Annotation Unit 10

As always, the passage firstly introduces the definition of the considered object. The differential dy of a function $f(x)$ is defined by $dy = f'(x)dx$ where dx is another real variable. Indeed, if the derivative is thought as the ratio of a small change of function dy to a small change of its argument dx, then the derivative of $f(x)$ is $\frac{dy}{dx}$ and thus indeed $dy = \frac{dy}{dx}dx$. One can also denote differential in another way: $df(x) = f'(x)dx$.

Every concept has its history and creators. The differential is not an exception. The author of the passage introduces the Leibniz notation for derivatives $\frac{dy}{dx}$, who first introduced an intuitive and heuristic definition of a differential.

Since most mathematicians ~~are boring nerds~~ seek the strictness of all defined concepts, this definition was widely criticized, in particular, by Cauchy. He reformulated some previous concepts and made equation $df(x) = f'(x)dx$ be rigorous defined. His approach significantly contributed to the further study of Calculus as quantities dy and dx can be manipulated in the same way as any other variables.

Next, the passage elaborates on how the notion of the differential might be extended.

Then the author introduces a more rigorous definition of a differential df of a function f. By definition, the differential df is a function of two variables x and Δx:

$$df(x, \Delta x) = f'(x)\Delta x$$

The concept of differential is widely used in many fields of Mathematics for the small change Δy of function $f(x)$ approximation, in which the value of argument increment Δx is also small enough:

$$\Delta y = f'(x)\Delta x + \varepsilon$$

Here the error of approximation ε tends to 0 as Δx approaches 0.

After the careful and explicit definition of a function differential that depends on one variable, subsequently comes the question: what if the function f depends on more than one variable?

The passage proceeds to answer this question. Suppose we have a function $f(x_1, \ldots, x_n)$. The differential of this function is defined as follows:

$$dy = \sum_{j=1}^{n} \frac{\partial f(x_1, \ldots, x_n)}{\partial x_j} dx_j$$

More precisely:

$$dy = \sum_{j=1}^{n} \frac{\partial f(x_1, \ldots, x_n)}{\partial x_j} \Delta x_j + \sum_{j=1}^{n} \varepsilon_j \Delta x_j$$

where each term ε_j tends to 0 as Δx_j approaches 0.

Finally, the author of the passage describes the properties of differential and draws parallels with other operations such as derivative, partial derivative, and total derivative. Namely:

1. **Linearity.** Suppose α, β are arbitrary real numbers and f, g are functions. Then, $d(\alpha f + \beta g) = \alpha df + \beta dg$.
2. **Product rule.** $d(fg) = f \cdot dg + df \cdot g$.

Lastly, the passage describes how the operation of taking a differential of a function is related to abstract algebra and how the chain rule works for this operation.