Mathematical analysis 2
Chapter 3 : Numerical series

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Course outline

Generalities

Definition

- *Let (u_n) be a sequence of real numbers. The expression*

$$u_0 + u_1 + \cdots + u_n + \cdots$$

  *is called **numerical series** of **general term** u_n.*

- *A series of general term u_n is denoted by $\sum_{n=0}^{+\infty} u_n$, $\sum_{n\geq 0} u_n$ or simply $\sum u_n$.*

Definition

- *The sum of the n first terms of the series is denoted by S_n and is called **partial sum***

$$S_n = u_0 + u_1 + \cdots + u_n = \sum_{k=0}^{n} u_k.$$

- *The sequence (S_n) is called **sequence of partial sum**.*

Course outline

Convergence of a series

Definition

- *A series of general term u_n is said to be* ***convergent*** *to S if the sequence of partial sum (S_n) is convergent. In this case we have*

$$S = \lim_{n\to\infty} S_n = \sum_{n=0}^{+\infty} u_n.$$

- *S is called* ***the sum of the series*** *and we have*

$$\sum u_n \text{ converges to } S \iff \lim_{n\to\infty} S_n = S$$

- *A series that is not convergent is called* ***divergent****.*

Remark

The nature of a series is by definition its convergence or divergence.

Geometric series

Example.

Let (u_n) be a geometric series with the first term $u_0 = a \neq 0$ and common ratio q. The general term is given by

$$u_n = aq^n \quad (a \neq 0).$$

The partial sum is given by

$$S_n = \begin{cases} a\left(\dfrac{1-q^{n+1}}{1-q}\right), & q \neq 1 \\ a(n+1), & q = 1 \end{cases}$$

Question. *When does a geometric series $\sum_{n=0}^{+\infty} aq^n$ converge?*

Geometric series

We have

$$S = \lim_{n\to\infty} S_n = \begin{cases} \dfrac{a}{1-q}, & \text{if } |q| < 1 \\ \text{The limit doesn't exist} & \text{if } q \leq -1 \\ \infty & \text{if } q \geq 1. \end{cases}$$

Consequently, the **geometric series**

- **Converges** if $|q| < 1$.
- **Diverges** if $|q| \geq 1$.

Telescopic series

Example.

Let $\sum_{n=1}^{+\infty} u_n$ *be the series defined by the general term*

$$u_n = \frac{1}{n(n+1)}, \ n \geq 1.$$

Telescopic series

Example.

Let $\sum_{n=1}^{+\infty} u_n$ *be the series defined by the general term*

$$u_n = \frac{1}{n(n+1)}, \; n \geq 1.$$

By decomposition to simple elements we can write the general term as follows

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Hence

$$\begin{aligned} S_n &= u_1 + u_2 + \cdots + u_n \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Telescopic series

Then

$$S_n = 1 - \frac{1}{n+1}.$$

And we have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (1 - \frac{1}{n+1}) = 1.$$

Therefor the series $\sum_{n\geq 1} \frac{1}{n(n+1)}$ converges to 1.

Telescopic series

Example.

Let $\sum_{n=1}^{+\infty} u_n$ *be the series defined by the general term*

$$u_n = \ln(1 + \frac{1}{n}), \ n \geq 1.$$

Telescopic series

Example.

Let $\sum_{n=1}^{+\infty} u_n$ *be the series defined by the general term*

$$u_n = \ln(1+\frac{1}{n}),\ n \geq 1.$$

We have

$$\forall n \geq 1 : \qquad \ln\left(1+\frac{1}{n}\right) = \ln(n+1) - \ln(n).$$

Then

$$S_n = (\ln(2)-\ln(1))+(\ln(3)-\ln(2))+\cdots+(\ln(n+1)-\ln(n)) = \ln(n+1)-\ln(1) = \ln(n+1)$$

The partial sum sequence is divergent then $\sum_{n=1}^{+\infty} \ln(1+\frac{1}{n})$ diverges.

Course outline

Divergence Test

Proposition

If $\lim\limits_{n\to\infty} u_n \neq 0$ *or* $\lim\limits_{n\to\infty} u_n$ *doesn't exist, then the series* $\sum u_n$ *diverges.*

Example.

The series $\sum\limits_{n\geq 0} \dfrac{n}{n+1}$ *is divergent since* $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1 \neq 0$

⚠ The divergence test provides a way of proving that a series diverges but there exist divergent series witch the general term go to zero.

Example.

Harmonic series $\sum\limits_{n=1}^{+\infty} \dfrac{1}{n}$ *is divergent but* $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0.$

Course outline

1. Generalities
   - Convergence of a series
   - Divergence Test
   - Propriétés et opérations sur les séries
2. Positive term series
   - Convergence criteria for Positive terms series
3. Arbitrary term series
4. Alternating series

Proposition

If the series $\sum u_n$ and $\sum v_n$ differ only for a finite number of terms, then the two series are of the same nature.

Remark

The nature of a series remains unchanged when adding or subtracting a finite number of terms.

Proposition (Operations on series)

Let $\sum u_n$ and $\sum v_n$ be two series convergent respectively to S and L then

1. *The series $\sum(u_n + v_n)$ is convernent to $S + L$ and we have*
$$\sum_{n=0}^{+\infty}(u_n + v_n) = \sum_{n=0}^{+\infty} u_n + \sum_{n=0}^{+\infty} v_n = S + L$$
2. *For all $\alpha \in \mathbb{R}$ the series $\sum(\alpha u_n)$ converges to (αS) and we have*
$$\sum_{n=0}^{+\infty}(\alpha u_n) = \alpha \sum_{n=0}^{+\infty} u_n = \alpha S.$$

Important remark

Remark

In the cases:

1. *If* $\sum u_n$ *is convergent and* $\sum v_n$ *is divergent then the series* $\sum (u_n + v_n)$ *is divergent.*
2. *If* $\sum u_n$ *and* $\sum v_n$ *diverge, their sum* $\sum (u_n + v_n)$ *is not necessary divergent.*

Example.

Study the nature of the series

$$\sum_{n=1}^{+\infty}\left(\frac{3}{2^n}+\frac{2}{n(n+1)}\right)$$

Example.

Study the nature of the series

$$\sum_{n=1}^{+\infty}\left(\frac{3}{2^n}+\frac{2}{n(n+1)}\right)$$

Solution. *We have*

- $\displaystyle\sum_{n=1}^{+\infty}\frac{1}{2^n}$ *is a geometric series with common ratio* $q=\frac{1}{2}\in]-1,1[$ *then it converges to* $\dfrac{1/2}{1-1/2}=1$.
- *The series* $\displaystyle\sum_{n=1}^{+\infty}\frac{1}{n(n+1)}$ *converges to* 1.

Therefore

$$\sum_{n=1}^{+\infty}\left(\frac{3}{2^n}+\frac{2}{n(n+1)}\right)=\sum_{n=1}^{+\infty}\frac{3}{2^n}+\sum_{n=1}^{+\infty}\frac{2}{n(n+1)}=3\sum_{n=1}^{+\infty}\frac{1}{2^n}+2\sum_{n=1}^{+\infty}\frac{1}{n(n+1)}=3+2=5.$$

Then the series $\displaystyle\sum_{n=1}^{+\infty}\left(\frac{3}{2^n}+\frac{2}{n(n+1)}\right)$ *is convergent to* 5.

Example.

The series $\sum \frac{1}{n(n+1)}$ *is* ***convergent****, even*

$$\sum \frac{1}{n(n+1)} = \sum \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

Where $\sum \frac{1}{n}$ ***diverges*** *and* $\sum \frac{1}{n+1}$ ***diverges*** *also.*

Course outline

Positive term series

Definition

A series $\sum u_n$ *is said to be a positive term series if* $u_n \geq 0 \ \forall n \geq n_0$; $n_0 \in \mathbb{N}$

Example.

The series $\displaystyle\sum_{n=1}^{+\infty} \frac{n+2}{n^2}$ *is a positive terms since:* $\forall n \geq 1, \ \dfrac{n+2}{n^2} \geq 0$

Remark

If a series $\sum u_n$ *is a positive term series then the sequence of partial sum* $(S_n)_n$ *is increasing.*

Positive term series

Proposition

Let $\sum u_n$ *be a positive term series*

$$\sum u_n \text{ converges } \iff (S_n) \text{ is upper bounded.}$$

Example.

Let's consider the positive term series $\sum \frac{1}{n(n+1)}$.

Positive term series

Proposition

Let $\sum u_n$ *be a positive term series*

$$\sum u_n \text{ converges } \iff (S_n) \text{ is upper bounded.}$$

Example.

Let's consider the positive term series $\sum \frac{1}{n(n+1)}$ *.We have*

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}.$$

For all $n \geq 1$*:* $S_n \leq 1$ *then* (S_n) *is upper bounded, therefore* $\sum \frac{1}{n(n+1)}$ *is convergent.*

Course outline

Comparison test

Theorem

Let $\sum u_n$ et $\sum v_n$ be two positive term series such that for all $n \geq n_0$, $n_0 \in \mathbb{N}$, we have

$$0 \leq u_n \leq v_n$$

Then

1. *If $\sum v_n$ converges $\Longrightarrow \sum u_n$ converges.*
2. *If $\sum u_n$ diverges $\Longrightarrow \sum v_n$ diverges.*

Example.

Study the nature of the series $\sum_{n=0}^{+\infty} \frac{|\cos n|}{5^n}$ using the comparison test.

Comparison test

In one hand $\sum_{n=0}^{+\infty} \frac{|\cos n|}{5^n}$ is a positive term series since

$$\frac{|\cos n|}{5^n} \geq 0 \quad \forall n \in \mathbb{N}$$

In the other hand we have

$$\forall n \in \mathbb{N}, \quad |\cos n| \leq 1 \implies \frac{|\cos n|}{5^n} \leq \frac{1}{5^n}.$$

In this case we choose $v_n = \frac{1}{5^n}$, wich is a convergent geometric series (since $q = 1/5 \in]-1, 1[$), Consequently the series $\sum_{n=0}^{+\infty} \frac{|\cos n|}{5^n}$ is convergent.

Comparison test

Example.

Study the nature of the series of general term $u_n = \dfrac{3+\sin(\ln n)}{n}$

Comparison test

Example.

Study the nature of the series of general term $u_n = \dfrac{3+\sin(\ln n)}{n}$

We have for all $n \geq 1$:

$$-1 \leq \sin(\ln n) \leq 1$$

$$2 \leq 3+\sin(\ln n) \leq 4$$

$$\frac{2}{n} \leq \frac{3+\sin(\ln n)}{n} \leq \frac{4}{n}$$

We can see that $\displaystyle\sum_{n=1}^{+\infty} \frac{3+\sin(\ln n)}{n}$ is a positive term series and

$$\sum_{n=1}^{+\infty} \frac{2}{n} \leq \sum_{n=1}^{+\infty} \frac{3+\sin(\ln n)}{n}$$

Since $\displaystyle\sum_{n=1}^{+\infty} \frac{2}{n}$ is divergent then $\displaystyle\sum_{n=1}^{+\infty} \frac{3+\sin(\ln n)}{n}$ is divergent.

Limit comparison test

Theorem

Let $\sum u_n$ *and* $\sum v_n$ *tow positive term series. If* $u_n \underset{\infty}{\sim} v_n$ *or* $\lim\limits_{\infty} \frac{u_n}{v_n} = \ell$, $\ell \neq 0$, $\ell \neq +\infty$ *then the two series have the same nature.*

Example.

Study the nature of the series $\sum\limits_{n \geq 0} \frac{n^3+1}{n^5+2n^3+2}$

Limit comparison test

Theorem

Let $\sum u_n$ and $\sum v_n$ tow positive term series. If $u_n \underset{\infty}{\sim} v_n$ or $\lim\limits_{\infty} \frac{u_n}{v_n} = \ell$, $\ell \neq 0$, $\ell \neq +\infty$ then the two series have the same nature.

Example.

Study the nature of the series $\sum\limits_{n\geq 0} \frac{n^3+1}{n^5+2n^3+2}$

We have for all $n \in \mathbb{N}$, $u_n > 0$ and

$$\frac{n^3+1}{n^5+2n^3+2} \underset{\infty}{\sim} \frac{n^3}{n^5} = \frac{1}{n^2}$$

Since $\sum\limits_{n=1}^{+\infty} \frac{1}{n^2}$ is convergent (Riemann Series) then $\sum\limits_{n\geq 0} \frac{n^3+1}{n^5+2n^3+2}$ is convergent.

Limit comparison test

Example.

Study the nature of the series defined by $\displaystyle\sum_{n=0}^{+\infty} \ln\left(1+\frac{1}{3^n}\right)$

Limit comparison test

Example.

Study the nature of the series defined by $\displaystyle\sum_{n=0}^{+\infty} \ln\left(1+\frac{1}{3^n}\right)$

We know that $\ln\left(1+\frac{1}{x}\right) \underset{\infty}{\sim} \frac{1}{x}$ then $\ln\left(1+\frac{1}{3^n}\right) \underset{\infty}{\sim} \frac{1}{3^n}$

Since $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{3^n}$ is a convergent geometric series then $\displaystyle\sum_{n=0}^{+\infty} \ln\left(1+\frac{1}{3^n}\right)$ is convergent.

Limit comparison test

Example.

Study the nature of the series defined by $\displaystyle\sum_{n=0}^{+\infty} \ln\left(1+\frac{1}{3^n}\right)$

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Since $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{3^n}$ is a convergent geometric series then $\displaystyle\sum_{n=0}^{+\infty} \ln\left(1+\frac{1}{3^n}\right)$ is convergent.

Example.

The series $\displaystyle\sum_{n=1}^{+\infty} \left|\sin\left(\frac{1}{n}\right)\right|$

Limit comparison test

Example.

Study the nature of the series defined by $\displaystyle\sum_{n=0}^{+\infty} \ln\left(1+\frac{1}{3^n}\right)$

We know that $\ln\left(1+\frac{1}{x}\right) \underset{\infty}{\sim} \frac{1}{x}$ then $\ln\left(1+\frac{1}{3^n}\right) \underset{\infty}{\sim} \frac{1}{3^n}$

Since $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{3^n}$ is a convergent geometric series then $\displaystyle\sum_{n=0}^{+\infty} \ln\left(1+\frac{1}{3^n}\right)$ is convergent.

Example.

The series $\displaystyle\sum_{n=1}^{+\infty} \left|\sin\left(\frac{1}{n}\right)\right|$ *is divergent since* $\left|\sin\left(\frac{1}{n}\right)\right| \underset{\infty}{\sim} \frac{1}{n}$ *since* ($\sin x \underset{0}{\sim} x$)*, and the series* $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n}$ *is divergent.*

Integral test

Theorem

Let $f : [a, +\infty[\to \mathbb{R}^+$ *be a* ***continuous positive decreasing*** *mapping. We set* $u_n = f(n)$ *for all* $n \in \mathbb{N}^*$, $(n \geq a)$ *Then*

$$\begin{aligned} \sum u_n converges \quad &\Longleftrightarrow \quad \int_a^{+\infty} f(x)dx \ exist \\ &\Longleftrightarrow \quad \lim_{t \to +\infty} \int_a^t f(x)dx = \ell. \quad (\ell \ finite\ ). \end{aligned}$$

Integral test

Example. (Harmonic series)

Study the nature of the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n}.$

Integral test

Example. (Harmonic series)

Study the nature of the series $\sum_{n=1}^{+\infty} \frac{1}{n}$.

We set $f(n) = \frac{1}{n}$ we consider the mapping $f : [1, +\infty[\to \mathbb{R}^+ / x \mapsto f(x) = \frac{1}{x}$.
The mapping f is continuous, positive and decreasing on $[1, +\infty[$

$$\int_1^t f(x)dx = \int_1^t \frac{1}{x} = \ln x\Big|_1^t = \ln t - \ln 1 = \ln t.$$

and

$$\lim_{t \to +\infty} \int_1^t f(x)dx = \lim_{t \to +\infty} \ln t = +\infty.$$

Then the series $\sum_{n=1}^{+\infty} \frac{1}{n}$ is divergent.

Integral test

Example.

Study the nature of the series $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$

Integral test

Example.

Study the nature of the series $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$

We set $f(n) = \frac{1}{n(n+1)}$ and consider the mapping

$$f : [1,+\infty[\to \mathbb{R}^+ / x \mapsto f(x) = \frac{1}{x(x+1)}.$$

The mapping f is continuous, positive and decreasing on $[1,+\infty[$

$$\begin{aligned} \int_1^t f(x)dx &= \int_1^t \frac{1}{x(x+1)} = \int_1^t \frac{1}{x}dx - \int_1^t \frac{1}{x+1}dx \quad \left(\text{since } \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}\right) \\ &= \ln x|_1^t - \ln(x+1)|_1^t = \ln t - \ln 1 - \ln(t+1) + \ln 2 \\ &= \ln\left(\frac{t}{t+1}\right) + \ln 2. \end{aligned}$$

Therefor $\lim_{t\to+\infty} \int_1^t f(x)dx = \cancel{\lim_{t\to+\infty} \ln\left(\frac{t}{t+1}\right)} + \ln 2 = \ln 2.$

Then the series $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$ is convergent.

Riemann series

Definition

Let $\alpha \in \mathbb{R}$, we call Riemann series all series with general term

$$u_n = \frac{1}{n^\alpha}, \quad n \geq 1, \quad \alpha \in \mathbb{R}.$$

Proposition

Reimann series $\sum \frac{1}{n^\alpha}$, $\alpha \in \mathbb{R}$ converges if and only if $\alpha > 1$ and diverges if $\alpha \leq 1$.

Riemann series

Example.

We have

- *The series* $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^3}}$ *is convergent*

  since

Riemann series

Example.

We have

- *The series* $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^3}}$ *is convergent* *since* $\sum \frac{1}{\sqrt{n^3}} = \sum \frac{1}{(n^3)^{1/2}} = \sum \frac{1}{n^{3/2}}$, *it is a Riemann series with* $\alpha = \frac{3}{2} > 1$ *then* $\sum \frac{1}{\sqrt{n^3}}$ *converges.*
- *The series* $\sum \sqrt{n}$ *is divergent since*

Riemann series

Example.

We have

- *The series* $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^3}}$ *is convergent* *since* $\sum \frac{1}{\sqrt{n^3}} = \sum \frac{1}{(n^3)^{1/2}} = \sum \frac{1}{n^{3/2}}$, *it is a Riemann series with* $\alpha = \frac{3}{2} > 1$ *then* $\sum \frac{1}{\sqrt{n^3}}$ *converges.*
- *The series* $\sum \sqrt{n}$ *is divergent since* $\sum \sqrt{n} = \sum n^{1/2} = \sum \frac{1}{n^{-1/2}}$, *it is a Riemann series with* $\alpha = \frac{-1}{2} \leq 1$ *then* $\sum \sqrt{n}$ *diverges.*

Riemann series

Proposition

Let $\sum u_n$ a positive term series

1. *If there exist $\alpha > 1$ such that the sequence $(n^\alpha u_n)$ is upper bounded by a constant $M > 0$ then $\sum u_n$ converges.*
2. *If there exist $\alpha \leq 1$ such that the sequence $(n^\alpha u_n)$ is lower bounded by a constant $m > 0$ then $\sum u_n$ diverges.*

Corollaire

Let $\sum u_n$ be a positive term series. We suppose that there exisit $\alpha \in \mathbb{R}$ such that

1. *If $\lim\limits_{n\to\infty} n^\alpha u_n = \ell$, $(\ell \neq 0$ et $\ell \neq +\infty)$ the series $\sum u_n$ and $\sum \frac{1}{n^\alpha}$ are of the same nature.*
2. *If $\lim\limits_{n\to\infty} n^\alpha u_n = 0$ and $\sum \frac{1}{n^\alpha}$ converges then $\sum u_n$ converges.*
3. *If $\lim\limits_{n\to\infty} n^\alpha u_n = \infty$ and $\sum \frac{1}{n^\alpha}$ diverges then $\sum u_n$ diverges.*

Riemann series

Example.

Study the nature of the series $\sum_{n=1}^{+\infty} e^{\frac{3}{n^2}} - 1$

Riemann series

Example.

Study the nature of the series $\sum_{n=1}^{+\infty} e^{\frac{3}{n^2}} - 1$

we know that

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \implies e^x - 1 \underset{0}{\sim} x$$

Consequently

$$e^{\frac{3}{n^2}} - 1 \underset{\infty}{\sim} \frac{3}{n^2}.$$

Then

$$\lim_{n \to +\infty} n^2 \left(e^{\frac{3}{n^2}} - 1 \right) = \lim_{n \to +\infty} n^2 \left(\frac{3}{n^2} \right) = 3$$

Therefore the series $\sum_{n=1}^{+\infty} e^{\frac{3}{n^2}} - 1$ converges.

Riemann series

Example.

study the nature of the series $\displaystyle\sum_{n\geq 2}\frac{1}{\ln n}$

Riemann series

Example.

study the nature of the series $\displaystyle\sum_{n\geq 2}\frac{1}{\ln n}$

We have $\displaystyle\lim_{n\to\infty} n\frac{1}{\ln n}=\infty$ and $\displaystyle\sum\frac{1}{n}$ diverges, then $\displaystyle\sum_{n\geq 2}\frac{1}{\ln n}$ diverges.

Riemann series

Example.

study the nature of the series $\sum_{n=0}^{+\infty} e^{-n}$

Riemann series

Example.

study the nature of the series $\sum_{n=0}^{+\infty} e^{-n}$

We have $\lim_{n\to\infty} n^2 e^{-n} = 0$ and $\sum \frac{1}{n^2}$ converges, then $\sum_{n=0}^{+\infty} e^{-n}$ converges.

D'Alembert ratio test

Proposition

Let $\sum u_n$ be a series of positive terms. We set

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = \ell$$

1. *If $\ell < 1 \implies \sum u_n$ converges.*
2. *If $\ell > 1 \implies \sum u_n$ diverges.*
3. *If $\ell = 1$ we can't say any thing of the nature of the series.*

D'Alembert ratio test

Example.

Study the nature of the series

$$\sum_{n=0}^{+\infty} \frac{1}{n!}$$

D'Alembert ratio test

Example.

Study the nature of the series

$$\sum_{n=0}^{+\infty} \frac{1}{n!}$$

We have

$$\frac{u_{n+1}}{u_n} = \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \frac{1}{n+1}$$

and

$$\lim_{n \to +\infty} \frac{u_{n+1}}{u_n} = \lim_{n \to +\infty} \frac{1}{n+1} = 0$$

Consequently $\sum_{n=0}^{+\infty} \frac{1}{n!}$ converges.

D'Alembert ratio test

Example.

Study the nature of the series

$$u_n = \frac{n^n}{n!}, \quad n \geq 0$$

D'Alembert ratio test

Example.

Study the nature of the series

$$u_n = \frac{n^n}{n!}, \quad n \geq 0$$

$$\forall n \in \mathbb{N}: \ \frac{u_{n+1}}{u_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = e^{n \ln\left(\frac{n+1}{n}\right)}$$

$$\lim_{n \to +\infty} \frac{u_{n+1}}{u_n} = \lim_{n \to +\infty} e^{n \ln\left(\frac{n+1}{n}\right)} = \lim_{n \to +\infty} e^{n\left(\frac{1}{n}\right)} = e > 1. \quad \left(car \ln\left(\frac{n+1}{n}\right) \underset{\infty}{\sim} \frac{1}{n}\right)$$

Therefore the series $\sum u_n = \frac{n}{n!}$ diverges.

D'Alembert ratio test

Example.

Study the nature of the series $$\sum_{n\geq 1} \frac{2^{2n}e^{-2n}}{n}$$

D'Alembert ratio test

Example.

Study the nature of the series $\sum_{n\geq 1} \frac{2^{2n} e^{-2n}}{n}$

We have

$$\forall n \in \mathbb{N}: \ \frac{u_{n+1}}{u_n} = \frac{2^{2(n+1)} e^{-2(n+1)}}{n+1} \cdot \frac{n}{2^{2n} e^{-2n}} = 2^2 e^{-2} \frac{n}{n+1}$$
$$= \left(\frac{2}{e}\right)^2 \frac{n}{n+1} \to \left(\frac{2}{e}\right)^2 < 1$$

Then the series $\sum_{n\geq 1} \frac{2^{2n} e^{-2n}}{n}$ converges.

Cauchy root test

Proposition

Let $\sum u_n$ be a positive term series. We set

$$\lim_{n \to +\infty} \sqrt[n]{u_n} = \ell$$

1. *If $\ell < 1 \implies \sum u_n$ converges.*
2. *If $\ell > 1 \implies \sum u_n$ diverges.*
3. *If $\ell = 1$ we can't say any thing on the nature of the series.*

Example.

Study the nature of the series $$\sum_{n=1}^{+\infty} \left(\frac{n+1}{n}\right)^{-n^2}$$

Cauchy root test

We have $\forall n \in \mathbb{N}^* \ u_n \geq 0$ and

$$\sqrt[n]{u_n} = \left(\frac{n+1}{n}\right)^{-n} = \left(1 + \frac{1}{n}\right)^{-n}$$

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} \left(\frac{n+1}{n}\right)^{-n} = \lim_{n\to+\infty} e^{-n\ln\left(1+\frac{1}{n}\right)} = \lim_{n\to+\infty} e^{-n\left(\frac{1}{n}\right)} = e^{-1} < 1$$

Then the series $\sum_{n=1}^{+\infty} \left(\frac{n+1}{n}\right)^{-n^2}$ converges.

Cauchy root test

Example.

Study the nature of the series $\displaystyle\sum_{n=0}^{+\infty}\left(\frac{n-1}{2n+3}\right)^n$

Cauchy root test

Example.

Study the nature of the series $\sum_{n=0}^{+\infty}\left(\frac{n-1}{2n+3}\right)^n$

$\sum_{n=0}^{+\infty}\left(\frac{n-1}{2n+3}\right)^n$ is a positive term series and we have

$$\lim_{n\to+\infty}\sqrt[n]{u_n}=\lim_{n\to+\infty}\frac{n-1}{2n+3}$$

$$=\frac{1}{2}<1$$

Then the series $\sum_{n=0}^{+\infty}\left(\frac{n-1}{2n+3}\right)^n$ converges.

Link between D'Alembert ratio test and Cauchy root test

Proposition

Let $\sum u_n$ *be a positive term series then*

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = l \Longrightarrow \lim_{n\to+\infty} \sqrt[n]{u_n} = \ell$$

Remark

The converse is false.

Course outline

Arbitrary term series

Definition

We call arbitrary term series all series $\sum u_n$ *which the general term can take positive or negative values.*

Example.

The series

$$\sum \frac{(-1)^n}{n^2}, \ \sum \sin\left(n\frac{\pi}{2}\right)$$

are arbitrary term series.

Absolutely convergent series

Definition

If $\sum |u_n|$ converges we say that the series $\sum u_n$ is absolutely convergent.

Remark

All convergent positive term series is absolutely convergent.

Theorem

Let $\sum u_n$ an arbitrary term series

- *If $\sum u_n$ is absolutely convergent then $\sum u_n$ is convergent. That is to say*
$$\sum |u_n| \text{ converges } \Longrightarrow \sum u_n \text{ converges.}$$
- *The converse is false.*
- *If $\sum u_n$ diverges then $\sum |u_n|$ diverges.*

Absolutely convergent series

Example.

Study the nature of the series $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$

Absolutely convergent series

Example.

Study the nature of the series $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$

We have

$$|u_n| = \left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$$

The series $\sum \frac{1}{n^2}$ is convergent then the series $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent then it converges.

Absolutely convergent series

Example.

Study the nature of the series $\displaystyle\sum_{n=1}^{+\infty}\frac{\cos(n\pi)}{n^3}$

Absolutely convergent series

Example.

Study the nature of the series $\displaystyle\sum_{n=1}^{+\infty} \frac{\cos(n\pi)}{n^3}$

We have

$$|u_n| = \left|\frac{\cos(n\pi)}{n^3}\right| \leq \frac{1}{n^3}$$

The series $\sum \frac{1}{n^3}$ is convergent then the series $\sum \left|\frac{\cos(n\pi)}{n^3}\right|$ converges therefore the series $\sum \frac{\cos(n\pi)}{n^3}$ converges.

Conditionally convergent series

Definition

A convergent series but non absolutely convergent is called conditionally convergent series.

Example.

The series $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ *is conditionally convergent.*

Conditionally convergent series

Definition

A convergent series but non absolutely convergent is called conditionally convergent series.

Example.

The series $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ *is conditionally convergent.*

$|u_n| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$ and the series $\sum \frac{1}{n}$ is divergent then the series $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ is not absolutely convergent but $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ is convergentthen $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

Abel test

Theorem

Let $\sum u_n$ *an arbitrary term series such that* $u_n = a_n \cdot b_n$, *où* a_n *et* b_n *two sequences satisfying*

1. *La suite* b_n *decreasing and positive.*
2. $\lim\limits_{n\to+\infty} b_n = 0$.
3. $\exists M > 0$ *such that* $\forall n \in \mathbb{N}$: $\left|\sum\limits_{k=0}^{n} a_k\right| \leq M$.

Then $\sum u_n$ *is convergent.*

Abel test

Example.

Show that the series $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ *is convergent*

Abel test

Example.

Show that the series $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ *is convergent*

Let $a_n = (-1)^n$ and $b_n = \frac{1}{n}$, we have

1. b_n is a positive decreasing sequence.
2. $\lim_{n\to+\infty} b_n = \lim_{n\to+\infty} \frac{1}{n} = 0.$
3. We have $\forall n \geq 1$

$$\sum_{k=0}^{n} a_k = \begin{cases} -1 \\ 0 \\ 1. \end{cases}$$

Then $\left|\sum_{k=0}^{n} a_k\right| \leq 1$, By Abel test $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ is convergent.

Course outline

Alternating series & Leibnitz test

Definition

A series $\sum u_n$ *is said to be alternating series if and only if for all* $n \geq n_0$, $(n_0 \in \mathbb{N})$ $u_n = (-1)^n v_n$ *or* $u_n = (-1)^{n+1} v_n$ *with* $v_n \geq 0$. *Then all series of the form* $\sum (-1)^n u_n$, $u_n \geq 0$ *is said to be alternating series.*

Example.

The series $\sum \dfrac{(-1)^n}{n}$, $\sum \dfrac{(-1)^n}{n^2+3}$, $\sum \dfrac{\cos(n\pi)}{e^n}$ *are alternating series.*

Alternating series & Leibnitz test

Definition

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Example.

The series $\sum \frac{(-1)^n}{n}$, $\sum \frac{(-1)^n}{n^2+3}$, $\sum \frac{\cos(n\pi)}{e^n}$ *are alternating series.*

Theorem

Let $\sum (-1)^n u_n$*be an alternating series . If* (u_n) *a positive decreasing convergent sequence to* 0 *then* $\sum (-1)^n u_n$ *is convergent.*

Alternating series & Leibnitz test

Example.

Study the nature of the series $\sum_{n=1}^{+\infty}(-1)^n \ln(1+\frac{1}{n})$

Alternating series & Leibnitz test

Example.

Study the nature of the series $\sum_{n=1}^{+\infty}(-1)^n \ln(1+\frac{1}{n})$

The series $\sum_{n=1}^{+\infty}(-1)^n \ln(1+\frac{1}{n})$ is a convergent alternating series, since we have:

1. $u_n = \ln(1+\frac{1}{n}) \geq 0.$
2. (u_n) is decreasing since

$$u_n = f(n), f'(n) = \frac{\frac{-1}{n^2}}{1+\frac{1}{n}} = \frac{-1}{n^2}\cdot\frac{n}{n+1} = \frac{-1}{n(n+1)} < 0.$$

3. $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \ln(1+\frac{1}{n}) = 0$

Then by Leibnitz test the series $\sum_{n=1}^{+\infty}(-1)^n \ln(1+\frac{1}{n})$is convergent.