<u>Exercise 1</u> (0,5 × 4)

1) c) If ν is a node of arity 2, then one of the following holds.

(i) ν has right-hand label $(\to E)$, and there are formulas ϕ and ψ such that ν has the left label ψ, and the left labels on the daughters of ν are, from left to right, ϕ and $(\phi \to \psi)$.

2) (ii) ν has right-hand label $(\wedge I)$, and there are formulas ϕ and ψ such that ν has the left label $(\phi \wedge \psi)$, and the left labels on the daughters of ν are, from left to right, ϕ and ψ.

(iii) ν has right-hand label $(\neg E)$, and there is a formula ϕ such that ν has the left label $\perp$, and the left labels on the daughters of ν are, from left to right, ϕ and $(\neg \phi)$.

(iv) ν has right-hand label $(\leftrightarrow I)$, and there are formulas ϕ and ψ such that ν has the left label $(\phi \leftrightarrow \psi)$, and the left labels on the daughters of ν are, from left to right, $(\phi \to \psi)$ and $(\psi \to \phi)$.

Exercise 2 (2 × 4)

1)

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\cancel{\phi}^{1} \quad (\phi \to (\psi \wedge \chi))}{(\psi \wedge \chi)}\,(\to E)}{\psi}\,(\wedge E)}{(\phi \to \psi)}\,(\to I)\ 1
\qquad
\dfrac{\dfrac{\dfrac{\cancel{\phi}^{2} \quad (\phi \to (\psi \wedge \chi))}{(\psi \wedge \chi)}\,(\to E)}{\chi}\,(\wedge E)}{(\phi \to \chi)}\,(\to I)\ 2
}{((\phi \to \psi) \wedge (\phi \to \chi))}\,(\wedge I)
$$

2)

$$
\dfrac{
(\phi \vee \chi)
\quad
\dfrac{\dfrac{\cancel{\phi}^{1} \quad (\phi \to \psi)}{\psi}\,(\to E)}{(\psi \vee \theta)}\,(\vee I)
\quad
\dfrac{\dfrac{\cancel{\chi}^{1} \quad (\chi \to \theta)}{\theta}\,(\to E)}{(\psi \vee \theta)}\,(\vee I)
}{(\psi \vee \theta)}\,(\vee E)\ 1
$$

3)

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\cancel{\phi}^{2} \quad \cancel{\psi}^{1}}{(\phi \wedge \psi)}\,(\wedge I) \quad (\neg(\phi \wedge \psi))}{\bot}\,(\neg E)}{(\neg \psi)}\,(\neg I)\ 1}{(\phi \to (\neg \psi))}\,(\to I)\ 2
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\cancel{\phi}^{3} \quad \cancel{\psi}^{4}}{(\phi \wedge \psi)}\,(\wedge I) \quad (\neg(\phi \wedge \psi))}{\bot}\,(\neg E)}{(\neg \phi)}\,(\neg I)\ 3}{(\psi \to (\neg \phi))}\,(\to I)\ 4
}{((\phi \to (\neg \psi)) \wedge (\psi \to (\neg \phi)))}\,(\wedge I)
$$

4)

$$
\dfrac{\dfrac{\dfrac{\dfrac{
\dfrac{\cancel{\phi}^{2} \quad \dfrac{\cancel{((\phi \to \psi) \wedge ((\neg \chi) \to (\neg \psi)))}^{3}}{(\phi \to \psi)}\,(\wedge E)}{\psi}\,(\to E)
\quad
\dfrac{\cancel{(\neg \chi)}^{1} \quad \dfrac{\cancel{((\phi \to \psi) \wedge ((\neg \chi) \to (\neg \psi)))}^{3}}{((\neg \chi) \to (\neg \psi))}\,(\wedge E)}{(\neg \psi)}\,(\to E)
}{\bot}\,(\neg E)}{\chi}\,(RAA)\ 1}{(\phi \to \chi)}\,(\to I)\ 2}{(((\phi \to \psi) \wedge ((\neg \chi) \to (\neg \psi))) \to (\phi \to \chi))}\,(\to I)\ 3
$$

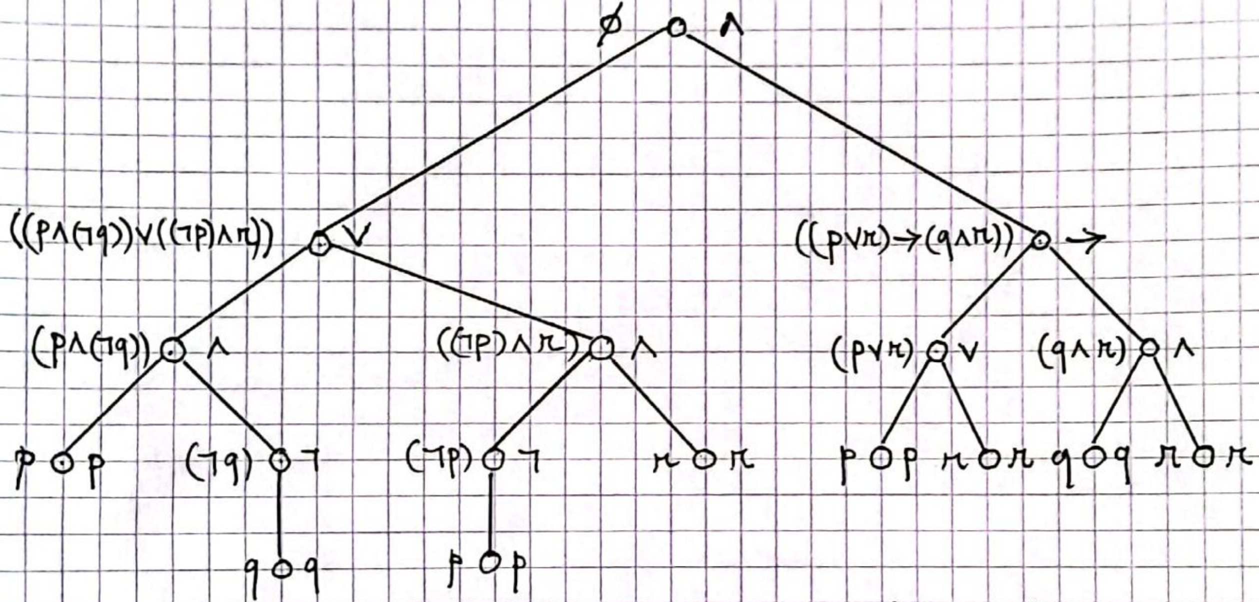
Exercise 3 $(1+2+(0{,}5+0{,}5)+1+1)$

1) We calculate the heads of the initial segments having a functor at their right end.

$d[(((p\wedge] = 3$, $d[(((p\wedge(\neg] = 4$, $d[(((p\wedge(\neg q))\vee] = 2$,
$d[(((p\wedge(\neg q))\vee((\neg] = 4$, $d[(((p\wedge(\neg q))\vee((\neg p)\wedge] = 3$,
$d[(((p\wedge(\neg q))\vee((\neg p)\wedge r))\wedge] = 1$.

Then the head of the formula is the third occurrence of $\wedge$.

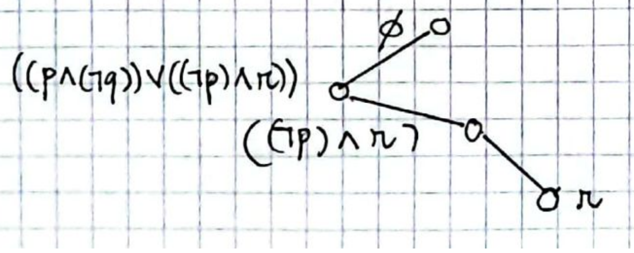
2)



3) There are 8 leaves. The number of subformulas equals the number of nodes, so there are 17 subformulas.

4) The complexity of the formula ϕ is the height of its parsing tree, which is the height of its root. Every leaf has height 0, and if μ is a node with daughters $\nu_1, \ldots, \nu_n$, then the height of μ is $\max(\text{height}(\nu_1), \ldots, \text{height}(\nu_n)) + 1$. By going upwards from leaves, we find that the complexity of ϕ is 4.

5) A branch is a path from the root to a leaf. We can take



Exercice 4 $((1,5+0,5)+(0,5+1,5))$

1) Let $\phi \in \mathcal{F}$. We will prove by induction on the complexity k of ϕ that $\phi \in \bigcup_{i\in\mathbb{N}} \mathcal{F}_i$.

If $k=0$, then ϕ is atomic, so $\phi \in \mathcal{F}_0$ and therefore $\phi \in \bigcup_{i\in\mathbb{N}} \mathcal{F}_i$.

Assume that the result holds for all formulas of complexity k, and let ϕ be a formula of complexity $k+1$. Then ϕ has one of the forms $(\neg\psi)$ or $(\psi \,\square\, \chi)$, where $\square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ and ϕ and ψ are formulas of complexities $\leq k$. Consider the case $\phi = (\psi \,\square\, \chi)$. By the induction assumption, there exist $i, j \in \mathbb{N}$ such that $\psi \in \mathcal{F}_i$ and $\chi \in \mathcal{F}_j$. Notice that since $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ for all i, then $\mathcal{F}_m \subseteq \mathcal{F}_n$ for all $m \leq n$. Therefore $\psi, \chi \in \mathcal{F}_\ell$ where $\ell = \max(i,j)$, and then $\phi = (\psi \,\square\, \chi) \in \mathcal{F}_{\ell+1}$, so $\phi \in \bigcup_{i\in\mathbb{N}} \mathcal{F}_i$.

The case $\phi = (\neg\psi)$ is similar. We have proved that $\mathcal{F} \subseteq \bigcup_{i\in\mathbb{N}} \mathcal{F}_i$.

Conversely, since each $\mathcal{F}_i$ is a set of formulas of $LP(\sigma)$, then $\bigcup_{i\in\mathbb{N}} \mathcal{F}_i \subseteq \mathcal{F}$. It follows that $\mathcal{F} = \bigcup_{i\in\mathbb{N}} \mathcal{F}_i$.

2) We will prove again by induction on the complexity of ϕ that $k(\phi) = h(\pi)$. If ϕ has complexity 0, then it is atomic, so we get $\phi \in \mathcal{F}_0$, hence $k(\phi) = 0 = h(\pi)$. Now, suppose that $k(\phi) = h(\pi)$ for all formulas of complexity $\leq k$. Let ϕ be a formula of $LP(\sigma)$ of complexity $k+1$. Then ϕ has one of the forms $(\neg\psi)$ or $(\chi_1 \,\square\, \chi_2)$. Consider the case $\phi = (\chi_1 \,\square\, \chi_2)$ (the case $\phi = (\neg\psi)$ is similar). Let π_1 and π_2 be the parsing trees associated to χ_1 and χ_2 respectively. Since χ_1 and χ_2 have complexities $\leq k$, then by induction assumption, $k(\chi_i) = h(\pi_i)$ for $1 \leq i \leq 2$. Set $h(\pi_1) = j$ and suppose $h(\pi_1) \geq h(\pi_2)$ (the case $h(\pi_2) \geq h(\pi_1)$ is similar). We have $h(\pi) = \max(h(\pi_1), h(\pi_2)) + 1 = j+1$. We will prove that $k(\phi) = j+1$. Since $k(\chi_1) = j$, then $\chi_1 \in \mathcal{F}_j \setminus \mathcal{F}_{j-1}$.

Set $r = k(X_2)$. Then $X_2 \in \mathcal{F}_r$, and since $r \leq j$ then $\mathcal{F}_r \subseteq \mathcal{F}_j$. Therefore we have $X_1, X_2 \in \mathcal{F}_j$ and $X_1 \notin \mathcal{F}_{j-1}$. This implies that $\phi = (X_1 \square X_2) \in \mathcal{F}_{j+1} \setminus \mathcal{F}_j$, which means that $k(\phi) = j+1$, as required.