Mathematical analysis 3
Chapter 3 : Special Functions
Gamma and Beta functions

R. KECHKAR

The National School of
Artificial Intelligence
المدرسة الوطنية للذكاء الاصطناعي

2023/2024

Course outline

Definition of Gamma function

In this chapter, we introduce the classical Gamma function, essentially understood as a generalized factorial. There are many applications of this function, for example, it is found in analysis, number theory, probability, and fractional calculus.

Definition

Let Γ *be the function defined on* $]0,+\infty[$ *by:*

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt.$$

Example:

$$\Gamma(1) = \int_0^{+\infty} t^{1-1}e^{-t}dt = \int_0^{+\infty} e^{-t}dt = -e^{-t}\Big|_0^{+\infty} = 1.$$

Properties of Gamma function

We have the following properties:

1. Γ is a defined and positive function on $]0, +\infty[$.
2. Γ is continuous on $]0, +\infty[$.
3. Γ is differentiable on $]0, +\infty[$ and

$$\forall x \in]0, +\infty[: \Gamma'(x) = \int_0^{+\infty} \ln t . t^{x-1} e^{-t} dt.$$

Properties of Gamma function

Proof: For all $t\in]0,+\infty[$:

$$t^{x-1}=e^{\ln(t^{x-1})}=e^{(x-1)\ln(t)}$$

By setting $\phi(x,t)=t^{x-1}e^{-t}$, we obtain:

$$\Gamma'(x)=\frac{d\Gamma}{dx}(x)=\int_0^{+\infty}\frac{\partial\phi(x,t)}{\partial x}dt=\int_0^{+\infty}\ln(t)e^{(x-1)\ln(t)}e^{-t}dt$$

$$=\int_0^{+\infty}\ln(t)t^{x-1}e^{-t}dt$$

4. Γ is of class C^∞ on $]0,+\infty[$ and for all $x\in]0,+\infty[$,

$$\forall k\in\mathbb{N}:\Gamma^{(k)}(x)=\int_0^{+\infty}(\ln(t))^k t^{x-1}e^{-t}dt.$$

5. Γ recurrence relation (the functional equation of Γ):

$$\forall x\in]0,+\infty[,\Gamma(x+1)=x\Gamma(x)$$

Properties of Gamma function

Proof: Integrating Γ by parts:

$$u(t) = e^{-t}, \quad v'(t) = t^{x-1} \Rightarrow u'(t) = -e^{-t}, \quad v(t) = \frac{1}{x}t^x,$$

for all $x \in]0, +\infty[$, we obtain

$$\Gamma(x) = \lim_{a\to+\infty}\left(\frac{1}{x}t^x e^{-t}\Big|_0^a\right) + \int_0^{+\infty}\frac{1}{x}t^x e^{-t}dt$$

$$= \lim_{a\to+\infty}\left(\frac{1}{x}a^x e^{-a}\right) + \frac{1}{x}\int_0^{+\infty} t^x e^{-t}dt$$

$$= \frac{1}{x}\Gamma(x+1).$$

Where:

$$\lim_{a\to+\infty}\frac{1}{x}a^x e^{-a} = \lim_{a\to+\infty}\frac{1}{x}e^{x\ln a - a} = \lim_{a\to+\infty}\frac{1}{x}e^{a(x-\ln a)-1} = 0.$$

Properties of Gamma function

6. The function Γ is the extension of the factorial function to $x \in]0, +\infty[$. For all $n \in \mathbb{N}$: $\Gamma(n+1) = n!$.

   Proof: For $n > 0$, we have

   $$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)$$

   $$= n(n-1)(n-2)\Gamma(n-2) = \ldots = n(n-1)(n-2)\ldots 3 \times 2 \times \Gamma(2) = n!.$$

   Since we agree that $0! = 1 = \Gamma(1) = \Gamma(0+1)$, we deduce

   $$\forall n \in \mathbb{N} : \Gamma(n+1) = n!,$$

   and we can write

   $$\forall n \in \mathbb{N} \setminus \{0\} : \Gamma(n) = (n-1)!.$$

Properties of Gamma function

7. We have $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
   Proof: $\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} t^{\frac{1}{2}-1} e^{-t} dt = \int_0^{\infty} t^{-\frac{1}{2}} e^{-t} dt$.
   Let $t^{\frac{1}{2}} = y \Rightarrow t = y^2$ and $dt = 2ydy$, thus

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} y^{-1} e^{-\frac{y^2}{2}} 2ydy = 2\int_0^{\infty} e^{-\frac{y^2}{2}} dy = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}.$$

Properties of Gamma function

8. For all $k \in \mathbb{N}$: $\Gamma\left(k+\frac{1}{2}\right) = \dfrac{(2k)!}{2^{2k}\cdot k!}\sqrt{\pi}$.

Proof: Knowing that $\Gamma(x+1) = x\Gamma(x)$, for all $k \in \mathbb{N}$, we obtain:

$$\begin{aligned}
\Gamma\left(k+\frac{1}{2}\right) &= \left(k-\frac{1}{2}\right).\Gamma\left(k-\frac{1}{2}\right) \\
&= \left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)\Gamma\left(k-\frac{3}{2}\right) \\
&\ \vdots \\
&= \left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)\cdots\frac{3}{2}.\frac{1}{2}.\Gamma(\frac{1}{2}) \\
&= \left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)\cdots\frac{3}{2}.\frac{1}{2}.\sqrt{\pi} \\
&= \left(\frac{2k-1}{2}\right)\left(\frac{2k-3}{2}\right)\cdots\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}
\end{aligned}$$

On the other hand, we have

$$k! = k(k-1)(k-2)\times\ldots\times 3\times 2\times 1 = \frac{2k}{2}\cdot\frac{2k-2}{2}\cdot\frac{2k-4}{2}\cdot\ldots\cdot\frac{4}{2}\cdot\frac{2}{2},$$

so

$$\Gamma\left(k+\frac{1}{2}\right) = \left(\frac{2k}{2}\cdot\frac{2k-1}{2}\cdot\frac{2k-2}{2}\cdot\ldots\cdot\frac{3}{2}\cdot\frac{2}{2}\cdot\frac{1}{2}\right)\cdot\frac{\sqrt{\pi}}{k!}$$

therefore

$$\Gamma\left(k+\frac{1}{2}\right) = \frac{(2k)!}{2^{2k}\cdot k!}\sqrt{\pi}.$$

The negative case

The negative case

1. Thanks to the recurrence relation $\Gamma(x+1) = x\Gamma(x)$, we can set by convention
$$\forall x \in]-1,0[: \Gamma(x) = \frac{\Gamma(x+1)}{x}.$$
Example:
$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)} = -2\sqrt{\pi}.$$

2. Let $n \in \mathbb{N}$, for any non-integer negative value of x, such that $x \in]-n, -n+1[$, we have
$$\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)(x+2)\ldots(x+n-1)},$$
and $\Gamma(x)$ has the sign of $(-1)^n$.

3. For all $n \in \mathbb{N}$: $\Gamma\left(-n+\frac{1}{2}\right) = (-1)^n \dfrac{2^{2n} \cdot n!}{(2n)!} \sqrt{\pi}$.

Course outline

Definition of Beta function

Definition

For all $x, y \in]0, +\infty[$, *we define the function Beta by:*

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

Example:

1. $\beta(1,1) = \int_0^1 t^{1-1}(1-t)^{1-1}dt = \int_0^1 1dt = 1.$
2. $\beta(2,1) = \int_0^1 t^{2-1}(1-t)^{1-1}dt = \int_0^1 tdt = \frac{1}{2}.$
3. $\beta(1,2) = \int_0^1 t^{1-1}(1-t)^{2-1}dt = \int_0^1 (1-t)dt = \frac{1}{2}.$
4. Show that $\beta(2,2) = \frac{1}{6}$.

Definition of Beta function

Example: Calculate $\beta\left(\frac{1}{2},\frac{1}{2}\right)$.
Solution: We have

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=\int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt,$$

using the change $t^{\frac{1}{2}}=x \Rightarrow t=x^2$ and $dt=2xdx$, then

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=\int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt=\int_0^1 x^{-1}(1-x^2)^{-\frac{1}{2}}2xdx$$

$$=2\int_0^1 \frac{1}{\sqrt{1-x^2}}dx=2[\arcsin x]_0^1=2\left(\frac{\pi}{2}\right)=\pi.$$

Proposition

We have

$$\forall x, y \in]0, +\infty[: \beta(x,y) = \beta(y,x)$$

Proof: We have

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

let's set $s = 1 - t$, then we obtain $t = 1 - s$, $dt = -ds$, $t = 0 \Rightarrow s = 1$ and $t = 1 \Rightarrow s = 0$, this gives us

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = -\int_1^0 (1-s)^{x-1}s^{y-1}ds$$

$$= \int_0^1 s^{y-1}(1-s)^{x-1}ds = \beta(y,x).$$

Proposition 2.2.2. Relation between the functions Γ and β
For all $x, y \in]0, +\infty[$:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \qquad (2.13)$$

Proof: Consider for x, y strictly positive:

$$\begin{aligned}
\Gamma(x)\Gamma(y) &= \int_0^{+\infty} t^{x-1}e^{-t}dt \int_0^{+\infty} s^{y-1}e^{-s}ds \\
&= \int_0^{+\infty} t^{x-1}e^{-t}dt \int_0^{+\infty} s^{y-1}e^{-s}ds \\
&= \int_0^{+\infty}\int_0^{+\infty} t^{x-1}s^{y-1}e^{-(t+s)}dtds \\
&= \iint_D t^{x-1}s^{y-1}e^{-(t+s)}dsdt,
\end{aligned}$$

where $D = \{(t,s) : t > 0, \text{ and } s > 0\}$.

Let's use the following change of variables:

$$u = t + s$$
$$v = \frac{t}{t+s},$$

which implies $t = u \cdot v$ and $s = u - u \cdot v$. We deduce that $\Delta = \{(u,v) : u > 0, \text{ and } 0 < v < 1\}$ and the Jacobian

$$J = \begin{vmatrix} \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \\ \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u$$

So, $dtds = |J|dudv = ududv$, which gives

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \iint_\Delta u^{x+y-1} v^{x-1} (1-v)^{y-1} e^{-u} ududv \\ &= \left(\int_0^{+\infty} u^{x+y-1} e^{-u} du \right) \left(\int_0^1 v^{x-1} (1-v)^{y-1} dv \right) \\ &= \Gamma(x+y) \times \beta(x,y). \end{aligned}$$

Proposition

We have

1. $\forall x, y > 0 : \beta(x,y) = \int_0^{+\infty} \frac{s^{x-1}}{(1+s)^{x+y}} ds.$
2. $\forall x, y > 0 : \beta(x,y) = \int_0^{1} \frac{s^{x-1} + s^{y-1}}{(1+s)^{x+y}} ds.$
3. $\beta(x,y) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1} d\theta.$
4. $\forall x, y > 0 : \beta(x,y+1) = \frac{y}{x+y}\beta(x,y)$, *and* $\beta(x+1,y) = \frac{x}{x+y}\beta(x,y).$
5. $\beta(x,y).\beta(x+y,1-y) = \frac{\pi}{x\sin(\pi y)}.$

Proposition

For $0 < x < 1$*, we have:*

$$\beta(x, 1-x) = \frac{\pi}{\sin(\pi x)}.$$

Proposition (Stirling formula)

For $x > 0$ *we have*

$$x! \sim x^x e^{-x}\sqrt{2\pi x}, \quad (x \to +\infty).$$

In other words

$$\Gamma(x+1) \sim x^x e^{-x}\sqrt{2\pi x}, \quad (x \to +\infty).$$

Form the previous proposition we can deduce the flowing limits:

$$\lim_{x\to+\infty}\Gamma(x)=+\infty$$

$$\lim_{x\to+\infty}\frac{\Gamma(x)}{x}=\lim_{x\to+\infty}\frac{(x-1)\Gamma(x-1)}{x}=+\infty$$

$$\lim_{x\to0}\Gamma(x)=\lim_{x\to0}\frac{\Gamma(x)}{x}=+\infty$$

Exercise

1. Show that
   - $\displaystyle\int_0^1 (\ln(\frac{1}{y}))^{n-1} dy = \Gamma(n)$.
   - $\displaystyle\int_0^{+\infty} x^n e^{-k^2x^2} dx = \frac{1}{2k^{n+1}} \Gamma\left(\frac{n+1}{2}\right)$.
2. Calculate the following integrals:
   - $\displaystyle\int_0^{+\infty} e^{-x^2} dx$.
   - $\displaystyle\int_0^{+\infty} \sqrt{x} e^{-3\sqrt{x}} dx$.