Notes on Vector Spaces

COMP90051 Statistical Machine Learning

Semester 2, 2018
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These slides

- Notes on linear algebra
  * Vectors and dot products
  * Hyperplanes and vector normals

Notes on Linear Algebra

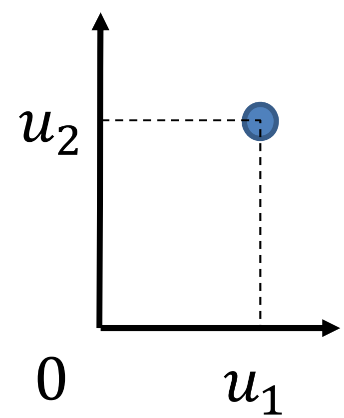
Link between geometric and algebraic interpretation of ML methods

What are vectors?

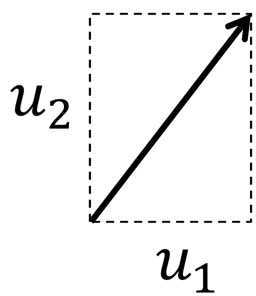
Suppose $\boldsymbol{u} = [u_1, u_2]'$. What does $\boldsymbol{u}$ really represent?

Ordered set of numbers $\{u_1, u_2\}$

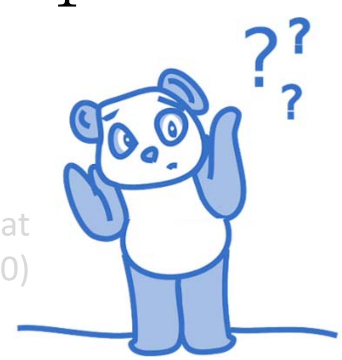
Cartesian coordinates of a point



A direction



art: OpenClipartVectors at pixabay.com (CC0)



Dot product: Algebraic definition

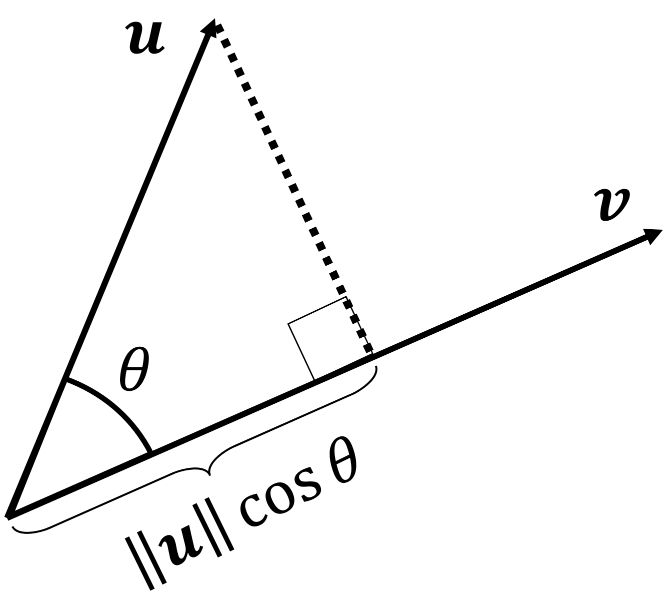
- Given two m-dimensional vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, their dot product is $\boldsymbol{u} \cdot \boldsymbol{v} \equiv \boldsymbol{u}'\boldsymbol{v} \equiv \sum_{i=1}^{m} u_i v_i$
  - E.g., weighted sum of terms is a dot product $\boldsymbol{x}'\boldsymbol{w}$
- If k is a scalar, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are vectors then

$$(k\boldsymbol{a})'\boldsymbol{b} = k(\boldsymbol{a}'\boldsymbol{b}) = \boldsymbol{a}'(k\boldsymbol{b})$$

$$\boldsymbol{a}'(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a}'\boldsymbol{b} + \boldsymbol{a}'\boldsymbol{c}$$

Dot product: Geometric definition

- Given two m-dimensional Euclidean vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, their dot product is $\boldsymbol{u} \cdot \boldsymbol{v} \equiv \boldsymbol{u}'\boldsymbol{v} \equiv \|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta$
  - $\|\boldsymbol{u}\|, \|\boldsymbol{v}\|$ are L_2 *norms* for $\boldsymbol{u}, \boldsymbol{v}$
  - θ is the angle between the vectors



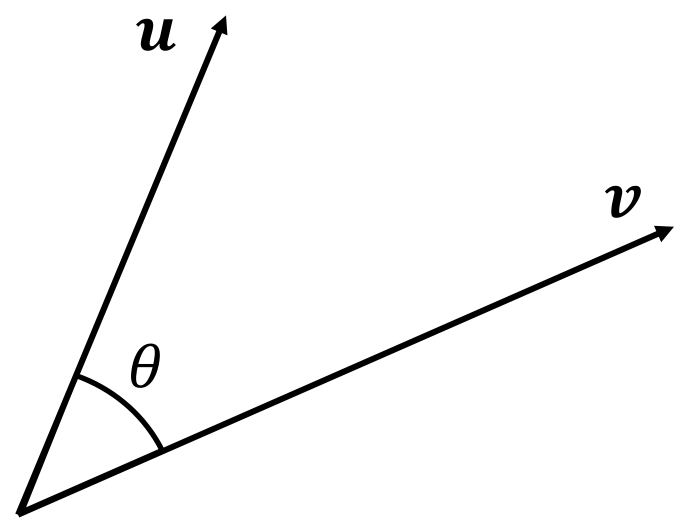
The *scalar projection* of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is given by

$$u_{\boldsymbol{v}} = \|\boldsymbol{u}\| \cos\theta$$

Thus dot product is
$\boldsymbol{u}'\boldsymbol{v} = u_{\boldsymbol{v}}\|\boldsymbol{v}\| = v_{\boldsymbol{u}}\|\boldsymbol{u}\|$

Geometric properties of the dot product

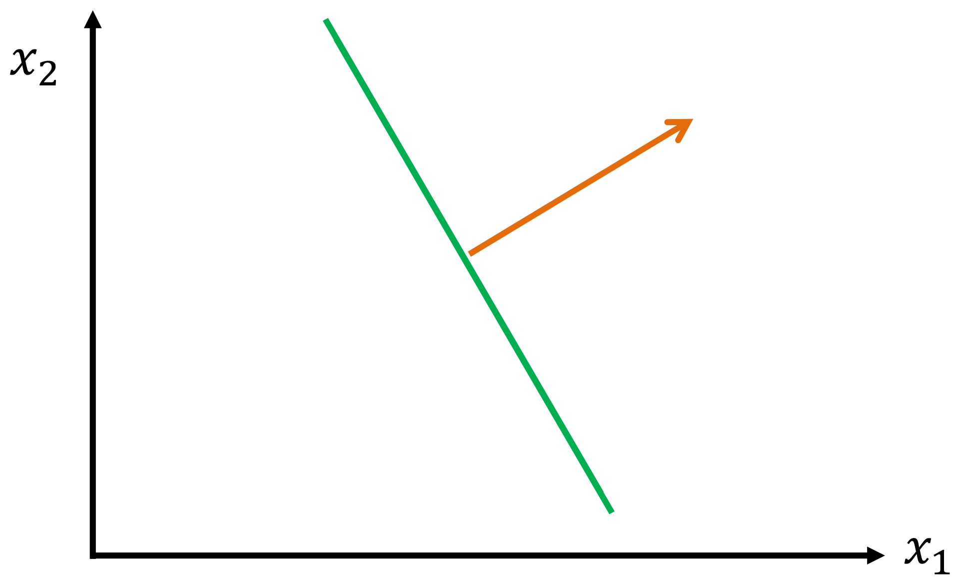
- If the two vectors are orthogonal then $\boldsymbol{u}'\boldsymbol{v} = 0$
- If the two vectors are parallel then $\boldsymbol{u}'\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|$, if they are anti-parallel then $\boldsymbol{u}'\boldsymbol{v} = -\|\boldsymbol{u}\|\|\boldsymbol{v}\|$
- $\boldsymbol{u}'\boldsymbol{u} = \|\boldsymbol{u}\|^2$, so $\|\boldsymbol{u}\| = \sqrt{u_1^2 + \cdots + u_m^2}$ defines the Euclidean vector length



Hyperplanes and normal vectors

- A ***hyperplane*** defined by parameters $\boldsymbol{w}$ and b is a set of points $\boldsymbol{x}$ that satisfy $\boldsymbol{x}'\boldsymbol{w} + b = 0$

- In 2D, a hyperplane is a line: a line is a set of points that satisfy $w_1x_1 + w_2x_2 + b = 0$



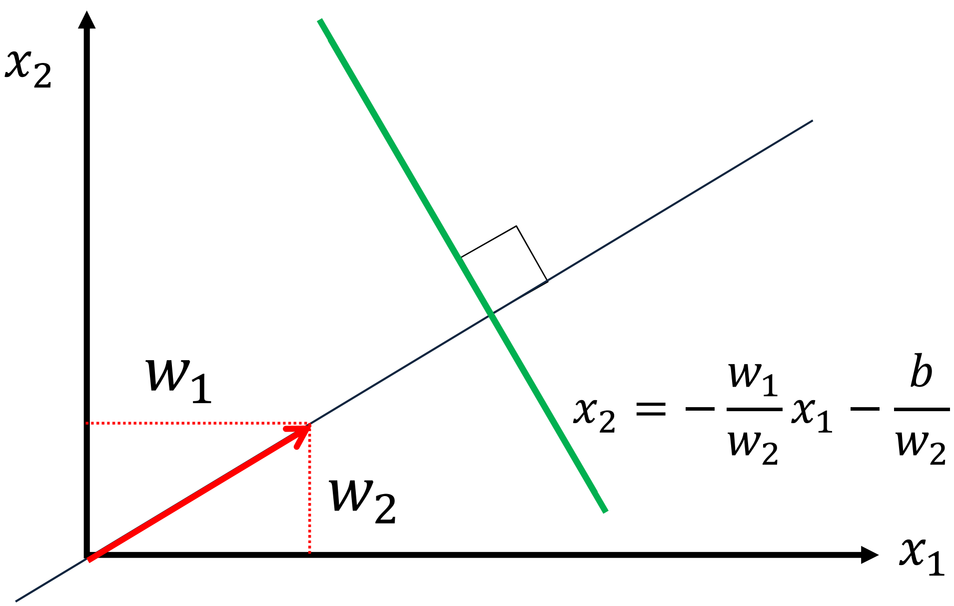
A ***normal vector*** for a hyperplane is a vector perpendicular to that hyperplane

Hyperplanes and normal vectors

- Consider a hyperplane defined by parameters $\boldsymbol{w}$ and b. Note that $\boldsymbol{w}$ is itself a vector
- Lemma: Vector $\boldsymbol{w}$ is normal to the hyperplane
- Proof sketch:
  - Choose any two points $\boldsymbol{u}$ and $\boldsymbol{v}$ on the hyperplane. Note that vector $(\boldsymbol{u}-\boldsymbol{v})$ lies on the hyperplane
  - Consider dot product $(\boldsymbol{u}-\boldsymbol{v})'\boldsymbol{w} = \boldsymbol{u}'\boldsymbol{w} - \boldsymbol{v}'\boldsymbol{w}$
    $$= (\boldsymbol{u}'\boldsymbol{w} + b) - (\boldsymbol{v}'\boldsymbol{w} + b) = 0$$
  - Thus $(\boldsymbol{u}-\boldsymbol{v})$ lies on the hyperplane, but is perpendicular to $\boldsymbol{w}$, and so $\boldsymbol{w}$ is a vector normal

Example in 2D

- Consider a line defined by w_1, w_2 and b
- Vector $\boldsymbol{w} = [w_1, w_2]'$ is a normal vector



Summary

- Notes on linear algebra
  - Vectors and dot products
  - Hyperplanes and vector normals