Academic year (2023/2024) **Mathematical analysis 3**	**Worksheet N 2** **Integral transformations** **$\mathcal{L}$aplace transform** **(Correction)**	ensia

Exercise 1: Calculate $L(f(t))$ with $f(t) = \begin{cases} (t-1)^2 & \text{if } t > 1, \\ 0 & \text{if } 0 \le t \le 1. \end{cases}$

Solution:

$$f(t) = \begin{cases} (t-1)^2 & \text{if } t > 1, \\ 0 & \text{if } 0 \le t \le 1. \end{cases}$$

Calculate its Laplace transform by definition:

$$L\{f(t)\} = F(x) = \int_0^{+\infty} e^{-xt} f(t)dt = \int_1^{+\infty} e^{-xt}(t-1)^2 dt;\ \text{where } x \in \mathbb{R}.$$

Let's use a change of variables: $u = t - 1$, i.e., $du = dt$:

$$L\{f(t)\} = F(x) = \int_0^{+\infty} e^{-x(u+1)} u^2 du = e^{-x} \int_0^{+\infty} e^{-xu} u^2 du = e^{-x} \int_0^{+\infty} e^{-xt} t^2 dt;$$

i.e., $L\{f(t)\} = e^{-x} \cdot L\{t^2\} = e^{-x} \cdot \dfrac{2}{x^3}, x > 0.$

Exercise 2: Determine the Laplace transforms of the following functions, indicating in each case the values of x for which these Laplace transforms exist.

1. $f_1(t) = 5t - 3;$
2. $f_2(t) = 3t^4 + 4e^{3t} - 2\sin(5t) + 3\cosh(2t);$
3. $f_3(t) = t^3 e^{-3t};$
4. $f_4(t) = 2e^{3t}\sin(t);$
5. $f_5(t) = \dfrac{\cos(at) - \cos(bt)}{t};$
6. $f_6(t) = \displaystyle\int_0^t \frac{\cos(au) - \cos(bu)}{u} du.$

Solution:

1. $f_1(t) = 5t - 3$, using linearity: $F_1(x) = \mathcal{L}\{5t - 3\} = 5\mathcal{L}\{t\} - 3\mathcal{L}\{1\}$. Since $\mathcal{L}\{t\} = \dfrac{1}{x^2}, x > 0$ and $\mathcal{L}\{1\} = \dfrac{1}{x}, x > 0$. So, $F_1(x) = \mathcal{L}\{5t - 3\} = \dfrac{5}{x^2} - \dfrac{3}{x}, x > 0.$

2. $f_2(t) = 3t^4 + 4e^{3t} - 2\sin(5t) + 3\cosh(2t)$, using linearity:
$$F_2(x) = \mathcal{L}\{3t^4\} + 4\mathcal{L}\{e^{3t}\} - 2\mathcal{L}\{\sin(5t)\} + 3\mathcal{L}\{\cosh(2t)\}.$$
Since $\mathcal{L}\{t^4\} = \dfrac{4!}{x^5}, \mathcal{L}\{e^{3t}\} = \dfrac{1}{x-3}, \mathcal{L}\{\sin(5t)\} = \dfrac{5}{x^2+25}$, and $\mathcal{L}\{\cosh(2t)\} = \dfrac{x}{x^2-4}$.
So,
$$F_2(x) = \frac{3 \cdot 4!}{x^5} + \frac{4}{x-3} - \frac{10}{x^2+25} + \frac{3x}{x^2-4}, x > 3.$$

3) $f_3(t) = t^3e^{-3t}$, let $g(t) = t^3$ and use the translation:

$$\mathcal{L}\{e^{-3t}t^3\} = G(x+3), \quad \forall x > -3.$$

Since $\mathcal{L}\{t^3\} = \frac{3!}{x^4}, \quad \forall x > 0$; then $F_3(x) = \mathcal{L}\{e^{-3t}t^3\} = \dfrac{3!}{(x+3)^4}, \quad \forall x > -3.$

4) $f_4(t) = 2e^{3t}\sin(t)$, let $g(t) = \sin(t)$ and use the translation property:

$$\mathcal{L}\{e^{3t}\sin(t)\} = G(x-3), \quad \forall x > 3.$$

Since $\mathcal{L}(\sin(t)) = \dfrac{1}{x^2+1}, \quad \forall x > 0$; then $\mathcal{L}\{e^{3t}\sin(t)\} = \dfrac{1}{(x-3)^2+1}, \quad \forall x > 3.$

Using linearity: $F_4(x) = 2\mathcal{L}\{e^{3t}\sin(t)\} = \dfrac{2}{(x-3)^2+1}, \quad \forall x > 3.$

5) $f_5(t) = \dfrac{\cos(at)-\cos(bt)}{t}$, let $g(t) = \cos(at) - \cos(bt)$,

as $\lim\limits_{t\to 0} \dfrac{g(t)}{t} = \lim\limits_{t\to 0} \dfrac{-a\sin(at) + b\sin(bt)}{1} = 0$ for $a, b \in \mathbb{R}$, then:

$$\mathcal{L}\{\frac{g(t)}{t}\} = \int_0^{+\infty} \frac{G(u)}{u}du, \quad \forall x > \gamma_g;$$

where $G(x) = \mathcal{L}(\cos(at) - \cos(bt))$ (using linearity):

$$G(x) = \mathcal{L}(\cos(at)) - \mathcal{L}(\cos(bt)) = \frac{x}{x^2+a^2} - \frac{x}{x^2+b^2}, \quad \forall x > 0;$$

thus,

$$\mathcal{L}\{\frac{g(t)}{t}\} = \int_0^{+\infty} \left(\frac{1}{2}\log\frac{x^2+b^2}{x^2+a^2}\right)\frac{1}{x}dx, \quad \forall x > 0;$$

i.e., $\mathcal{L}(f_5(t)) = \dfrac{1}{2}\log\dfrac{x^2+b^2}{x^2+a^2}, \quad \forall x > 0$

6) $f_6(t) = \displaystyle\int_0^t \frac{\cos(au)-\cos(bu)}{u}du$; we notice that $f_6(t) = \displaystyle\int_0^t f_5(u)du$ then:

$$\mathcal{L}\left(\int_0^t f_5(u)du\right) = F_5(x)\frac{1}{x}, \quad \forall x > 0,$$

from the previous question, we have: $F_5(x) = \mathcal{L}(f_5(t)) = \frac{1}{2}\log\dfrac{x^2+b^2}{x^2+a^2}, \quad \forall x > 0$; thus,

$$\mathcal{L}(f_6(t)) = \frac{1}{2x}\log\frac{x^2+b^2}{x^2+a^2}, \quad \forall x > 0.$$

Exercise 3:

1) Show that $\mathcal{L}\left\{\dfrac{\sin(2t)}{t}\right\} = \dfrac{1}{4}\log\left(\dfrac{x^2+4}{x^2}\right)$.

2) Evaluate:

$$\int_0^{+\infty} \frac{e^{-t}\sin(2t)}{t}dt$$

Solution:

1) Show that $\mathcal{L}\left\{\dfrac{\sin(2t)}{t}\right\} = \dfrac{1}{4}\log\left(\dfrac{x^2+4}{x^2}\right)$, let $f(t) = \sin^2 t$, as $\lim\limits_{t\to 0}\dfrac{f(t)}{t} = \lim\limits_{t\to 0}\dfrac{\sin t}{t} = 0 \in \mathbb{R}$, then:

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_x^{+\infty} \frac{F(u)}{u}du, \quad \forall x > \gamma_f$$

where $F(x) = \mathcal{L}\{\sin^2 t\} = \frac{1}{2}(\mathcal{L}\{1\} - \mathcal{L}\{\cos(2t)\}) = \dfrac{1}{2}\left(\dfrac{1}{x} - \dfrac{x}{x^2+4}\right)$. Thus,

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \frac{1}{2}\int_x^{+\infty}\left(\frac{1}{u} - \frac{u}{u^2+4}\right)du = \frac{1}{2}\left[\log(u) - \frac{1}{2}\log(u^2+4)\right]_x^{+\infty}$$

$$= \frac{1}{4}\left[\log(u^2) - \log(u^2+4)\right]_x^{+\infty} = \frac{1}{4}\log\left(\frac{u^2+4}{u^2}\right)$$

$$= \frac{1}{4}\log\left(\frac{x^2+4}{x^2}\right), \quad \forall x > 0$$

2) Evaluate: $I = \displaystyle\int_0^{+\infty}\frac{e^{-xt}\sin(2t)}{t}dt$, returning to the definition:

$$\mathcal{L}\left\{\frac{\sin(2t)}{t}\right\} = \int_0^{+\infty}\frac{e^{-xt}\sin(2t)}{t}dt = \frac{1}{4}\log\left(\frac{x^2+4}{x^2}\right), \quad \forall x > 0$$

thus for $x = 1$ we find the value of I; hence, $I = \frac{1}{4}\log(5)$.

Exercise 4: Determine the inverse transforms of the following functions:

$$1)\frac{x^2}{(x+3)^3}, \qquad 2)\frac{1}{x(x+1)^3}, \qquad 3)\frac{x}{(x^2+1)^2}, \qquad 4)\frac{x^3-2x^2+1}{x^4+x^2}.$$

Solution:

1) $F_1(x) = \dfrac{x^2}{(x+3)^3} = \dfrac{(x+3-3)^2}{(x+3)^3}$,

knowing that

$$\mathcal{L}^{-1}\{G(x+3)\} = e^{-3t}g(t) \text{ where } g(t) = \mathcal{L}^{-1}\{G(x)\}.$$

then $G(x) = \dfrac{x^2-6x+9}{x^3} = \dfrac{1}{x} - \dfrac{6}{x^2} + \dfrac{9}{2x^3}$,

i.e., $g(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{x}\right\} - 6\mathcal{L}^{-1}\left\{\dfrac{1}{x^2}\right\} + \dfrac{9}{2}\mathcal{L}^{-1}\left\{\dfrac{1}{x^3}\right\} = 1 - 6t + \dfrac{9}{2}t^2$,

therefore $\mathcal{L}^{-1}\{F_1(x)\} = e^{-3t}(1 - 6t + \dfrac{9}{2}t^2)$, for $\forall t \geq 0$..

2) $F_2(x) = \dfrac{1}{x(x+1)^3}$.

Let's assume: $f(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{x(x+1)^3}\right\} = \mathcal{L}^{-1}\left\{\dfrac{\frac{1}{(x+1)^3}}{x}\right\} = \displaystyle\int_0^t g(u)du$ where $g(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{(x+1)^3}\right\}$,

and $G(x) = \dfrac{1}{(x+1)^3}$. Therefore: $g(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{(x+1)^3}\right\} = e^{-t}h(t) = h(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{x^3}\right\} = \dfrac{1}{2}t^2$.

Thus, $g(t) = e^{-t}\cdot\dfrac{1}{2}t^2 = \dfrac{1}{2}\displaystyle\int_0^t u^2e^{-u}du$, by performing two integration by parts we get:

$$f(t) = 1 - e^{-t}(1 + t + \frac{1}{2}t^2) \quad \forall t \geq 0.$$

3) $F_3(x) = -\dfrac{1}{2}\dfrac{2x}{(x^2+1)^2} = -\dfrac{1}{2}\dfrac{1}{x^2+1}$. Therefore, $\mathcal{L}^{-1}(F_3(x)) = -\dfrac{1}{2}\cdot\sin(t)\ \forall t \geq 0$..

4) $F_4(x) = \dfrac{x^3-2x^2+1}{x^4+x^2} = \dfrac{x}{x^2+1} - 3\dfrac{1}{x^2+1} + \dfrac{1}{x^2}$.

Since $\dfrac{1}{u(u+1)} = \dfrac{1}{u} - \dfrac{1}{u+1}$,
we have:
$F_4(x) = \dfrac{x}{x^2+1} - 3\dfrac{1}{x^2+1} + \dfrac{1}{x^2} = \dfrac{x}{x^2+1} - 3\dfrac{1}{x^2+1} + \dfrac{1}{x^2}$.
Therefore,

$$\mathcal{L}^{-1}(F_4(x)) = \cos(t) - 3\sin(t) + t \quad \forall t \geq 0.$$

Exercise 5: Solve the following differential equation using Laplace transform:

$$y''(t) + y(t) = \sin(2t) \text{ for } t > 0; \quad y(0) = 2; \quad y'(0) = 0$$

Solution: Using Laplace transform to solve the differential equation:

$$y''(t) + y(t) = \sin(2t); \quad y(0) = 2; \quad y'(0) = 0$$

Which is equivalent to solving using Laplace transform:

$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(\sin(2t)); \quad y(0) = 2; \quad y'(0) = 0 :$
Let's define: $Y = L(y)$:
$\Rightarrow x^2Y - xy(0) - y'(0) + Y = \dfrac{2}{x^2+4}; \quad x > 0 :$
$\Rightarrow x^2Y - 2x + Y = \dfrac{2}{x^2+4}; \quad x > 0 :$
$\Rightarrow (x^2+1)Y = 2x + \dfrac{2}{x^2+4}; \quad x > 0 :$
Therefore, $Y = \dfrac{2}{(x^2+4)(x^2+1)} + \dfrac{2x}{x^2+1}$
Given that $\dfrac{2}{(u+1)(u+4)} = \dfrac{2}{3}(\dfrac{1}{u+1} - \dfrac{1}{u+4})$,
we have $Y = \dfrac{2}{3}\dfrac{1}{x^2+1} - \dfrac{1}{3}\dfrac{2}{x^2+4} + 2\dfrac{x}{x^2+4}$
$y = \mathcal{L}^{-1}(Y) = \dfrac{2}{3}\sin(t) - \dfrac{1}{3}\sin(2t) + 2\cos(2t) \quad \forall t \geq 0.$