Substitution

Chapter 2, Section 7

Definition

Definition 3.7.1

A *substitution* S (for LP) is a function whose domain is a finite set $\{q_1, \cdots, q_k\}$ of propositional symbols, and which assigns to each symbol q_j $(1 \leq j \leq k)$ a formula ψ_j of LP. We write this function S as

$$\psi_1/q_1, \cdots, \psi_k/q_k \tag{3.53}$$

We apply the substitution (3.53) to a formula ϕ by simultaneously replacing every occurrence of each propositional symbol q_j in ϕ by ψ_j $(1 \leq j \leq k)$,and we write the resulting expression as $\phi[S]$, that is,

$$\phi[\psi_1/q_1, \cdots, \psi_k/q_k] \tag{3.54}$$

Example

Example 3.7.2

Let ϕ be the formula

$$((p_1 \to (p_2 \wedge (\neg p_3))) \leftrightarrow p_3)$$

Let ψ_1 be $(\neg(\neg p_3))$, let ψ_2 be p_0 and let ψ_3 be $(p_1 \to p_2)$. Then the expression

$$\phi[\psi_1/p_1, \psi_2/p_2, \psi_3/p_3]$$

is

(3.55) $$(((\neg(\neg p_3)) \to (p_0 \wedge (\neg(p_1 \to p_2)))) \leftrightarrow (p_1 \to p_2))$$

Recursive definition

Definition 3.7.3

Let $\{q_1, \cdots, q_k\}$ be propositional symbols, $\psi_1, \cdots, \psi_k$ formulas and ϕ a formula of LP. We define $\phi[\psi_1/q_1, \cdots, \psi_k/q_k]$ by recursion on the complexity of ϕ as follows.

If ϕ is atomic then

$$\phi[\psi_1/q_1, \cdots, \psi_k/q_k] = \begin{cases} \psi_i & \text{if } \phi \text{ is } q_i \ (1 \leq i \leq k) \\ \phi & \text{otherwise} \end{cases}$$

If $\phi = (\neg\chi)$ where χ is a formula, then

$$\phi[\psi_1/q_1, \cdots, \psi_k/q_k] = (\neg\chi[\psi_1/q_1, \cdots, \psi_k/q_k])$$

If $\phi = (\chi_1 \boxdot \chi_1)$, where χ_1 and χ_2 are formulas and $\boxdot \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then

$$\phi[\psi_1/q_1, \cdots, \psi_k/q_k] = (\chi_1[\psi_1/q_1, \cdots, \psi_k/q_k]) \boxdot (\chi_2[\psi_1/q_1, \cdots, \psi_k/q_k])$$

Substitution and Truth Values

Definition 3.7.4

If A is a σ- structure and S is the substitution $\psi_1/q_1, \cdots, \psi_k/q_k$, we define a σ-structure $A[S]$ by

$$A[S](p) = \begin{cases} A^*(\psi_j) \text{ if } p \text{ is } q_j \ (1 \leq j \leq k) \\ A(p) \qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

Lemma 3.7.5 *Let A be a σ-structure and S the substitution $\psi_1/q_1, \cdots, \psi_k/q_k$ with $\psi_1, \cdots, \psi_k$, in* LP*(σ). Then, for all formulas ϕ of* LP$(\sigma \cup \{q_1, \cdots, q_k\})$,

$$A^*(\phi[S]) = A[S]^*(\phi).$$

Substitution and Replacement Theorems

Theorem 3.7.6

*(a) (**Substitution Theorem**) Let* S *be a substitution and* ϕ_1, ϕ_2 *logically equivalent formulas of* LP. *Then*

$$\phi_1[S] \text{ eq } \phi_2[S].$$

*(b) (**Replacement Theorem**) Let* S_1 *and* S_2 *be the following substitutions :*

$$\psi_1/q_1, \cdots, \psi_k/q_k, \qquad \psi'_1/q_1, \cdots, \psi'_k/q_k$$

where for each j $(1 \leq j \leq k)$, ψ_j eq ψ'_j. *Then for every formula* ϕ,

$$\phi[S_1] \text{ eq } \phi[S_2].$$

Example of Substitution

Example 3.7.7

A formula is a tautology if and only if it is logically equivalent to $(\neg\bot)$ (cf. Exercise 3.6.5), and $(\neg\bot)[S]$ is just $(\neg\bot)$. Hence the Substitution Theorem implies that if ϕ is a tautology then so is $\phi[S]$. For example, the formula $(p \to (q \to p))$ is a tautology. By applying the substitution

$$(p_1 \wedge (\neg p_2))/p, \qquad (p_0 \leftrightarrow \bot)/q$$

we deduce that

$$((p_1 \wedge (\neg p_2)) \to ((p_0 \leftrightarrow \bot) \to (p_1 \wedge (\neg p_2))))$$

is a tautology. More generally, we could substitute any formula ϕ for p and any formula ψ for q, and so by the Substitution Theorem any formula of the form

$$(\phi \to (\psi \to \phi))$$

is a tautology.

Example of Replacement

Example 3.7.8

We know that

$$(3.58) \quad (p_1 \land p_2) \quad eq \quad (\neg((\neg p_1) \lor (\neg p_2)))$$

We would like to be able to put the right-hand formula in place of the left-hand one in another formula, for example, $((p_1 \land p_2) \to p_3)$. The trick for doing this is to choose another propositional symbol, say *r*, that does not occur in the formulas in front of us. (This may involve expanding the signature, but that causes no problems.) Then

$$((p_1 \land p_2) \to p_3) \text{ is } (r \to p_3)[(p_1 \land p_2)/r]$$

$$((\neg((\neg p_1) \lor (\neg p_2))) \to p_3) \text{ is } (r \to p_3)[(\neg((\neg p_1) \lor (\neg p_2)))/r]$$

Then the Replacement Theorem tells us at once that

$$((p_1 \land p_2) \to p_3) \quad \text{eq} \quad ((\neg((\neg p_1) \lor (\neg p_2))) \to p_3).$$

Another Example of Substitution

Example 3.7.9

Starting with the same logical equivalence (3.58) as in the previous example, we can change the symbols p_1 and p_2, provided that we make the same changes in both formulas of (3.58). Let ϕ and ψ be any formulas, and let S be the substitution

$$\phi\ /p_1,\ \psi\ /p_2$$

Then $(p_1 \land p_2)[S]$ is $(\phi \land \psi)$, and $(\neg\big((\neg p_1) \lor (\neg p_2)\big))[S]$ is $(\neg\big((\neg\phi) \lor (\neg\psi)\big))$. So we infer from the Substitution Theorem that

$$(\phi \land \psi) \quad \text{eq} \quad (\neg\big((\neg\phi) \lor (\neg\psi)\big)).$$