Optimization Notes

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The problem we are trying to optimize is as follows.

$$\max_{c_1(s),c_2(s),l_1(s),l_2(s),x'(s),R'(s)} \sum_s P(s_-,s)\left[\sum_i \alpha_i\left(\frac{\psi c_i(s)^{1-\sigma}}{1-\sigma}+(1-\psi)\log(1-l_i(s))\right)+\beta V(x'(s),R'(s),s)\right]$$

subject to the constraints

$$\psi c_2(s)^{-\sigma}(c_2(s)-c_1(s))+x'(s)+\left(\frac{-(1-\psi)l_2(s)}{1-l_2(s)}+R'(s)\frac{(1-\psi)l_1(s)}{1-l_1(s)}\right)=\frac{\psi c_2(s)^{-\sigma}}{\beta\mathbb{E}[\psi c_2^{-\sigma}]} \tag{1}$$

$$\frac{\sum_s P(s_-,s)\psi c_2(s)^{-\sigma}}{\sum_s P(s_-,s)\psi c_1(s)^{-\sigma}}=R \tag{2}$$

$$\frac{(1-\psi)}{\theta_1(1-l_1(s))\psi c_1(s)^{-\sigma}}=\frac{(1-\psi)}{\theta_2(1-l_2(s))\psi c_2(s)^{-\sigma}} \tag{3}$$

$$n_1c_1(s)+n_2c_2(s)+g(s)=\theta_1 l_1(s)+\theta_2 l_2(s) \tag{4}$$

$$R'(s)=\frac{\psi c_2(s)^{-\sigma}}{\psi c_1(s)^{-\sigma}} \tag{5}$$

conditional on s_-, R and x. For $s=2$, given a guess of $z=(c_1(1),c_1(2),c_2(1)$ it is possible to find the variables $c_2(2),l_1,l_2,x',R'$ such that the equations above our satisfied and their derivatives with respect to z. Taking derivatives we can the perform an unconstrained optimization. From now on ∇ will refer to the vector of partial derivatives with respect to z, $u_{c,i}(s)$ will refer to $\psi c_i(s)^{-\sigma}$ and $u_{l,i}(s)=-\frac{1-\psi}{1-l_i(s)}$. Also, for the purposes of these notes, we will denote $P_s=P(s_-,s)$. These notes will follow same order as the computations in the code. To begin with, taking equation (2) we can solve for $c_2(2)$ as

$$c_2(2)=\left(\frac{P(s_-,1)c_1(1)^{-\sigma}+P(s_-,2)c_1(2)^{-\sigma}-P(s_-,1)c_2(1)^{-\sigma}}{P(s_-,2)}\right)^{-1/\sigma} \tag{6}$$

Taking derivatives we obtain

$$\nabla c_2(2)=c_2(s)^{1+\sigma}\begin{bmatrix}\frac{RP_1}{P_2}c_1(1)^{-\sigma-1}\\ Rc_1(2)^{-\sigma-1}\\ \frac{-P_1}{P_2}c_2(1)^{-\sigma-1}\end{bmatrix} \tag{7}$$

We compute $R'(s)$ directly from equation (5). Taking derivatives we obtain

$$\nabla R'(s)=\sigma c_2(s)^{-\sigma}c_1(s)^{\sigma-1}\nabla c_1(s)-\sigma c_2(s)^{-\sigma-1}c_1(s)^{\sigma}\nabla c_2(s). \tag{8}$$

The wage equation, equation (3), gives us the following relationship between l_1 and l_2

$$l_1(s)=1-(1-l_2(s))\frac{\theta_2}{\theta_1}R'(s) \tag{9}$$

substituting into the resource constraint, equation (4), we quickly obtain

$$l_2(s)=\frac{n_1c_1(s)+n_2c_2(s)+g(s)+n_1\theta_2R'(s)-\theta_1n_1}{\theta_2(n_2+R(s)n_1)}\textbf{??} \tag{10}$$

which, when differentiated, gives us

$$\nabla l_2(s) = \frac{n_1 \nabla R'(s)}{n_2 + R'(s) n_2} - \frac{n_1 l_2(s) \nabla R'(s)}{n_2 + R'(s) n_1} + \frac{n_1 \nabla c_1(s)}{\theta_2 (n_2 + R'(s) n_1)} + \frac{n_2 \nabla c_2(s)}{\theta_2 (n_2 + R'(s) n_1)} \tag{11}$$

we then also get

$$\nabla l_1(s) = \nabla l_2(s) \frac{\theta_2}{\theta_1} R'(s) - (1 - l_2(s)) \frac{\theta_2}{\theta_1} \nabla R'(s) \tag{12}$$

Finally, rearranging equation (1),

$$x'(s) = \frac{\psi c_2(s)^{-\sigma} x}{\beta \sum_{s'} P_{s'} \psi c_2(s')^{-\sigma}} + \frac{(1-\psi) l_2(s)}{1 - l_2(s)} - R'(s) \frac{(1-\psi) l_1(s)}{1 - l_1(s)} + \psi c_1(s) c_2(s)^{-\sigma} - \psi c_2(s)^{1-\sigma} \tag{13}$$

Taking the gradient,

$$\begin{aligned}
\nabla x'(s) = & \left(\frac{-\sigma \psi c_2(s)^{-\sigma-1}}{\beta \mathbb{E}[\psi c_2^{-\sigma}]} + \frac{\sigma \psi c_2(s)^{-2\sigma-1} \beta P_s}{(\beta \mathbb{E}[\psi c_2^{-\sigma}])^2} - \sigma \psi c_2(s)^{-\sigma-1} c_1(s) - (1-\sigma) \psi c_2(s)^{-\sigma} \right) \nabla c_2(s) \\
& + \frac{\sigma \psi c_2(s)^{-\sigma} c_2(s^-)^{-\sigma-1} \beta P_{s^-}}{(\beta \mathbb{E}[\psi c_2^{-\sigma}])^2} \nabla c_2(s^-) + \psi c_2(s)^{-\sigma} \nabla c_1(s) + \frac{1-\psi}{(1 - l_2(s))^2} \nabla l_2(s) \\
& - R'(s) \frac{1-\psi}{(1 - l_1(s))^2} \nabla l_1(s) - \frac{(1-\psi) l_1(s)}{1 - l_1(s)} \nabla R'(s)
\end{aligned} \tag{14}$$

With the notation that s^- is 2 when $s = 1$ and 1 when $s = 2$. The FOC necessary conditions that the optimizer must satisfy can then be written as

$$\sum_s P_s \left[\sum_i \alpha_i \left(u_{c,i}(s) \nabla c_i(s) + u_{l,i}(s) \nabla l_i(s) \right) + \beta \left(V_x(x'(s), R'(s)) \nabla x'(s) + V_R(x'(s), R'(s)) \nabla R'(s) \right) \right] \tag{15}$$