Soundness for Propositional Logic

Chapter 2, Section 9

Introduction

In Definition 3.4.4 of Section 3.4 we defined exactly what is meant by the sequent

(3.65) $$\Gamma \vdash_{\sigma} \psi$$

where σ is a signature, Γ is a set of formulas of LP(σ) and ψ is a formula of LP(σ). It will be convenient to write

$$\Gamma \nvdash_{\sigma} \psi$$

to express that (3.65) is *not* correct.

We also described a programme, proposed by Hilbert, for checking that the provable sequents (3.65) are exactly those that on grounds of truth and falsehood we ought to be able to prove. In this section and the next, we make the programme precise and carry it through for propositional logic.

Model

Definition 3.9.1

Let σ be a signature, Γ a set of formulas of LP(σ) and ψ a formula of LP(σ).

(a) We say that a σ-structure A is a *model of* Γ if it is a model of every formula in Γ, that is, if $A^*(\phi) = \mathrm{T}$ for every $\phi \in \Gamma$.

(b) We write

(3.66) $$\Gamma \vDash_\sigma \psi$$

to mean that for every σ-structure A, if A is a model of Γ then A is a model of ψ. The expression (3.66) is called a *semantic sequent*.

(c) We write

$$\Gamma \nvDash_\sigma \psi$$

to mean that (3.66) is not true.

Model

Definition 3.9.1

Let σ be a signature, Γ a set of formulas of LP(σ) and ψ a formula of LP(σ).

(a) We say that a σ-structure A is a *model of* Γ if it is a model of every formula in Γ, that is, if $A^*(\phi) = \mathrm{T}$ for every $\phi \in \Gamma$.

(b) We write

$$\Gamma \vDash_\sigma \psi \tag{3.66}$$

to mean that for every σ-structure A, if A is a model of Γ then A is a model of ψ. The expression (3.66) is called a *semantic sequent*.

(c) We write

$$\Gamma \nvDash_\sigma \psi$$

to mean that (3.66) is not true.

When the context allows, we will ignore the subscript and write $\Gamma \vDash \psi$ instead of $\Gamma \vDash_\sigma \psi$.

Problematic

Our aim will be to establish that for all Γ and ψ,

(3.67) $$\Gamma \vdash_\sigma \psi \Leftrightarrow \Gamma \vDash_\sigma \psi.$$

The two directions in (3.67) say very different things. Going from left to right, (3.67) says that if there is a derivation with undischarged assumptions in Γ and conclusion ψ, then every model of Γ is a model of ψ. What would it mean for this to fail? It would mean that there is such a derivation D, and there is also a structure A in which all the formulas in Γ are true but ψ is false. Hence we would have derived a formula that is false in A from formulas that are true in A. This would be a devastating breakdown of our rules of proof: they should never derive something false from something true. So the left-to-right direction in (3.67) is verifying that we did not make some dreadful mistake when we set up the rules of natural deduction. The second half of this section will be devoted to this verification.

Problematic

The direction from right to left in (3.67) says that if the truth of Γ guarantees the truth of ψ, then our natural deduction rules allow us to derive ψ from Γ. That is to say we do not need any more natural deduction rules besides those that we already have. We return to this in the next section.

Soundness, Adequacy and Completeness

Both directions of (3.67) are best read as saying something about our system of natural deduction. There are other proof calculi, and they have their own versions of (3.67). We can write $\Gamma \vdash_{\mathcal{C}} \psi$ to mean that ψ is derivable from Γ in the proof calculus $\mathcal{C}$. Then

$$\Gamma \vdash_{\mathcal{C}} \psi \quad \Rightarrow \quad \Gamma \vDash \psi$$

is called *Soundness* of the calculus $\mathcal{C}$, and the converse

$$\Gamma \vDash \psi \quad \Rightarrow \quad \Gamma \vdash_{\mathcal{C}} \psi$$

is called *Adequacy* of the calculus $\mathcal{C}$. The two directions together are called *Completeness* of $\mathcal{C}$.

Soundness Theorem

Theorem 3.9.2 (Soundness of Natural Deduction for Propositional Logic)

Let σ be a signature, Γ a set of formulas of LP*(σ) and ψ a formula of* LP*(σ). If $\Gamma \vdash_\sigma \psi$, then $\Gamma \vDash_\sigma \psi$.*

The theorem states that

(3.68) If D is a σ-derivation whose conclusion is ψ and whose undischarged assumptions all lie in Γ, then every σ-structure that is a model of Γ is also a model of ψ.