Chapitre 1. Random Variables

Prof. REMITA Mohamed Riad

National School of Artificial Intelligence.

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Definitions and notations

Example

A coin is tossed twice. The possible results are $\{PP, PF, FP \text{ et } FF\}$. We define variable X representing the number of tails P obtained. Then the values of X are $\{0, 1 \text{ and } 2\}$.

Example

A die is rolled until a 6 is rolled. The possible outcomes are $\{6, (1,6), (2,6), \cdots, (5,6), (1,1,6), \cdots, (5,5,6), \cdots\}$. We define a variable X representing the number of throws needed until a 6 is obtained. Then the values of X are $\{1, 2, 3, \cdots\} = \mathbb{N}^*$.

Example

The service life of a spare part can be represented by a r.r.v.

Definitions and notations

We have a probabilized space $(\Omega, \mathcal{A}, \mathbb{P})$ and a probabilizable space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

Definition

We call a real random variable, noted r.r.v. any application X of $(\Omega, \mathcal{A})$ in $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that:
$\forall B \in \mathcal{B}_{\mathbb{R}}, X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{A}$.

B can be presented in several forms

- If $B =]a, b] : X^{-1}(B) = \{a < X \leq b\}$
- If $B = \{a\} : X^{-1}(B) = \{X = a\}$
- If $B = [a, +\infty[: X^{-1}(B) = \{X \geq a\}$
- If $B =]-\infty, a] : X^{-1}(B) = \{X \leq a\}$

Definitions and notations

Since the Borel $\sigma-$algebra is generated by all the intervals $]-\infty, x]$, then a random variable can be defined by the following definition:

Definition

The application X of $(\Omega, \mathcal{A})$ in $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is a real random variable if for all $x \in \mathbb{R}$ and $\forall B \in \mathcal{B}_{\mathbb{R}}$ the subset $A_x = X^{-1}(]-\infty, x]) = \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{A}$.

Example

We throw two coins and let X be the number of tails obtained. We know that $\Omega = \{(F, F); (F, P); (P, F); (P, P)\}$ and the values of X are $\{0, 1, 2\}$.

1. Show that X is a random variable on Ω endowed with the algebra $\mathcal{P}(\Omega)$.
2. Show that X is not a random variable on Ω endowed with the algebra $\mathcal{A}_1 = \{\Omega, \emptyset, \{(F, F)\}, \{(F, F); (F, P); (P, F)\}\}$.

Definitions and notations

Let Ω be the space of trials associated to a Bernoulli random experiment and let $I_A(\cdot)$ be the function from Ω to $\{0,1\}$ defined by

$$I_A(\cdot) = \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ if } \omega \notin A \end{cases}.$$

$I_A(\cdot)$ is called indicator function (or Dirac measure) of the event A.

We can show easily that $I_A(\cdot)$ is a random variable for the algebra $\mathcal{A}_{I_A(\cdot)} = \left\{\Omega, \emptyset, A, \overline{A}\right\}$.

Properties. The indicator function $I_A(\cdot)$ satisfies the following properties:

1. $I_A(\omega) = 1 - I_{\overline{A}}(\omega), \forall A \in \mathcal{A}$,
2. $I_{\cap A_i}(\omega) = \prod_i I_{A_i}(\omega), \forall A_i \in \mathcal{A}$,
3. $\mathbb{P}(I_A(\omega) = 1) = \mathbb{P}(A), \forall A \in \mathcal{A}$,
4. $\mathbb{P}(I_A(\omega) = 0) = 1 - \mathbb{P}(A) = \mathbb{P}\left(\overline{A}\right), \forall A \in \mathcal{A}$.

Induced probability

Theorem

Let X be a r.r.v. defined on probabilized space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in a probabilizable space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. The application $\mathbb{P}_X$ of $\mathcal{B}_{\mathbb{R}}$ in $\mathbb{R}$ defined by $\mathbb{P}_X(B) = \mathbb{P}\left(X^{-1}(B)\right)$, is a probability on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

Remark

The definition is due to the existence of $\mathbb{P}$ on $(\Omega, \mathcal{A})$, hence the notion of induced probability.

Induced probability

Proof.

It is obvious that $\mathbb{P}_X$ is an application with values in $[0,1]$.
Moreover $\mathbb{P}$ verifies these conditions:

$$\mathbb{P}_X(\mathbb{R}) = \mathbb{P}\left(X^{-1}(\mathbb{R})\right) = \mathbb{P}(\Omega) = 1$$

Let $(B_i)_{i\geq 1}$ be two by two incompatible borelean sequences. Then

$$\begin{aligned}
\mathbb{P}_X\left(\underset{i\geq 1}{\cup} B_i\right) &= \mathbb{P}\left(X^{-1}\left(\underset{i\geq 1}{\cup} B_i\right)\right) = \mathbb{P}\left(\underset{i\geq 1}{\cup} X^{-1}(B_i)\right) \\
&= \sum_{i\geq 1} \mathbb{P}\left(X^{-1}(B_i)\right) = \sum_{i\geq 1} \mathbb{P}_X(B_i),
\end{aligned}$$

noting that $X^{-1}(B_i)$ and $X^{-1}(B_j)$ are incompatible $\forall i \neq j$. $\square$

Cumulative distribution function of a random variable

Definition

The cumulative ditribution function of a r.r.v. is the function F or F_X defined by:

$$F(x) = F_X(x) = \mathbb{P}(X \leq x).$$

Properties of a cumulative distribution function

Definition

A sequence of events $(A_n)_{n\geq 1}$ is increasing (resp. decreasing) if $A_n \subset A_{n+1}$ (resp. $A_{n+1} \subset A_n$) for all $n \geq 1$.
$(A_n)_{n\geq 1}$ is said to be monotonic if it is increasing or decreasing.
In this case we put $\lim\limits_{n\to\infty} A_n = \bigcup\limits_{n\geq 1} A_n$ if it is increasing (resp. $\lim\limits_{n\to\infty} A_n = \bigcap\limits_{n\geq 1} A_n$ if it is decreasing).

Cumulative distribution function of a random variable

Remark

$\lim\limits_{n\to\infty} A_n$ *exists if and only if the sequence* $\left(A_n\right)_{n\geq 1}$ *is monotonic.*

Lemma

(Property of the continuity of $\mathbb{P}$*)*
If $\left(A_n\right)_{n\geq 1}$ *is a monotonic sequence of events, then we have:*

$$\lim_{n\to\infty} \mathbb{P}\left(A_n\right) = \mathbb{P}\left(\lim_{n\to\infty} A_n\right).$$

Cumulative distribution function of a random variable

Theorem

If F is the cumulative distibution function of X then

1. $\forall x \in \mathbb{R}\ 0 \leq F(x) \leq 1$;
2. *F is an increasing function;*
3. *F is right continuous;*
4. $\lim_{x \to \infty} F(x) = 1$ *and* $\lim_{x \to -\infty} F(x) = 0$.

Cumulative distribution function of a random variable

Proof.

1. Obvious because $F(X) = \mathbb{P}(X \leq x)$ so $0 \leq F(X) \leq 1$.
2. Suppose that $x_1 \leq x_2$, hence $]-\infty, x_1] \subset]-\infty, x_2]$ and $X^{-1}(]-\infty, x_1]) \subset X^{-1}(]-\infty, x_2])$. It follows that $\mathbb{P}\left(X^{-1}(]-\infty, x_1])\right) \leq \mathbb{P}\left(X^{-1}(]-\infty, x_2])\right)$ hence $F(x_1) \leq F(x_2)$.
3. Let us show that for any real sequence (ε_n) decreasing and converging to 0, $\lim_{n\to\infty} F(x+\varepsilon_n) = F(x)$. We set $A_n =]x, x+\varepsilon_n]$. The (A_n) are decreasing and $\lim_{n\to\infty} A_n = \emptyset$, hence from the lemma $\lim_{n\to\infty} \mathbb{P}_X(A_n) = \mathbb{P}_X(\lim_{n\to\infty} A_n) = \mathbb{P}_X(\emptyset) = 0$. Since

$$\begin{aligned}\mathbb{P}_X(A_n) &= \mathbb{P}(x < X \leq x+\varepsilon_n) = \mathbb{P}(X \leq x+\varepsilon_n) - \mathbb{P}(X \leq x) \\ &= F(x+\varepsilon_n) - F(x),\end{aligned}$$

then $\lim_{n\to\infty} F(x+\varepsilon_n) = F(x)$.

Cumulative distribution function of a random variable

Proof.

4. Let $B_n =]-\infty, x_n]$ where the x_n is decreasing and converging to $-\infty$. We deduce that (B_n) is a decreasing sequence and $\lim B_n = \emptyset$, and according to the lemma

$$\lim_{n\to\infty} \mathbb{P}_X(B_n) = \mathbb{P}_X\left(\lim_{n\to\infty} B_n\right) = \mathbb{P}_X(\emptyset) = 0$$

or $\lim_{n\to\infty} \mathbb{P}_X(B_n) = \lim_{n\to\infty} \mathbb{P}_X(X \leq x_n) = \lim_{n\to\infty} F(x_n) = 0$.
Let us consider the sequence defined by $C_n =]-\infty, y_n]$ where (y_n) is an increasing real sequence such that $\lim_{n\to\infty} y_n = +\infty$. We deduce that the (C_n) are increasing and $\lim_{n\to\infty} C_n = \mathbb{R}$.
We have $\lim_{n\to+\infty} F(y_n) = \lim_{n\to+\infty} \mathbb{P}(X \leq y_n) = \lim_{n\to+\infty} \mathbb{P}_X(C_n) = \mathbb{P}_X(\lim_{n\to+\infty} C_n) = \mathbb{P}_X(\mathbb{R}) = 1$.

Support of a real random variable

We call the support of an r.v. X the set $X(\Omega)$. This support comes in several forms:

- If $X(\Omega)$ is finite or infinite countable X is said to be a discrete (or discontinuous) random variable, denoted d.r.v.
- If $X(\Omega)$ is infinite uncountable X is said to be a continuous random variable, denoted c.r.v.
  Moreover a c.r.v. is said to be absolutely continuous if it admits a continuous and derivable distribution function (except possibly at some points).

Discrete random variables

Definition

The random variable X is said to be discrete if it takes a finite or infinite countable number of values.

Notation: When the r.v. X takes the value x we write $\{X = x\}$ to describe the event $\{\omega \in \Omega, X(\omega) = x\}$.

Example

In the example of tossing a coin twice $X = 0$ correspond to the case where there is no tail P, this means that $\{X = 0\} = \{FF\}$. In the same way we have $\{X = 1\} = \{PF, FP\}$ and $\{X = 2\} = \{PP\}$.

Probability distribution of a discrete random variable

Definition

Let X be a d.r.v. one calls probability distribution or mass function of the r.v. X the application

$$\begin{aligned} p \quad : \quad & \mathbb{R} \longrightarrow [0,1] \\ & x \longmapsto p(x) = \mathbb{P}(X = x). \end{aligned}$$

Properties:

1.$\forall x \in \mathbb{R}, p(x) \geq 0;$

2. $\sum_{x \in \mathbb{R}} p(x) = 1.$

Example. When we throw a coin twice, we have
$p(0) = \mathbb{P}(X = 0) = \mathbb{P}(\{FF\}) = \frac{1}{4}$;
$p(1) = \mathbb{P}(X = 1) = \mathbb{P}(\{PF, FP\}) = \frac{2}{4} = \frac{1}{2}$;
$p(2) = \mathbb{P}(X = 2) = \mathbb{P}(\{PP\}) = \frac{1}{4}$.
The distribution is usually written in the following form

x	0	1	2	$\sum_{x=0}^{x=2} p(x)$
$\mathbb{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

Discrete random variables

Cumulative distribution function

1. If X is a discrete r.v. then

$$F_X(x) = \sum_{x_i \leq x} \mathbb{P}(X = x_i) = \sum_{x_i \leq x} p(x_i).$$

2. The cumulative distribution function allows to determine the probability law of the r.v. X.

Indeed, $\forall x_j \in X(\Omega)$

$$\mathbb{P}(X = x_j) = \sum_{i=1}^{j} \mathbb{P}(X = x_i) - \sum_{i=1}^{j-1} \mathbb{P}(X = x_i) = F_X(x_j) - F_X(x_{j-1}).$$

Example

We throw 3 dice and define the r.v. X as the number of 6 obtained.
The probability law of X is

x	0	1	2	3	$\sum_{x=0}^{x=2} p(x)$
$\mathbb{P}(X = x)$	$\frac{5^3}{6^3}$	$\frac{3 \cdot 5^2}{6^3}$	$\frac{3 \cdot 5}{6^3}$	$\frac{1}{6^3}$	1

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \frac{125}{216} & \text{if } 0 \leq x < 1 \\ \frac{200}{216} & \text{if } 1 \leq x < 2 \\ \frac{215}{216} & \text{if } 2 \leq x < 3 \\ 1 & \text{if } 3 \leq x \end{cases}$$

Definition

A real random variable X is said to be absolutely continuous if its cumulative distribution function $F_X(\cdot)$ satisfy the two following conditions:

1. F_X is continuous on $\mathbb{R}$;
2. F_X is derivable in every point $x \in \mathbb{R}$ except perhaps on a finite set D.

Continuous random variables

Theorem

Let X be an absolutely continuous random variable, with cumulative distribution function F_X, then for any pair $(a, b) \in \mathbb{R}^2$ such that $a < b$, we have

1. $\mathbb{P}(X = a) = 0.$
2. $\mathbb{P}(X \in]a, b]) = \mathbb{P}(X \in]a, b[) = \mathbb{P}(X \in [a, b[) = \mathbb{P}(X \in [a, b]) = F_X(b) - F_X(a).$
3. $\mathbb{P}(X \in]a, \infty[) = \mathbb{P}(X \in [a, \infty[) = 1 - F_X(a).$
4. $\mathbb{P}(X \in]-\infty, b]) = \mathbb{P}(X \in]-\infty, b[) = F_X(b).$

Continuous random variables

Definition

A real random variable X defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with cumulative distribution function F_X is said to be absolutely continuous random variable, if there exists a real function f_X satisfying the following conditions:

1. $f_X(x) \geq 0; \forall x \in \mathbb{R}$;
2. f_X is continuous on $\mathbb{R}$, except perhaps on a finite number of points where it has a finite left limit and finite right limit.
3. The integral $\int_{-\infty}^{+\infty} f_X(x)\,dx$ exists and is equal to 1.
4. The cumulative distribution function F_X can be written, for all $x \in \mathbb{R}$ in the form

$$F_X(x) = \int_{-\infty}^{x} f_X(s)\,ds.$$

Continuous random variables

Definition

A function f that satisfies the four previous conditions is called a probability density function or distribution function of an absolutely continuous random variable X.

Continuous random variables

Example

Let X be a random variable with cumulative distribution function F_X given by

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - \frac{1}{2}(x+2)e^{-\frac{x}{2}} & \text{if } x \geq 0 \end{cases}$$

1. Show that the random variable X is absolutely continuous.
2. Find the constant C such that the function f defined by

$$f(x) = \begin{cases} Cxe^{-\frac{x}{2}} & \text{if } \ x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

   be the probability density of the random variable X.
3. Verify that

$$F_X(x) = \int_{-\infty}^{x} f(s)\, ds.$$

Solution

1. F_X is continuous on $]-\infty, 0[$ and on $]0, +\infty[$ show that it is continuous in 0. We have $\lim_{x\to 0}\left(1-\frac{1}{2}(x+2)\,e^{-\frac{x}{2}}\right)=0$ hence F_X is continuous in 0.
F_X is derivable on $]-\infty, 0[$ and on $]0, +\infty[$ show that it is derivable in 0. We have $\lim_{x\to 0}\left(\frac{F_X(x)-F_X(0)}{x}\right)=0$
F_X is derivable on $\mathbb{R}$, hence X is an absolutely continuous variable.

Solution

2. To show that f is a density function we determine first the constant C using the condition 3 of the definition i.e. $\int_{-\infty}^{+\infty} f(x)\, dx = 1$, then we verify the other conditions.
We have

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)\, dx &= \int_0^{+\infty} Cxe^{-\frac{x}{2}} dx = C\left(\left[-2xe^{-\frac{x}{2}}\right]_0^{\infty} + 2\int_0^{+\infty} e^{-\frac{x}{2}} dx\right) \\ &= C\left[-4e^{-\frac{x}{2}}\right]_0^{\infty} = 4C = 1.\end{aligned}$$

hence $C = \frac{1}{4}$ and

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{4}e^{-\frac{x}{2}} & \text{if } x \geq 0 \end{cases}$$

We have $f_X(x) \geq 0; \forall x \in \mathbb{R}$.
It is a continuous fuction in 0 and then continuous on $\mathbb{R}$.
Then f is a probability density function of the random variable X.

Solution

3. If $x < 0$, $\int_{-\infty}^{x} f(s)\,ds = 0$ since on $]-\infty, 0[$, $F_X(x) = 0$
If $x \geq 0$,

$$\begin{aligned}
\int_{-\infty}^{x} f(s)\,ds &= \int_{-\infty}^{0} f(s)\,ds + \int_{0}^{x} f(s)\,ds = 0 + \int_{0}^{x} \frac{s}{4} e^{-\frac{s}{2}}\,ds \\
&= \frac{1}{4}\left(\left[-2se^{-\frac{s}{2}}\right]_0^x + 2\int_0^x e^{-\frac{s}{2}}\,ds\right) \\
&= \frac{1}{4}\left(-2xe^{-\frac{x}{2}} - 4\left[e^{-\frac{s}{2}}\right]_0^x\right) \\
&= \frac{1}{4}\left(-2xe^{-\frac{x}{2}} - 4e^{-\frac{x}{2}} + 4\right) = 1 - \frac{1}{2}(x+2)\,e^{-\frac{x}{2}}
\end{aligned}$$

hence $F_X(x) = \int_{-\infty}^{x} f(s)\,ds$.

Mathematical expectation and variance

Definition

Let X be a d.r.v. with possible values $x_1, x_2, \cdots$ and mass function $p(x)$. The mathematical expectation of X is

$$\mathbb{E}[X] = \sum_{i \geq 1} x_i p(x_i) = \sum_{i \geq 1} x_i \mathbb{P}(X = x_i)$$

provided that the above serie is absolutely convergent, otherwise we will say that X does not have a mathematical expectation.

Remark

If X has a finite number of values then $\mathbb{E}[X]$ exists.

Mathematical expectation and variance

Definition

Let X be a c.r.v. with distribution function f, the mathematical expectation of X is

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} xf(x)\,dx$$

provided that the above integral is absolutely convergent, otherwise we will say that X does not have a mathematical expectation.

Example

Let T be a c.r.v. with distribution function f defined by

$$f(t) = \begin{cases} \frac{1}{t^2} & \text{if } t > 1 \\ 0 & \text{elswhere} \end{cases}$$

Determine $\mathbb{E}[T]$.

Mathematical expectation and variance

Solution : We have

$$\begin{aligned}\int_{-\infty}^{+\infty} |tf(t)|\, dt &= \int_{1}^{+\infty} \left|\frac{1}{t}\right| dt \\ &= \lim_{x\to\infty} \int_{1}^{x} \left|\frac{1}{t}\right| dt = \lim_{x\to\infty} \log x - \log 1 = +\infty\end{aligned}$$

hence the expectation doesn't exist.

Definition

Let G be a function of a random variable X, the expectation of $G(X)$ is given by

$$\mathbb{E}[G(X)] = \begin{cases} \sum_{x\in\mathbb{R}} G(x)\, p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{+\infty} G(x)\, f(x)\, dx & \text{if } X \text{ is continuous} \end{cases}$$

provided that the above serie and integral are absolutely convergent.

Mathematical expectation and variance

Theorem

Let X be a random variable, then

1. $\mathbb{E}\left[c\right] = c$ *where* c *is a constant,*
2. $\mathbb{E}\left[\alpha H\left(X\right) + \beta G\left(X\right)\right] = \alpha\mathbb{E}\left[H\left(X\right)\right] + \beta\mathbb{E}\left[G\left(X\right)\right]$ *where* H *and* G *are functions of* X *and* α, β *are reals. Provided that the different expectations exist.*

Definition

Let X be a random variable, we call moment of order $k\left(k \in \mathbb{N}\right)$ the following value

$$\mathbb{E}\left[X^k\right] = \begin{cases} \sum_{x\in\mathbb{R}} x^k p\left(x\right) \text{ if } X \text{ is } \text{ discrete} \\ \int_{-\infty}^{+\infty} x^k f\left(x\right) dx \text{ if } X \text{ is continuous} \end{cases}$$

provided that the above serie and integral are absolutely convergent.

Mathematical expectation and variance

Definition

Let X be a random variable, the variance of X, noted σ_X^2 or $Var(X)$ is

$$\sigma_X^2 = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 .$$

We call standard deviation of X the number

$$\sigma_X = \sqrt{Var(X)}.$$

If $\mathbb{E}[X] = 0$ we say that the random variable is centrend.
If $Var(X) = 1$ we say that the random variable is reduced.

Mathematical expectation and variance

Theorem

Let X be a random variable with expectation $\mathbb{E}[X]$ and variance σ_X^2. If $Y = aX + b$ where a and b are real constants, then

$$\mathbb{E}[Y] = a\mathbb{E}[X] + b \quad \textit{and } \sigma_Y^2 = a^2\sigma_X^2.$$