Exercise 4 (3 + 5 + 3 + 4)

1) Let $\sigma = \{P_1, \dots, P_m\}$. Let $n \in \mathbb{N}^*$ and $\phi_1, \dots, \phi_n \in \mathcal{F}$

By straightforward induction we obtain

$$\mu(\phi_1 \vee \dots \vee \phi_n) = \mu(\phi_1) \wedge \dots \wedge \mu(\phi_n)$$

and

$$\mu(\phi_1 \wedge \dots \wedge \phi_n) = \mu(\phi_1) \vee \dots \vee \mu(\phi_n)$$

Let $\phi \in \mathcal{F}$ a DNF. Then ϕ has the form

$$\phi_1 \vee \dots \vee \phi_n,$$

where each $\phi_i = \phi_{i,1} \wedge \dots \wedge \phi_{i,s}$ $(1 \leq i \leq n)$ is a conjunction of literals.

Since for all j, $1 \leq j \leq m$, $\mu(P_j) = P_j$ and $\mu(\neg P_j) = \neg \mu(P_j) = \neg P_j$, then for all i, $1 \leq i \leq n$,

$$\mu(\phi_i) = \mu(\phi_{i,1}) \vee \dots \vee \mu(\phi_{i,s})$$
$$= \phi_{i,1} \vee \dots \vee \phi_{i,s}$$

is a disjunction of literals, hence

$$\mu(\phi) = \mu(\phi_1 \vee \dots \vee \phi_n)$$
$$= \mu(\phi_1) \wedge \dots \wedge \mu(\phi_n)$$

is CNF.

The vice versa case is similar.

2) This is the proof of Post's theorem, see page 80.

3) First notice that $\overline{(\overline{A})} = A$ since for any $p \in \sigma$ we have

$$\overline{(\overline{A})}(p) = T \iff \overline{A}(p) = F \iff A(p) = T.$$

We have, for any σ-structure A,

$$(f^u)^u(A) = T \iff f^u(\overline{A}) = F$$
$$\iff f(\overline{(\overline{A})}) = T$$
$$\iff f(A) = T.$$

Since the codomain of the functions $(f^u)^u$ and f has only two elements, we deduce that

$$(f^u)^u = f.$$

4) Suppose $f = |\phi|$. We will prove that $f^u = |u(\phi)|$. For this, it suffices to prove that, for any $A \in \Sigma$, we have

$$\overline{A}^*(\phi) = F \iff A^*(u(\phi)) = T \qquad (1)$$

Indeed, if (1) is satisfied, then we have for any $A \in \Sigma$:

$$f^u(A) = T \iff f(\overline{A}) = F$$
$$\iff |\phi|(\overline{A}) = F$$
$$\iff \overline{A}^*(\phi) = F$$
$$\iff A^*(u(\phi)) = T$$
$$\iff |u(\phi)|(A) = T,$$

that is, $f^u = |u(\phi)|$

We will prove (1) by induction on the complexity of ϕ.

If ϕ has complexity 0, then $\phi = p \in \sigma$, so

$$\overline{A}^*(\phi) = F \iff \overline{A}(p) = F \iff A(p) = T$$
$$\iff A(u(p)) = T.$$

Therefore, (1) is satisfied.

Assume that (1) is satisfied for all formulae in $\mathcal{F}$ with complexities $\leq k$, and let $\phi \in \mathcal{F}$ with complexity $k+1$. Then ϕ has one of the forms $(\neg\psi)$, $(\psi \vee \chi)$ or $(\psi \wedge \chi)$.

i) $\phi = (\neg\psi)$. Then we have

$$\begin{aligned}\overline{A}^*(\neg\psi) = F &\iff \overline{A}^*(\psi) = T \\ \text{(by induction assumption)} &\iff A^*(u(\psi)) = F \\ &\iff A^*(\neg u(\psi)) = T \\ &\iff A^*(u(\phi)) = T,\end{aligned}$$

so (1) is true.

ii) $\phi = (\psi \vee \chi)$. Then we have

$$\begin{aligned}\overline{A}^*(\phi) = F &\iff \overline{A}^*(\psi) = F \text{ and } \overline{A}^*(\chi) = F \\ \text{(by induction assumption)} &\iff A^*(u(\psi)) = T \text{ and } A^*(u(\chi)) = T \\ &\iff A^*(u(\psi) \wedge u(\chi)) = T \\ &\iff A^*(u(\psi \vee \chi)) = T \\ &\iff A^*(\phi) = T,\end{aligned}$$

So again (1) is true.

iii) $\phi = (\psi \wedge \chi)$. Then we have

$$\begin{aligned}\overline{A}^*(\phi) = F &\iff \overline{A}^*(\psi) = F \text{ or } \overline{A}^*(\chi) = F \\ \text{(by induction assumption)} &\iff A^*(u(\psi)) = F \text{ or } A^*(u(\chi)) = F \\ &\iff A^*(u(\psi) \vee u(\chi)) = F \\ &\iff A^*(u(\psi \wedge \chi)) = F \\ &\iff A^*(u(\phi)) = F.\end{aligned}$$

Therefore (1) is satisfied, and it follows that $f^u = |u(\phi)|$.