

Homework #2

Problem 456

Find the limit:

$$L = \lim_{x o 1} rac{\prod_{k=2}^n (1 - x^{1/k})}{(1 - x)^{n-1}}$$

Solution

Firstly, we introduce the substitution $z=1-x \implies$ when $x \to 1$, $z \to 0$. Thus:

$$L = \lim_{z o 0} \left(rac{1}{z^{n-1}} \prod_{k=2}^n (1-(1-z)^{1/k})
ight)$$

Let us consider the term $1-(1-z)^{1/k}$. Notice that $(1-z)^{1/k}=1-z/k+\overline{o}(z),z\to 0$. Thus $1-(1-z)^{1/k}=1-(1-z/k+\overline{o}(z))=z/k-z\overline{o}(1)$. Hence finally:

$$L = \lim_{z o 0} rac{\prod_{k=2}^n (z/k - \overline{o}(1)z)}{z^{n-1}} = \lim_{z o 0} rac{z^{n-1}}{z^{n-1}} \prod_{k=2}^n \left(rac{1}{k} - \overline{o}(1)
ight) = \lim_{z o 0} \prod_{k=2}^n \left(rac{1}{k} - \overline{o}(1)
ight)$$

Now notice that we have finite number of terms and hence since $\overline{o}(1) \to 0$ as $z \to 0$ we have:

$$L = \prod_{k=2}^{n} \frac{1}{k} = \frac{1}{n!}$$

Problem 457

Find the limit:

$$L=\lim_{x o +\infty}(\sqrt{(x+a)(x+b)}-x),\; a,b\in \mathbb{R}$$

Solution.

1st method. Multiply both numerator and denominator by $\sqrt{(x+a)(x+b)} + x$:

$$L = \lim_{x o +\infty} rac{(x+a)(x+b) - x^2}{\sqrt{(x+a)(x+b)} + x}$$

After simplifications:

$$L=\lim_{x
ightarrow+\infty}rac{x(a+b)+ab}{\sqrt{(x+a)(x+b)}+x}=\lim_{x
ightarrow+\infty}rac{a+b+rac{ab}{x}}{\sqrt{(1+rac{a}{x})(1+rac{b}{x})}+1}=rac{a+b}{2}$$

2nd method. Let us make some manipulations with the initial expression:

$$L = \lim_{x o +\infty} x \left(\sqrt{1 + rac{a+b}{x} + rac{ab}{x^2}} - 1
ight)$$

Now use the fact that:

$$\sqrt{1+rac{a+b}{x}+rac{ab}{x^2}}=1+rac{1}{2}\left(rac{a+b}{x}+rac{ab}{x^2}
ight)+\left(rac{a+b}{x}+rac{ab}{x^2}
ight)\overline{o}(1),\;x
ightarrow+\infty$$

Thus:

$$L = \lim_{x o +\infty} \left(rac{a+b}{2} + rac{ab}{2x} + \left(a+b + rac{ab}{x}
ight) \overline{o}(1)
ight) = rac{a+b}{2}$$

Problem 501

Find the limit:

$$L = \lim_{x o 0} rac{(\cos x)^{1/2} - (\cos x)^{1/3}}{\sin^2 x}$$

Solution

Firstly let us consider $(\cos x)^\mu, \mu \in \mathbb{R}$. Let $lpha(x) = -x^2/2 + \overline{o}(x^2)$. As x o 0 we have:

$$\cos x=1-rac{x^2}{2}+\overline{o}(x^2)=1+lpha(x),\;x o 0$$

One can clearly see that $\lim_{x \to 0} \alpha(x) = 0$. Thus:

$$(\cos x)^\mu = (1+lpha(x))^\mu = 1+\mulpha(x)+\overline{o}(lpha(x)) = 1-rac{\mu x^2}{2}+\overline{o}(x^2)+\overline{o}(lpha(x))$$

Again, it is not that complicated to notice that $\overline{o}(x^2)+\overline{o}(-x^2/2+\overline{o}(x^2))=\overline{o}(x^2)$. Thus:

$$(\cos x)^\mu=1-rac{\mu x^2}{2}+\overline{o}(x^2)$$

Now let us consider $\sin^2 x$. Notice that $\sin x = x + \overline{o}(x)$. Thus:

$$\sin^2 x = (x + \overline{o}(x))^2 = x^2 (1 + \overline{o}(1))^2$$

Finally:

$$L = \lim_{x o 0} rac{1 - rac{x^2}{4} + \overline{o}(x^2) - 1 + rac{x^2}{6} - \overline{o}(x^2)}{x^2(1 + \overline{o}(1))^2} = \lim_{x o 0} rac{x^2(-1/12 + \overline{o}(1))}{x^2(1 + \overline{o}(1))^2}$$

After dividing nominator and denominator by x^2 we obtain:

$$L = \lim_{x \to 0} \frac{-1/12 + \overline{o}(1)}{(1 + \overline{o}(1))} = -\frac{1}{12}$$

Problem 502

Find the limit:

$$L = \lim_{x \to 0} \frac{\sqrt{1 - \cos x^2}}{1 - \cos x}$$

Solution

First of all:

$$\cos x = 1 - rac{x^2}{2} + \overline{o}(x^2), x
ightarrow 0$$

Thus:

$$1-\cos x=rac{x^2}{2}-\overline{o}(x^2)=x^2\left(rac{1}{2}-\overline{o}(1)
ight)$$

Now let us move to the simplifying numerator. We have:

$$\cos x^2 = 1 - rac{x^4}{2} + \overline{o}(x^4), \; x^2 o 0 \; as \; x o 0$$

Thus:

$$\sqrt{1-\cos x^2}=\sqrt{rac{x^4}{2}-\overline{o}(x^4)}=x^2\sqrt{rac{1}{2}-\overline{o}(1)}$$

Substituting our results into initial limit we yield:

$$L = \lim_{x o 0} rac{x^2 \sqrt{1/2 - \overline{o}(1)}}{x^2 (1/2 - \overline{o}(1))} = \lim_{x o 0} rac{\sqrt{1/2 - \overline{o}(1)}}{1/2 - \overline{o}(1)} = \sqrt{2}$$

Problem 504

Find the limit:

$$L = \lim_{x o 0} rac{1 - \cos x (\cos 2x)^{1/2} (\cos 3x)^{1/3}}{x^2}$$

Solution

Firstly, we use:

$$\cos x = 1 - rac{x^2}{2} + \overline{o}(x^2), x
ightarrow 0$$

Similarly:

$$\cos 2x=1-2x^2+\overline{o}(x^2),\;\cos 3x=1-rac{9x^2}{2}+\overline{o}(x^2),\;x
ightarrow 0$$

As we proved before:

$$(\cos 2x)^{1/2} = (1-2x^2+\overline{o}(x^2))^{1/2} = 1-x^2+\overline{o}(x^2), x o 0 \ (\cos 3x)^{1/3} = 1-rac{3x^2}{2}+\overline{o}(x^2), \ x o 0$$

Thus we have:

$$L = \lim_{x o 0} rac{1 - (1 - x^2/2 + \overline{o}(x^2))(1 - x^2 + \overline{o}(x^2))(1 - 3x^2/2 + \overline{o}(x^2))}{x^2}$$

After simplifications we have:

$$L = \lim_{x o 0} rac{3x^2 - 9x^4/4 + \overline{o}(x^2)}{x^2} = 3$$

Problem 495

Find the limit:

$$L=\lim_{x o\pi/3}rac{\sin(x-\pi/3)}{1-2\cos x}$$

Solution

First of all, we use the substitution $heta=x-\pi/3 \implies heta o 0$ as $x o \pi/3$:

$$L = \lim_{ heta o 0} rac{\sin heta}{1 - 2\cos(heta + \pi/3)}$$

Now use:

$$\cos(heta+\pi/3)=rac{1}{2}\cos heta-rac{\sqrt{3}}{2}\sin heta$$

Thus:

$$L = \lim_{ heta o 0} rac{\sin heta}{(1 - \cos heta) + \sqrt{3} \sin heta}$$

Now we use the fact that:

$$\sin \theta = \theta + \overline{o}(\theta), \theta \to 0$$

$$\cos heta = 1 - rac{ heta^2}{2} + \overline{o}(heta^2)$$

Thus:

$$L = \lim_{ heta o 0} rac{ heta + \overline{o}(heta)}{ heta^2/2 + \sqrt{3} heta + \overline{o}(heta^2) + \overline{o}(heta)} = \lim_{ heta o 0} rac{ heta(1 + \overline{o}(1))}{ heta(\sqrt{3} + \overline{o}(1)) + heta^2(1/2 + \overline{o}(1))}$$

After dividing both sides by θ we obtain:

$$L = \lim_{ heta o 0} rac{1 + \overline{o}(1)}{\sqrt{3} + \overline{o}(1) + heta(1/2 + \overline{o}(1))} = rac{1}{\sqrt{3}}$$

Problem 450

Find the limit:

$$L = \lim_{x o 0} rac{(1 + rac{x}{3})^{1/3} - (1 + rac{x}{4})^{1/4}}{1 - (1 - rac{x}{2})^{1/2}}$$

Solution:

We use the following:

$$\left(1+rac{x}{3}
ight)^{1/3} = 1+rac{x}{9}+\overline{o}(x),\; x o 0 \ \left(1+rac{x}{4}
ight)^{1/4} = 1+rac{x}{16}+\overline{o}(x),\; x o 0 \ \left(1-rac{x}{2}
ight)^{1/2} = 1-rac{x}{4}+\overline{o}(x),\; x o 0$$

Thus:

$$L = \lim_{x o 0} rac{x(7/144 + \overline{o}(1))}{x(1/4 + \overline{o}(1))} = rac{7}{36}$$

Problem 459

Find the limit:

$$L=\lim_{x
ightarrow+\infty}x(\sqrt{x^2+2x}-2\sqrt{x^2+x}+x)$$

Solution:

Firstly, let us make some manipulations with the initial expression:

$$L = \lim_{x o +\infty} x^2 \left(\sqrt{1 + rac{2}{x}} - 2\sqrt{1 + rac{1}{x}} + 1
ight)$$

First of all, it is essential to note that the first order of smallness would not suffice to solve this problem. Indeed, if we put $(1+2/x)^{1/2}=1+1/x+\overline{o}(1/x),\ 1/x\to 0$ and $(1+1/x)^{1/2}=1+1/2x+\overline{o}(1/x),\ 1/x\to 0$:

$$L = \lim_{x o +\infty} x^2 \overline{o}\left(rac{1}{x}
ight) = \lim_{x o +\infty} x \overline{o}(1)$$

we cannot evaluate this limit for the obvious reasons. Thus we have to use the second order of smallness, i.e.:

$$(1+x)^{lpha}=1+lpha x+rac{lpha(lpha-1)}{2}x^2+\overline{o}(x^2)$$

Thus:

$$\sqrt{1+rac{2}{x}}=1+rac{1}{x}-rac{1}{2x^2}+\overline{o}(x^{-2}),\;rac{1}{x} o 0$$

$$\sqrt{1+rac{1}{x}}=1+rac{1}{2x}-rac{1}{8x^2}+\overline{o}(x^{-2}),\ rac{1}{x} o 0$$

Thus:

$$L = \lim_{x o +\infty} x^2 \left(1 + rac{1}{x} - rac{1}{2x^2} + \overline{o}(x^{-2}) - 2 - rac{1}{x} + rac{1}{4x^2} + \overline{o}(x^{-2}) + 1
ight)$$

After simplifications:

$$L=\lim_{x o +\infty}x^2\left(-rac{1}{4x^2}+\overline{o}\left(rac{1}{x^2}
ight)
ight)=\lim_{x o +\infty}\left(-rac{1}{4}+\overline{o}(1)
ight)=-rac{1}{4}$$

Problem 453

Find the limit:

$$L = \lim_{x o 0} rac{(1+lpha x)^{1/m}(1+eta x)^{1/n}-1}{x}$$

Solution:

Firstly, we use:

$$(1+lpha x)^{1/m}=1+rac{lpha x}{m}+\overline{o}(x),\;x o 0$$

$$(1+eta x)^{1/n}=1+rac{eta x}{n}+\overline{o}(x),\;x o 0$$

Thus our limit can be written as follows:

$$L = \lim_{x o 0} rac{(1 + lpha x/m + \overline{o}(x))(1 + eta x/n + \overline{o}(x)) - 1}{x}$$

After simplifications we obtain:

$$L = \lim_{x o 0} rac{eta x/n + lpha x/m + \overline{o}(x)}{x} = rac{lpha}{m} + rac{eta}{n}$$