7. Hyperbolic Functions

$$\cosh x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \text{valid for all } x$$

$$\sinh x = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{valid for all } x$$

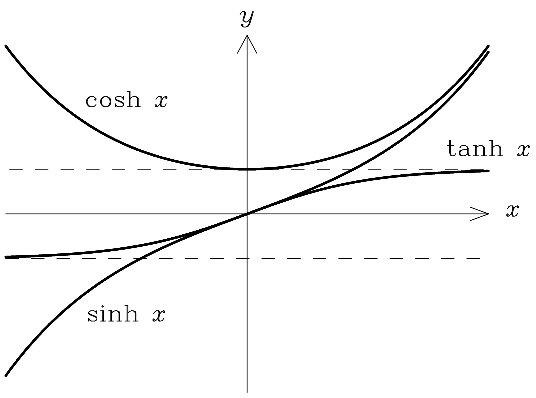
$\cosh \mathrm{i}x = \cos x$ $\qquad$ $\cos \mathrm{i}x = \cosh x$

$\sinh \mathrm{i}x = \mathrm{i}\sin x$ $\qquad$ $\sin \mathrm{i}x = \mathrm{i}\sinh x$

$\tanh x = \dfrac{\sinh x}{\cosh x}$ $\qquad$ $\operatorname{sech} x = \dfrac{1}{\cosh x}$

$\coth x = \dfrac{\cosh x}{\sinh x}$ $\qquad$ $\operatorname{cosech} x = \dfrac{1}{\sinh x}$

$\cosh^2 x - \sinh^2 x = 1$



For large positive x:

$\cosh x \approx \sinh x \to \dfrac{\mathrm{e}^x}{2}$

$\tanh x \to 1$

For large negative x:

$\cosh x \approx -\sinh x \to \dfrac{\mathrm{e}^{-x}}{2}$

$\tanh x \to -1$

Relations of the functions

$\sinh x = -\sinh(-x)$ $\qquad$ $\operatorname{sech} x = \operatorname{sech}(-x)$

$\cosh x = \cosh(-x)$ $\qquad$ $\operatorname{cosech} x = -\operatorname{cosech}(-x)$

$\tanh x = -\tanh(-x)$ $\qquad$ $\coth x = -\coth(-x)$

$$\sinh x = \frac{2\tanh(x/2)}{1-\tanh^2(x/2)} = \frac{\tanh x}{\sqrt{1-\tanh^2 x}} \qquad \cosh x = \frac{1+\tanh^2(x/2)}{1-\tanh^2(x/2)} = \frac{1}{\sqrt{1-\tanh^2 x}}$$

$\tanh x = \sqrt{1-\operatorname{sech}^2 x}$ $\qquad$ $\operatorname{sech} x = \sqrt{1-\tanh^2 x}$

$\coth x = \sqrt{\operatorname{cosech}^2 x + 1}$ $\qquad$ $\operatorname{cosech} x = \sqrt{\coth^2 x - 1}$

$\sinh(x/2) = \sqrt{\dfrac{\cosh x - 1}{2}}$ $\qquad$ $\cosh(x/2) = \sqrt{\dfrac{\cosh x + 1}{2}}$

$$\tanh(x/2) = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

$\sinh(2x) = 2\sinh x \cosh x$ $\qquad$ $\tanh(2x) = \dfrac{2\tanh x}{1+\tanh^2 x}$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

$\sinh(3x) = 3\sinh x + 4\sinh^3 x$ $\qquad$ $\cosh 3x = 4\cosh^3 x - 3\cosh x$

$$\tanh(3x) = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

18. Statistics

Mean and Variance

A random variable X has a distribution over some subset x of the real numbers. When the distribution of X is discrete, the probability that $X = x_i$ is P_i. When the distribution is continuous, the probability that X lies in an interval δx is $f(x)\delta x$, where $f(x)$ is the probability density function.

$$\text{Mean } \mu = E(X) = \sum P_i x_i \text{ or } \int x\, f(x)\, \mathrm{d}x.$$

$$\text{Variance } \sigma^2 = V(X) = E[(X-\mu)^2] = \sum P_i (x_i - \mu)^2 \text{ or } \int (x-\mu)^2 f(x)\, \mathrm{d}x.$$

Probability distributions

Error function: $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x \mathrm{e}^{-y^2}\, \mathrm{d}y$

Binomial: $f(x) = \dbinom{n}{x} p^x q^{n-x}$ where $q = (1-p)$, $\quad \mu = np$, $\sigma^2 = npq$, $p < 1$.

Poisson: $f(x) = \dfrac{\mu^x}{x!}\mathrm{e}^{-\mu}$, and $\sigma^2 = \mu$

Normal: $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[-\dfrac{(x-\mu)^2}{2\sigma^2}\right]$

Weighted sums of random variables

If $W = aX + bY$ then $E(W) = aE(X) + bE(Y)$. If X and Y are independent then $V(W) = a^2V(X) + b^2V(Y)$.

Statistics of a data sample $x_1, \ldots, x_n$

$$\text{Sample mean } \overline{x} = \frac{1}{n}\sum x_i$$

$$\text{Sample variance } s^2 = \frac{1}{n}\sum (x_i - \overline{x})^2 = \left(\frac{1}{n}\sum x_i^2\right) - \overline{x}^2 = E(x^2) - [E(x)]^2$$

Regression (least squares fitting)

To fit a straight line by least squares to n pairs of points (x_i, y_i), model the observations by $y_i = \alpha + \beta(x_i - \overline{x}) + \epsilon_i$, where the ϵ_i are independent samples of a random variable with zero mean and variance σ^2.

Sample statistics: $s_x^2 = \dfrac{1}{n}\sum (x_i - \overline{x})^2$, $\quad s_y^2 = \dfrac{1}{n}\sum (y_i - \overline{y})^2$, $\quad s_{xy}^2 = \dfrac{1}{n}\sum (x_i - \overline{x})(y_i - \overline{y})$.

Estimators: $\widehat{\alpha} = \overline{y}$, $\widehat{\beta} = \dfrac{s_{xy}^2}{s_x^2}$; $E(Y \text{ at } x) = \widehat{\alpha} + \widehat{\beta}(x - \overline{x})$; $\widehat{\sigma}^2 = \dfrac{n}{n-2}$(residual variance),

where residual variance $= \dfrac{1}{n}\sum \{y_i - \widehat{\alpha} - \widehat{\beta}(x_i - \overline{x})\}^2 = s_y^2 - \dfrac{s_{xy}^4}{s_x^2}$.

Estimates for the variances of $\widehat{\alpha}$ and $\widehat{\beta}$ are $\dfrac{\widehat{\sigma}^2}{n}$ and $\dfrac{\widehat{\sigma}^2}{ns_x^2}$.

Correlation coefficient: $\widehat{\rho} = r = \dfrac{s_{xy}^2}{s_x s_y}$.