Disjunctive and Conjunctive Normal Forms

Chapter 2, Section 8

Conjunction

Definition 3.8.1

(a) A *conjunction* of formulas is a formula

$$(\cdots(\phi_1 \wedge \phi_2) \wedge \cdots) \wedge \phi_n) \tag{3.59}$$

where $\phi_1, \cdots, \phi_n$ are formulas; these n formulas are called the *conjuncts* of the conjunction. We allow n to be 1, so that a single formula is a conjunction of itself. We abbreviate (3.59) to

$$(\phi_1 \wedge \cdots \wedge \phi_n)$$

leaving out all but the outside parentheses.

Disjunction

Definition 3.8.1

(b) A *disjunction* of formulas is a formula

$$(3.60) \qquad (\cdots (\phi_1 \vee \phi_2) \vee \cdots) \vee \phi_n)$$

where $\phi_1, \cdots, \phi_n$ are formulas; these n formulas are called the *disjuncts* of the disjunction. We allow n to be 1, so that a single formula is a disjunction of itself. We abbreviate (3.60) to

$$(\phi_1 \vee \cdots \vee \phi_n)$$

leaving out all but the outside parentheses.

Negation and Literal

Definition 3.8.1

(c) The *negation* of a formula ϕ is the formula

(3.61) $$(\neg\phi)$$

We abbreviate (3.61) to

$$\neg\phi$$

A formula that is either an atomic formula or the negation of an atomic formula is called a *literal* .

Generalization

Remark 3.8.2

It is easily checked that (d) and (e) of Definition 3.5.6 generalize as follows :

(d) $A^*(\phi_1 \wedge \cdots \wedge \phi_n) = \mathrm{T}$ if and only if $A^*(\phi_1) = \cdots = A^*(\phi_n) = \mathrm{T}$.

(e) $A^*(\phi_1 \vee \cdots \vee \phi_n) = \mathrm{T}$ if and only if $A^*(\phi_i) = \mathrm{T}$ for at least one i.

The Function $|\phi|$

Definition 3.8.3

Let σ be a signature and ϕ a formula of LP(σ). Then ϕ determines a function $|\phi|$ from the set of all σ-structures to the set $\{\mathrm{T}, \mathrm{F}\}$ of truth values, by :

$$|\phi|(A) = A^*(\phi) \quad \text{for each } \sigma\text{-structure } A.$$

This function $|\phi|$ is really the same thing as the head column of the truth table of ϕ, if you read a T or F in the i-th row as giving the value of $|\phi|$ for the σ-structure described by the i-th row of the table.

Post's Theorem

The following theorem can be read as 'Every truth table is the truth table of some formula'.

Theorem 3.8.4 (Post's Theorem)

Let σ be a finite non-empty signature and g a function from the set of σ-structures to $\{\mathrm{T}, \mathrm{F}\}$. Then there exits a formula ψ of LP(σ) such that

$$g = |\psi|.$$

Example

Example 3.8.5

We find a formula to complete the truth table

p_1	p_2	p_3	?
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

Example (continued)

Example 3.8.5

There are three rows with value T. The formula ψ_{A_1} is $p_1 \wedge p_2 \wedge \neg p_3$. The formula ψ_{A_2} is $p_1 \wedge \neg p_2 \wedge p_3$. The formula ψ_{A_3} is $\neg p_1 \wedge p_2 \wedge p_3$.

So the required formula is

$$(p_1 \wedge p_2 \wedge \neg p_3) \vee (p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge p_2 \wedge p_3).$$

Disjunctive and Conjunctive Normal Forms

Definition 3.8.6

- A *basic conjunction* is a conjunction of one or more literals, and a *basic disjunction* is a disjunction of one or more literals. A single literal counts as a basic conjunction and a basic disjunction.
- A formula is in *disjunctive normal form* (DNF) if it is a disjunction of one or more basic conjunctions.
- A formula is in *conjunctive normal form* (CNF) if it is a conjunction of one or more basic disjunctions.

Example

Example 3.8.7

(1)

$$p_1 \wedge \neg p_1$$

is a basic conjunction, so it is in DNF. But also p_1 and $\neg p_1$ are basic disjunctions, so the formula is in CNF too.

(2)

$$(p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_2 \wedge p_3)$$

is in DNF.

Example (continued)

Example 3.8.7

(3) Negating the formula in (2), applying the De Morgan Laws and removing double negations gives

$$\neg((p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_2 \wedge p_3))$$

eq $$\neg(p_1 \wedge \neg p_2) \wedge \neg(\neg p_1 \wedge p_2 \wedge p_3)$$

eq $$(\neg p_1 \vee \neg\neg p_2) \wedge (\neg\neg p_1 \vee \neg p_2 \vee \neg p_3)$$

eq $$(\neg p_1 \vee p_2) \wedge (p_1 \vee \neg p_2 \vee \neg p_3)$$

This last formula is in CNF.

Existence of DNF and CNF

Theorem 3.8.8

Let σ be a non-empty finite signature. Every formula ϕ of LP(σ) is logically equivalent to a formula ϕ^{DNF} of LP(σ) in DNF, and to a formula ϕ^{CNF} of LP(σ) in CNF.

Corollary 3.8.9

Let σ be any signature (possibly empty). Every formula ϕ of LP(σ) is logically equivalent to a formula of LP(σ) in which no truth function symbols appear except $\wedge, \neg$ and $\bot$.

Satisfiability of Formulas in DNF

A formula in DNF is satisfiable if and only if at least one of its disjuncts is satisfiable. Consider any one of these disjuncts; it is a basic conjunction

$$\phi_1 \wedge \cdots \wedge \phi_m,$$

This conjunction is satisfiable if and only if there is a σ-structure A such that

$$A^*(\phi_1) = \cdots = A^*(\phi_m) = \mathrm{T}.$$

Since the ϕ_i are literals, we can find such an A unless there are two literals among $\phi_1, \cdots, \phi_m$ which are respectively p and $\neg p$ for the same propositional symbol p. We can easily check this condition by inspecting the formula. So checking the satisfiability of a formula in DNF and finding a model, if there is one, are trivial. (See Exercise 3.8.3(b) for a test of this.)

Satisfiability of Formulas in CNF

The situation with formulas in CNF is completely different. Many significant mathematical problems can be written as the problem of finding a model for a formula in CNF. The general problem of determining whether a formula in CNF is satisfiable is known as SAT. Many people think that the question of finding a fast algorithm for solving SAT, or proving that no fast algorithm solves this problem, is one of the major unsolved problems of twenty-first century mathematics. (It is the ‘P = NP’ problem.)

Coloring Problem

Example 3.8.10 A *proper m-colouring* of a map is a function assigning one of m colours to each country in the map, so that no two countries with a common border have the same colour as each other. A map is *m-colourable* if it has a proper m-colouring.

Suppose a map has countries $c_1, \cdots, c_m$. Write p_{ij} for 'Country c_i has the j-th colour'. Then finding a proper m-colouring of the map is equivalent to finding a model of a certain formula θ in CNF. Namely, take θ to be the conjunction of the following formulas :

$$p_{i1} \vee p_{i2} \vee \cdots \vee p_{im} \text{ (for } 1 \le i \le n\text{);}$$

$$\neg p_{ik} \vee \neg p_{jk} \text{ (for all } i, j, k \text{ where } c_i, c_j \text{ have a common border).}$$

More precisely, if A is a model of θ, then we can colour each country c_i with the first colour j such that $A^*(p_{ij}) = \mathrm{T}$.