Work sheet N 2
Integral transformations
Part I: $\mathcal{F}$ourier transform

Exercise 1. Calculate the Fourier transform of

$$f(t) = \begin{cases} e^{-|t|} + 1 & \text{if } -3 < t < 3, \\ e^{-|t|} & \text{otherwise,} \end{cases} \qquad g(t) = \begin{cases} x^2 & \text{if } -1 < t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2. Let

$$f(t) = \begin{cases} 1 - |t| & \text{if } -1 < t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

1. Plot the graph of the function f.
2. Calculate the Fourier transform of f.
3. Use the Fourier inversion theorem to deduce the value of the integral

$$\int_0^{+\infty} \frac{\cos x(1 - \cos x)}{x^2} dx.$$

Exercise 2: Let f be the function defined on $\mathbb{R}$ by $f(t) = e^{-t^2}$:

1. Show that f is a solution of the ODE $y' + 2ty = 0$ for all $t \in \mathbb{R}$.
2. By applying the Fourier transform to this ODE, deduce another ODE satisfied by $F(f)$.
3. Knowing that $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$, calculate $F(f)(0)$.
4. Deduce that $F(f)(x) = \sqrt{\pi} e^{-\frac{x^2}{4}}$.

Exercise 3:

1. Given that $F(e^{-|t|})(x) = \frac{2}{x^2+1}$, deduce $F(e^{-at})(x)$ for $a > 0$.
2. Show that if $f, f' \in \mathcal{L}^1(\mathbb{R})$, then $F(f')(x) = ixF(f)(x)$.
3. By applying the Fourier transform to the following differential equation: $-2y'' + 6y = e^{-3|t|}$ for all $t \in \mathbb{R}$:
   (a) Show that if a function g such that $g, g', g'' \in \mathcal{L}^1(\mathbb{R})$ is a solution of this equation, then

   $$F(g)(x) = \frac{1}{2} \left(\frac{1}{x^2 + 3} - \frac{1}{x^2 + 9} \right).$$

   (b) Assuming that g is continuous on $\mathbb{R}$ and differentiable to the left and right of every $t \in \mathbb{R}$, deduce the expression of g.