Exercise 27

(a) We show that

$$(\neg((p_1 \wedge p_2) \wedge p_3)) \text{ eq } ((\neg(p_1 \wedge p_2)) \vee (\neg p_2)).$$

Set $\phi_1 := (\neg(p_1 \wedge p_2))$ and

$$\phi_2 := ((\neg p_1) \vee (\neg p_2))$$

By De Morgan law, we have

$$\phi_1 \text{ eq } \phi_2 .$$

Let S be the substitution

$$S : [(p_1 \wedge p_2)/p_1 \, , \, p_3/p_2] .$$

Then we have

$$\phi_1[S] = (\neg((p_1 \wedge p_2) \wedge p_3)) \text{ and}$$

$$\phi_2[S] = ((\neg(p_1 \wedge p_2)) \vee (\neg p_3)) .$$

By the Substitution Theorem, we get

$$\phi_1[S] \text{ eq } \phi_2[S] , \qquad (1)$$

which is the desired result.

Now we show that

$$((\neg(p_1 \wedge p_2)) \vee (\neg p_3)) \text{ eq } (((\neg p_1) \vee (\neg p_2)) \vee (\neg p_3)).$$

We expand the signature by adding the propositional symbol r, and we set

$$\phi = (r \vee (\neg p_3)).$$

Consider the substitutions :

$S_1 : \psi_1 / r$, where $\psi_1 = (\neg(p_1 \wedge p_2))$

$S_2 : \psi_1' / r$, where $\psi_1' = ((\neg p_1) \vee (\neg p_2))$.

We have, by De Morgan law, that

$$\psi_1 \text{ eq } \psi_1',$$

then, by the Replacement Theorem, we obtain

$$\phi[S_1] \text{ eq } \phi[S_2], \text{ that is}$$

$$(r \vee (\neg p_3))[(\neg(p_1 \wedge p_2))/r] \text{ eq } (r \vee (\neg p_3))[((\neg p_1) \vee (\neg p_2))/r]$$

$$\Leftrightarrow ((\neg(p_1 \wedge p_2)) \vee (\neg p_3)) \text{ eq } (((\neg p_1) \vee (\neg p_2)) \vee (\neg p_3)). \qquad (2)$$

From (1) and (2) and the transitivity of the relation eq, we deduce that

$$(\neg((p_1 \wedge p_2) \wedge p_3)) \text{ eq } ((\neg(p_1 \wedge p_2)) \vee (\neg p_3))$$

$$\text{eq } (((\neg p_1) \vee (\neg p_2)) \vee (\neg p_3)).$$

(b) We show, by induction on n, that for any formulae $\phi_1, \dots, \phi_n$ we have

$$(\neg(\cdots(\phi_1 \wedge \phi_2) \wedge \cdots) \wedge \phi_n))$$

$$\text{eq} \quad (\cdots((\neg\phi_1) \vee (\neg\phi_2)) \vee \cdots) \vee (\neg\phi_n)).$$

For $n=2$, this is just the De Morgan law.

We suppose that the property is true for n, and we prove it for $n+1$.

Let $\phi_1, \dots, \phi_{n+1}$ be any formulae. We introduce the propositional symbols r and s and we set

$$\chi_1 := (\neg(r \wedge s)) \text{ and } \chi_2 := ((\neg r) \vee (\neg s)).$$

We have, by De Morgan law,

$$\chi_1 \quad \text{eq} \quad \chi_2.$$

Consider the substitution defined by

$$S : [(\cdots(\phi_1 \wedge \phi_2) \wedge \cdots) \wedge \phi_n)/r,\ \phi_{n+1}/s]$$

By the substitution Theorem, we have

$$\chi_1[S] \quad \text{eq} \quad \chi_2[S],$$

that is,

$$(\neg((\cdots(\phi_1 \wedge \phi_2) \wedge \cdots) \wedge \phi_n) \wedge \phi_{n+1}))$$

$$\text{eq} \quad ((\neg(\cdots(\phi_1 \wedge \phi_2) \wedge \cdots) \wedge \phi_n)) \vee (\neg\phi_{n+1})).$$

Now, we introduce the propositional symbol t, and we set

$$\theta := (t \vee (\neg\phi_{n+1})).$$

Consider the following substitutions

$$S_1 : (\neg((\cdots(\phi_1 \wedge \phi_2) \wedge \cdots) \wedge \phi_n)) / t$$

$$S_2 : (\cdots((\neg\phi_1) \vee (\neg\phi_2)) \vee \cdots) \vee (\neg\phi_n)) / t.$$

By the induction hypothesis, we can apply the Replacement Theorem, so that

$$\theta[S_1] \quad eq \quad \theta[S_2],$$

It follows that

$$(\neg(\cdots(\phi_1 \wedge \phi_2) \wedge \cdots) \wedge \phi_{n+1}))$$

$$eq \quad (\cdots((\neg\phi_1) \vee (\neg\phi_2)) \vee \cdots) \vee (\neg\phi_{n+1})).$$

This completes the proof.

For part (c), see the reference book.

Exercise 30

e) We construct the truth table of the given formula

$\neg$	(p	$\wedge$	q)	$\rightarrow$	(q	$\leftrightarrow$	r) $:= \phi$
F	T	T	T	T	T	T	T
F	T	T	T	T	T	F	F
T	T	F	F	F	F	F	T
T	T	F	F	T	F	T	F
T	F	F	T	T	T	T	T
T	F	F	T	F	T	F	F
T	F	F	F	F	F	F	T
T	F	F	F	T	F	T	F

$\Uparrow$

From the proof of Post's Theorem, the formula in DNF which is logically equivalent to ϕ is

$$\phi^{DNF} = (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r).$$

(this corresponds to the σ-structures where ϕ is true.)

To find ϕ^{CNF}, notice that $(\neg\phi)$ is true if and only if ϕ is false, and set $\theta : (\neg\phi)^{DNF}$. Then

$$\theta \text{ eq } \neg\phi$$

So

$$\neg\theta \text{ eq } \neg\neg\phi \text{ eq } \phi$$

We have

$$\theta = (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$$

Then

$$\neg\theta \text{ eq } (\neg p \vee q \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee \neg r)$$

which is a formula in CNF that is logically equivalent to ϕ.

j) Using successively the relation

$$(\phi \to \Psi) \text{ eq } ((\neg\phi) \vee \Psi),$$

we obtain

$$p_1 \to (p_2 \to (p_3 \to p_4))$$

$$\text{eq } p_1 \to (p_2 \to (\neg p_3 \vee p_4))$$

$$\text{eq } p_1 \to (\neg p_2 \vee (\neg p_3 \vee p_4))$$

$$\text{eq } p_1 \to (\neg p_2 \vee \neg p_3 \vee p_4)$$

$$\text{eq } (\neg p_1 \vee \neg p_2 \vee \neg p_3 \vee p_4)$$

this last formula is in DNF.

It is also in CNF.

Exercise 31

(a) Set $\phi := (p_1 \wedge \neg p_2 \wedge p_3 \wedge \neg p_4) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4)$.

Using commutativity and associativity properties, we obtain

$$(p_1 \wedge \neg p_2 \wedge p_3 \wedge \neg p_4) \text{ eq } (p_1 \wedge \neg p_2 \wedge \neg p_4 \wedge p_3)$$

and

$$(p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4) \text{ eq } (p_1 \wedge \neg p_2 \wedge \neg p_4 \wedge \neg p_3).$$

So

$$\phi \text{ eq } (p_1 \wedge \neg p_2 \wedge \neg p_4 \wedge p_3) \vee (p_1 \wedge \neg p_2 \wedge \neg p_4 \wedge \neg p_3).$$

(use Replacement Theorem with $(r \vee s)$)

By distributivity and transitivity, we get

$$\phi \text{ eq } ((p_1 \wedge \neg p_2 \wedge \neg p_4) \wedge (p_3 \vee \neg p_3)).$$

Since $(p_3 \vee \neg p_3)$ is a tautology, we deduce

$$\phi \text{ eq } (p_1 \wedge \neg p_2 \wedge \neg p_4).$$

(b) From part (a), we have

$$(p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge p_4) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4) \vee (p_1 \wedge p_2 \wedge p_3 \wedge \neg p_4)$$

$$\text{eq } (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge p_3 \wedge \neg p_4)$$

which is a shorter formula in DNF that is logically equivalent to the given formula.

Exercise 32

(a) A formula in DNF has a model if and only if at least one of its disjuncts is satisfiable. Since $(\neg p_1 \wedge p_3)$ is obviously satisfiable, we can take as a model for the formula in (i) the σ-structure given by $A(p_1)=F$, $A(p_2)=T$ and $A(p_3)=T$.

There is no model for the formula in (ii) because its two disjuncts are not satisfiable. Indeed, each disjunct contains p_i and $\neg p_i$ for some i.

(b) We can suggest the following instructions:

1) If each disjunct contains p_i and $\neg p_i$ for some i, then the formula has no model.

2) Otherwise, choose a disjunct that doesn't contain p_i and $\neg p_i$ for some i. Then a model for the formula is given by the σ-structure defined by:

$A(p_i)=T$ if the formula contains p_i
$A(p_i)=F$ if the formula contains $\neg p_i$
$A(p_i)=T$ otherwise