Mathematical analysis 2
Chapter 1: Multivariable and vectorial functions
Part 2: Limits and continuity

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Plan

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Numerical function

Definition

- *We call* ***numerical function*** *of* n *variables the mapping* f *defined by*
$$\begin{array}{rcl} f: D \subseteq \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ (x_1, \cdots x_n) & \longmapsto & f(x_1, \cdots x_n) \end{array}$$
- *We call* ***domain*** *of* f *and we note* D_f *the set of all* $X \in \mathbb{R}^n$ *such that* f *exists, that is*
$$D_f = \{X \in \mathbb{R}^n : f(X) \text{ exists}\}.$$
- *We call* ***range*** *of* f *the set of values that f takes on* D_f*, that is*
$$\{f(X) \text{ such that } X \in D_f\}$$
- *We call* ***graph*** *of a function* f *of* n *variables the set*
$$\{(X, f(X)) \text{ such that } X \in D_f\}$$

Numerical functions: example

Remark

Let $f : \mathbb{R}^n \to \mathbb{R}$. $X \mapsto f(X)$. $X \in \mathbb{R}^n$ *is called* ***independent variables*** *and* $f(X)$ ***dependent variable***.

Example.

Let us determine the domain of the following functions

$$f_1 : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & f_1(x,y) = \dfrac{xy}{x^2+y^2} \end{array} \qquad f_2 : \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ (x,y,z) & \longmapsto & f_2(x,y,z) = \dfrac{\ln(z)}{x+y} \end{array}$$

- $D_{f_1} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 0\} \Longrightarrow D_{f_1} = \mathbb{R}^2 \setminus (0,0)$.
- $D_{f_2} = \{(x,y,z) \in \mathbb{R}^3 : x+y \neq 0, z > 0\} \Rightarrow D_{f_2} = \{(x,y,z) \in \mathbb{R}^3 : y \neq x, z > 0\}$.

Level curves

Definition

*The **level curves** of a function f of two variables are the curves with equations $f(x,y)=k$, where k is a constant.*

Plan

Limit of numerical function of several variables

Definition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $X = (x_1, \cdots, x_n) \mapsto f(X)$ *a function and* $a = (a_1, \cdots a_n)$ *an accumulation point of* D. *We say that the limit of* f *as* X *approches* a is L *if*

$$\forall \varepsilon > 0, \quad \exists \delta > 0 \text{ such that } (\forall X \in D \text{ and } ||X - a|| < \delta) \implies |f(X) - L| < \varepsilon$$

Then we write

$$\lim_{X \to a} f(X) = L$$

Remark

The limit of a function doesn't depend on the chosen norm because on $\mathbb{R}^n$ *all norms are equivalent.*

Limit of numerical function of several variables

Proposition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, a function and a an accumulation point of D. If limit of f at a exists then this limit is unique.

Proposition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ a function and a an accumulation point of D. We have the following equivalence

$$\lim_{X \to a} f(X) = L \Leftrightarrow \lim_{X \to a} |f(X) - L| = 0$$

Limit of numerical function of several variables

Proposition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ *a function and* a *an accumulation point of* D.

- $\lim\limits_{X \to a} f(X) = +\infty \Leftrightarrow \forall M > 0, \exists \delta > 0 : (\forall X \in D : ||X - a|| < \delta) \Rightarrow f(X) > M.$
- $\lim\limits_{X \to a} f(X) = -\infty \Leftrightarrow \forall M > 0, \exists \delta > 0 : (\forall X \in D : ||X - a|| < \delta) \Rightarrow f(X) < -M.$

Properties of limit of numerical function of several variables

Proposition

Let f *et* $g : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ *and* a *an accumulation point de* D. *Suppose that* $\lim_{X\to a} f(X) = l_1$ *and* $\lim_{X\to a} g(X) = l_2$. *Then*

- $\forall \alpha, \beta \in \mathbb{R}$, $\lim_{X\to a} f(X)(\alpha f + \beta g)(X) = \alpha l_1 + \beta l_2$.
- $\lim_{X\to a} f(X)(f.g)(X) = l_1 \cdot l_2$.
- *If* $l_2 \neq 0$, $\lim_{X\to a} f(X)\left(\frac{f}{g}\right)(X) = \frac{l_1}{l_2}$.

Different methods to find limit

- **Subtitution** when the limit is trivial to calculate we plug in directly the the given value.

Example.

Hier we just have to plug in the given points

- $\lim_{(x,y)\to(0,0)} e^{x^2+y^2} = e^{0+0} = 1$
- $\lim_{(x,y)\to(\pi,\pi)} \frac{x^2\cos y}{x+y} = \frac{-\pi^2}{2\pi} = -\frac{\pi}{2}$

Different methods to find limit

- **Path approach** We use generally this method to show that the limit doesn't exist

Proposition

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ *and* (a,b) *an accumulation point of* D*. If there exist tow continuous functions* $y = \varphi_1(x)$, $y = \varphi_2(x)$ *that pass through* (a,b) *such that*

$$\lim_{x\to a} f(x, \varphi_1(x)) \neq \lim_{x\to a} f(x, \varphi_2(x))$$

then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ *doesn't exist.*

Different methods to find limit

Example.

Let f a function defined by $f(x,y)=\frac{x^2}{y}$. find $\lim_{(x,y)\to(0,0)} f(x,y)$

- If we set $y=x^2$ then $\lim_{(x,y)\to(0,0)} f(x,y)=1$
- if we set $y=2x^2$ then $\lim_{(x,y)\to(0,0)} f(x,y)=\frac{1}{2}$

So, we have found two different paths such that we have two different limits; therefore, the limit does not exist.

Different methods to find limit

- **Double limit**

Proposition

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ *such that* $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$. *Suppose that*

- $\forall x \in \mathbb{R}$, $\lim\limits_{y\to b} f(x,y)$ *exist.*
- $\forall y \in \mathbb{R}$, $\lim\limits_{x\to a} f(x,y)$ *exist.*

Then $\lim\limits_{y\to b}\left(\lim\limits_{x\to a} f(x,y)\right) = \lim\limits_{x\to a}\left(\lim\limits_{y\to b} f(x,y)\right)$

Different methods to find limit

Example.

Let find $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin x^2 \cos y}{x^2+y^2}$

$$\lim_{y\to0}\lim_{x\to0}\frac{\sin x^2 \cos y}{x^2+y^2} = \lim_{y\to0}\frac{0}{y^2} = 0 = l_1$$

$$\lim_{x\to0}\lim_{y\to0}\frac{\sin x^2 \cos y}{x^2+y^2} = \lim_{x\to0}\frac{\sin x^2}{x^2} = 1 = l_2$$

Since $l_1 \neq l_2$ *we deduce that* $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin x^2 \cos y}{x^2+y^2}$ *doesn't exist.*

Different methods to find limit

- **Sandwich approach** by using squeeze theorem

Theorem

*(**Squeeze theorem**) Let* f *et* $g : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ *and* a *an accumulation point of* D*. If*

- $\forall X \in v(a) \cap D, f(X) \leq h(X) \leq g(X)$
- $\lim\limits_{X\to a} f(X) = \lim\limits_{X\to a} g(X) = l$

Then $\lim\limits_{X\to a} h(X) = l$.

Corollaire

Let f *et* $g : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ *and* a *an accumulation point of* D*. If*

- $\forall X \in v(a) \cap D, |f(X)| \leq |g(X)|$
- $\lim\limits_{X\to a} g(X) = 0$

Then $\lim\limits_{X\to a} f(X) = 0$.

Different methods to find limit

Example.

Let $f(x,y) = (x+y)\cos\left(\frac{1}{x^2+y^2}\right)$, $\lim_{(x,y)\to(0,0)} f(x,y)$?

We have

$$-1 \leq \cos\left(\frac{1}{x^2+y^2}\right) \leq 1 \Longrightarrow -(x+y) \leq (x+y)\cos\left(\frac{1}{x^2+y^2}\right) \leq (x+y).$$

Since $(x,y) \to (0,0)$ *then*

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Different methods to find limit

- **Change of variable**

Example.

Let $f(x,y)=\frac{|x-1||y-2|}{|x-1|+|y-2|}$. *We want to find* $\lim_{(x,y)\to(1,2)} f(x,y)$ *We use the change of variable :* $x'=x-1$ *et* $y'=y-2$. *Then when* $(x,y)\to(1,2)$, $(x',y')\to(0,0)$. *We have,*

$$\begin{cases} |x'| & \leq |x'|+|y'| \\ |y'| & \leq |x'|+|y'| \end{cases} \Rightarrow |x'||y'| \leq (|x'|+|y'|)^2$$

$$f(x,y)=\frac{|x'|\,|y'|}{|x'|+|y'|} \leq |x|+|y| \to 0$$

Different methods to find limit

Example.

Let $f(x,y) = \dfrac{x^3 y}{x^2+y^2}$. $\lim\limits_{(x,y)\to(0,0)} f(x,y)$?

Using polar coordinates and set $x = r\cos(\theta)$ *and* $y = r\sin(\theta)$ *with* $r \geq 0$ *and* $\theta \in [0, 2\pi[$. *Then,*

$$|f(x,y)| = \frac{|r^4\cos(\theta)^3\sin(\theta)\,|}{r^2} = r^2|\cos(\theta)|^3|\sin(\theta)| \leq r^2 \to 0$$

Then we get $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0.$

Different methods to find limit

Example.

Let $f(x,y) = \dfrac{xy}{x^2+y^2}$. $\lim\limits_{(x,y)\to(0,0)} f(x,y)$?

Using polar coordinates and set $x = r\cos(\theta)$ *et* $y = r\sin(\theta)$ *with* $r \geq 0$ *and* $\theta \in [0, 2\pi[$. *Then we have,*

$$f(x,y) = \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta.$$

Then if $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ *exists, it will be equal to*

$$\lim_{r\to 0}\cos\theta\sin\theta = \cos\theta\sin\theta$$

Then it is not unique (since it depends on the value of θ *).*
Consequently $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ *doesn't exist.*

Different methods to find limit

- **Taylor expansion**

Example.

Let $f(x,y)=\dfrac{x(\sin y-y)}{x^2+y^2}$. *We have* $\sin(y)=y-y^2\varepsilon(1)$ *with* $\lim\limits_{y\to 0}\varepsilon(1)=0$, *then*

$$0\leq|f(x,y)|=\frac{\left|xy^2\varepsilon(1)\right|}{x^2+y^2}\leq|x\varepsilon(1)|.$$

We have also $x\varepsilon(1)\to 0$ *when* $(x,y)\to(0,0)$ *then by squeeze theorem we get* $\lim_{(x,y)\to(0,0)}f(x)=0$.

Plan

Continuity of numerical function of several variables

Definition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ *and* $a = (a_1, \cdots a_n) \in D$.

- *We say that* f *is* ***continuous*** *at the point* a *if an only if* $\lim_{X \to a} f(X) = f(a)$.

  That is to say

  $\forall \varepsilon > 0, \quad \exists \delta > 0$ *such that* $(\forall X \in D$ *and* $\|X - a\| < \delta) \implies |f(X) - f(a)| < \varepsilon$
- *We say that* f *is continuous on* D *if* f *is continuous at all points in* D.

Definition

Let $f : D \subseteq \mathbb{R}^n$ *and* $a = (a_1, \cdots a_n) \in \mathbb{R}^n$. *We say that* f *is continuous at* a *if the three conditions are satisfied*

- f *is defined at* a $(a \in D)$.
- $\lim_{X \to a} f(X)$ *exists and is finite.*
- $\lim_{X \to a} f(X) = f(a)$.

Properties of continuous functions

Proposition

Let $f, g : D \subseteq \mathbb{R}^n \to \mathbb{R}$ *and* $a \in D$. *Suppose that* f, g *are continuous at* a *than*

- $\forall \alpha, \beta \in \mathbb{R}$, $(\alpha f + \beta g)$ *is continous at* a.
- $f.g$ *is continous at* a.
- *If* $g(a) \neq 0$, $\dfrac{f}{g}$ *is continuous at* a.

Proposition

Let $f : D \subseteq \mathbb{R}^n \to A \subseteq \mathbb{R}$ *and* $g : A \subseteq \mathbb{R} \to \mathbb{R}$. *Suppose that* f *is continuous at* a *and* g *is continuous at* $f(a)$. *Then the function* $f \circ g : D \subseteq \mathbb{R}^n \to \mathbb{R}$ *is continuous at* a.

Example

Example.

We want to study the continuity on $\mathbb{R}^2$ *of*

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Solution : On $\mathbb{R}^2 - \{(0.0)\}, f$ *est continous since it is the quotient of tow polynomials. At* (0.0)*,We have*

$$x^2 \leq x^2 + y^2 \Rightarrow \frac{x^2}{x^2+y^2} \leq 1$$

Then

$$\forall (x,y) \in \mathbb{R}^2_*, \quad f_1(x,y) = \frac{x^2y^2}{x^2+y^2} = y^2 \frac{x^2}{x^2+y^2} \leq y^2 \to 0.$$

Consequentely

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Conclusion : f *is continous at* $(0,0)$ *then on* $\mathbb{R}^2$.

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Vectorial functions of several variables

Definition

(Vectorial function)

- *We call **vectorial function** of n varables the mapping f defined by*

$$\begin{array}{rcl} f : D \subseteq \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \ \ (m \geq 2) \\ X = (x_1, \cdots x_n) & \longmapsto & f(X) = (f_1(X), \cdots, f_m(X)) \end{array}$$

- *We call **domain** of f and we note D_f the set given by*

$$D_f = \bigcap_{1 \leq j \leq m} D_{f_j} = \{X \in \mathbb{R}^n : f_j \ exist, \ \forall 1 \leq j \leq m\}.$$

Vectorial functions: Example

Example.

$$\begin{aligned} f : D \subseteq \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto f(x,y) = \left(\tfrac{xy}{x^2+y^2}, \sqrt{1-x^2-y^2}\right). \end{aligned}$$

We have

$\bullet f_1(x,y) = \dfrac{xy}{x^2+y^2}$ $\qquad D_{f_1} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 0\} = \mathbb{R}^2 \setminus (0,0)$

$\bullet f_2(x,y) = \sqrt{1-x^2-y^2}$

$$\begin{aligned} D_{f_2} &= \{(x,y) \in \mathbb{R}^2 : 1-x^2-y^2 \geq 0\} \\ &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\} \\ &= \bar{\mathscr{D}}((0,0),1) \end{aligned}$$

$$D_f = D_{f_1} \cap D_{f_2} = \bar{\mathscr{D}}((0,0),1) \setminus (0,0).$$

Plan

Limit of vectorial function of several variables

Definition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *a function and* $a = (a_1, \cdots a_n)$ *an accumulation point of* D. *We say that the limit of* f *as* X *approches* a *is* $L = (l_1, \cdots, l_2)$ *if*

$$\forall \varepsilon > 0, \quad \exists \delta > 0 \text{ such that } (\forall X \in D \text{ and } ||X - a|| < \delta) \implies ||f(X) - L|| < \varepsilon$$

Then we write

$$\lim_{X \to a} f(X) = L$$

Proposition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *a function and* $a = (a_1, \cdots a_n)$ *an accumulation point of* D. *Suppose that* $L = (l_1, \cdots, l_2)$ *than we have*

$$\lim_{X \to a} f(X) = L \iff \forall i = 1, \cdots, m, \ \lim_{X \to a} f_i(X) = l_i$$

Limit of vectorial functions: Example

Example.

Let

$$f(x,y) = \left(xy\log\left(x^2+y^2\right), \frac{\sin xy}{y}, \frac{e^{x^2y}-1}{xy} \right).$$

Give the domaine of definition of f *and find* $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

Solution

1) Domaine of definition of f :
- $f_1(x,y) = xy\log\left(x^2+y^2\right), D_{f_1} = \left\{(x,y) \in \mathbb{R}^2 / x \neq 0 \vee y \neq 0\right\} = \mathbb{R}^2_*$.
- $f_2(x,y) = \frac{\sin xy}{y}, D_{f_2} = \left\{(x,y) \in \mathbb{R}^2 / y \neq 0\right\} = \mathbb{R} \times \mathbb{R}^*$.
- $f_3(x,y) = \frac{e^{x^2y}-1}{xy}, D_{f_3} = \left\{(x,y) \in \mathbb{R}^2 / x \neq 0 \wedge y \neq 0\right\} = \mathbb{R}^* \times \mathbb{R}^*$. Then,
$D_f = D_{f_1} \cap D_{f_2} \cap D_{f_3} = \mathbb{R}^* \times \mathbb{R}^*$.

Solution

2) Let's find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$:

We know that $\begin{cases} |x| \leq \sqrt{x^2+y^2} \\ |y| \leq \sqrt{x^2+y^2} \end{cases} \Rightarrow |x||y| \leq x^2+y^2$

Then $\forall (x,y) \in \mathbb{R}^* \times \mathbb{R}^*$ $\left|f_1(x,y)\right| = \left|xy\log\left(x^2+y^2\right)\right| \leq \left(x^2+y^2\right)\left|\log\left(x^2+y^2\right)\right|$
Since $\lim\limits_{(x,y)\to(0,0)} \left(x^2+y^2\right)\left|\log\left(x^2+y^2\right)\right| = 0$, then from squeeze theorem we have

$$\lim_{(x,y)\to(0,0)} f_1(x,y) = 0$$

.

Solution

$$\lim_{(x,y)\to(0,0)} f_2(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin xy}{y} = \lim_{(x,y)\to(0,0)} x\frac{\sin xy}{xy} = 0.$$

- $$\lim_{(x,y)\to(0,0)} f_3(x,y) = \lim_{(x,y)\to(0,0)} \frac{e^{x^2y}-1}{xy} = \lim_{(x,y)\to(0,0)} x\frac{e^{x^2y}-1}{x^2y} = 0.$$

On en conclut : $\lim_{(x,y)\to(0,0)} f(x,y) = (0,0,0)$.
Note that

$$\lim_{u\to 0^+} u\log u = 0, \quad \lim_{u\to 0}\frac{e^u-1}{u} = 1, \quad \lim_{u\to 0}\frac{\sin u}{u} = 1$$

Properties of continuoues vectorial functions

Proposition

Let $f, g : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *and* a *an accumulation point of* D. *Suppose that* $\lim\limits_{X \to a} f(X) = l_1$ *and* $\lim\limits_{X \to a} g(X) = l_2$ *then*

- $\forall \alpha, \beta \in \mathbb{R}, \lim\limits_{X \to a} (\alpha f + \beta g) = \alpha l_1 + \beta l_2$

Remark

Product and quotion of vectorial functions are not defined.

Plan

Continity of vectorial function of several variables

Definition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *and* $a = (a_1, \cdots a_n) \in D$.

- *We say that* f *is* ***continuous*** *at the point* a *if an only if* $\lim_{X \to a} f(X) = f(a)$.

  That is to say

  $$\forall \varepsilon > 0, \quad \exists \delta > 0 : (\forall X \in D : 0 < ||X - a|| < \delta) \implies ||f(X) - f(a)|| < \varepsilon$$

- *We say that* f *is continuous on* D *if* f *is continuous at all points in* D.

Proposition

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *and* $a = (a_1, \cdots a_n) \in D$. *We say that* f *is continuous at* a *if and only if all* f_i, $i = 1, \cdots, m$ *are continuous at* a. *That is to say*

$$f \text{ continuous at } a \iff \forall i = 1, \cdots, m \quad f_i \text{ continuous at } a$$

Properties of Continity vectorial function of several variables

Proposition

Let $f, g : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *and* $a \in D$. *Suppose that* f, g *are continuous at* a *than*

- $\forall \alpha, \beta \in \mathbb{R}$, $(\alpha f + \beta g)$ *is continous at* a.

Proposition

Let $f : D \subseteq \mathbb{R}^n \to A \subseteq \mathbb{R}^m$ *and* $g : A \subseteq \mathbb{R}^m \to \mathbb{R}^p$. *Suppose that* f *is continuous at* a *and* g *is continuous at* $f(a)$. *Then the function* $f \circ g : D \subseteq \mathbb{R}^n \to \mathbb{R}^p$ *is continuous at* a.