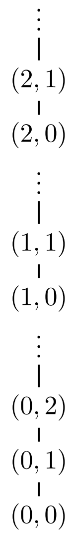
Selected Tutorial Solutions, Week 8



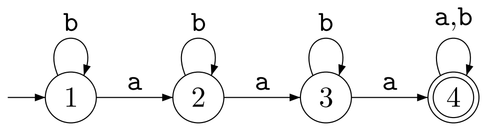
59. (b) is not well-founded, as we can have infinite strictly decreasing sequences in $\mathbb{Q}$, such as $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. But (a), (c) and (d) are all well-founded. For (d) it may help to look at the Hasse diagram for $\mathbb{N} \times \mathbb{N}$ ordered by $\prec$ (shown here in the margin).

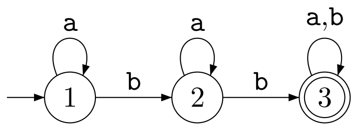
60. We can describe the contents of the bag with a triple $(w, b, r) \in \mathbb{N}^3$. The triple stands for w white, b blue, and r red marbles being present in the bag. We claim that each round decreases the content of the bag, according to the lexicographic ordering of triples. Namely, look at what happens in one round, assuming the current state is described by (w, b, r):

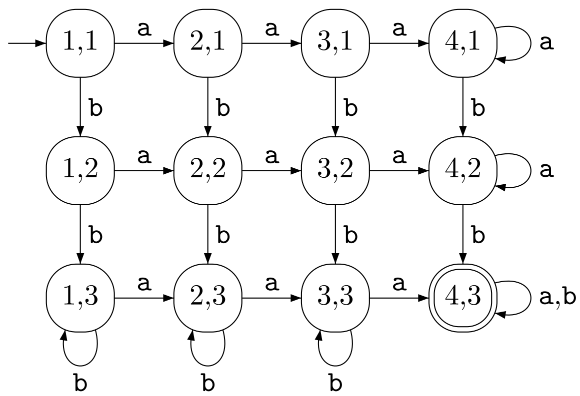
   - In case (b) we will be left with either $(w-1, b, r-1)$, $(w, b-1, r-1)$, or $(w, b, r-2)$, depending on whether the red marble's companion is white, blue, or red, respectively.
   - In case (c) we are left with $(w-1, b+5, r)$.
   - In case (d) one marble is blue and the other is white or blue. We are left with either $(w-1, b-1, r+10)$ or $(w, b-2, r+10)$, according as the blue marble's companion is white or blue, respectively.

   Each of those six triples is strictly smaller than (w, b, r) in the lexicographic ordering. Since this ordering is well-founded on $\mathbb{N}^3$, the process must halt after a finite number of rounds.

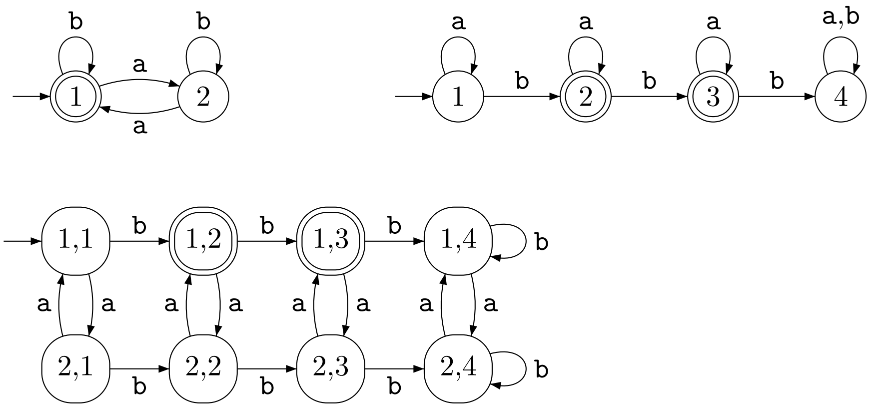
61. (a) $\{w \mid w \text{ has at least three } \texttt{a}\text{s}\} \cap \{w \mid w \text{ has at least two } \texttt{b}\text{s}\}$



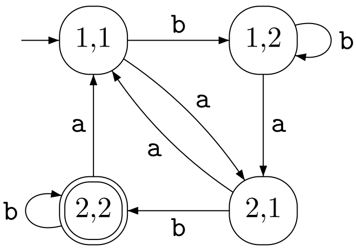




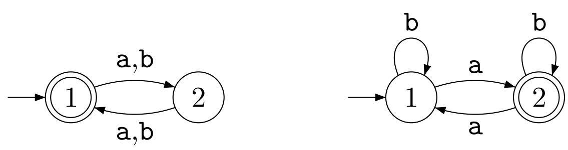
(b) $\{w \mid w \text{ has an even number of } \texttt{a}\text{s}\} \cap \{w \mid w \text{ has one or two } \texttt{b}\text{s}\}$

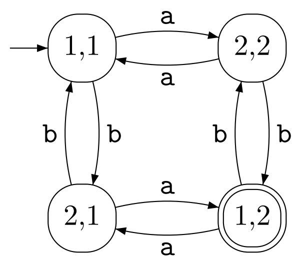


(c) $\{w \mid w \text{ has an odd number of } \texttt{a}\text{s}\} \cap \{w \mid w \text{ ends with } \texttt{b}\}$

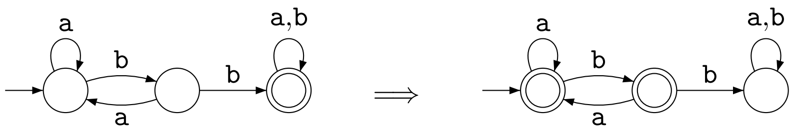


(d) $\{w \mid w \text{ has an even length}\} \cap \{w \mid w \text{ has an odd number of } \texttt{a}\text{s}\}$

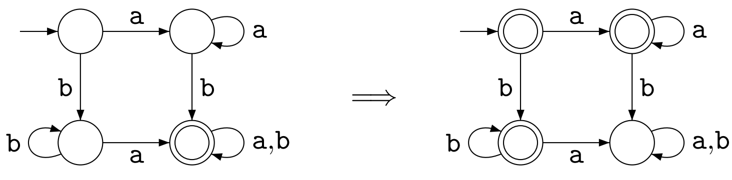




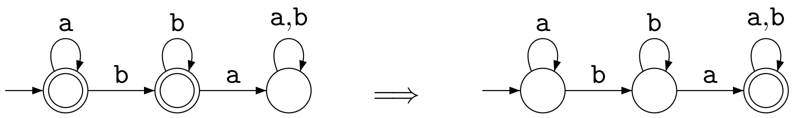
62. (a) $\{w \mid w \text{ does not contain the substring } \texttt{bb}\}$



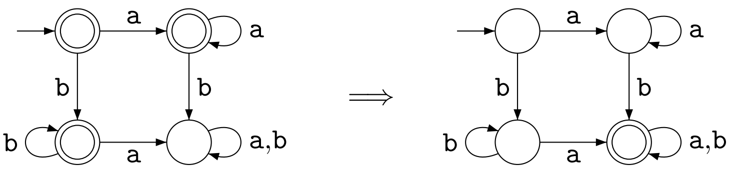
(b) $\{w \mid w \text{ contains neither the substring } \texttt{ab} \text{ nor } \texttt{ba}\}$



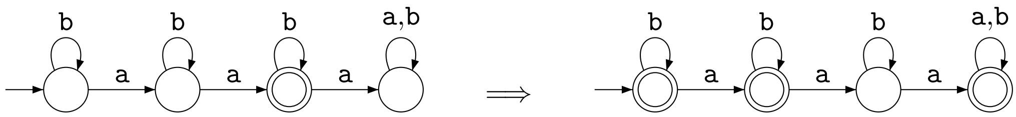
(c) $\{w \mid w \text{ is any string not in } \texttt{a}^*\texttt{b}^*\}$



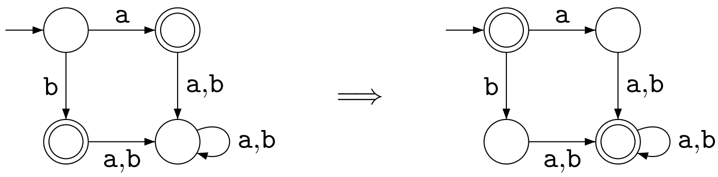
(d) $\{w \mid w$ is any string not in $\mathtt{a}^* \cup \mathtt{b}^*\}$ (compare to (b)!)



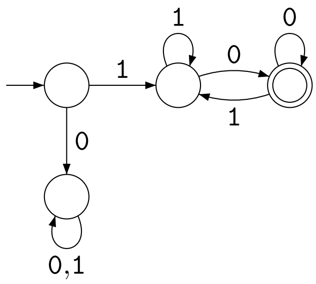
(e) $\{w \mid w$ is any string that doesn't contain exactly two $\mathtt{a}$s$\}$



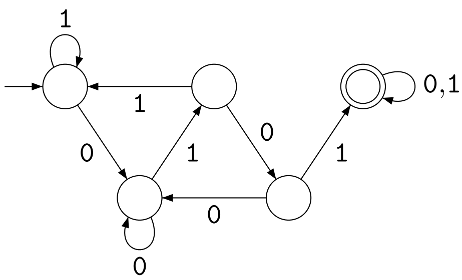
(f) $\{w \mid w$ is any string except $\mathtt{a}$ and $\mathtt{b}\}$



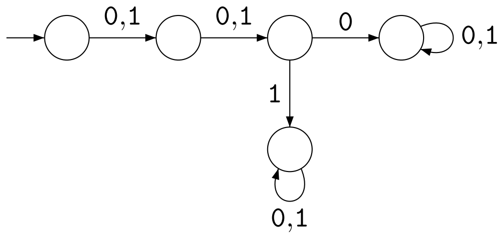
63. (a) $\{w \mid w$ begins with a $\mathtt{1}$ and ends with a $\mathtt{0}\}$



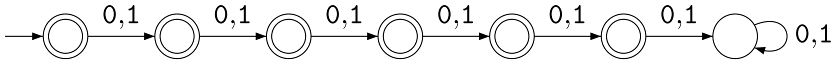
(b) $\{w \mid w$ contains the substring $\mathtt{0101}\}$



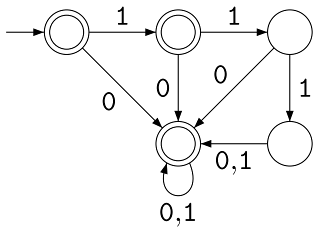
(c) $\{w \mid w$ has length at least 3 and its third symbol is $\mathtt{0}\}$



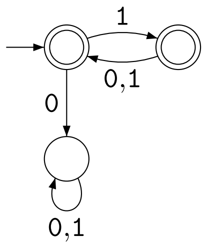
(d) $\{w \mid \text{the length of } w \text{ is at most } 5\}$



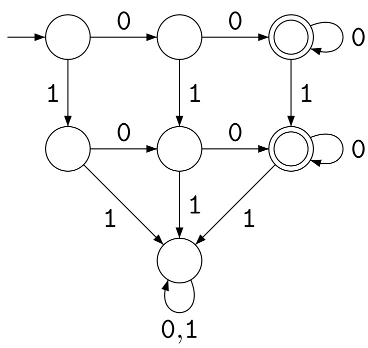
(e) $\{w \mid w \text{ is any string except } \texttt{11} \text{ and } \texttt{111}\}$



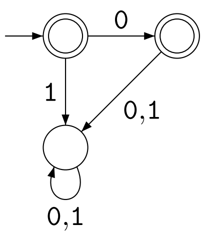
(f) $\{w \mid \text{every odd position of } w \text{ is a } \texttt{1}\}$



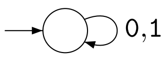
(g) $\{w \mid w \text{ contains at least two } \texttt{0}\text{s and at most one } \texttt{1}\}$



(h) $\{\epsilon, \texttt{0}\}$



(i) The empty set



(j) All strings except the empty string