Corrigé Test 2 (Section 1). 23/24

$$f_X(x) = \frac{1}{a}\left(1 - \frac{|x|}{a}\right) \mathbb{1}_{[-a,a]}(x) \quad ; \quad a > 0.$$

1) On a $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$, dPns.

$$F_X(x) = \begin{cases} 0 & \text{si } x < -a. \\ \frac{1}{a}\int_{-a}^{x}\left(1+\frac{t}{a}\right)dt = \frac{1}{a}\left(\frac{x^2}{2a}+x\right)+\frac{1}{2} & \text{si } -a \leq x < 0 \\ \frac{1}{a}\int_{-a}^{0}\left(1+\frac{t}{a}\right)dt + \frac{1}{a}\int_{0}^{x}\left(1-\frac{t}{a}\right)dt = \frac{1}{2}+\frac{1}{a}\left(x-\frac{x^2}{2a}\right) & \text{si } 0 \leq x < a \\ 1 & \text{si } x \geq a. \end{cases}$$

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D'où
$$\boxed{F_X(x) = \begin{cases} 0 & \text{si } x < -a \\ \frac{1}{2}\left(1+\frac{x}{a}\right)^2 & \text{si } -a \leq x < 0 \\ 1 - \frac{1}{2}\left(1-\frac{x}{a}\right)^2 & \text{si } 0 \leq x < a \\ 1 & \text{si } x \geq a. \end{cases}}$$

2) On a :

$$\boxed{P(X > \theta) = 1 - F_X(\theta) = \begin{cases} 1 & \text{si } \theta < -a. \\ 1 - \frac{1}{2}\left(1+\frac{\theta}{a}\right)^2 & \text{si } -a \leq \theta < 0 \\ \frac{1}{2}\left(1-\frac{\theta}{a}\right)^2 & \text{si } 0 \leq \theta < a. \\ 0 & \text{si } \theta \geq a. \end{cases}}$$

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3) $P(|X| > \theta_k) = 1 - P(|X| \leq \theta_k) = 1 - \left(F_X(\theta_k) - F_X(-\theta_k)\right)$

Comme $\theta_k = \frac{a}{2^k} < a$ pour tout $k \geq 1$.

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Alors $P(|X| > \theta_k) = 1 - \left[1 - \frac{1}{2}\left(1-\frac{\theta_k}{a}\right)^2 - \frac{1}{2}\left(1-\frac{\theta_k}{a}\right)^2\right]$

$$= \left(1 - \frac{\theta_k}{a}\right)^2 = \boxed{\left(1 - \frac{1}{2^k}\right)^2}$$

4) $\mathbb{E}[X^r] = \int_{-\infty}^{+\infty} x^r f_X(x)\,dx = \frac{1}{a}\left(\int_{-a}^{0} x^r\left(1+\frac{x}{a}\right)dx + \int_{0}^{a} x^r\left(1-\frac{x}{a}\right)dx\right)$

En changeant x en $-x$ dans l'intégrale $\int_{-a}^{0} x^r\left(1+\frac{x}{a}\right)dx$.

on obtient $\int_{0}^{a}(-x)^r\left(1-\frac{x}{a}\right)dx$

D'où $\mathbb{E}[X^r] = \frac{1}{a}\int_{0}^{a}\left((-x)^r + x^r\right)\left(1-\frac{x}{a}\right)dx$.

$$\underset{u=\frac{x}{a}}{=} \int_{0}^{1}\left((-au)^r + (au)^r\right)(1-u)\,du \quad 2$$

$$= \left(1+(-1)^r\right)a^r\int_{0}^{1} u^r(1-u)\,du = \frac{1+(-1)^r}{(r+1)(r+2)}a^r$$

$$\Rightarrow \boxed{\mathbb{E}\left[X^{2r+1}\right] = 0} \quad \text{et} \quad \boxed{\mathbb{E}\left[X^{2r}\right] = \frac{a^{2r}}{(r+1)(2r+1)}}$$

En particulier $\quad Var(X) = \mathbb{E}[X^2] = \frac{a^2}{6}$.

5) $\varphi_X(t) = \mathbb{E}[e^{itX}] = \int_{-\infty}^{+\infty} e^{itx} f_X(x)\,dx$.

$$= \frac{1}{a}\left(\int_{-a}^{0} e^{itx}\left(1+\frac{x}{a}\right)dx + \int_{0}^{a} e^{itx}\left(1-\frac{x}{a}\right)dx\right)$$

$$= \frac{1}{a}\left(\int_{0}^{a} e^{-itx}\left(1-\frac{x}{a}\right)dx + \int_{0}^{a} e^{itx}\left(1-\frac{x}{a}\right)dx\right).$$

$$= \frac{2}{a}\int_{0}^{a}\cos(tx)\left(1-\frac{x}{a}\right)dx = \boxed{\left(\frac{\sin\left(\frac{at}{2}\right)}{\left(\frac{at}{2}\right)}\right)^2} \quad 1$$

$$= \frac{2}{(at)^2}\left(1-\cos(at)\right)$$