SCUDEM Problem 1 Presentation

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November 13, 2022

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Plan

Single crew model

Model assumptions

- We consider the crew as a whole and indivisible unit with some parameters describing its state.
- The crew's state is described via 2 parameters: the capability level c and the stress level s, which both somehow depend on the number of tasks left to solve n.
- Capability is some fixed function of s. That is, $c = f(s)$.
- Capability is the speed of solving tasks: $c = -\frac{dn}{dt}$.

Using these facts, our problem consists in solving the equation

$$\dot{n} = -f(s)$$

Stress change dynamics

Set of assumptions

- The stress increase rate is proportional to the number of tasks left: $\frac{ds}{dt} \propto n$.
- The more stress the crew undergoes, the more resistant it becomes to the new upcoming stress.

Using these assumptions, we can write down the stress change dynamics equation as follows:

$$\frac{ds}{dt} = -\alpha s + \beta n$$

Which leads us to the following system of differential equations:

$$\begin{cases} \dot{n} = -f(s) \\ \dot{s} = -\alpha s + \beta n \end{cases}$$

Addition to a stress change dynamics

We will add one more important assumption:

If the crew *feels* they are making everything in time, its level of stress decreases and vice versa.

Suppose that at certain time moment t_0 the speed of solving tasks is $\frac{dn}{dt}\Big|_{t=t_0}$ and the deadline is scheduled on the time moment τ. That means that the crew at this moment of time feels that it can solve $-\left(\frac{dn}{dt}\right)\Big|_{t=t_0}(\tau - t_0)$ tasks if it maintains its speed. If $n(t_0)$ tasks left at this point, it means that the change of stress is

$$\Delta s = \beta \cdot \left(n(t_0) + \frac{dn}{dt}\Big|_{t=t_0}(\tau - t_0) \right) \Delta t$$

Geometric intuition

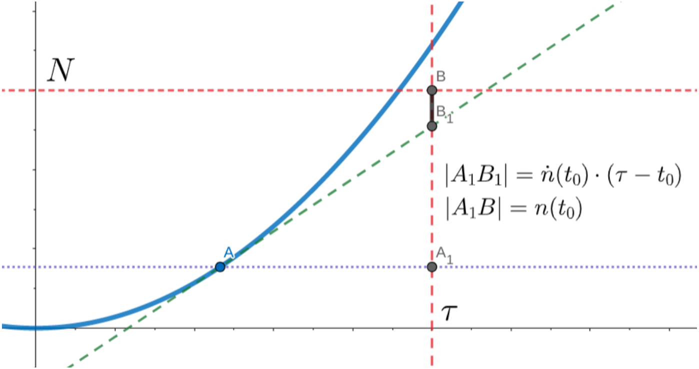

Figure: B is a point of deadline where N tasks must be solved. A_1B is a number of tasks left to be solved and A_1B_1 how many the crew manages to solve if they maintain their speed.

Result Differential Equation

All things considered, we end up with equation

$$\begin{cases} \dot{n} = -f(s) \\ \dot{s} = -\alpha s + \beta(n + \dot{n}(\tau - t)) \end{cases}$$

We won't generalize the solution, but will sketch the solution outline.
Firstly, we find $s = f^{-1}(-\dot{n}) := \varphi(\dot{n})$. From here we can simply find $\dot{s}$ by differentiating s: $\dot{s} = \phi(\dot{n}, \ddot{n})$.
Then we substitute the result into the second equation, yielding:

$$\phi(\dot{n}, \ddot{n}) + \alpha\varphi(\dot{n}) - \beta(n + \dot{n}(\tau - t)) = 0$$

Which is a second order differential equation with respect to $n(t)$.

Dependence of capability on stress

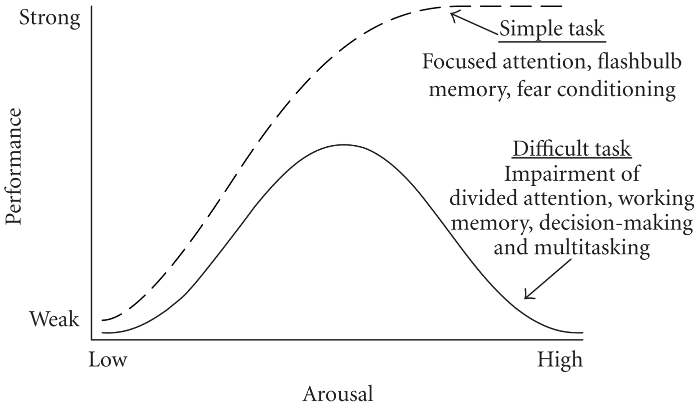

Figure: Original Yerkes-Dodson curve based on the original evidence from Yerkes and Dodson 1908

Dependence of capability on stress

This dependence can be approximated using relation:

$$c = f(s) = e^{-\sigma(s-\mu)^2}$$

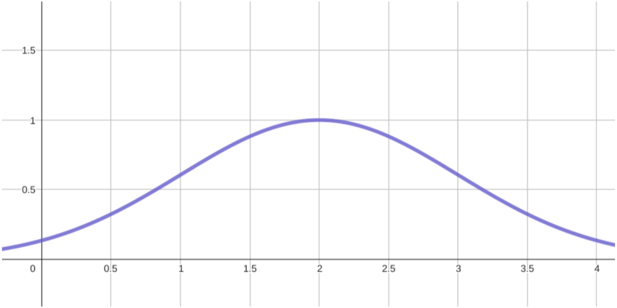

Figure: Approximation of Yerkes-Dodson curve using $f(s) = e^{-\sigma(s-\mu)^2}$ for $\sigma = 0.5, \ \mu = 2$

Solving differential equation

According to the relation $\dot{n} = -f(s)$, we obtain $\dot{n} = -e^{-\sigma(s-\mu)^2}$. From here we get:

$$s = \mu - \sqrt{-\frac{\ln(-\dot{n})}{\sigma}} \implies \dot{s} = \frac{\ddot{n}}{2\dot{n}\sqrt{-\sigma\ln(-\dot{n})}}$$

By substituting it into the relation

$$\dot{s} = -\alpha s + \beta(n + \dot{n}(\tau - t)),$$

we obtain (with initial conditions $n(0) = n_0, \dot{n}(0) = -f(s_0)$):

$$\ddot{n} = -2\alpha\dot{n}\sqrt{-\sigma\ln(-\dot{n})}\left(\mu - \sqrt{-\frac{\ln(-\dot{n})}{\sigma}}\right) + 2\beta\dot{n}\sqrt{-\sigma\ln(-\dot{n})}(n + \dot{n}(\tau - t)) \tag{1}$$

Results

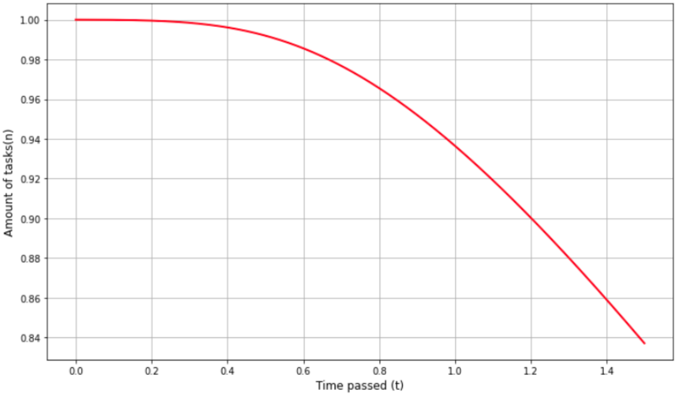

Figure: Dependence of amount of tasks left to solve on time passed. Curve was built for parameters
$n_0 = 1, s_0 = 6, \alpha = 2, \beta = 1, \sigma = 0.5, \mu = 2, \tau = 2$

Analysis

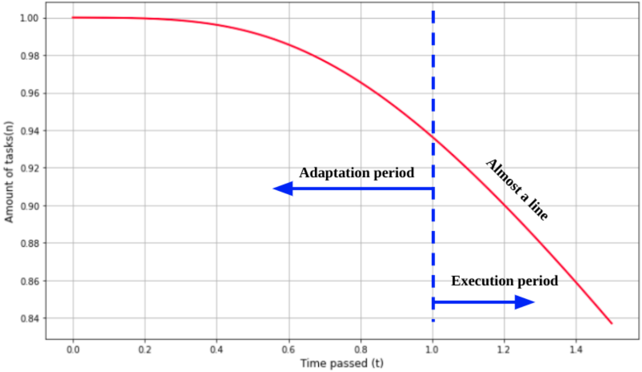

Figure: Process periods classification

Application of a single crew model

Firstly, we apply the model we used before. That means that if two groups considered independently, their *DE*s would look like:

$$\begin{cases} \dot{n} = -f(s_1) - f(s_2) \\ \dot{s_k} = -\alpha_k s_k + \beta_k (n + \dot{n}(\tau - t)), \ k = 1, 2 \end{cases}$$

First equation here means that the speed of solving tasks equals to the sum of two crews' capabilities.
That is great, but it does not give us any information about interaction between two crews. And this interaction does occur since they are working together.

Additional assumption

We will add one more assumption:

Stresses relationship

The greater stress one group undergoes, the greater stress another group undergoes as well. In other words, rate of stress increase of one crew is proportional to the level of stress of another crew ($\dot{s}_1 \propto s_2, \ \dot{s}_2 \propto s_1$)

Theoretically, with this assumption, the stress of both crews must increase which would result in a smaller capability. Yet, now their capability is being summed up and therefore it is not obvious whether such cooperation would fasten the task solving process.

Differential equation

With the previous assumption, let us modify the equation

$$\begin{cases} \dot{n} = -f(s_1) - f(s_2) \\ \dot{s}_1 = -\alpha_1 s_1 + \gamma_1 s_2 + \beta_1(n + \dot{n}(\tau - t)) \\ \dot{s}_2 = -\alpha_2 s_2 + \gamma_2 s_1 + \beta_2(n + \dot{n}(\tau - t)) \end{cases}$$

If we denote $\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$, $\dot{\mathbf{s}} = \begin{bmatrix} \dot{s}_1 \\ \dot{s}_2 \end{bmatrix}$ together with $\mathbf{M} = \begin{bmatrix} -\alpha_1 & \gamma_1 \\ \gamma_2 & -\alpha_2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$ we get the more concise form:

$$\begin{cases} \dot{n} = -f(s_1) - f(s_2) \\ \dot{\mathbf{s}} = \mathbf{M}\mathbf{s} + (n + \dot{n}(\tau - t))\mathbf{b} \end{cases}$$

Modelling the whole process

Suppose that we need to solve N tasks in certain amount of time T. At the beginning, first crew starts working, and the process is described via *Single Crew model*:

$$\begin{cases} \dot{n} = -f(s_1) \\ \dot{s_1} = -\alpha_1 s_1 + \beta_1(n + \dot{n}(T-t)) \end{cases}$$

$$n(0) = N, \ \dot{n}(0) = -f(s_{0,1})$$

here $s_{0,1}$ denotes the initial stress of a first crew. Suppose that at point τ the second crew comes in. Suppose that at this point $n(\tau) = n_\tau, \ s_1(\tau) = s_\tau, \ f(s_1(\tau)) = f_\tau$.

Modelling the whole process

Then, the second crew joins in with some value of initial stress $s_{0,2}$. Now we will apply *Two Crews model* DE:

$$\begin{cases} \dot{n} = -f(s_1) - f(s_2) \\ \dot{s}_1 = -\alpha_1 s_1 + \gamma_1 s_2 + \beta_1(n + \dot{n}(T - \tau - t)) \\ \dot{s}_2 = -\alpha_2 s_2 + \gamma_2 s_1 + \beta_2(n + \dot{n}(T - \tau - t)) \end{cases}$$

with initial conditions:

$$n(0) = n_\tau,\ \dot{n}(0) = -f_\tau - f(s_{2,0})$$

$$s_1(0) = s_\tau,\ s_2(0) = s_{2,0}$$

By combining the solutions of these two equations we can now describe the whole process.

Result

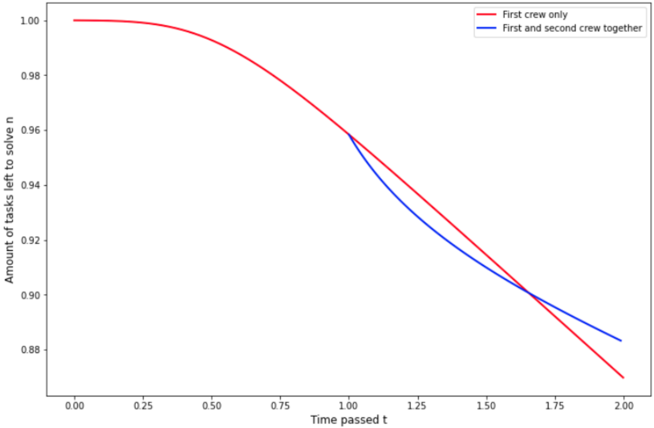

Figure: Dependence of $n(t)$

Result

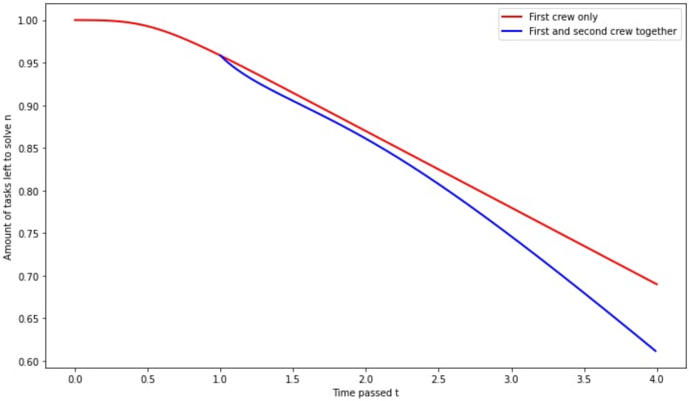

Figure: Dependence of $n(t)$ for different set of parameters

Analysis

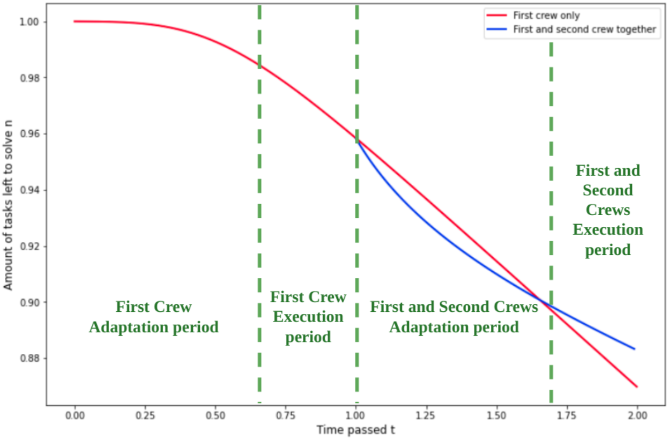

Figure: Periods classification for *Two Crews model* DE

Schedule Factor

Schedule factor

- We did not consider that the crews need to rest. In order to add such component to our DE, notice, that for now we consider dependence of capability from stress only. Now, suppose that it is also time-dependent, namely: $c = f(s,t)$. But time-dependence does not fully fix the issue. Let us go even further: suppose that capability is a function $c = f(s)H(t)$ where $H(t)$ is a schedule factor which is a periodic function values of which range from 0 to 1. This way, relation $\dot{n} = -c$ might be rewritten as:

$$\dot{n} = -f(s)H(t)$$

Schedule Factor

Example

For example, we might take $H(t) = |\cos \omega t|$. This schedule factor corresponds to the situation where after each period of $H(t)$ (which is $\frac{\pi}{\omega}$) productivity gradually increases until it reaches its maximum after half a period and then gradually decreases to 0. Another example is $H(t) = \lfloor \frac{t}{T} \rfloor - \frac{t}{T} + 1$. In this case, at the beginning of each period (which is T) the productivity is highest and then it linearly decreases until it reaches 0.

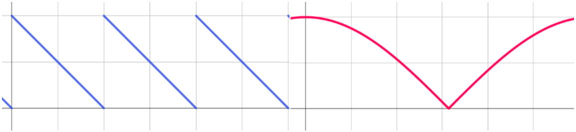

Figure: Graphs $H(t) = |\cos t|$ and $H(t) = \lfloor t \rfloor - t + 1$.

Schedule Factor

Example

Finally, we can put the following function $H(t)$, as described in the problem statement:

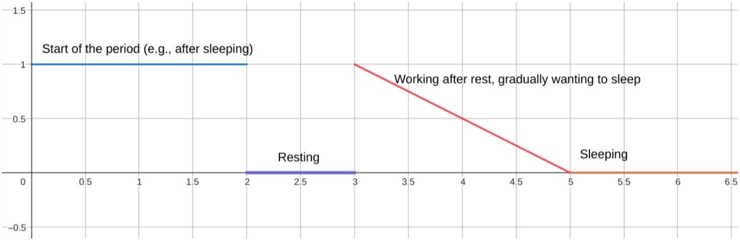

Figure: Schedule factor $H(t)$ according to the problem statement

That being said, now our model can predict process considering crews' schedule.

Solutions for different schedule factors

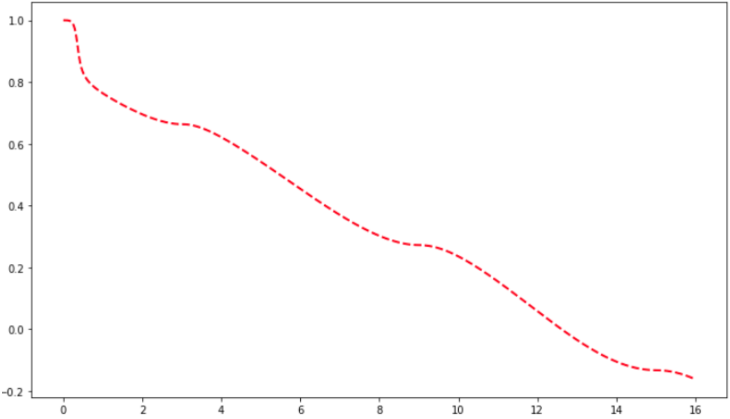

Figure: Dependence $n(t)$ for a single crew model using $H(t) = |\cos \omega t|$.

Solutions for different schedule factors

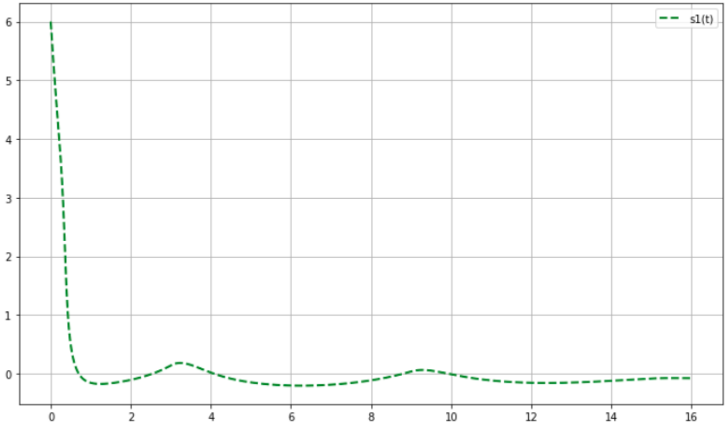

Figure: Dependence $s(t)$ for a single crew model using $H(t) = |\cos \omega t|$.

Solutions for different schedule factors

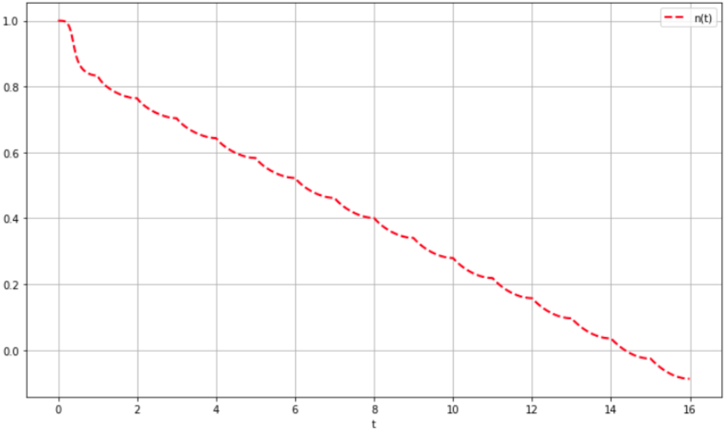

Figure: Dependence $n(t)$ for a single crew model using $H(t) = \lfloor t/T \rfloor - t/T + 1$.

Solutions for different schedule factors

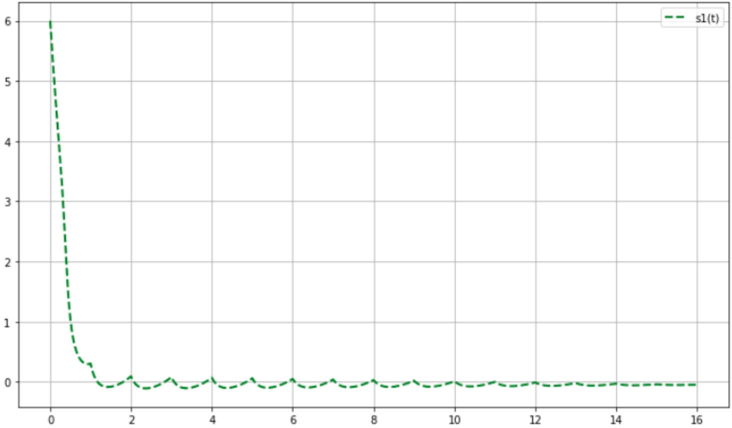

Figure: Dependence $s(t)$ for a single crew model using $H(t) = \lfloor t/T \rfloor - t/T + 1$.

Solutions for different schedule factors

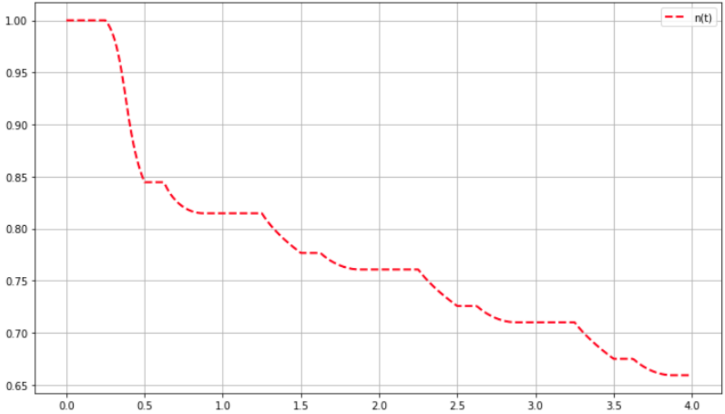

Figure: Dependence $n(t)$ for a single crew model using schedule factor that resembles a real schedule with sleeping, resting etc.

Solutions for different schedule factors

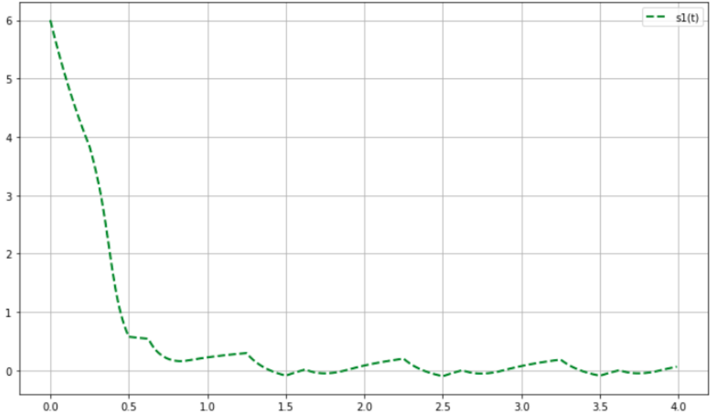

Figure: Dependence $s(t)$ for a single crew model using schedule factor that resembles a real schedule with sleeping, resting etc.

Different stress dependencies

- Two different crews might have different dependencies on stress. So we might write the first equation in our system of equations as follows:

$$\dot{n} = -H(t)\left(f_1(s_1) - f_2(s_2)\right)$$

Here we consider that both crews have the same schedule. Otherwise, the equation can be written as

$$\dot{n} = -H_1(t)f_1(s_1) - H_2(t)f_2(s_2)$$

p-Crews Model

- The model might take in consideration arbitrary number of crews. The system of DE for p crews will look as follows:

$$\begin{cases} \dot{n} = -H(t)\sum_{j=1}^{p} f_j(s_j) \\ \dot{s}_k = \sum_{j=1}^{p} \alpha_{k,j} s_j + \beta_k(n + \dot{n}(\tau - t)), \; k = \overline{1,p} \end{cases}$$

where $\alpha_{k,k} < 0, \beta_k > 0 \; \forall k$ and $\alpha_{k,j} \geq 0 \; \forall j \neq k$.

Questions for the further research

- When building Single Crew Model, almost for any dependence $c = f(s)$ we tried to substitute, we obtained a linear segment on the plot $n(t)$. Is it possible to determine the slope of this segment and time moment when it occurs without finding numerically or explicitly $n(t)$?
- Is it possible to determine the slope of linear segment when considering Two Crews Model and compare it to the slope of a linear segment when considering one crew only (again, without solving explicitly $n(t)$)?
- Based on the schedule factor $H(t)$ we can get dependencies $n(t), s_k(t)$, but is it possible to approximate $H(t)$ analytically which would minimize the stress and maximize the speed of solving tasks?

Conclusion

- We managed to build a model that can describe a task solving process of a single crew depending on a huge variety of different parameters: initial stress, dependence of capability on stress, adaptance to a current level of stress etc.
- We expanded a single crew model to describe the task solving process for two crews interacting with each other.
- We analyzed obtained results.
- We made possible to include a schedule factor and predict process based on it. Mentioned several improvements such as different stress dependencies and wrote down the DE that can describe a process for an arbitrary number of interacting crews.

Thank you for your attention!

Corbett, Martin (2015-08-10). "From law to folklore: work stress and the Yerkes-Dodson Law". Journal of Managerial Psychology. 30 (6): 741–752. doi:10.1108/jmp-03-2013-0085