Section 1. Euler Methods

Section 1.1. Simple Euler Method

In[1]:=
```
f[x_, y_] = -2 y^2 Log[x] - y/x; (* RHE of DE *)
r[x_] = 1/(x (1 + Log[x]^2)); (* Solution to the Cauchy DE *)
x0 = 1;
y0 = 1;
b = 2;
n = 10; (* Parameters used in the problem statement *)
```

Step 1. Verify that r[x] is indeed a solution to y' = f(x, y) with y(1) = 1

In[4]:=
```
Simplify[r[x0] - y0] (* Must be 0 *)
Simplify[r'[x] - f[x, r[x]]] (* Must be 0 *)
```

Out[4]= 0

Out[5]= 0

Alternatively, we can simply solve the equation via DSolve method...

In[6]:=
```
DSolve[{y'[x] == f[x, y[x]], y[x0] == y0}, y[x], x]
```

Out[6]= $\left\{\left\{y[x] \to \frac{1}{x\,(1 + \mathrm{Log}[x]^2)}\right\}\right\}$

Step 2. Executing the Euler Method for finding the solution

In[7]:=
```
h = (b - x0)/n; (* Step Size *)
xC = x0 + 0.0; yC = y0 + 0.0; (* Current values of x and y *)
simpleEulerMethod = Table[
 m = f[xC, yC];
 y1 = yC + h m;
 x1 = xC + h;
 xC = x1;
 yC = y1;
 {x1, y1}, (* Store (x1, y1) in the list *)
 {i, 1, n}
];
```

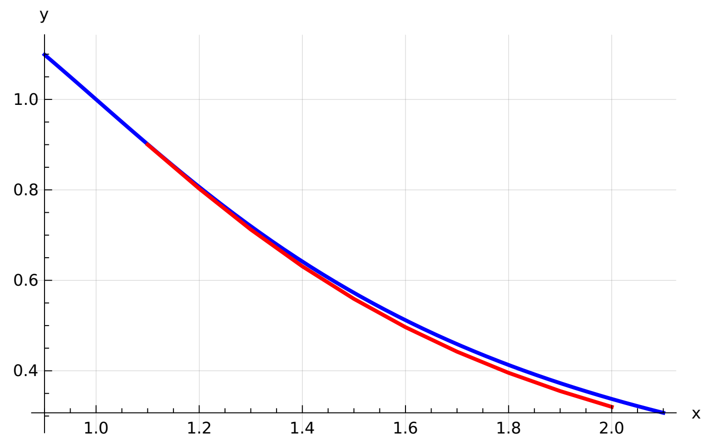
In[10]:=
```
simpleEulerMethod
```

Out[10]=
```
{{1.1, 0.9}, {1.2, 0.802742}, {1.3, 0.712349}, {1.4, 0.630926}, {1.5, 0.559072},
 {1.6, 0.496454}, {1.7, 0.442258}, {1.8, 0.395485}, {1.9, 0.355127}, {2., 0.320246}}
```

Step 3. Validating that the result is correct.

In[19]:=
```
Show[
 Plot[r[x], {x, x0 - 0.1, b + 0.1}, PlotStyle → {Blue, Thick}], (* Function curve *)
 ListPlot[simpleEulerMethod, PlotStyle → {Red, PointSize[Large]}, Joined → True],
 (* Numerical points *)
 PlotTheme → "Detailed",
 AxesLabel → {"x", "y"},
 PlotLegends → {"y[x]", "Computed Points"},
 GridLines → Automatic
]
```

Out[19]=



Plotting differences between real and approximated results

In[12]:=
```
t = Table[Abs[simpleEulerMethod[[i]][[2]] - r[simpleEulerMethod[[i]][[1]]]], {i, 1, n}]
Max[t]
```

Out[12]=

```
{0.000907042, 0.00378199, 0.00734179, 0.0107169,
 0.0134678, 0.0154619, 0.0167394, 0.0174158, 0.0176243, 0.017488}
```

Out[13]=

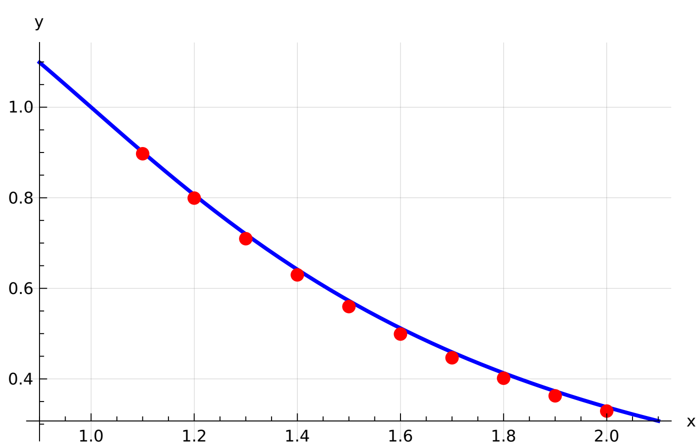
```
0.0176243
```

Section 1.2. Enhanced Euler Method

In[20]:=
```
n = 10;
h = (b - x0) / n; (* Step Size *)
xC = x0 + 0.0; yC = y0 + 0.0; (* Current values of x and y *)
enhancedEulerMethod = Table[
 x1 = xC + h;
 y1 = yC + h m;
 y2 = yC + (h / 2) (f[xC, yC] + f[x1, y1]);
 xC = x1;
 yC = y2;
 {x1, y2}, (* Store (x1, y1) in the list *)
 {i, 1, n}
];
```

In[24]:=
```
Show[
 Plot[r[x], {x, x0 - 0.1, b + 0.1}, PlotStyle → {Blue, Thick}], (* Function curve *)
 ListPlot[enhancedEulerMethod, PlotStyle → {Red, PointSize[Large]}, Joined → False],
 (* Numerical points *)
 PlotTheme → "Detailed",
 AxesLabel → {"x", "y"},
 PlotLegends → {"y[x]", "Computed Points"},
 GridLines → Automatic
]
```

Out[24]=



Again, plotting differences between real and approximated results. Also print the maximum deviation.

In[25]:=
```
t = Table[Abs[enhancedEulerMethod[[i]][[2]] - r[enhancedEulerMethod[[i]][[1]]]], {i, 1, n}]
Max[t]
```

Out[25]=
```
{0.00365384, 0.00721888, 0.0100709, 0.0119443,
 0.0128453, 0.0129251, 0.0123824, 0.0114077, 0.0101599, 0.00876047}
```

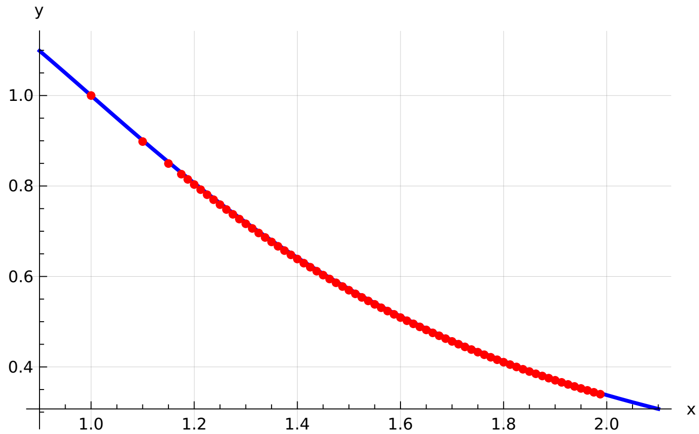
Out[26]=
```
0.0129251
```

Section 2. Runge–Kutta Method

In[27]:=

```
h = (b - x0) / n; (* Step Size *)
ε = 10 ^ (-5) + 0.0;
xC = x0; (* x value for step=h *)
yC = y0 ;(* y value for step=h *)
rungeKutta = {};

rkStep[x_, y_, h_] := Module[{result},
k1 = h * f[x, y] + 0.0;
k2 = h * f[x + h / 2, y + h * k1 / 2];
k3 = h * f[x + h / 2, y + h * k2 / 2];
k4 = h * f[x + h, y + h * k3];
Return[y + (k1 + 2 k2 + 2 k3 + k4) / 6]
];

While[xC < b,
(* Store the computed (x, y) points *)
AppendTo[rungeKutta, {xC, yC}];

(* Compute RK4 coefficients for h *)
y1 = rkStep[xC, yC, h];
y2 = rkStep[xC, yC, h / 2];
y3 = rkStep[xC + h / 2, y2, h / 2];

(* Update x values *)
xC += h;
yC = y3;

If[Abs[y1 - y3] ≥ 15 ε, h = h / 2];
If[h < 10 ^ (-6), Break[]];
]
Show[
 Plot[r[x], {x, x0 - 0.1, b + 0.1}, PlotStyle → {Blue, Thick}], (* Function curve *)
 ListPlot[rungeKutta, PlotStyle → {Red, PointSize[Medium]}], (* Numerical points *)
 PlotTheme → "Detailed",
 AxesLabel → {"x", "y"},
 PlotLegends → {"y[x]", "Computed Points"},
 GridLines → Automatic
]
```

Out[34]=



Printing differences and the maximal difference:

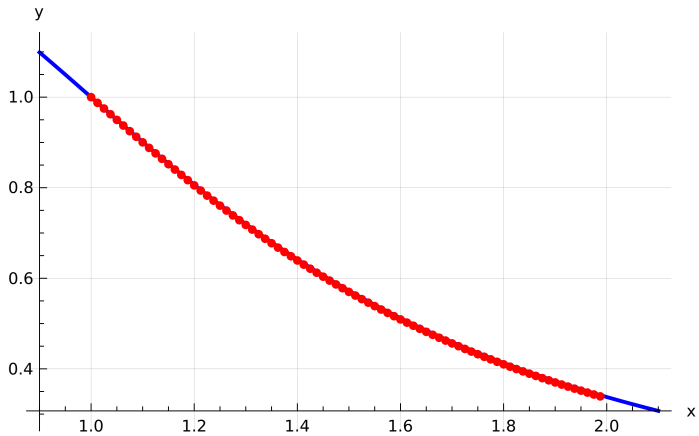
In[35]:=
```
t = Table[Abs[rungeKutta[[i]][[2]] - r[rungeKutta[[i]][[1]]]], {i, 5, 50}]
Max[t]
```

Out[35]=
```
{0.00331736, 0.00330905, 0.00330025, 0.00329097, 0.00328118, 0.00327089, 0.00326009,
 0.00324878, 0.00323697, 0.00322466, 0.00321184, 0.00319854, 0.00318474,
 0.00317048, 0.00315574, 0.00314055, 0.00312492, 0.00310887, 0.0030924, 0.00307552,
 0.00305827, 0.00304065, 0.00302267, 0.00300436, 0.00298573, 0.0029668, 0.00294758,
 0.00292809, 0.00290836, 0.00288839, 0.0028682, 0.00284781, 0.00282723,
 0.00280648, 0.00278558, 0.00276455, 0.00274339, 0.00272212, 0.00270075,
 0.0026793, 0.00265779, 0.00263622, 0.00261461, 0.00259297, 0.00257131, 0.00254964}
```

Out[36]=
```
0.00331736
```

In[37]:=
```
h = 1/80; (* Step Size *)
ε = 10^(-5) + 0.0;
xC = x0; (* x value for step=h *)
yC = y0 ;(* y value for step=h *)
rungeKutta = {};

rkStep[x_, y_, h_] := Module[{result},
k1 = h*f[x, y] + 0.0;
k2 = h*f[x + h/2, y + h*k1/2];
k3 = h*f[x + h/2, y + h*k2/2];
k4 = h*f[x + h, y + h*k3];
Return[y + (k1 + 2 k2 + 2 k3 + k4)/6]
];

While[xC < b,
(* Store the computed (x, y) points *)
AppendTo[rungeKutta, {xC, yC}];

(* Compute RK4 coefficients for h *)
y1 = rkStep[xC, yC, h];

(* Update x and y values *)
xC += h;
yC = y1;
]
Show[
 Plot[r[x], {x, x0 - 0.1, b + 0.1}, PlotStyle → {Blue, Thick}], (* Function curve *)
 ListPlot[rungeKutta, PlotStyle → {Red, PointSize[Medium]}], (* Numerical points *)
 PlotTheme → "Detailed",
 AxesLabel → {"x", "y"},
 PlotLegends → {"y[x]", "Computed Points"},
 GridLines → Automatic
]
```

Out[44]=



In[45]:=
```
t = Table[Abs[rungeKutta[[i]][[2]] - r[rungeKutta[[i]][[1]]]], {i, 1, 80}]
Max[t]
```

Out[45]=

{0, 0.0000790486, 0.000159762, 0.000241852, 0.00032504, 0.000409051, 0.000493622, 0.000578499, 0.000663439, 0.00074821, 0.000832595, 0.000916388, 0.000999395, 0.00108144, 0.00116235, 0.00124198, 0.00132019, 0.00139685, 0.00147186, 0.0015451, 0.00161648, 0.00168594, 0.00175341, 0.00181882, 0.00188213, 0.00194331, 0.00200232, 0.00205915, 0.00211378, 0.00216621, 0.00221644, 0.00226447, 0.00231031, 0.002354, 0.00239554, 0.00243496, 0.0024723, 0.00250758, 0.00254085, 0.00257214, 0.00260149, 0.00262895, 0.00265456, 0.00267836, 0.00270041, 0.00272074, 0.00273942, 0.00275648, 0.00277198, 0.00278596, 0.00279848, 0.00280958, 0.00281931, 0.00282772, 0.00283485, 0.00284077, 0.00284549, 0.00284909, 0.00285159, 0.00285305, 0.0028535, 0.00285299, 0.00285156, 0.00284925, 0.00284609, 0.00284212, 0.00283739, 0.00283193, 0.00282576, 0.00281893, 0.00281147, 0.00280341, 0.00279477, 0.0027856, 0.00277591, 0.00276573, 0.0027551, 0.00274404, 0.00273256, 0.0027207}

Out[46]=

0.0028535