

# Optimization Notes

December 19, 2012

The problem we are trying to optimize is as follows.

$$\max_{c_1(s), c_2(s), l_1(s), l_2(s), x'(s), R'(s)} \sum_s P(s, s) \left[ \sum_i \alpha_i \left( \frac{\psi c_i(s)^{1-\sigma}}{1-\sigma} + (1-\psi) \log(1-l_i(s)) \right) + \beta V(x'(s), R'(s), s) \right]$$

subject to the constraints

$$\psi c_2(s)^{-\sigma} (c_2(s) - c_1(s)) + x'(s) + \left( \frac{-(1-\psi)l_2(s)}{1-l_2(s)} + R'(s) \frac{(1-\psi)l_1(s)}{1-l_1(s)} \right) = \frac{\psi c_2(s)^{-\sigma}}{\beta \mathbb{E}[\psi c_2^{-\sigma}]} \quad (1)$$

$$\frac{\sum_s P(s, s) \psi c_2(s)^{-\sigma}}{\sum_s P(s, s) \psi c_1(s)^{-\sigma}} = R \quad (2)$$

$$\frac{(1-\psi)}{\theta_1(1-l_1(s))\psi c_1(s)^{-\sigma}} = \frac{(1-\psi)}{\theta_2(1-l_2(s))\psi c_2(s)^{-\sigma}} \quad (3)$$

$$n_1 c_1(s) + n_2 c_2(s) + g(s) = \theta_1 l_1(s) + \theta_2 l_2(s) \quad (4)$$

$$R'(s) = \frac{\psi c_2(s)^{-\sigma}}{\psi c_1(s)^{-\sigma}} \quad (5)$$

conditional on  $s_-$ ,  $R$  and  $x$ . For  $s = 2$ , given a guess of  $z = (c_1(1), c_1(2), c_2(1))$  it is possible to find the variables  $c_2(2), l_1, l_2, x', R'$  such that the equations above are satisfied and their derivatives with respect to  $z$ . Taking derivatives we can then perform an unconstrained optimization. From now on  $\nabla$  will refer to the vector of partial derivatives with respect to  $z$ ,  $u_{c,i}(s)$  will refer to  $\psi c_i(s)^{-\sigma}$  and  $u_{l,i}(s) = -\frac{1-\psi}{1-l_i(s)}$ . Also, for the purposes of these notes, we will denote  $P_s = P(s, s)$ . These notes will follow same order as the computations in the code. To begin with, taking equation (2) we can solve for  $c_2(2)$  as

$$c_2(2) = \left( \frac{P(s_-, 1) c_1(1)^{-\sigma} + P(s_-, 2) c_1(2)^{-\sigma} - P(s_-, 1) c_2(1)^{-\sigma}}{P(s_-, 2)} \right)^{-1/\sigma} \quad (6)$$

Taking derivatives we obtain

$$\nabla c_2(2) = c_2(s)^{1+\sigma} \begin{bmatrix} \frac{RP_1}{P_2} c_1(1)^{-\sigma-1} \\ \frac{R}{P_2} c_1(2)^{-\sigma-1} \\ \frac{-P_1}{P_2} c_2(1)^{-\sigma-1} \end{bmatrix} \quad (7)$$

We compute  $R'(s)$  directly from equation (5). Taking derivatives we obtain

$$\nabla R'(s) = \sigma c_2(s)^{-\sigma} c_1(s)^{\sigma-1} \nabla c_1(s) - \sigma c_2(s)^{-\sigma-1} c_1(s)^{\sigma} \nabla c_2(s). \quad (8)$$

The wage equation, equation (3), gives us the following relationship between  $l_1$  and  $l_2$

$$l_1(s) = 1 - (1 - l_2(s)) \frac{\theta_2}{\theta_1} R'(s) \quad (9)$$

substituting into the resource constraint, equation (4), we quickly obtain

$$l_2(s) = \frac{n_1 c_1(s) + n_2 c_2(s) + g(s) + n_1 \theta_2 R'(s) - \theta_1 n_1}{\theta_2 (n_2 + R(s) n_1)} \quad (10)$$

which, when differentiated, gives us

$$\nabla l_2(s) = \frac{n_1 \nabla R'(s)}{n_2 + R'(s)n_2} - \frac{n_1 l_2(s) \nabla R'(s)}{n_2 + R'(s)n_1} + \frac{n_1 \nabla c_1(s)}{\theta_2(n_2 + R'(s)n_1)} + \frac{n_2 \nabla c_2(s)}{\theta_2(n_2 + R'(s)n_1)} \quad (11)$$

we then also get

$$\nabla l_1(s) = \nabla l_2(s) \frac{\theta_2}{\theta_1} R'(s) - (1 - l_2(s)) \frac{\theta_2}{\theta_1} \nabla R'(s) \quad (12)$$

Finally, rearranging equation (1),

$$x'(s) = \frac{\psi c_2(s)^{-\sigma} x}{\beta \sum_{s'} P_{s'} \psi c_2(s')^{-\sigma}} + \frac{(1 - \psi) l_2(s)}{1 - l_2(s)} - R'(s) \frac{(1 - \psi) l_1(s)}{1 - l_1(s)} + \psi c_1(s) c_2(s)^{-\sigma} - \psi c_2(s)^{1-\sigma} \quad (13)$$

Taking the gradient,

$$\begin{aligned} \nabla x'(s) = & \left( \frac{-\sigma \psi c_2(s)^{-\sigma-1}}{\beta \mathbb{E}[\psi c_2^{-\sigma}]} + \frac{\sigma \psi c_2(s)^{-2\sigma-1} \beta P_s}{(\beta \mathbb{E}[\psi c_2^{-\sigma}])^2} - \sigma \psi c_2(s)^{-\sigma-1} c_1(s) - (1 - \sigma) \psi c_2(s)^{-\sigma} \right) \nabla c_2(s) \\ & + \frac{\sigma \psi c_2(s)^{-\sigma} c_2(s^-)^{-\sigma-1} \beta P_{s^-}}{(\beta \mathbb{E}[\psi c_2^{-\sigma}])^2} \nabla c_2(s^-) + \psi c_2(s)^{-\sigma} \nabla c_1(s) + \frac{1 - \psi}{(1 - l_2(s))^2} \nabla l_2(s) \\ & - R'(s) \frac{1 - \psi}{(1 - l_1(s))^2} \nabla l_1(s) - \frac{(1 - \psi) l_1(s)}{1 - l_1(s)} \nabla R'(s) \end{aligned} \quad (14)$$

With the notation that  $s^-$  is 2 when  $s = 1$  and 1 when  $s = 2$ . The FOC necessary conditions that the optimizer must satisfy can then be written as

$$\sum_s P_s \left[ \sum_i \alpha_i (u_{c,i}(s) \nabla c_i(s) + u_{l,i}(s) \nabla l_i(s)) + \beta (V_x(x'(s), R'(s)) \nabla x'(s) + V_R(x'(s), R'(s)) \nabla R'(s)) \right] \quad (15)$$