GUARANTEED PERFORMANCE HEURISTICS FOR THE BOTTLENECK TRAVELING SALESMAN PROBLEM

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Received August 1983
Revised October 1983

We consider constant-performance, polynomial-time, nonexact algorithms for the minimax or bottleneck version of the Traveling Salesman Problem. It is first shown that no such algorithm can exist for problems with arbitrary costs unless P = NP. However, when costs are positive and satisfy the triangle inequality, we use results pertaining to the squares of biconnected graphs to produce a polynomial-time algorithm with worst-case bound 2 and show further that, unless P = NP, no polynomial alternative can improve on this value.

Graphs, combinatorics, traveling salesman, heuristic

1. Introduction

Let $G(\mathscr{V}, \mathscr{E})$ be a complete undirected graph of order $|\mathscr{V}| \geqslant 3$ with weights c_{ij} on every edge (i, j) in $\mathscr{E}$. Traveling Salesman Problems are defined over hamiltonian cycles in G (i.e. simple cycles including all vertices). The classic minisum version of the problem is

$$\min\Big\{ \sum_{(i,j)\in\mathscr{H}} c_{ij}: \mathscr{H} \text{ is the edge set of a hamiltonian cycle of } G \Big\}.$$

Its cousin, the minimax or *Bottleneck Traveling Salesman Problem* (BTSP) is

$$\min\Big\{ \max_{(i,j)\in\mathscr{H}} c_{ij}: \mathscr{H} \text{ is the edge set of a hamiltonian cycle of } G \Big\}.$$

It is easy to see that a polynomial-time algorithm for (BTSP) would provide a polynomial-time mechanism for testing whether arbitrary graphs are hamiltonian. Since the latter is a classic and formally difficult problem, exact polynomial-time algorithms for (BTSP) cannot exist unless P = NP.

It is natural, then, to seek *polynomial-time*, nonexact algorithms with *constant performance bounds*, i.e. worst-case bounds independent of problem parameters. In spite of the wide literature of such algorithms for the minisum Traveling Salesman Problem (see for example Parker and Rardin [5]), and the treatment of heuristic algorithms for (BTSP) in Garfinkel and Gilbert [4], we know of no previous constant-performance-bound, polynomial-time heuristic for (BTSP).

In this note we investigate such algorithms. Our main result is a procedure with worst-case bound 2 holding when costs are positive and satisfy the triangle inequality. We also show that it is not likely that this bound will be reduced by any alternative, polynomial algorithm.

2. Arbitrary costs

Sahni and Gonzales [7] demonstrated that, unless P = NP, the minisum Traveling Salesman

The work reported was partially supported by the National Science Foundation under Grant ECS-8018954

Problem admits no constant-performance-bound, polynomial-time algorithm when costs are arbitrary. A corresponding result holds for (BTSP).

Theorem 1. *There can exist no polynomial-time, constant-performance-bound algorithm for an arbitrary instance of* (BTSP), *unless* P = NP.

Proof. We proceed by showing that if the indicated algorithm, A, with finite bound ρ did exist, it could be employed to test hamiltonicity in arbitrary graphs – proving P = NP. Assume $\Omega_A/\Omega^* \leqslant \rho < +\infty$ where Ω_A is the value produced by algorithm A and Ω^* is an optimal value. Now, for an arbitrary graph $G(\mathscr{V}, \mathscr{E})$, we can construct a corresponding instance of (BTSP) by completing the graph and assigning weights

$$c_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \mathscr{E}, \\ \rho + 1 & \text{if } (i,j) \notin \mathscr{E}. \end{cases}$$

Suppose G is hamiltonian. Then the corresponding instance of (BTSP) we have $\Omega^* = 1$ and hence $\Omega_A \leqslant \rho$. Conversely, if G is not hamiltonian, then $\Omega^* = \rho + 1$ which implies that $\Omega_A > \rho$. Thus G is hamiltonian precisely when Ω_A is not greater than ρ, and algorithm A provides a polynomial-time procedure for deciding which graphs are hamiltonian. □

3. An algorithm

The negative result of Theorem 1 makes very unlikely a polynomial time, constant-performance-bound algorithm for arbitrary instances of (BTSP). However, we can derive one under more restricted costs.

3.1. Biconnected subgraphs

A graph is said to be *biconnected* if every pair of its vertices belongs to at least one common cycle. For a given biconnected graph $G(\mathscr{V}, \mathscr{E})$ we can define the *Bottleneck Biconnected Subgraph* problem (BBS) as

$$\min\left\{ \max_{(i,j)\in\mathscr{S}} c_{i,j}\colon G(\mathscr{V},\mathscr{S}) \text{ is biconnected}, \mathscr{S} \subset \mathscr{E} \right\}.$$

It is easy to see that (BBS) provides a lower bound on (BTSP).

Lemma 1. *For* Ω^* = *the optimal value of* (BTSP) *and* Ω_{BB} *optimal in* (BBS), $\Omega_{BB} \leqslant \Omega^*$.

Proof. Immediate from the fact that every hamiltonian cycle of a $G(\mathscr{V}, \mathscr{E})$ is a biconnected subgraph. □

Problem (BBS) is also very easily solved. A straight-forward greedy procedure gives a polynomial-time algorithm:

Algorithms BB (*weighted biconnected graph* $G(\mathscr{V}, \mathscr{E})$)

Step 0: Initialization. Sort edges of $\mathscr{E}$ into non-decreasing order by edge weight c_{ij} and initialize solution set $\mathscr{E}_{BB} \leftarrow \emptyset$.

Step 1: Augmentation. Select the next edge in order of the sorted list and place it in $\mathscr{E}_{BB}$.

Step 2: Stopping. Test whether $G(\mathscr{V}, \mathscr{E}_{BB})$ is biconnected. If so, compute

$$\Omega_{BB} \leftarrow \max\{ c_{ij}\colon (i,j) \in \mathscr{E}_{BB} \}$$

and stop. Otherwise, repeat Step 1. □

Lemma 2. *Algorithm* BB *correctly computes an* $\mathscr{E}_{BB}$ *optimal in* (BBS) *in time bounded by a polynomial in* $|\mathscr{E}|$.

Proof. The $\mathscr{E}_{BB}$ solution obtained from Algorithm BB is obviously optimal because $G(\mathscr{V}, \mathscr{E}_{BB})$ is biconnected and construction shows every subgraph with lesser bottleneck cost is not. For polynomiality, note that Step 0 is a sort requiring $(|\mathscr{E}| \log |\mathscr{E}|)$ time. Steps 1 and 2 are executed on at most $|\mathscr{E}|$ occasions, and the required check of biconnectedness at Step 2 can be done in $O(|\mathscr{E}|)$ time (see e.g. Aho, Hopcroft and Ullman [1]). Thus, the algorithm completes in at most $O(|\mathscr{E}|^2)$ time. □

3.2. Hamiltonian cycles in the squares of graphs

For an arbitrary graph $G(\mathscr{V}, \mathscr{E})$ the *square* $G^2(\mathscr{V}, \mathscr{E}^2)$ is the graph formed by adding 'short cut' edges for every two edge path. That is, $G^2(\mathscr{V}, \mathscr{E}^2)$ has the same vertex set as G, and edge set

$$\mathscr{E}^2 \triangleq \mathscr{E} \cup \{(i,k)\colon (i,j,k) \text{ is a path of } G(\mathscr{V},\mathscr{E}) \text{ for some } j \in \mathscr{V}\}.$$

The two graphs in Figure 1 illustrate the concept.

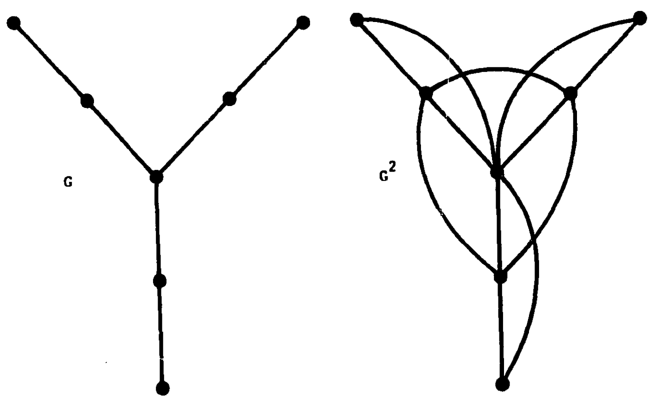

Fig. 1. A graph and its square.

Neither the first graph in Figure 1 nor its square are hamiltonian. In fact, the tree shown establishes that connectivity in a graph is not enough to guarantee hamiltonicity of its square. If we require G to be biconnected however, the matter is different.

Lemma 3 (Fleischner [3]). *The square of any biconnected graph is hamiltonian.* □

The fact that Lemma 3 holds was conjectured by Nash-Williams and later proved by Fleischner. Fleischner's proof is an existence one, but it yields algorithmic insights. In Rardin and Parker [6], we show explicitly how an algorithm can be devised from those insights to exhibit a hamiltonian circuit in the square of any biconnected graph.

Details of the procedure are far to bulky to include here. However, the approach is to derive from the given biconnected graph a particular connected and spanning subgraph possessing structural properties sufficient to make easy the construction of a hamiltonian cycle in its square. These subgraphs are defined by the edge-disjoint union of an Euler subgraph and a foreset of vertex-disjoint paths. Fleischner [2] proved that any biconnected, bridgeless graph possesses such a subgraph and outlined how to identify a hamiltonian cycle in its (and thus the original graph's) square when, in addition, every edge meets at least one degree-2 vertex. The companion paper [3] inductively treats a large number of cases in demonstrating that subgraphs with the needed degree-2 property can be obtained via suitable contraction.

Discussion in Rardin and Parker [6] shows that at each step of these constructions, the cardinality of at least one specified edge on vertex subset is reduced. Since steps themselves involve only polynomial exercises such as identifying the biconnected blocks of a graph, finding shortest paths and exhibiting Euler traversals of given Euler subgraphs, polynomiality of the entire algorithm is guaranteed. We summarize:

Lemma 4. *Given any biconnected graph $G(\mathscr{V}, \mathscr{E})$, a hamiltonian cycle $\mathscr{H} \subset \mathscr{E}^2$ can be produced in the square $G^2(\mathscr{V}, \mathscr{E}^2)$ of G in time bounded by a polynomial in $|\mathscr{V}|$ and $|\mathscr{E}|$.* □

3.3. *The algorithm*

We are now ready to specify our nonexact algorithm for (BTSP).

Algorithm BT (*weighted complete graph* $G(\mathscr{V}, \mathscr{E})$)

Step 1: Bottleneck-optimal Biconnected Subgraph. Apply Algorithm BB above to obtain $G(\mathscr{V}, \mathscr{E}_{BB})$, a bottleneck-optimal biconnected subgraph of $G(\mathscr{V}, \mathscr{E})$.

Step 2: Tour. Identify an approximate optimal tour for (BTSP) by tracing a hamiltonian cycle, $\mathscr{H}_{BT}$, in the square $G^2(\mathscr{V}, \mathscr{E}^2_{BB})$ of the result from Step 1, and define

$$\Omega_{BT} \triangleq \max\{c_{ij}: (i, j) \in \mathscr{H}_{BT}\}. \quad \square$$

The algorithm certainly produces a feasible solution to (BTSP). Moreover, its polynomiality follows from Lemmas 2 and 4.

4. Performance bounds under the triangle inequality

Costs satisfy the *triangle inequality* if $c_{ij} + c_{jk} \geqslant c_{ik}$ for all $i, j, k \in \mathscr{V}$. Results of the previous section allow us to establish a constant worst-case bound on the performance of Algorithm BT in the presence of the triangle inequality.

Theorem 2. *Let $G(\mathscr{V}, \mathscr{E})$ be a complete undirected graph with positive weights c_{ij} satisfying the triangle inequality. Then, if Ω^* is the optimal value of* (BTSP) *on G, and Ω_{BT} the value produced by applying Algorithm* BT *to G,*

$$\Omega_{BT}/\Omega^* \leqslant 2.$$

Proof. By Lemma 1, Ω_{BB}, the value of the bottleneck-optimal biconnected subgraph produced at Step 1 of Algorithm BT satisfies $\Omega_{BB} \leqslant \Omega^*$ or $2\Omega_{BB}/\Omega^* \leqslant 2$. But edges of $\mathscr{H}_{BT}$, the hamiltonian cycle obtained from Algorithm BT, either belong to $\mathscr{E}_{BB}$, the optimal edge set from Algorithm BB, or correspond to two-edge paths of $\mathscr{E}_{BB}$. Under the triangle inequality no edge of $\mathscr{H}_{BT}$ can thus cost more than $2\Omega_{BB}$. That is, $\Omega_{BT} \leqslant 2\Omega_{BB}$ and the proof is complete. □

One needs only to assign weights 1 and 2 suitably to show the bound of Theorem 2 is realizable. Naturally, of course, we would prefer a smaller value than 2. Our last result shows none is likely.

Theorem 3. *Let A be any polynomial-time algorithm yielding nonexact solutions for* (BTSP) *and Ω_A the value of solutions produced by A. If there exists a constant ρ such that $\Omega_A/\Omega^* \leqslant \rho$ for all* (BTSP) *instances satisfying the hypothesis of Theorem* 2, *unless* P = NP, $\rho \geqslant 2$.

Proof. As with Theorem 1, we show that an Algorithm A with worst-case performance bound $\rho < 2$ could be used to test hamiltonicity of arbitrary graphs – proving P = NP. Here we choose costs

$$c_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \mathscr{E}, \\ 2 & \text{otherwise} \end{cases}$$

in completing the graph. Clearly, the indicated c_{ij} satisfy the triangle inequality. Over these costs an Algorithm A with bound $\rho < 2$ would yield $\Omega_A < 2$ precisely when the given graph is hamiltonian and $\Omega_A \geqslant 2$ otherwise. □

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