The University of Melbourne
School of Computing and Information Systems
COMP30026 Models of Computation

Sample Answers to Tutorial Exercises, Week 4

18. (a)

$$\begin{array}{ll}
\neg(A \wedge \neg(B \wedge C)) & \\
\neg A \vee (B \wedge C) & \text{(push negation in)} \\
(\neg A \vee B) \wedge (\neg A \vee C) & \text{(distribute } \vee \text{ over } \wedge\text{)}
\end{array}$$

The result is now in reduced CNF.

(b)

$$\begin{array}{ll}
A \vee (\neg B \wedge (C \vee (\neg D \wedge \neg A))) & \\
(A \vee \neg B) \wedge (A \vee C \vee (\neg D \wedge \neg A)) & \text{(distribute } \vee \text{ over } \wedge\text{)} \\
(A \vee \neg B) \wedge (A \vee C \vee \neg D) \wedge (A \vee C \vee \neg A) & \text{(distribute } \vee \text{ over } \wedge\text{)}
\end{array}$$

The result is in CNF but not RCNF. To get RCNF we need to eliminate the last clause which is a tautology, and we end up with $(A \vee \neg B) \wedge (A \vee C \vee \neg D)$.

(c)

$$\begin{array}{ll}
(A \vee B) \Rightarrow (C \wedge D) & \\
\neg(A \vee B) \vee (C \wedge D) & \text{(unfold } \Rightarrow\text{)} \\
(\neg A \wedge \neg B) \vee (C \wedge D) & \text{(de Morgan)} \\
(\neg A \vee (C \wedge D)) \wedge (\neg B \vee (C \wedge D)) & \text{(distribute } \vee \text{ over } \wedge\text{)} \\
(\neg A \vee C) \wedge (\neg A \vee D) \wedge (\neg B \vee C) \wedge (\neg B \vee D) & \text{(distribute } \wedge \text{ over } \vee\text{)}
\end{array}$$

The result is in RCNF. We could have chosen different orders for the distributions.

(d)

$$\begin{array}{ll}
A \wedge (B \Rightarrow (A \Rightarrow B)) & \\
A \wedge (\neg B \vee \neg A \vee B) & \text{(unfold both occurrences of } \Rightarrow\text{)} \\
A & \text{(rightmost clause is tautological: remove it)}
\end{array}$$

19. Let us follow the method given in a lecture, except we do the double-negation elimination aggressively, as soon as opportunity arises:

$$\begin{array}{ll}
\neg((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)) & \\
\neg(\neg(B \vee \neg A) \vee \neg(B \vee A) \vee B) & \text{(unfold } \Rightarrow \text{ and eliminate double negation)} \\
(B \vee \neg A) \wedge (B \vee A) \wedge \neg B & \text{(de Morgan for outermost neg; elim double neg)}
\end{array}$$

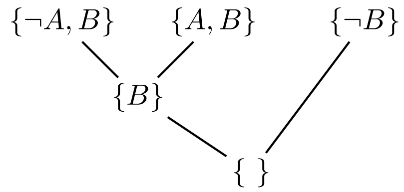
This is RCNF without further reductions.

We could also have used other transformations—sometimes this can shorten the process. For example, we could have rewritten the sub-expression $\neg B \Rightarrow \neg A$ as $A \Rightarrow B$ (the contraposition principle). You may want to check that this does not change the result.

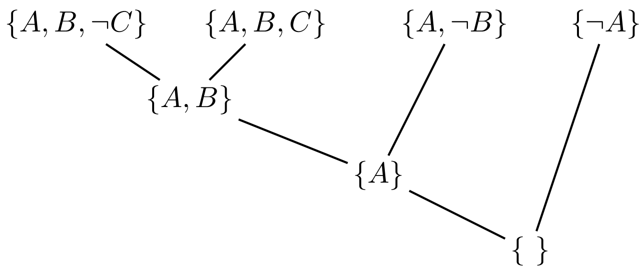
The resulting formula, written as a set of sets of literals:

$$\{\{\neg A, B\}, \{A, B\}, \{\neg B\}\}$$

We can now construct the refutation:



20. Here is a refutation:



From this we conclude that $(A \vee B \vee \neg C) \wedge \neg A \wedge (A \vee B \vee C) \wedge (A \vee \neg B)$ is unsatisfiable.

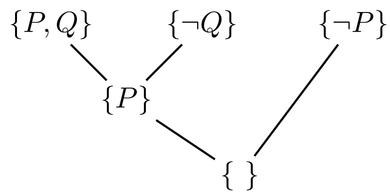
21. (a) $(P \vee Q) \Rightarrow (Q \vee P)$. First negate the formula (why?), to get $\neg((P \vee Q) \Rightarrow (Q \vee P))$. Then we can use the usual techniques to convert the negated proposition to RCNF. Here is a useful shortcut, combining $\Rightarrow$-elimination with one of de Morgan's laws:

$$\neg(A \Rightarrow B) \equiv A \wedge \neg B.$$

So:

$$\begin{array}{ll} \neg((P \vee Q) \Rightarrow (Q \vee P)) & \\ (P \vee Q) \wedge \neg(Q \vee P) & \text{(shortcut)} \\ (P \vee Q) \wedge \neg Q \wedge \neg P & \text{(de Morgan)} \end{array}$$

The result allows for a straight-forward refutation:



(b) $(\neg P \Rightarrow P) \Rightarrow P$. Again, first negate the formula, to get $\neg((\neg P \Rightarrow P) \Rightarrow P)$. Then turn the result into RCNF:

$$\begin{array}{ll} \neg((\neg P \Rightarrow P) \Rightarrow P) & \\ (\neg P \Rightarrow P) \wedge \neg P & \text{(shortcut from above)} \\ (\neg\neg P \vee P) \wedge \neg P & \text{(unfold } \Rightarrow\text{)} \\ (P \vee P) \wedge \neg P & \text{(eliminate double negation)} \\ P \wedge \neg P & (\vee\text{-absorption)} \end{array}$$

The resolution proof is immediate; we will leave it out.

(c) $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$. Again, negate the formula, to get $\neg(((P \Rightarrow Q) \Rightarrow P) \Rightarrow P)$. Then turn the result into RCNF:

$$\begin{array}{ll}
\neg(((P \Rightarrow Q) \Rightarrow P) \Rightarrow P) & \\
((P \Rightarrow Q) \Rightarrow P) \wedge \neg P & \text{(shortcut, outermost } \Rightarrow) \\
((\neg P \vee Q) \Rightarrow P) \wedge \neg P & \text{(unfold } \Rightarrow) \\
(\neg(\neg P \vee Q) \vee P) \wedge \neg P & \text{(unfold } \Rightarrow) \\
((\neg\neg P \wedge \neg Q) \vee P) \wedge \neg P & \text{(de Morgan)} \\
((P \wedge \neg Q) \vee P) \wedge \neg P & \text{(double negation)} \\
(P \vee P) \wedge (\neg Q \vee P) \wedge \neg P & \text{(distribution)} \\
P \wedge (\neg Q \vee P) \wedge \neg P & \text{(absorption)}
\end{array}$$

Again this gives an immediate refutation: just resolve $\{P\}$ against $\{\neg P\}$.

(d) $P \Leftrightarrow ((P \Rightarrow Q) \Rightarrow P)$. Negating the formula, we get $P \oplus ((P \Rightarrow Q) \Rightarrow P)$. Let us turn the resulting formula into RCNF:

$$\begin{array}{ll}
P \oplus ((P \Rightarrow Q) \Rightarrow P) & \\
(P \vee ((P \Rightarrow Q) \Rightarrow P)) \wedge (\neg P \vee \neg((P \Rightarrow Q) \Rightarrow P)) & \text{(eliminate } \oplus) \\
(P \vee ((P \Rightarrow Q) \Rightarrow P)) \wedge (\neg P \vee ((P \Rightarrow Q) \wedge \neg P)) & \text{(shortcut from above)} \\
(P \vee (\neg(\neg P \vee Q) \vee P)) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) & (\Rightarrow\text{-elimination}) \\
(P \vee (\neg\neg P \wedge \neg Q) \vee P) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) & \text{(de Morgan)} \\
(P \vee (P \wedge \neg Q) \vee P) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) & \text{(double negation)} \\
P \wedge (P \vee \neg Q) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) & (\vee\text{-absorption, distribution}) \\
P \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge \neg P & (\vee\text{-absorption, distribution})
\end{array}$$

Once again, now just resolve $\{P\}$ against $\{\neg P\}$.

22. (a) $\{\{A, B\}, \{\neg A, \neg B\}, \{\neg A, B\}\}$ stands for the formula $(A \vee B) \wedge (\neg A \vee \neg B) \wedge (\neg A \vee B)$. This is satisfiable by $\{A \mapsto \mathbf{f}, B \mapsto \mathbf{t}\}$.

(b) $\{\{A, \neg B\}, \{\neg A\}, \{B\}\}$ stands for $(A \vee \neg B) \wedge \neg A \wedge B$. A refutation is easy:

$$\begin{tikzcd}[column sep=small]
\{A, \neg B\} \arrow[dr, dash] & & \{\neg A\} \arrow[dl, dash] & & \{\neg B\} \arrow[ddl, dash] \\
& \{\neg B\} \arrow[drr, dash] & & & \\
& & & \{\ \} &
\end{tikzcd}$$

(c) $\{\{A\}, \emptyset\}$ stands for $A \wedge \mathbf{f}$, which is clearly not satisfiable.

(d) We have $\{\{A, B\}, \{\neg A, \neg B\}, \{B, C\}, \{\neg B, \neg C\}, \{A, C\}, \{\neg A, \neg C\}\}$. This set is not satisfiable, as a proof by resolution shows:

$$\begin{tikzcd}[column sep=tiny]
\{A, B\} \arrow[dr, dash] & & \{\neg B, \neg C\} \arrow[dl, dash] & & \{A, C\} \arrow[ddl, dash] & & \{\neg A, \neg B\} \arrow[dr, dash] & & \{B, C\} \arrow[dl, dash] & & \{\neg A, \neg C\} \arrow[ddl, dash] \\
& \{A, \neg C\} \arrow[dr, dash] & & & & & & \{\neg A, C\} \arrow[dr, dash] & & & \\
& & \{A\} \arrow[drrr, dash] & & & & & & \{\neg A\} \arrow[dlll, dash] & & \\
& & & & & \{\ \} & & & & &
\end{tikzcd}$$

23. Let us give names to the propositions:

    - C: Ann clears 2 meters
    - F: Ann gets the flu
    - K: The selectors are sympathetic
    - S: Ann is selected
    - T: Ann trains hard

    The four assumptions then become:

    (a) $C \Rightarrow S$
    (b) $T \Rightarrow (F \Rightarrow K)$
    (c) $(T \wedge \neg F) \Rightarrow C$
    (d) $K \Rightarrow S$

    It is easy to see that S is not a logical consequence of these, as we can give all five variables the value *false*, and all the assumptions will thereby be true.

    To see that $T \Rightarrow S$ is a logical consequence of the assumptions, we can negate it, obtaining $T \wedge \neg S$. Then, translating everything to clausal form, we can use resolution to derive an empty clause.

    Alternatively, note that $T \Rightarrow (F \Rightarrow K)$ is equivalent to $(T \wedge F) \Rightarrow K$. Since also $K \Rightarrow S$, we have $(T \wedge F) \Rightarrow S$. Similarly, $(T \wedge \neg F) \Rightarrow C$ together with $C \Rightarrow S$ gives us $(T \wedge \neg F) \Rightarrow S$.

    But from $(T \wedge F) \Rightarrow S$ and $(T \wedge \neg F) \Rightarrow S$ we get $T \Rightarrow S$. (You may want to check that by massaging the conjunction of the two formulas.)

24. Let us give names to the propositions:

    - A: The commissioner apologises
    - F: The commissioner can attend the function
    - R: The commissioner resigns

    The four statements then become

    (a) $F \Rightarrow (A \wedge R)$
    (b) $(R \wedge A) \Rightarrow F$
    (c) $R \Rightarrow F$
    (d) $F \Rightarrow A$

    The first translation may not be obvious. But to say "X does not happen unless Y happens" is the same as saying "it is not possible to have X happen and at the same time Y does not happen." That is, $\neg(X \wedge \neg Y)$, which is equivalent to $X \Rightarrow Y$. Note that (a) entails (d) and (c) entails (b).

29. These are the clauses generated:

    (a) For each node i generate the clause $B_i \vee G_i \vee R_i$. That's $n+1$ clauses of size 3 each.
    (b) For each node i generate three clauses: $(\neg B_i \vee \neg G_i) \wedge (\neg B_i \vee \neg R_i) \wedge (\neg G_i \vee \neg R_i)$. That comes to $3n+3$ clauses of size 2 each.
    (c) For each pair (i, j) of nodes with $i < j$ we want to express $E_{ij} \Rightarrow (\neg(B_i \wedge B_j) \wedge \neg(G_i \wedge G_j) \wedge \neg(R_i \wedge R_j)$. This means for each pair (i, j) we generate three clauses: $(\neg E_{ij} \vee \neg B_i \vee \neg B_j) \wedge (\neg E_{ij} \vee \neg G_i \vee \neg G_j) \wedge (\neg E_{ij} \vee \neg R_i \vee \neg R_j)$. There are $n(n+1)/2$ pairs, so we generate $3n(n+1)/2$ clauses, each of size 3.

    Altogether we generate $3n + 3 + 6n + 6 + 9n(n+1)/2$ literals, that is, $9(n+1)(n/2+1)$.