Correction exam Analysis 2 / 2023 - 202

Exercise 1: $\sum_{n\geq 0} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n\geq 0} \left(\frac{1}{1+x^2}\right)^n.$

1/- Determination of domain of convergence: (1pt)

- For $x=0$ the series is convergent.
- Let see the domain of convergence of the series $\sum_{n\geq 0}\left(\frac{1}{1+x^2}\right)^n$

wich is a geometric series. It converges if and only if

$\left|\frac{1}{1+x^2}\right| < 1$. $\left|\frac{1}{1+x^2}\right| < 1 \Rightarrow x \in \mathbb{R}^*$

Therefore $\boxed{D = \mathbb{R}^* \cup \{0\} = \mathbb{R}}$.

2/- Continuity of the sum: (1,5 pts)

We set $S_n(x) = x^2 \sum_{k=0}^{n}\left(\frac{1}{1+x^2}\right)^n$ and $S(x) = \lim_{n\to+\infty} S_n(x)$.

then $\forall x \in D$:

$$S_n(x) = \begin{cases} 1+x^2 - \left(\frac{1}{1+x^2}\right)^n ; & x \neq 0 \\ 0 ; & x = 0 \end{cases}, \quad S(x) = \begin{cases} 1+x^2 ; & x \neq 0 \\ 0 ; & x = 0 \end{cases}.$$

the sum S is not continuous on D since $\lim_{x\to 0} S(x) = 1 \neq 0 = S(0)$.

3/- The uniform convergence on D: (1pt)

- the functions $x \longmapsto \frac{x^2}{(1+x^2)^n}$ $\forall n \in \mathbb{N}$ are continuous on D but $x \longmapsto S(x)$ is not continuous on D then $\sum_{n\geq 0} \frac{x^2}{(1+x^2)^n}$ doesn' converge uniformally on D.

4/- Uniform convergence on $[a, +\infty[$, $a>0$: (1pt)

We have $\sup_{[a,+\infty[} |S_n(x) - S(x)| = \sup_{[a,+\infty[} \left(\frac{1}{1+x^2}\right)^n = \left(\frac{1}{1+a^2}\right)^n$

and $\lim_{n\to+\infty}\left(\frac{1}{1+a^2}\right)^n = 0$, so $\sum_{n\geq 0} \frac{x^2}{(1+x^2)^n}$ converges uniformally on $[a,$

$[.$

Exe 2: a) (E) $\begin{cases} y'' + xy' + y = 1 \\ y(0) = y'(0) = 0 \end{cases}$; $y(x) = \sum_{n\geq 0} a_n x^n$ on $]-R, R[$

1/- Determination of the recursive relation satisfied by (a_n) **1pt**

We have $\forall x \in]-R, R[$:

$$y(x) = \sum_{n\geq 0} a_n x^n$$

$$xy'(x) = x\sum_{n\geq 1} n a_n x^{n-1} = \sum_{n\geq 0} n a_n x^n,$$

$$y''(x) = \sum_{n\geq 2} n(n-1) a_n x^{n-2} = \sum_{n\geq 0} (n+2)(n+1) a_{n+2} x^n$$

Inded: $(E) \Leftrightarrow \sum_{n\geq 0} \left((n+2)(n+1) a_{n+2} + (n+1) a_n \right) x^n = 1 \quad \forall n \geq 0$

$$\Leftrightarrow \begin{cases} (n+2)(n+1) a_{n+2} + (n+1) a_n = 0 \ ; \ \forall n \geq 1 \\ 2a_2 + a_0 = 1 \ ; \ n = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a_{n+2} = -\dfrac{1}{n+2} a_n \ ; \ n \geq 1 \\ a_2 = \dfrac{1 - a_0}{2} \end{cases} \quad (E)'$$

2/- Express a_n in function of n and calculate R:

We have $a_0 = y(0) = 0$; $a_1 = y'(0) = 0$

Then from this and $(E)'$ we deduce that $a_{2n+1} = 0 \ \forall n \in \mathbb{N}$

and $a_{2n} = \left(\dfrac{-1}{2n}\right) \times \left(-\dfrac{1}{2n-2}\right) \times \dots \times \left(-\dfrac{1}{4}\right) a_2 = \dfrac{(-1)^{n+1}}{2\times 4 \times \dots \times 2n}$

then $\boxed{a_{2n} = \dfrac{(-1)^{n+1}}{2^n n!} \ , \forall n \geq 1}$ **1pt** $\boxed{y(x) = \sum_{n\geq 1} \dfrac{(-1)^{n+1}}{2^n n!} x^{2n}}$

Set $b_n = \dfrac{(-1)^{n+1}}{2^n n!}$, we have

$$\left| \frac{b_{n+1}}{b_n} \right| = \frac{2^n n!}{2^{n+1}(n+1)!} = \frac{1}{2(n+1)} \xrightarrow[n\to+\infty]{} 0, \text{ then } \boxed{R = +\infty}$$

1pt

The series converges on $\mathbb{R}$.

Expression of the series using elementary functions:

$$y(x) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{2^n n!} x^{2n} = - \sum_{n\geq 1} \frac{1}{n!}\left(-\frac{x^2}{2}\right)^n = 1 - e^{-x^2/2}$$

then $\boxed{f(x) = 1 - e^{-x^2/2}}$ (1pt)

b) Study the nature of the numerical series:

1) $\sum_{n\geq 1} \frac{\cos\sqrt{n}}{n\sqrt{n}}$, we have $0 \leq \left|\frac{\cos\sqrt{n}}{n\sqrt{n}}\right| \leq \frac{1}{n\sqrt{n}} \xrightarrow[n\to+\infty]{} 0$

then $\frac{\cos\sqrt{n}}{n\sqrt{n}} \xrightarrow[n\to+\infty]{} 0$ (1,5 pts)

in the other hand $\sum_{n\geq 1} \frac{1}{n\sqrt{n}}$ is convergent (Riemann series, $\alpha =$

then $\sum_{n\geq 1} \frac{\cos\sqrt{n}}{n\sqrt{n}}$ is absolutly convergent then it is convergent

2) $\sum_{n\geq 2} \frac{(-1)^n}{n^2+\sin n^2}$; we have $0 < \left|\frac{(-1)^n}{n^2+\sin n^2}\right| \leq \frac{1}{n^2-1} \xrightarrow[n\to+\infty]{}$

then $\lim_{n\to+\infty} \frac{(-1)^n}{n^2+\sin n^2} = 0$. (1,5 pts)

$$\sum_{n\geq 1} \left|\frac{(-1)^n}{n^2+\sin n^2}\right| \leq \sum_{n\geq 1} \frac{1}{n^2-1} \underset{+\infty}{\sim} \sum \frac{1}{n^2}$$

(convergent Rieman series $\alpha = 2$

then the series is absolutly convergent then it is convergent.

3) $\sum_{n\geq 0} \frac{ch(n)}{ch(2n)}$. we have $ch\, n \underset{+\infty}{\sim} \frac{e^n}{2}$, $ch\, 2n \underset{+\infty}{\sim} \frac{e^{2n}}{2}$

$\frac{ch\, n}{ch\, 2n} \underset{+\infty}{\sim} e^{-n} \xrightarrow[n\to+\infty]{} 0$. and $\sum e^{-n}$ is a convergent geometric series; with commun ratio, $q = \frac{1}{e} \in]-1, 1[$.

so $\sum_{n\geq 0} \frac{ch\, n}{ch\, 2n}$ is convergent. (1,5 pts)

$f, u, v \in C(\mathbb{R}^2)$.

$f : \mathbb{R}^2 \to \mathbb{R}$; $u : \mathbb{R}^2 \to \mathbb{R}$; $v : \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \mapsto f(x,y)$ $\quad (x,y) \mapsto u(x,y)$ $\quad (x,y) \mapsto v(x,y)$.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \ (1) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ (2) \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \ (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ (4) \qquad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \ (5)$$

$g : \mathbb{R}^2 \to \mathbb{R}$; $g(x,y) = f(u(x,y), v(x,y))$

Show that $\Delta g = 0$ on $\mathbb{R}^2$: $\Delta g = 0 \Leftrightarrow \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} =$

$$\begin{cases} \frac{\partial g}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} & \text{0,5 pts} \\ \frac{\partial g}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} & \text{0,5 pts} \end{cases}$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 f}{\partial v^2} \cdot \left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial f}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2}$$
$$+ \frac{\partial f}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2} + 2 \frac{\partial^2 f}{\partial u \partial v} \cdot \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \quad \text{0,5pt}$$

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial^2 f}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial f}{\partial u} \cdot \frac{\partial^2 u}{\partial y^2} + \frac{\partial f}{\partial v} \cdot \frac{\partial^2 v}{\partial y^2}$$
$$+ 2 \frac{\partial^2 f}{\partial u \partial v} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial y} \quad \text{0,5pt}$$

taking into account that $f \in C^2$ on $\mathbb{R}^2$ then:

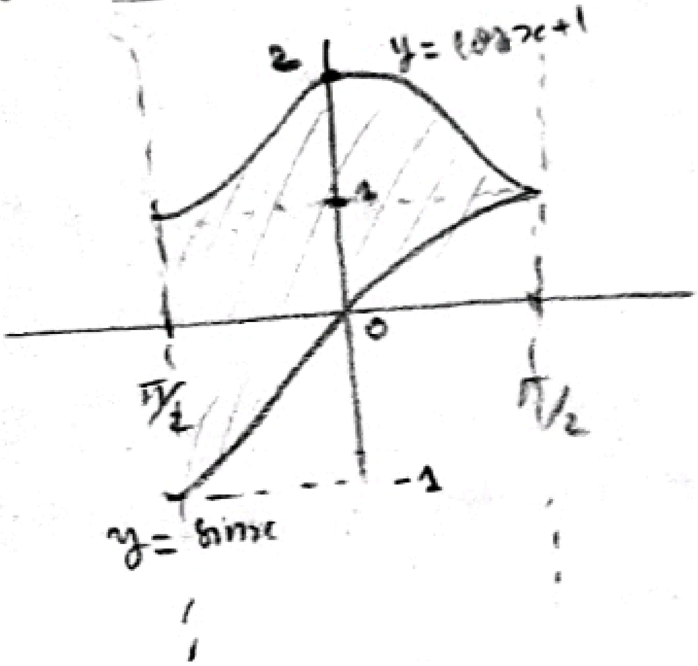
$$\Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \frac{\partial^2 f}{\partial u^2}\Big(\underbrace{\left(\frac{\partial u}{\partial x}\right)^2}_{\text{use(4)}} + \underbrace{\left(\frac{\partial u}{\partial y}\right)^2}_{\text{use(5)}}\Big) + \frac{\partial^2 f}{\partial v^2}\left(\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right)$$
$$+ 2 \frac{\partial^2 f}{\partial u \partial v}\Big(\underbrace{\frac{\partial u}{\partial x}}_{\text{use(4)}} \frac{\partial v}{\partial x} + \underbrace{\frac{\partial u}{\partial y}}_{\text{use(5)}} \cdot \frac{\partial v}{\partial y}\Big) + \frac{\partial f}{\partial u}\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{\text{use(2)}} + \frac{\partial f}{\partial v}\underbrace{\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}_{\text{use(3)}}$$

$$\Rightarrow \Delta g = \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \underbrace{\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)}_{\text{use(1)}} \Rightarrow \Delta g = 0 \quad \text{1pt}$$

Ex04: $D \subset \mathbb{R}^2$ delimited by:

$y = \cos x + 1$; $y = \sin x$; $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$.

① Represent D



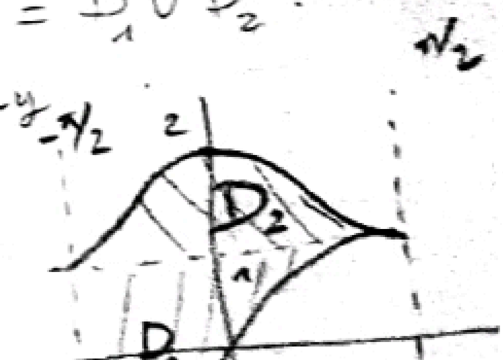
(1pt)

② i) D can be written as: (1pt)

$D = \{(x,y) \in \mathbb{R}^2 : -\frac{\pi}{2} \le x \le \frac{\pi}{2} ;\ \sin x \le y \le \cos x + 1\}$.

ii) $D = \{(x,y) \in \mathbb{R}^2 : c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$.

In this case the domain is devided ~~into~~ two sub domains D_1 and D_2 ; $D = D_1 \cup D_2$.



• $-1 \le y \le 1$ and $-\frac{\pi}{2} \le x \le \mathrm{Arcsin}\, y$

since $x \mapsto \sin x$ is bijective on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

• $1 \le y \le 2$: $y = \cos x + 1 \iff x = g(y)$

- for $0 \le x \le \frac{\pi}{2}$ we have $g(y) = \arccos(y-1)$

since $x \mapsto \cos x + 1$ is bijective on $[0, \pi/2]$

- for $-\frac{\pi}{2} \le x \le 0$: $-x \in [0, \frac{\pi}{2}]$ and $\cos(-x) = \cos x \Rightarrow \cos(-x) = y - 1$

$\Rightarrow -x = \mathrm{Arccos}(y-1) \Rightarrow x = -\mathrm{Arccos}(y-1)$.

then for $1 \leq y \leq 2$: $-\operatorname{Arccos}(y-1) \leq x \leq \operatorname{Arccos}(y-1)$

1pt

and

$D_2 = \{ (x,y) \in \mathbb{R}^2 : \quad 1 \leq y \leq 2 \; ; \; -\operatorname{Arccos}(y-1) \leq x \leq \operatorname{Arccos}(y-1)$

3) $\displaystyle\iint_D dx\,dy = \int_{-\pi/2}^{\pi/2} \int_{\sin x}^{\cos x + 1} dy\, dx = \int_{-\pi/2}^{\pi/2} (\cos x + 1 - \sin x)\, dx$

$\displaystyle = 2 \int_{-\pi/2}^{\pi/2} (\cos x + 1)\, dx = 2 + \pi$

$$\boxed{\text{Area } D = 2 + \pi}$$

1pt