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- If $X(\Omega)$ is infinite uncountable X is said to be a continuous random variable, denoted c.r.v.
  Moreover a c.r.v. is said to be absolutely continuous if it admits a continuous and derivable distribution function (except possibly at some points).

Discrete random variables

Definition

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Let X be a d.r.v. one calls probability distribution or mass function of the r.v. X the application

$$\begin{array}{rcl} p & : & \mathbb{R} \longrightarrow [0,1] \\ & & x \longmapsto p(x) = \mathbb{P}(X = x). \end{array}$$

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Properties:

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The distribution is usually written in the following form

x	0	1	2	$\sum_{x=0}^{x=2} p(x)$
$\mathbb{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

Discrete random variables

Cumulative distribution function

1. If X is a discrete r.v. then

$$F_X(x) = \sum_{x_i \leq x} \mathbb{P}(X = x_i) = \sum_{x_i \leq x} p(x_i).$$

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2. The cumulative distribution function allows to determine the probability law of the r.v. X.
Indeed, $\forall x_j \in X(\Omega)$

$$\mathbb{P}(X = x_j) = \sum_{i=1}^{j} \mathbb{P}(X \leq x_i) - \sum_{i=1}^{j-1} \mathbb{P}(X \leq x_i) = F_X(x_j) - F_X(x_{j-1}).$$

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x	0	1	2	3	$\sum_{x=0}^{x=2} p(x)$
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$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{125}{216} & \text{if } 0 \leq x < 1 \\ \frac{200}{216} & \text{if } 1 \leq x < 2 \\ \frac{215}{216} & \text{if } 2 \leq x < 3 \\ 1 & \text{if } 3 \leq x \end{cases}$$

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3. $\mathbb{P}(X \in]a, \infty[) = \mathbb{P}(X \in [a, \infty[) = 1 - F_X(a).$
4. $\mathbb{P}(X \in]-\infty, b]) = \mathbb{P}(X \in]-\infty, b[) = F_X(b).$

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4. The cumulative distribution function F_X can be written, for all $x \in \mathbb{R}$ in the form

$$F_X(x) = \int_{-\infty}^{x} f_X(s)\, ds.$$

Continuous random variables

Definition

A function f that satisfies the four previous conditions is called a probability density function or distribution function of an absolutely continuous random variable X.

Continuous random variables

Example

Let X be a random variable with cumulative distribution function F_X given by

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - \frac{1}{2}(x+2)\, e^{-\frac{x}{2}} & \text{if } x \geq 0 \end{cases}$$

1. Show that the random variable X is absolutely continuous.
2. Find the constant C such that the function f defined by

$$f(x) = \begin{cases} Cxe^{-\frac{x}{2}} & \text{if } \; x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

   be the probability density of the random variable X.
3. Verify that

$$F_X(x) = \int_{-\infty}^{x} f(s)\, ds.$$

Solution

1. F_X is continuous on $]-\infty, 0[$ and on $]0, +\infty[$ show that it is continuous in 0. We have $\lim_{x\to 0}\left(1-\frac{1}{2}(x+2)\,e^{-\frac{x}{2}}\right)=0$ hence F_X is continuous in 0.
F_X is derivable on $]-\infty, 0[$ and on $]0, +\infty[$ show that it is derivable in 0. We have $\lim_{x\to 0}\left(\frac{F_X(x)-F_X(0)}{x}\right)=0$
F_X is derivable on $\mathbb{R}$, hence X is an absolutely continuous variable.

Solution

2. To show that f is a density function we determine first the constant C using the condition 3 of the definition i.e. $\int_{-\infty}^{+\infty} f(x)\,dx = 1$, then we verify the other conditions.
We have

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)\,dx &= \int_{0}^{+\infty} Cxe^{-\frac{x}{2}}\,dx = C\left(\left[-2xe^{-\frac{x}{2}}\right]_0^{\infty} + 2\int_{0}^{+\infty} e^{-\frac{x}{2}}\,dx\right) \\ &= C\left[-4e^{-\frac{x}{2}}\right]_0^{\infty} = 4C = 1.\end{aligned}$$

hence $C = \frac{1}{4}$ and

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{4}e^{-\frac{x}{2}} & \text{if } x \geq 0 \end{cases}$$

We have $f_X(x) \geq 0; \forall x \in \mathbb{R}$.
It is a continuous fuction in 0 and then continuous on $\mathbb{R}$.
Then f is a probability density function of the random variable X.

Solution

3. If $x < 0$, $\int_{-\infty}^{x} f(s)\, ds = 0$ since on $]-\infty, 0[$, $F_X(x) = 0$
If $x \geq 0$,

$$\begin{aligned}
\int_{-\infty}^{x} f(s)\, ds &= \int_{-\infty}^{0} f(s)\, ds + \int_{0}^{x} f(s)\, ds = 0 + \int_{0}^{x} \frac{s}{4} e^{-\frac{s}{2}}\, ds \\
&= \frac{1}{4}\left(\left[-2se^{-\frac{s}{2}} \right]_0^x + 2\int_0^x e^{-\frac{s}{2}}\, ds \right) \\
&= \frac{1}{4}\left(-2xe^{-\frac{x}{2}} - 4\left[e^{-\frac{s}{2}} \right]_0^x \right) \\
&= \frac{1}{4}\left(-2xe^{-\frac{x}{2}} - 4e^{-\frac{x}{2}} + 4 \right) = 1 - \frac{1}{2}(x+2)\, e^{-\frac{x}{2}}
\end{aligned}$$

hence $F_X(x) = \int_{-\infty}^{x} f(s)\, ds$.

Mathematical expectation and variance

Definition

Let X be a d.r.v. with possible values $x_1, x_2, \cdots$ and mass function $p(x)$.

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Let X be a d.r.v. with possible values $x_1, x_2, \cdots$ and mass function $p(x)$ .The mathematical expectation of X is

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provided that the above serie is absolutely convergent, otherwise we will say that X does not have a mathematical expectation.

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Remark

If X has a finite number of values then $\mathbb{E}[X]$ exists.

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Example

Let T be a c.r.v. with distribution function f defined by

$$f\left(t\right) = \begin{cases} \frac{1}{t^2} & \text{if } t > 1 \\ 0 & \text{elswhere} \end{cases}$$

Determine $\mathbb{E}\left[T\right]$.

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Definition

Let G be a function of a random variable X, the expectation of $G(X)$ is given by

$$\mathbb{E}[G(X)] = \begin{cases} \sum_{x\in\mathbb{R}} G(x)\, p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{+\infty} G(x)\, f(x)\, dx & \text{if } X \text{ is continuous} \end{cases}$$

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Definition

Let X be a random variable, we call moment of order k $(k \in \mathbb{N})$ the following value

$$\mathbb{E}\left[X^k\right] = \begin{cases} \sum_{x \in \mathbb{R}} x^k p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{+\infty} x^k f(x)\, dx & \text{if } X \text{ is continuous} \end{cases}$$

provided that the above serie and integral are absolutely convergent.

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Let X be a random variable, the variance of X, noted σ_X^2 or $Var\left(X\right)$ is

$$\sigma_X^2 = \mathbb{E}\left[\left(X - \mathbb{E}\left[X\right]\right)^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2.$$

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We call standard deviation of X the number

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If $\mathbb{E}[X] = 0$ we say that the random variable is centrend.
If $Var(X) = 1$ we say that the random variable is reduced.

Mathematical expectation and variance

Theorem

Let X be a random variable with expectation $\mathbb{E}[X]$ and variance σ_X^2. If $Y = aX + b$ where a and b are real constants, then

$$\mathbb{E}[Y] = a\mathbb{E}[X] + b$$

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Discrete uniform distribution

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The r.v. X has the discrete uniform distribution on the set of real numers $\{x_1, \cdots, x_n\}$

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The r.v. X has the discrete uniform distribution on the set of real numers $\{x_1, \cdots, x_n\}$if $\mathbb{P}_X$ is the equiprobability on this set i.e.: $X \in X(\Omega) = \{x_1, \cdots, x_n\}$ and $\forall k \in \Omega, \mathbb{P}(X = k) = \frac{1}{n}$

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$\mathbb{E}(X) = \frac{n+1}{2}$; $Var(X) = \frac{n^2-1}{12}$.

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The r.v. X has the discrete uniform distribution on the set of real numers $\{x_1, \cdots, x_n\}$if $\mathbb{P}_X$ is the equiprobability on this set i.e.:
$X \in X(\Omega) = \{x_1, \cdots, x_n\}$ and $\forall k \in \Omega, \mathbb{P}(X = k) = \frac{1}{n}$
We note $X \rightsquigarrow \mathcal{U}(n)$.
$\mathbb{E}(X) = \frac{n+1}{2}$; $Var(X) = \frac{n^2-1}{12}$.

Example

When we throw a dice, the number obtained follow the uniforme distribution on $\{1, \cdots, 6\}$ with $\mathbb{P}_X(x) = \frac{1}{6}, \forall x \in \{1, \cdots, 6\}$.

Discrete probability distribution (finite case)

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Example

In the toss of an unbalanced coin, the probability of getting "heads" is $p \neq \frac{1}{2}$. X the r.v. defined by $X = 1$ if we get "heads" and $X = 0$ if we get "tails". $X \rightsquigarrow \mathcal{B}(p)$ with the probability distribution

$$\mathbb{P}(X=x) = \begin{cases} p, & \text{if } x = 1 \\ q, & \text{if } x = 0 \end{cases}$$

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Binomiale distribution

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Remark

The r.v. X can be defined as a sum of n independent Bernoulli r.v. $X_1, X_2, \cdots, X_n$ $(X = X_1 + X_2 + \cdots + X_n)$. Such that $\mathbb{P}(X_i = 1) = p$.

Definition

A r.v. X follows a binomial distribution of parameters (n, p) where $n \geq 0$ and $(p \in [0, 1])$ if $X(\Omega) = \{0, 1, \cdots, n\}$ and $\mathbb{P}(X = k) = C_n^k p^k (1 - p)^{n-k}, \forall k = 0, 1, \cdots, n$ (with: $p + q = 1$).

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$\mathbb{E}(X) = np$; $Var(X) = np(1 - p)$.

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Example

Let $X \rightsquigarrow \mathcal{B}(n, p)$.

1. Determine n such that $\mathbb{P}(X = 0) \leq 0,01$;
2. Determine n such that $\mathbb{P}(X \geq 1) \geq 0,90$.

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Remark

The possible values of X are $\max(0, n - N_q) \leq k \leq \min(n, N_p)$

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We note $X \rightsquigarrow \mathcal{H}(N, n, p)$, with $p = \frac{N_p}{N}, p + q = 1$.The a.v. X follows the hypergeometric law of parameters

$\mathbb{E}(X) = np; Var(X) = npq\frac{N-n}{N-1}$.

Discrete probability distribution (finite case)

Geometric distribution

The geometric distribution is the law of expectation of the first success of a sequence of independent trials each of which has a probability p of success, i.e. $\mathbb{P}(X = k)$ is the probability that the k^{th} trial is the first success.

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Example

We play heads or tails with a rigged coin such that the probability of getting tails is $\frac{1}{3}$. Let X be the r.v. representing the number of

Discrete probability distribution (finite case)

Pascal (Negative Binomial or Polya) distribution

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