Annotation Unit 11

The topic discussed in the passage is how we can describe functions and what properties they have.

The author of the passage begins by enumerating all different ways of defining functions: piecewise definition, induction, recursion, limits, infinite series, as a solution of integral or differential equations, etc. Roughly speaking, we define function either by giving formula or giving some sort of recipe or rather algorithm using which we can find this function.

The passage then focuses on three ways of representing or defining the function.

Graph. Essentially, the graph is a set of ordered pairs $\{a, b\}$ where the first element of a pair corresponds to the input of the function and the second one to its output when the first element is inputted. We can illustrate this abstraction for functions f : $\mathbb{R} \to \mathbb{R}$ by drawing a graph where we take a "steps" from the origin to the left/right and b steps up or down.

Formulas or algorithms. It is important to note that there is no single unique way of expressing function. For example, both $(x+1)(x-1)$ and $x^2 - 1$ are the same functions. To further illustrate this idea, the author of the passage gives an example of the factorial function. It can be explicitly defined using the formula $n! = \prod_{k=1}^{n} k$ or using a recursion: $0! = 1$ and $n! = n \cdot (n-1)!$ Both methods define the same object.

Furthermore, not necessarily functions can be expressed using formulas or algorithms and do not always have to deal with numbers. For example, consider the function φ that takes some text as an input and gives the last letter of a text as an output (e.g., $\varphi(\text{I love English}) = \text{h}$). After all, functions work with arbitrary sets of inputs and outputs, which are not always numbers.

Computability. Sometimes functions can be specified using algorithms. For example let us define function $\phi(n)$: $\mathbb{Z} \to \mathbb{Z}$ as follows:

$$\phi(n) = \begin{cases} 1, n \text{ is prime} \\ n^2,\ \text{n is even} \\ \sin \pi n \text{ otherwise} \end{cases}$$

Essentially, we described the function $\phi(n)$ using an algorithm: we first check whether n is prime. If it is prime, the output is 1. If not, we check whether n is even, and if it is, the output is n^2. Otherwise, the output is 0 (as $\sin \pi n = 0$ holds for any integers n. Besides, that proves that any function can be expressed in a multitude of different ways). For example, if $n = 5$ the output is 1 and if $n = 15$ the output is $\sin 15\pi = 0$.

Such functions are called computable functions (which can be defined by an algorithm). The passage then mentions some interesting enough properties of computable functions. For example, if we consider all functions $h : \mathbb{Z} \to \mathbb{Z}$, than a very limited number of them are computable (moreover, it is known that the number of such functions is countable).

Next, the author of the passage switches the topic to the properties of functions.

As an example, the passage proposes to consider the function $f(x) = x^3 - 9x^2 + 23x - 15$ and interval $A = [3.5, 4.25]$ which is a subset of function's domain. The image of A under f is, by definition, the set of possible outputs that f can take by substituting different elements of A into the function. For example, if $f(n) = n + 1$ and $A = \{2, 3\}$ then the image of A is $\{3, 4\}$. Returning to the function proposed by the passage, we can find that the image of A is approximately interval $[-3.08, -1.88]$. To get this interval, we need to project the part of the graph $f(x)$ bounded between lines $x = 3.5$ and $x = 4.25$ to the Oy axis.

The preimage is another crucial concept described in the passage. The preimage of set B under f is, by definition, a set of values that, by substituting into f, yields values from the set B. For instance, if we consider $B = [1, 2.5]$ and the polynomial from the passage, the preimage of B consists of three parts. To obtain them, one needs to take the part of the f bounded between lines $y = 1$ and $y = 2.5$ and project it to the Ox axis.

Finally, the passage focuses on describing real-valued functions, that is, functions whose codomain is either set or a subset of real numbers $\mathbb{R}$. If the domain is also either a set or a subset of real numbers, then f is, in addition, a function of a real variable. The study of such functions is called real analysis. One crucial property of such functions is that given any two real-valued functions $f, g : X \to Y$, their sum $f + g$ and product $f \cdot g$ are functions with the same domain and codomain. They are defined as follows:

$$(f + g)(x) = f(x) + g(x),\ (f \cdot g)(x) = f(x) \cdot g(x)$$

Similarly, complex analysis studies functions whose inputs/outputs are complex numbers.

In the end, the passage provides the most essential types of real-valued functions.