The University of Melbourne
School of Computing and Information Systems
COMP30026 Models of Computation

Selected Tutorial Solutions, Week 7

46. (a) We can negate both sides of the biimplication, so we just need to show:

$$A \subseteq B \Leftrightarrow A \setminus B = \emptyset$$

The left-hand side is, by definition: $\forall x(x \in A \Rightarrow x \in B)$. The right-hand side can be written: $\neg\exists y(y \in A \wedge y \notin B)$. Pushing the negation in, we get $\forall y(y \notin A \vee y \in B)$, or equivalently, $y \in A \Rightarrow y \in B$.

(b) It is easier to look at the logical expressions. The left-hand side is $\{x \mid x \in A \wedge x \in B\}$. The right-hand side is

$$\begin{aligned} & \{x \mid x \in A \wedge x \notin A \setminus B\} \\ = \; & \{x \mid x \in A \wedge \neg(x \in A \wedge x \notin B)\} \\ = \; & \{x \mid x \in A \wedge (x \notin A \vee x \in B)\} \\ = \; & \{x \mid x \in A \wedge x \in B\} \end{aligned}$$

47. These are simpler expressions:

(a) $A \oplus B = A$ is equivalent to $B = \emptyset$.
(b) $A \oplus B = A \setminus B$ is equivalent to $B \subseteq A$.
(c) $A \oplus B = A \cup B$ is equivalent to $A \cap B = \emptyset$.
(d) $A \oplus B = A \cap B$ is equivalent to $A \cup B = \emptyset$.
(e) $A \oplus B = A^c$ is equivalent to $B = X$, assuming a universal set X.

48. The statement is false, as we have, for example, $\{42\} \times \emptyset = \emptyset \times \{42\} = \emptyset$, but $\emptyset \neq \{42\}$.

49. Assume that R is transitive. Let (x, z) be in $R \circ R$. That means there is some y, such that $R(x, y)$ and $R(y, z)$ hold. By transitivity, $R(x, z)$ holds, so $(R \circ R) \subseteq R$.

Conversely, assume that $R \circ R \subseteq R$. Consider x, y, z such that $R(x, y)$ and $R(y, z)$ hold. Clearly (x, z) is in $R \circ R$, and hence, by assumption, in R. But that means R is transitive.

As an example of a transitive relation for which $R \circ R = R$ does not hold, consider $<$ on $\mathbb{Z}$. It is transitive, but $< \circ <$ does not contain, say (2,3). Since (2,3) is in $<$, $<$ is different from $< \circ <$.

50. Here is the complete table:

Property	Reflexivity	Symmetry	Transitivity
preserved under $\cap$?	yes	yes	yes
preserved under $\cup$?	yes	yes	no
preserved under inverse?	yes	yes	yes
preserved under complement?	no	yes	no

To see how transitivity fails to be preserved under union, consider two relations on $\{a, b, c\}$, namely $R = \{(a, a), (a, b), (b, b)\}$ and $S = \{(c, a)\}$, both transitive. $R \cup S$ is not transitive, because in the union we have (c, a) and (a, b), but not (c, b). And R's complement, $\{(a, c), (b, a), (b, c), (c, a), (c, b), (c, c)\}$ is not transitive either, as it contains, for example, (a, c) and (c, a), but not (a, a).

51. From the first row of the last question's table, it follows that, if R and S are equivalence relations, then so is their intersection. But their union may not be. As an example, take the reflexive, symmetric, transitive closures of R and S from the previous answer, to get these two equivalence relations:

$$R' = \{(a,a),(a,b),(b,a),(b,b),(c,c)\} \quad \text{and} \quad S' = \{(a,a),(a,c),(b,b),(c,a),(c,c)\}.$$

Their union fails to be transitive, as it contains (c,a) and (a,b) but not (c,b).

52. From $f(g(y)) = y$ we conclude that g is injective. Namely, if B has cardinality 1 then g is trivially injective. Otherwise, consider $y, y' \in B$, with $y \neq y'$. Suppose $g(y) = g(y')$. Then, applying f to both, we have $y = f(g(y)) = f(g(y')) = y'$, contradicting $y \neq y'$. So we must have $g(y) \neq g(y')$, that is, g is injective.

    Similarly we can show that f is surjective. To do this, we must show that for each $y \in B$ there is some $x \in A$ such that $f(x) = y$. But that is easy—that x is $g(y) \in A$.

53. We have: $h(h(h(x))) = x$ for all $x \in X$. First, let us show that h must be injective. If $h(x) = h(y)$, then, applying h twice on each side, we have $h(h(h(x))) = h(h(h(y)))$, whence $x = y$. So h is injective. Second, let us show that h must be surjective. Consider an arbitrary element $x \in X$. We have $x = h(h(h(x)))$, that is, h maps $h(h(x))$ to x. Since x was arbitrary, h is surjective.

    For the counter-example, take $X = \{a,b,c\}$ and let h map a to b, b to c, and c to a. Then h is not the identity function on X, but $h \circ h \circ h$ is.

54. We certainly do not have $A \times A = A$. In fact, no member of A is a member of $A \times A$, and no member of $A \times A$ is a member if A. So $\times$ is not absorptive.

    Neither is it commutative. Let $A = \{0\}$ and $B = \{1\}$. Then $A \times B = \{(0,1)\}$ while $B \times A = \{(1,0)\}$, and those singleton sets are different, because the members are.

    If we also define $C = \{2\}$ then $A \times (B \times C) = \{(0,(1,2))\}$ while $(A \times B) \times C = \{((0,1),2)\}$. Again, these are different. However, it is not uncommon to identify both of $(0,(1,2))$ and $((0,1),2)$ with the triple $(0,1,2)$ ("flattening" the nested pairings). If we agree to do that then $\times$ is associative, and we can simply write $A \times B \times C$ for the set of triples.

55. The conjecture is false. For a counter-example, take A to be $\{0,1\}$ and $R = \{(0,0)\}$. Then R is symmetric, and also anti-symmetric, but R is not reflexive, as it does not include $(1,1)$.

56. If f is injective then B has at least 42 elements. If f is surjective then B has at most 42 elements. (So if f is bijective, B has exactly 42 elements.)

57. Here are some functions that satisfy the requirements. We show $f_i(x)$ in the table's row x, column i:

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
a	a	a	b	b	b	a	c	b
b	b	a	b	d	b	a	b	a
c	c	a	c	d	c	a	d	d
d	d	a	d	d	c	c	d	c

    Maybe you skipped this optional exercise; but you may still want to verify, for each of these eight functions, that it really does satisfy its specification.