# Mathematical analysis 2
# Chapter 6 : Fourier series

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# Course outline

## Some definitions

### Definition

*We say that a function defined on a subset $D$ of $\mathbb{R}$ is $T$-periodic if:*

$$\forall x \in D: \quad x+T \in D \ \text{ and } \ f(x+T) = f(x).$$

### Lemma

- *If $f$ is $T$-periodic, then $f(x+nT) = f(x)$ for any integer $n$.*
- *The functions $\cos(mx)$ and $\sin(mx)$ have a period $T = \dfrac{2\pi}{m}$ $(m \neq 0)$.*

### Lemma

*Let $f : \mathbb{R} \to \mathbb{R}$ be a $T$-periodic function where $T > 0$, integrable over the interval $[0, T]$. Then:*

$$\forall \alpha \in \mathbb{R}, \quad \int_{\alpha}^{\alpha+T} f(t)\,dt = \int_0^T f(t)\,dt.$$

# Course outline

1. Generalities
2. Trigonometric series
3. Fourier series

# Trigonometric series

## Definition

- *We call trigonometric series all series of functions* $\sum f_n$ *whose general term is of the form:*

$$f_0(x) = \frac{a_0}{2} \quad \text{and} \quad f_n(x) = a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \quad \forall n \geq 1,$$

*where* $l \neq 0$, *and* $(a_n)_n$ *and* $(b_n)_n$ *are two sequences called the coefficients of the trigonometric series.*

- *A trigonometric series is denoted as:*

$$\frac{a_0}{2} + \sum_{n\geq 1} (a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l})$$

# Trigonometric series

**Example.**

- *The series* $\sum_{n\geq 1} \frac{\cos(nx)}{n^2}$ *is a trigonometric series with* $l=\pi$, $a_0=0$, *and for all* $n \in \mathbb{N}^*$, $a_n = \frac{1}{n^2}$ *and* $b_n = 0$.

- *The series* $2 + \sum_{n\geq 1} \frac{\cos(nx)}{n^2} + \frac{\sin(nx)}{n}$ *is a trigonometric series with* $l=\pi$, $a_0=4$, *and for all* $n \in \mathbb{N}^*$, $a_n = \frac{1}{n^2}$ *and* $b_n = \frac{1}{n}$.

# Convergence of trigonometric series

## Theorem

*If the numerical series* $\sum a_n$ *and* $\sum b_n$ *converge absolutely, then the trigonometric series:*

$$\frac{a_0}{2} + \sum_{n\geq 1} (a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l})$$

*converges normally (and then absolutely and uniformly) over* $\mathbb{R}$.

# Convergence of trigonometric series

## Proposition

*Let $(a_n)_n$ and $(b_n)_n$ be two sequences of positive real numbers, decreasing, and converging to 0. The trigonometric series:*

$$\frac{a_0}{2} + \sum_{n\geq 1}(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l})$$

- *converges pointwise for all $x \in \mathbb{R}$ such that $x \neq 2kl$, $k \in \mathbb{Z}$.*
- *converges uniformly on every interval $[2kl+\alpha, 2(k+1)l-\alpha]$ where $k \in \mathbb{Z}$ and $\alpha \in ]0, l[$.*

# Properties of the sum of a trigonometric series

**Proposition (Continuity)**

*If the trigonometric series* $\frac{a_0}{2} + \sum_{n\geq 1}(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l})$ *converges uniformly over an interval* $I$*, then its sum is a continuous function on* $I$.

**Proof.**

- *All functions* $f_n$ *such that* $f_n(x) = a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$ *are continuous on* $\mathbb{R}$;
- *The series* $\frac{a_0}{2} + \sum_{n\geq 1}(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l})$ *converges uniformly on* $I$; *thus, the sum function is continuous on* $I$.

# Properties of the sum of a trigonometric series

## Proposition (Periodicity)

*If the trigonometric series* $a_0 + \sum_{n\geq 1}(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l})$ *converges, then its sum is a periodic function with period* $2l$.

Suppose the series $a_0 + \sum_{n\geq 1}(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l})$ converges and its sum is equal to $S(x)$, i.e., $S(x) = a_0 + \sum_{n\geq 1}(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l})$.
Let's demonstrate that $S(x+2l) = S(x)$:

$$\begin{aligned}
S(x+2l) &= a_0 + \sum_{n\geq 1}(a_n \cos(\frac{n\pi}{l}(x+2l)) + b_n \sin(\frac{n\pi}{l}(x+2l))) \\
&= a_0 + \sum_{n\geq 1}(a_n \cos(\frac{n\pi x}{l} + 2n\pi) + b_n \sin(\frac{n\pi x}{l} + 2n\pi)) \\
&= a_0 + \sum_{n\geq 1}(a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})) = S(x)
\end{aligned}$$

# Links between the sum of a trigonometric series and the coefficients of the series

## Proposition

*If the trigonometric series:* $a_0 + \sum_{n\geq 1} (a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l})$ *converges uniformly to* $f$ *on* $[-l, l]$, *then its coefficients are given by:*

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx, \quad \forall n \geq 1$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx, \quad \forall n \geq 1$$

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# Expansion of a function into a Fourier series

Given a function $f$ from $\mathbb{R}$ to $\mathbb{R}$, $2l$-periodic, we would like to find coefficients $(a_n)_n$ and $(b_n)_n$ such that $f$ can be expanded into a trigonometric series, that is:

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

The purpose of this section is to address the following two questions:

- What properties must the function $f$ have for the coefficients $(a_n)_n$ and $(b_n)_n$ to exist?
- If $(a_n)_n$ and $(b_n)_n$ exist, what properties must the function $f$ have for its trigonometric series to converge and for its sum to be equal to $f(x)$?

# Fourier coefficients

## Definition

*Let $f$ be a function that is $2l$-**periodic**, **integrable over any bounded closed interval in** $\mathbb{R}$.*

- *The **Fourier coefficients** of $f$ are defined as follows:*
$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$
$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \quad \forall n \geq 1$$
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \quad \forall n \geq 1$$
- *The **Fourier series** associated with $f$ is the trigonometric series given by:*
$$Ff(x) = \frac{a_0}{2} + \sum_{n\geq 1}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$$

# Remarks

## Remark

- *We can replace the boundaries of the integrals in the previous definition by* $\int_0^{2l}$ *or to any interval of length* $2l$.
- *It is not evident that* $Ff(x)$ *converges, and even if it converges, its sum is not necessarily* $f(x)$.

# Definitions

## Definition

- *A function $f$ is said to be **even** if $f(x) = f(-x)$ $\forall x \in D_f$.*
- *The graph of $f$ is symmetric with respect to the $y$-axis.*
- *In this case, $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ for all $a$.*

## Definition

- *A function $f$ is said to be **odd** if $f(x) = -f(-x)$ $\forall x \in D_f$.*
- *The graph of $f$ is symmetric with respect to the origin $O$.*
- *In this case, $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$ for all $a$.*

# Fourier series of odd an even functions

## Proposition

*Let $f$ be a function that is $2l$-**periodic** and **integrable over any closed and bounded interval** in $\mathbb{R}$.*

1. *If $f$ is **even**, then for all $n \in \mathbb{N}^*$, $b_n = 0$ and*

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx; \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx, \quad for\ n \geq 1.$$

2. *If $f$ is **odd**, then for all $n \in \mathbb{N}$, $a_n = 0$ and*

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx, \quad for\ n \geq 1.$$

# Case of functions 2-$\pi$ periodic ($l = \pi$)

## Proposition

*Let $f$ be a $2\pi$-periodic function, integrable over any closed and bounded interval in $\mathbb{R}$.*

1. *The Fourier coefficients of $f$ are given by:* $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \quad \forall n \geq 1.$$
2. *The Fourier series associated with $f$ is:*
$$Ff(x) = \frac{a_0}{2} + \sum_{n\geq 1} (a_n\cos(nx) + b_n\sin(nx))$$
3. *If $f$ is even, then for all $n \in \mathbb{N}^*$, $b_n = 0$ and $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$*
$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx, \quad \forall n \geq 1$$
4. *If $f$ is odd, then for all $n \in \mathbb{N}$, $a_n = 0$ and*
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx, \quad \forall n \geq 1$$

## Case of functions 2-$\pi$ periodic ($l = \pi$)

**Example:** Consider the function $f : [-\pi, \pi] \to \mathbb{R}$, a $2\pi$-periodic and even function defined by $f(x) = x$ if $x \in [0, \pi]$.

- Plot the graph of $f$ in the interval $[-2\pi, 2\pi]$.
- Calculate the Fourier coefficients of $f$ and provide its Fourier series.

**Solution**

1) Graph of $f$ in the interval $[-2\pi, 2\pi]$:

2) Calculation of the Fourier coefficients of $f$:

$f$ is $2\pi$-periodic over $\mathbb{R}$. $f$ is continuous over $\mathbb{R}$ , thus it is integrable over any bounded closed set of $\mathbb{R}$. Then $Ff$ exists.

$f$ is even, so $b_n = 0, \forall n \geq 1$. And

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi} x\, dx = \frac{2}{\pi} \left[ \frac{x^2}{2} \right]_0^{\pi} = \pi.$$

$$\forall n \geq 1, a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

By integrating by parts,

$$\begin{cases} u = x \\ v' = \cos(nx) \end{cases} \quad \Longrightarrow \quad \begin{cases} u' = 1, \\ v = \frac{1}{n}\sin(nx) \end{cases}$$

$$a_n = \frac{2}{n\pi}\left[x\sin(nx)\right]_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi}\sin(nx)\,dx = \frac{2}{n^2\pi}(\cos(n\pi) - 1) = \frac{2}{n^2\pi}((-1)^n - 1).$$

We obtain:

$$Ff(x) = \frac{\pi}{2} + \sum_{n\geq 1}\frac{2((-1)^n - 1)}{n^2\pi}\cos(nx).$$

We notice that

$$a_n = \begin{cases} -\frac{4}{(2k+1)^2\pi}, & \text{if } n = 2k+1, k \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

$$Ff(x) = \frac{\pi}{2} - \sum_{n\geq 0}\frac{4}{(2n+1)^2\pi}\cos((2n+1)x).$$

## Piecewise continuity and piecewise differentiability

### Definition

*We say that the function $f$ is* ***piecewise continuous*** *on $[a,b]$ if there exists a subdivision $a = x_0 < x_1 < \ldots < x_i < \ldots < x_n = b$ such that:*

- *For every $i$, $f$ is continuous on each $]x_i, x_{i+1}[$,*
- *and $\lim\limits_{x \to x_i^+} f(x)$ and $\lim\limits_{x \to x_{i+1}^-} f(x)$ exist and are finite, $i := 0; \ldots; n-1$.*

### Definition

*We say that the function $f$ is $C^1$* ***piecewise*** *on $[a,b]$ if there exists a subdivision $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ such that:*

- *For every $i$, $f$ is $C^1$ on each $]x_i, x_{i+1}[$,*
- *and $\lim\limits_{x \to x_i^+} f'(x)$ and $\lim\limits_{x \to x_{i+1}^-} f'(x)$ exist and are finite, $i := 0; \ldots; n-1$.*

## Dirichlet theorem

What properties must the function $f$ possess for its Fourier series to converge and for its sum to be equal to $f$?

### Theorem

*Let $f$ be a function that is $2l$-periodic, integrable over any bounded closed set in $\mathbb{R}$, and let $x_0 \in \mathbb{R}$ satisfy:*

1. $\lim\limits_{x \to x_0^+} f(x) = f(x_0^+)$, *and* $\lim\limits_{x \to x_0^-} f(x) = f(x_0^-)$, *exist and are finite.*
2. $\lim\limits_{x \to x_0^+} \dfrac{f(x) - f(x_0^+)}{x - x_0}$ *and* $\lim\limits_{x \to x_0^-} \dfrac{f(x) - f(x_0^-)}{x - x_0}$ *exist and are finite.*

*Then, $Ff(x_0)$ converges, and its sum is equal to $\frac{1}{2}(f(x_0^+) + f(x_0^-))$, i.e.,*

$$Ff(x_0) = \frac{1}{2}(f(x_0^+) + f(x_0^-)).$$

*Additionally, if $f$ is continuous at $x_0$ (i.e., $f(x_0^+) = f(x_0^-) = f(x_0)$), then:*

$$Ff(x_0) = f(x_0).$$

# Dirichlet theorem

## Corollary

*If* $f$ *is a* $2l$*-periodic function, piecewise* $C^1$ *on* $[-l,l]$*, then the Fourier series* $Ff$ *of* $f$ *converges on* $\mathbb{R}$*, and its sum is given by:*

$$\forall x \in \mathbb{R},\ Ff(x) = \frac{1}{2}\left(f(x^+) + f(x^-)\right)$$

*Moreover, if* $f$ *is continuous on* $E \subseteq \mathbb{R}$*, then:* $\forall x \in E,\ Ff(x) = f(x)$.

# Parseval formula

## Theorem

*Let $f$ be a $2l$-periodic function, integrable over any bounded closed set of $\mathbb{R}$, with Fourier coefficients $(a_n)_n$ and $(b_n)_n$. Then:*

1. $\sum\limits_{n\geq 1}(a_n^2+b_n^2)$ *is a convergent numerical series.*
2. $\dfrac{1}{l}\displaystyle\int_{-l}^{l} f^2(x)\,dx = \dfrac{a_0^2}{2} + \sum_{n\geq 1}(a_n^2+b_n^2)$ *(Parseval's Formula).*

## Corollary

*If $f$ is a $2l$-periodic function, integrable over any bounded closed set of $\mathbb{R}$, then its Fourier coefficients tend toward zero as $n \to +\infty$, which means that*

$$\lim_{n\to+\infty} a_n = 0 \quad \textit{and} \quad \lim_{n\to+\infty} b_n = 0$$

**Example:** Consider the function $f : \mathbb{R} \to \mathbb{R}$, a $2\pi$-periodic and odd function defined by $f(x) = 1, \forall x \in [0, \pi]$.

- Plot the graph of $f$ in the interval $[-2\pi, 2\pi]$.
- Calculate the Fourier coefficients of $f$ and provide its Fourier series.
- Study the convergence of of $Ff$.
- Derive the value of the series:

$$\sum_{n\geq 0} (-1)^n \frac{1}{2n+1}, \sum_{n\geq 0} \frac{1}{(2n+1)^2}, \sum_{n\geq 1} \frac{1}{n^2}, \sum_{n\geq 1} (-1)^{n+1} \frac{1}{n^2}.$$

**Solution:**
1. **The graph of** $f$:
2. **Calculating the Fourier coefficients of** $f$: $f$ is $2\pi$-periodic over $\mathbb{R}$. $f$ is piecewise continuous over $\mathbb{R}$ , hence it is integrable over any closed and bounded set of $\mathbb{R}$. Thus, $Ff$ exists.

As $f$ is odd, $a_n = 0, \forall n \geq 0$. And

$$\forall n \geq 1,\ b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi} = -\frac{2}{n\pi}(\cos(n\pi) - 1)$$

Then

$$\forall n \geq 1,\ b_n = -\frac{2((-1)^n - 1)}{n\pi}.$$

Thus,

$$Ff(x) = \frac{a_0}{2} + \sum_{n\geq 1}(a_n\cos(nx) + b_n\sin(nx)) = \sum_{n\geq 1}\frac{2(1-(-1)^n)}{n\pi}\sin(nx).$$

We can see that:

$$b_n = \begin{cases} \frac{4}{(2k+1)\pi}, & \text{if } n = 2k+1, k \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}.$$

Hence,

$$Ff(x) = \sum_{n\geq 0}\frac{4}{(2n+1)\pi}\sin((2n+1)x).$$

3. **Study of the convergence of $Ff$:** Since $f$ is odd and $2\pi$-periodic, it suffices to apply the Dirichlet's Theorem on $[0,\pi]$ to $f$. $f$ is of class $C^1$ piecewise over $[0,\pi]$. Indeed, $f$ is $C^1$ on $]0,\pi[$ as it's constant. And

$$\lim_{x\to\pi^-} f'(x) = \lim_{x\to\pi^-} 0 = 0 \in \mathbb{R}, \quad \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} 0 = 0 \in \mathbb{R}.$$

Thus, $Ff$ converges for all $x\in\mathbb{R}$, and at points of continuity of $f$, its sum equals:

$$Ff(x) = \sum_{n\geq 0} \frac{4}{(2n+1)\pi} \sin((2n+1)x) = f(x), \quad \forall x \in \mathbb{R} - \{n\pi\}.$$

And at points of discontinuity of $f$, its sum equals:

$$\forall x \in \{n\pi\}, \quad Ff(x) = \frac{f(x+)+f(x-)}{2} = 0 \neq f(x).$$

**Conclusion:** $Ff$ converges toward $f$ on $\mathbb{R} - \{n\pi\}$.

1. **Calculating** $\sum_{n\geq 0}(-1)^n\frac{1}{2n+1}$: for $x=\frac{\pi}{2}$:

$$Ff\left(\frac{\pi}{2}\right)=f\left(\frac{\pi}{2}\right)\Rightarrow\sum_{n\geq 0}\frac{4}{(2n+1)\pi}\sin\left((2n+1)\frac{\pi}{2}\right)=1\Rightarrow\sum_{n\geq 0}\frac{4(-1)^n}{(2n+1)\pi}=1$$

This implies:

$$\sum_{n\geq 0}(-1)^n\frac{1}{(2n+1)}=\frac{\pi}{4}.$$

2. **Calculating** $\sum_{n\geq 0}\frac{1}{(2n+1)^2}$: applying Parseval's formula: $f$ is $2\pi$-periodic and integrable over any closed and bounded set of $\mathbb{R}$, thus:

$$\frac{a_0^2}{2}+\sum_{n\geq 1}(a_n^2+b_n^2)=\frac{2}{\pi}\int_0^{\pi}1dx$$

$$\sum_{n\geq 0}\frac{16}{(2n+1)^2\pi^2}=\frac{2}{\pi}\int_0^{\pi}1\,dx=2$$

This implies:

$$\sum_{n\geq 0}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}.$$

3. **Calculating** $S_3 = \sum_{n\geq 1} \frac{1}{n^2}$: As the series $\sum_{n\geq 1} \frac{1}{n^2}$ converges absolutely, we can write:

$$\sum_{n\geq 1} \frac{1}{n^2} = \sum_{k\geq 1} \frac{1}{(2k)^2} + \sum_{k\geq 0} \frac{1}{(2k+1)^2} = \frac{1}{4} \sum_{k\geq 1} \frac{1}{k^2} + \sum_{k\geq 0} \frac{1}{(2k+1)^2} = \frac{1}{4} S_3 + \frac{\pi^2}{8}$$

This implies: $S_3 = \frac{1}{4} S_3 + \frac{\pi^2}{8}$, thus $S_3 = \frac{\pi^2}{6}$.

4. **Calculating** $S_4 = \sum_{n\geq 1} (-1)^{n+1} \frac{1}{n^2}$: Similarly, as the series $\sum_{n\geq 1} (-1)^{n+1} \frac{1}{n^2}$ converges absolutely, we can write:

$$\sum_{n\geq 1} (-1)^{n+1} \frac{1}{n^2} = \sum_{k\geq 1} (-1)^{2k+1} \frac{1}{(2k)^2} + \sum_{k\geq 0} (-1)^{2k+2} \frac{1}{(2k+1)^2}$$

$$= -\frac{1}{4} \sum_{k\geq 1} \frac{1}{k^2} + \sum_{k\geq 0} \frac{1}{(2k+1)^2} = -\frac{1}{4} S_3 + \frac{\pi^2}{8}$$

This implies: $S_4 = -\frac{1}{4} \frac{\pi^2}{6} + \frac{\pi^2}{8}$, thus $S_4 = \frac{\pi^2}{12}$.