Taxation, redistribution, and debt with aggregate shocks*

Mikhail Golosov
Princeton

Thomas J. Sargent
NYU

July 16, 2012

Abstract

We study optimal income taxes and transfers in an economy with heterogeneous agents and aggregate shocks. An optimal equilibrium determines agents' net asset positions, but not their absolute levels. The distribution of debt holdings across agents influences optimal allocations and taxes, but the level of government debt does not. Higher correlations of debt holdings and labor incomes imply more distortions and lower welfare. In incomplete markets economies, the government has no precautionary incentive to accumulate assets to smooth government expenditures, an outcome that implies different optimal taxes and government debt than representative agent Ramsey models like Aiyagari et al. (2002). Imposing exogenous borrowing constraints can actually increase welfare. When markets are incomplete, the government optimally responds to an increase in government expenditures by increasing taxes and decreasing transfers. These responses are persistent and history dependent.

*We thank Mark Aguiar, Anmol Bhandari, V.V. Chari, David Evans, Guy Laroque, Robert E. Lucas, Jr., Ali Shourideh, and seminar participants at Bocconi, Chicago, EIEF, the Federal Reserve Bank of Minneapolis, Princeton, and UCL for helpful comments. We thank Anmol Bhandari and David Evans for excellent help with the computations.

1 Introduction

When a government commits to paying its debts,

- Are higher levels of government debt more distortionary than lower levels?
- How should debt and taxes respond to random fluctuations in government expenditures?

In the macroeconomics literature, widely asserted answers to these questions are:

- Higher levels of government debt are more distortionary because the deadweight cost of taxation is convex in tax rates; and
- If markets are incomplete, a precautionary motive induces the government to accumulate assets. As time passes, the government finances an increasing fraction of its expenditures from earnings on its assets.

Papers that offer these answers typically use a representative agent framework that restricts a government to levy linear taxes on labor income and perhaps lump-sum transfers but not lump-sum taxes. Typical justifications for excluding lump-sum taxes are either that some agents cannot afford to pay lump-sum taxes or that a government that cares about redistribution would not want to impose them. These justifications are not modeled explicitly in the typical representative agent Ramsey model.

In this paper, we study a heterogenous agent economy in which agents differ in their productivities. A government optimally chooses taxes and transfers to finance stochastic expenditures and to redistribute goods. *A priori*, we impose no restrictions on the sign of transfers. If some agents are sufficiently poor or if the government wants enough redistribution, the government optimally chooses only positive transfers.

We start by considering an incomplete markets economy without capital in which agents can trade only a one-period risk-free bond. The government imposes an affine income tax that consists of a proportional tax on labor income and a lump-sum tax or transfer. Our main result shows that an optimum determines agents' net asset positions, defined as a list of assets of agent i minus assets of agents j. The absolute level of the distribution of assets is indeterminate, which means that the level of the debt for one agent can be normalized arbitrarily. All gross distributions of assets that imply the same optimal net asset positions for private agents support the same optimal equilibrium allocation. We also show how this result extends to more general structures of shocks, asset markets (complete or incomplete), taxes, and the presence or absence of physical capital.

The finding that an optimal policy determines only relative debt levels has implications about the welfare costs of government debt and optimal responses of taxes and transfers to expenditure shocks. First, the absolute *level* of the initial government debt *per se* has no implications about allocations that can be achieved in equilibrium, but the *relative distribution* of asset holdings does. In an economy with a high level of government debt that is distributed equally across households, the government can reduce transfers and achieve the same allocation as in an economy with no government debt. For the same quantity of initial debt, the more concentrated the distribution of debt holdings is in the right tail of the labor income distribution, the higher are optimal taxes and the lower is welfare. Secondly, since the absolute level of government debt is indeterminate, there is an optimal equilibrium that sets government debt to zero always. Since this outcome prevails even in incomplete market economies, it means that generally the government has no precautionary saving motive, in contrast with results in representative agent economies obtained by Aiyagari et al. (2002, aka AMSS), Faraglia et al. (2012), and others.

The absence of a government precautionary saving motive leads to starkly different properties of allocations and optimal taxes vis-a-vis a representative agent Ramsey model like AMSS's. When agents' preferences are quasi-linear (i.e., linear in consumption), the AMSS model has a government accumulate assets until eventually it can finance all of its expenditures from earnings on its assets. Meanwhile, distortionary taxes stochastically converge to zero. In contrast, when agents are heterogenous and transfers are set optimally, both taxes and debt are constant across all dates and states. We show that this difference is not the outcome of turning on and off the restriction that transfers must be non-negative but instead comes from the government's redistributive motive. Even when they are imposed, non-negativity constraints on transfers do not bind if agents are sufficiently heterogeneous. The lack of heterogeneity across agents makes those non-negativity constraints bind in representative agent models, except when the first-best allocation is achievable. In a representative agents model with incomplete markets, the prospect that those constraints will bind gives the government a precautionary motive that impels it to accumulate assets. That motive endures so long as there remain future states in which those non-negativity constraints might bind.

Our results are even starker when agents face exogenous borrowing constraints and government expenditure shocks. We show that any competitive equilibrium allocation in an economy in which agents face the weakest possible borrowing constraints is also a competitive equilibrium in another economy in which agents face arbitrarily tighter borrowing constraints. The reason is that in response to an expenditure shock, the government can structure transfers so that individual borrowing constraints never bind, which implies that welfare in the economy with exogenous borrowing constraints is weakly higher than in the unconstrained economy. We also show that exogenously tighter borrowing constraints may allow the government strictly to increase welfare. It may be optimal to structure transfers so that some agents hit their borrowing limits because changes in interest rates have different effects on constrained and unconstrained agents in ways that can facilitate redistribution. As in the economy with no borrowing constraints, by itself the size of the initial government debt has no effect on achievable allocations; only agents' net initial asset positions determine optimal taxes and allocations.

We provide analytical examples and numerical simulations that show that in response to a positive shock to government spending, it is optimal to increase taxes and decrease transfers. Moreover, optimal taxes and transfers are history dependent, with longer sequences of realizations of positive government expenditure shocks implying larger tax increases and transfer decreases.

The last part of this paper proceeds in the spirit of the New Dynamic Public Finance literature by allowing general taxes subject to explicitly defined informational restrictions. We assume that skills of agents are unobserved and consider two economies, one in which agents' assets are perfectly observable and the other in which they are unobservable. We show that in the first economy, optimal distortions depend only on the current realization of government expenditures, not on history, and that any distribution of assets in each period can be supported as an optimal competitive equilibrium. In contrast, the optimal allocations in the second economy pin down agents' relative, but not absolute, asset positions. Moreover, when markets are incomplete, optimal allocations and distortions are history dependent. Thus, not being able to observe individuals' asset positions emerges as a key friction shaping stochastic properties of optimal assets and distortions.

Our paper is closely related to several strands of literature. A large literature, both with incomplete markets, such as AMSS, Farhi (2010), Faraglia et al. (2012), and with complete markets, such as Lucas and Stokey (1983), Chari, Christiano and Kehoe (1994), and Chari and Kehoe (1999), studies optimal taxation with aggregate shocks in representative agent economies. Papers by da Costa and Lutz (2011), Golosov, Tsyvinski, and Werning (2007) and Kocherlakota (2005) study optimal taxation in the NDPF framework, but their focus differs from ours.

Three papers that are perhaps most closely related to ours are Bassetto (1999), Shin (2006), and Werning (2007). Like us, those authors depart from a representative agent assumption by allowing heterogeneity and considering distributional consequences of alternative tax and borrowing policies. Bassetto extends the Lucas-Stokey (1983) environment to include I types of agents who are heterogeneous in their time-invariant labor productivities w^i. There are complete markets and a Ramsey planner who puts Pareto weight ω^i on type i agents and who has access only to proportional taxes on labor income and history-contingent borrowing and lending. Bassetto studies how the Ramsey planner's vector of Pareto weights influence how he responds to government expenditures and other shocks by adjusting the proportional labor tax and government borrowing to cover expenses while manipulating prices in ways that redistribute wealth between, on the one hand, low-w_i types ('rentiers') whose main income source is their history-dependent income from assets, and on the other hand, high-w_i types 'workers' whose main income source is their labor income.

Shin (2006) extends the AMSS economy to have two households who face idiosyncratic income risk. When that idiosyncratic income risk is big enough relative to aggregate government expenditure risk, the Ramsey planner chooses to issue debt in order to help households engage in precautionary saving, thereby overturning the AMSS result that the Ramsey planners sets taxes to zero eventually. Shin emphasizes that the government does this at the cost of imposing tax distortions. Shin's Ramsey planner balances two competing self-insurance motives: aggregate tax smoothing and individual consumption smoothing.

Werning (2007) also allows heterogeneity and studies optimal transfers. He focuses on complete market economies and obtains counterparts to some of our results, including that government assets can be set to zero in all periods. He allows unrestricted taxation of initial assets, so that in his economy the initial distribution of assets plays no role. Our main result, Theorem 1 and its corollaries, substantially generalize his results by showing that any allocation of assets among agents and the government that implies the same optimal net asset position leads to the same optimal allocations. This conclusion holds for very general market structures. We also explore the role of borrowing constraints. Werning (2007) provides an extensive characterization of optimal allocations and distortions in complete market economies, while we focus on incomplete market economies and the role of precautionary savings motives for private agents and the government that are not present when markets are complete.

Our paper is organized as follows. Section 2 lays out our benchmark environment and derives our main results about the distribution of the net asset holdings and its implications. Section 3 characterizes optimal affine taxes and allocations in an economy with incomplete markets and then analyzes the government's precautionary savings motive. Section 4 extends our analysis to economies with arbitrary borrowing constraints and capital. Section 5 studies optimal taxes subject to explicit informational constraints. Section 6 offers concluding remarks.

2 Environment

There are I types of infinitely lived agents. There is a mass π_i of a type $i \in I$ agent, with $\sum_{i=1}^{I} \pi_i = 1$. Preferences of an agent of type i over stochastic processes for consumption $\{c_{i,t}\}$ and labor supply $\{l_{i,t}\}$ are ordered by

$$\mathbb{E}_0 \sum_{t=1}^{\infty} \beta^t U^i \left(c_{i,t}, l_{i,t}\right) \tag{1}$$

where $\mathbb{E}_t$ is a mathematical expectations operator conditioned on time t information and $\beta \in (0,1)$ is a discount factor. We assume that $l_i \in \left[0, \bar{l}_i\right]$ for some $\bar{l}_i < \infty$. For most of our main results, such as those in this section and also sections 4 and 5, we need no additional assumptions on U^i, such as differentiability or convexity, so that our setup allows both extensive and intensive responses of labor.

An agent of type i who supplies l_i units of labor produces $\theta_i l_i$ units of output, where $\theta_i \in \Theta$ is a nonnegative scalar. Feasible allocations satisfy

$$\sum_{i=1}^{I} \pi_i c_{i,t} + g_t = \sum_{i=1}^{I} \pi_i \theta_i l_{i,t}, \tag{2}$$

where g_t denotes government expenditures g_t that follow an irreducible finite-state Markov process. Let $s_t = g_t$ and $s^t = (s_0, ..., s_t)$. We use two notations to track histories. Most of the time we use a notation z_t to denote a random variable with a time t conditional distribution that is a function of the history g^t. In some places, we use a more explicit notion $z\left(s^t\right)$ to denote a realization of the stochastic process z_t at a particular history s^t.

The government's preferences over stochastic process for consumption and work are ordered by

$$\mathbb{E}_0 \sum_{i=1}^{I} \pi_i \alpha_i \sum_{t=0}^{\infty} \beta^t U^i \left(c_{i,t}, l_{i,t}\right) \tag{3}$$

where $\alpha_i \geq 0$, $\sum_{i=1}^{I} \alpha_i = 1$ is a set of Pareto weights.

We start with particular structures of asset markets and taxes. We assume that all agents can trade only a one-period risk-free bond, and that the government can impose only an affine tax on labor income. We do this to make close contact with the earlier literature on Ramsey optimal taxation with incomplete markets. As we show later, Theorem 1, our main result in this section, remains true under much more general assumptions about the market structure. In particular, it remains true if agents can trade debt of more maturities or if they have access to a full set of history-contingent claims. It is also straightforward to extend our analysis to allow more general forms of taxes.

Under an affine tax system, agent i's budget constraint at t is

$$c_{i,t} + b_{i,t} = (1 - \tau_t)\,\theta_i l_{i,t} + R_{t-1} b_{i,t-1} + T_t, \tag{4}$$

where $b_{i,t}$ denotes asset holdings of agent i at time $t \geq 0$, R_{t-1} is a gross risk-free one-period interest rate from time $t-1$ to time t for $t \geq 1$, and $R_{-1} \equiv 1$. For $t \geq 0$, R_t is measurable with respect to time t information. To rule out Ponzi schemes, we assume that $b_{i,t}$ must be bounded from below. In this section, we impose no further constraints on agents abilities to borrow and lend. In section 4, we extend our analysis to economies with arbitrary borrowing constraints.

The government budget constraint is

$$g_t + B_t = \tau_t \sum_{i=1}^{I} \pi_i \theta_i l_{i,t} - T_t + R_{t-1} B_{t-1}, \tag{5}$$

where B_t denotes the government's assets at time t. We assume that government assets are bounded from below. Our assumptions about preferences imply that the government can collect finite revenues in each period, so this restriction rules out government Ponzi schemes.

We assume that agents and the government start with initial assets $\{b_{i,-1}\}_{i=1}^{I}$ and B_{-1}, respectively, and that asset holdings satisfy the market clearing condition

$$\sum_{i=1}^{I} \pi_i b_{i,t} + B_t = 0 \text{ for all } t \geq -1. \tag{6}$$

Since all $b_{i,t}$ and B_t are bounded from below, constraint (6) implies that they must also be bounded from above.

We allow the government to choose a feasible sequence of transfers $\{T_t\}$. We do not restrict the sign of T_t at any particular date or history. This means that our analysis covers the following example: if type i has $\theta_i = 0$ and no initial wealth, the present value of transfers to agent i must necessarily be nonnegative. All results in the present paper include this example as a special case.

We now define the components of a competitive of equilibrium.

Definition 1 *An* allocation *is a sequence* $\{c_{i,t}, l_{i,t}\}_{i,t}$. *An* asset profile *is a sequence* $\{\{b_{i,t}\}_i, B_t\}_t$. *A* price system *is an interest rate sequence* $\{R_t\}_t$. *A* tax policy *is a sequence* $\{\tau_t, T_t\}_t$.

Definition 2 *For given initial asset positions* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$ *and tax policy* $\{\tau_t, T_t\}_t$, *a* <u>*competitive equilibrium with affine taxes*</u> *is a sequence* $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ *such that the allocation and the private components* $\{b_{i,t}\}_{i,t}$ *of the asset profile maximize (1) subject to (4), the asset profile* $\{\{b_{i,t},\}_i, B_t\}_t$ *is bounded, and constraints (2), (5) and (6) are satisfied.*

We will focus on optimal competitive equilibria throughout this paper.

Definition 3 *Given $(\{b_{i,-1}\}_i, B_{-1})$, an optimal competitive equilibrium with affine taxes is a tax policy $\{\tau_t^*, T_t^*\}_t$, an allocation $\left\{\left\{c_{i,t}^*, l_{i,t}^*, b_{i,t}^*\right\}_i, B_t^*\right\}_t$, and a price system $\{R_t^*\}_t$ such that (i) given $\left(\{b_{i,-1}\}_i, B_{-1}\right)$ and the price system $\{R_t^*\}_t$, $\left\{\left\{c_{i,t}^*, l_{i,t}^*, b_{i,t}^*\right\}_i, B_t^*\right\}_t$ is a competitive equilibrium allocation given $\left(\{b_{i,-1}\}_i, B_{-1}\right)$ and $\{\tau_t^*, T_t^*\}_t$; and (ii) there are no other taxes $\{\tau_t, T_t\}_t$ such that a competitive equilibrium given $\left(\{b_{i,-1}\}_i, B_{-1}\right)$ and $\{\tau_t, T_t\}_t$ has strictly higher value of (3).*

We will call taxes $\{\tau_t^*, T_t^*\}_t$ an *optimal tax policy*, allocation $\{c_{i,t}^*, l_{i,t}^*\}_{i,t}$ an *optimal allocation*, and asset profile $\left\{\left\{b_{i,t}^*\right\}_i, B_t^*\right\}_t$ an *optimal asset profile.*

A key object in our analysis will be agents' *net assets positions* $\left\{\tilde{b}_{i,t}\right\}_{t,i\geq 2}$ defined as

$$\tilde{b}_{i,t} = b_{i,t} - b_{1,t} \text{ for all } t \geq -1, i \geq 2.$$

The significance of this object is explained in the following theorem, a main result of the paper.[1]

Theorem 1 *Let $\left\{\left\{c_{i,t}^*, l_{i,t}^*, b_{i,t}^*\right\}_i, B_t^*, R_t^*\right\}_{t\geq 0}$ be a competitive equilibrium given $\left(\left\{b_{i,-1}^*\right\}_i, B_{-1}^*\right)$ and $\{\tau_t^*, T_t^*\}_t$. For any bounded sequences $\left\{\hat{b}_{i,t}\right\}_{i,t\geq -1}$ that satisfy*

$$\hat{b}_{i,t} - \hat{b}_{1,t} = \tilde{b}_{i,t}^* \equiv b_{i,t}^* - b_{1,t}^* \textit{ for all } t \geq -1, i \geq 2,$$

there exists a sequence $\left\{\hat{T}_t\right\}_{t\geq 0}$ and $\left\{\hat{B}_t\right\}_{t\geq -1}$ that satisfies (6) such that $\left\{\left\{c_{i,t}^*, l_{i,t}^*, \hat{b}_{i,t}\right\}_i, \hat{B}_t, R_t^*\right\}_t$ is a competitive equilibrium given $\left(\left\{\hat{b}_{i,-1}\right\}_i, \hat{B}_{-1}\right)$ and $\left\{\tau_t^*, \hat{T}_t\right\}_t$.

Proof. Let

$$\hat{T}_t = T_t^* + \left(\hat{b}_{1,t} - b_{1,t}^*\right) - R_{t-1}^*\left(\hat{b}_{1,t-1} - b_{1,t-1}^*\right) \text{ for all } t \geq 0. \tag{7}$$

Given taxes $\left\{\tau_t^*, \hat{T}_t\right\}_t$, the allocation $\left\{c_{i,t}^*, l_{i,t}^*, \hat{b}_{i,t}\right\}_t$ is a feasible choice for consumer i since it satisfies

$$\begin{aligned}
c_{i,t}^* &= (1-\tau_t^*)\,\theta_i l_{i,t}^* + R_{t-1}^*\hat{b}_{i,t-1} - \hat{b}_{i,t} + \hat{T}_t \\
&= (1-\tau_t^*)\,\theta_i l_{i,t}^* + R_{t-1}^*\left(\hat{b}_{i,t-1} - \hat{b}_{1,t-1}\right) - \left(\hat{b}_{i,t} - \hat{b}_{1,t}\right) + T_t^* + R_{t-1}^* b_{1,t-1}^* - b_{1,t}^* \\
&= (1-\tau_t^*)\,\theta_i l_{i,t}^* + R_{t-1}^*\left(b_{i,t-1}^* - b_{1,t-1}^*\right) - \left(b_{i,t}^* - b_{1,t}^*\right) + T_t^* + R_{t-1}^* b_{1,t-1}^* - b_{1,t}^* \\
&= (1-\tau_t^*)\,\theta_i l_{i,t}^* + R_{t-1}^* b_{i,t-1}^* - b_{i,t}^* + T_t^*.
\end{aligned}$$

[1] Wallace's (1981) Modigliani-Miller theorem for a class of government open market operations has a similar flavor. Sargent (1987) describes the structure of a set of related Modigliani-Miller theorems for government finance.

Suppose it is not the optimal choice, in the sense that there exists some other sequence $\left\{\hat{c}_{i,t}, \hat{l}_{i,t}, \hat{b}_{i,t}\right\}_t$ that gives strictly higher utility to consumer i. Then the choice $\left\{\hat{c}_{i,t}, \hat{l}_{i,t}, b^*_{i,t}\right\}_t$ is feasible given taxes $\{\tau^*_t, T^*_t\}_t$, which contradicts the assumption that $\left\{c^*_{i,t}, l^*_{i,t}, b^*_{i,t}\right\}_t$ was the optimal choice for the consumer given taxes $\{\tau^*_t, T^*_t\}_t$. The new allocation satisfies all feasibility constraints and therefore is an equilibrium. ■

Theorem 1 shows that in general there are many different transfer schemes $\{T_t\}_t$ and asset profiles $\{b_{i,t}, B_t\}_{i,t}$ that support the same equilibrium allocations. As a result there are many optimal transfer and government debt sequences $\{T_t, B_t\}_t$.

Next we list some important implications of theorem 1.

Corollary 1 ***(Irrelevance of the level of government debt)*** *For any B'_{-1}, B''_{-1} there are sequences $\left\{b'_{i,-1}\right\}_i$ and $\left\{b''_{i,-1}\right\}_i$ such that optimal equilibrium allocations given $\left(\left\{b'_{i,-1}\right\}_i, B'_{-1}\right)$ and given $\left(\left\{b''_{i,-1}\right\}_i, B''_{-1}\right)$ are the same.*

Corollary 1 shows that it is not the level of government debt that matters for equilibrium allocations, but rather who owns government debt. The following example provides some intuition.

Example 1 *Suppose that we increase the initial level of government debt from 0 to some level B'_{-1}. If transfers $\{T_t\}_t$ were to be held fixed, the government would want to increase taxes $\{\tau_t\}_t$ to collect a present value of revenues sufficient to repay B'_{-1}. Since deadweight losses are convex in τ, higher levels of debt would then impose disproportionately larger distortions on the economy, which makes higher levels of debt particularly bad. But now consider how this conclusion would change if we were to allow the government to adjust transfers. To find optimal transfers, we need to know how the holdings of the debt B'_{-1} are distributed across agents. Suppose that agents hold equal amounts of the new debt. In this case, each unit of debt repayment does exactly as much redistribution as one unit of transfers. Since the original level of transfers at zero government debt was optimal, the best policy for the government with the level of debt B'_{-1} is to reduce transfers by exactly the amount of the increase in per capita debt. As a result, distorting taxes $\{\tau_t\}$ and allocations remain unchanged.*

The calculation in example 1 is sensitive to the assumption that holdings of the government debt are equal across agents. Suppose instead that the new debt is owned disproportionately by richer households. That implies that inequality is effectively initially higher in the economy with higher debt. As a result the government would need to increase both taxes $\{\tau_t\}$ and transfers $\{T_t\}$ to offset the increase in inequality associated with the increase in government debt. The conclusion would be the opposite if the new debt were to be disproportionately owned by poorer households. We further investigate welfare consequences of the initial distribution of government debt in some numerical examples in section 3.5.

Example 1 highlights several conclusions that come out of our analysis. It is not the level of the government debt that matters for achievable optimal allocations but rather how it is distributed across households. In general, government debts that are widely distributed across households (e.g., implicit Social Security debt) are less distortionary than ones concentrated in the right tail of the income distribution (e.g., government debt held by hedge funds).[2]

Corollary 2 ***(No government precautionary accumulation of assets)*** *Without loss of generality we can set* $B_t^* = 0$ *(or* $b_{i,t}^* = 0$ *for some* i*) for all* t*.*

Corollary 2 shows that outcomes from our model differ starkly from incomplete market models with representative agents, e.g., AMSS or Faraglia et al. (2012). In those models, market incompleteness gives the government a precautionary motive to accumulate assets to assist smoothing distortions due to taxes. As government assets increase, the government decreases taxes and relies more on income from its assets to finance its expenditures. Corollary 2 shows that this force evaporates when the government can and wants to redistribute. We investigate the relationship between results in AMSS and our model further in section 3.3.

Corollary 3 ***(No need for lump-sum taxes)*** *There are many parameter values for which* $T_t^* \geq 0$ *for all* t*.*

Corollary 3 identifies cases in which we do not need to rely on lump-sum taxes for our arguments to go through. The easiest way to see this is to consider an economy in which $\theta_1 = 0$ and $b_{1,-1} = 0$, so that one of the agents cannot work and has no labor income. By Corollary 2, there is an optimal equilibrium that has $b_{1,t}^* = 0$ for all t. Since consumption is non-negative, this implies that the only feasible transfers that satisfy (4) are non-negative. Moreover, from Theorem 1, we can choose many other initial conditions $\{b_{i,-1}\}$ that also would imply that optimal transfers must be positive. Alternatively, in an economy with any distribution of types $\{\theta_i\}_i$, starting with any optimal sequence of transfers, we can use equation (7) to construct an alternative, strictly positive, sequence of transfers, provided that the initial assets of at least one type of agent are sufficiently low.

[2] It is straightforward to extend our analysis to open economy with foreign holdings of domestic debt. The more government debt is owned by the foreigners, the higher are the distortions the government will need to impose.

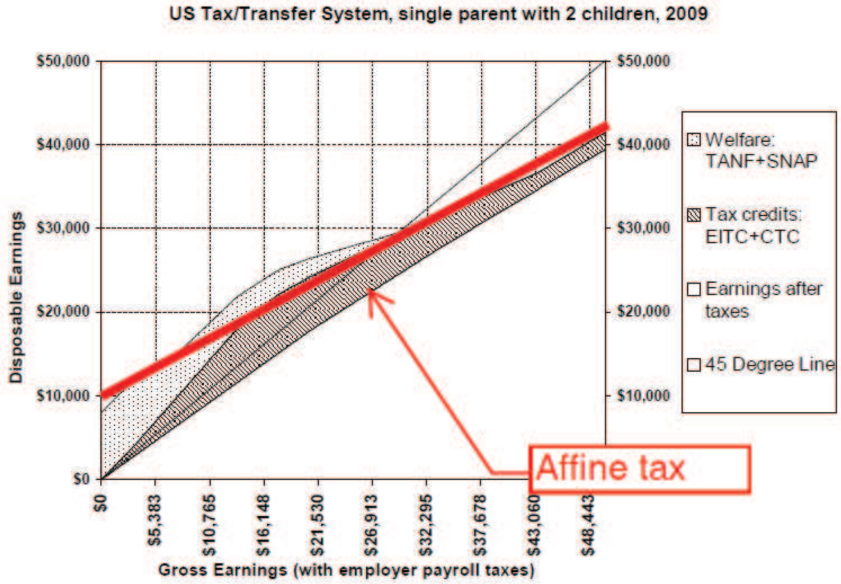

Figure 1: The U.S. tax-transfer system is poorly approximated by a linear function, better by an affine function.

More generally, corollary 3 does not require that one type of agent have no income. If the government puts a sufficiently high Pareto weight on a low-skilled type, optimal transfers are typically positive.

However, we do not view an a priori restriction of transfers to be non-negative as desirable. Rather, taxes and transfers should be set optimally given the government's distributional objectives and agents' abilities to pay. The fact that modern governments choose strictly positive transfers, as Figure 1 that summarizes the U.S. tax and transfer system indicates, suggests to us that the empirically relevant range of parameters governing inequality and the government's distributional objectives set optimal transfers to be strictly positive.[3] But if one has strong reasons to believe that the structure of an economy is such that a poll tax is feasible and desirable, we do want to rule out such a tax by assumption.

Finally, we offer a comment about aggregate shocks. We assumed that the only uncertainty comes from the aggregate shock g_t and that agents can trade only one-period risk-free debt. These are unnecessarily restrictive. We can define an economy with a general structure of shocks and assets in which we allow $\theta_{i,t}$ to be stochastic and subject to both aggregate and idiosyncratic shocks. We can also introduce additional preference shocks $\iota_{i,t}$ to agents' utility

[3]Figure 1 shows that while the U.S. tax-and-transfer system is poorly described by a linear tax on labor earnings, it is pretty well approximated by an affine tax-transfer system with a positive intercept indicative of lump-sum transfers, not taxes.

functions, $U^i(c_{i,t}, l_{i,t}, \iota_{i,t})$ (which again may be aggregate or idiosyncratic) and allow agents to trade risk-free debts of different maturities or even a complete set of Arrow securities. For this more general economy, the proof of Theorem 1 remains unchanged. As a result, we have

Corollary 4 *(**General assets and shocks**) Theorem 1 remains true for an economy with a general structure of shocks and assets.*

Theorem 1 also has powerful and surprising implications for economies in which agents face exogenous borrowing constraints, as we explore in section 4.

Finally, the proof of Theorem 1 goes through if the government can use more general than affine taxes.

Corollary 5 *(**General income taxes**) Theorem 1 holds for economies in which the government can impose an arbitrary non-linear tax $T_t(y_t)$ or $T_t(y_t, ..., y_0)$.*

3 Characterization of the optimum with affine taxes

In this section, we provide a partial characterization of optimal equilibria. We proceed in several steps. First, we derive general implementability conditions and a Bellman equation associated with a planning problem. Then we discuss two special cases that we can solve analytically. We also contrast our findings with those from representative agent models. Finally, we provide numerical simulations for more general cases.

In this section, we impose further assumptions on the utility function to simplify our analysis. We assume that $U^i : \mathbb{R}^2_+ \to \mathbb{R}$ is concave in $(c, -l)$ and twice continuously differentiable. Let $U^i_{x,t}$ or $U^i_{xy,t}$ denote first and second derivatives of U^i with respect to $x, y \in \{c, l\}$ in period t and assume that $\lim_{x\to\bar{l}_i} U^i_l(c, x) = \infty$, $\lim_{x\to 0} U^i_l(c, x) = 0$ for all c and i.

3.1 Implementability conditions

We focus on an interior equilibrium. Necessary conditions for consumer optimality are

$$(1 - \tau_t)\,\theta_i U^i_{c,t} = -U^i_{l,t} \tag{8}$$

and

$$U^i_{c,t} = \beta R_t \mathbb{E}_t U^i_{c,t+1}. \tag{9}$$

To characterize an equilibrium, we require

Lemma 1 *Any sequence* $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\}_t$ *is part of a competitive equilibrium with affine taxes if and only if it satisfies (2), (4), (8), and (9) and* $b_{i,t}$ *is bounded for all* i *and* t.

Proof. Necessity is obvious. In the appendix we use arguments of Magill and Quinzii (1994) and Constantinides and Duffie (1996) to show that any $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ that satisfies (4), (8), and (9) is a solution to consumer i's maximization problem. Equilibrium B_t is determined by (6) and constraint (5) is then implied by Walras' Law ■

To find the optimal equilibrium, by Lemma 1 we can choose $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, R_t, \tau_t, T_t\}_t$ to maximize (3) subject to (2), (8), and (9). We follow steps similar to ones taken by Lucas and Stokey (1983) and AMSS and apply a first-order approach. Substituting consumers' first-order conditions (8) and (9) into the budget constraints (4) yields implementability constraints of the form

$$c_{i,t} + b_{i,t} = -\frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} + T_t + \frac{U^i_{c,t-1}}{\beta \mathbb{E}_{t-1} U^i_{c,t}} b_{i,t-1} \text{ for all } i, t. \tag{10}$$

For $I \geq 2$, we can use constraint (10) for one of the agents, e.g. $i = 1$, to eliminate T_t from (10) for all other agents. Define $\tilde{b}_{i,t} \equiv b_{i,t} - b_{1,t}$ and represent the implementability constraints as

$$\begin{aligned} & (c_{i,t} - c_{1,t}) + \tilde{b}_{i,t} \\ = \ & -\frac{U^i_{l,t}}{\theta_i U^i_{c,t}} (\theta_i l_{i,t} - \theta_1 l_{1,t}) + \frac{U^i_{c,t-1}}{\beta \mathbb{E}_{t-1} U^i_{c,t}} \tilde{b}_{i,t-1} \text{ for all } i > 1, t. \end{aligned} \tag{11}$$

With this representation of the implementability constraints, the maximization problem depends only on the $I - 1$ variables $\tilde{b}_{i,t-1}$. Thus, we reduced the dimensionality of the problem from I to $I - 1$. This is another consequence of Theorem 1.

Though economic outcomes differ markedly, the mathematical structure of the Ramsey problem in our heterogeneous agent incomplete markets economy with affine labor taxes resembles that for representative agent economies with linear taxes studied by AMSS and Farhi (2010). We can follow their steps of analysis. Multiply (11) by $U^i_{c,t}$ and define $\tilde{a}_{i,t} \equiv \tilde{b}_{i,t} U^i_{c,t}$ to obtain

$$U^i_{c,t} (c_{i,t} - c_{1,t}) + \frac{U^i_{l,t}}{\theta_i} (\theta_i l_{i,t} - \theta_1 l_{1,t}) + \tilde{a}_{i,t} = \frac{U^i_{c,t}}{\beta \mathbb{E}_{t-1} U^i_{c,t}} \tilde{a}_{i,t-1} \text{ for all } i > 1, t. \tag{12}$$

Let $\beta^t \Pr(s^t) \psi_i(s^t)$ be a multiplier on this constraint in a Lagrangian for the Ramsey planner. First-order conditions with respect to $\tilde{a}_i(s^t)$ imply

$$\begin{aligned} \psi_{i,t} &= \left(\mathbb{E}_t \left[U^i_{c,t+1}\right]\right)^{-1} \mathbb{E}_t \left[U^i_{c,t+1} \psi_{i,t+1}\right] \\ &= \mathbb{E}_t \psi_{i,t+1} + \left(\mathbb{E}_t \left[U^i_{c,t+1}\right]\right)^{-1} \mathrm{Cov}_t \left(U^i_{c,t+1}, \psi_{i,t+1}\right). \end{aligned} \tag{13}$$

This is a multi-agent counterpart of equation (17) of AMSS (2002) for their representative agent economy with a linear tax on labor. The Lagrange multiplier $\psi_{i,t}$ measures the distortion of the tax system: when $\psi_{i,t} = 0$ for all i, the distorting tax τ_t equals zero. Equations like (13) are often taken to imply that distortions follow a "random-walk-like" process, partly confirming Barro's (1979) insight about tax smoothing. In representative agent economies with linear taxes, they also can imply that asymptotically optimal distortions decline (possibly to zero) while government asset holdings grow.[4] We have seen from Theorem 1 that with heterogeneous agents and affine taxes, government assets are indeterminate and therefore play no necessary role in shaping equilibrium allocations and distortions. Nevertheless, the next section reveals a Barro-AMSS-like insight about smoothing distortions that holds with affine taxation despite the fact that salient outcomes about long run levels of government owned assets and tax distortions obtained by AMSS and Farhi (2010) under linear taxes evaporate.

3.2 Bellman equations

In the spirit of Kydland and Prescott (1980) and Farhi (2010), we can formulate the planner's problem recursively. Let $\tilde{\mathbf{b}} = \left(\tilde{b}_2, ..., \tilde{b}_I\right)$ and $\mathbf{u} = \left(U_c^1, ..., U_c^I\right)$. For $t \geq 1$, the planner's Bellman equation is

$$V(\tilde{\mathbf{b}}, \mathbf{u}, g_-) = \max_{\tilde{\mathbf{b}}', \mathbf{c}, \mathbf{l}} \sum_g \Pr(g|g_-) \left[\sum_i \pi_i \alpha_i U^i(g) + \beta V\left(\tilde{\mathbf{b}}', \mathbf{U}_c^i(g), g\right)\right] \tag{14}$$

$$\left(c_i(g) - c_1(g)\right) + \tilde{b}_i'(g) + \frac{U_l^i(g)}{\theta_i U_c^i(g)}\left(\theta_i l_i(g) - \theta_1 l_1(g)\right) = \frac{u_i}{\beta \mathbb{E}_{s_-} U_c^i(g)} \tilde{b}_i \tag{15}$$

$$\forall g, \forall i > 1$$

$$\frac{\sum_s \Pr(g|g_-) U_c^i(g)}{u_i} = \frac{\sum_s \Pr(g|g_-) U_c^j(g)}{u_j} \; \forall i, j, g, \tag{16}$$

$$\frac{U_l^i(g)}{\theta_i U_c^i(g)} = \frac{U_l^j(g)}{\theta_j U_c^j(g)} \; \forall i, j, g, \tag{17}$$

$$\sum_i \pi_i c_i(g) + g = \sum_i \pi_i \theta_i l_i(g) \; \forall g. \tag{18}$$

Policies for $\tilde{\mathbf{b}}', \mathbf{c}, \mathbf{l}$ expressed as functions of $(\tilde{\mathbf{b}}, \mathbf{u}, g_-)$ that attain the value function $V(\tilde{\mathbf{b}}, \mathbf{u}, g_-)$ that solves Bellman equation (14) describe the Ramsey plan for $t \geq 1$. The time 0 component of the Ramsey plan is then determined as follows. Given $\tilde{b}_{-1}$ and g_0, the planner chooses $(c_0, \tilde{b}_0)$ that solve

$$V_0(\tilde{b}_{-1}, g_0) = \max_{c_0, \tilde{b}_0, l_0} \sum_{i=1}^{I} \alpha_i u(c_{i,0}) + \beta V(\tilde{b}_0, u_{c,0}, g_0)$$

[4] See, e.g. Aiyagari et al. (2002) and Faraglia et al. (2012).

subject to

$$
\begin{aligned}
(c_{i,0} - c_{1,0}) + \tilde{b}_{i,0} + \frac{U^i_{\ell,0}}{\theta_i U^i_{c,0}}(\theta_i \ell_{i,0} - \theta_1 \ell_{1,0}) &= \tilde{b}^i_{-1}, \quad \forall i > 1 \\
\frac{U^i_{\ell,0}}{\theta_i U^i_{c,0}} &= \frac{U^j_{\ell,0}}{\theta_j U^j_{c,0}}, \quad \forall i, j \\
\sum_{i=1}^{I} \pi_i c_{i,0} + g_0 &= \sum_{i=1}^{I} \pi_i \theta_i \ell_{i,0}.
\end{aligned}
$$

We will use this formulation to calculate Ramsey plans numerically. Before doing that, we characterize some special cases that highlight the main economic forces determining optimal taxes and allocations.

3.3 Quasi-linear preferences

We start our analysis with quasi-linear preferences of the form

$$
U^i(c, l) = c - h_i(l), \tag{19}
$$

for some strictly increasing function $h_i(\cdot)$, where consumption is restricted to be non-negative

$$
c \geq 0. \tag{20}
$$

These preferences have been used extensively before.[5] Equilibrium dynamics when all households are identical and the government can use linear taxes and non-negative transfers are known from AMSS. The government accumulates assets until it can finance all future government expenditures from interest income. Taxes on labor income follow a random walk-like process that converges to zero.

We use these preferences to indicate how dramatically equilibrium dynamics change *vis a vis* AMSS when there are heterogeneously situated agents and a government that wants to redistribute. We have already shown (Corollary 2 and 3) that when transfers are set optimally, the government has no precautionary motive to accumulate assets even when the distribution of types is such that optimal transfers are always positive. We show next (Proposition 1) that the dynamics of distortions and output are also very different: under mild assumptions, optimal labor taxes are constant across all dates and states.

To contrast outcomes in our environment with those in AMSS's, we restrict our attention to parameters for which an equilibrium is interior in the sense that constraint (20) does not bind.

[5] See, e.g. AMSS, Farhi (2010), Battaglini and Coate (2007, 2008), Yared (2010), Faraglia et al. (2012).

It is easy to show that there is a large set of parameters $\{\alpha_i, \theta_i, g\}$ that verify interiority. In the next section, we will not assume interiority. To simplify notation we assume that the initial debt is $\left\{\beta^{-1} b_{i,-1}\right\}_i$.

Proposition 1 *Suppose that preferences are quasi-linear for all i and that an equilibrium is interior. Then the optimal tax, τ_t^*, satisfies $\tau_t^* = \tau^*$. An optimum asset profile $\left\{b_{i,t}^*, B_t^*\right\}_{i,t}$ can be chosen to satisfy $b_{i,t}^* = b_{i,-1}$ for all $i,\ t \geq 0$ and $B_t^* = B_{-1}$ for all $t \geq 0$.*[6]

Proof. When an equilibrium allocation is interior, the first-order condition (8) becomes $(1-\tau_t)\,\theta_i = h_i'(l_{i,t})$. For our purposes it is more convenient to express the labor supply component of the allocation as a function of $(1-\tau)$ and optimize with respect to τ rather than $\{l_i\}_i$. We can invert function $h_i'(\cdot)$ to express labor supply l_i as a function of $(1-\tau)$. Call this function $H_i(1-\tau)$. When the equilibrium allocation is interior, $R = 1/\beta$ and the implementability constraint (10) becomes

$$c_{i,t} + b_{i,t} - (1-\tau_t)\, H_i(1-\tau_t) = T_t + \beta^{-1} b_{i,t-1}. \tag{21}$$

We can find the optimal allocation by maximizing

$$\max_{\{c_{i,t}, b_{i,t}, \tau_t, T_t\}_{i,t}} \mathbb{E}_0 \sum_{t=0}^{\infty} \sum_{i=1}^{I} \alpha_i \pi_i \beta^t \left[c_{i,t} - h_i\left(H_i(1-\tau)\right)\right]$$

subject to $\{b_{i,t}\}_{i,t}$ being bounded, (21), and

$$\sum_{i=1}^{I} \pi_i c_{i,t} + g_t = \sum_{i=1}^{I} \pi_i \theta_i H_i(1-\tau_t).$$

Note that since $\{b_{i,t}\}_{i,t}$ is bounded,

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\beta^{-1} b_{i,t-1} - b_{i,t}\right] = \beta^{-1} b_{i,-1} + \lim_{\mathcal{T}\to\infty} \mathbb{E}_0 \left(\sum_{t=0}^{\mathcal{T}} \beta^t \left[b_{i,t} - b_{i,t}\right] - \beta^{\mathcal{T}+1} b_{i,\mathcal{T}+1} \right) = \beta^{-1} b_{i,-1}.$$

Use (21) to eliminate $c_{i,t}$ together with the expression above to get

$$\max_{\{b_{i,t}, \tau_t, T_t\}_{i,t}} \mathbb{E}_0 \sum_{t=0}^{\infty} \sum_{i=1}^{I} \alpha_i \pi_i \beta^t \left[T_t + (1-\tau_t)\, H_i(1-\tau_t) - h_i\left(H_i(1-\tau)\right)\right] + \beta^{-1} \sum_{i=1}^{I} b_{i,-1}. \tag{22}$$

subject to

$$\sum_{i=1}^{I} \pi_i \left[T_t + \beta^{-1} b_{i,t-1} - b_{i,t} + (1-\tau_t)\, H_i(1-\tau_t)\right] + g_t = \sum_{i=1}^{I} \pi_i \theta_i H_i(1-\tau_t). \tag{23}$$

[6] We thank Guy Laroque for suggesting the idea for this proof.

Let $\beta^t \lambda_t$ be the Lagrange multiplier on the time t feasibility constraint. The first-order condition with respect to T_t implies that $\lambda_t = \sum_{i=1}^{I} \alpha_i \pi_i$ is constant and independent of t. Therefore, optimal taxes τ_t^* are also constant and independent of t. The last part of the Proposition follows from Theorem 1. ■

In the quasi-linear economy, fluctuations in lump-sum taxes and transfers do all the work. Furthermore, as we pointed out in Corollary 3, if some agents are sufficiently poor or if the planner wants enough redistribution, the lump-sum component can be positive at all dates and states. In such an economy, the planner always uses lump-sum *transfers* and never uses lump-sum *taxes*, so even if we had imposed the AMSS constraint $T_t \geq 0$, it would never bind.

The Lucas and Stokey (1983) and AMSS (2002) representative agent models impose $T_t \geq 0$, a constraint that always binds in the Lucas and Stokey model and binds until the government acquires enough assets to finance all future expenditures from its asset income in the AMSS model. In those models, the government would like to impose lump-sum *taxes*, not transfers. Distributional motives render the situation very different in our model.

Comparison with representative agent economies

Proposition 1 shows that optimal equilibrium dynamics of taxes and allocations in our economy are starkly different from those in representative agent Ramsey models like AMSS. To understand the economic mechanisms that lead to such different outcomes, recall that representative agent models impose an additional constraint on taxes, namely,

$$T_t \geq 0 \text{ for all } t. \tag{24}$$

Although this constraint need not bind in $I > 1$ economies like ours with more than one agent, when $I = 1$ it almost *always* binds.[7] To shed light on economic forces, in this section we impose the additional constraint (24). When we impose (24), our setup is identical to one commonly used in the literature on representative agent Ramsey models.

When constraint (24) is imposed, there are still many sequences of equilibrium gross debts $\{b_{i,t}\}$ that imply the same net debts $\{\tilde{b}_{i,t}\}$. However, there emerge some additional restrictions on feasible paths for $\{b_{i,t}\}$. We can also prove some of the Corollaries of Theorem 1 but under more stringent conditions (for example, we need to impose that one of the types has a sufficiently low skill and that its initial assets are sufficiently small). More generally, the gross distribution of assets affects equilibrium outcomes. This means that we cannot obtain implementability

[7] The exception is when the first-best outcome is eventually achieved in AMSS when the government has accumulated a sufficiently large stock of assets to finance all subsequent expenditures.

constraints like (12). Instead, we have an implementability constraint for each type, namely,

$$c_{i,t} + b_{i,t} = h_i' (l_{i,t}) \, l_{i,t} + \beta^{-1} b_{i,t-1} + T_t. \tag{25}$$

The altered optimal tax problem is

$$\max_{\{c_{i,t}, b_{i,t}, l_{i,t}, T_t\}_{i,t}} \mathbb{E}_0 \sum_{t=0}^{\infty} \sum_{i=1}^{I} \alpha_i \pi_i \beta^t \left[c_{i,t} - h_i (l_{i,t}) \right] \tag{26}$$

subject to (2), (24), (25), and

$$h_i' (l_{i,t}) / \theta_i = h_j' (l_{j,t}) / \theta_j \text{ for all } i, j.$$

Proposition 2 *Suppose g can take more than one value. Let $\beta^t \chi_t$ be the Lagrange multiplier on constraint (24) in maximization problem (26). Then $\chi_t \to 0$ a.s.*

Proof. Maximization problem (26) can be equivalently written as maximization problem (22) with an additional constraint (24). The first-order conditions for T_t yield $\sum_{i=1}^{I} \alpha_i \pi_i = \lambda_t + \chi_t$ while the first-order condition for $b_{i,t}$ implies $\lambda_t = \mathbb{E}_t \lambda_{t+1}$. Since $\chi_t \geq 0$, these two conditions imply that χ_t is a nonnegative martingale and therefore χ_t must converge to a constant a.s. This, in turn, implies that λ_t must converge to a constant a.s. Then the first-order conditions for τ_t also imply that τ_t must converge a.s. to some τ^*.

Suppose $\chi_t \to \chi^* > 0$. This implies that $T_t \to 0$ and (23) becomes

$$-\beta^{-1} B_{i,t-1} + B_{i,t} + \sum_{i=1}^{I} \pi_i (1 - \tau^*) H_i (1 - \tau^*) + g_t = \sum_{i=1}^{I} \pi_i \theta_i H_i (1 - \tau^*),$$

where we used (6) to substitute for $\sum_{i=1}^{I} \pi_i b_{i,t}$. Since g_t can take more than one value and follows an irreducible Markov process, for any bound on B_t, we can find a sequence of government expenditures g_t for which this bound will eventually be violated, leading to a contradiction. This implies that $\chi_t \to 0$. ■

This proposition highlights the key force driving the long-run results of AMSS (2002). Since the risk-free interest rate equals the discount rate, a Ramsey planner who faces constraint (24) always wants to save a bit more to relax future constraints (24). This motive endures before the planner has saved enough to render all future constraints (24) slack. In the representative agent economy of AMSS, constraint (24) binds every period until the government has acquired enough assets that it never again has to use distortionary taxes τ_t. This explains the AMSS (2002) result that the government collects no taxes in the long run. When agents are heterogenous and the government cares about redistribution, outcomes can be very different. As we have discussed earlier in this section, in some settings with heterogenous agents, constraints (24) do not bind, and as a result the government has no reason to accumulate assets or to smooth distortions imperfectly over time.

3.4 Distortion smoothing with risk-aversion

Our quasi-linear example is limiting because agents are indifferent to fluctuations that leave the ex-ante present value of consumption unaltered. With risk-averse agents, things becomes more complicated because it is more difficult to isolate a pure "labor distortion", since the distortion generated by τ_t depends partly on agents' diverse accumulations of assets. Their asset accumulations will in turn depend on their aversions to consumption risk and the associated precautionary motives. In general, analysis of such economies requires numerical computations. To highlight the main economic forces, it is possible to construct special economies that still allow us to separate "labor distortion" and "risk aversion" effects. We accomplish this by assuming that some agents' decisions are influenced only by a fluctuating distorting labor tax, while others are influenced only by their aversion to consumption risk.

Here is a simple example of such an economy. There are two types. A type 1 agent has quasilinear preferences as in the previous section with $\theta_1 = 1$ and $\lim_{l \to \bar{l}_1} h'(l) = \infty$, while a type 2 agent is risk averse and has $\theta_2 = 0$; his preferences can be represented with a strictly concave, twice differentiable utility function $u(c_{2,t})$ that satisfies Inada conditions. We call this an *AMSS-like economy*. Higher curvature in u makes fluctuations in $c_{2,t}$, and hence in transfers T_t, more costly.

This simple AMSS-like economy highlights key forces governing optimal taxes and transfers with incomplete markets. Optimal allocations and taxes are generally *history-dependent*, in the sense that optimal allocations at time t depend not only on the current realization of government expenditures g_t, but also on the history of expenditures. This result contrasts sharply both with the complete market economies of Lucas and Stokey (1983) and Werning (2007) and with the constrained optimal allocations in period t that depend only on g_t to be discussed in section 5.1. We also use this economy to highlight different ways that taxes, transfers, and debts adjust to aggregate shocks. We show these same forces again in numerical examples for more general economies.

Proposition 3 *Suppose that there is unique $\hat{l}$ that solves $h''(\hat{l})\hat{l} + h'(\hat{l}) = 1$. Let $c^*_{1,t}$ be an optimal allocation of consumption to the risk-neutral agent 1 in the AMSS-like economy. Then $c^*_{1,t} = 0$ infinitely often almost surely.*

Proof. We show this result by contradiction. Suppose that *(20)* does not bind after some period $\bar{\mathcal{T}}$. Then the gross risk-free interest rate that period is β^{-1} and, since u satisfies the Euler equation,

$$u'(c_t) = \mathbb{E}_t u'(c_{t+1}).$$

Since the optimal allocation in period t is recursive in $\left(\tilde{b}_{2,t-1}, u_c(c_{t-1})\right)$, the optimal allocation after $\bar{\mathcal{T}}$ can be found by solving the following optimization problem

$$\max_{\{c_1,c_2,l_1,\tilde{b}\}} \mathbb{E}_{\mathcal{T}} \sum_{t=\bar{\mathcal{T}}+1}^{\infty} \beta^{t-\bar{\mathcal{T}}-1} \left[\alpha_1 \left(c_{1,t} - h(l_{1,t})\right) + \alpha_2 u(c_{2,t})\right] \tag{27}$$

subject to the constraints that the sequence $\left\{\tilde{b}_{i,s}\right\}_{i,s>\bar{\mathcal{T}}}$ is bounded and

$$c_{2,t} - c_{1,t} + \tilde{b}_t + h'(l_{1,t})l_{1,t} = \frac{1}{\beta}\tilde{b}_{t-1}, \tag{28}$$

$$c_{1,t} + c_{2,t} + g_t = l_{1,t}, \tag{29}$$

$$u_{c,t} = \mathbb{E}_t u_{c,t+1}, \tag{30}$$

$$u_{c,\mathcal{T}} = \bar{u}. \tag{31}$$

Equation (30) implies that $u_{c,t}$ is a supermartingale and therefore converges. It cannot converge to zero, since then $c_{2,t}$ would diverge to infinity and violate (29) (this follows since $c_{1,t}$ is bounded from below and $l_{1,t}$ cannot diverge to infinity). Therefore, $u_{c,t}$ and $c_{2,t}$ both converge to finite values, so $c_{2,t} \to c_2^*$. Consumption of agent 1, $c_{1,t}$, is then determined as a residual from (29) and follows the same Markov process as g_t.

In the appendix, we show that the Lagrange multiplier η_t on constraint (28) must converge to some finite value η^*. Then the first-order conditions for $c_{1,t}$ imply that the multiplier on the feasibility constraint (29), λ_t, must also converge to a finite value $\lambda^* = \alpha_1 - \eta^*$. The first-order condition for l_1,

$$h'(l_{1,t})\left(\alpha_1 - \eta_t\left(1 + \frac{h''(l_{1,t})l_{1,t}}{h'(l_{1,t})}\right)\right) = \lambda_t,$$

implies that l_1 converges to some l^*.

At this stage it is helpful to consider particular histories s^t explicitly. Pick any s^t such that $s^{\bar{\mathcal{T}}} < s^t$, by which we denotes histories s^t that can occur after a history $s^{\bar{\mathcal{T}}}$. Choose any $s^\infty > s^t$ and substitute repeatedly into (28) for all s^k that satisfy $s^t \leq s^k < s^\infty$ to get

$$\sum_{k=0}^{\infty} \beta^k \left[2c_2\left(s^{t+k}\right) + \left(h'(l_1(s^{t+k})) - 1\right) l_1(s^{t+k})\right] + \sum_{k=0}^{\infty} \beta^k g\left(s^{t+k}\right) + \lim_{\mathcal{T}\to\infty} \beta^{\mathcal{T}} \tilde{b}\left(s^{t+\mathcal{T}+1}\right) = \frac{1}{\beta}\tilde{b}(s^t). \tag{32}$$

If we choose t sufficiently large, the first integral is sufficiently close to a constant for almost all possible paths s^k. But different paths of s^k lead to different values of $\sum_{k=0}^{\infty} \beta^k g\left(s^k\right)$, which implies that for some $s^{\infty} > s^t$, $\lim_{\mathcal{T}\to\infty, s^{t+\mathcal{T}+1} < s^{\infty}} \beta^{\mathcal{T}} \tilde{b}\left(s^{t+\mathcal{T}+1}\right) \neq 0$. This implies that $\tilde{b}\left(s^{t+\mathcal{T}+1}\right)$ is unbounded along that history, which leads to a contradiction. ■

Proposition 3 reveals key forces. The government wants (a) to smooth labor supply distortions caused by taxes, and (b) to smooth consumption of the risk-averse agent. To smooth labor supply distortions, the government should keep the marginal tax rate on labor constant across all realizations of s_t. To smooth consumption of the risk averse type 2 agent, the government could (i) keep transfers T_t constant by borrowing from or lending to the risk-neutral type 1 agent in response to shocks to g_t; or (ii) let T_t fluctuate and have the risk-averse agent borrow from or lend to the risk-neutral type 1 agent to smooth consumption. With incomplete markets, the government cannot do either of these things perfectly. There is always a long enough sequence of bad shocks so that either in case (i) the government runs into its borrowing limit and must adjust the distorting tax rate to raise more revenues; or in case (ii) the risk-averse type 2 agent runs into his borrowing limit and can no longer smooth his consumption, in which case the government must adjust the distorting tax rate to help the type 2 risk-averse agent smooth consumption.

This example indicates that optimal allocations are generally *history dependent*, in the sense that allocations in period t will depend not only on the current realization of g_t but also on their history g^{t-1}. In the next section, we show that the same insights continue to hold with flexible nonlinear tax systems, and that this implies that the optimal distortions τ_t are generally history-dependent as in AMSS and Barro (1979).

3.5 Numerical examples

In this section we present two numerical examples. First, we consider a non random economy with no shocks in which allocations are constant over time. In this economy we further explore the implications of Corollary 1 by considering how transfers and taxes are determined as functions of the distribution of agents' net assets. Second, we consider a stochastic economy and study how taxes, transfers, and agents' net asset positions respond to government expenditure shocks.

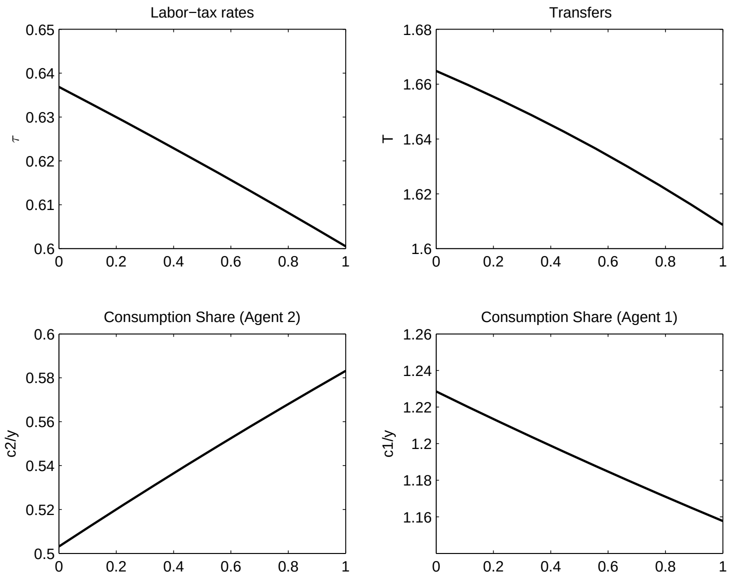

Figure 2: Distorting taxes τ, transfers T, consumption of high θ type, and consumption of low θ type, all as functions of the fraction of initial government debt owned by low θ type 2 agent.

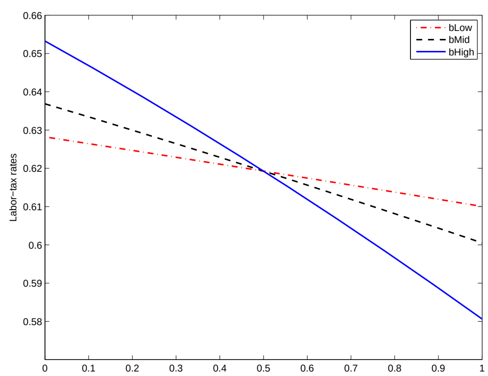

Figure 3: Distorting labor tax for different levels of initial government debt, all as functions of the fraction of initial government debt owned by low θ type 2 agent.

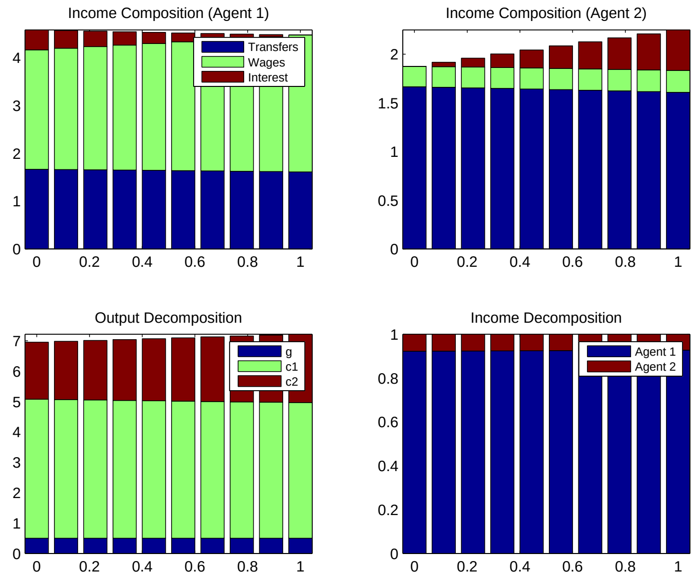

Figure 4: Top panels: sources of income for high θ and low θ types; lower panels, decomposition of output by use and distribution of income across types, all as functions of the fraction of initial government debt owned by low θ type 2 agent.

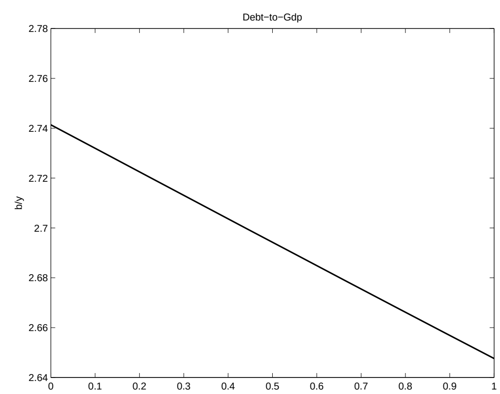

Figure 5: Debt to GDP ratio as function of the fraction of initial government debt owned by low θ type 2 agent.

Nonstochastic stationary example

Figures 2, 3, 4, and 5 show outcomes for a simple nonstochastic $I = 2$ two-types stationary economy. We have set parameters so that government purchases are about 10 percent of what would be GDP without distorting taxes, and we have set productivities and Pareto weights so that after-tax wages for the high type are about 5 times what they are for the low type, an approximation to the 90-10 decile ratio in the U.S. There are two types with $\theta_1 = 8$ and $\theta_2 = 1$. The measures of the two types of workers are $\pi_1 = .5$ and $\pi_2 = .5$. Preferences of both types are ordered by (1) with $\beta = .98$ and

$$U^i = \frac{c_{it}^{1-\sigma}}{1-\sigma} - \frac{l_{it}^{1+\gamma}}{1+\gamma}$$

with $\sigma = 1$ and $\gamma = .3$. We set $g_t = 1$ for all t. Initial government debt is $B = 10$, which is double undistorted full employment GDP. The relatively unskilled type 2 agents initially own a fraction $x \in [0, 1]$ of the government debt. Pareto weights are $\alpha_1 = .5, \alpha_2 = .5$. To normalize, we assume that the two types of agents do not borrow or lend with each other, only with the government. We take $\tilde{b}_2$ as an initial condition. Because g is nonstochastic and constant, there are complete markets.

Pareto optimal allocations have constant c_1, c_2, τ, T, all of which are functions of $g, \tilde{b}_2$; $R = \beta^{-1}$. Figure 2 shows τ and T as functions of x as well as the consumption levels of the two types as functions of the fraction x of government debt initially in the hands of the low θ type 2 agent. Figure 3 shows τ as a function of x for three levels of initial government debt $-B$. Figure 2 shows the two agents' sources of income as functions of x.

The government sets affine taxes to finance g and to transfer from the high θ type to the low θ type through two types of transfers: T, the constant in the affine tax schedule, and $\pi_1 RxB$, the interest payments on the government debt received by the low θ type. Figures 2, 4, and 5 show outcomes. Figure 2 shows that the government sets a lower distorting tax rate τ and a lower explicit transfer T_t when x is higher. Higher levels of initial government debt steepen the slopes of the τ on x curve because the larger is B, the more potent interest payments become as a means of subsidizing the low θ type.

Remark: A downward slope of the labor tax rate as a function of the fraction x of initial government debt in the hands of the low θ type agent prevails so long as the Pareto weight α_1 attached to the high θ agent is sufficiently high (.15 or above with the other parameters set at the values for our figures). For a fixed α_1, the consumption share of the low θ agent 2 rises with his share x of initial government debt. The downward slope of the distorting tax function requires that his interest earnings can rise enough as his share of initial assets rises. When α_1 is too low, his interest earnings can be too low, meaning that rising consumption of a type 2 agent as a function of x might have to be achieved with higher transfers and hence higher labor taxes.

Aggregate shocks

In this $I = 2$ example, we keep all the parameters of preferences and the skill distribution as in the previous section. We assume that g_t can take values $g_H = 1.2$ and $g_L = 1$ with equal probability in each period. We set $\tilde{b}_{-1} = 0$. We simplify Bellman equation (14) to compute optimal allocations. For our two-type example, we can take as state variables (x, ρ, g_-), where $x = u_c(c_2)\{b_2 - b_1\}$, interpretable as marginal-utility-adjusted net asset positions, and $\rho = \frac{u_c(c_2)}{u_c(c_1)}$ or the ratio of marginal utilities across agents. The resulting Bellman equation is

$$V(x, \rho_-, g_-) = \max_{x'(g),c_1(g),c_2(g),l_1(g),l_2(g)} \sum_g Pr(g|g_-) \left(\left[\sum_i \pi_i \alpha_i U^i(g) \right] + \beta V(x'(g), \rho(g), g) \right) \quad (33)$$

subject to, $\forall g$,

$$U_{c,2}[c_2 - c_1] + x' + \left(U_{l,2} l_2 - U_{c,2} \frac{U_{l,1}}{U_{c,1}} l_1 \right) = \frac{x U_{c,2}}{\beta \mathbb{E}_{g_-} \mathbf{U}_{c,2}} \quad (34a)$$

$$\frac{\mathbb{E}_{g_-} \mathbf{U}_{c,2}}{\mathbb{E}_{g_-} \mathbf{U}_{c,1}} = \rho_- \quad (34b)$$

$$\frac{U_{l,2}}{\theta_2 U_{c,2}} = \frac{U_{l,1}}{\theta_1 U_{c,1}} \quad (34c)$$

$$\pi_1 c_1 + \pi_2 c_2 + g = \pi_1 \theta_1 l_1 + \pi_2 \theta_2 l_2 \quad (34d)$$

$$\rho = \frac{U_{c,2}}{U_{c,1}}. \quad (34e)$$

We use cubic splines to approximate $V(g_-)$ over an *arbitrary* grid $[\underline{x}, \bar{x}] \times [\underline{\rho}, \bar{\rho}]$ for the continuous state variables. The choice of grid is intended to assure that along any path associated with the law of motion for the states implied by the policy rules, the bounds on the grid are never breached.

Figure **??** shows realizations of taxes τ_t, transfers T_t, net asset positions $\tilde{b}_t$, and GDP in this economy for a particular history of government expenditures g_t. By Theorem 1, without further assumptions on $\{b_{i,t}\}_{i,t}$ only the present value of optimal transfers is determined. To compute transfers in Figure **??**, we set $b_{2,t} = 0$. Shaded areas on the graph represent times of low government expenditures, and light areas represent times of high government expenditures.

Several general insights emerge from these figures. When its expenditures are high, the government increases taxes τ_t and decreases transfers T_t. These responses are consistent with mechanisms described in Section 3.4. Changes in both transfers and taxes are costly for the

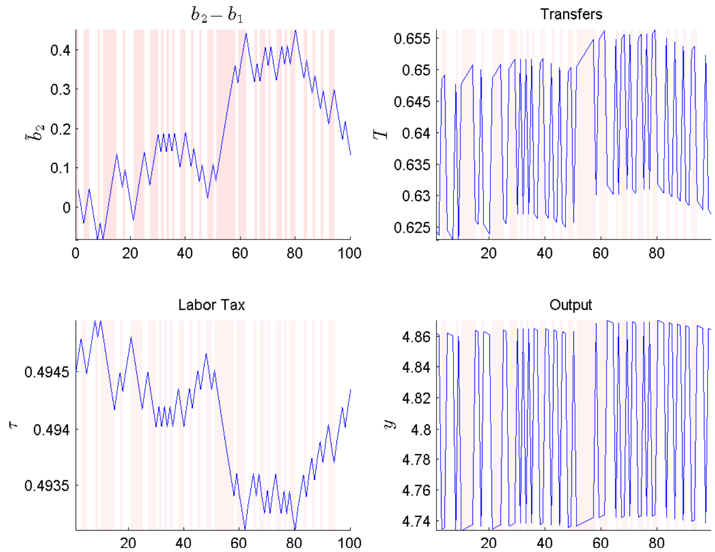

Figure 6: Outcomes with general preferences.

government and it is optimal to use some of both to spread out costs. Moreover, distortions and allocations are history-dependent. Longer spells of high government expenditures result in higher taxes and lower transfers.

The dynamics of relative assets positions $\tilde{b}_t$ are determined by two conflicting forces. Agent 1's labor income is more volatile than agent 2's labor income, so agent 1 faces more risk. At the same time, agent 1 is richer, so with CES preferences his level of absolute risk aversion is smaller. With CES preferences the second force dominates, and agent 2 de-accumulates assets faster (slower) during periods of high (low) government expenditures than agent 1.

Finally, output is higher during periods of high government expenditures although consumption and welfare are lower.

4 Extensions: borrowing constraints and capital

In this section, we discuss two extensions, namely, to economies with borrowing constraints and with capital.

4.1 Borrowing constraints

Our discussion so far has focused on situations in which agents are constrained in their abilities to borrow and lend only to the extent that they cannot run up debts too big to repay. In this section, we show that many of our conclusions are actually *strengthened* if agents face exogenous borrowing limits.[8] In economies with exogenous borrowing constraints, agents' maximization problems have the additional constraint

$$b_{i,t} \geq \underline{b}_i \tag{35}$$

for some exogenously given $\{\underline{b}_i\}_i$. We define an equilibrium similarly to Definition 2.

Definition 4 *For given* $\left(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1}\right)$ *and* $\{\tau_t, T_t\}_t$ *, a competitive equilibrium with affine taxes and exogenous borrowing constraints is a sequence* $\{\{c_{i,t}, l_{i,t}, b_{i,t}\}_i, B_t, R_t\}_t$ *such that* $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ *maximizes (1) subject to (4) and (35),* $\{\{b_{i,t}\}_i, B_t\}_t$ *are bounded, and constraints (2), (5) and (6) are satisfied.*

We define an *optimal* competitive equilibrium with exogenous borrowing constraints by extending Definition 3 in the obvious way.

Our first result is that exogenous borrowing constraints do not restrict a government's ability to smooth its expenditures.

Theorem 2 *Let* $\left\{c^*_{i,t}, l^*_{i,t}\right\}_{i,t}$ *and* $\{R^*_t\}$ *be a competitive equilibrium allocation and interest rate sequence in an economy without exogenous borrowing constraints given initial asset distribution* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$ *. Then for any exogenous constraints* $\{\underline{b}_i\}_i$ *there is a government tax policy* $\{\tau_t, T_t\}_t$ *such that* $\left\{c^*_{i,t}, l^*_{i,t}\right\}_{i,t}$ *is a competitive equilibrium allocation in the economy with exogenous borrowing constraints* $\left(\{b_{i,-1}, \underline{b}_i\}_i, B_{-1}\right)$ *and* $\{\tau_t, T_t\}_t$ *.*

Proof. Let $\left\{c^*_{i,t}, l^*_{i,t}, b^*_{i,t}\right\}_{i,t}$ be a competitive equilibrium allocation without exogenous borrowing constraints. Let $\Delta_t = \max_i \left\{\underline{b}_i - b^*_{i,t}\right\}$. Define $\hat{b}_{i,t} \equiv b^*_{i,t} + \Delta_t$ for all $t \geq 0$ and $\hat{b}_{i,-1} = b_{-1}$. By Theorem 1, $\left\{c^*_{i,t}, l^*_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is also a competitive equilibrium allocation without exogenous borrowing constraints. Moreover, by construction $\hat{b}_{i,t} - \underline{b}_i = b^*_{i,t} + \Delta_t - \underline{b}_i \geq 0$, therefore $\hat{b}_{i,t}$ satisfies (35). Since agents' budget sets are smaller in the economy with exogenous borrowing constraints, and $\left\{c^*_{i,t}, l^*_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ are feasible at interest rate process $\{R^*_t\}_t$, then $\left\{c^*_{i,t}, l^*_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is also an optimal choice for agents in the economy with exogenous borrowing constraints $\{\underline{b}_i\}_i$. Since all market clearing conditions are satisfied, $\left\{c^*_{i,t}, l^*_{i,t}, \hat{b}_{i,t}\right\}_{i,t}$ is a competitive equilibrium allocation and asset profile. ■

The intuition for theorem 2 is as follows. Suppose that the exogenous borrowing constraints restricted the ability of the government to achieve the desired allocations. This means that

[8] Bryant and Wallace (1984) describe how a government can use borrowing constraints as part of a welfare-improving policy to finance exogenous government expenditures. Sargent and Smith (1987) describe Modigliani-Miller theorems for government finance in a collection of economies in which borrowing constraints on classes of agents produce the kind of rate of return discrepancies that Bryant and Wallace manipulate.

ideally the government would like to increase the amount of its overall borrowing and to repay agents in the future. The government can instead reduce transfers today and increase them tomorrow, which achieves the same outcomes without violating the exogenous borrowing constraints.

Corollary 6 *Given initial* $\left(\{b_{i,-1}\}_i, B_{-1}\right)$, *welfare is higher in the corresponding economy with exogenous borrowing constraints* $\{\underline{b}_i\}_i$ *than in the economy without such constraints.*

Interestingly, welfare may be strictly higher in the economy with exogenous borrowing constraints. Intuitively, it may be optimal for the government to push agents against their borrowing limit. When some agents are borrowing constrained, their shadow interest rate on borrowing differs from the interest rate that unconstrained agents face. When the government structures transfers to change the interest rate, constrained and unconstrained agents are affected differently. This can improve welfare by facilitating redistribution. In the next several paragraphs, we will formalize this intuition.

When agents face borrowing constraint, their intertemporal optimality condition (9) is replaced by

$$U^i_{c,t} \geq \beta R_t \mathbb{E}_t U^i_{c,t+1} \tag{36}$$

and

$$\left(U^i_{c,t} - \beta R_t \mathbb{E}_t U^i_{c,t+1}\right)\left(b_{i,t} - \underline{b}_i\right) = 0. \tag{37}$$

Then we can write the implementability constraint (10) as

$$c_{i,t} + b_{i,t} = -\frac{U^i_{l,t}}{U^i_{c,t}} l_{i,t} + T_t + R_{t-1} b_{i,t-1} \tag{38}$$

and maximize (3) with respect to $\{c_{i,t}, l_{i,t}, b_{i,t}, T_t, R_t\}_{i,t}$ subject to (2), (35), (36), (37), (38), and

$$\frac{U^i_{l,t}}{\theta_i U^i_{c,t}} = \frac{U^j_{l,t}}{\theta_j U^j_{c,t}} \text{ for } \forall i, j, t.$$

All $\{c_{i,t}, l_{i,t}\}$ that satisfy the set of constraints (2) and (10) also satisfy the new set of constraints, but not vice versa, so that the planner is choosing from a strictly larger set when agents face borrowing constraints.

Next, we construct an example in which a social planner can achieve strictly higher welfare with borrowing constraints than without them.

Consider a version of the section 3.4 economy. Throughout, we assume that the non-negativity constraint (20) does not bind for consumption of agent 1, (or, alternatively, that agent 1 is allowed to have negative consumption). Suppose that $g_t = 0$ for all t, so that the economy is deterministic. In addition, assume that $\underline{b}_1 = 0$ and $\underline{b}_2 = -\infty$. Suppose that initial asset positions are $\left(\beta^{-1}\bar{b}_{1,-1}, \beta^{-1}\bar{b}_{2,-1}, \beta^{-1}\bar{B}_{-1}\right)$. Under these assumptions, we can write the planner's maximization problem as

$$\max_{\{c_{1,t},c_{2,t},l_{1,t},b_{1,t},b_{2,t},R_t\}_t} \sum_{t=0}^{\infty} \beta^t \left[\alpha_1 \left(c_t - h(l_{1,t})\right) + \alpha_2 u(c_{2,t})\right] \tag{39}$$

subject to (2), (30),

$$c_{2,t} - c_{1,t} + b_{2,t} - b_{1,t} + h'(l_{1,t})l_{1,t} = R_{t-1}\left(b_{2,t-1} - b_{1,t-1}\right), \tag{40}$$

$$b_{1,t} \geq 0, \tag{41}$$

$$\left(1 - \beta R_t\right) b_{1,t} = 0,$$

and

$$1 \geq \beta R_t. \tag{42}$$

Think of solving this maximization problem in two stages. First, drop constraint (42) and solve the resulting relaxed problem of maximizing (39) subject to the remaining constraints for a fixed sequence of $\{R_t\}_t$. Denote the value of the objective function for the reduced problem by $W\left(\{R_t\}_t\right)$. Second, solve the original problem by choosing $\{R_t\}_t$ to maximize

$$W\left(\{R_t\}_t\right)$$

subject to (42). When $R_t = 1/\beta$ for all t, the solution of the reduced problem is an optimal allocation for an economy in which agents face no borrowing constraints. Moreover, by Theorem 1, we can set $b_{1,t} = 0$ for all t, so that constraint (41) does not bind. If the function h is sufficiently well behaved, the solution $\left\{\bar{c}_{1,t}, \bar{c}_{2,t}, \bar{l}_{1,t}, \bar{b}_{1,t}, \bar{b}_{2,t}\right\}_t$ is time invariant and satisfies

$$\bar{b}_{2,t} - \bar{b}_{1,t} = \bar{b}_{2,t} = \bar{b}_{2,-1} - \bar{b}_{1,-1}.$$

Let $\beta^t \bar{\eta}_t$ be a Lagrange multiplier on constraint (40). It can be shown that $\bar{\eta}_t < 0$ when $\bar{b}_{2,-1} - \bar{b}_{1,-1} = 0$, and therefore, by continuity, $\bar{\eta}_t < 0$ if $\bar{b}_{2,-1} - \bar{b}_{1,-1}$ is negative but sufficiently small. It also easy to guess and verify that constraint (30) does not bind.

Let $\frac{\partial}{\partial R_1} W\left(\{R_t\}_t\right)\Big|_{\{R_t\}=\boldsymbol{\beta}^{-1}}$ be the derivative of $W\left(\{R_t\}_t\right)$ with respect to R_1 evaluated at $R_t = \beta^{-1}$ for all t. Our observations above imply that

$$\frac{\partial}{\partial R_1} W\left(\{R_t\}_t\right)\Big|_{\{R_t\}=\boldsymbol{\beta}^{-1}} = -\bar{\eta}_1 \bar{b}_{2,t} = -\bar{\eta}_1 \left(\bar{b}_{2,-1} - \bar{b}_{1,-1}\right) < 0$$

and therefore $R_t = \beta^{-1}$ for all t is not the optimal equilibrium sequence. Therefore, welfare in the economy with exogenous borrowing constraints is strictly higher than in the economy without exogenous borrowing constraints.

The outcome that welfare can be strictly higher with exogenous borrowing constraints depends on our assumption that agents are subject only to aggregate shocks. If agents were also subject to idiosyncratic shocks, exogenous borrowing constraints would have the additional effect of limiting agents' ability to self-insure against those idiosyncratic shocks.[9] Nevertheless, the insight from the example carries through that exogenous borrowing constraints can help a government smooth distortions with respect to aggregate shocks like government expenditure shocks.

When agents face exogenous borrowing constraints, Theorem 1 may no longer be true: equilibrium allocations will in general depend not only on net asset positions $\left\{\tilde{b}_{i,t}\right\}_{t,i\geq 2}$ but also on gross positions $\{b_{i,t}\}_{t,i}$. However, *optimal* allocations in an economy with exogenous borrowing constraints still depend only on the net initial asset positions, $\left\{\tilde{b}_{i,-1}\right\}_i$ and not on the level of the initial government debt B_{-1}, thus extending Corollary 1 to economies with exogenous borrowing limits.

Proposition 4 *For any $\left(\left\{b'_{i,-1}\right\}_i, B'_{-1}\right)$ and $\left(\left\{b''_{i,-1}\right\}_i, B''_{-1}\right)$ that satisfy (35), if $b'_{i,-1} - b'_{1,-1} = b''_{i,-1} - b''_{1,-1}$ for all i, then the optimal equilibrium allocations given $\left(\left\{b'_{i,-1}, \underline{b}_i\right\}_i, B'_{-1}\right)$ and given $\left(\left\{b''_{i,-1}, \underline{b}_i\right\}_i, B''_{-1}\right)$ coincide.*

Proof. Suppose $\{\tau'_t, T'_t\}_t$ are the optimal taxes in the $\cdot'$ economy. Let $T''_0 = T'_0 + \left(b'_{1,-1} - b''_{1,-1}\right)$ and $T''_t = T'_t$ for $t > 0$. Following the same steps as the proof of Theorem 1 we can verify that $\{\tau'_t, T''_t\}_t$ are the optimal taxes in the second economy. ■

4.2 Capital

Theorem 1 and its corollaries are unchanged if we add capital to our setup whether or not we allow the government to tax it.

Several papers showed that relative to economies without capital, the government has more flexibility to 'complete markets' in economies with capital. For example, Zhu (1992) and Chari, Christiano, and Kehoe (1994) showed that if a government can set a state-contingent tax on the capital stock, it can replicate a complete-markets equilibrium allocation. Similarly, Farhi (2010) showed that by holding claims on the capital stock in addition to a risk-free bond, the

[9]See Aiyagari and McGrattan (1998) and Heathcote (2005) for details.

government can expand the set of achievable allocations. Whether the government can or cannot do that is immaterial to our results since, as we argued in Corollary 4, they extend so easily to economies with more general asset structures.

Farhi (2010) also showed that when its ability to complete markets through capital taxation is imperfect, the government has a precautionary savings motive. He obtains a result like one of AMSS's: allocations in an economy with capital and quasi-linear preferences economy converge to undistorted allocations in the long run (Farhi (2010), Claim 1). If we were to add heterogeneity and transfers to an economy like Farhi's, optimal outcomes would be very different, for the same reasons discussed in Section 3.3.

5 Constrained optimum and more general taxes

Several insights emerge from the above analysis. We showed that while absolute levels of debts are not determined in an optimal equilibrium, agents' relative asset positions are. Moreover, when markets are incomplete, optimal allocations are generally history dependent, meaning that allocations in period t depend both on the realization of government expenditures g_t and on the history g^{t-1}.

A recent literature on optimal taxation, sometimes advertised as a New Dynamic Public Finance (NDPF), approaches taxes from a different angle than does a Ramsey analysis.[10] The NDPF literature characterizes optimal allocations that respect information gaps between agents and the government. Then it studies taxes and other arrangements that can decentralize an optimal allocation.

In this section, we consider two informationally constrained economies. In the first, agents' skills are the only private information; the government observes everything else, including each agent's assets. In the second economy, we assume that all agents' assets are also unobservable. We will show that while in the first economy allocations are history independent and that *any* distribution of assets is consistent with an optimal competitive equilibrium, in the second economy allocations are history-dependent (unless agents can trade a full set of state-contingent securities) and that only agents' net asset positions are pinned down.

5.1 Constrained optimum with observable assets

In the spirit of the NDPF, we assume that θ is private information and that the government observes output $y \equiv \theta l$ and c for each agent. Constrained optimal allocations solve the mechanism

[10] For surveys, see Golosov, Tsyvinski, and Werning (2007) and Kocherlakota (2010).

design problem

$$\max_{\{c_{i,t},y_{i,t}\}} \mathbb{E}_0 \sum_{i=1}^{I} \alpha_i \pi_i \sum_{t=0}^{\infty} \beta^t U^i \left(c_{i,t}, \frac{y_{i,t}}{\theta_i} \right) \tag{43}$$

subject to incentive constraints

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U^i \left(c_{i,t}, \frac{y_{i,t}}{\theta_i} \right) \geq \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U^i \left(c_{j,t}, \frac{y_{j,t}}{\theta_i} \right) \text{ for all } i, j \tag{44}$$

and the feasibility constraint

$$\sum_{i=1}^{I} \pi_i c_{i,t} + g_t = \sum_{i=1}^{I} \pi_i y_{i,t}. \tag{45}$$

Let $\eta_{i,j}$ be Lagrange multipliers on (44). Define

$$W^i \left(c_{i,t}, y_{i,t} \right) = \left(\alpha_i \pi_i + \eta_{i,i} \right) U^i \left(c_{i,t}, \frac{y_{i,t}}{\theta_i} \right) - \sum_{j \neq i} \eta_{j,i} U^j \left(c_{j,t}, \frac{y_{j,t}}{\theta_i} \right).$$

Then we can rewrite the mechanism design problem as

$$\max_{\{c_{i,t},y_{i,t}\}} \min_{\{\eta_{ij}\}} \mathbb{E}_0 \sum_{i=1}^{I} \sum_{t=0}^{\infty} \beta^t W^i \left(c_{i,t}, y_{i,t} \right)$$

subject to (45). This problem is equivalent to a sequence of static problems for each realization of g_t. Therefore, the optimal allocation depends only on the realization of g_t and not on the time period t or the history g^{t-1}.

Now we investigate the type of taxes that can decentralize a constrained optimal allocation as a competitive equilibrium. We consider a general non-linear tax $T_t \left(y_t, X_t \right)$, where X_t is a vector of additional agent-specific variables, and ask what information X_t should contain. The vector X_t will not be unique. For example, past labor incomes and asset returns are equivalent ways of tracking labor income and consumption rates. Still, we can identify common features that a "fully optimal" tax system $T_t \left(y_t, X_t \right)$ must have.

As in the previous section, we assume that agents begin with initial debt holdings $\{b_{i,-1}\}_i$ and that each period they are able to trade a one-period bond with non-state-contingent return R. An agent's budget constraint is the same as (4) except that $-T_t + \tau y_t$ is replaced by $T_t \left(y_t, X_t \right)$. We modify the definition of a competitive equilibrium with these more general taxes accordingly.

We have the following result:

Proposition 5 *(i) Constrained optimal allocations can be decentralized as a competitive equilibrium with tax function* $T_t \left(y_t, b_{t-1}, F \left(\{y_s\}_{s=0}^{t-1} \right) \right)$*, where* $F \left(\{y_s\}_{s=0}^{t-1} \right)$ *is a function of previous labor earnings.*

(ii) The marginal tax on debt $\frac{\partial T_t\left(y_t, b_{t-1}, F\left(\{y_s\}_{s=0}^{t-1}\right)\right)}{\partial b}$ *must be either a function of* $\left(y_t, F\left(\{y_s\}_{s=0}^{t-1}\right)\right)$ *or a non-linear function of* b_{t-1}. *We are free to set* $T_t\left(y_t, b_{t-1}, F\left(\{y_s\}_{s=0}^{t-1}\right)\right) = y_t + \max\{R_t b_t, 0\}$ *if* $b_t \neq 0$.

Most of the results are easy to see. Let $\left\{c_{i,t}^{sp}, y_{i,t}^{sp}\right\}_{i,t}$ be a constrained optimal allocation. In general, there are many tax systems that implement this allocation, for example, one that sets $T(y_t, b_{t-1}, y_{t-1}, ..., y_0) = y_{it}^{sp} - c_{i,t}^{sp}$ if the vector $(y_t, b_{t-1}, y_{t-1}, ..., y_0) = \left(y_{i,t}^{sp}, 0, y_{i,t-1}^{sp}, ..., y_{i,0}^{sp}\right)$ and an arbitrarily high value of $T(\cdot)$ for all other $(y_t, b_{t-1}, y_{t-1}, ..., y_0)$. This assures that an agent must choose only among sequences $\left\{c_{i,t}^{sp}, y_{i,t}^{sp}\right\}_{i,t}$ that are incentive compatible by construction.

The optimal allocation $\left\{c_{i,t}^{sp}, y_{i,t}^{sp}\right\}_{i,t}$ generally has the property that agents' marginal rates of substitution are not equalized:

$$\frac{\beta \mathbb{E}_t U_c^i\left(c_{i,t+1}^{sp}, y_{i,t+1}^{sp}/\theta_i\right)}{U_c^i\left(c_{i,t}^{sp}, y_{i,t}^{sp}/\theta_i\right)} \neq \frac{\beta \mathbb{E}_t U_c^j\left(c_{j,t+1}^{sp}, y_{j,t+1}^{sp}/\theta_j\right)}{U_c^j\left(c_{j,t}^{sp}, y_{j,t}^{sp}/\theta_j\right)}.$$

As a result, marginal returns on assets $\frac{\partial T_t\left(y_t, b_{t-1}, F\left(\{y_s\}_{s=0}^{t-1}\right)\right)}{\partial b}$ cannot be linear in asset holdings and must either be non-linear or else depend on an agent's income, which typically will make them state-contingent. Debt plays no significant role, since, as part (ii) of Proposition 5 indicates, the government can implement the optimum by taxing away *all* of an agent's income from assets if his debt differs from zero.

Given taxes $T_t\left(y_t, b_{t-1}, F\left(\{y_s\}_{s=0}^{t-1}\right)\right)$, we can define an optimal competitive equilibrium analogously to Definition 3. A much stronger version of Theorem 1 emerges in this economy.

Corollary 7 *Any sequence of assets* $\{b_{i,t}\}_{i,t}$ *is a part of an optimal competitive equilibrium allocation for some optimal non-linear tax* $T_t\left(y_t, b_{t-1}, F\left(\{y_s\}_{s=0}^{t-1}\right)\right)$.

5.2 Constrained optimum with unobservable assets

We contrast allocations in the economy described in the previous section with allocations in an economy in which the government does not observe agents' assets. To define optimal allocations, we need to take a stand on what assets agents can trade. In line with our previous analysis, we assume that they can trade only a one-period risk-free bond. Later we will explain how our conclusions would change with more general asset structures.

We borrow a general formulation of the constrained optimal problem from Golosov and Tsyvinski (2007). The mechanism designer collects agents' reports about their types and chooses allocations of labor services and consumption. The agents can re-trade consumption as they choose. Asset prices are determined by market clearing conditions. Let $\{x_{i,t}, y_{i,t}\}_{i,t}$ be the allocation of consumption and labor services that the planner assigns to agents. The utility of agent i who chooses basket $\{x_{j,t}, y_{j,t}\}_t$ and faces interest rates $\{R_t\}_t$ is

$$\mathcal{V}_i\left(\{x_{j,t}, y_{j,t}, R_t\}_t\right) = \max_{\left\{c^j_{i,t}, b^j_{i,t}\right\}_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U_i\left(c^j_{i,t}, \frac{y_{j,t}}{\theta_i}\right)$$

subject to

$$c^j_{i,t} + b^j_{i,t} = x_{j,t} + R_{t-1} b^j_{i,t-1}.$$

The mechanism designer chooses an allocation such that no agent wants to re-trade along the equilibrium path. The optimal allocation solves[11]

$$\max_{\{c_{i,t}, y_{i,t}, R_t\}_{i,t}} \mathbb{E}_0 \sum_{i=1}^{I} \alpha_i \pi_i \sum_{t=0}^{\infty} \beta^t U^i\left(c_{i,t}, \frac{y_{i,t}}{\theta_i}\right)$$

subject to (2), (9) and

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U^i\left(c_{i,t}, \frac{y_{i,t}}{\theta_i}\right) \geq \mathcal{V}_i\left(\{c_{j,t}, y_{j,t}, R_t\}_t\right) \text{ for all } i, j.$$

In the Appendix we construct an example that shows that optimal allocations are generally history dependent, in contrast to the history independent optimal allocations that emerge for an economy where the government can observe all agents' asset holdings.

An optimal allocation can be decentralized by a non-linear tax $\{T_t(y_t, ..., y_0)\}_t$. We can define a competitive equilibrium in a fashion that parallels Definition 3. To maintain consistency with the informational requirement of the mechanism design problem, we assume that the government does not know who holds government debt (if government debt were to be issued in an equilibrium). As Corollary 4 indicates, Theorem 1 continues to hold in this economy. Thus, again the distribution of net assets $\left\{\tilde{b}_{i,t}\right\}$ is uniquely determined, but not agents' gross assets. These results show that dynamic properties of optimal taxes and allocations depend crucially on whether the government can observe and tax individuals' asset holdings.

6 Concluding remarks

The analysis in the previous section highlights the following general features of incomplete markets models. When a government's ability to tax assets is limited, the distribution of assets $\left\{\tilde{b}_{i,t}\right\}_{i=2}^{I}$ is a key component of a 'state vector' that influences optimal allocations. When a

[11] For details see Golosov and Tsyvinski (2007).

government expenditure shock occurs, the distribution $\left\{\tilde{b}_{i,t}\right\}_{i=2}^{I}$ responds endogenously. Government expenditures g_t affect agents' decisions about how much to save in period t, $\left\{\tilde{b}_{i,t}\right\}_{i=2}^{I}$. When markets are incomplete, the *ex post* return on these assets in period $t+1$ does not depend on g_{t+1}. As a result, there is history dependence in optimal allocations and there are distortions not present under the optimal mechanism design allocation of section 5. This outcome is robust to a variety of assumptions about the tax structure so long as the government's ability to tax assets state-contingently or nonlinearity is limited. By way of contrast, if agents have access to the full set of state-contingent assets, as in Werning (2007), optimal allocations will be history independent. But the conclusions of Theorem 1 will continue to hold with respect to an appropriately defined bundle of Arrow securities.

7 Appendix

7.1 Proof of Lemma 1

We prove a slight more general version of our result. Consider an infinite horizon, incomplete markets economy in which an agent maximizes utility function $U : \mathbb{R}^n_+ \to \mathbb{R}$ subject to an infinite sequence of budget constraints. We assume that U is concave and differentiable. Let $\mathbf{x}(s^t)$ be a vector of n goods and let $\mathbf{p}(s^t)$ be a price vector in state s^t with $p_i(s^t)$ denoting the price of good i. We use a normalization $p_1\left(s^t\right) = 1$ for all s^t. There is a risk-free bond.

Let $b(s^t)$ be the agent's bond holdings, and let $\mathbf{e}\left(s^t\right)$ be a stochastic vector of endowments.

Consumer maximization problem

$$\max_{\mathbf{x}_t, b_t} \sum_{t=0}^{\infty} \beta^t \Pr\left(s^t\right) U(\mathbf{x}\left(s^t\right)) \tag{46}$$

subject to

$$\mathbf{p}\left(s^t\right)\mathbf{x}\left(s^t\right) + q(s^t)b\left(s^t\right) = \mathbf{p}\left(s^t\right)\mathbf{e}\left(s^t\right) + b\left(s^{t-1}\right) \tag{47}$$

and $\left\{b\left(s^t\right)\right\}$ is bounded and $\left\{q(s^t)\right\}$ is the price of the risk-free bond.

The Euler conditions are

$$\begin{aligned} \mathbf{U}_x(s^t) &= U_1(s^t)\mathbf{p}(s^t) \\ \Pr\left(s^t\right) U_1\left(s^t\right) q(s^t) &= \beta \sum_{s^{t+1} > s^t} \Pr\left(s^{t+1}\right) U_1\left(s^{t+1}\right). \end{aligned} \tag{48}$$

Proposition 6 *Consider an allocation $\{\mathbf{x}_t, b_t\}$ that satisfies (47), (48) and $\{b_t\}_t$ is bounded. Then $\{\mathbf{x}_t, b_t\}$ is a solution to (46).*

Proof. The proof follows closely Constantinides and Duffie (1996). Suppose there is another budget feasible allocation $\mathbf{x} + \mathbf{h}$ that maximizes (46). Since U is strictly concave,

$$\begin{aligned} &\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(\mathbf{x}_t + \mathbf{h}_t) - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(\mathbf{x}_t) \\ \leq\ &\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbf{U}_x(\mathbf{x}_t)\mathbf{h}_t \end{aligned} \tag{49}$$

To attain $\mathbf{x} + \mathbf{h}$, the agent must deviate by φ_t from his original portfolio b_t such that $\{\varphi_t\}_t$ is bounded, $\varphi_{-1} = 0$ and

$$\mathbf{p}(s^t)\mathbf{h}\left(s^t\right) = \varphi(s^{t-1}) - q(s^t)\varphi(s^t)$$

Multiply by $\beta^t \Pr\left(s^t\right) U_1(s^t)$ to get:

$$\begin{aligned}
\beta^t \Pr\left(s^t\right) U_1(s^t)\mathbf{p}(s^t)\mathbf{h}\left(s^t\right) &= \beta^t \Pr\left(s^t\right) U_1(s^t)\varphi(s^{t-1}) - q(s^t)\beta^t \Pr\left(s^t\right) U_1(s^t)\varphi(s^t) \\
&= \beta^t \Pr\left(s^t\right) U_1(s^t)\varphi(s^{t-1}) - \beta^{t+1} \sum_{s^{t+1}>s^t} \Pr\left(s^{t+1}\right) U_1\left(s^{t+1}\right)\varphi(s^t)
\end{aligned}$$

where we used the second part of (48) in the second equality. Sum over the first T periods and use the first part of (48) to eliminate $\mathbf{U}_x(\mathbf{x}_t) = U_1(s^t)\mathbf{p}(s^t)$

$$\sum_{t=0}^{T} \beta^t \Pr\left(s^t\right) \mathbf{U}_x(\mathbf{x}_t)\mathbf{h}\left(s^t\right) = -\sum_{s^{T+1}>s^T} \beta^{T+1} \Pr\left(s^{T+1}\right) U_1\left(s^{T+1}\right)\varphi(s^T).$$

Since $\{\varphi_t\}_t$ is bounded there must exist $\bar{\varphi}$ s.t. $|\varphi_t| \leq \bar{\varphi}$ for all t. By Theorem 5.2 of Magill and Quinzii (1994), this equilibrium with debt constraints implies a transversality condition on the right hand side of the last equation, so by transitivity we have

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \Pr\left(s^t\right) \mathbf{U}_x(\mathbf{x}_t)\mathbf{h}\left(s^t\right) = 0.$$

Substitute this into (49) to show that $\mathbf{h}$ does not improve utility of consumer. ■

7.2 Proof of technical details of Proposition 3

In this appendix we show that $c_{2,t}$ and η_t must converge to finite values.

Equation (30) implies that $u_{c,t}$ is a supermartingale and therefore converges. It cannot converge to zero, since then $c_{2,t}$ would diverge to infinity. Since $c_1 \geq 0$, constraint (29) can only be satisfied if $l_{1,t} \to \infty$, violating boundedness. Therefore, $u_{c,t}$ and $c_{2,t}$ both converge to finite values, so $c_{2,t} \to c_2^*$.

The first-order conditions for $\tilde{b}$ imply that

$$\eta_t = \mathbb{E}_t \eta_{t+1}.$$

Thus, η_t is a martingale. But η_t does not necessarily have to be bounded, so we cannot apply a standard martingale convergence result. We use a different argument to prove our result.

Let $\beta^t \Pr\left(s^t\right)\eta(s^t)$, $\beta^t \Pr\left(s^t\right)\lambda(s^t)$, and $\beta^t \Pr\left(s^t\right)\zeta(s^t)$ be Lagrange multipliers on (28), (29), and (30), respectively. The first-order conditions for $c_{1,t}$ and $l_{1,t}$ are

$$\alpha_1 - \eta(s^t) = \lambda(s^t)$$

and

$$\alpha_1 h'(s^t) - \eta\left(s^t\right)\left[h''(s^t)l(s^t) + h'(s^t)\right] = \lambda(s^t).$$

These conditions imply that if $l(s^t)$ converges, so does $\eta(s^t)$. We shall show that if $\eta(s^t)$ does not converge, then $l(s^t)$ must converge, which will establish a contradiction.

Combine the first-order conditions for $c_{1,t}$ and $c_{2,t}$ to get

$$\alpha_2 u_c(s^t) + \left(\zeta(s^{t-1}) - \zeta(s^t)\right) u_{cc}(s^t) = \alpha_1 - 2\eta(s^t). \tag{50}$$

Since $u_c(s^t)$ and $u_{cc}(s^t)$ converge to finite values, if $\eta(s^t)$ does not converge to a constant, neither does $\zeta(s^{t-1}) - \zeta(s^t)$. Rewrite (50) as

$$\zeta(s^t) = \zeta(s^{t-1}) + \frac{\alpha_2 u_c(s^t) - \alpha_1}{u_{cc}(s^t)} + 2\frac{\eta(s^t)}{u_{cc}(s^t)}. \tag{51}$$

Choose a t sufficiently large that $\frac{\alpha_2 u_c(s^t) - \alpha_1}{u_{cc}(s^t)}$ and $u_{cc}(s^t)$ are close to being constants. Since $\eta(s^t)$ is a martingale that does not converge, we can find an $\varepsilon > 0$ and a history $\hat{s}^\infty$ such that

$$\left| \frac{\alpha_2 u_c(s^k) - \alpha_1}{u_{cc}(s^k)} + 2\frac{\eta(s^k)}{u_{cc}(s^k)} \right| > \varepsilon \text{ for } s^k \in \hat{s}^\infty$$

for infinitely many k. Then from (51) $|\zeta(s^t)| \to \infty$ for any $s^t \in \hat{s}^\infty$.

Rewrite (27) as

$$\max_{\{c_1, c_2, l_1, \tilde{b}\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pr\left(s^t\right) \left[\alpha_1 \left(c\left(s^t\right) - h(l_1\left(s^t\right))\right) + \alpha_2 u(c_2\left(s^t\right)) + \left(\zeta\left(s^{t-1}\right) - \zeta\left(s^t\right)\right) u_c\left(s^{t+1}\right)\right] \tag{52}$$

subject to constraints (28), (29) and a requirement that sequence $\left\{\tilde{b}_t\right\}$ is bounded. This problem is recursive. We can find the optimal allocations $\left\{c_1\left(s^t\right), c_2\left(s^t\right), l_1\left(s^t\right), \tilde{b}\left(s^t\right)\right\}$ for $s^t > s^m$ for some s^m if we take the sequence of multipliers $\left\{\zeta\left(s^t\right)\right\}_{s^t > s^m}$ and $\tilde{b}(s^m)$ as given to solve

$$\max_{\{c_1, c_2, l_1, \tilde{b}\}} \sum_{t=m+1}^{\infty} \sum_{s^t > s^m} \beta^t \Pr\left(s^t\right) \left[\frac{\alpha_1}{\zeta\left(s^{t-1}\right)} \left(c\left(s^t\right) - h(l_1\left(s^t\right))\right) + \frac{\alpha_2}{\zeta\left(s^{t-1}\right)} u(c_2\left(s^t\right)) + \left(1 - \frac{\zeta\left(s^t\right)}{\zeta\left(s^{t-1}\right)}\right) u_c\left(s^{t+1}\right)\right] \tag{53}$$

subject to constraints (28), (29) and a requirement that sequence $\left\{\tilde{b}_t\right\}$ is bounded. Denote by $\beta^t \Pr\left(s^t\right) \hat{\eta}(s^t)$ and $\beta^t \Pr\left(s^t\right) \hat{\lambda}(s^t)$ the Lagrange multipliers of this re-normalized problem. The first-order conditions for this problem with respect to c_1, l_1, and c_2 are

$$\frac{\alpha_1}{\zeta\left(s^{t-1}\right)} - \hat{\eta}(s^t) = \hat{\lambda}(s^t), \tag{54}$$

$$\frac{\alpha_1}{\zeta\left(s^{t-1}\right)} h'\left(s^t\right) - \hat{\eta}(s^t)\left[h''(s^t) l(s^t) + h'(s^t)\right] = \hat{\lambda}(s^t). \tag{55}$$

Combine these equations to get

$$\frac{\alpha_1}{\zeta\left(s^{t-1}\right)} h'\left(s^t\right) - \hat{\eta}(s^t)\left[h''(s^t) l(s^t) + h'(s^t)\right] = \frac{\alpha_1}{\zeta\left(s^{t-1}\right)} - \hat{\eta}(s^t).$$

Consider a history $\hat{s}^\infty$ and choose $s^m \in \hat{s}^\infty$. We know that $|\zeta\left(s^{t-1}\right)| \to \infty$. Suppose that $h'(s^t)$ remains bounded. Then the above equation implies that $l(s^t)$ converges to a value $\hat{l}$ that satisfies

$$h''(\hat{l})\hat{l} + h'(\hat{l}) = 1.$$

The assumptions of the theorem imply that this value is unique. This establishes that $\eta(s^t)$ converges. Alternatively, suppose that $h'(s^t) \to \infty$. Substitute (29) into (28):

$$2c_2\left(s^t\right) + g(s^t) + \left(h'(l_1(s^t)) - 1\right) l_1(s^t) = \frac{1}{\beta}\tilde{b}(s^{t-1}) - \tilde{b}\left(s^t\right).$$

Since $\tilde{b}(s^t)$ is bounded, the right side of this expression is bounded. Both $g(s^t)$ and $c_2(s^t)$ are finite, so $\left(h'(l_1(s^t)) - 1\right) l_1(s^t)$ must remain bounded, which rules out the possibility that $h'(s^t) \to \infty$. This completes the proof.

7.3 Additional details for Section 5.2

In this appendix, we provide example to the claim in Section 5.2 that when assets are not observable, the optimal allocations are generally history dependent. We consider a version of the AMSS-like economy set up in Section 3.4. Unlike that section, we assume that the risk averse agent can work, his productivity is θ_2 and his utility is given by $u(c) - h_2\left(\frac{y}{\theta_2}\right)$. We assume that the planner assigns a sufficiently high Pareto weight on agent 1 so that it is agent 2 incentive constraint which binds.

Similarly to the discussion of Section 3.4, two cases are possible. The equilibrium can either be interior, in which case $c_{1,t}$ is always above $\underline{c}$, or constraint (20) eventually binds. The latter case automatically implies history-dependence of allocations, so we consider the former case. The optimal tax problem is

$$\max_{\left\{c^j_{i,t}, b^j_{i,t}, y_{i,t}, R_t\right\}_{t,i,j}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\alpha_1\left(c_{1,t} - h_1\left(\frac{y_{1,t}}{\theta_1}\right)\right) + \alpha_2\left(u(c_{2,t}) - h_2\left(\frac{y_{2,t}}{\theta_2}\right)\right)\right]$$

subject to

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_{2,t}) - h_2\left(\frac{y_{2,t}}{\theta_2}\right)\right) \geq \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c^1_{2,t}) - h_2\left(\frac{y_{1,t}}{\theta_2}\right)\right)$$

$$c_{2,t} + b_{2,t} = x_{2,t} + \frac{1}{\beta} b_{2,t-1}$$

$$u'\left(c_{2,t+1}\right) = \mathbb{E}_t u'\left(c_{2,t+1}\right)$$

$$c^1_{2,t} + b^1_{2,t} = x_{1,t} + \frac{1}{\beta} b^1_{2,t-1}$$

$$u'\left(c^1_{2,t+1}\right) = \mathbb{E}_t u'\left(c^1_{2,t+1}\right)$$

$$c_{1,t} + b_{1,t} = x_{1,t} + \frac{1}{\beta} b_{1,t-1}$$

$$c_{1,t} + c_{2,t} + g_t = y_{1,t} + y_{2,t}$$

$$b^j_{i,t} \geq \underline{B}.$$

There are a few redundant equations here. Without loss of generality we can set $b_{1,t} = 0$ for all t in which case $c_{1,t} = x_{1,t}$. Moreover, it is clear that the Lagrange multiplier on the second constraint must be zero. Therefore, we have

$$\max_{\left\{c^j_{i,t}, b^j_{i,t}, y_{i,t}\right\}_{t,i,j}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\alpha_1 \left(c_{1,t} - h_1\left(\frac{y_{1,t}}{\theta_1}\right)\right) + \alpha_2 \left(u(c_{2,t}) - h_2\left(\frac{y_{2,t}}{\theta_2}\right)\right)\right]$$

s.t.

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_{2,t}) - h_2\left(\frac{y_{2,t}}{\theta_2}\right)\right) \geq \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c^1_{2,t}) - h_2\left(\frac{y_{1,t}}{\theta_2}\right)\right)$$

$$c^1_{2,t} + b^1_{2,t} = c_{1,t} + \frac{1}{\beta} b^1_{2,t-1}$$

$$u'\left(c_{2,t+1}\right) = \mathbb{E}_t u'\left(c_{2,t+1}\right)$$

$$u'\left(c^1_{2,t+1}\right) = \mathbb{E}_t u'\left(c^1_{2,t+1}\right)$$

$$c_{1,t} + c_{2,t} + g_t = y_{1,t} + y_{2,t}$$

Guess that neither Euler equation constraint binds. Take the first-order conditions

$$\alpha_1 - \eta^1_{2,t} = \lambda_t \tag{56}$$

$$u'\left(c_{2,t}\right)\left(\alpha_2 + \mu\right) = \lambda_t \tag{57}$$

$$\mu u'\left(c^1_{2,t}\right) = \eta^1_{2,t} \tag{58}$$

$$\eta^1_{2,t} = \mathbb{E}_t \eta^1_{2,t+1} \tag{59}$$

From (59) $\eta^1_{2,t}$ is a martingale, therefore from (56) , λ_t is a martingale, and therefore $u'\left(c_{2,t}\right)$ and $u'\left(c^1_{2,t}\right)$ are martingales, which confirms our guess. Moreover, $u'\left(c_{2,t}\right)$, $u'\left(c^1_{2,t}\right)$, λ_t and $\eta^1_{2,t}$ all must converge. Next we discuss what they must converge to.

Note that since λ_t converges to a constant, the first-order conditions for $y_{2,t}$ and $y_{1,t}$ imply that they also converge to constants. Since $c_{2,t}$ converges to a constant, $c_{1,t}$ must fluctuate to offset fluctuations in g_t. If $c_{1,t}$ fluctuates, then $u'\left(c^1_{2,t}\right) \to 0$ and therefore $\eta^1_{2,t} \to 0$. This implies that in the long run this economy converges to the constrained optimal allocations discussed in Section 5.1. Intuitively, what happens is that as $c_{1,t}$ fluctuates, the agent 2, if he deviates, accumulates infinitely large amount of assets. When he does that, smoothing fluctuations in $c_{1,t}$ has no effect on his welfare, and we get back to the fully constrained optimum allocations.

Note that $c_{2,t}$ cannot be constant in all t. If it were, then $c^1_{2,t}$ would also have to be constant in all t, which is impossible since $c_{1,t}$ must fluctuate.

References

Aiyagari, S. Rao (1995), "Optimal Capital Income Taxation with Incomplete Markets, Borrowing Constraints, and Constant Discounting," *Journal of Political Economy*, 103, 1158-1175.

Aiyagari, S. Rao and Ellen R. (1998),"The optimum quantity of debt," *Journal of Monetary Economics*, 42(3), 447-469.

Aiyagari, S Rao, Albert Marcet, Thomas Sargent, and Juha Seppala (2002), "Optimal Taxation without State-Contingent Debt" *Journal of Political Economy*, 110, 1220-1254.

Barro, Robert (1974), "Are Government Bonds Net Wealth?" *Journal of Political Economy*, 82 (6): 1095–1117.

Bassetto, Marco (1999), "Optimal Fiscal Policy with Heterogeneous Agents " manuscript, Northwestern University.

Barro, Robert (1979), "On the Determination of Public Debt", *Journal of Political Economy*, 87, 940-971.

Battaglini, Marco and Stephen Coate (2007), "Inefficiency in Legislative Policymaking: A Dynamic Analysis," *American Economic Review*, 97(1), 118-149.

Battaglini, Marco and Stephen Coate (2008), "A Dynamic Theory of Public Spending, Taxation, and Debt," *American Economic Review*, 98(1), 201-36.

Bryant, John and Neil Wallace (1984), "A Price Discrimination Analysis of Monetary Policy," *Review of Economic Studies*, 51 (2), 279-288.

Chari, V.V., Lawrence Christiano, and Patrick. Kehoe (1994), "Optimal Fiscal Policy in a Business Cycle Model", *Journal of Political Economy*, 102, 617-652.

Chari, V.V. and Patrick Kehoe (1999), "Optimal Fiscal and Monetary Policy," in J. Taylor and M. Woodford, eds., *Handbook of Macroeconomics* (Amsterdam: Elsevier).

Constantinides, George and Darrell Duffie (1996), "Asset Pricing with Heterogenous Consumers" *Journal of Political Economy*, 104(2), 219-240.

da Costa, Carlos and Vitor Lutz (2011), "The Private Memory of Aggregate Shocks", mimeo.

Golosov Mikhail and Aleh Tsyvinski (2007), "Optimal Taxation with Endogenous Insurance Markets," *Quarterly Journal of Economics*, 122(2), 487-534.

Golosov, Mikhail, Aleh Tsyvinski, and Ivan Werning (2007), "New Dynamic Public Finance: A User's Guide," in D. Acemoglu, K. Rogoff, and M. Woodford, eds., *NBER Macroe-*

conomics Annual 2007.

Faraglia, Elisa, Albert Marcet, and Andrew Scott (2012), "Dealing with Maturity: Optimal Fiscal Policy with Long Bonds" mimeo.

Farhi Emmanuel (2010), "Capital Taxation and Ownership When Markets Are Incomplete," *Journal of Political Economy*, 118(5), 908-948.

Heathcote, Jonathan (2005), "Fiscal Policy with Heterogeneous Agents and Incomplete Markets," *Review of Economic Studies*, 72(1), 161-188.

Kocherlakota, Narayana (2005), "Zero Expected Wealth Taxes: A Mirrlees Approach to Dynamic Optimal Taxation", Econometrica, 73(5), 1587-1621.

Kocherlakota, Narayana (2010), *The New Dynamic Public Finance* Princeton University Press.

Kydland, Finn E. and Edward C. Prescott (1980), "Dynamic optimal taxation, rational expectations and optimal control," *Journal of Economic Dynamics and Control*, Vol. 2(1), 79-91.

Lucas, Robert and Nancy Stokey (1983), "Optimal Fiscal and Monetary Policy in an Economy without Capital," *Journal of Monetary Economics* 12, 55-93.

Magill, Michael and Martine Quinzii (1994), "Infinite Horizon Incomplete Markets", *Econometrica*, 62(4), 853-880.

Sargent, Thomas J. (1987), *Dynamic Macroeconomic Theory*, Harvard University Press, Cambridge, Ma.

Sargent, Thomas J. and Bruce D. Smith (1987) ,"Irrelevance of Open Market Operations in Some Economies with Government Currency Being Dominated in Rate of Return," *American Economic Review*, 77(1), 78-92.

Shin, Yongseok (2006), "Ramsey Meets Bewley: Optimal Government Financing with Incomplete Markets, "manuscript, University of Wisconsin.

Wallace, Neil (1981), "A Modigliani-Miller Theorem for Open-Market Operations," *American Economic Review*, 71(3), 267-74.

Werning, Ivan (2007), "Optimal Fiscal Policy with Redistribution," *Quarterly Journal of Economics*, 122(3), 925-967.

Werning, Ivan (2012), "Nonlinear Capital Taxation" mimeo MIT.

Yared, Pierre (2010), "Politicians, Taxes, and Debt," *Review of Economic Studies*, 77, 806-840.

Zhu, Xiaodong (1992), "Optimal Fiscal Policy in a Stochastic Growth Model", *Journal of Economic Theory*, 58, 250-289.