

Exercise 1. Let $u = (x, y) \in \mathbb{R}^2$. We define the following norms:

$$\|u\|_1 = |x| + |y|, \quad \|u\|_2 = \sqrt{x^2 + y^2}, \quad \|u\|_3 = \max(|x|, |y|).$$

Show that $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_3$ are equivalent norms over \mathbb{R}^2 .

Exercise 2. Show that the union of a family of open sets is an open set. Deduce that an intersection of closed sets is closed set.

Exercise 3. Show that any open subset of \mathbb{R}^2 is the union of open balls.

Exercise 4. Represent the definition sets of the following functions

$$\begin{aligned} f_1(x, y) &= \ln(2x + y - 2), & f_2(x, y) &= \sqrt{1 - xy}, \\ f_3(x, y) &= \frac{\ln(y - x)}{x}, & f_4(x, y) &= \frac{1}{\sqrt{x^2 + y^2 - 1}} + \sqrt{4 - x^2 - y^2}. \end{aligned}$$

Exercise 5. Represent the level curves (Solutions of the equation $f(x, y) = k$)

$$f_1(x, y) = y^2, \text{ with } k = -1 \text{ and } k = 1, \quad f_2(x, y) = \frac{x^4 + y^4}{8 - x^2 y^2}, \text{ with } k = 2.$$

Exercise 6. Find the limit of each function at the indicated point.

$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 + y^4}{x^2 + y^2}$	$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 + 3xy + y^2}{x^2 + 5y^2}$	$\lim_{(x,y) \rightarrow (0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2}$
$\lim_{(x,y) \rightarrow (0,0)} \frac{(x+y)^2}{x^2 + y^2}$	$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2}$	$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 \sin^2(y)}{x^2 + 3y^2}$
$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$	$\lim_{(x,y) \rightarrow (0,0)} \frac{\arctan((x+y)^2)}{x^2}$	$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y e^y}{x^4 + 4y^2}$
$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$	$\lim_{(x,y,z) \rightarrow (\pi, 0, 1/3)} e^{y^2} \tan(xz)$	$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^3}{2x^2 + y^2}$
$\lim_{(x,y) \rightarrow (0,0)} \frac{ x + y }{x^2 + y^2}$	$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2}$	$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$
$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^2}{x^2 + y^2}$	$\lim_{(x,y) \rightarrow (1,2)} \frac{y \sin(x+1)}{x^2 - 2x + 1}$	$\lim_{(x,y) \rightarrow (0,0)} x e^{x/y};$
$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$	$\lim_{(x,y) \rightarrow (0,1)} \frac{x \ln y}{\sqrt{x^2 + (y-1)^2}}$	$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 y}{x^2 + y^2}$
$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$	$\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$	$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{\sqrt{x^2 + y^2}}$
$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - x^3 + y^2 + y^3}{x^2 + y^2}$	$\lim_{(x,y) \rightarrow (0,0)} \frac{x - y}{x + y}$	$\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \ln(x^2 + y^2) + 5$
$\lim_{(x,y) \rightarrow (0,0)} \frac{y^2 \sin^2(x)}{x^4 + y^4}$		

Exercise 7. Examine the continuity of functions defined by

$$f(0,0) = 0 \text{ and } f(x,y) = \frac{x^4 + y^4}{x^2 + xy + y^2}, (x,y) \neq (0,0)$$

$$f(0,0) = 0 \text{ and } f(x,y) = \frac{xy^2}{x^2 + y^4}, (x,y) \neq (0,0)$$

$$f(0,0) = 0 \text{ and } f(x,y) = \frac{y \sin x^2 - x \sin y^2}{x^2 + y^2}, (x,y) \neq (0,0)$$

$$f(0,0) = 0 \text{ and } f(x,y) = \frac{x^2 \ln(1 + |y|)}{\sqrt{x^2 + y^2}}, (x,y) \neq (0,0)$$

Exercise 8. Find the first partial derivatives of the following functions

$$\begin{array}{lll} (x,y) \mapsto x^4 + 5xy^3 & (x,y) \mapsto x^2y - 3y^4 & (x,y) \mapsto x^3 \sin y \\ (x,t) \mapsto e^{xt} & (x,t) \mapsto \ln(x + t^2) & (u,v) \mapsto \frac{u}{v^2} \\ (x,y) \mapsto ye^{xy} & (x,y) \mapsto (x^2 + xy)^3 & (x,y) \mapsto y(x + x^2y)^5 \\ (x,y) \mapsto \frac{x}{x + y^2} & (x,y) \mapsto \frac{ax + by}{cx + dy} & (u,v) \mapsto \frac{e^v}{u + v^2} \\ (u,v) \mapsto (u^2v - v^3)^5 & (r,\theta) \mapsto \sin(r \cos \theta) & (p,q) \mapsto \tan^{-1}(pq^2) \\ (x,y) \mapsto xy & (x,y) \mapsto \int_y^x \cos(e^t) dt & (\alpha,\beta) \mapsto \int_\alpha^\beta \sqrt{t^3 + 1} dt \\ (x,y,z) \mapsto x^3yz^2 + 2yz & (x,y,z) \mapsto xy^2 \exp(-xz) & (x,y,z) \mapsto \ln(x + 2y + 3z) \\ (x,y,z) \mapsto y \tan(x + 2z) & (t,u,v) \mapsto \sqrt{t^4 + u^2 \cos v} & (x,y,z) \mapsto \frac{xy}{z} \\ (x,y,z) \mapsto x^2y \cos\left(\frac{z}{t}\right) & (x,y,z,t) \mapsto \frac{\alpha x + \beta y^2}{\gamma z + \delta t^2} & \end{array}$$

Exercise 9. Find all the second partial derivatives

$$\begin{array}{lll} f(x,y) = x^4y - 2x^3y^2 & f(x,y) = \ln(ax + by) & f(x,y) = \frac{y}{x + 3y} \\ f(r,\theta) = e^{-2r} \cos \theta & f(s,t) = \sin(s^2 - t^2) & f(x,y) = \arctan\left(\frac{x+y}{1-xy}\right) \end{array}$$

Exercise 10. Study the differentiability of the following functions.

$$f(x,y) = \begin{cases} \frac{(x+y) \sin(xy)}{\sqrt{x^2 + y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \quad f(x,y) = \begin{cases} (x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Exercise 11. Assume that all the given functions are differentiable.

1. If $z = f(x,y)$, where $x = r \cos(\theta)$ and $y = r \sin(\theta)$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ and show that:

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2$$

2. If $u = f(x,y)$, where $x = e^s \cos(t)$ and $y = e^s \sin(t)$, show that:

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s} \left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 \right]$$

Exercise 12. Assume that all the given functions have continuous second-order partial derivatives. 1. Show that any function of the form

$$z = f(x + at) + g(x - at)$$

is a solution of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

[Hint: Let $u = x + at$ and $v = x - at$.]

2. If $z = f(x, y)$, where $x = r \cos(\theta)$ and $y = r \sin(\theta)$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r} \frac{\partial z}{\partial r}.$$

Exercise 13. (Homogeneous Functions) A function f is called homogeneous of degree n if it satisfies the equation

$$f(tx, ty) = t^n f(x, y)$$

for all t , where n is a positive integer and f has continuous second-order partial derivatives.

1. Verify that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.
2. Show that if g is homogeneous of degree n , then

$$x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = ng(x, y).$$

[Hint: Use the Chain Rule to differentiate $g(tx, ty)$ with respect to t .]

Also, show that

$$x^2 \frac{\partial^2 g}{\partial x^2} + 2xy \frac{\partial^2 g}{\partial x \partial y} + y^2 \frac{\partial^2 g}{\partial y^2} = n(n-1)g(x, y).$$

3. If h is homogeneous of degree n , show that

$$\frac{\partial xh(tx, ty)}{\partial x} = t^{n-1} \frac{\partial xh(x, y)}{\partial x}.$$

Exercise 14.

1. Give the Taylor expansion up to order 2 of the function $f(x, y) = e^x \sin(x + y)$ around the point $(0, 0)$.
2. Give the Taylor expansion up to order 3 of the function $g(x, y) = -x^2 + 2xy + 3y^2 - 6x - 2y + 4$ around the point $(-2, 1)$.

Exercise 15. Use Implicit Differentiation to find dy/dx

$$1) y \cos(x) = x^2 + y^2 \quad 2) \cos(xy) = 1 + \sin(y) \quad 3) \tan^{-1}(x^2y) = x + xy^2 \quad 4) e^y \sin(x) = x + xy.$$

Exercise 16. Use Implicit Differentiation to find $\partial z / \partial x$ and $\partial z / \partial y$

$$1) x^2 + 2y^2 + 3z^2 = 1 \quad 2) x^2 - y^2 + z^2 - 2z = 4 \quad 3) e^z = xyz \quad 4) yz + x \ln(y) = z^2.$$

Additional exercises

Exercise 1. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by $f(x, y) = \frac{x + y}{1 + x^2 + y^2}$.

1. Determine and represent its level curves.
2. Calculate its first partial derivatives.
3. Write the equation of the tangent plane to f at $(0, 0)$.

Exercise 2. Let f be the function defined on \mathbb{R}^2 by

$$f(x, y) = \begin{cases} \frac{xy}{x^2 - y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

1. Show that f is continuous on \mathbb{R}^2 .
2. Calculate $\nabla f(x, y)$.
3. Show that ∇f is continuous on \mathbb{R}^2 .
4. Demonstrate that f has second partial derivatives at every point.
5. What can you deduce from the calculation of $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$ and $\frac{\partial^2 f}{\partial y \partial x}(0, 0)$?

Exercise 3. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined as follows:

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

1. Is it continuous on \mathbb{R}^2 ?
2. Calculate $\nabla f(x, y)$.
3. Is the function f of class $C^1(\mathbb{R}^2)$?
4. What can be concluded about the differentiability of the function f on \mathbb{R}^2 ?

Exercise 4. Suppose that the equation $F(x, y, z) = 0$ implicitly defines each of the three variables x , y , and z as functions of the other two: $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$. If F is differentiable and $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, and $\frac{\partial F}{\partial z}$ are all nonzero, show that

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = -1.$$
