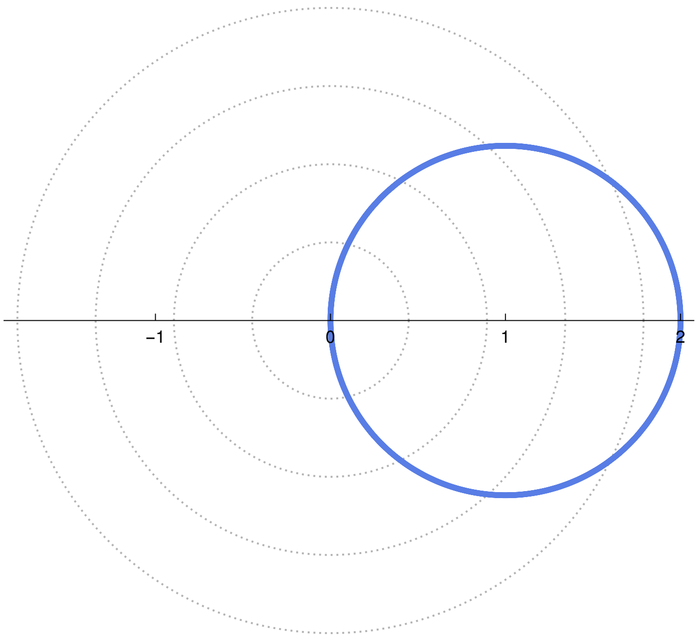
Problem 1.

Problem Statement. Find length of a curve defined in polar coordinates $\rho = 2\cos\theta$.

In[∘]:=
```
ρ[θ_] = 2 Cos[θ];
PolarPlot[ρ[θ], {θ, 0, 2 π}, PlotTheme → "Business"]
```

Out[∘]=



First derivative:

In[∘]:=
```
dρ[θ_] = Simplify[D[ρ[θ], θ]]
```

Out[∘]= $-2\,\mathrm{Sin}[\theta]$

Arc Length:

In[∘]:=
```
l[a_, b_] = Integrate[Sqrt[ρ[θ]^2 + dρ[θ]^2], {θ, a, b}]
```

Out[∘]= $-2\,a + 2\,b$

Full curve length:

In[∘]:=
```
L = l[0, 2 π]
```

Out[∘]= $4\,\pi$

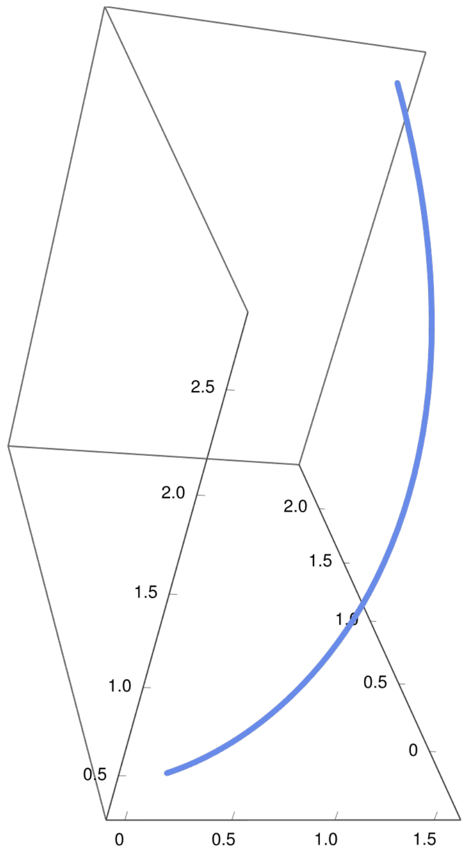
Problem 2.

Problem Statement. Find tangent Frenet’s basis equations (lines and planes) of curve

$r(t) = e^t \{\cos t, \sin t, 1\}$ at point $M(1, 0, 1)$

In[76]:=
```
Clear[r, t]
r[t_] = {Exp[t] Cos[t], Exp[t] Sin[t], Exp[t]};
ParametricPlot3D[r[t], {t, -1, 1}, PlotTheme → "Business"]
```

Out[78]=



First derivative:

In[79]:=
```
dr[t_] = Simplify[D[r[t], t]]
```

Out[79]= $\{e^t (\mathrm{Cos}[t] - \mathrm{Sin}[t]),\ e^t (\mathrm{Cos}[t] + \mathrm{Sin}[t]),\ e^t\}$

Normalized First Derivative:

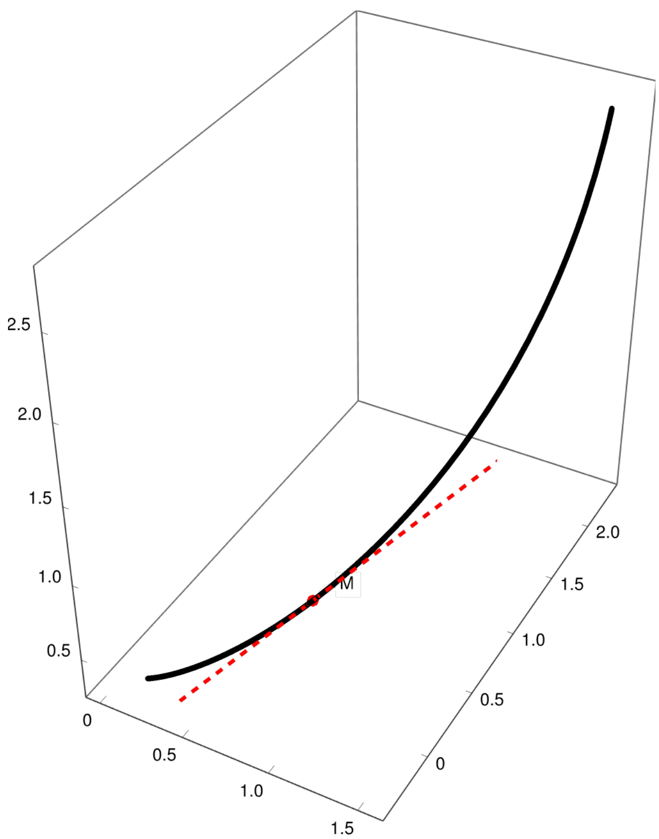
In[80]:=
```
Clear[τ]
τ[t_] = Assuming[t ∈ Reals, Simplify[dr[t] / Sqrt[dr[t].dr[t]]]]
```

Out[81]= $\left\{\frac{\mathrm{Cos}[t] - \mathrm{Sin}[t]}{\sqrt{3}},\ \frac{\mathrm{Cos}[t] + \mathrm{Sin}[t]}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right\}$

Tangent Line Equation:

In[82]:=
```
lτ[t_, u_] = r[t] + τ[t] u;
Show[
 ParametricPlot3D[r[t], {t, -1, 1}, PlotTheme → "Business", PlotStyle → Black],
 ListPointPlot3D[{r[0]} → {"M"}, PlotStyle → Red],
 ParametricPlot3D[lτ[0, u], {u, -5, 5}, PlotStyle → {Red, Dashed}]
]
```

Out[83]=



Second Derivative

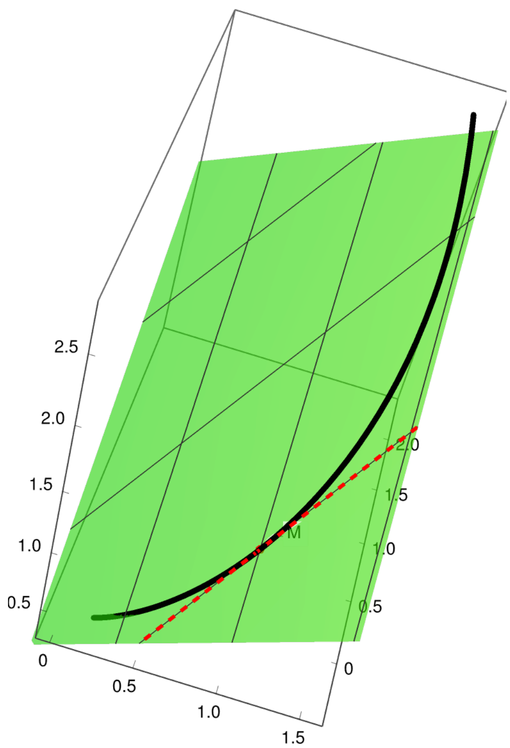
In[84]:=
```
d2r[t_] = Simplify[D[dr[t], t]]
```

Out[84]= $\{-2\, e^t \,\text{Sin}[t],\ 2\, e^t \,\text{Cos}[t],\ e^t\}$

Plane Equation

In[85]:=
```
πτ[t_, u_, v_] = r[t] + dr[t] u + d2r[t] v;
Print["Parametric plane equation is ", πτ[0, u, v]]
Show[
 ParametricPlot3D[r[t], {t, -1, 1}, PlotTheme → "Business", PlotStyle → Black],
 ListPointPlot3D[{r[0]} → {"M"}, PlotStyle → Red],
 ParametricPlot3D[lτ[0, u], {u, -5, 5}, PlotStyle → {Red, Dashed}],
 ParametricPlot3D[πτ[0, u, v], {u, -5, 5}, {v, -5, 5},
  PlotStyle → Directive[Green, Opacity[0.6], Specularity[White, 20]]]
]
```

Parametric plane equation is {1 + u, u + 2 v, 1 + u + v}

Out[87]=


Binormal

In[88]:=
```
Clear[B, t]
B[t_] = Simplify[Cross[dr[t], d2r[t]]]
```

Out[89]= $\left\{e^{2t}(-\mathrm{Cos}[t] + \mathrm{Sin}[t]),\ -e^{2t}(\mathrm{Cos}[t] + \mathrm{Sin}[t]),\ 2 e^{2t}\right\}$

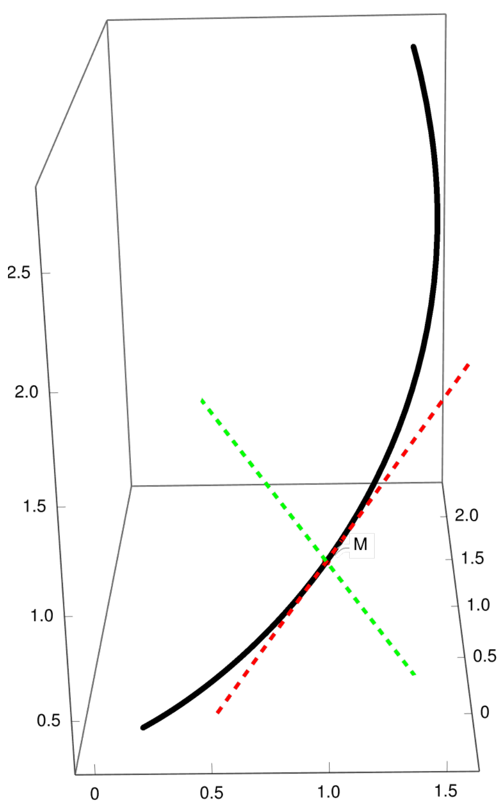
In[90]:= $\beta[t\_] = \mathrm{Assuming}\left[t \in \mathrm{Reals},\ \mathrm{Simplify}\left[\frac{B[t]}{\mathrm{Sqrt}[B[t].B[t]]}\right]\right]$

Out[90]= $\left\{\frac{-\mathrm{Cos}[t] + \mathrm{Sin}[t]}{\sqrt{6}},\ -\frac{\mathrm{Cos}[t] + \mathrm{Sin}[t]}{\sqrt{6}},\ \sqrt{\frac{2}{3}}\right\}$

In[91]:=
```
B[0]
```
Out[91]=
```
{-1, -1, 2}
```

In[92]:=
```
lβ[t_, u_] = r[0] + B[0] u;
Print["Binormal line is ", lβ[0, u]]
Show[
 ParametricPlot3D[r[t], {t, -1, 1}, PlotTheme → "Business", PlotStyle → Black],
 ListPointPlot3D[{r[0]} → {"M"}, PlotStyle → Red],
 ParametricPlot3D[lτ[0, u], {u, -5, 5}, PlotStyle → {Red, Dashed}],
 ParametricPlot3D[lβ[0, u], {u, -5, 5}, PlotStyle → {Green, Dashed}]
]
```
Binormal line is $\{1-u, -u, 1+2u\}$

Out[94]=



Main normal

In[95]:=
```
V[t_] = Cross[dr[t], B[t]]
```
Out[95]= $\{3e^{3t}\text{Cos}[t] + 3e^{3t}\text{Sin}[t],\ -3e^{3t}\text{Cos}[t] + 3e^{3t}\text{Sin}[t],\ 0\}$

In[96]:=
```
v[t_] = Assuming[t ∈ Reals, Simplify[V[t]/Sqrt[V[t].V[t]]]]
```

Out[96]= $\left\{\frac{\text{Cos}[t]+\text{Sin}[t]}{\sqrt{2}}, \frac{-\text{Cos}[t]+\text{Sin}[t]}{\sqrt{2}}, 0\right\}$

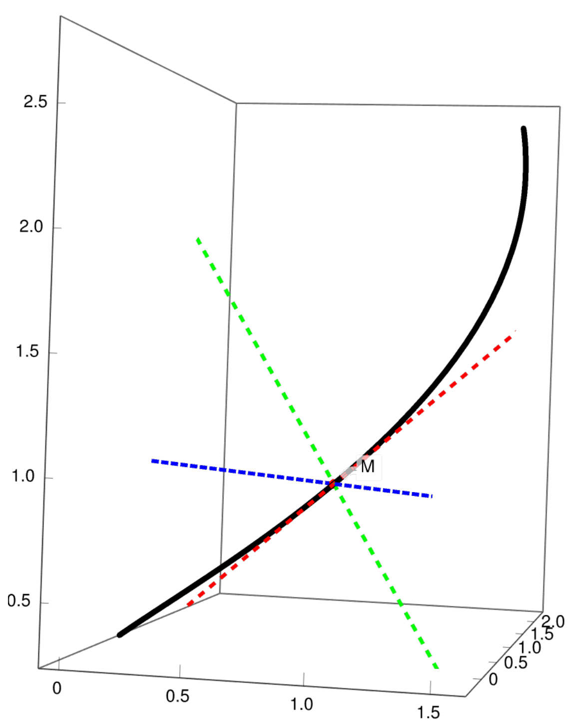
In[97]:=
```
V[0]
```

Out[97]= {3, -3, 0}

In[98]:=
```
lv[t_, u_] = r[0] + V[0] u;
Print["Binormal line is ", lv[0, u]]
Show[
 ParametricPlot3D[r[t], {t, -1, 1}, PlotTheme → "Business", PlotStyle → Black],
 ListPointPlot3D[{r[0]} → {"M"}, PlotStyle → Red],
 ParametricPlot3D[lτ[0, u], {u, -5, 5}, PlotStyle → {Red, Dashed}],
 ParametricPlot3D[lβ[0, u], {u, -5, 5}, PlotStyle → {Green, Dashed}],
 ParametricPlot3D[lv[0, u], {u, -5, 5}, PlotStyle → {Blue, Dashed}]
]
```

Binormal line is $\{1+3u, -3u, 1\}$

Out[100]=



Planes

In[101]:=
```
πv[x_, y_, z_] = Simplify[Dot[V[0], {x, y, z} - r[0]]]
πτ[x_, y_, z_] = Simplify[Dot[dr[0], {x, y, z} - r[0]]]
```

Out[101]=

$3(-1 + x - y)$

Out[102]=

$-2 + x + y + z$

Problem 3.

Problem Statement. Find curvature and torsion of curve with radius-vector $f(t) = \{a\cos^2 t, a\cos t\sin t, b\,t\}$.

In[∘]:=
```
u[t_] = {a Cos[t]^2, a Cos[t] Sin[t], b t};
du[t_] = Simplify[D[u[t], t]]
```

Out[∘]= {-2 a Cos[t] Sin[t], a Cos[2 t], b}

In[∘]:=
```
d2u[t_] = Simplify[D[du[t], t]]
```

Out[∘]= {-2 a Cos[2 t], -2 a Sin[2 t], 0}

In[∘]:=
```
dud2u[t_] = Simplify[Cross[du[t], d2u[t]]]
```

Out[∘]= $\{2\,a\,b\,\mathrm{Sin}[2\,t], -2\,a\,b\,\mathrm{Cos}[2\,t], 2\,a^2\}$

In[∘]:=
```
dud2uLen[t_] = Simplify[Sqrt[dud2u[t].dud2u[t]]]
```

Out[∘]= $2\sqrt{a^2(a^2 + b^2)}$

In[∘]:=
```
d3u[t_] = Simplify[D[d2u[t], t]]
```

Out[∘]= {4 a Sin[2 t], -4 a Cos[2 t], 0}

In[∘]:=
```
d1ud2ud3u[t_] = Simplify[Det[{du[t], d2u[t], d3u[t]}]]
```

Out[∘]= $8\,a^2\,b$

In[119]:=

```
x[t_] = a Cos[t]^2; y[t_] = a Cos[t] Sin[t]; z[t_] =  b t;
torsion = Simplify[(D[x[t], {t, 3}] (D[y[t], t] × D[z[t], {t, 2}] - D[y[t], {t, 2}] × D[z[t], t]) +
      D[y[t], {t, 3}] (D[x[t], {t, 2}] × D[z[t], t] - D[x[t], t] × D[z[t], {t, 2}]) +
      D[z[t], {t, 3}] (D[x[t], t] × D[y[t], {t, 2}] - D[x[t], {t, 2}] × D[y[t], t])) /
    ((D[y[t], t] × D[z[t], {t, 2}] - D[y[t], {t, 2}] × D[z[t], t])^2 +
      (D[x[t], {t, 2}] × D[z[t], t] - D[x[t], t] × D[z[t], {t, 2}])^2 +
      (D[x[t], t] × D[y[t], {t, 2}] - D[x[t], {t, 2}] × D[y[t], t])^2)]
```

Out[120]=

$$\frac{2\ b}{a^2 + b^2}$$