Homework #2

Problem 456

Find the limit:

$$L = \lim_{x \to 1} \frac{\prod_{k=2}^{n}(1 - x^{1/k})}{(1-x)^{n-1}}$$

Solution

Firstly, we introduce the substitution $z = 1 - x \implies$ when $x \to 1$, $z \to 0$. Thus:

$$L = \lim_{z \to 0} \left(\frac{1}{z^{n-1}} \prod_{k=2}^{n} (1 - (1-z)^{1/k}) \right)$$

Let us consider the term $1 - (1-z)^{1/k}$. Notice that $(1-z)^{1/k} = 1 - z/k + \overline{o}(z), z \to 0$. Thus $1 - (1-z)^{1/k} = 1 - (1 - z/k + \overline{o}(z)) = z/k - z\overline{o}(1)$. Hence finally:

$$L = \lim_{z \to 0} \frac{\prod_{k=2}^{n}(z/k - \overline{o}(1)z)}{z^{n-1}} = \lim_{z \to 0} \frac{z^{n-1}}{z^{n-1}} \prod_{k=2}^{n} \left(\frac{1}{k} - \overline{o}(1) \right) = \lim_{z \to 0} \prod_{k=2}^{n} \left(\frac{1}{k} - \overline{o}(1) \right)$$

Now notice that we have finite number of terms and hence since $\overline{o}(1) \to 0$ as $z \to 0$ we have:

$$L = \prod_{k=2}^{n} \frac{1}{k} = \frac{1}{n!}$$

Problem 457

Find the limit:

$$L = \lim_{x \to +\infty} (\sqrt{(x+a)(x+b)} - x), \ a, b \in \mathbb{R}$$

Solution.

1st method. Multiply both numerator and denominator by $\sqrt{(x+a)(x+b)} + x$:

$$L = \lim_{x \to +\infty} \frac{(x+a)(x+b) - x^2}{\sqrt{(x+a)(x+b)} + x}$$

After simplifications:

$$L = \lim_{x \to +\infty} \frac{x(a+b) + ab}{\sqrt{(x+a)(x+b)} + x} = \lim_{x \to +\infty} \frac{a + b + \frac{ab}{x}}{\sqrt{(1+\frac{a}{x})(1+\frac{b}{x})} + 1} = \frac{a+b}{2}$$

2nd method. Let us make some manipulations with the initial expression:

$$L = \lim_{x \to +\infty} x \left(\sqrt{1 + \frac{a+b}{x} + \frac{ab}{x^2}} - 1 \right)$$

Now use the fact that:

$$\sqrt{1 + \frac{a+b}{x} + \frac{ab}{x^2}} = 1 + \frac{1}{2}\left(\frac{a+b}{x} + \frac{ab}{x^2}\right) + \left(\frac{a+b}{x} + \frac{ab}{x^2}\right)\overline{o}(1), \ x \to +\infty$$

Thus:

$$L = \lim_{x \to +\infty} \left(\frac{a+b}{2} + \frac{ab}{2x} + \left(a + b + \frac{ab}{x}\right)\overline{o}(1) \right) = \frac{a+b}{2}$$

Problem 501

Find the limit:

$$L = \lim_{x \to 0} \frac{(\cos x)^{1/2} - (\cos x)^{1/3}}{\sin^2 x}$$

Solution

Firstly let us consider $(\cos x)^{\mu}, \mu \in \mathbb{R}$. Let $\alpha(x) = -x^2/2 + \overline{o}(x^2)$. As $x \to 0$ we have:

$$\cos x = 1 - \frac{x^2}{2} + \overline{o}(x^2) = 1 + \alpha(x),\ x \to 0$$

One can clearly see that $\lim_{x\to 0} \alpha(x) = 0$. Thus:

$$(\cos x)^\mu = (1+\alpha(x))^\mu = 1 + \mu\alpha(x) + \overline{o}(\alpha(x)) = 1 - \frac{\mu x^2}{2} + \overline{o}(x^2) + \overline{o}(\alpha(x))$$

Again, it is not that complicated to notice that $\overline{o}(x^2) + \overline{o}(-x^2/2 + \overline{o}(x^2)) = \overline{o}(x^2)$. Thus:

$$(\cos x)^\mu = 1 - \frac{\mu x^2}{2} + \overline{o}(x^2)$$

Now let us consider $\sin^2 x$. Notice that $\sin x = x + \overline{o}(x)$. Thus:

$$\sin^2 x = (x + \overline{o}(x))^2 = x^2(1 + \overline{o}(1))^2$$

Finally:

$$L = \lim_{x\to 0} \frac{1 - \frac{x^2}{4} + \overline{o}(x^2) - 1 + \frac{x^2}{6} - \overline{o}(x^2)}{x^2(1+\overline{o}(1))^2} = \lim_{x\to 0} \frac{x^2(-1/12 + \overline{o}(1))}{x^2(1+\overline{o}(1))^2}$$

After dividing nominator and denominator by x^2 we obtain:

$$L = \lim_{x\to 0} \frac{-1/12 + \overline{o}(1)}{(1+\overline{o}(1))} = -\frac{1}{12}$$

Problem 502

Find the limit:

$$L = \lim_{x\to 0} \frac{\sqrt{1-\cos x^2}}{1-\cos x}$$

Solution

First of all:

$$\cos x = 1 - \frac{x^2}{2} + \overline{o}(x^2), x \to 0$$

Thus:

$$1 - \cos x = \frac{x^2}{2} - \overline{o}(x^2) = x^2\left(\frac{1}{2} - \overline{o}(1)\right)$$

Now let us move to the simplifying numerator. We have:

$$\cos x^2 = 1 - \frac{x^4}{2} + \overline{o}(x^4),\ x^2 \to 0\ as\ x \to 0$$

Thus:

$$\sqrt{1 - \cos x^2} = \sqrt{\frac{x^4}{2} - \overline{o}(x^4)} = x^2\sqrt{\frac{1}{2} - \overline{o}(1)}$$

Substituting our results into initial limit we yield:

$$L = \lim_{x \to 0} \frac{x^2\sqrt{1/2 - \overline{o}(1)}}{x^2(1/2 - \overline{o}(1))} = \lim_{x \to 0} \frac{\sqrt{1/2 - \overline{o}(1)}}{1/2 - \overline{o}(1)} = \sqrt{2}$$

Problem 504

Find the limit:

$$L = \lim_{x \to 0} \frac{1 - \cos x(\cos 2x)^{1/2}(\cos 3x)^{1/3}}{x^2}$$

Solution

Firstly, we use:

$$\cos x = 1 - \frac{x^2}{2} + \overline{o}(x^2), x \to 0$$

Similarly:

$$\cos 2x = 1 - 2x^2 + \overline{o}(x^2),\ \cos 3x = 1 - \frac{9x^2}{2} + \overline{o}(x^2),\ x \to 0$$

As we proved before:

$$(\cos 2x)^{1/2} = (1 - 2x^2 + \overline{o}(x^2))^{1/2} = 1 - x^2 + \overline{o}(x^2), x \to 0$$

$$(\cos 3x)^{1/3} = 1 - \frac{3x^2}{2} + \overline{o}(x^2),\ x \to 0$$

Thus we have:

$$L = \lim_{x \to 0} \frac{1 - (1 - x^2/2 + \overline{o}(x^2))(1 - x^2 + \overline{o}(x^2))(1 - 3x^2/2 + \overline{o}(x^2))}{x^2}$$

After simplifications we have:

$$L = \lim_{x \to 0} \frac{3x^2 - 9x^4/4 + \overline{o}(x^2)}{x^2} = 3$$

Problem 495

Find the limit:

$$L = \lim_{x \to \pi/3} \frac{\sin(x - \pi/3)}{1 - 2\cos x}$$

Solution

First of all, we use the substitution $\theta = x - \pi/3 \implies \theta \to 0$ as $x \to \pi/3$:

$$L = \lim_{\theta \to 0} \frac{\sin\theta}{1 - 2\cos(\theta + \pi/3)}$$

Now use:

$$\cos(\theta + \pi/3) = \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$$

Thus:

$$L = \lim_{\theta \to 0} \frac{\sin\theta}{(1 - \cos\theta) + \sqrt{3}\sin\theta}$$

Now we use the fact that:

$$\sin\theta = \theta + \overline{o}(\theta), \theta \to 0$$

$$\cos\theta = 1 - \frac{\theta^2}{2} + \overline{o}(\theta^2)$$

Thus:

$$L = \lim_{\theta \to 0} \frac{\theta + \overline{o}(\theta)}{\theta^2/2 + \sqrt{3}\theta + \overline{o}(\theta^2) + \overline{o}(\theta)} = \lim_{\theta \to 0} \frac{\theta(1 + \overline{o}(1))}{\theta(\sqrt{3} + \overline{o}(1)) + \theta^2(1/2 + \overline{o}(1))}$$

After dividing both sides by θ we obtain:

$$L = \lim_{\theta \to 0} \frac{1 + \overline{o}(1)}{\sqrt{3} + \overline{o}(1) + \theta(1/2 + \overline{o}(1))} = \frac{1}{\sqrt{3}}$$

Problem 450

Find the limit:

$$L=\lim_{x\to 0}\frac{(1+\frac{x}{3})^{1/3}-(1+\frac{x}{4})^{1/4}}{1-(1-\frac{x}{2})^{1/2}}$$

Solution:

We use the following:

$$\left(1+\frac{x}{3}\right)^{1/3}=1+\frac{x}{9}+\overline{o}(x),\ x\to 0$$

$$\left(1+\frac{x}{4}\right)^{1/4}=1+\frac{x}{16}+\overline{o}(x),\ x\to 0$$

$$\left(1-\frac{x}{2}\right)^{1/2}=1-\frac{x}{4}+\overline{o}(x),\ x\to 0$$

Thus:

$$L=\lim_{x\to 0}\frac{x(7/144+\overline{o}(1))}{x(1/4+\overline{o}(1))}=\frac{7}{36}$$

Problem 459

Find the limit:

$$L=\lim_{x\to +\infty}x(\sqrt{x^2+2x}-2\sqrt{x^2+x}+x)$$

Solution:

Firstly, let us make some manipulations with the initial expression:

$$L=\lim_{x\to +\infty}x^2\left(\sqrt{1+\frac{2}{x}}-2\sqrt{1+\frac{1}{x}}+1\right)$$

First of all, it is essential to note that the first order of smallness would not suffice to solve this problem. Indeed, if we put $(1+2/x)^{1/2}=1+1/x+\overline{o}(1/x),\ 1/x\to 0$ and $(1+1/x)^{1/2}=1+1/2x+\overline{o}(1/x),\ 1/x\to 0$:

$$L=\lim_{x\to +\infty}x^2\overline{o}\left(\frac{1}{x}\right)=\lim_{x\to +\infty}x\overline{o}(1)$$

we cannot evaluate this limit for the obvious reasons. Thus we have to use the second order of smallness, i.e.:

$$(1+x)^{\alpha}=1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\overline{o}(x^2)$$

Thus:

$$\sqrt{1+\frac{2}{x}} = 1+\frac{1}{x}-\frac{1}{2x^2}+\overline{o}(x^{-2}),\ \frac{1}{x}\to 0$$

$$\sqrt{1+\frac{1}{x}} = 1+\frac{1}{2x}-\frac{1}{8x^2}+\overline{o}(x^{-2}),\ \frac{1}{x}\to 0$$

Thus:

$$L = \lim_{x\to+\infty} x^2\left(1+\frac{1}{x}-\frac{1}{2x^2}+\overline{o}(x^{-2})-2-\frac{1}{x}+\frac{1}{4x^2}+\overline{o}(x^{-2})+1\right)$$

After simplifications:

$$L = \lim_{x\to+\infty} x^2\left(-\frac{1}{4x^2}+\overline{o}\left(\frac{1}{x^2}\right)\right) = \lim_{x\to+\infty}\left(-\frac{1}{4}+\overline{o}(1)\right) = -\frac{1}{4}$$

Problem 453

Find the limit:

$$L = \lim_{x\to 0}\frac{(1+\alpha x)^{1/m}(1+\beta x)^{1/n}-1}{x}$$

Solution:

Firstly, we use:

$$(1+\alpha x)^{1/m} = 1+\frac{\alpha x}{m}+\overline{o}(x),\ x\to 0$$

$$(1+\beta x)^{1/n} = 1+\frac{\beta x}{n}+\overline{o}(x),\ x\to 0$$

Thus our limit can be written as follows:

$$L = \lim_{x\to 0}\frac{(1+\alpha x/m+\overline{o}(x))(1+\beta x/n+\overline{o}(x))-1}{x}$$

After simplifications we obtain:

$$L = \lim_{x\to 0}\frac{\beta x/n+\alpha x/m+\overline{o}(x)}{x} = \frac{\alpha}{m}+\frac{\beta}{n}$$