Computer Mathematics Course Work

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Variant **13**

1. Introduction to Wolfram Mathematica

1.1. Arithmetical Operations

Task: Find value of expression $\frac{0.134+0.05}{\left(18+\frac{1}{6}\right)-\left(1+\frac{11}{14}\right)-\frac{2}{15}\left(2+\frac{6}{7}\right)}$

In[∘]:=
```
Clear[Answer]
Answer = (0.134 + 0.05)/((18 + 1/6) - (1 + 11/14) - 2/15 (2 + 6/7));
Print["Answer is ", Answer]
```

```
Answer is 0.0115
```

1.2. Evaluating expression containing solely elementary functions

Task: Find the value of expression $y = \frac{1}{3}\cos\left(\frac{1}{\tan 3}\right) + \frac{1}{5}\frac{\sin^2\left(5\sqrt{2}\right)}{\cos 10 e^3}$

In[∘]:=
```
Clear[y, Answer]
y = 1/3 Cos[1/Tan[3]] + 1/5 (Sin[5 Sqrt[2]])^2/Cos[10 ⅇ^3];
Answer = N[y];
Print["Answer is ", Answer]
```

```
Answer is 0.350611
```

2. Algebraic Evaluation

2.1. Algebraic Manipulations

Task: Simplify expression $\dfrac{2\,a^2\,(b+c)^{2\,n} - \frac{1}{2}}{a\,n^2 - a^3 - 2\,a^2 - a} : \dfrac{2\,a\,(b+c)^n - 1}{a^2\,c - a\left(n\,c - c\right)}$

In[∘]:= Clear[a, b, c, n, Result]

$$\texttt{Result = Simplify}\left[\left(\frac{2\ a^{ɣ}\ (b+c)^{ɣ\ p} - \frac{G}{ɣ}}{a\ n^{ɣ} - a^{ɤ} - 2\ a^{ɣ} - a}\right)\Big/\left(\frac{2\ a\ (b+c)^{p} - 1}{a^{ɣ}\ c - a\left(n\ c - c\right)}\right)\right];$$

Print["Result is ", Result]

$$\texttt{Result is } -\frac{c\left(1 + 2\ a\ (b+c)^n\right)}{2\ (1 + a + n)}$$

2.2. Finding values of algebraic expressions

Task: Calculate $\left(\sqrt{a\,b} - \dfrac{a\,b}{a + \sqrt{a\,b}}\right) : \dfrac{\sqrt[4]{a\,b} - \sqrt{b}}{a - b}$ for $a = 8$, $b = 2$

In[∘]:= Remove[a, b, Answer]

$$\texttt{Expr = }\left(\texttt{Sqrt}[a\ b] - \frac{a\ b}{a + \texttt{Sqrt}[a\ b]}\right)\Big/\left(\frac{\texttt{Surd}[a\ b,\ 4] - \texttt{Sqrt}[b]}{a - b}\right);$$

Answer = Simplify[Expr /. {a → 8, b → 2}];

Print["The answer is ", Answer]

The answer is $8\left(2 + \sqrt{2}\right)$

2.3. Solving equations

Task: Solve equation $\dfrac{1}{a} + \dfrac{1}{a + x} + \dfrac{1}{a + 2\,x} = 0$ for x

In[∘]:=
```
Clear[a, x]
Solve[1/a + 1/(a + x) + 1/(a + 2 x) == 0, x]
```

Out[∘]= $\left\{\left\{x \to \frac{1}{2}\left(-3\,a - \sqrt{3}\,a\right)\right\}, \left\{x \to \frac{1}{2}\left(-3\,a + \sqrt{3}\,a\right)\right\}\right\}$

3. Elements of Analytical Geometry

3.1. Operating with vectors

Task: For vectors $a = [-2, 7, -1]^T$, $b = [-3, 5, 2]^T$, $c_1 = 2\,a + 3\,b$, $c_2 = 3\,a + 2\,b$ determine whether c_1 and c_2 are collinear. Find lengths of vectors a and b and angle between them. Draw vectors a and b.

In[∘]:=
```
Clear[a, b, c1, c2]
a = {-2, 7, -1};
b = {-3, 5, 2};
c1 = 2 a + 3 b;
c2 = 3 a + 2 b;
ratio = c1/c2;
Print["Ratio is ", ratio]
```

Ratio is $\left\{\frac{13}{12}, \frac{29}{31}, 4\right\}$

We thus conclude that c_1 and c_2 are not collinear.

In[1]:=
```
LengthOfA = Norm[a];
LengthOfB = Norm[b];
AngleAB = N[VectorAngle[a, b]] 180/π;
Print["Length of a is ", LengthOfA, " and length of b is ", LengthOfB]
Print["Angle between a and b (in degrees) is ", AngleAB]
```

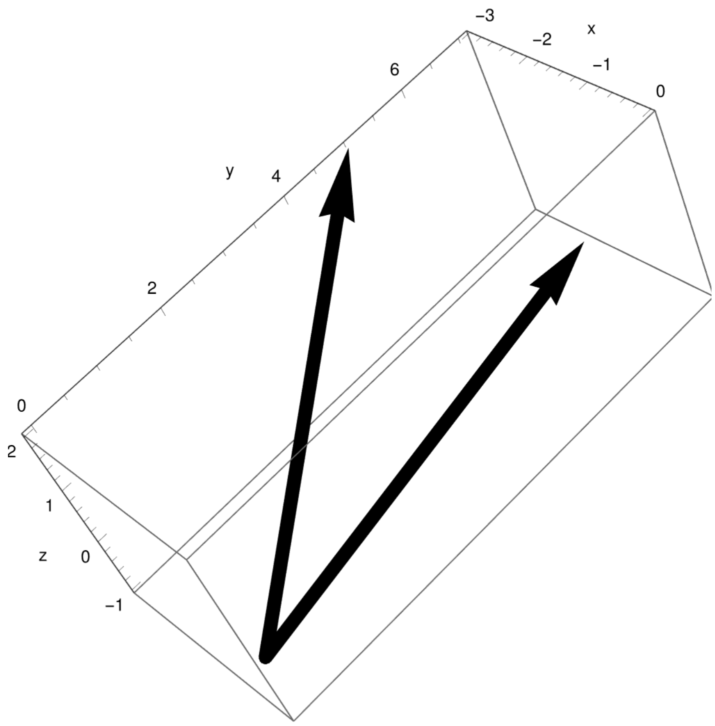
Length of a is Norm[a] and length of b is Norm[b]

Angle between a and b (in degrees) is $\frac{180\,\text{VectorAngle}[a, b]}{\pi}$

In[∘]:=
```
z = {0, 0, 0}
Graphics3D[{Thickness[0.02], Arrowheads[0.1], Arrow[{z, a}], Arrow[{z, b}]},
 Axes → True, AxesLabel → {"x", "y", "z"}]
```

Out[∘]= {0, 0, 0}

Out[∘]=

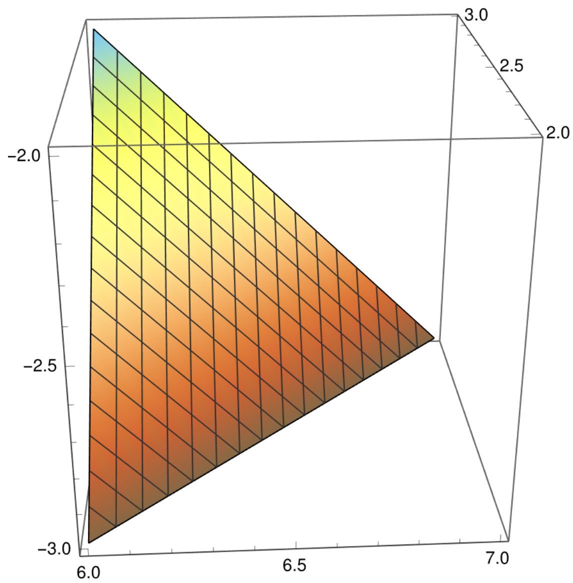


3.2. Geometric calculations in Triangle

Task: Given triangle with vertices A_1 (6, 2, −3), A_2 (6, 3, −2), A_3 (7, 3, −3), find its sides, angles and area.

In[∘]:=
```
A1 = {6, 2, -3}; A2 = {6, 3, -2}; A3 = {7, 3, -3};
ListPlot3D[{A1, A2, A3}, BoxRatios → Automatic, ColorFunction → "SouthwestColors"]
```

Out[∘]=



In[∘]:=
```
A1A2 = A2 - A1; A1A3 = A3 - A1; A2A3 = A3 - A2;
LengthA1A2 = Norm[A1A2]; LengthA1A3 = Norm[A1A3]; LengthA2A3 = Norm[A2A3];
angleA1 = VectorAngle[A1A2, A1A3];
angleA2 = VectorAngle[-A1A2, A2A3]; angleA3 = VectorAngle[A1A3, A2A3];
Print["Lengths of sides AGAγ, AGAɤ, and AγAɤ are ",
 LengthA1A2, ", ", LengthA1A3, ", ", LengthA2A3]
Print["Angles AG, Aγ, and Aɤ are ", angleA1, ", ", angleA2, ", ", angleA3]
```

Lengths of sides A_1A_2, A_1A_3, and A_2A_3 are $\sqrt{2}$, $\sqrt{2}$, $\sqrt{2}$

Angles A_1, A_2, and A_3 are $\frac{\pi}{3}$, $\frac{\pi}{3}$, $\frac{\pi}{3}$

In[∘]:=
```
S = 1/2 Norm[Cross[A1A2, A1A3]];
Print["Area of triangle is ", S]
```

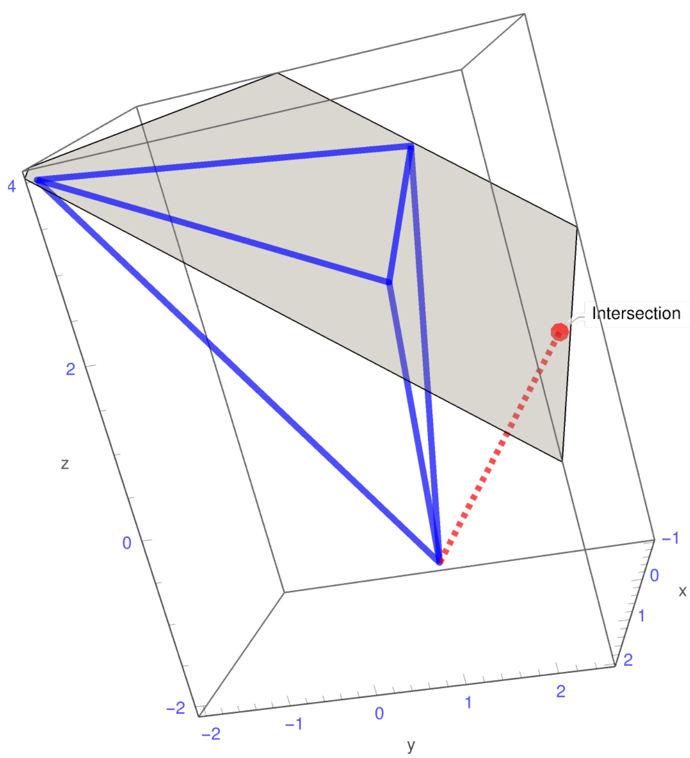
Area of triangle is $\frac{\sqrt{3}}{2}$

3.3. Calculations in Tetrahedron

Task. Given Tetrahedron with vertices $A_1\,(1, 1, 2)$, $A_2\,(-1, 1, 3)$, $A_3\,(2, -2, 4)$, $A_4\,(-1, 0, -2)$, find its volume, height h from vertex A_4 on side $A_1 A_2 A_3$. Display this tetrahedron, height h and point $B = h \cap \left(A_1 A_2 A_3\right)$.

In[∘]:=
```
A1 = {1, 1, 2}; A2 = {-1, 1, 3}; A3 = {2, -2, 4}; A4 = {-1, 0, -2};
A1A2 = A2 - A1; A1A3 = A3 - A1; A1A4 = A4 - A1;
V = 1/6 Abs[Det[{A1A2, A1A3, A1A4}]];
n = -Cross[A1A2, A1A3];
S = 1/2 Norm[n];
h = (3 V)/S;
B = A4 - n/(2 S) h;
graph1 = Graphics3D[{Thickness[0.01], Line[{A1, A2, A3, A1, A4, A2, A3, A4}]},
    AxesLabel → {"x", "y", "z"}, Axes → True, BaseStyle → {Blue, Opacity[.7]}];
graph2 =
   ListPointPlot3D[{B} → {"Intersection"}, PlotStyle → {Directive[PointSize[0.03], Red]}];
graph3 = Graphics3D[{Thickness[0.01], {Dashed, Red, Line[{A4, B}]}}];
graph4 = Graphics3D[{Opacity[0.3], Gray, InfinitePlane[{A1, A2, A3}]}];
Show[graph1, graph2, graph3, graph4]
```

Out[∘]=



4. Matrix Calculations

4.1. Working with matrices

Task: Given matrices $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, find $M = 2A^2 + 3A + E$ where E is an identity matrix. Find inverse matrix A^{-1}. Find $\det A$. Find sum and difference of A, B. Find $A \cdot B$ and $A \otimes B$, find difference between these values. Solve matrix equation $A X = B$ with respect to $X \in \mathrm{R}^{3\times 3}$.

In[∘]:=
```
A = {{1, 2, 1}, {0, 2, 0}, {-1, 1, 1}};
B = {{1, 1, 2}, {0, 2, 1}, {1, -1, 0}};
M = 2 * A.A + 3 A + 5 IdentityMatrix[3];
Print["Matrix polynomial equals ", M // MatrixForm]
```

Matrix polynomial equals $\begin{pmatrix} 8 & 20 & 7 \\ 0 & 19 & 0 \\ -7 & 5 & 8 \end{pmatrix}$

In[∘]:=
```
InverseA = Inverse[A];
Print["Inverse of matrix A is ", InverseA // MatrixForm]
```

Inverse of matrix A is $\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{3}{4} & \frac{1}{2} \end{pmatrix}$

In[∘]:=
```
DetA = Det[A];
Print["det(A) = ", DetA]
```

det(A) = 4

In[∘]:=
```
Print["Sum of matrices A and B is ", (A + B) // MatrixForm]
Print["Difference between matrices A and B is ", (A - B) // MatrixForm]
```

Sum of matrices A and B is $\begin{pmatrix} 2 & 3 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Difference between matrices A and B is $\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & -1 \\ -2 & 2 & 1 \end{pmatrix}$

In[∘]:=
```
MatProduct = A.B;
ElemProduct = A * B;
Print["Matrix product of A and B is ", MatProduct // MatrixForm]
Print["Elementwise product of A and B is ", ElemProduct // MatrixForm]
Print["Difference between corresponding products is",
 (MatProduct - ElemProduct) // MatrixForm]
```

Matrix product of A and B is $\begin{pmatrix} 2 & 4 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

Elementwise product of A and B is $\begin{pmatrix} 1 & 2 & 2 \\ 0 & 4 & 0 \\ -1 & -1 & 0 \end{pmatrix}$

Difference between corresponding products is $\begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix}$

In[∘]:=
```
X = InverseA.B;
Print["Solution to matrix equation AX=B w/ respect to X is ", X // MatrixForm]
```

Solution to matrix equation AX=B w/ respect to X is $\begin{pmatrix} 0 & \frac{1}{2} & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} \\ 1 & -\frac{3}{2} & \frac{1}{4} \end{pmatrix}$

4.2. Working with both matrices and vectors

Task: Given matrices $A = \begin{pmatrix} 0 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix}$, $x = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ calculate $y = (A - B)^2 x$ and $z = (A B - B A) x$. Solve equation $A \cdot u = x$ with respect to u using inverse matrix and Cramer's rule. For each row and column of A find sum of elements. For matrix B find max and min elements.

In[∘]:=
```
A = {{0, 2, 3}, {4, 1, 0}, {2, -1, -2}};
B = {{1, 2, 0}, {3, 0, -1}, {2, 1, -1}};
x = {2, 4, -1};
y = (A - B).(A - B).x;
Print["Vector y is ", MatrixForm[y]]
z = (A.B - B.A).x;
Print["Vector z is ", MatrixForm[z]]
```

Vector y is $\begin{pmatrix} -16 \\ -7 \\ -3 \end{pmatrix}$

Vector z is $\begin{pmatrix} 12 \\ 34 \\ -25 \end{pmatrix}$

To solve equation $A \cdot u = x$ we firstly use inverse matrix, that is, $u = A^{-1} x$:

In[∘]:=
```
u = Inverse[A].x;
Print["Solution to Au=x w/ respect to u is ", MatrixForm[u]]
```

Solution to Au=x w/ respect to u is $\begin{pmatrix} -\frac{ɤ}{ɣ} \\ 10 \\ -6 \end{pmatrix}$

Now, using Cramer’s rule:

In[∘]:=
```
A1 = Join[Transpose[{x}], A[[All, 2 ;; 3]], 2];
Print["After putting vector x to the first column we get ", MatrixForm[A1]]
```

After putting vector x to the first column we get $\begin{pmatrix} 2 & 2 & 3 \\ 4 & 1 & 0 \\ -1 & -1 & -2 \end{pmatrix}$

In[∘]:=
```
A2 = Join[A[[All, 1 ;; 1]], Transpose[{x}], A[[All, 3 ;; 3]], 2];
Print["After putting vector x to the second column we get ", MatrixForm[A2]]
```

After putting vector x to the second column we get $\begin{pmatrix} 0 & 2 & 3 \\ 4 & 4 & 0 \\ 2 & -1 & -2 \end{pmatrix}$

In[∘]:=
```
A3 = Join[A[[All, 1 ;; 2]], Transpose[{x}], 2];
Print["After putting vector x to the third column we get ", MatrixForm[A3]]
```

After putting vector x to the third column we get $\begin{pmatrix} 0 & 2 & 2 \\ 4 & 1 & 4 \\ 2 & -1 & -1 \end{pmatrix}$

In[∘]:=
```
cramerU = {Det[A1], Det[A2], Det[A3]} / Det[A];
Print["Vector u calculated using Cramer's rule is ", cramerU // MatrixForm]
```

Vector u calculated using Cramer's rule is $\begin{pmatrix} -\frac{3}{2} \\ 10 \\ -6 \end{pmatrix}$

In[∘]:=
```
Print["Sum of columns of matrix A is ", Total[A]]
Print["Sum of rows of matrix A is ", Total[A, {2}]]
Print["Max element of B is ", Max[B]]
Print["Min element of B is ", Min[B]]
```

```
Sum of columns of matrix A is {6, 2, 1}

Sum of rows of matrix A is {5, 5, -1}

Max element of B is 3

Min element of B is -1
```

4.3. Creating matrices

Task 1: Generate a one-dimensional array a of size 16 containing random real numbers. Sort it in descending order and convert to a matrix A. Create matrix B of size 4×4 containing 5's. Generate matrix F of size 4×4, elements $F_{i,j}$ of which are calculated according to formula $F_{i,j} = i^2 - j^2$. Generate diagonal array G with elements of diagonal [1, −1, 2, −2]. Calculate $M = A + B - C - G + E - 3$ where E is identity matrix of size 4.

In[∘]:=
```
a = RandomReal[1, 16];
Print["Random list a is ", a]
aSorted = Sort[a, Greater];
Print["Sorted list a is ", aSorted]
A = ArrayReshape[aSorted, {4, 4}];
Print["Reshaped list a as a matrix A is ", A // MatrixForm]
B = ConstantArray[5, {4, 4}];
Print["Matrix B is ", B // MatrixForm]
F = Table[i^2 - j^2, {i, 1, 4}, {j, 1, 4}];
Print["Matrix F is ", F // MatrixForm]
G = DiagonalMatrix[{1, -1, 2, -2}];
Print["Matrix G is ", G // MatrixForm];
M = A + B - F - G + IdentityMatrix[4] - 3;
Print["Expression A+B-F-G+E-3 equals ", M // MatrixForm]
```

```
Random list a is {0.35427, 0.699043, 0.911459, 0.477622, 0.589589, 0.200876, 0.154128,
  0.557092, 0.580649, 0.758855, 0.807006, 0.961893, 0.419614, 0.473136, 0.863843, 0.452541}

Sorted list a is {0.961893, 0.911459, 0.863843, 0.807006, 0.758855, 0.699043, 0.589589,
  0.580649, 0.557092, 0.477622, 0.473136, 0.452541, 0.419614, 0.35427, 0.200876, 0.154128}
```

Reshaped list a as a matrix A is $\begin{pmatrix} 0.961893 & 0.911459 & 0.863843 & 0.807006 \\ 0.758855 & 0.699043 & 0.589589 & 0.580649 \\ 0.557092 & 0.477622 & 0.473136 & 0.452541 \\ 0.419614 & 0.35427 & 0.200876 & 0.154128 \end{pmatrix}$

Matrix B is $\begin{pmatrix} 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \\ 5 & 5 & 5 & 5 \end{pmatrix}$

Matrix F is $\begin{pmatrix} 0 & -3 & -8 & -15 \\ 3 & 0 & -5 & -12 \\ 8 & 5 & 0 & -7 \\ 15 & 12 & 7 & 0 \end{pmatrix}$

Matrix G is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

Expression A+B−F−G+E−3 equals $\begin{pmatrix} 2.96189 & 5.91146 & 10.8638 & 17.807 \\ -0.241145 & 4.69904 & 7.58959 & 14.5806 \\ -5.44291 & -2.52238 & 1.47314 & 9.45254 \\ -12.5804 & -9.64573 & -4.79912 & 5.15413 \end{pmatrix}$

Task 2. Create matrix Z elements of which are calculated according to relation $Z_{i,j} = x_i^2 - y_j^2$ where $x = \{-3, 0, 1, 2, 5\}$, $y = \{-5, 0, -4, 1, 3\}$. Add to Z a column containing {1, 2, 3, 4, 5}. Then, between columns 1 and 2 insert another column containing 1's.

In[ ]:=
```
x = {-3, 0, 1, 2, 5}; y = {-5, 0, -4, 1, 3};
Z = Table[(x[[i]])^2 - (y[[j]])^2, {i, 1, 5}, {j, 1, 5}];
Print["Initial matrix Z is ", Z // MatrixForm]
```

Initial matrix Z is $\begin{pmatrix} -16 & 9 & -7 & 8 & 0 \\ -25 & 0 & -16 & -1 & -9 \\ -24 & 1 & -15 & 0 & -8 \\ -21 & 4 & -12 & 3 & -5 \\ 0 & 25 & 9 & 24 & 16 \end{pmatrix}$

In[ ]:=
```
joinedZ = Join[Z, Transpose[{{1, 2, 3, 4, 5}}], 2];
Print["After adding columns {1,2,3,4,5} to Z we get ", joinedZ // MatrixForm]
```

After adding columns {1,2,3,4,5} to Z we get $\begin{pmatrix} -16 & 9 & -7 & 8 & 0 & 1 \\ -25 & 0 & -16 & -1 & -9 & 2 \\ -24 & 1 & -15 & 0 & -8 & 3 \\ -21 & 4 & -12 & 3 & -5 & 4 \\ 0 & 25 & 9 & 24 & 16 & 5 \end{pmatrix}$

In[∘]:=
```
joinedWithOnesZ =
  Join[joinedZ[[All, 1 ;; 1]], Transpose[{ConstantArray[1, 5]}], joinedZ[[All, 3 ;; 6]], 2];
Print["After adding column {1,1,1,1,1} between first and second solumns we get ",
 joinedWithOnesZ // MatrixForm]
```

```
After adding column {1,1,1,1,1} between first and second solumns we get
```

$$\begin{pmatrix} -16 & 1 & -7 & 8 & 0 & 1 \\ -25 & 1 & -16 & -1 & -9 & 2 \\ -24 & 1 & -15 & 0 & -8 & 3 \\ -21 & 1 & -12 & 3 & -5 & 4 \\ 0 & 1 & 9 & 24 & 16 & 5 \end{pmatrix}$$

5. Solving equations and systems of equations

5.1. Solving equations

Task. Solve equation $\dfrac{\sqrt{1+x\ /a\ } - x/a}{\sqrt{1+x\ /a\ } + x/a} = \dfrac{1}{4}$ w/ respect to x.

In[∘]:=
```
Clear[x, a]
Solve[(Sqrt[1 + x^2/a^2] - x/a)/(Sqrt[1 + x^2/a^2] + x/a) == 1/4, x]
```

$$\text{Solve}\left[\frac{\text{Sqrt}\left[1+\frac{x^2}{a^2}\right]-\frac{x}{a}}{\text{Sqrt}\left[1+\frac{x^2}{a^2}\right]+\frac{x}{a}} == \frac{1}{4},\ x\right]$$

Out[∘]= $\left\{\left\{x \to \frac{3\,a}{4}\right\}\right\}$

5.2. Solving system of linear equations

Task. Solve system of linear equations $A\,x = b$ given my matrix $A = \begin{pmatrix} 1 & 2 & -3 & 4 & -1 \\ 2 & -1 & 3 & -4 & 2 \\ 3 & 1 & -1 & 2 & -1 \\ 4 & 3 & 4 & 2 & 2 \\ 1 & -1 & -1 & 2 & -3 \end{pmatrix}$ and vector $b = \begin{pmatrix} -1 \\ 8 \\ 3 \\ -2 \\ -3 \end{pmatrix}$.

5.2.1. Inverse matrix method

In[∘]:=
```
Clear[A, b, x];
A = {{1, 2, -3, 4, -1}, {2, -1, 3, -4, 2},
    {3, 1, -1, 2, -1}, {4, 3, 4, 2, 2}, {1, -1, -1, 2, -3}};
b = {-1, 8, 3, -2, -3};
x = Inverse[A].b;
Print["x = ", x]
```

```
x = {2, 0, -2, -2, 1}
```

5.2.2. Using Solve function

In[∘]:=
```
Solve[x1 + 2 x2 - 3 x3 + 4 x4 - x5 == -1 && 2 x1 - x2 + 3 x3 - 4 x4 + 2 x5 == 8 &&
  3 x1 + x2 - x3 + 2 x4 - x5 == 3 && 4 x1 + 3 x2 + 4 x3 + 2 x4 + 2 x5 == -2 &&
  x1 - x2 - x3 + 2 x4 - 3 x5 == -3, {x1, x2, x3, x4, x5}]
```

Out[∘]=
```
{{x1 → 2, x2 → 0, x3 → -2, x4 → -2, x5 → 1}}
```

5.2.3. Using LinearSolve function

In[∘]:=
```
Clear[A, b, x];
A = {{1, 2, -3, 4, -1}, {2, -1, 3, -4, 2},
    {3, 1, -1, 2, -1}, {4, 3, 4, 2, 2}, {1, -1, -1, 2, -3}};
b = {-1, 8, 3, -2, -3};
x = LinearSolve[A, b];
Print["x = ", x]
```

```
x = {2, 0, -2, -2, 1}
```

5.2.4. Using Cramer's method

In[∘]:=
```
Clear[A, b, x];
A = {{1, 2, -3, 4, -1}, {2, -1, 3, -4, 2},
   {3, 1, -1, 2, -1}, {4, 3, 4, 2, 2}, {1, -1, -1, 2, -3}};
b = {-1, 8, 3, -2, -3};
A1 = Join[Transpose[{b}], A[[All, 2 ;; 5]], 2];
A2 = Join[A[[All, 1 ;; 1]], Transpose[{b}], A[[All, 3 ;; 5]], 2];
A3 = Join[A[[All, 1 ;; 2]], Transpose[{b}], A[[All, 4 ;; 5]], 2];
A4 = Join[A[[All, 1 ;; 3]], Transpose[{b}], A[[All, 5 ;; 5]], 2];
A5 = Join[A[[All, 1 ;; 4]], Transpose[{b}], 2];
x = {Det[A1], Det[A2], Det[A3], Det[A4], Det[A5]} / Det[A];
Print["x = ", x]
```

```
x = {2, 0, -2, -2, 1}
```

5.2.5. Using RowReduce method

In[∘]:=
```
Clear[A, b, x];
A = {{1, 2, -3, 4, -1}, {2, -1, 3, -4, 2},
   {3, 1, -1, 2, -1}, {4, 3, 4, 2, 2}, {1, -1, -1, 2, -3}};
b = {-1, 8, 3, -2, -3};
RowReduce[Join[A, Transpose[{b}], 2]] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

5.3. Solving non-linear equations

Task. Solve system of non-linear equations $x^3 + y^3 = 35$ and $x\,y\,(x + y) = 30$.

In[∘]:=
```
Clear[x, y]
Solve[x^3 + y^3 == 35 && x y (x + y) == 30, {x, y}, Reals]
```

Out[∘]=
```
{{x → 2, y → 3}, {x → 3, y → 2}}
```

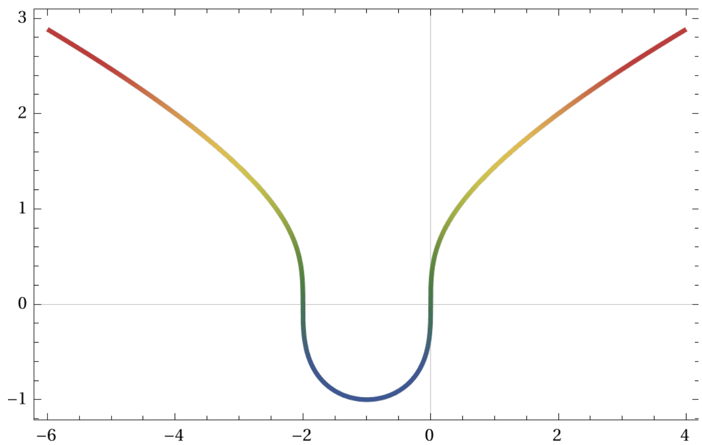
6. Functions and their plots

6.1. Plotting function by a points list

Task. Plot function $y = \sqrt[3]{x(x+2)}$ and its table of values. Plot ListPlot based on this table.

In[∘]:=
```
Remove[x, y];
y[x_] = CubeRoot[x (x + 2)];
p1 = Plot[y[x], {x, -6, 4}, PlotTheme → "Scientific",
  PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"]
```

Out[∘]=



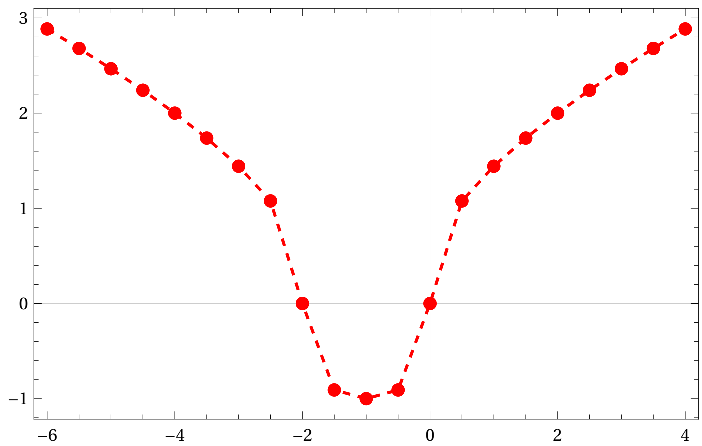
In[∘]:=
```
points = Table[{x, N[y[x]]}, {x, -6, 4, 0.5}];
points // TableForm
```

Out[∘]//TableForm=

-6.	2.8845
-5.5	2.68005
-5.	2.46621
-4.5	2.2407
-4.	2.
-3.5	1.73801
-3.	1.44225
-2.5	1.07722
-2.	0.
-1.5	-0.90856
-1.	-1.
-0.5	-0.90856
0.	0.
0.5	1.07722
1.	1.44225
1.5	1.73801
2.	2.
2.5	2.2407
3.	2.46621
3.5	2.68005
4.	2.8845

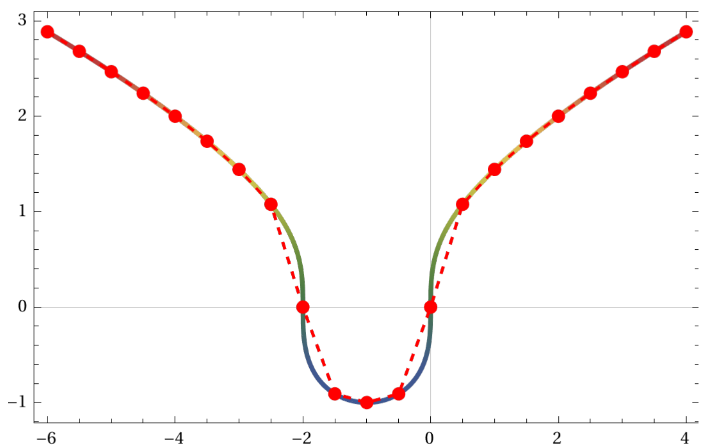
In[∘]:=
```
p2 = ListPlot[points, Joined → True, Mesh → Full,
  PlotStyle → {PointSize[0.02], Red, Dashed}, PlotTheme → "Scientific"]
```

Out[∘]=



In[∘]:=
```
Show[p1, p2]
```

Out[∘]=

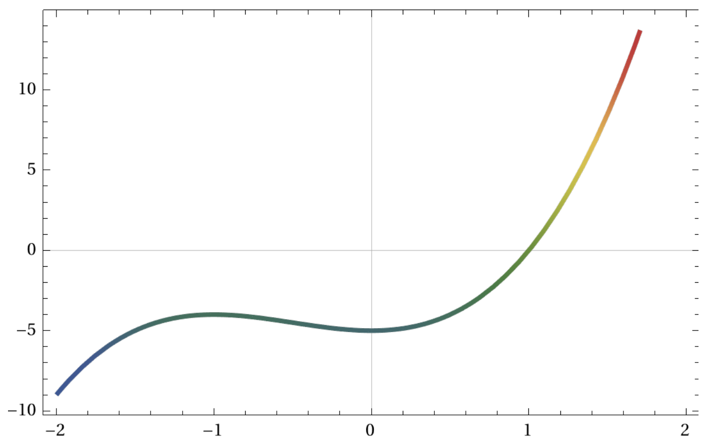


6.2. Building polynomial plot

Task. Plot polynomial $y = -5 + 3\,x^2 + 2\,x^3$.

In[ ]:=
```
Remove[x, y];
y[x_] = -5 + 3 x^2 + 2 x^3;
Plot[y[x], {x, -2, 2}, PlotTheme → "Scientific",
 PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"]
```

Out[ ]=

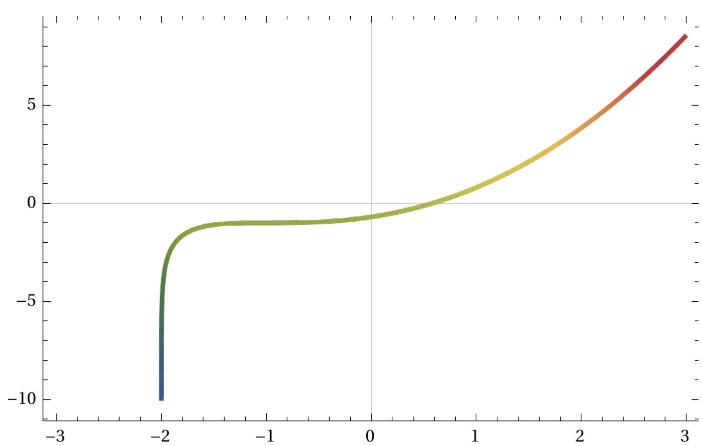


6.3. Build arbitrary explicitly defined function

Task. Plot $y(x) = 2x + x^2 - (x+1)\ln(x+2)$

In[ ]:=
```
Clear[x, y];
y[x_] = 2 x + x^2 - (x + 1) Log[x + 2];
Plot[y[x], {x, -3, 3}, PlotTheme → "Scientific",
 PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"]
```

Out[ ]=

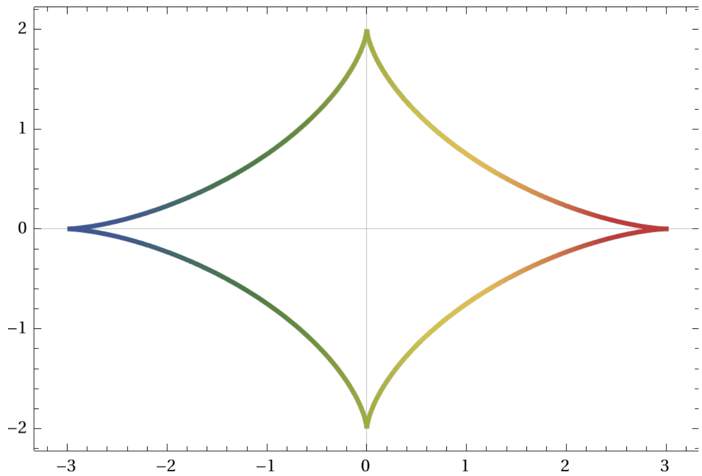


6.4. Drawing Parametric Plots

Task. Build function $x(t) = 3\cos^3 t$, $y(t) = 2\sin^3 t$. Build table of values on interval $[0, 2\pi]$ and corresponding list plot.

In[∘]:=
```
Remove[x, y, t];
x[t_] = 3 (Cos[t])^3;
y[t_] = 2 (Sin[t])^3;
p1 = ParametricPlot[{x[t], y[t]}, {t, 0, 2 π}, PlotTheme → "Scientific",
  PlotStyle → {Thickness[0.007], Blue}, ColorFunction → "DarkRainbow"]
```

Out[∘]=

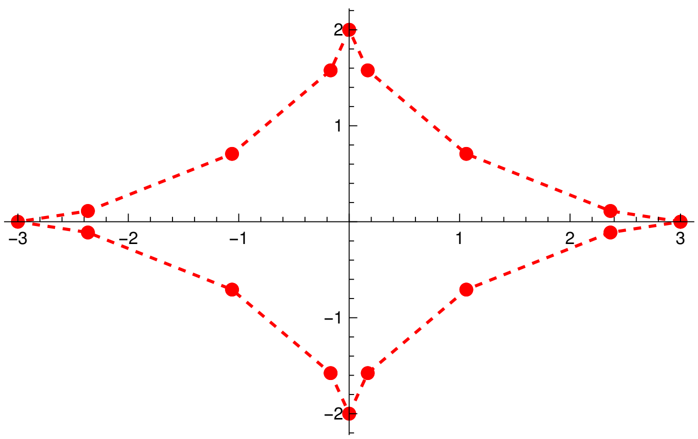


In[∘]:=
```
points = Table[{x[t], y[t]}, {t, 0, 2 π, π/8}];
npoints = N[points];
npoints // TableForm
```

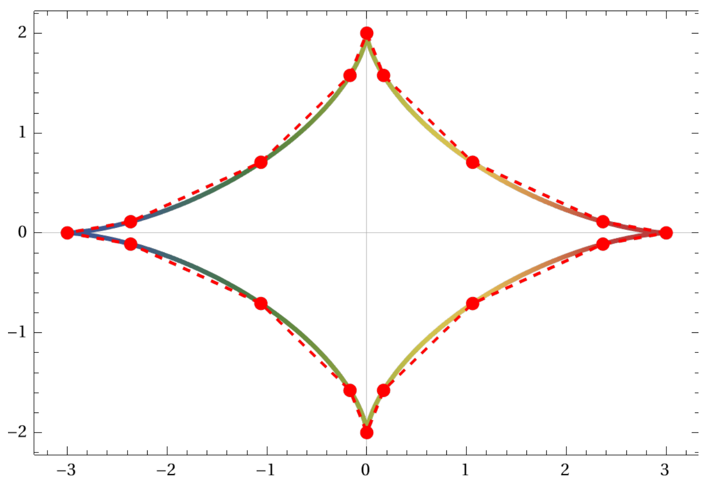
Out[∘]//TableForm=

3.	0.
2.36574	0.112085
1.06066	0.707107
0.168128	1.57716
0.	2.
-0.168128	1.57716
-1.06066	0.707107
-2.36574	0.112085
-3.	0.
-2.36574	-0.112085
-1.06066	-0.707107
-0.168128	-1.57716
0.	-2.
0.168128	-1.57716
1.06066	-0.707107
2.36574	-0.112085
3.	0.

In[ ]:=
```
p2 = ListPlot[npoints, Joined → True,
  Mesh → Full, PlotStyle → {PointSize[0.02], Red, Dashed}]
```

Out[ ]=


In[ ]:=
```
Show[p1, p2]
```

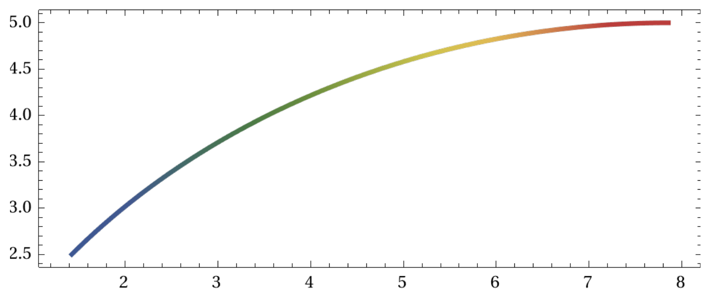
Out[ ]=


6.5. Drawing parametric plots

Task. Build plot $x(t) = \frac{5}{2}\left(t - \sin t\right)$, $y(t) = \frac{5}{2}\left(1 - \cos t\right)$ for interval $t \in \left[\frac{\pi}{2}, \pi\right]$.

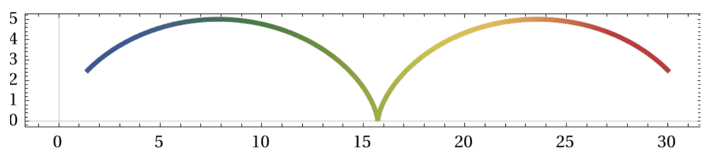
In[∘]:=
```
Clear[x, y, t]
x[t_] = 5/2 (t - Sin[t]);
y[t_] = 5/2 (1 - Cos[t]);
p1 = ParametricPlot[{x[t], y[t]}, {t, π/2, π}, PlotTheme → "Scientific",
  PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"]
```

Out[∘]=



In[∘]:=
```
p2 = ParametricPlot[{x[t], y[t]}, {t, π/2, 7 π/2}, PlotTheme → "Scientific",
  PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"]
```

Out[∘]=

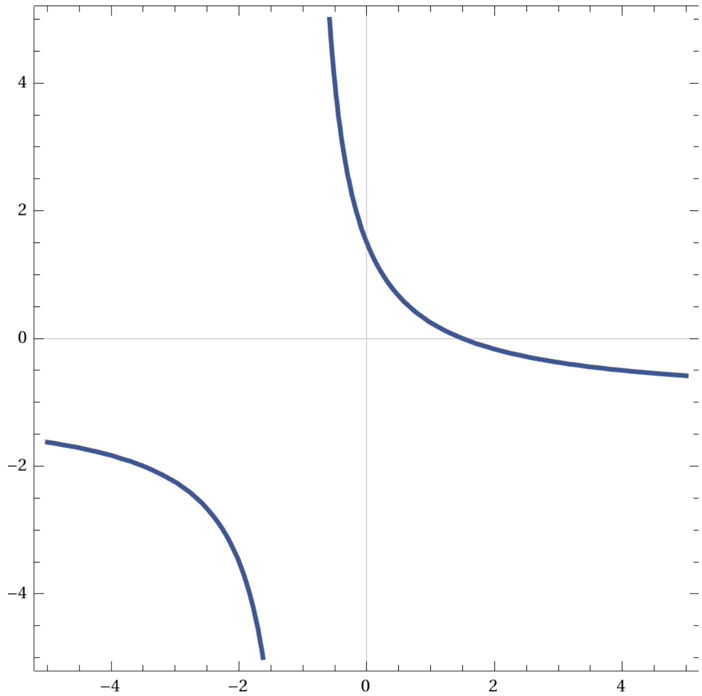


6.6. Build unexplicitly defined functions

Task. Build curve $2\,x\,y + 2\,x + 2\,y - 3 = 0$

In[ ]:=
```
Remove[x, y, F];
F[x_, y_] = 2 x y + 2 x + 2 y - 3;
ContourPlot[F[x, y] == 0, {x, -5, 5}, {y, -5, 5}, PlotTheme → "Scientific",
 ColorFunction → "DarkRainbow", ContourStyle → {Thickness[0.007]}]
```

Out[ ]=



7. Geometric objects

7.1. Building a plane based on three points

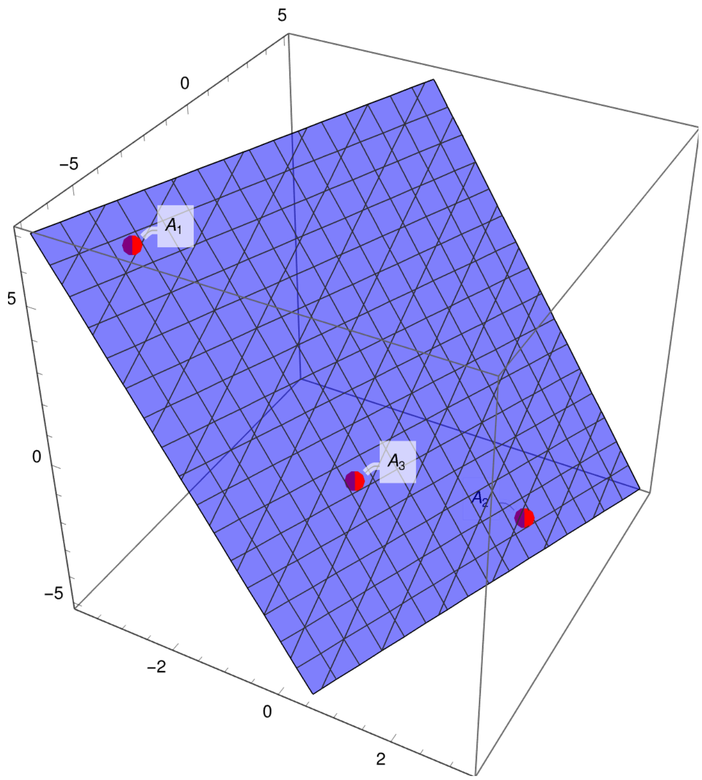
Task: Write down an equation of plane intersecting points A_1 (−3, −5, 6), A_2 (2, 1, −4), A_3 (0, −3, −1). Display this plane and three points.

In[ ]:=
```
Clear[A1, A2, A3, x, y, z];
A1 = {-3, -5, 6}; A2 = {2, 1, -4}; A3 = {0, -3, -1};
p1 = ListPointPlot3D[{A1, A2, A3} → {"A1", "A2", "A3"},
    PlotStyle → {Directive[PointSize[0.03], Red]}];
r = {x, y, z};
Eq = Det[{r - A1, A2 - A1, A3 - A1}];
Print["Plane equation is ", Eq == 0]
```

```
Plane equation is 7 - 22 x + 5 y - 8 z == 0
```

In[∘]:=
```
p2 = ContourPlot3D[Eq == 0, {x, -7, 5}, {y, -7, 5},
    {z, -5, 7}, ContourStyle → Directive[Opacity[0.5], Blue]];
Show[p2, p1, PlotRange → All, AspectRatio → Automatic]
```

Out[∘]=



7.2. Parallelogram in R^3

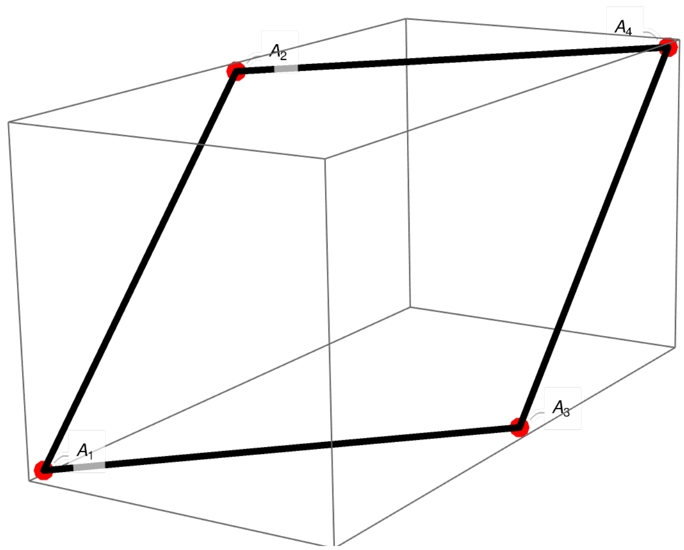
Task: In R three vertices of parallelogram A (6, 2, −3), A (6, 3, −2), A (7, 3, −3) are given. Find coordinate of the fourth vertex A which is opposite to the side A . Find area of this parallelogram. Display contours of this parallelogram and its vertices.

In[∘]:=
```
A1 = {6, 2, -3}; A2 = {6, 3, -2}; A3 = {7, 3, -3};
A1A2 = A2 - A1; A1A3 = A3 - A1;
A4 = A1 + (A1A2 + A1A3);
Print["Coordinate of vertex A₄ is ", A4]
```

```
Coordinate of vertex A₄ is {7, 4, -2}
```

In[∘]:=
```
p1 = Graphics3D[{Thickness[0.01], Line[{A1, A2, A4, A3, A1}]}];
p2 = ListPointPlot3D[{A1, A2, A3, A4} → {"A₁", "A₂", "A₃", "A₄"},
   PlotStyle → {Directive[PointSize[0.03], Red]}];
Show[p1, p2]
```

Out[∘]=



In[∘]:=
```
S = Norm[Cross[A1A2, A1A3]];
Print["Area of this parallelogram is ", S]
```

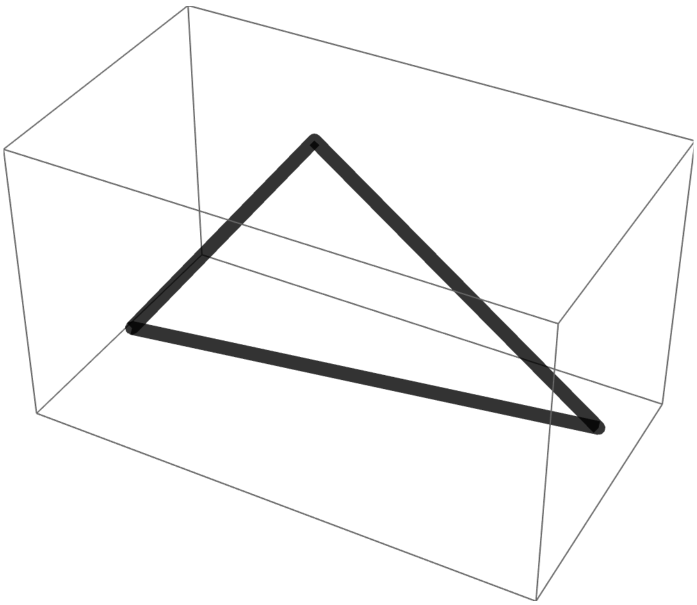
Area of this parallelogram is $\sqrt{3}$

7.3. Calculations in R^3 triangle

Task: Given a triangle with vertices A_1 (−4, −2, 0), A_2 (−1, −2, 4), A_3 (3, −2, 1) find length of height $h = \| \overrightarrow{A_2 H} \|$ from vertex A_2 on side $A_1 A_3$. Find length of median $m = \| \overrightarrow{A_2 M} \|$ from vertex A_2 on side $A_1 A_3$. Display triangle's contour and its median.

In[∘]:= A1 = {-4, -2, 0};
A2 = {-1, -2, 4};
A3 = {3, -2, 1};
p1 = Graphics3D[{Thickness[0.02], Opacity[0.8], Black, Line[{A1, A2, A3, A1}]}]

Out[∘]=



In[∘]:= A1A2 = A2 - A1; A1A3 = A3 - A1;
S = $\frac{1}{2}$ Norm[Cross[A1A2, A1A3]];
Print["Area of this triangle is ", S]

Area of this triangle is $\frac{25}{2}$

In[∘]:= LengthA1A3 = Norm[A1A3];
h = $\frac{2\,S}{\text{LengthA1A3}}$;
Print["Length of height is ", h]

Length of height is $\frac{5}{\sqrt{2}}$

In[∘]:= A2A1 = A1 - A2;
A2A3 = A3 - A2;
A2M = $\frac{\text{A2A1} + \text{A2A3}}{2}$;
LengthA2M = Norm[A2M];
Print["Length of median A_2M is ", LengthA2M]

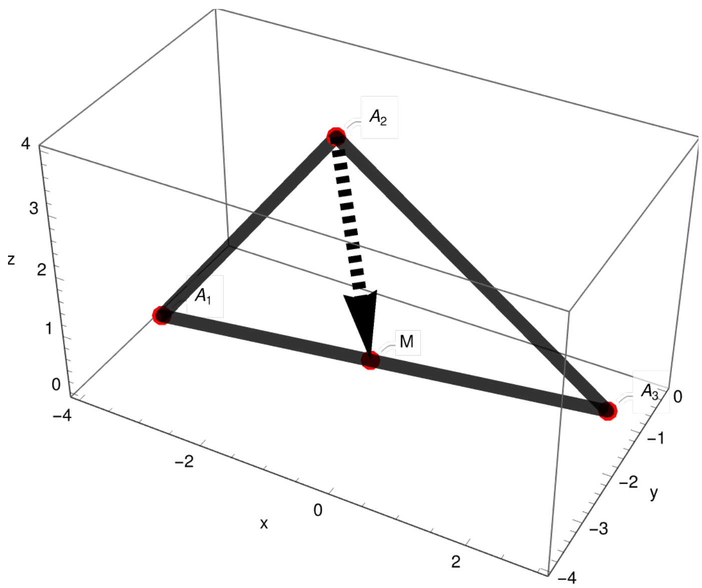
Length of median A_2M is $\frac{5}{\sqrt{2}}$

In[∘]:=
```
M = A2 + A2M;
Print["Coordinate of median is ", M]
```

Coordinate of median is $\left\{-\frac{1}{2}, -2, \frac{1}{2}\right\}$

In[∘]:=
```
p2 = Graphics3D[
    {Thickness[0.02], Black, Dashed, Opacity[1.0], Arrowheads[0.1], Arrow[{A2, M}]}];
p3 = ListPointPlot3D[{A1, A2, A3, M} → {"A₁", "A₂", "A₃", "M"},
    PlotStyle → {Directive[PointSize[0.03], Red]}];
Show[p1, p2, p3, Axes → True, AxesLabel → {"x", "y", "z"}]
```

Out[∘]=



7.4. Line and plane in R^3

Task: Find intersection point of line $l: \dfrac{x+2}{-1} = \dfrac{y-1}{1} = \dfrac{z+3}{2}$ and plane $\pi: x + 2y + 3z - 2 = 0$.

Display plane, line, and intersection point.

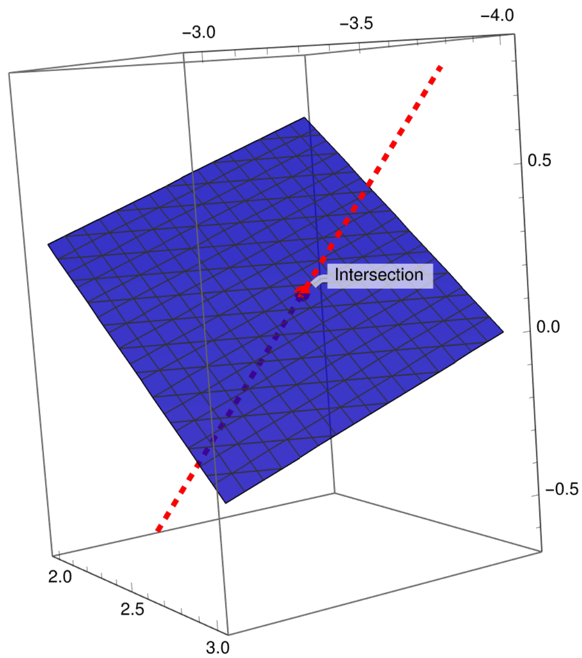
In[∘]:=
```
Remove[x, y, z, t];
x[t_] = -2 - t;
y[t_] = 1 + t;
z[t_] = -3 + 2 t;
sol = Solve[x[t] + 2 y[t] + 3 z[t] - 2 == 0, t];
t0 = t /. sol[[1]];
Print["Parameter t at which plane and line intersect is ", t0]
```

Parameter t at which plane and line intersect is $\frac{11}{7}$

```
In[]:= G = {x[t0], y[t0], z[t0]};
       Print["Intersection point has coordinates ", G]
```

Intersection point has coordinates $\left\{-\frac{25}{7}, \frac{18}{7}, \frac{1}{7}\right\}$

```
In[]:= pLine = ParametricPlot3D[{x[t], y[t], z[t]},
          {t, 1.2, 1.9}, PlotStyle → {Red, Thickness[0.01], Dashed}];
       pPlane = ContourPlot3D[x + 2 y + 3 z - 2 == 0, {x, -4, -3}, {y, 2, 3}, {z, -1, 2},
          ContourStyle → Directive[Blue, Opacity[0.8], Specularity[White, 30]]];
       pG = ListPointPlot3D[{G} → {"Intersection"},
          PlotStyle → {Directive[PointSize[0.03], Red]}];
       Show[pLine, pPlane, pG, PlotRange → All, PlotTheme → "Scientific"]
```

Out[]=


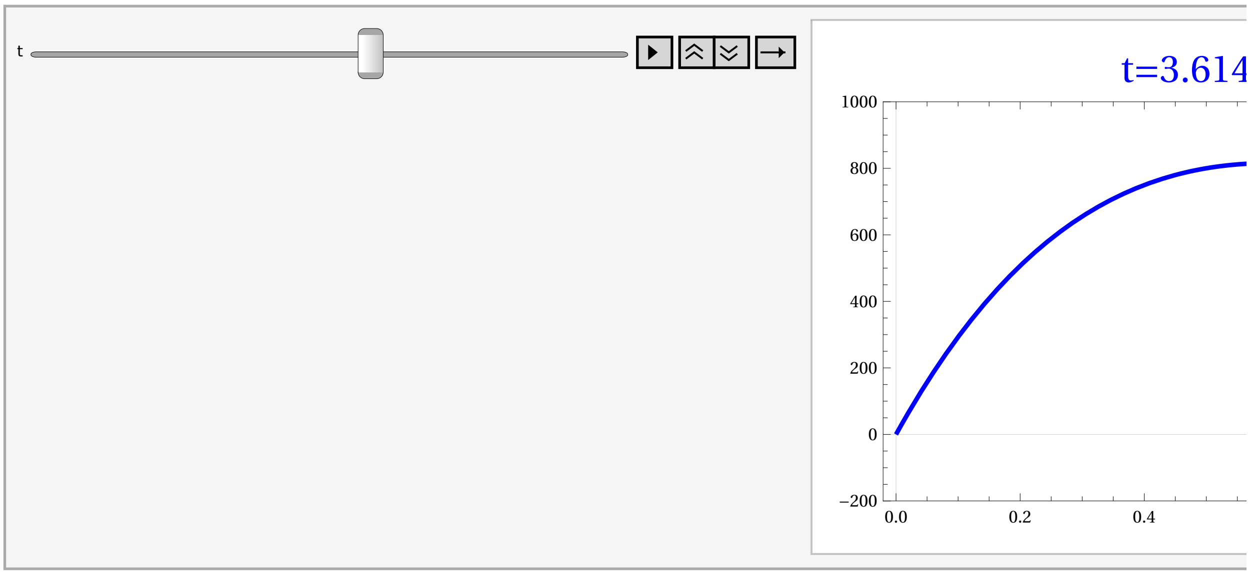
8. Animations

8.1. Curve animation

Task: Build animation of curve $u(x, t) = 12\, x\,(x - t)^5 \sin t$ movement on the given time interval $x \in [0, 1]$, $t \in [0, 2\pi]$.

In[150]:=
```
Remove[x, t];
Animate[
 Plot[12 x (x - t)^5 Sin[t], {x, 0, 1}, PlotRange → {-200, 1000},
  PlotLabel → Style["t=" <> ToString[t], Blue, 20],
  PlotTheme → "Scientific", PlotStyle → {Thickness[0.007], Blue}],
 {t, 0, 2 π}, AnimationRunning → False
]
```

Out[151]=

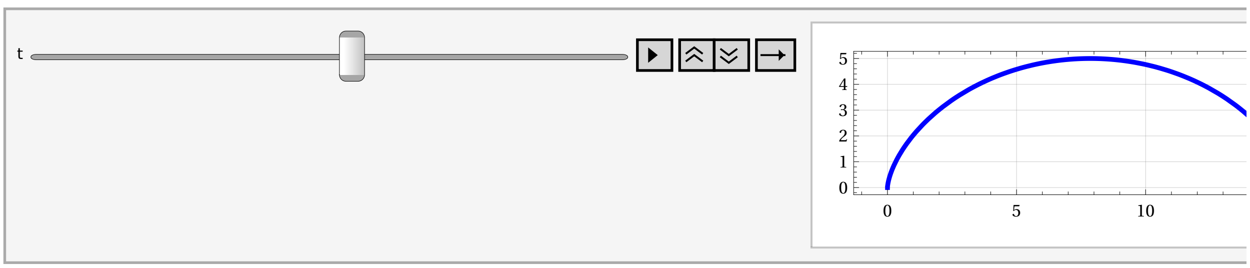


8.2. Animation of point movement along the curve

Task: Build point’s movement animation along the curve
$x(t) = \frac{5}{2}\left(t - \sin t\right)$, $y(t) = \frac{5}{2}\left(1 - \cos t\right)$, $t \in \left[\frac{\pi}{2}, \pi\right]$.

In[156]:=
```
Remove[x, y, t];
x[t_] = 5/2 (t - Sin[t]);
y[t_] = 5/2 (1 - Cos[t]);
Animate[
 Show[
  ParametricPlot[{x[τ], y[τ]}, {τ, 0, 3 π},
   PlotTheme → "Scientific", PlotStyle → {Thickness[0.007], Blue}],
  ListPlot[{{x[t], y[t]}} → {"Point"}, PlotStyle → {Black, PointSize[0.03]}],
  GridLines → Automatic],
 {t, 0, 3 π}]
```

Out[159]=



9. Derivatives and Limits

9.1. Rational Sequences

Task: Given a sequence $x_n = \frac{(n+3)^3 + (n+4)^3}{(n+3)^4 - (n+4)^4}$,find $\lim_{n\to\infty} x_n$ and print 8 first elements of it.

In[∘]:=
```
Clear[x, n]
x[n_] = ((n + 3)^3 + (n + 4)^3)/((n + 3)^4 - (n + 4)^4);
Table[N[{n, x[n]}], {n, 1, 8}] // TableForm
```

$$x[n\_] = \frac{(n+3)^3 + (n+4)^3}{(n+3)^4 - (n+4)^4};$$

Out[∘]//TableForm=

1.	-0.512195
2.	-0.508197
3.	-0.505882
4.	-0.504425
5.	-0.503448
6.	-0.502762
7.	-0.502262
8.	-0.501887

In[∘]:=
```
L = Limit[x[n], n → ∞];
Print["Limit of xn is ", L]
```

Limit of x_n is $-\frac{1}{2}$

9.2. Irrational Sequences

Task: Given a sequence $x_{Æ} = \dfrac{\sqrt{n\ +3} - \sqrt{n-3}}{\sqrt{n\ +3} + \sqrt{n-3}}$, find $\lim\limits_{Æ\to\infty} x_{Æ}$, print first 8 elements of this sequence, build a plot for first 50 elements.

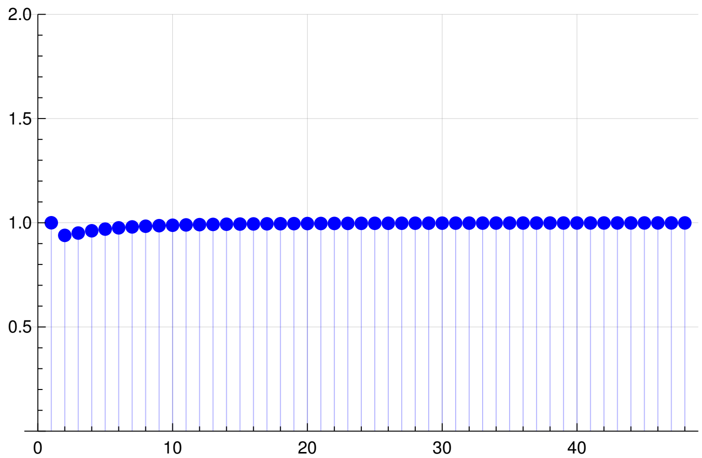
In[∘]:=
```
Clear[x, n]
x[n_] = (Sqrt[n^5 + 3] - Sqrt[n - 3])/(Sqrt[n^5 + 3] + Sqrt[n - 3]);
Table[N[x[n], 3], {n, 3, 9}]
```

$$x[n\_] = \frac{\text{Sqrt}[n^5 + 3] - \text{Sqrt}[n - 3]}{\text{Sqrt}[n^5 + 3] + \text{Sqrt}[n - 3]};$$

Out[∘]= {1.00, 0.939, 0.951, 0.961, 0.970, 0.976, 0.980}

```
In[∘]:= ListPlot[Table[x[n], {n, 3, 50}], PlotStyle → Directive[PointSize[0.02], Blue],
         PlotRange → {0, 2}, GridLines → Automatic, Filling → Axis]
```

Out[∘]=




```
In[∘]:= L = Limit[x[n], n → ∞];
        Print["Limit of sequence xₙ is, as expected, ", L]
```

```
Limit of sequence xₙ is, as expected, 1
```

9.3. Limit of functions

Task: Given $f(x) = \dfrac{1 - \cos 2x + \tan\ x}{x \sin 3x}$, find $L = \lim\limits_{x\to} f(x)$. Make sure values of function approaches to L when x approaches 0 by building a table of values around $x = 0$.

```
In[∘]:= Clear[f, x]
        f[x_] = (1 - Cos[2 x] + (Tan[x])^2) / (x Sin[3 x]);
        x0 = 0;
        y0 = Limit[f[x], x → x0];
        Print["Limit of f(x) as x approaches 0 is ", N[y0]]
```

```
Limit of f(x) as x approaches 0 is 1.
```

In[ ]:=
```
imax = 10;
dx = Sort[Table[(-1)^i 10^-Floor[i/2], {i, 0, imax}]];
Table[N[{x0 + dx[[i]], f[x0 + dx[[i]]]}], {i, 1, imax}] // TableForm
```

Out[ ]//TableForm=

-1.	27.2227
-0.1	1.01517
-0.01	1.00015
-0.001	1.
-0.0001	1.
0.00001	1.
0.0001	1.
0.001	1.
0.01	1.00015
0.1	1.01517

9.4. Limit of rational function at special point

Task: Given rational function $f(x) = \dfrac{x\ \ + 4x\ \ + 5x + 2}{x\ \ - 3x - 2}$, find limit $L = \lim\limits_{x \to -} f(x)$. Make sure $x = -1$ is a critical point of $f(x)$. Make sure values of function f approaches L as x approaches -1 by plotting $f(x)$.

In[ ]:=
```
Clear[f, x]
f[x_] = (x^3 + 4 x^2 + 5 x + 2)/(x^3 - 3 x - 2);
x0 = -1;
y0 = Limit[f[x], x → x0];
Print["Limit of f(x) as x approaches -1 is ", y0]
```

Limit of f(x) as x approaches -1 is $-\frac{1}{3}$

In[∘]:=

```
imax = 10;
dx = Sort[Table[(-1)^i 10^-Floor[i/2], {i, 0, imax}]];
Table[N[{x0 + dx[[i]], f[x0 + dx[[i]]]}], {i, 1, imax}] // TableForm
```

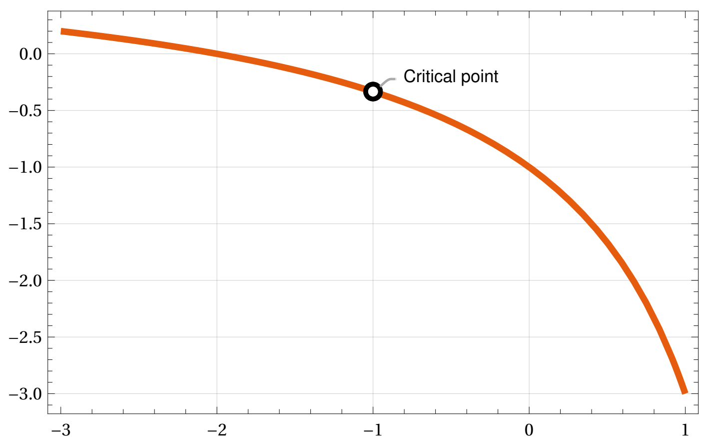
Out[∘]//TableForm=

-2.	0.
-1.1	-0.290323
-1.01	-0.328904
-1.001	-0.332889
-1.0001	-0.333289
-0.99999	-0.333338
-0.9999	-0.333378
-0.999	-0.333778
-0.99	-0.337793
-0.9	-0.37931

In[∘]:=

```
p1 = ListPlot[{{x0, y0}} → {"Critical point"}, PlotStyle → {Black, PointSize[0.03]}];
p2 = ListPlot[{{x0, y0}}, PlotStyle → {White, PointSize[0.015]}];
pf =
  Plot[f[x], {x, x0 - 2, x0 + 2}, PlotStyle → Thickness[0.01], PlotTheme → "Scientific"];
Show[pf, p1, p2, AxesOrigin → {0, 0}, GridLines → Automatic]
```

Out[∘]=



9.5. Calculating derivatives

Task: Given $f(x) = \dfrac{1+x}{2\sqrt{1+2x}}$ find $\dfrac{df}{dx}$. Build $f(x)$ and $\dfrac{df(x)}{dx}$. Find value of derivative at

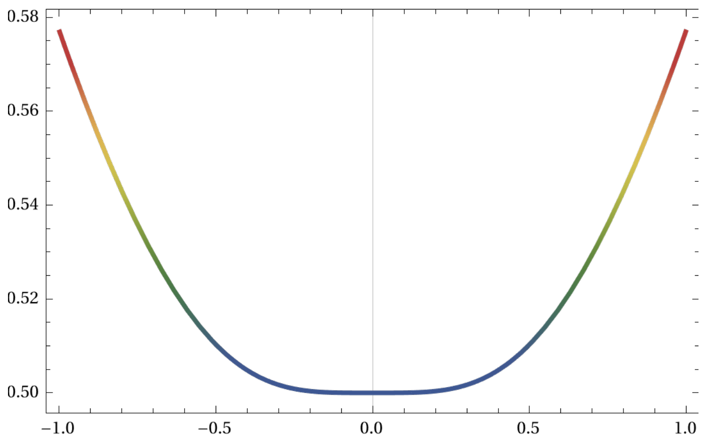
$x = 0.5,\ 1.0,\ 3.0.$

In[∘]:=
```
y[x_] = (1 + x)/(2 Sqrt[1 + 2 x]);
y1[x_] = Simplify[D[y[x], x]];
Print["Derivative of ", y[x], " is ", y1[x]]
```

Derivative of $\frac{1+x^2}{2\sqrt{1+2x^2}}$ is $\frac{x^3}{\left(1+2x^2\right)^{3/2}}$

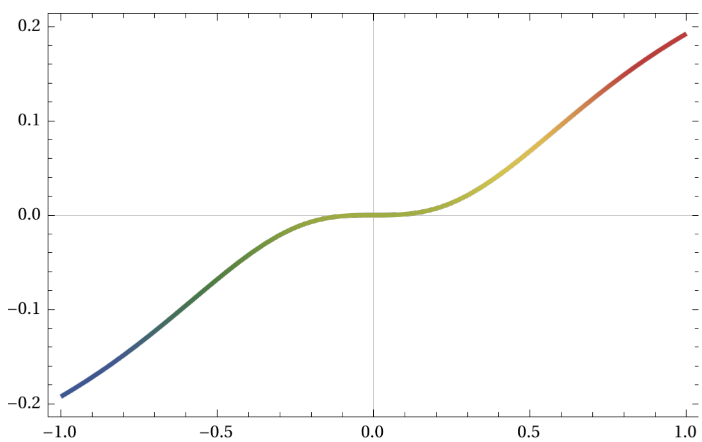
In[∘]:=
```
Plot[y[x], {x, -1, 1}, PlotTheme → "Scientific",
 PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"]
```

Out[∘]=



In[∘]:=
```
Plot[y1[x], {x, -1, 1}, PlotTheme → "Scientific",
 PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"]
```

Out[∘]=



In[∘]:=
```
Print["Derivatives at x=0,0.5,3.0 are ", y1[{0, 0.5, 3.0}]]
```

Derivatives at x=0,0.5,3.0 are {0, 0.0680414, 0.326012}

9.6. Derivatives of expressions containing trigonometric functions

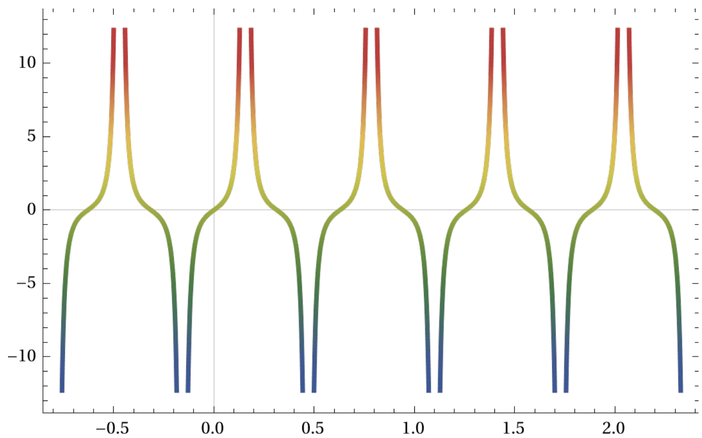
Task: Find derivative of $y\,(x) = 8 \sin\left(\dfrac{1}{\tan 3}\right) + \dfrac{1}{5} \cdot \dfrac{\sin^2 5\,x}{\cos 10\,x}$ and plot its function.

In[∘]:=
```
y[x_] = 8 Sin[1/Tan[3]] + 1/5 (Sin[5 x])^2/Cos[10 x];
y1[x_] = Simplify[D[y[x], x]];
Print["Derivative of ", y[x], " is ", y1[x]]
```

```
Derivative of 1/5 Sec[10 x] Sin[5 x]^2 + 8 Sin[Cot[3]] is Sec[10 x] Tan[10 x]
```

In[∘]:=
```
Plot[y1[x], {x, -π/4, 3π/4}, PlotTheme → "Scientific",
 PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"]
```

Out[∘]=



9.7. Derivatives of higher degrees

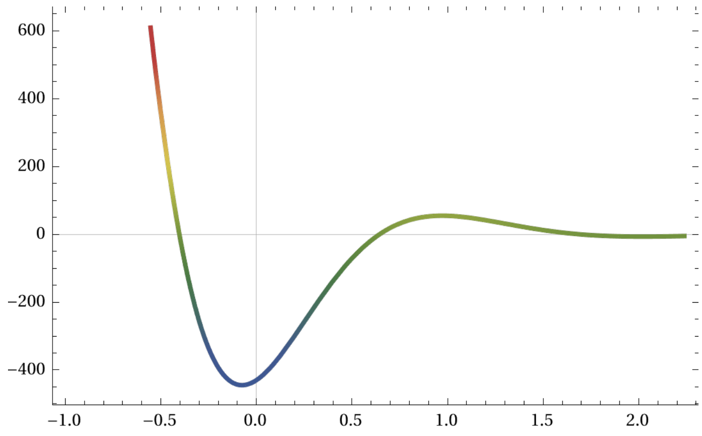
Task. Find 4th order derivative of $f\,(x) = \sin\left(2 + 3\,x\right) e^{1-2\,x}$ and plot it.

In[∘]:=
```
y[x_] = Sin[2 + 3 x] Exp[1 - 2 x];
y4[x_] = Simplify[D[y[x], {x, 4}]];
Print["4th order derivative of ", y[x], " is ", y4[x]]
```

```
4th order derivative of e^(1-2 x) Sin[2 + 3 x] is e^(1-2 x) (120 Cos[2 + 3 x] - 119 Sin[2 + 3 x])
```

In[∘]:=
```
Plot[y4[x], {x, -1, 2.25}, PlotTheme → "Scientific",
  PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"]
```

Out[∘]=



9.8. Using derivatives in equations

Task: Show that function $y = \frac{1-x}{1+x}$ satisfies $y - x\frac{dy}{dx} = 1 + x\frac{dy}{dx}$.

In[∘]:=
```
y[x_] = (1 - x)/(1 + x);
y1[x_] = Simplify[D[y[x], x]];
Print["Derivative of ", y[x], " is ", y1[x]]
```

Derivative of $\frac{1-x}{1+x}$ is $-\frac{2}{(1+x)^2}$

In[∘]:=
```
Simplify[y[x] - x y1[x] == 1 + x^2 y1[x]]
```

Out[∘]= True

10. Using Differential Calculus

10.1. Tangent line

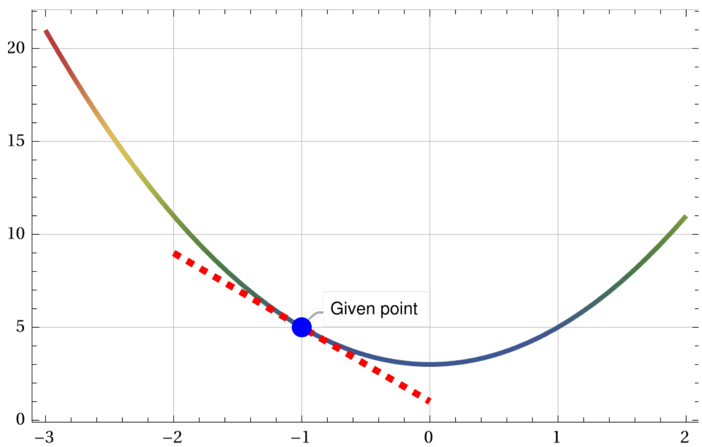
Task: Given curve $y = 2x^2 + 3$, write equation of tangent at $x_0 = -1$. Plot this curve and tangent line.

In[∘]:=
```
y[x_] = 2 x^2 + 3;
y1[x_] = Simplify[D[y[x], x]];
x0 = -1;
τ[x_] = y[x0] + y1[x0] (x - x0);
Print["Equation of tangent is y(x)=", τ[x]]
```

```
Equation of tangent is y(x)=5 - 4 (1 + x)
```

In[∘]:=
```
p1 = Plot[y[x], {x, -3, 2}, PlotTheme → "Scientific",
    PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"];
p2 = Plot[τ[x], {x, x0 - 1, x0 + 1}, PlotStyle → {Red, Thickness[0.01], Dashed}];
p3 = ListPlot[{{x0, y[x0]}} → {"Given point"}, PlotStyle → {Blue, PointSize[0.03]}];
Show[p1, p2, p3, PlotRange → All, GridLines → Automatic]
```

Out[∘]=



10.2. Tangent line and normal for a curve defined parametrically

Task: Given parametrically defined curve $x(t) = \dfrac{\sin t}{1+t^4}$, $y(t) = \dfrac{\cos t}{1+t^4}$, find tangent and normal line equations at $t_0 = \dfrac{\pi}{12}$. Build these lines and curve, display considered point.

In[ ]:=
```
x[t_] = Sin[t]/(1 + t^4); x1[t_] = Simplify[D[x[t], t]];
y[t_] = Cos[t]/(1 + t^4); y1[t_] = Simplify[D[y[t], t]];
t0 = N[π/12];
xτ[t_] = x[t0] + t x1[t0];
yτ[t_] = y[t0] + t y1[t0];
Print["Tangent line has a parametric equation x(t)=", xτ[t], " and y(t)=", yτ[t]]
```

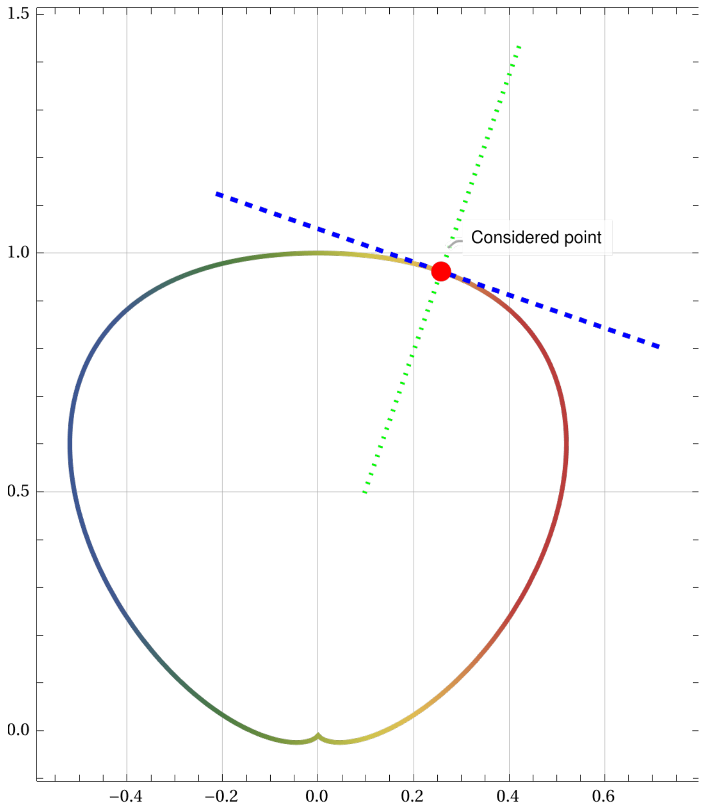
Tangent line has a parametric equation x(t)=0.257609 + 0.943006 t and y(t)=0.96141 - 0.32629 t

In[ ]:=
```
xn[t_] = x[t0] + t y1[t0];
yn[t_] = y[t0] - t x1[t0];
Print["Normal has a parametric equation x(t)=", xn[t], " and y(t)=", yn[t]]
```

Normal has a parametric equation x(t)=0.257609 - 0.32629 t and y(t)=0.96141 - 0.943006 t

In[∘]:=
```
p1 = ParametricPlot[{x[t], y[t]}, {t, -π, π}, PlotTheme → "Scientific",
    PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"];
p2 = ListPlot[{{x[t0], y[t0]}} → {"Considered point"}, PlotStyle → {Red, PointSize[0.03]}];
pτ = ParametricPlot[{xτ[t], yτ[t]},
    {t, -0.5, 0.5}, PlotStyle → {Blue, Dashed, Thickness[0.007]}];
pn = ParametricPlot[{xn[t], yn[t]},
    {t, -0.5, 0.5}, PlotStyle → {Green, Dotted, Thickness[0.007]}];
Show[p1, pτ, pn, p2, PlotRange → All, GridLines → Automatic]
```

Out[∘]=



10.3. Asymptotes

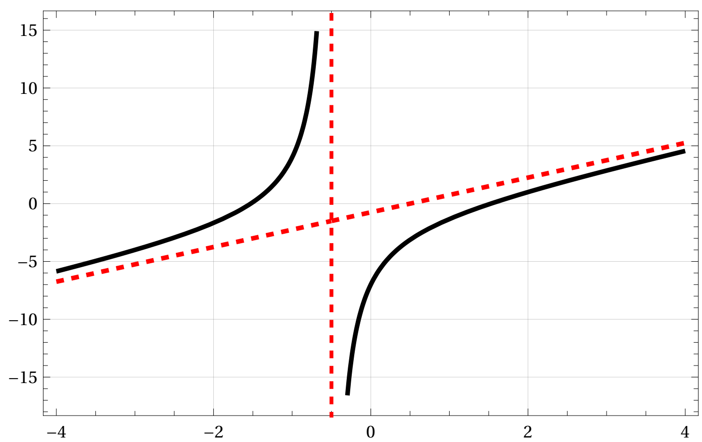
Task. Find asymptotes of function $f(x) = \dfrac{3x^2 - 7}{2x + 1}$. Plot this function and its asymptotes.

In[∘]:=
```
f[x_] = (3 x^2 - 7)/(2 x + 1);

a = Limit[f[x]/x, x → ∞];

b = Limit[f[x] - a x, x → ∞];

y[x_] = a x + b;

Print["Equation of asymptote is y(x)=", y[x]]
```

Equation of asymptote is y(x)= $-\frac{3}{4}+\frac{3x}{2}$

In[∘]:=
```
verticalLine = Line[{{-1/2, -20}, {-1/2, 20}}];

p1 = Plot[{f[x], y[x]}, {x, -4, 4}, PlotTheme → "Scientific",
  PlotStyle → {{Black, Thickness[0.007]}, {Red, Thickness[0.007], Dashed}},
  GridLines → Automatic, Epilog → {Directive[{Thick, Red, Dashed}], verticalLine}]
```

Out[∘]=



10.4. Normal and tangent line to a curve defined unexplicitly

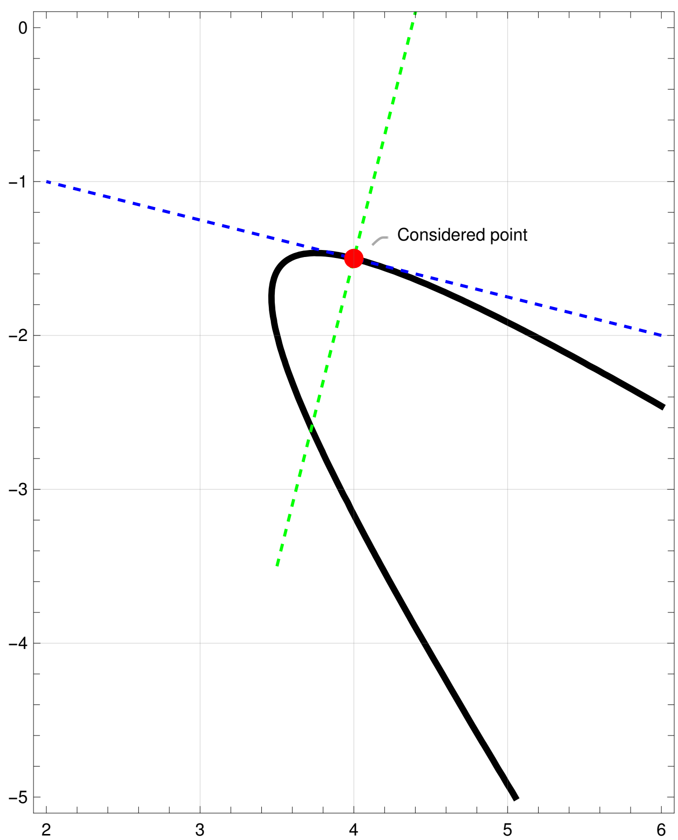
Task. Find tangent and normal line equations for $\Phi(x, y) = 12x^2 + 26xy + 12y^2 - 52x - 48y + 73 = 0$. Plot this curve and line equations for arbitrary chosen point.

```
In[∘]:= Φ[x_, y_] = 12 x^2 + 26 x y + 12 y^2 - 52 x - 48 y + 73;
pΦ =
  ContourPlot[Φ[x, y] == 0, {x, 2, 6}, {y, -5, 0}, ContourStyle → {Black, Thickness[0.01]}];
x0 = 4;
sol = Solve[Φ[x0, y] == 0, y];
y0 = N[sol[[2]][[1, 2]]];
Print["We consider point (", x0, ",", y0, ")"]
```

```
We consider point (4,-1.5)
```

```
In[∘]:= τ = Derivative[1, 0][Φ][x0, y0] (x - x0) + Derivative[0, 1][Φ][x0, y0] (y - y0);
n = Derivative[1, 0][Φ][x0, y0] (y - y0) - Derivative[0, 1][Φ][x0, y0] (x - x0);
pLines = ContourPlot[{τ == 0, n == 0}, {x, x0 - 2, x0 + 2},
   {y, y0 - 2, y0 + 2}, ContourStyle → {{Blue, Dashed}, {Green, Dashed}}];
pPoint = ListPlot[{{x0, y0}} → {"Considered point"}, PlotStyle → {Red, PointSize[0.03]}];
Show[pΦ, pLines, pPoint, AspectRatio → Automatic, GridLines → Automatic]
```

Out[∘]=



10.5. Maximum and minimum points

Task: Find maximum and minimal values of $f(x) = \frac{x^5 - 3x}{\sqrt{16 + x^4}}$ on the interval $x \in [-1.5, 5.5]$. Plot this function and find max and min using derivatives. Check this solution using function that solve this equation automatically.

In[∘]:=
```
f[x_] = (x^5 - 3 x)/Sqrt[16 + x^4];
a = -1.5; b = 5.5;
s = NSolve[f'[x] == 0, x, Reals]
```

Out[∘]= {{x → -0.867639}, {x → 0.867639}}

In[∘]:=
```
x0 = N[s[[1]][[1, 2]]];
y0 = f[x0];
P0 = {x0, y0};
Print["We will consider a critical point ", P0]
```

```
We will consider a critical point {-0.867639, 0.5187}
```

In[∘]:=
```
f2[x_] = Simplify[D[f[x], x, {x, 2}]];
secondDerivative0 = f2[x0];
Print["Second  derivative at point ", P0, " is ", secondDerivative0]
```

```
Second  derivative at point {-0.867639, 0.5187} is 10.5986
```

Thus we see that $\frac{d^2 f}{d x^2}(x_0) > 0$ which means we have a local minima.

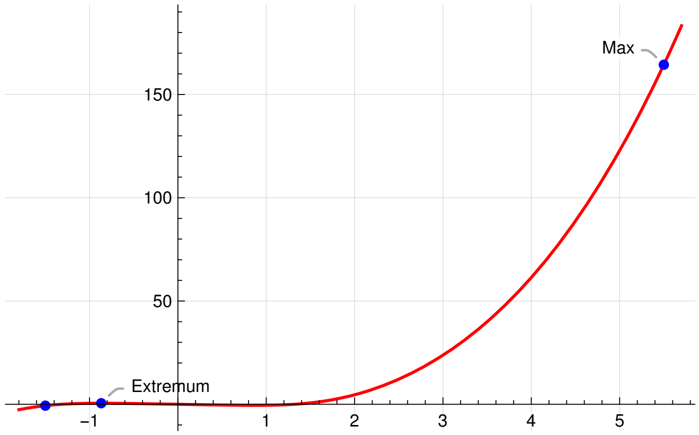
In[∘]:=
```
intervalValues = {f[a], f[b]};
Print["Values on the edge of interval are ", intervalValues]
```

```
Values on the edge of interval are {-0.674109, 164.399}
```

Therefore, we see that the maximum value occurs at $x = b = 5.5$ where $f(b) \approx 164.4$ whereas minimum at $x = a = 1.5$ where $f(a) \approx -0.67$. Value at our critical point equals $f(x_0) \approx 0.52$ which is neither maximum nor minimum.

In[∘]:=
```
p1 = Plot[f[x], {x, a - 0.3, b + 0.2}, PlotStyle → Red];
pPoint = ListPlot[{{a, f[a]}, {b, f[b]}, {x0, f[x0]}} → {"Min", "Max", "Extremum"},
   PlotStyle → {Blue, PointSize[0.015]}];
Show[p1, pPoint, GridLines → Automatic]
```

Out[∘]=



In[∘]:=
```
mn = Minimize[{f[x], a ≤ x ≤ b}, x]
mx = Maximize[{f[x], a ≤ x ≤ b}, x]
```

Out[∘]= {-0.674109, {x → -1.5}}

Out[∘]= {164.399, {x → 5.5}}

10.6. Partial Derivatives

Task. Find first and second order partial derivatives of $z(x, y) = \dfrac{x^2 - y^2}{1 + x^2}$. Plot function and normal vector at (–0.5, 0.7).

In[∘]:=
```
z[x_, y_] = (x^2 - y^2)/(1 + x^2);
zx[x_, y_] = Simplify[D[z[x, y], x]];
zy[x_, y_] = Simplify[D[z[x, y], y]];
Print["First order derivatives are: ", {zx[x, y], zy[x, y]}]
```

First order derivatives are: $\left\{\dfrac{2\,x\,(1+y^2)}{(1+x^2)^2}, -\dfrac{2\,y}{1+x^2}\right\}$

In[∘]:=
```
zxx[x_, y_] = Simplify[D[z[x, y], {x, 2}]];
zxy[x_, y_] = Simplify[D[z[x, y], x, y]];
zyy[x_, y_] = Simplify[D[z[x, y], {y, 2}]];
Print["Second order derivatives are: ", {zxx[x, y], zxy[x, y], zyy[x, y]}]
```

Second order derivatives are: $\left\{-\frac{2\left(-1+3x^2\right)\left(1+y^2\right)}{\left(1+x^2\right)^3}, \frac{4xy}{\left(1+x^2\right)^2}, -\frac{2}{1+x^2}\right\}$

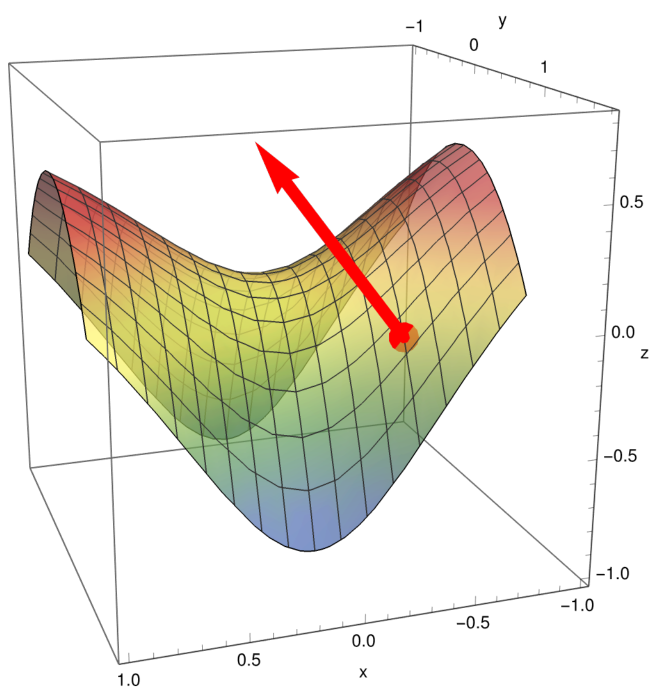
In[∘]:=
```
x0 = -0.5; y0 = 0.7;
P = {x0, y0, z[x0, y0]};
Print["Considered point has coordinates ", P]
```

Considered point has coordinates {-0.5, 0.7, -0.192}

In[∘]:=
```
n = -{zx[x0, y0], zy[x0, y0], -1};
Print["Normal vector has coordinates ", n]
p1 = Plot3D[z[x, y], {x, -1, 1}, {y, -1, 1}, PlotRange → All,
   ColorFunction → "DarkRainbow", PlotStyle → Opacity[0.7]];
pt = ListPointPlot3D[{P} → "Point", PlotStyle → {Red, PointSize[0.05]}];
p2 = Graphics3D[{Red, Thickness[0.02], Arrowheads[0.05], Arrow[{P, P + n}]}];
Show[p1, p2, pt, BoxRatios → {1, 1, 1}, AxesLabel → {"x", "y", "z"}]
```

Normal vector has coordinates {0.9536, 1.12, 1}

Out[∘]=



11. Definite and indefinite integrals. Multiple integrals.

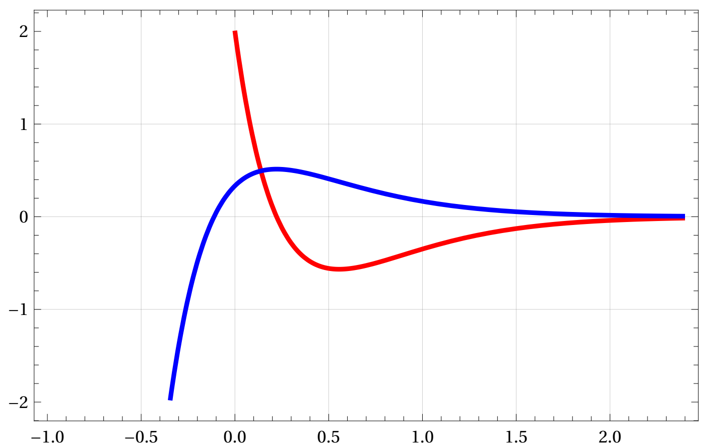
11.1. Calculating indefinite integrals.

Task: Calculate indefinite integral $\int e^{-3x}\left(2-9x\right)dx$. Build plot of an integrand and antiderivative.

In[∘]:=
```
f[x_] = Exp[-3 x] (2 - 9 x);
F[x_] = Integrate[f[x], x];
Print["Integral equals ", F[x]]
```

Integral equals $e^{-3x}\left(\frac{1}{3}+3x\right)$

In[∘]:=
```
Plot[{f[x], F[x]}, {x, -1, 2.4}, PlotTheme → "Scientific",
 PlotStyle → {{Red, Thickness[0.007]}, {Blue, Thickness[0.007]}}, GridLines → Automatic]
```

Out[∘]=


11.2. Indefinite integral of arbitrary functions

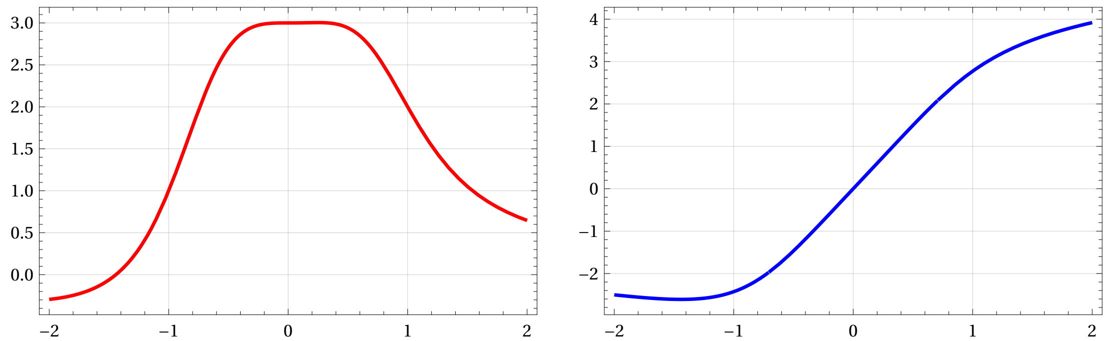
Task: Calculate indefinite integral $\int \frac{x^3+3}{x^4+1}dx$. Build integrand and antiderivative plots.

In[∘]:=
```
f[x_] = (x^3 + 3)/(x^4 + 1);
F[x_] = Simplify[∫ f[x] ⅆx];
Print["Integral equals to ", F[x]]
```

Integral equals to $\frac{1}{8}\left(-6\sqrt{2}\,\text{ArcTan}\left[1-\sqrt{2}\,x\right]+6\sqrt{2}\,\text{ArcTan}\left[1+\sqrt{2}\,x\right]-3\sqrt{2}\,\text{Log}\left[1-\sqrt{2}\,x+x^2\right]+3\sqrt{2}\,\text{Log}\left[1+\sqrt{2}\,x+x^2\right]+2\,\text{Log}\left[1+x^4\right]\right)$

In[∘]:=
```
p1 = Plot[f[x], {x, -2, 2}, PlotTheme → "Scientific",
    PlotStyle → {Red, Thickness[0.007]}, GridLines → Automatic];
p2 = Plot[F[x], {x, -2, 2}, PlotTheme → "Scientific",
    PlotStyle → {Blue, Thickness[0.007]}, GridLines → Automatic];
GraphicsRow[{p1, p2}]
```

Out[∘]=



11.3. Indefinite integral of rational functions

Task: Calculate indefinite integral $\int \frac{x^3 - 6x^2 + 10x - 10}{(x+1)(x-2)^3}\, d\,x$. Plot integrand and its antiderivative, excluding special points

In[∘]:=
$$f[x\_] = \frac{x^3 - 6\,x^2 + 10\,x - 10}{(x+1)\,(x-2)^3};$$

```
q[x_] = Apart[f[x]];
Print["After expanding fraction into simpler fractions, we get ", q[x]]
```

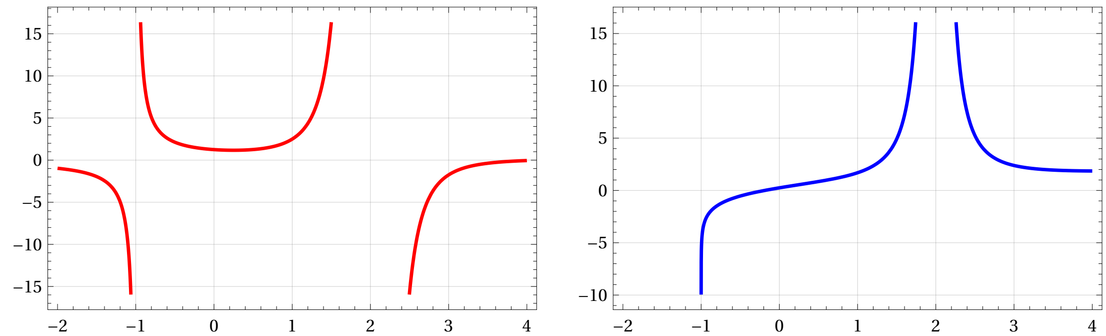
After expanding fraction into simpler fractions, we get $-\frac{2}{(-2+x)^3} + \frac{1}{1+x}$

In[∘]:=
```
F[x_] = Integrate[q[x], x];
Print["Integral equals ", F[x]]
```

Integral equals $\frac{1}{(-2+x)^2} + \mathrm{Log}[1+x]$

```
In[∘]:= p1 = Plot[f[x], {x, -2, 4}, PlotTheme → "Scientific",
            PlotStyle → {Red, Thickness[0.007]}, GridLines → Automatic];
        p2 = Plot[F[x], {x, -2, 4}, PlotTheme → "Scientific",
            PlotStyle → {Blue, Thickness[0.007]}, GridLines → Automatic];
        GraphicsRow[{p1, p2}]
```

Out[∘]=



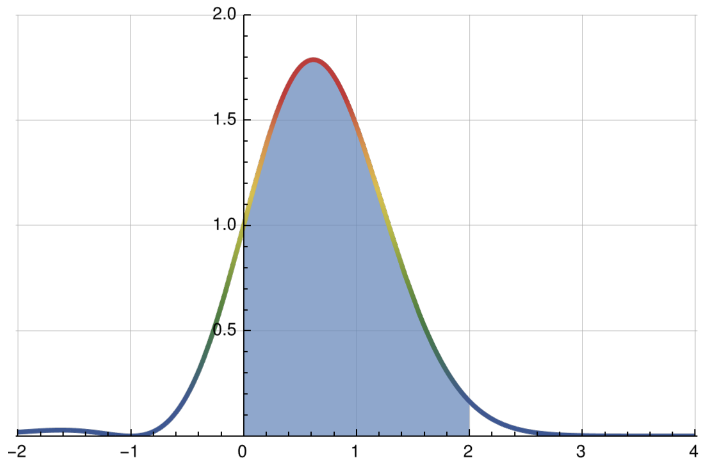
11.4. Definite integrals.

Task: Calculate definite integral $\int_0^2 (x^2 + 2x + 1)\, e^{-x^2}\, dx$. Plot integrand and highlight an integrated area.

```
In[∘]:= f[x_] = (x^2 + 2 x + 1) Exp[-x^2];
        a = 0; b = 2;
        s = N[Integrate[f[x], {x, a, b}]];
        Print["Value of integral is ", s]

        Value of integral is 2.28649
```

In[∘]:=
```
p1 = Plot[f[x], {x, a, b}, Filling → Axis, FillingStyle → Opacity[0.7], PlotRange → {0, 8}];
p2 = Plot[f[x], {x, a - 2, b + 2}, PlotTheme → "Scientific",
   PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"];
Show[p1, p2, PlotRange → {{a - 1.9, b + 1.9}, {0, 2}},
 AxesOrigin → {0, 0}, GridLines → Automatic]
```

Out[∘]=



11.5. Definite Integrals of arbitrary functions

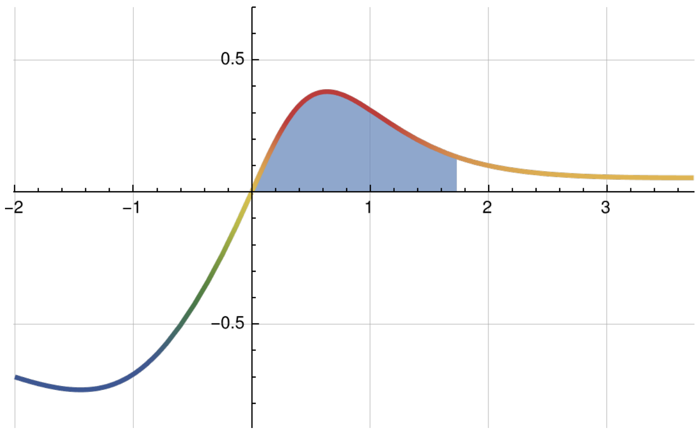
Task: Find definite integral $\int^{\sqrt{}} \frac{x - (\arctan x)}{1 + x}\, d\, x$. Plot integrand and area of integration.

In[∘]:=
```
f[x_] = (x - (ArcTan[x])^4) / (1 + x^2);
a = 0; b = Sqrt[3];
s = Integrate[f[x], {x, a, b}];
Print["Integral equals ", s, " and numerically ", N[s, 3]]
```

Integral equals $-\frac{\pi^{\hbar}}{1215}$ + Log[2] and numerically 0.441

In[∘]:=
```
p1 = Plot[f[x], {x, a, b}, Filling → Axis, FillingStyle → Opacity[0.7], PlotRange → {0, 8}];
p2 = Plot[f[x], {x, a - 2, b + 2}, PlotTheme → "Scientific",
   PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"];
Show[p1, p2, PlotRange → {{a - 1.9, b + 1.9}, {-0.9, 0.7}},
 AxesOrigin → {0, 0}, GridLines → Automatic]
```

Out[∘]=



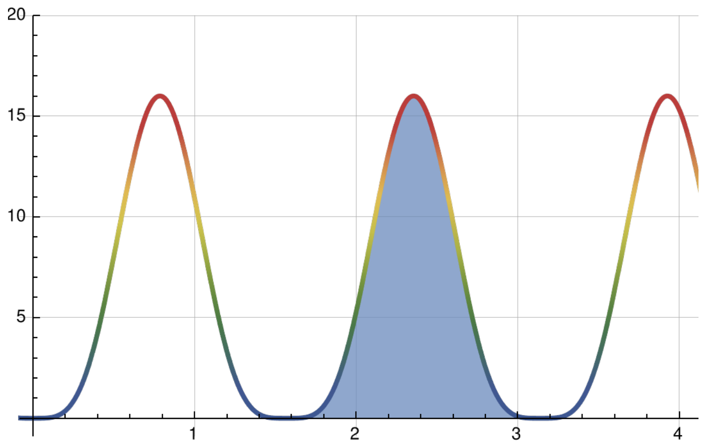
11.6. Definite integrals of trigonometric functions

Task: Calculate integral $\int_{\pi/}^{\pi} 2 \ \sin \ \cos \ x\,x\,d\,x$. Plot integrand and area of integration,

In[∘]:=
```
f[x_] = 2^8 (Sin[x])^4 (Cos[x])^4;
a = π/2; b = π;
s = Integrate[f[x], {x, a, b}];
Print["Value of this integral is ", s]
```

```
Value of this integral is 3 π
```

```
p1 = Plot[f[x], {x, a, b}, Filling → Axis,
    FillingStyle → Opacity[0.7], PlotRange → {0, 30}];
p2 = Plot[f[x], {x, -2, b + 1}, PlotTheme → "Scientific",
    PlotStyle → {Thickness[0.007]}, ColorFunction → "DarkRainbow"];
Show[p1, p2, PlotRange → {{0, b + 0.9}, {-0.9, 20.0}},
 AxesOrigin → {0, 0}, GridLines → Automatic]
```

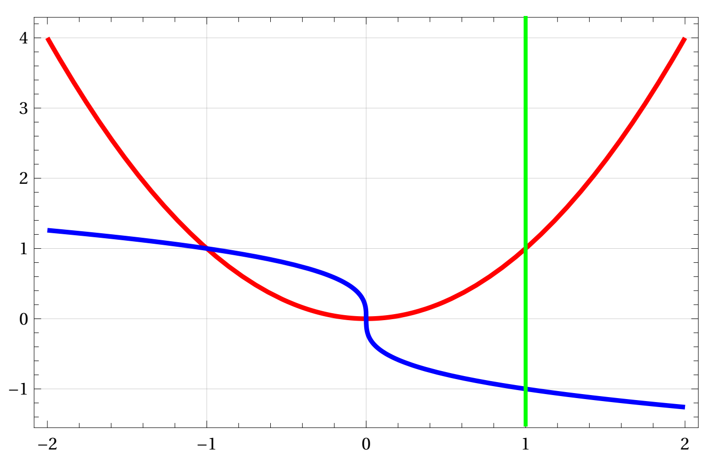


11.7. Multiple Integral

Task: Calculate double integral $\iint_g \left(\frac{4}{5}\,x\,y + \frac{9}{11}\,x\ \ y\right) d\,x\,d\,y$ for region D: $x = 1$, $y = x^2$, $y = -\sqrt[3]{x}$.

Plot region D.

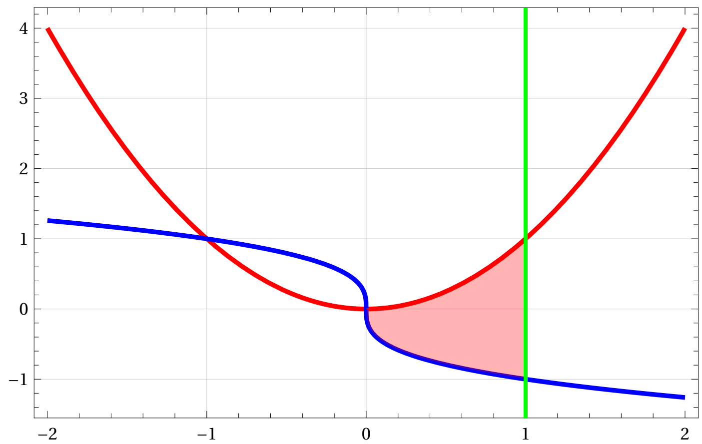
```
y1[x_] = x^2;
y2[x_] = -CubeRoot[x];
l = Line[{{1, -10}, {1, 10}}];
p1 = Plot[{y1[x], y2[x]}, {x, -2, 2}, PlotTheme → "Scientific",
   PlotStyle → {{Red, Thickness[0.007]}, {Blue, Thickness[0.007]}},
   GridLines → Automatic, Epilog → {Directive[{Thick, Green}], l}]
```



Therefore, we see that region D is bounded between $x = 0$ and $x = 1$

In[∘]:=
```
p2 = Plot[{y1[x], y2[x]}, {x, 0, 1}, PlotTheme → "Scientific",
   PlotStyle → {{Red, Thickness[0.002]}, {Blue, Thickness[0.002]}},
   Filling → {1 → {2}}, GridLines → Automatic];
Show[p1, p2]
```

Out[∘]=



That being said, we can rewrite our integral as $\int_0^1 \int_{-\sqrt[3]{x}}^{x^2} \left(\left(\frac{4}{5}\, x\, y + \frac{9}{11}\, x^2\, y^2\right) d\,y\right) d\,x.$

In[∘]:=
```
f[x_, y_] = 4/5 x y + 9/11 x^2 y^2;
s = Integrate[f[x, y], {x, 0, 1}, {y, -CubeRoot[x], x^2}];
Print["Integral equals ", s]
```

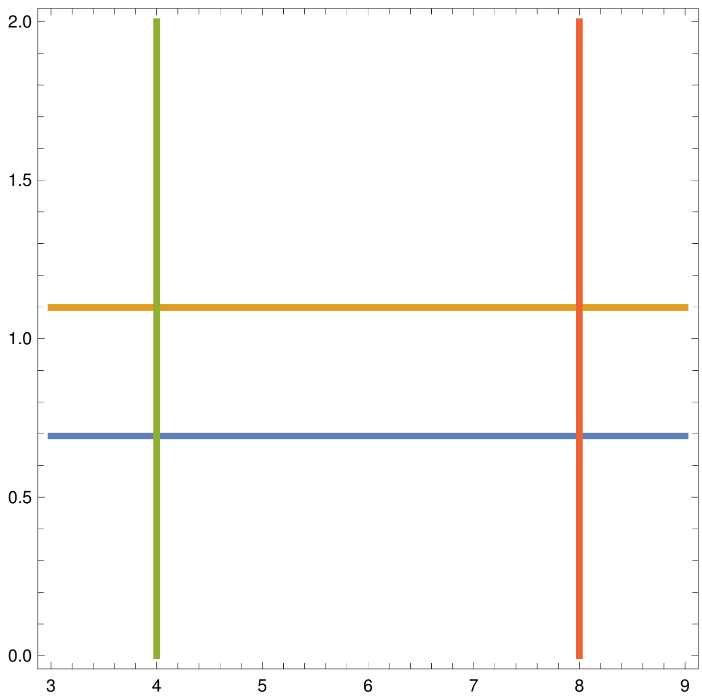
```
Integral equals 1/66
```

11.8. Calculating double integrals

Task: Evaluate integral $\iint_D y\, e^{x\,y/4}\, d\,x\, d\,y$ for region $D: y = \ln 2,\ y = \ln 3,\ x = 4,\ x = 8.$

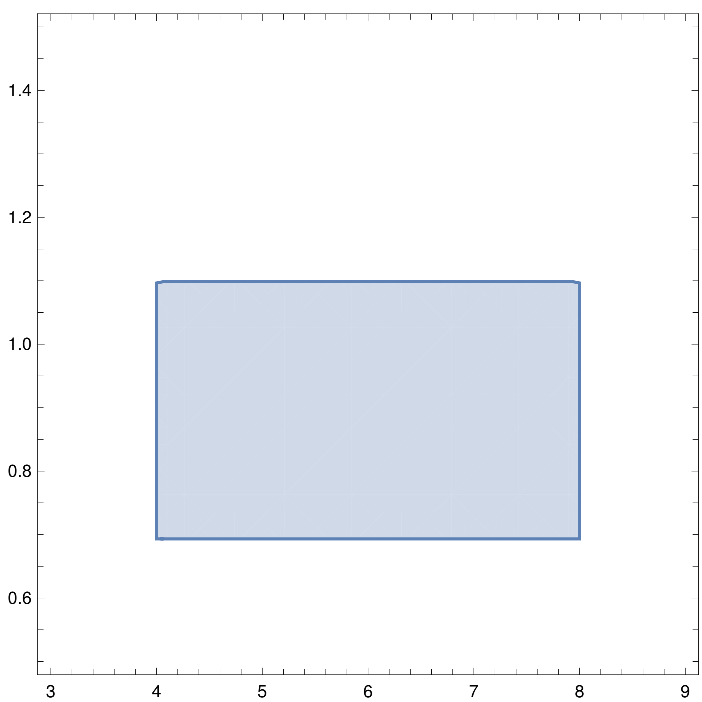
In[∘]:=
```
p1 = ContourPlot[{y == Log[2], y == Log[3], x == 4, x == 8},
  {x, 3, 9}, {y, 0, 2}, ContourStyle → Thickness[0.01]]
```

Out[∘]=



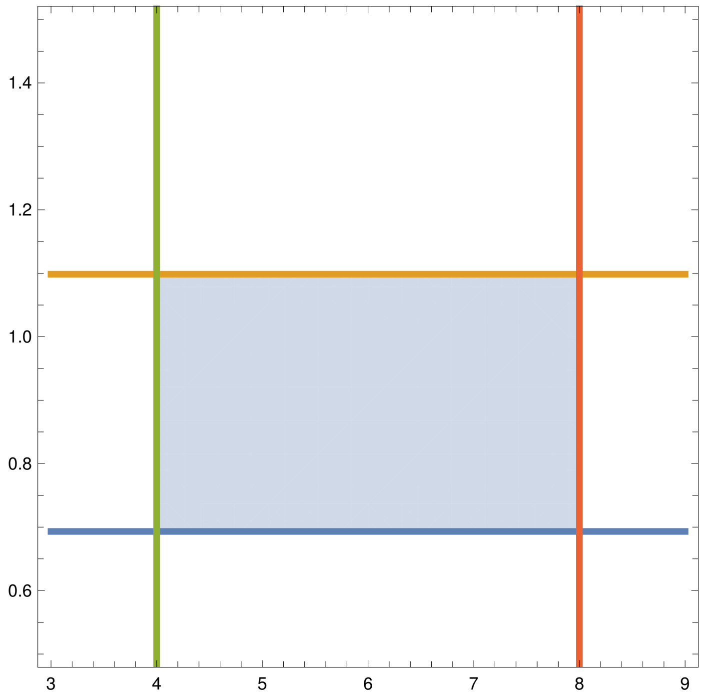
In[∘]:=
```
p2 = RegionPlot[y ≥ Log[2] && y ≤ Log[3] && x ≥ 4 && x ≤ 8, {x, 3, 9}, {y, 0.5, 1.5}]
```

Out[∘]=



In[∘]:=
```
Show[p2, p1]
```

Out[∘]=



Therefore, our integral becomes $\int_4^8 \left(\int_{\log 2}^{\log 3} y\, e^{x y/4}\, d y \right) d x.$

In[∘]:=
```
f[x_, y_] = y Exp[x y/4];
F = Chop[∫_4^8 ∫_Log[2]^Log[3] f[x, y] ⅆy ⅆx];
Print["Value of integral is ", F]
```

```
Value of integral is 6
```

12. Integral Calculus Application

12.1. Area between two explicitly defined curves

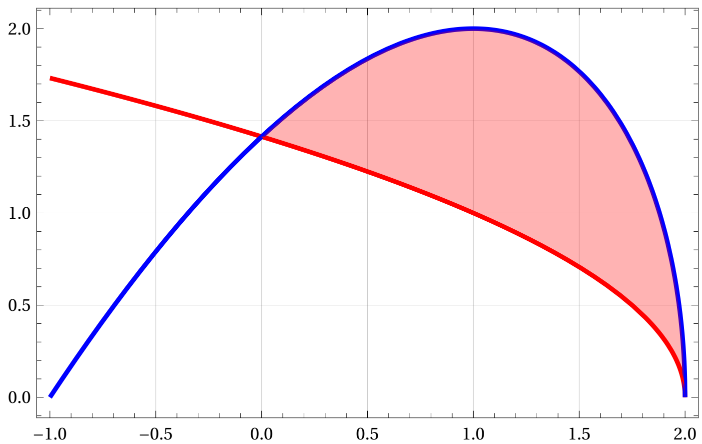
Task: Find area of region bounded by two curves $f_1(x) = \sqrt{2 - x}$,$f_2(x) = (x + 1)\sqrt{2 - x}$ between their two intersection points. Plot these curves and target area.

In[∘]:=
```
f1[x_] = Sqrt[2 - x];
f2[x_] = (x + 1) Sqrt[2 - x];
s = Solve[f1[x] == f2[x], x, Reals];
x1 = x /. s[[1]];
x2 = x /. s[[2]];
Print["Two intersection points are ", x1, " and ", x2]
```

```
Two intersection points are 0 and 2
```

In[∘]:=
```
p1 = Plot[{f1[x], f2[x]}, {x, -1, 2}, PlotTheme → "Scientific", PlotStyle →
    {{Red, Thickness[0.007]}, {Blue, Thickness[0.007]}}, GridLines → Automatic];
p2 = Plot[{f1[x], f2[x]}, {x, 0, 2}, PlotTheme → "Scientific",
   PlotStyle → {{Red, Thickness[0.002]}, {Blue, Thickness[0.002]}},
   Filling → {1 → {2}}, GridLines → Automatic];
Show[p1, p2]
```

Out[∘]=



In[∘]:=
```
s = Integrate[f2[x] - f1[x], {x, x1, x2}];
Print["Area of region equals ", s]
```

Area of region equals $\frac{16\sqrt{2}}{15}$

12.2. Finding length of explicitly defined curve

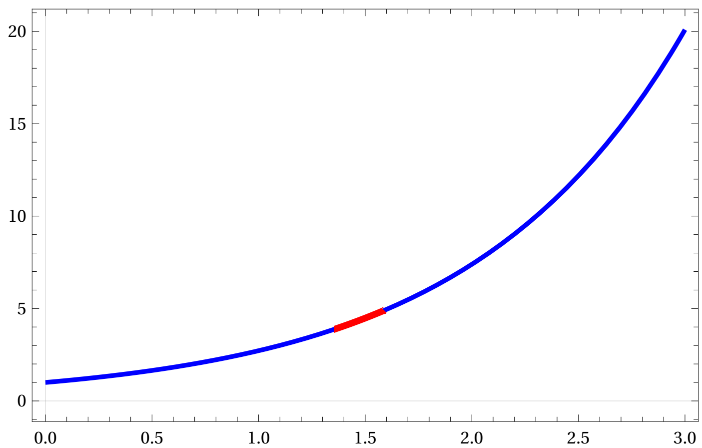
Task: Find length of a curve segment $y = e^x$, $\ln\sqrt{15} \le x \le \ln\sqrt{24}$. Build a curve and display segment on it.

In[∘]:=
```
y[x_] = Exp[x];
x1 = Log[Sqrt[15]]; x2 = Log[Sqrt[24]];
```
$L = \int_{x1}^{x2} \sqrt{1 + y'[x]^2}\, dx;$
```
Print["Length of a segment is ", L]
```

Length of a segment is 1 + ArcTanh[4] − ArcTanh[5]

In[∘]:=
```
p1 = Plot[y[x], {x, 0, 3}, PlotTheme → "Scientific", PlotStyle → {Blue, Thickness[0.007]}];
p2 = Plot[y[x], {x, x1, x2}, PlotTheme → "Scientific",
    PlotStyle → {Red, Thickness[0.01]}, GridLines → Automatic];
Show[p1, p2]
```

Out[∘]=



12.3. Finding length of a parameterically defined curve

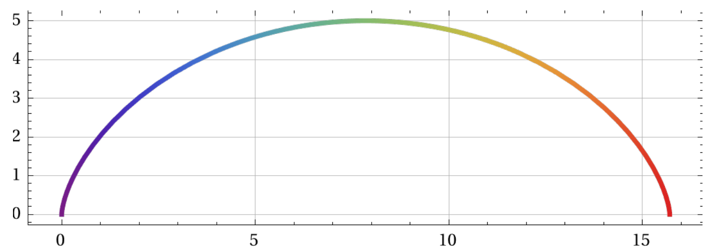
Task: Find length of curve $x(t) = \frac{5}{2}\left(t - \sin t\right)$, $y(t) = \frac{5}{2}\left(1 - \cos t\right)$, $t \in \left[0, 2\pi\right]$. Plot this segment.

In[∘]:=
$x[t\_] = \frac{5}{2}(t - \mathrm{Sin}[t]);$

$y[t\_] = \frac{5}{2}(1 - \mathrm{Cos}[t]);$
```
a = 0; b = 2 π;
L = Integrate[Sqrt[D[x[t], t]^2 + D[y[t], t]^2], {t, a, b}];
Print["Length of this segment is ", L]
```

Length of this segment is 20

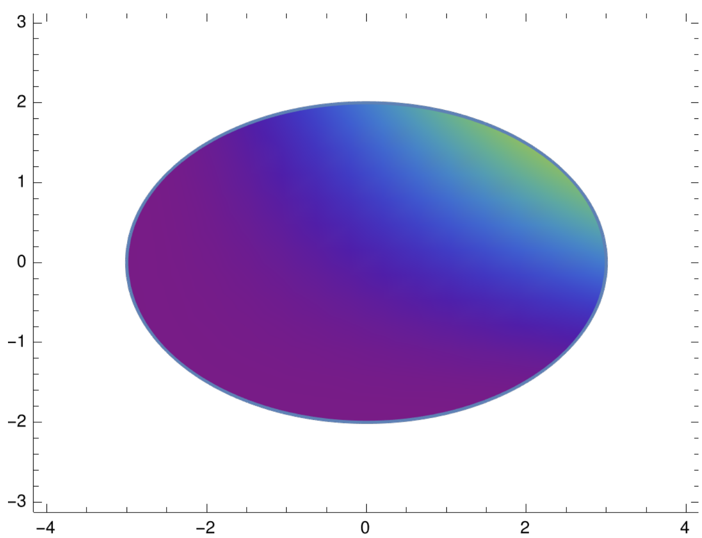
```
ParametricPlot[{x[t], y[t]}, {t, 0, 2 π},
 PlotTheme → "Scientific", ColorFunction → "Rainbow",
 PlotStyle → {Thickness[0.007], Blue}, GridLines → Automatic]
```



12.4. Finding plate’s mass

Task: Given plate’s surface density $\mu(x, y) = x \;\; y$ and region it takes on a plane $\frac{x}{9} + \frac{y}{4} \leq 1$, display it and find its mass.

```
R = x^2/9 + y^2/4 ≤ 1;
μ[x_, y_] = x^2 y^2;
RegionPlot[R, {x, -4, 4}, {y, -3, 3}, AspectRatio → Automatic,
 ColorFunction → Function[{x, y}, ColorData["Rainbow"][μ[x, y]]]]
```




```
mass = Integrate[μ[x, y] Boole[R], {x, -4, 4}, {y, -3, 3}];
Print["Plate's mass is ", mass]
```

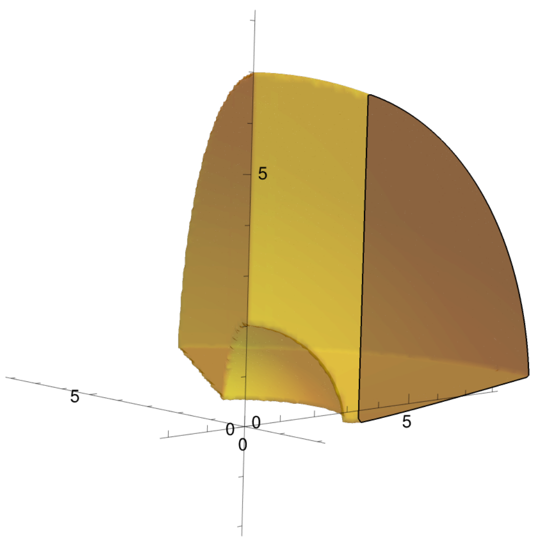
Plate's mass is 9 π

12.5. Finding body's volume

Task: Find volume of body $4 \le x^2 + y^2 + z^2 \le 49,\ -\sqrt{3}\ x < y < \sqrt{3}\ x,\ z \ge \sqrt{\frac{x^2 + y^2}{99}}$ and display it.

In[∘]:=
```
Clear[x, y, z]
R = (4 ≤ x^2 + y^2 + z^2 ≤ 49) && (-Sqrt[3] x < y < Sqrt[3] x) && (z ≥ Sqrt[(x^2 + y^2)/99]);
RegionPlot3D[R, {x, -2, 8}, {y, -2, 8}, {z, -2, 8}, PlotPoints → 100, Boxed → False,
 AxesOrigin → {0, 0, 0}, Mesh → None, PlotStyle → Directive[Yellow, Opacity[0.5]]]
```

Out[∘]=



In[∘]:=
```
v = Chop[NIntegrate[Boole[R], {x, -∞, +∞}, {y, -∞, +∞}, {z, -∞, +∞}]];
Print["Value of integral is ", v]
```

```
Value of integral is 210.487
```

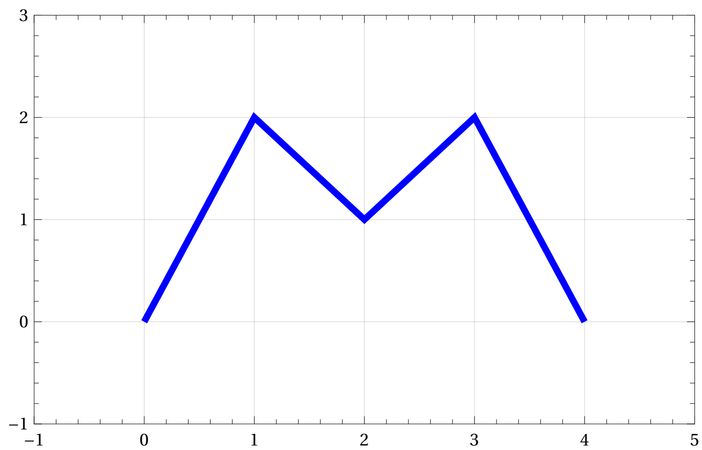
13. Interpolation and approximation

13.1. Curved lines

Task: Build a curved line consisting of vertices A_1 (0, 0), A_2 (1, 2), A_3 (2, 1), A_4 (3, 2), A_5 (4, 0)

In[24]:=
```
A = {{0, 0}, {1, 2}, {2, 1}, {3, 2}, {4, 0}};
ListPlot[A, PlotRange → {{-1, 5}, {-1, 3}}, PlotStyle → {Thickness[0.01], Blue},
 Joined → True, GridLines → Automatic, PlotTheme → "Scientific"]
```

Out[25]=



14. Solving Differential Equations

14.1. First order differential equations

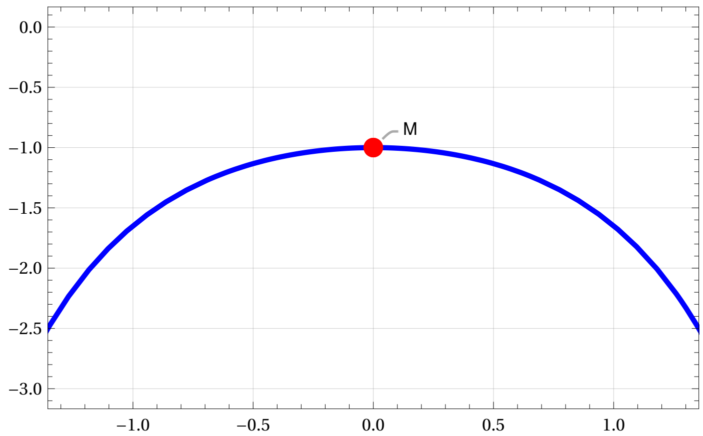
Task: Using *DSolve*, find integral curve of the differential equation $\frac{dy}{dx} = x\,y$ passing through $M(0, -1)$.

In[29]:=
```
Remove[x, y];
r = DSolve[{y'[x] == x y[x], y[0] == -1}, y[x], x];
y[x_] = Simplify[y[x] /. r[[1]]];
Print["Solution to differential equation y'=xy is ", y[x]]
```

Solution to differential equation y'=xy is $-e^{\frac{x^2}{2}}$

In[75]:=
```
p1 = Plot[y[x], {x, -2, 2}, PlotStyle → {Blue, Thickness[0.008]},
    PlotTheme → "Scientific", GridLines → Automatic];
p2 = ListPlot[{{0, -1}} → {"M"}, PlotStyle → {Red, PointSize[0.03]}];
Show[p1, p2, PlotRange → {{-1.3, 1.3}, {-3, 0}}]
```

Out[77]=



14.2. Second order differential equations with constant coefficients

Task: Using DSolve, find the general solution to DE $\frac{d^2 y}{d x^2} + 2\frac{d y}{d x} = 10\, e^x \left(\sin x + \cos x\right)$. Set to arbitrary constants values $C_1 = 1$, $C_2 = -1$ and build a plot for this particular case.

In[78]:=
```
Remove[x, y];
r = DSolve[y''[x] + 2 y'[x] == 10 Exp[x] (Sin[x] + Cos[x]), y[x], x]
```

Out[79]= $\left\{\left\{y[x] \to -\frac{1}{2} e^{-2x} c_1 + c_2 - e^x \text{Cos}[x] + 3 e^x \text{Sin}[x]\right\}\right\}$

In[80]:=
```
f = y[x] /. r[[1]];
Print["Solution to the given DE is ", f]
```

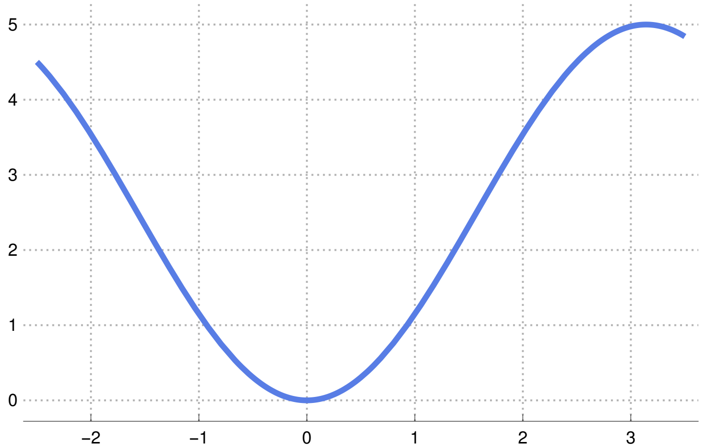
Solution to the given DE is $-\frac{1}{2} e^{-2x} c_1 + c_2 - e^x \text{Cos}[x] + 3 e^x \text{Sin}[x]$

In[82]:=
```
y[x_] = f /. {C[1] → 1, C[2] → -1};
Print["When C₁=1,C₂=-1 we obtain ", y[x]]
```

When C_1=1,C_2=-1 we obtain $-1 - \frac{e^{-2x}}{2} - e^x \text{Cos}[x] + 3 e^x \text{Sin}[x]$

In[160]:=
```
Plot[y[x], {x, -2.5, 3.5}, PlotTheme → "Business", GridLines → Automatic]
```

Out[160]=



14.3. Third order differential equation with constant coefficients

Task: Using function *DSolve*, find general solution of equation $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = (6x - 11)e^{-D}$.

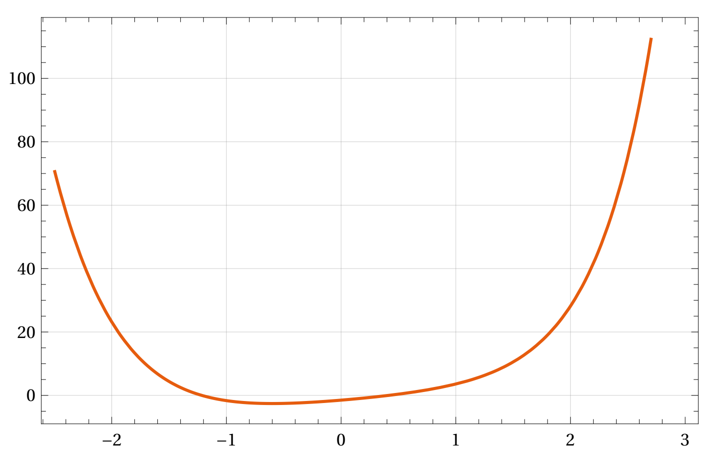
Set all constants to 1 and plot of solution.

In[136]:=
```
Remove[x, y, r, f];
r = DSolve[y'''[x] - y''[x] - 2 y'[x] == (6 x - 11) Exp[-x], y[x], x];
yc[x_] = y[x] /. r[[1]];
y[x_] = yc[x] /. {C[1] → 1, C[2] → 1, C[3] → 1};
Print["Answer to the given question is ", y[x]]
```

Answer to the given question is $1 + \frac{e^{2x}}{2} + e^{-x}(-3 - x + x^2)$

In[145]:=
```
Plot[y[x], {x, -2.5, 3}, PlotTheme → "Scientific", GridLines → Automatic]
```

Out[145]=



14.4. Cauchy problem with constant coefficients

Task: Using *DSolve*, solve and plot a solution to $\frac{d^2 y}{d x^2} + 9 y = \frac{9}{\sin 3 x}$, $y(0) = 1$, $\frac{d y}{d x}(0) = 0$.

In[148]:=
```
Remove[x, y];
r = DSolve[{y''[x] + 9 y[x] == 9/Sin[3 x], y[0] == 1, y'[0] == 0}, y[x], x]
```

Out[149]=
```
{}
```