1 Simple line intersection

We want to find whether or not two lines intersect, there are plenty of ways to do that, but my preferred way is to find two scalars t and u and check their range to know if two lines/rays do intersect.

$$\vec{A_s} + t \cdot \vec{A_d} = \vec{B_s} + u \cdot \vec{B_d}$$

Where $\vec{A_s}$ is the starting position of our line and $\vec{A_d}$ is its <u>not normalized</u> direction, similarly for $\vec{B}$. Rearranging the equation:

$$t \cdot \vec{A_d} - u \cdot \vec{B_d} = \vec{B_s} - \vec{A_s}$$

Rewriting it in matrix/vector form:

$$t \begin{bmatrix} A_{dx} \\ A_{dy} \end{bmatrix} - u \begin{bmatrix} B_{dx} \\ B_{dy} \end{bmatrix} = \begin{bmatrix} B_{sx} \\ B_{sy} \end{bmatrix} - \begin{bmatrix} A_{sx} \\ A_{sy} \end{bmatrix}$$

$$\begin{bmatrix} tA_{dx} - uB_{dx} \\ tA_{dy} - uB_{dy} \end{bmatrix} = \begin{bmatrix} B_{sx} - A_{sx} \\ B_{sy} - A_{sy} \end{bmatrix}$$

$$\begin{bmatrix} t \\ u \end{bmatrix} \begin{bmatrix} A_{dx} & -B_{dx} \\ A_{dy} & -B_{dy} \end{bmatrix} = \begin{bmatrix} B_{sx} - A_{sx} \\ B_{sy} - A_{sy} \end{bmatrix}$$

Since we're looking for two scalars t and u, we can do the following:

$$A = \begin{bmatrix} A_{dx} & -B_{dx} \\ A_{dy} & -B_{dy} \end{bmatrix} \qquad det(A) = -A_{dx}B_{dy} + A_{dy}B_{dx}$$

$$\begin{bmatrix} t \\ u \end{bmatrix} = \begin{bmatrix} B_{sx} - A_{sx} \\ B_{sy} - A_{sy} \end{bmatrix} \begin{bmatrix} A_{dx} & -B_{dx} \\ A_{dy} & -B_{dy} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} t \\ u \end{bmatrix} = \begin{bmatrix} B_{sx} - A_{sx} \\ B_{sy} - A_{sy} \end{bmatrix} \cdot \frac{1}{det(A)} \begin{bmatrix} -B_{dy} & B_{dx} \\ -A_{dy} & A_{dx} \end{bmatrix}$$

The rest of the work is simple algebra and rearranging terms:

$$\begin{bmatrix} t \\ u \end{bmatrix} = \frac{1}{det(A)} \begin{bmatrix} -B_{dy}(B_{sx} - A_{sx}) + B_{dx}(B_{sy} - A_{sy}) \\ -A_{dy}(B_{sx} - A_{sx}) + A_{dx}(B_{sy} - A_{sy}) \end{bmatrix}$$

Through this we obtain the following formulas for our scalars:

$$t = \frac{1}{det(A)} \cdot [-B_{dy}(B_{sx} - A_{sx}) + B_{dx}(B_{sy} - A_{sy})]$$
$$u = \frac{1}{det(A)} \cdot [-A_{dy}(B_{sx} - A_{sx}) + A_{dx}(B_{sy} - A_{sy})]$$

2 Some explanation

Going through all of this the first time, I was a little confused as to why it all works, that's why I wrote this section to explain some of the logic behind the above formulas.

Why is our direction vector not normalized? We can think of our $\vec{A_s}$ (starting point) as a translation of the origin point $(0; 0)$ (moving the origin point towards where $\vec{A_s}$ is). Our direction vector $\vec{A_d}$ then is a step from our new origin point, because of that we do not need to normalize it, we just care about the scalar t being in range between $[0; 1]$, that way, if it exceeds 1 or goes below 0, we know that the intersection is impossible, because the line would have to either swap its direction or grow in size.

What is the first equation actually telling us? For that I highly encourage anyone interested to play with the equation in 'desmos calculator', visualisation like that helps tremendously when learning new math concepts or when trying to figure out a new problem. This equation is telling us that given two lines ($\vec{A}$ and $\vec{B}$) we can find two scalars (t and u) so that when we perform the operation $\vec{A_s} + t \cdot \vec{A_d}$ or $\vec{B_s} + u \cdot \vec{B_d}$ we can find their [lines'] intersection point.