Relations and Functions

Chapter 3, Section 2

Arity

Definition 5.2.1 The *arity* (also called *rank*) of a statement or a term is the number of variables that have free occurrences in it. (If a variable has two or more free occurrences, it still counts for just one in the arity.) We say that a statement or term of arity n is *n-ary*; *binary* means 2-ary. A statement of arity ≥ 1 is called a *predicate*.

Example

Example 5.2.2 The statement

$$\text{If } p > -1 \text{ then } (1+p)^n \geq 1 + np$$

is a binary predicate, with free variables p and n. The term

$$\int_x^y \tan z \, dz$$

is also binary, with free variables x and y.

n – ary Relation

Definition 5.2.3 Suppose X is a set and n is a positive integer. Then an *n-ary relation on X* is a set of ordered n-tuples of elements of X. (And henceforth we abbreviate 'ordered n-tuple' to 'n-tuple'.)

$n-$ary Relation

There are two main ways of naming a relation. The first is to list the n-tuples that are in it. For example, here is a binary relation on the set $\{0, 1, 2, 3, 4, 5\}$:

(5.6) $\{(0,1),(0,2),(0,3),(0,4),(0,5),(1,2),(1,3),(1,4),$ $(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\}$

This is the ‘less than’ relation on $\{0, 1, 2, 3, 4, 5\}$.

$n-$ ary Relation

This method is of no use when the relation has too many n-tuples to list. In such cases, we name the relation by giving an n-ary predicate. We need a convention for fixing the order of the variables. For example, if we take the predicate

$$y \geq x$$

and ask what relation it defines on $\{0,1,2,3\}$, does x go before y (so that $(0,1)$ is in the relation) or the other way round (so that $(1,0)$ is in the relation)?

One standard solution to this problem is to give the definition of the relation as follows :

(5.7) $$R(x,y) \Leftrightarrow y \geq x$$

$n-$ary Relation

More precisely, we test whether a pair $(3,2)$ is in the relation R by putting 3 for x and 2 for y in the defining predicate :

(5.8) $$2 \geq 3$$

This statement (5.8) is false, so the pair $(3,2)$ is not in the relation R. But '$2 \geq 2$' is true, so $(2,2)$ is in R. In view of this and similar examples, if R is a binary predicate, we often write 'xRy' instead of '$R(x,y)$'.

Strict Linear Order

Definition 5.2.4 Suppose X is a set. A *strict linear order* on X is a binary relation R on X such that

(1) for all $x \in X$, xRx is false ;

(2) for all x, y and $z \in X$, if xRy and yRz then xRz ;

(3) for all $x, y \in X$, either xRy or yRx or $x = y$.

Example

As an example, let X be a subset of $\mathbb{R}$, and define

$$xRy \Leftrightarrow x < y$$

on X (where $<$ is the usual linear order of the real numbers).

If R is a strict linear order on X, and Y is a subset of X, then the restriction of R to Y is a strict linear order on Y.

Least element and Greatest element

Definition 5.2.5 Suppose R is a strict linear order on a set X. An element $y \in X$ is called a *least element of* X if yRx whenever $x \in X$ and $x \neq y$. An element $z \in X$ is called a *greatest element of* X if xRz whenever $x \in X$ and $x \neq z$.

It is easy to see that there can be at most one least element and at most one greatest element in a strict linear order.

Existence of Least element

Example 5.2.6 We consider strict linear orders on a finite set. Suppose X is a finite non-empty set and R is a strict linear order on X. Then we claim that there is a least element in X.

$n -$ ary Function

Definition 5.2.7 Suppose n is a positive integer, X is a set and R is an $(n+1)$-ary relation on X. We say that R is an *n-ary function* on X if the following holds :

(5.10) For all $a_1, \cdots, a_n$ in X there is exactly one b in X such that $(a_1, \cdots, a_n, b)$ is in R

When this condition holds, the unique b is called the *value* of the function at $(a_1, \cdots, a_n)$.

Example

For example, the relation (5.6) is not a function on $\{0,1,2,3,4,5\}$: there is no pair with 5 on the left, and there are five pairs with 0 on the left. But the following is a 1-ary function on $\{0,1,2,3,4,5\}$:

(5.11) $$\{(0,1),(1,2),(2,3),(3,4),(4,5),(5,0)\}$$

This function is sometimes known as 'plus one mod 6'.

Example

Since a function is a kind of relation, we can define particular functions in the same ways as we define particular relations. But there is another way that is particularly appropriate for functions.

Instead of an $(n + 1)$-ary predicate, we use an n-ary term, as in the following example :

(5.12) $F(x, y)$ = the remainder when $y - x$ is divided by 6

Overloading

Remark 5.2.8 *Overloading* means using an expression with two different arities. For example, in ordinary arithmetic we have

$$-6$$

with − of arity 1, and

$$8 - 6$$

with − of arity 2. So the symbol − is overloaded.In this course overloading is likely to cause confusion, so we avoid it. A relation or function symbol has only one arity. There is another kind of overloading. Compare

(a) $\cos(\pi) = -1.$

(b) cos is differentiable.

In the first case, cos is a 1-ary function symbol; in the second it is a term. The difference is between applying the function and talking about it. Again we will not allow this in our formal languages. In the terminology of the next section, cos in (a) is a function symbol and not a constant symbol, and vice versa in (b).

Substitutions

To test whether a pair (a, b) is in the relation R of (5.7), we substituted a name of a for x and a name of b for y in the predicate '$y \geq x$', and we asked whether the resulting sentence is true. Normally this method works. But here is a case where we get strange results by substituting a name for a variable in a predicate.

Consider the equation

(5.13) $$\int_1^2 2(x + y)dx = 5$$

which is a 1-ary predicate with the variable y. To integrate (5.13), we regard y as constant and we get

$$5 = [x^2 + 2xy]_1^2 = (4 + 4y) - (1 + 2y) = 3 + 2$$

which you can solve to get $y = 1$.

Substitutions

Now suppose we try to say that 1 is a solution for y in equation (5.13), by writing x in place of y. We get

$$5 = \int_1^2 2(x + x)dx = \int_1^2 4xdx = 8 - 2 = 6$$

So $5 \ = \ 6$. What went wrong?

Our mistake was that when we put x in place of the free occurrence of y in the equation, x *became bound* by the integration over x.

Substitutions

Definition 5.2.9 When a term t can be substituted for all free occurrences of a variable y in an expression E, without any of the variables in t becoming bound by other parts of E, we say that t is *substitutable for y* in E. (Some logicians say ‘free for’ rather than ‘substitutable for’.)

The problem with our integral equation (5.13) was that the term x is not substitutable for y in the equation.

Substitutions

Definition 5.2.10 Suppose E is an expression, $y_1, \cdots, y_n$ are distinct variables and $t_1, \cdots, t_n$ are terms such that each y_i is substitutable for y_i in E. Then we write

(5.14) $$E[t_1/y_1, \cdots, t_n/y_n]$$

for the expression got from E by simultaneously replacing each free occurrence of each y_i in E by t_i. If some t_i is not substitutable for y_i in E, we count the expression $E[t_1/y_1, \cdots, t_n/y_n]$ as meaningless. The expression

$$t_1/y_1, \cdots, t_n/y_n$$

in (5.14) is called a *substitution for variables*, or a *substitution* for short.