Homework 2.

Task 1. Equation Definition

First, define matrices of our dynamical system

In[1497]:=

```
Clear[A, b, ω1, ω2, η]
A = {
    {0, 0, 1, 0},
    {0, 0, 0, 1},
    {-(ω1^2 + η ω2^2), η ω2^2, 0, 0},
    {ω2^2, -ω2^2, 0, 0}
  };
b = {{0}, {0}, {0}, {1}};
```

Task 2. Hamiltonian and Conjugated Equation

Next, define the Hamiltonian

In[1555]:=

```
Clear[w, ψ1, ψ2, ψ3, ψ4, t]
w = {{w1}, {w2}, {w3}, {w4}};
f = A.w + b u;
ψ = {{ψ1}, {ψ2}, {ψ3}, {ψ4}};
H = Transpose[ψ].f;
H = H[[1]][[1]];
Print["Sought Hamiltonian is ", H]
```

```
Sought Hamiltonian is w3 ψ1 + w4 ψ2 + (-6 w1 + w2) ψ3 + (u + w1 - w2) ψ4
```

Find the gradient to write the coefficients equation (in both forms):

In[1566]:=

```
Print["Negative Gradient of Hamiltonian (w.r.t. vector w) is ",
  -Grad[H, {w1, w2, w3, w4}]]
Print["-Conjugate[A]ψ is ", -Transpose[A].ψ]
Print["Two expressions are the same!"]
```

```
Negative Gradient of Hamiltonian (w.r.t. vector w) is {6 ψ3 - ψ4, -ψ3 + ψ4, -ψ1, -ψ2}
-Conjugate[A]ψ is {{6 ψ3 - ψ4}, {-ψ3 + ψ4}, {-ψ1}, {-ψ2}}
Two expressions are the same!
```

Now, let us substitute numbers. We have $k_1 = 5$, $k_2 = 1$, $l_1 = l_2 = 1$, $m_1 = 1$, $m_2 = 1$ and for that reason we have ${\omega_1}^2 = k_1 / m_1 = 5$ and ${\omega_2}^2 = k_2 / m_2 = 1$. Also, $\eta = m_2 / m_1 = 1$.

In[1508]:=

```
ω1 = Sqrt[5]; ω2 = 1; η = 1;
-Transpose[A].ψ
```

Out[1509]=

$$\{\{6\,\psi3-\psi4\},\{-\psi3+\psi4\},\{-\psi1\},\{-\psi2\}\}$$

In[1592]:=

```
DSolve[{ψ1'[t] == 6 ψ3[t] - ψ4[t], ψ2'[t] == -ψ3[t] + ψ4[t],
  ψ3'[t] == -ψ1[t], ψ4'[t] == -ψ2[t]}, {ψ1[t], ψ2[t], ψ3[t], ψ4[t]}, t]
```

Out[1592]=

$$\Big\{\Big\{\psi1[t]\to\frac{1}{2}\,c_2\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{e^{t\,\#1}}{7+2\,\#1^2}\,\&\Big]-\frac{1}{2}\,c_4\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{e^{t\,\#1}\,\#1}{7+2\,\#1^2}\,\&\Big]+\frac{1}{2}\,c_1\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{e^{t\,\#1}+e^{t\,\#1}\,\#1^2}{7+2\,\#1^2}\,\&\Big]+\frac{1}{2}\,c_3\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{5\,e^{t\,\#1}+6\,e^{t\,\#1}\,\#1^2}{7\,\#1+2\,\#1^3}\,\&\Big],$$

$$\psi2[t]\to\frac{1}{2}\,c_1\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{e^{t\,\#1}}{7+2\,\#1^2}\,\&\Big]-\frac{1}{2}\,c_3\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{e^{t\,\#1}\,\#1}{7+2\,\#1^2}\,\&\Big]+\frac{1}{2}\,c_2\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{6\,e^{t\,\#1}+e^{t\,\#1}\,\#1^2}{7+2\,\#1^2}\,\&\Big]+\frac{1}{2}\,c_4\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{5\,e^{t\,\#1}+e^{t\,\#1}\,\#1^2}{7\,\#1+2\,\#1^3}\,\&\Big],$$

$$\psi3[t]\to\frac{1}{2}\,c_4\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{e^{t\,\#1}}{7+2\,\#1^2}\,\&\Big]+\frac{1}{2}\,c_3\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{e^{t\,\#1}+e^{t\,\#1}\,\#1^2}{7+2\,\#1^2}\,\&\Big]-\frac{1}{2}\,c_2\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{e^{t\,\#1}}{7\,\#1+2\,\#1^3}\,\&\Big]-\frac{1}{2}\,c_1\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{e^{t\,\#1}+e^{t\,\#1}\,\#1^2}{7\,\#1+2\,\#1^3}\,\&\Big],$$

$$\psi4[t]\to\frac{1}{2}\,c_3\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{e^{t\,\#1}}{7+2\,\#1^2}\,\&\Big]+\frac{1}{2}\,c_4\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{6\,e^{t\,\#1}+e^{t\,\#1}\,\#1^2}{7+2\,\#1^2}\,\&\Big]-\frac{1}{2}\,c_1\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{e^{t\,\#1}}{7\,\#1+2\,\#1^3}\,\&\Big]-\frac{1}{2}\,c_2\,\text{RootSum}\Big[5+7\,\#1^2+\#1^4\,\&,\ \frac{6\,e^{t\,\#1}+e^{t\,\#1}\,\#1^2}{7\,\#1+2\,\#1^3}\,\&\Big]\Big\}\Big\}$$

Task 3. Feldbaum Theorem

First, let us check the controllability

In[1594]:=

```
B = {0, 0, 0, 1};
AB = A.B;
AAB = A.A.B;
AAAB = A.A.A.B;
K = Transpose[{B, AB, AAB, AAAB}];
Print["Kalman Matrix is ", K]
Print["Rank of this matrix is ", MatrixRank[K]]
```

```
Kalman Matrix is {{0, 0, 0, 1}, {0, 1, 0, -1}, {0, 0, 1, 0}, {1, 0, -1, 0}}
Rank of this matrix is 4
```

In[1601]:=

```
Print["Eigenvalues of A is ", Eigenvalues[A]]
Print["Thus, the theorem cannot be applied"]
```

Eigenvalues of A is

$$\left\{i\sqrt{\frac{1}{2}\left(7+\sqrt{29}\right)}, -i\sqrt{\frac{1}{2}\left(7+\sqrt{29}\right)}, i\sqrt{\frac{1}{2}\left(7-\sqrt{29}\right)}, -i\sqrt{\frac{1}{2}\left(7-\sqrt{29}\right)}\right\}$$

Thus, the theorem cannot be applied