Discrete probability distribution (finite case)

Discrete uniform distribution

Definition

The r.v. X has the discrete uniform distribution on the set of real numers $\{x_1, \cdots, x_n\}$ if $\mathbb{P}_X$ is the equiprobability on this set i.e.: $X \in X(\Omega) = \{x_1, \cdots, x_n\}$ and $\forall k \in \Omega, \mathbb{P}(X = k) = \frac{1}{n}$

We note $X \rightsquigarrow \mathcal{U}(n)$.

$\mathbb{E}(X) = \frac{n+1}{2}$; $Var(X) = \frac{n^2-1}{12}$.

Example

When we throw a dice, the number obtained follow the uniforme distribution on $\{1, \cdots, 6\}$ with $\mathbb{P}_X(x) = \frac{1}{6}, \forall x \in \{1, \cdots, 6\}$.

Discrete probability distribution (finite case)

Bernoulli distribution

Definition

The r.v. X follow the Bernoulli distribution of parameter p, ($p \in [0,1]$) if it takes only two values 0 and 1 with $\mathbb{P}(X=1) = p$ and $\mathbb{P}(X=0) = 1-p = q$ (with: $p+q=1$).
We note $X \rightsquigarrow \mathcal{B}(p)$.
$\mathbb{E}(X) = p$; $Var(X) = p(1-p) = pq$.

Example

In the toss of an unbalanced coin, the probability of getting "heads" is $p \neq \frac{1}{2}$. X the r.v. defined by $X = 1$ if we get "heads" and $X = 0$ if we get "tails". $X \rightsquigarrow \mathcal{B}(p)$ with the probability distribution

$$\mathbb{P}(X=x) = \begin{cases} p, & \text{if } x = 1 \\ q, & \text{if } x = 0 \end{cases}$$

Let be an urn containing:

- white balls W in proportion p;
- red balls R in proportion $q = 1 - p$.

One carries out n successive draws of a ball with delivery. We define the r.v. X as the number of white balls obtained during the n draws (It can take the values: $0, 1, \cdots, n$).

Remark

The r.v. X can be defined as a sum of n independent Bernoulli r.v. $X_1, X_2, \cdots, X_n$ $(X = X_1 + X_2 + \cdots + X_n)$. Such that $\mathbb{P}(X_i = 1) = p$.

Discrete probability distribution (finite case)

Binomiale distribution

Definition

A r.v. X follows a binomial distribution of parameters (n, p) where $n \geq 0$ and $(p \in [0, 1])$ if $X(\Omega) = \{0, 1, \cdots, n\}$ and $\mathbb{P}(X = k) = C_n^k p^k (1-p)^{n-k}, \forall k = 0, 1, \cdots, n$ (with: $p + q = 1$).

We note $X \rightsquigarrow \mathcal{B}(n, p)$.

$\mathbb{E}(X) = np$; $Var(X) = np(1-p)$.

Example

Let $X \rightsquigarrow \mathcal{B}(n, p)$.

1. Determine n such that $\mathbb{P}(X = 0) \leq 0,01$;
2. Determine n such that $\mathbb{P}(X \geq 1) \geq 0,90$.

Discrete probability distribution (finite case)

Hypergeometric distribution

One carries out n successive drawings of a ball, without handing-over, which is the same as when one takes a sample of n balls in only one blow, in an urn containing N balls of two categories:

- N_p white balls W in proportion p;
- N_q red balls R in proportion $q = 1 - p$.

Let be the r.v. X, representing the number of balls W obtained.

Remark

The possible values of X are $\max(0, n - N_q) \leq k \leq \min(n, N_p)$

Discrete probability distribution (finite case)

Hypergeometric distribution

Definition

The r.v. X follows the hypergeometric distribution of parameters N, n, p, where $n \leq N$, if $X(\Omega) = \{0, 1, \cdots, n\}$ we have $\forall k \in X(\Omega), \mathbb{P}(X = k) = \frac{C_{N_p}^{k} C_{N_q}^{n-k}}{C_N^n}$

We note $X \rightsquigarrow \mathcal{H}(N, n, p)$, with $p = \frac{N_p}{N}, p + q = 1$.The a.v. X follows the hypergeometric law of parameters $\mathbb{E}(X) = np; Var(X) = npq\frac{N-n}{N-1}$.

Discrete probability distribution (infinite case)

Geometric distribution

The geometric distribution is the law of expectation of the first success of a sequence of independent trials each of which has a probability p of success, i.e. $\mathbb{P}(X = k)$ is the probability that the k^{th} trial is the first success.

Definition

A r.v. X follows a geometric distribution of parameter p, where $0 \leq p \leq 1$ if

- $X(\Omega) = \mathbb{N}^*$;
- $\mathbb{P}(X = k) = pq^{k-1}$ with $p + q = 1$.

We note $X \rightsquigarrow \mathcal{G}(p)$.
$\mathbb{E}(X) = \frac{1}{p}$; $Var(X) = \frac{q}{p^2}$.

Example

We play heads or tails with a rigged coin such that the probability of getting tails is $\frac{1}{3}$. Let X be the r.v. representing the number of

Discrete probability distribution (infinite case)

Pascal (Negative Binomial or Polya) distribution

If a r.v. represents the number of fails before the r^{th} success of a sequence of independent Bernoulli trials each of which has a probability p of success.

Definition

A r.v. X follows a Pascal distribution of parameters r and p, where $0 \leq p \leq 1$ if

- $X(\Omega) = \mathbb{N}$;
- $\mathbb{P}(X = k) = C_{k+r-1}^{k} p^r \left(1 - p\right)^k$ with $p + q = 1$.

We note $X \rightsquigarrow \mathcal{BN}(r, p)$.
$\mathbb{E}(X) = \frac{rq}{p}$; $Var(X) = \frac{rq}{p^2}$.

Discrete probability distribution (infinite case)

Poisson Distribution

We observe the realization of random events in time and space obeying the following conditions:

- The probability of realization in a small period Δt is proportional to Δt.
- It is independent of what has happened previously.

Definition

X follows a Poisson Distribution of parameter $\lambda (\lambda > 0)$, noted $\mathcal{P}(\lambda)$ if its values are in $\mathbb{N}$ and if:

$$\forall k \in \mathbb{N}, \mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

We note $X \rightsquigarrow \mathcal{P}(\lambda)$.
$\mathbb{E}(X) = \lambda$; $Var(X) = \lambda$.

Definition

X follows a continuous uniform distribution on $[a, b]$ if it has the following density

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{elswhere} \end{cases}.$$

We note $X \rightsquigarrow \mathcal{U}([a, b])$.
The cumulative distribution function:

$$F(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a}, & x \in [a, b] \\ 1 & \text{if } x > b \end{cases}.$$

$\mathbb{E}(X) = \frac{a+b}{2}; \quad Var(X) = \frac{(b-a)^2}{12}.$

Definition

X follows an exponential distribution of parameter $\lambda > 0$ if it has the following density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{elswhere} \end{cases}.$$

We set $X \rightsquigarrow \mathcal{E}(\lambda)$.
The cumulative distribution function:

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{elswhere} \end{cases}.$$

$\mathbb{E}(X) = \frac{1}{\lambda}; \quad Var(X) = \frac{1}{\lambda^2}.$

Definition

X follows a Gamma distribution of parameters $\alpha > 0$ and $\beta > 0$ if it has the following density

$$f(x) = \begin{cases} \frac{\beta^{\alpha} e^{-\beta x} x^{\alpha-1}}{\Gamma(\alpha)} & \text{if } x \geq 0 \\ 0 & \text{elswhere} \end{cases}.$$

Where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$.
We set $X \rightsquigarrow \mathcal{G}amma(\alpha, \beta)$.
The cumulative distribution function:

$$F(x) = \frac{\beta^{\alpha} \int_0^{x} t^{\alpha-1} e^{-\beta t} dt}{\Gamma(\alpha)}.$$

$\mathbb{E}(X) = \frac{\alpha}{\beta}; \quad Var(X) = \frac{\alpha}{\beta^2}.$

Remark: If $\alpha = 1$ we find the exponential distribution with parameter $\lambda = 1$.

Properties of the function $\Gamma(\alpha)$

a. $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.
b. $\Gamma(1) = 1$.
c. $\Gamma(n) = (n - 1)!$.

Definition

X follows the Gaussian (Normal) distribution of parameters μ and σ if it has the following density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

We note $X \rightsquigarrow \mathcal{N}(\mu, \sigma)$.
$\mathbb{E}(X) = \mu; \quad Var(X) = \sigma^2$.

The standard normal distribution

The random variable $U = \frac{X-\mu}{\sigma}$ follows the normal distributuion $\mathcal{N}(0, 1)$.

Any problem concerning X is reduced to U and we have several tables concerning the standard normal distribution.

Approximations

Approximation of the hypergeometric distribution by a binomial distribution

The hypergeometric distribution can be approximated by the binomial distribution as soon as the size N of the population is large compared with the size n of the sample.

Approximation of the binomial distribution by a Poisson distribution

If n is large and p small enough (in practice if $n \geq 30$ and $p \leq 0,1$ with $np \leq 10$) we can replace the binomial distribution $\mathcal{B}(n, p)$ with the Poisson distribution $\mathcal{P}(np)$, $(\lambda = np)$.

Approximation of the binomial distribution by a normal distribution

If n is large and $\not p$ not too close to 0 and 1 (in practice if $n \geq 30$, $np \geq 5$ and $nq \geq 5$) we can replace the binomial distribution $\mathcal{B}(n, p)$ with the normal distribution $\mathcal{N}(np, \sqrt{npq})$.