# Annotation Unit 9

The topic of the passage is “continuity.” It starts with a simple, “non-mathematician-readable” explanation of the concept. Suppose we have a function $f(x)$. As you may know, we can put some $x$ inside the function and get the corresponding value $y = f(x)$. Now try to increase $x$ a bit (if $x$ is a real number, try to increase it by $0.0001$, for instance). If the function evaluated at a new point also increased a bit, chances are, you are dealing with a continuous function at this point. Then the author gives two functions to illustrate the definition: function $h(t)$, which describes the height of the growing flower at time $t$ and $m(t)$ — the amount of money in a bank account at the time $t$. The function $h(t)$ is continuous everywhere, whereas the function $m(t)$ is not (since certain “jumps” occur when the owner of the wallet makes transactions with the money in the account).

Then the passage proceeds to describe the history of the concept development. It outlines the $\varepsilon - \delta$ definition of continuity and mentions essential scientists who dealt with continuous functions.

Since “continuity” is a broad term used for describing different sorts of functions, the passage decides to focus on functions $f : D \to \mathbb{R}$, $D \subset \mathbb{R}$ (that is, both input and output are real numbers). It begins with the most efficient way of teaching students new concepts: giving a rough definition 🙂. In this case, a function is continuous if its representation on the graph in the Cartesian plane has no “cliffs”, “jumps”, or “holes.”

Next, the writer of the passage introduces the strict definition of continuity. Suppose we have a function $f(x)$ and some point $x_0 \in D$. Then $f(x)$ is said to be continuous at point $x_0$ if $\lim_{x \to x_0} f(x) = f(x_0)$. $f(x)$ is continuous on a set $S \subset D \subset \mathbb{R}$ if, for any point $s$ in the set $S$ (that is, $s \in S$), $f(x)$ is continuous at point $s$. This rule can be rewritten as follows:

$$\forall s \in S \subset D : \lim_{x \to s} f(x) = f(s)$$

Finally, the function $f(x)$ is said to be continuous if $f(x)$ is continuous on its domain, that is, $f(x)$ is continuous on $D$.

Similarly to the definition of the limit, continuity can be determined using sequences. If one takes any sequence $\{x_n\}_{n=0}^{\infty}$ that converges to $x_0 \in D$ (that is,

$\lim_{n\to\infty} x_n = x_0$), the sequence $\{f(x_n)\}_{n=0}^{\infty}$ must converge to $f(x_0)$.

Furthermore, the first definition (using limit of the function itself) can be interpreted as follows: for any positive real number $\varepsilon$ we can find another positive real number $\delta$ (which in most cases is the function which depends on $\varepsilon$) such that for any $x \in D$ such that $|x - x_0| < \delta$, the inequality $|f(x) - f(x_0)| < \varepsilon$ holds.

Remember the intuitive explanation of continuity at the beginning of the passage? To make this definition rigorous, one may define the hyperreal numbers. The author of the passages then gives a detailed explanation of how it works based on two examples.

Finally, the passage enumerates several properties of continuity. Firstly, if we have two continuous functions $f, g$, their composition $f \circ g = f(g(x))$ is continuous as well. Secondly, the author gives several functions to illustrate the concept of function discontinuity.

Next, the passage introduces several theorems, the core of which is the concept of continuity. First one is the intermediate value theorem that states that for any function $f$ continuous on the interval $[a, b]$ the following statement always holds: if we take any number $\beta$ between $f(a)$ and $f(b)$, then we can always find number $c$ between $a$ and $b$ such that $f(c) = \beta$.

To better understand this idea, the passage yet again introduces an example from real life: if a child grows from $1\ m$ to $1.5\ m$, then at some point, the child's height should have been $1.25\ m$.

The second essential theorem is called the extreme value theorem. It states that if some function $f$ is continuous on a closed interval $[a, b]$ then this function reaches both maximum and minimum values at some points between $a$ and $b$.

Finally, the relation to differentiability and integrability is considered. Consider some continuous function $f$ on the closed interval $[a, b]$. It is always integrable but not always differentiable (yet, the converse holds).