COMP30026 Models of Computation

Pushdown Automata

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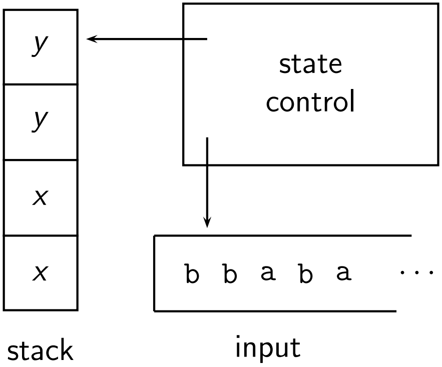
Pushdown Automata

The automata we saw so far were limited by their lack of memory.

A pushdown automaton (PDA) is a finite-state automaton, equipped with a stack.

The language $\{a^i b^i \mid i \geq 0\}$ is not recognised by any DFA, since it requires the ability of a recogniser to remember how many consecutive as have been consumed from the input.



Fine but Important Points

We shall consider the non-deterministic version of a PDA.

It may, in one transition step, read a symbol from input and pop the top stack symbol.

Based on these, and the current state, it will decide on which state to go to next, as well as what to push on the stack.

It may choose to leave out the popping, or the pushing, or both.

It may also ignore the input.

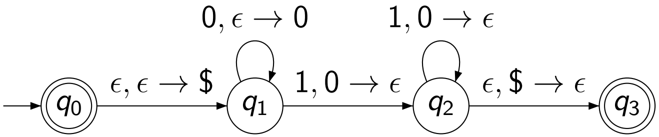
Pushdown Automata Formally

A pushdown automaton is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where

- Q is a finite set of states,
- Σ is the finite input alphabet,
- Γ is the finite stack alphabet,
- $\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \to \mathcal{P}(Q \times \Gamma_\epsilon)$ is the transition function,
- $q_0 \in Q$ is the start state, and
- $F \subseteq Q$ are the accept states.

PDA Example 1

This PDA recognises $\{0^n1^n \mid n \geq 0\}$:



Acceptance Precisely

The PDA $(Q, \Sigma, \Gamma, \delta, q_0, F)$ accepts input w iff $w = v_1 v_2 \cdots v_n$ with each $v_i \in \Sigma_\epsilon$, and there are states $r_0, r_1, \ldots, r_n \in Q$ and strings $s_0, s_1, \ldots, s_n \in \Gamma^*$ such that

1. $r_0 = q_0$ and $s_0 = \epsilon$.
2. $(r_{i+1}, b) \in \delta(r_i, v_{i+1}, a)$, $s_i = at$, $s_{i+1} = bt$ with $a, b \in \Gamma_\epsilon$ and $t \in \Gamma_\epsilon^*$.
3. $r_n \in F$.

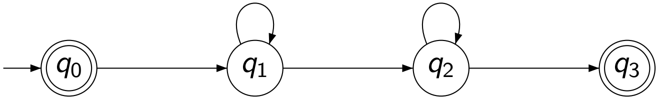
Note 1: There is no requirement that $s_n = \epsilon$, so the stack may be non-empty when the machine stops (even when it accepts).

Note 2: Trying to pop an empty stack leads to rejection of input, rather than "runtime error".

PDA Example 2

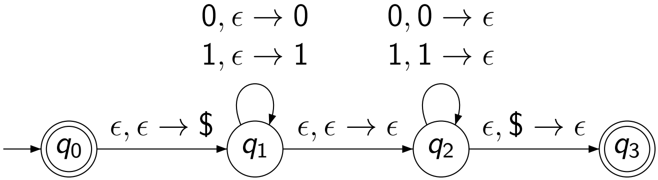
Let $w^{\mathcal{R}}$ denote the string w reversed.

Let us design a PDA to recognise $\{ww^{\mathcal{R}} \mid w \in \{0,1\}^*\}$, the set of even-length binary palindromes:



PDA Example 2

This PDA recognises $\{ww^{\mathcal{R}} \mid w \in \{0,1\}^*\}$:


$$\begin{tikzcd}
\arrow[r] & q_0 \arrow[r, "{\epsilon, \epsilon \to \$}"] & q_1 \arrow[loop above, "{0, \epsilon \to 0 \atop 1, \epsilon \to 1}"] \arrow[r, "{\epsilon, \epsilon \to \epsilon}"] & q_2 \arrow[loop above, "{0, 0 \to \epsilon \atop 1, 1 \to \epsilon}"] \arrow[r, "{\epsilon, \$ \to \epsilon}"] & q_3
\end{tikzcd}$$

CFLs Have PDAs as Recognisers

Given a context-free language L (in the form of a grammar), we can find a PDA which recognises L.

And, every PDA recognises a context-free language.

We won't prove the second claim, but the first claim can easily be seen to hold.

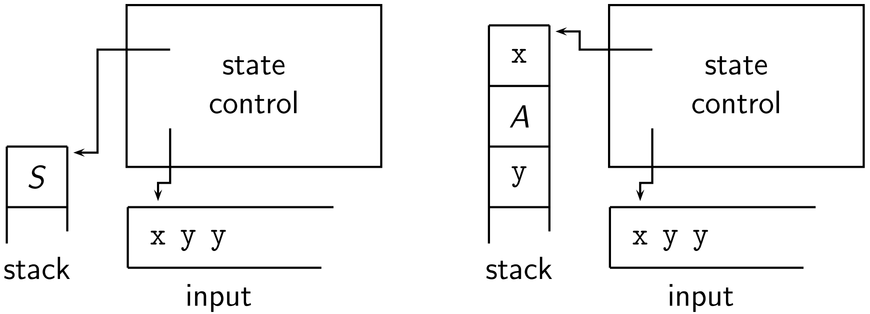
Namely, given a CFG G, we show how to construct a PDA P such that $L(P) = L(G)$.

The idea is to let the PDA use its stack to store a list of "pending" recogniser tasks.

The construction does not give the cleverest PDA, but it always works.

From Context-Free Grammars to PDAs

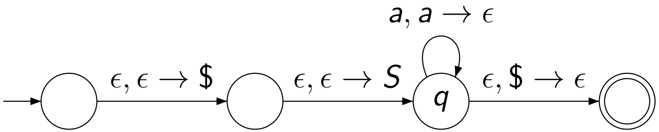
Say $S \rightarrow xAy$ is a rule in G, and the PDA finds the symbol S on top of its stack, it may pop S and push y, A, and x, in that order.



If it finds the terminal x on top of the stack, and x is the next input symbol, it may consume the input and pop x.

From Context-Free Grammars to PDAs

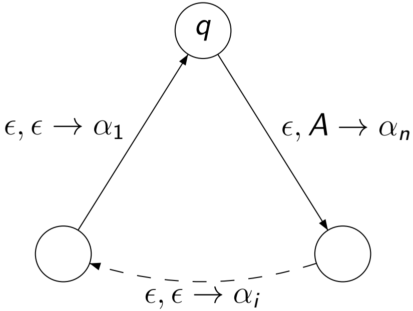
Construct the PDA like this:



$a, a \rightarrow \epsilon$
$\epsilon, \epsilon \rightarrow \$$
$\epsilon, \epsilon \rightarrow S$
q
$\epsilon, \$ \rightarrow \epsilon$

with a self-loop from q for each terminal a.

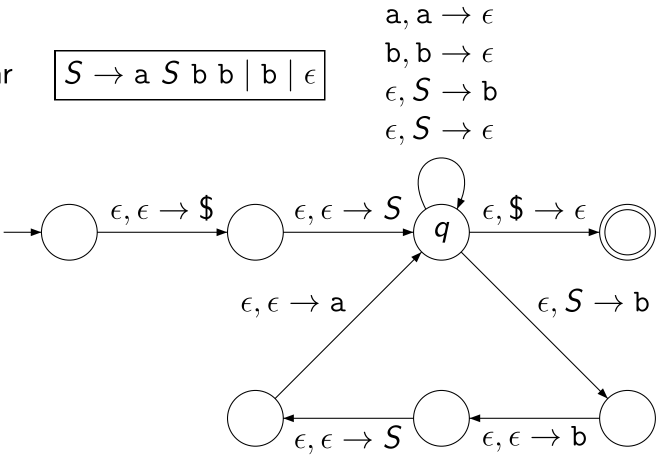
Add, for each rule $A \rightarrow \alpha_1 \ldots \alpha_n$,
the self-loop shown here:



q
$\epsilon, \epsilon \rightarrow \alpha_1$
$\epsilon, A \rightarrow \alpha_n$
$\epsilon, \epsilon \rightarrow \alpha_i$

Example Recogniser

For the grammar $\boxed{S \rightarrow \texttt{a}\ S\ \texttt{b}\ \texttt{b} \mid \texttt{b} \mid \epsilon}$



$\texttt{a}, \texttt{a} \rightarrow \epsilon$
$\texttt{b}, \texttt{b} \rightarrow \epsilon$
$\epsilon, S \rightarrow \texttt{b}$
$\epsilon, S \rightarrow \epsilon$
$\epsilon, \epsilon \rightarrow \$$
$\epsilon, \epsilon \rightarrow S$
q
$\epsilon, \$ \rightarrow \epsilon$
$\epsilon, \epsilon \rightarrow \texttt{a}$
$\epsilon, S \rightarrow \texttt{b}$
$\epsilon, \epsilon \rightarrow S$
$\epsilon, \epsilon \rightarrow \texttt{b}$

Pumping Lemma for CFLs

There are languages that are not context-free, and again there is a pumping lemma that can be used to show (some) languages non-context-free:

If A is context-free then there is a number p such that for any string $s \in A$ with $|s| \geq p$, s can be written as $s = uvxyz$, satisfying

1. $uv^ixy^iz \in A$ for all $i \geq 0$
2. $|vy| > 0$
3. $|vxy| \leq p$

We won't prove this lemma, but we give two examples of its use.

Pumping Example 1

$A = \{ww \mid w \in \{0,1\}^*\}$ is not context-free.

Assume it is, let p be the pumping length, take $0^p1^p0^p1^p$.

By the pumping lemma, $0^p1^p0^p1^p = uvxyz$, with uv^ixy^iz in A for all $i \geq 0$, and $|vxy| \leq p$.

There are three ways that vxy can be part of

$$00\ldots0011\ldots1100\ldots0011\ldots11$$

If it straddles the midpoint, it has form 1^n0^m, so pumping down, we are left with $0^p1^i0^j1^p$, with $i < p$, or $j < p$, or both.

If it is in the first half, uv^2xy^2z will have pushed a 1 into the first position of the second half.

Similarly if vxy is in the second half.

Pumping Example 2

$B = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \in \mathbb{N}\}$ is not context-free.

Assume it is, let p be the pumping length, and take $\mathtt{a}^p\mathtt{b}^p\mathtt{c}^p \in B$.

By the pumping lemma, $\mathtt{a}^p\mathtt{b}^p\mathtt{c}^p = uvxyz$, with uv^ixy^iz in B for all i.

Either v or y is non-empty (or both are).

If one of them contains two different symbols from $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ then uv^2xy^2z has symbols in the wrong order, and so cannot be in B.

So both v and y must contain only one kind of symbol. But then uv^2xy^2z can't have the same number of as, bs, and cs.

In all cases we have a contradiction.

Closure Properties for CFLs

The class of context-free languages is closed under

- union,
- concatenation,
- Kleene star,
- reversal.

Closure Properties for CFLs

The class of context-free languages is not closed under intersection!

Hence it is not closed under complement either (why?)

Consider these two CFLs:

$$\begin{array}{rcl} C & = & \{\mathtt{a}^m\mathtt{b}^n\mathtt{c}^n \mid m, n \in \mathbb{N}\} \\ D & = & \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^m \mid m, n \in \mathbb{N}\} \end{array}$$

Exercise: Prove that they are context-free!

But $C \cap D$ is the language $B = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \in \mathbb{N}\}$ which we just showed is not context-free.

However, we do have: If A is context-free and R is regular then $A \cap R$ is context-free.

Regular Grammars

There is a grammar notion that corresponds to the regular languages.

If we restrict the kind of rules allowed in CFGs, so they must be either of form

$$A \to w$$

or

$$A \to w\ B$$

with $w \in \Sigma^*$ and $A, B \in V$, then we have "regular grammars".

These generate exactly the regular languages.

(Here we chose the so-called right-linear form — we could have said $A \to B\ w$ instead of $A \to w\ B$.)

Deterministic PDAs

Is a deterministic PDA (a DPDA) as powerful as a PDA?

No. A DPDA can recognise the context-free

$$\{wcw^{\mathcal{R}} \mid c \in \Sigma, w \in (\Sigma \setminus \{c\})^*\}$$

but not the context-free $\{ww^{\mathcal{R}} \mid w \in \Sigma^*\}$.

Intuitively a deterministic machine cannot know when the middle of the input has been reached. Suppose it gets input

000011000000000110000

A deterministic machine won't know when to start popping the stack.