtractions.
- For some architectures, 1 multiplication is more expensive than *b* additions and *b* subtractions.
- Ideally:
  - programme all alternative algorithms
  - run them on the target machine
  - find which is more/most efficient!

Machine-Independent Analysis

- We will assume that every basic operation takes constant time
- Example of Basic Operations:
  – Addition, Subtraction, Multiplication, Memory Access
- Non-basic Operations:
  – Sorting, Searching
- Efficiency of an algorithm is thus measured in terms of the number of basic operations it performs
  – We do not distinguish between the basic operations.

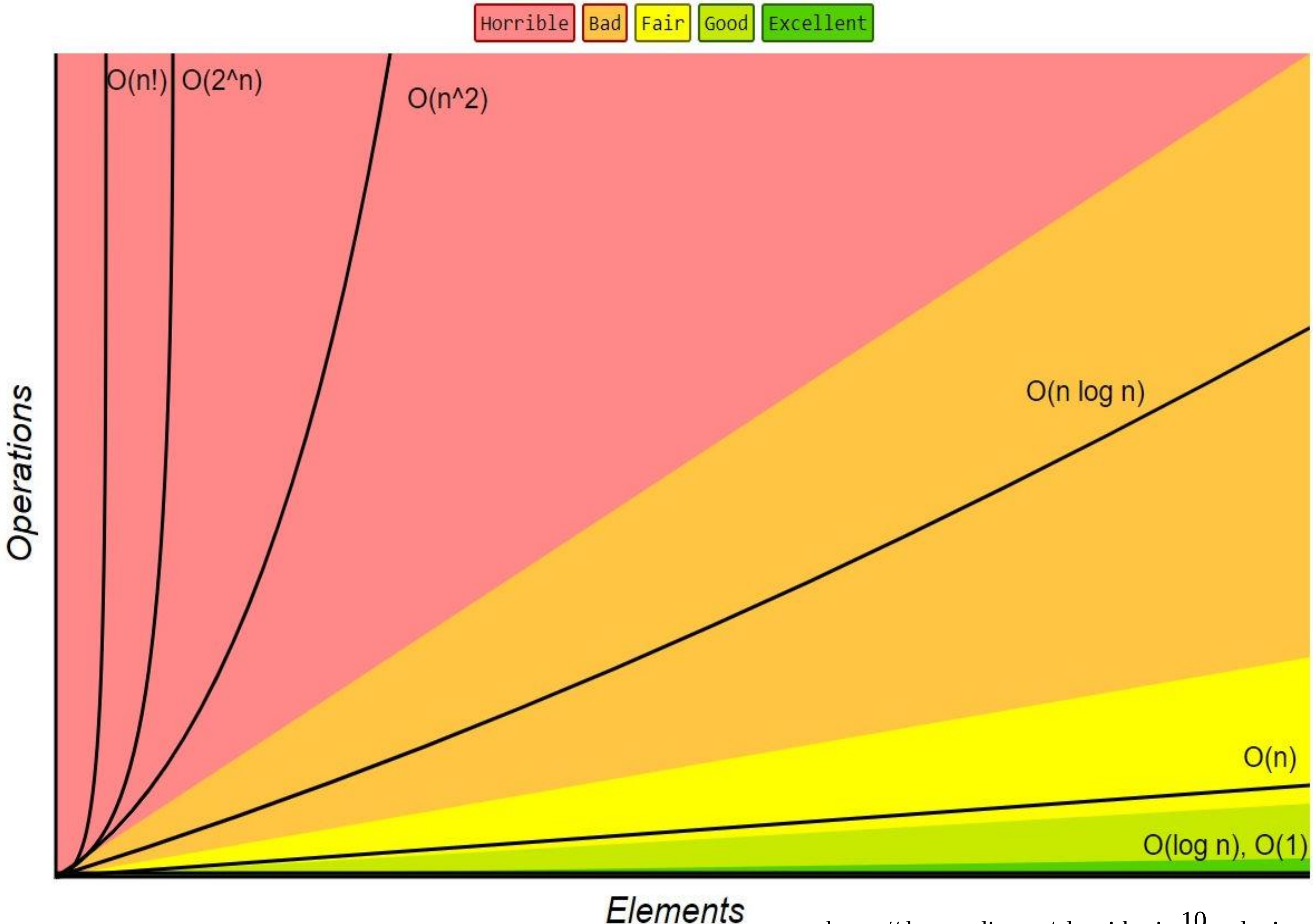
- Subroutine 1 uses 1 basic operation
- Subroutine 2 uses $2b$ basic operations

  ➔ Subroutine 1 is more efficient.

- This measure is good for all large input sizes

- **In fact**, we will not worry about the exact number of operations, but will look at ``broad classes' of values.
  - Let there be n inputs.
  - If an algorithm needs n basic operations and another needs 2n basic operations, we will consider them to be in the same efficiency category.
  - However, we distinguish between exp(n), n, log(n)

- We worry about algorithms speed for large input sizes (i.e. rate of growth of algorithm speed based on input size)



https://devopedia.org/algorithmic-complexity

Weak ordering

- In this chapter, f and g are functions from the set of natural numbers to itself.
- Consider the following definitions:
  - We will consider two functions to be equivalent, $f \sim g$, if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ where $0 < c < \infty$
  - We will state that $f < g$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
- For the functions we are interested in, these define a weak ordering.

Weak ordering

Let $f(n)$ and $g(n)$ describe the run-time of two algorithms. In general, there are functions s.t.:

- If $f(n) \sim g(n)$, then it is always possible to improve the performance of one function over the other by purchasing a faster computer.
- If $f(n) < g(n)$, then you can <u>never</u> purchase a computer fast enough so that the second function **always** runs in less time than the first.

- Note that for small values of n, it may be reasonable to use an algorithm that is asymptotically more expensive, but we will consider these on a one-to-one basis.

Function Orders: Big Oh Notation

Definition 2.1

$f(n) = O(g(n))$ if there are positive *constants* c and n_0 such that $0 \le f(n) \le c * g(n)$ when $n \ge n_0$

i.e. A function $f(n)$ is $O(g(n))$ if the rate of growth of $f(n)$ is not faster than that of $g(n)$.

i.e. if $lim_{n\to\infty} f(n)/g(n)$ exists and is finite, then $f(n)$ is $O(g(n))$

Intuitively, (not exactly) $f(n)$ is $O(g(n))$ means $f(n) \le g(n)$ for all n beyond some value n_0; i.e. $g(n)$ is an *upper bound* for $f(n)$.

Remark

Before we begin, let us make some assumptions:

- Our functions will describe the time or memory required to solve a problem of size n
- Restrictions are put on the functions:
  - They are defined for $n \geq 0$
  - They are strictly positive for all n
    - In fact, $\mathrm{f}(n) > c$ for some value $c > 0$
    - That is, any problem requires at least one instruction and byte
  - They are increasing (monotonic increase)

Examples of Functions

sqrt(n) , n, 2n, ln (n), exp(n), n + sqrt(n) , n + n^2

$\lim_{n\to\propto}$ sqrt(n) /n = 0,	sqrt(n) is O(n)
$\lim_{n\to\propto}$ n/sqrt(n) = infinity,	n is not O(sqrt(n))
$\lim_{n\to\propto}$ n /2n = 1/2,	n is O(2n)
$\lim_{n\to\propto}$ 2n /n = 2,	2n is O(n)

Other examples:

$\lim_{n \to \propto} \ln(n) / n = 0$	$\ln(n)$ is $O(n)$
$\lim_{n \to \propto} n / \ln(n) =$ infinity	n is not $O(\ln(n))$
$\lim_{n \to \propto} \exp(n) / n =$ infinity	$\exp(n)$ is not $O(n)$
$\lim_{n \to \propto} n / \exp(n) = 0$	n is $O(\exp(n))$
$\lim_{n \to \propto} (n + \mathrm{sqrt}(n)) / n = 1$	$n + \mathrm{sqrt}(n)$ is $O(n)$
$\lim_{n \to \propto} n / (\mathrm{sqrt}(n) + n) = 1$	n is $O(n + \mathrm{sqrt}(n))$
$\lim_{n \to \propto} (n + n^2) / n =$ infinity	$n + n^2$ is not $O(n)$
$\lim_{n \to \propto} n / (n + n^2) = 0$	n is $O(n + n^2)$

Implication of Big Oh notation

- Suppose we know that our algorithm uses at most $O(f(n))$ basic steps for any n inputs, and n is sufficiently large,
  - then we know that our algorithm will terminate after executing at most $f(n)$ basic steps.
- We also know that a basic step takes a constant time on a machine.

  ➔ Our algorithm will terminate in at most $f(n)$ units of time, for all large n.

Other Complexity: Big Ω Notation

Now a lower bound notation, $\Omega(n)$

Definition 2.2

$f(n) = \Omega(g(n))$ if there are positive *constants* c and n_0 such that $f(n) \geq c * g(n)$ when $n \geq n_0$.

i.e. $\lim_{n\to\infty} (f(n)/g(n)) > 0$, if $\lim_{n\to\infty} (f(n)/g(n))$ exists.

We say *g(n)* is a *lower bound* on *f(n)*, i.e. no matter what specific inputs we have, <u>the algorithm will not run faster than this lower bound</u>.

Interpretation of the Ω Notation

- Suppose, an algorithm has complexity Ω(f(n)) .
- This means that there exists a positive constant c such that for all sufficiently large n, there exists at least one input for which the algorithm consumes at least c * f(n) steps.
- Big-O vs. Big-Omega:

➢ **Big-Ω** notation: gives a lower bound of the running time of an algorithm. Big-Ω(n) means the algorithms runs at least in *n* time but could actually take a lot longer.

➢ **Big-O** notation: gives an upper bound so O(n) would mean the algorithm runs in its worst case in *n* (i.e. linear) time.

Other Complexity: Big θ Notation

Definition 2.3

$f(n) = \theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

- $\theta(g(n))$ is referred to as ``asymptotic equality"
- $lim_{n \to \infty}(f(n)/g(n))$ is a finite, positive constant, if it exists.
- $f(n)$ has **a rate of growth** equal to that of $g(n)$

Other Complexity: Little-oh Notation

Definition 2.4:

$f(n) = o(g(n))$ if, for all positive constants c, there exists an n_0 such that $f(n) < c * g(n)$ when $n > n_0$. (``asymptotic strict inequality'')

- Less formally, $f(n) = o(g(n))$ if $f(n) = O(g(n))$ and $f(n) \neq \theta(g(n))$.
- $f(n)$ is $o(g(n))$ if given any positive constant c, there exists some m such that $f(n) < c * g(n)$ for all $n \geq m$

  $lim_{n \to \propto} (f(n)/g(n)) = 0$, if $lim_{n \to \propto} (f(n)/g(n))$ exists

- Little *oh* means: "is ultimately smaller than"

Rates of gowth

Suppose that $\mathrm{f}(n)$ and $\mathrm{g}(n)$ satisfy $\lim_{n\to\infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)} = c$

If $\lim_{n\to\infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)} = c$ where $0 < c < \infty$, it follows that $\mathrm{f}(n) = \boldsymbol{\Theta}(\mathrm{g}(n))$

If $\lim_{n\to\infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)} = c$ where $0 \le c < \infty$, it follows that $\mathrm{f}(n) = \mathbf{O}(\mathrm{g}(n))$

If $\lim_{n\to\infty} \frac{\mathrm{f}(n)}{\mathrm{g}(n)} = 0$, we will say $\mathrm{f}(n) = \mathbf{o}(\mathrm{g}(n))$, i.e. $\mathrm{f}(n)$ has a rate of growth less than that of $\mathrm{g}(n)$

Terminology

Asymptotically less than or equal to	**O**
Asymptotically greater than or equal to	**Ω**
Asymptotically equal to	**θ**
Asymptotically strictly less	**o**

Recap

$$f(n) = \mathbf{o}(g(n)) \qquad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$$f(n) = \mathbf{O}(g(n)) \qquad \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$$f(n) = \mathbf{\Theta}(g(n)) \qquad 0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$$f(n) = \mathbf{\Omega}(g(n)) \qquad \lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

Example Functions

sqrt(n) , n, 2n, ln n, exp(n), n + sqrt(n) , n + n^2

$\lim_{n\to\infty}$ sqrt(n) /n = 0 sqrt(n) is o(n), O(n)

$\lim_{n\to\infty}$ n/sqrt(n) = infinity n is Ω(sqrt(n))

$\lim_{n\to\infty}$ n/ln(n) = infinity n is Ω(ln(n))

$\lim_{n\to\infty}$ 2n /n = 2 2n is θ(n)

$\lim_{n\to\infty}$ n /2n = 1/2 n is θ(2n)

Terminology

The most common classes are given names:

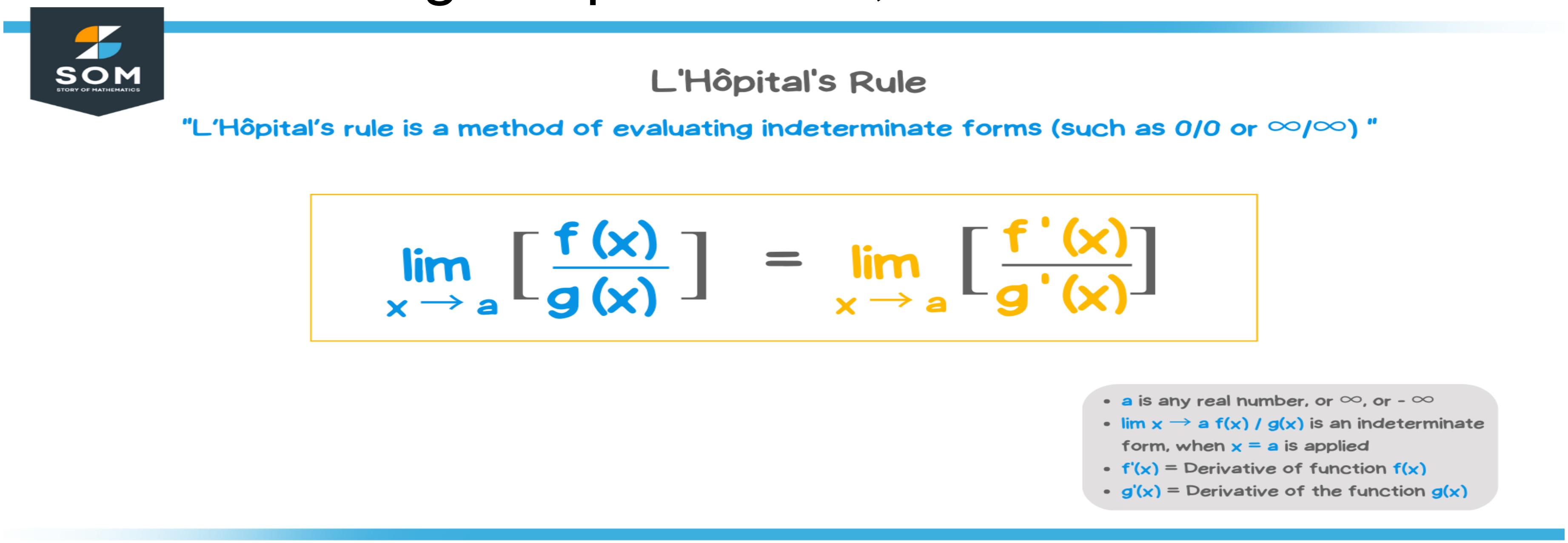
$\Theta(1)$	constant
$\Theta(\ln(n))$	logarithmic
$\Theta(n)$	linear
$\Theta(n \ln(n))$	"n log n"
$\Theta(n^2)$	quadratic
$\Theta(n^3)$	cubic
$2^n, e^n, 4^n, \ldots$	exponential

Little-o as a Weak Ordering

- We can show that, for example

$$\ln(n) = \mathbf{o}(n^p) \qquad \text{for any } p > 0$$

- Proof: Using l'Hôpital's rule,



- So we have:

$$\lim_{n\to\infty} \frac{\ln(n)}{n^p} = \lim_{n\to\infty} \frac{1/n}{pn^{p-1}} = \lim_{n\to\infty} \frac{1}{pn^p} = \frac{1}{p} \lim_{n\to\infty} n^{-p} = 0$$

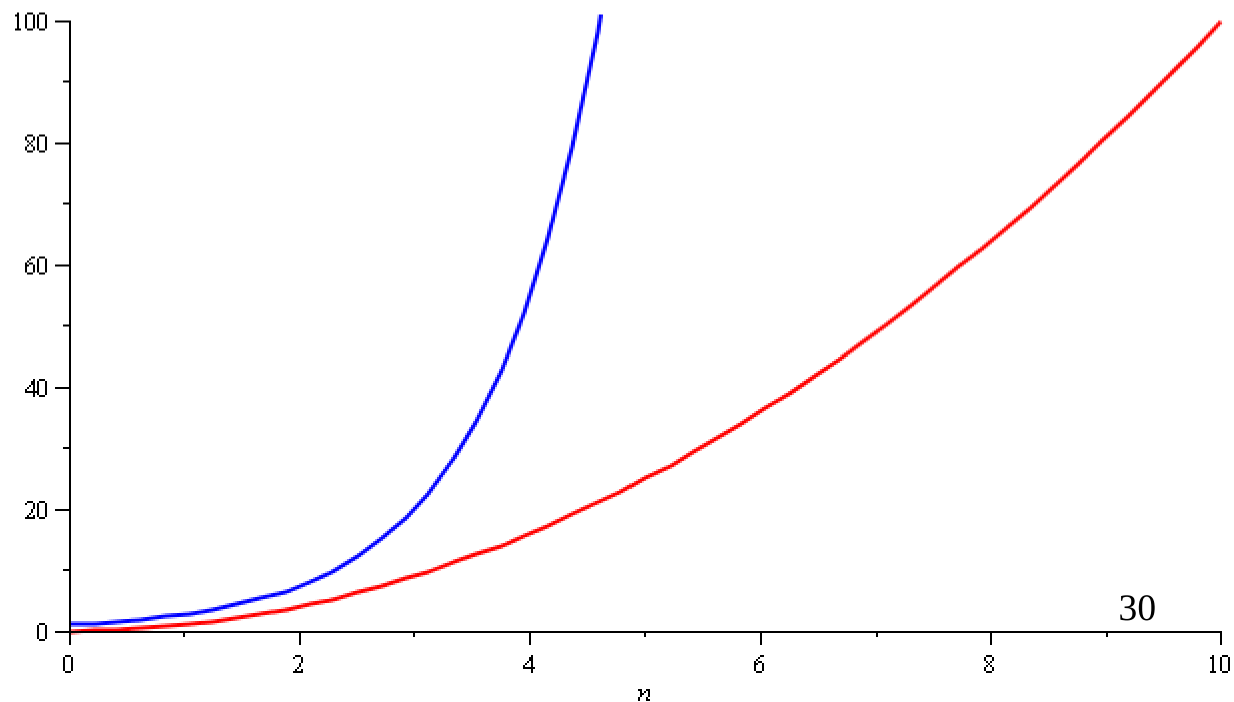
Algorithms Analysis

- An algorithm is said to have *polynomial time complexity* if its run-time may be described by $O(n^d)$ for some fixed $d \geq 0$.
  - We will consider such algorithms to be **efficient.**
- Problems that have no known polynomial-time algorithms are said to be intractable.
  - Traveling salesman problem: find the shortest path that visits n cities
  - Best run time: $\Theta(n^2 2^n)$

Algorithm Analysis

In general, you don't want to implement exponential-time or exponential-memory algorithms

– Warning: don't call a quadratic curve "exponential".



Rules for arithmetic with big-O symbols

- If $T_1(n) = O(f(n))$ and $T_2(n) = O(g(n))$, then
  - (a) $T_1(n) + T_2(n) = O(f(n) + g(n))$ (intuitively and less formally it is $O(\max(f(n), g(n)))$),
  - (b) $T_1(n) * T_2(n) = O(f(n) * g(n))$.
- If $T(n)$ is a polynomial of degree k, then $T(n) = \theta(n^k)$.
- $log^k\, n = O(n)$ for any constant k.

  This tells us that logarithms grow very slowly.

Rules for arithmetic with big-O symbols

- If $f(n) = O(g(n))$, then $c * f(n) = O(g(n))$ for any constant c.
- If $f_1(n) = O(g(n))$ but $f_2(n) = o(g(n))$, then $f_1(n) + f_2(n) = O(g(n))$.
- If $f(n) = O(g(n))$, and $g(n) = o(h(n))$, then $f(n) = o(h(n))$. (complexity of $g \circ h$)
- These are not all of the rules, but they're enough for most purposes.

Complexity of a Problem vs Complexity of an Algorithm

A problem is $O(f(n))$ means there is some $O(f(n))$ algorithm to solve the problem.

A problem is $\Omega(f(n))$ means every algorithm that can solve the problem is $\Omega(f(n))$

Algorithm Complexity Analysis

- We define $T\text{avg}(N)$ and $T\text{worst}(N)$, as the average and worst-case running time, resp., used by an algorithm on input of size N. Clearly, $T\text{avg}(N) \leq T\text{worst}(N)$.
- Occasionally, the best-case performance of an algorithm is analyzed.
  - but of little interest: does not represent the typical behavior.
- Average-case performance often reflects typical behavior
- **Worst-case performance represents a guarantee for performance on any possible input**
- We are interested in algorithm analysis not programme analysis: implementation issues/details/inefficiencies, etc.

Algorithm Complexity Analysis

Consider the following algorithm

```
diff = sum = 0;
For (k=0: k < N; k++)
      sum → sum + 1;
      diff → diff - 1;
For (k=0: k < 3N; k++)
      sum → sum - 1;
```

- First line takes 2 basic steps
- Every iteration of first loop takes 2 basic steps.
- First loop runs N times
- Every iteration of second loop takes 1 basic step
- Second loop runs for 3N times
- Overall, 2 + 2N + 3N steps
- This is O(N)

Rules for Complexity Analysis

Complexity of a loop:

O(Number of iterations in a loop * maximum complexity of each iteration)

Nested Loops:

Analyze the innermost loop first,

Complexity of next outer loop =

number of iterations in this loop * complexity of inner loop, etc.

```
sum = 0;
For (i=0; i < N; i++)
    For (j=0; j < N; j++)   sum → sum + 1;
```

Inner loop: O(N)

Outer loop: N iterations

Overall: $O(N^2)$

```
for( i = 0; i < n; ++i )
    a[ i ] = 0;
for( i = 0; i < n; ++i )
    for( j = 0; j < n; ++j )
        a[ i ] += a[ j ] + i + j;
```

First loop: O(N)

Inner loop: O(N)

Outer loop: N iterations

Overall: $O(N^2) + O(N)$
$\rightarrow O(N^2)$

If (Condition)

S1

Else S2

Maximum of the two complexities

If (yes)

Algo 1

else Ago 2

O(Algo 1)

Analysis of recursion

- Suppose we have the code (not a good one):

```
Long fib (int n) {
        if (n == 0 || n == 1)                    1
                return 1;                        2
        else
                return fib(n – 1) + fib(n – 2);  3
}
```

T(0) = T(1) = 1;

n>=2 T(n) = cost of constant op at 1 + cost of line 3 work

T(n) = 1 op + (addition + 2 function calls) = O(1) + (addition + cost of fib(n-1) + cost fib(n-2)) = 2 + T(n-1) + T(n-2)

Analysis of recursion

- Easy to show by induction T(n) >= fib(n)
- Can prove for n>4, fib(n) >= $(3/2)^n$

➔Running time of the programme grows exponentially

By using an array and a for loop, the programme running time can be reduced substantially.

<u>Ex:</u> Do it and analyse your algorithm.

Maximum Subsequence Problem

Given an array of N elements

Need to find i, j such that the sum of all elements between the ith and jth positions is maximum for all such sums

Running time of 4 algorithms for max subsequence sum (in seconds) [Weiss, Fig 2.2]

	Algorithm Time			
Input Size	1 $O(N^3)$	2 $O(N^2)$	3 $O(N \log N)$	4 $O(N)$
$N = 100$	0.000159	0.000006	0.000005	0.000002
$N = 1{,}000$	0.095857	0.000371	0.000060	0.000022
$N = 10{,}000$	86.67	0.033322	0.000619	0.000222
$N = 100{,}000$	NA	3.33	0.006700	0.002205
$N = 1{,}000{,}000$	NA	NA	0.074870	0.022711

Algorithm 1

```
/**
 * Cubic maximum contiguous subsequence sum algorithm.
 */
int maxSubSum1( const vector<int> & a )
{
    int maxSum = 0;

    for( int i = 0; i < a.size( ); ++i )
        for( int j = i; j < a.size( ); ++j )
        {
            int thisSum = 0;

            for( int k = i; k <= j; ++k )
                thisSum += a[ k ];

            if( thisSum > maxSum )
                maxSum = thisSum;
        }

    return maxSum;
}
```

Analysis of Algorithm 1

Precise analysis is obtained from the sum

$\sum_{i=0}^{N-1} \sum_{j=i}^{N-1} \sum_{k=i}^{j} 1$ # of times

thisSum += a[k]; # is executed

1. First: $\sum_{k=i}^{j} 1 = j - i + 1$
2. Inner loop: # sum of first N-i elements

$$\sum_{j=i}^{N-1} (j-i+1) = (N-i+1)(N-i)/2$$

3. Outer Loop:

$$\sum_{i=0}^{N-1} (N-i+1)(N-i)/2 = (N^3 + 3N^2 + 2N)/6$$

Analysis of Algorithm 1

Overall: $O(N^3)$

Note: The innermost loop can be made more efficient leading to $O(N^2)$

Algorithm 2

```
/**
 * Quadratic maximum contiguous subsequence sum algorithm.
 */
int maxSubSum2( const vector<int> & a )
{
    int maxSum = 0;

    for( int i = 0; i < a.size( ); ++i )
    {
        int thisSum = 0;
        for( int j = i; j < a.size( ); ++j )
        {
            thisSum += a[ j ];

            if( thisSum > maxSum )
                maxSum = thisSum;
        }
    }

    return maxSum;
}
```

Divide and Conquer

- Break a big problem into two small sub-problems
- Solve each of them efficiently.
- Combine the two solutions

Maximum subsequence sum by divide and conquer

- Divide the array into two parts: left part, right part each to be solved recursively
- Max. subsequence lies completely in left, or completely in right or spans the middle.
- If it spans the middle, then it includes the max subsequence in the left ending at the last element and the max subsequence in the right starting from the center

4 –3 5 –2 -1 2 6 -2

Max subsequence sum for first half = 6

second half = 8

Max subsequence sum for first half ending at the last element (4^{th} elements included) is 4

Max subsequence sum for second half starting at the first element (5^{th} element included) is 7

Max subsequence sum spanning the middle is 4 + 7 = 11

Max subsequence spans the middle

Algorithm 3: Divide & Conquer

```
int maxSumRec( const vector<int> & a, int left, int right )
{
    if( left == right )  // Base case
        if( a[ left ] > 0 )
            return a[ left ];
        else
            return 0;

    int center = ( left + right ) / 2;
    int maxLeftSum  = maxSumRec( a, left, center );
    int maxRightSum = maxSumRec( a, center + 1, right );

    int maxLeftBorderSum = 0, leftBorderSum = 0;
    for( int i = center; i >= left; --i )
    {
        leftBorderSum += a[ i ];
        if( leftBorderSum > maxLeftBorderSum )
            maxLeftBorderSum = leftBorderSum;
    }
```

```
    int maxRightBorderSum = 0, rightBorderSum = 0;
    for( int j = center + 1; j <= right; ++j )
    {
        rightBorderSum += a[ j ];
        if( rightBorderSum > maxRightBorderSum )
            maxRightBorderSum = rightBorderSum;
    }

    return max3( maxLeftSum, maxRightSum,
                    maxLeftBorderSum + maxRightBorderSum );
}

/**
 * Driver for divide-and-conquer maximum contiguous
 * subsequence sum algorithm.
 */
int maxSubSum3( const vector<int> & a )
{
    return maxSumRec( a, 0, a.size( ) - 1 );
}
```

Algorithm 3 Analysis

- If N=1; lines 8 to 12 executed; taken to be one unit
  ➔ $T(1) = 1$
- N>1: 2 recursive calls + 2 for loops + some

  bookkeeping ops (e.g. lines 14, 34)
  - The 2 for loops (lines 19 to 32): clearly $O(N)$
  - Lines 8, 14, 18, 26, 34: constant time; ignored compared to $O(N)$
  - Recursive calls made on half the array size each.

    ➔ $2 * T(N/2)$

SO: programme time is $2 * T(N/2) + O(N)$ with $T(1) = 1$

Complexity Analysis

$T(1) = 1$

$T(n) = 2T(n/2) + c.n$

$= 2.c.n/2 + 4T(n/4) + c.n$

$= 4T(n/4) + 2c.n$

$= 8T(n/8) + 3c.n$

=…………..

$= 2^i T(n/2^i) + i.c.n$

=………………… (reach a point when $n = 2^i$ $i = \log n$

$= n.T(1) + c\, n \log n$

$n + c\, n \log n =$ **$O(n \log n)$**

Algorithm 4

```
/**
 * Linear-time maximum contiguous subsequence sum algorithm.
 */
int maxSubSum4( const vector<int> & a )
{
    int maxSum = 0, thisSum = 0;

    for( int j = 0; j < a.size( ); ++j )
    {
        thisSum += a[ j ];

        if( thisSum > maxSum )
            maxSum = thisSum;
        else if( thisSum < 0 )
            thisSum = 0;
    }

    return maxSum;
}
```

Algorithm 4 Analysis

- $T(N) = O(N)$ Obvious!
- What is not obvious at first sight is the logic of the algorithm.

Exercise: convince yourself that the 4 algorithms do the required work. (Trace them!)

Binary Search

- You have a sorted list of numbers
- You need to search the list for a specific number
- If the number exists
  - Then find its position
  - Else detect that
- In Binary search, subdivide the list into 2
- If number is in the center, done;
  else: **double-recursively** search into the left one and into the **right** one.

```
Search(num, A[],left, right)
  {
        if (left == right)
         {
              if (A[left ] == num)     return(left) and exit;
              else conclude NOT PRESENT and exit;
          }
      center = ⌊(left + right)/2⌋;
       If (A[center] < num)
          Search(num, A[], center + 1, right);
      If (A[center] > num)
        Search(num, A[], left, center );
     If (A[center] == num) return(center) and exit;
}
```

Complexity Analysis

$T(n) = T(n/2) + c$

Similar logic as for the analysis of Algorithm 3 above ➔

O(log n) complexity

Master Theorem

- **Master Theorem**: used to calculate time complexity of divide-and-conquer algorithms.
- Simple and quick way to calculate the time complexity of a recursive relation.
- It applies to recurrence relations of the form:
  $T(n) = aT(n/b) + f(n)$

where

- n is the size of the input;
- a is the number of subproblems in the recursion;
- n/b is the size of each subproblem (all assumed to have the same size);
- f(n): cost of work done outside recursive calls; includes cost of dividing the problem and solutions merge cost.

Theorem 4.1 (Master theorem)

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n),$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

This slide and the following two are taken from:
Su, J. CS 161 Lecture 3, University of Stanford,
/https://web.stanford.edu/class/archive/cs/cs161/cs161.1168/lecture3.pdf , (Retrieved 02/10/2023)

Examples of Master Theorem

- <u>Example 1</u>: $T(n) = 9\,T(n/3) + n$.
  Here $a = 9$, $b = 3$, $f(n) = n$, and $n^{\log_b a} = n^{\log_3 9} = \theta(n^2)$.

Since $f(n) = O(n^{\log_3 9-\varepsilon})$ for $\varepsilon = 1$, case 1 of the Master Theorem applies, so $T(n) = \theta(n^2)$.

- <u>Example 2</u>: $T(n) = T(2n/3)+1$. Here $a = 1$, $b = 3/2$, $f(n) = 1$, and $n^{\log_b a} = n^0 = 1$.
  Since $f(n) = \theta(n^{\log_b a})$, case 2 of the Master Theorem applies, so $T(n) = \theta(\log n)$.

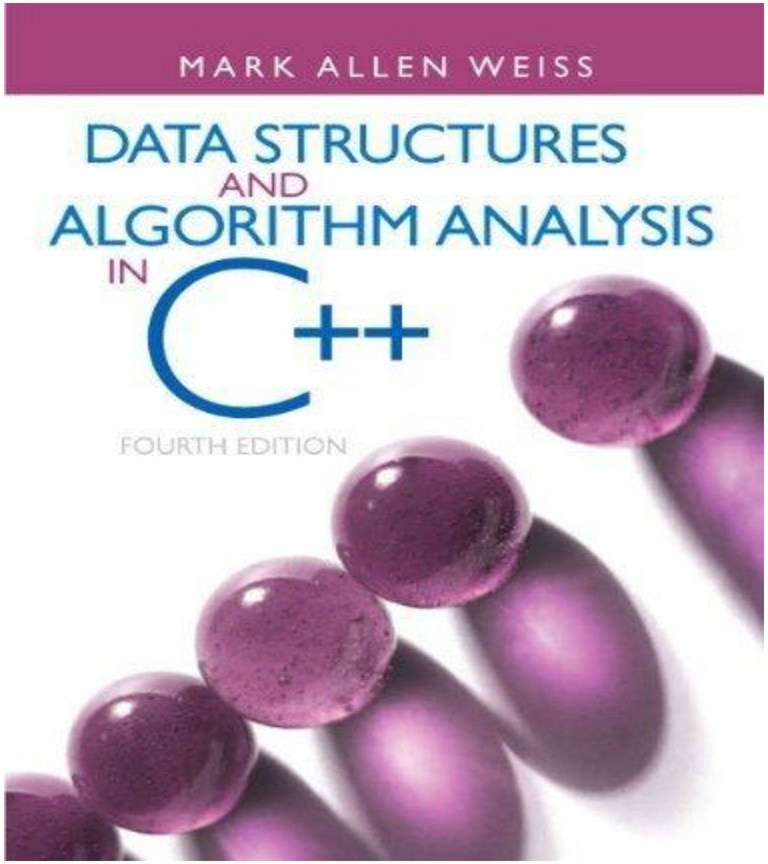
- Example 3: $T(n) = 3T(n/4) + n \log n$. Here $n^{\log b\ a} = n^{\log_4 3} = O(n^{0.793})$. For $\varepsilon = 0.2$, we have $f(n) = \boldsymbol{\Omega}(n^{\log_4 3+\varepsilon})$.

So case 3 applies if we can show that $a\ f(n/b) \leq c\ f(n)$ for some $c < 1$ and all sufficiently large n. This would mean $3\ n/4 \log(n/4) \leq c\ n \log n$. Setting $c= 3/4$ would cause this condition to be satisfied. Hence $T(n) = \theta(n \log n)$.

- Example 4: $T(n) = 2T(n/2)+n \log n$. Here the Master Theorem does not apply.

$n^{\log_b a} = n^{\log_2 2} = n$, and $f(n) = n \log n$. Case 3 does not apply because $f(n)$ is not $\boldsymbol{\Omega}(n^{\log_b a+\varepsilon})$ since $\lim (f(n) / n^{\log_b a+\varepsilon})$ tends towards 0 when n tends towards infinity.

Slides based on the textbook



Mark Allen Weiss, (2014) Data Structures and Algorithm Analysis in C++, 4th edition, Pearson.

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