Part I

Underwater Acoustics

Chapter 1

Propagation of acoustic waves

1.1 Propagation equation

1.1.1 Basic laws

Let us consider a fluid at rest, characterized by its density $\rho_0(\mathbf{r})$ and the pressure field $p_0(\mathbf{r})$. An acoustic wave propagating through it makes the fluid particles move, with velocity $\mathbf{u}$, and generates a small perturbation in density and pressure, $\rho(\mathbf{r},t) = \rho_0(\mathbf{r}) + \delta\rho(\mathbf{r},t)$, $p(\mathbf{r},t) = p_0(\mathbf{r}) + \delta p(\mathbf{r},t)$.
If we neglect the frictional forces and Coriolis force, applying Newton's law to a fluid particle leads to

$$\frac{d}{dt}(\rho\mathbf{u}) = \rho\mathbf{g} - \nabla p \tag{1.1}$$

Assuming $\mathbf{u}$, $\delta\rho$ and δp are small quantities, and noticing that $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}.\nabla$, at first order the previous equation becomes

$$\rho_0\frac{\partial\mathbf{u}}{\partial t} = \delta\rho\ \mathbf{g} - \nabla\delta p \tag{1.2}$$

Assuming also that the response of the fluid is linearly linked to the (small) constraint, the change in volume of the fluid particle is proportional to the pressure field δp

$$\frac{\delta v}{v} = -\frac{\delta\rho}{\rho} = -\chi\delta p \tag{1.3}$$

where χ represents the compressibility of the fluid. The negative sign ensures the compressibility to be positive, since an increase of pressure generally comes with a decrease of volume. Eq. (**??**) also allows one to compare the order of magnitude of the two terms of the right-hand side of eq. (**??**). The ratio $\frac{\nabla\delta p}{g\delta\rho} = \frac{1}{\rho_0\chi g}\frac{\nabla\delta p}{\delta p} \simeq 10^5\frac{\nabla\delta p}{\delta p}$ is much greater than 1 if δp varies at length scales smaller than a kilometer, which will be the case here.

In the frame of the linear approximation (**??**), the displacements of the boundaries of the fluid particle are proportional to the pressure they are facing, like a spring subjected to a force. Denoting by $\xi(x)$ the displacement along the x direction at abscissa x, the change in length of the fluid particle along this direction is $\xi(x+dx) - \xi(x)$. The relative volume change thus writes

$$\frac{\delta v}{v} = \frac{\left(1+\frac{\partial\xi}{\partial x}\right)dx\left(1+\frac{\partial\eta}{\partial y}\right)dy\left(1+\frac{\partial\zeta}{\partial z}\right)dz - dx\ dy\ dz}{dx\ dy\ dz} \tag{1.4}$$

which, at first order becomes

$$\frac{\delta v}{v} = \nabla.\mathbf{d} \tag{1.5}$$

where $\mathbf{d}(\xi,\eta,\zeta)$ denotes the displacement vector.
Taking the time derivative of eq.(**??**), and noticing that $\mathbf{u} = \frac{d\mathbf{d}}{dt}$, provides the second relationship between $\mathbf{u}$ and δp

$$-\chi\frac{\partial\delta p}{\partial t} = \nabla.\mathbf{u} \tag{1.6}$$

A third law that may also be of interest is the local conservation of mass, which, at first order, writes as

$$\nabla.(\rho_0 \mathbf{u}) + \frac{\partial \delta\rho}{\partial t} = 0 \tag{1.7}$$

1.1.2 Propagation equation

Combining equations (**??**) and (**??**) allows us to derive the so-called propagation equation of acoustic waves for the pressure field. Derivating eq.(**??**) with respect to time and replacing $\partial\mathbf{u}/\partial t$ according to eq.(**??**), where the gravity force is now neglected, leads to the following equation for δp

$$\nabla.\left(\frac{1}{\rho_0}\nabla\delta p\right) - \chi\frac{\partial^2\delta p}{\partial t^2} = 0 \tag{1.8}$$

If the fluid is **homogeneous**, *i.e.* the density at rest ρ_0 is the same everywhere, one gets the classical wave equation, outside the support of the source

$$\Delta\delta p - \rho_0\chi\frac{\partial^2\delta p}{\partial t^2} = 0 \tag{1.9}$$

$\rho_0\chi$ is a positive number which has the dimension of the inverse of the square of a velocity. Therefore, $1/\sqrt{\rho_0\chi}$ can be interpreted as the propagation velocity c of acoustic waves in such a medium.

At this step, let us come back to the definition of compressibility χ and let us consider a perfect gas, obeying the state equation $pv = nRT$. At constant temperature T, the change in pressure and volume occurs such that the product pv remains constant. It is also shown in thermodynamics courses that for a transformation with no heat exchange, the invariant quantity is pv^γ where γ is the ratio of the specific heat capacities at constant pressure and constant volume, c_p/c_v, respectively, which is equal to 1.4 for diatomic gases like air. Therefore, the same perturbation δp does not generate the same volume change depending on whether the propagation of the acoustic wave creates an isotherm or adiabatic transformation of the fluid. Assuming, like Newton did, that the transformation is isotherm leads to sound speed in air of about 290 m/s in standard conditions, while it is $\sqrt{1.4}$ greater, about 340 m/s, if no heat exchange occurs. The first experiments during the eighteenth century have proved that the latter is the right value and it has been explained later with the help of statistical thermodynamics theory. The same holds in water, and, at usual frequencies, the velocity of acoustic waves is definitely

$$c = \frac{1}{\sqrt{\rho_0\chi_S}} \tag{1.10}$$

where χ_S denotes the isentropic compressibility.

If one looks for time-harmonic solutions of eq.(**??**) with pulsation ω, $\delta p(\mathbf{r}, t) = \Re\left(\delta\tilde{p}(\mathbf{r})\exp(-i\omega t)\right)$, the wave equation transforms into Helmholtz equation for the complex amplitude $\delta\tilde{p}$

$$\Delta\delta\tilde{p} + k^2\delta\tilde{p} = 0 \tag{1.11}$$

The solutions of this equation are superpositions of time-harmonic plane waves of the form $\delta\tilde{p} = a_0\exp(i\mathbf{k}.\mathbf{r})$, where $\mathbf{k} = (k_x, k_y, k_z)$ represents the complex wavevector, of which real part $\mathbf{k}'$ and imaginary part $\mathbf{k}''$ must satisfy $k'^2 - k''^2 = \omega^2/c^2$ and $\mathbf{k}'.\mathbf{k}'' = 0$. The ratio ω/c is the wavenumber and is denoted by k.

1.1.3 Dissipation effects

In pure water, the main mechanism responsible for the damping of acoustic waves is the friction of two neighboring fluid elements moving at different velocities. This refers to as the viscosity of the fluid.
In sea water, sound attenuation also results from the dissociation of boric acid and of magnesium sulfate, which are the main source of damping, up to 1 kHz for $B(OH)_3$ and from 1 to 10 kHz for $MgSO_4$, for typical concentrations. Part of the energy of the acoustic wave is transferred to the vibration modes of the molecules.
The attenuation A is expressed in dB/km, such that $Ar = -10\log_{10}\left(P(r)/P_0\right)$, r in km.

1.2 Reflexion at the water-air interface

Here, the water-air interface is assumed to be flat. When the acoustic wave hits a flat interface, it splits into a reflected wave, which travels back into water (half-space $z < 0$, sound velocity c_w in the vicinity to the surface), and a transmitted wave, which goes through into the air (half-space $z > 0$, sound velocity c_a). The characteristics of these two waves are derived from the so-called boundary conditions, which ensure the continuity of the pressure and of the vertical component of the velocity.
Let us consider an incident monochromatic plane wave with pulsation ω and wavevector $\mathbf{k}$, impinging on the interface $z = 0$

$$\delta p^{inc}(\mathbf{r}, t) = \Re\left(a \exp(i\mathbf{k}^{inc}.\mathbf{r} - i\omega t)\right) \tag{1.12}$$

The continuity of δp at $z = 0$ at any time implies that the reflected and transmitted waves have the same pulsation ω and the same horizontal components of the wavevector (k_x, k_y) (Snell's law). Therefore, the reflected and transmitted waves may be written as

$$\delta p^r(\mathbf{r}, t) = \Re\left(b \exp(i\kappa_0.\mathbf{r} - iq_0 z - i\omega t)\right) \tag{1.13}$$

$$\delta p^t(\mathbf{r}, t) = \Re\left(b' \exp(i\kappa_0.\mathbf{r} + iq_0' z - i\omega t)\right) \tag{1.14}$$

where $\kappa_0 = (k_x, k_y, 0)$ is the projection of $\mathbf{k}^{inc}$ on the plane $z = 0$ and q_0 its vertical component. Since each wave is a solution of the propagation equation eq.(**??**), one gets the vertical components of the wavevectors

$$k_z^r = -q_0 = -\sqrt{\frac{\omega^2}{c_w^2} - \kappa_0^2} = -\frac{\omega}{c_w}\cos\theta^{inc} \tag{1.15}$$

$$k_z^t = q_0' = \sqrt{\frac{\omega^2}{c_a^2} - \kappa_0^2} = \frac{\omega}{c_a}\cos\theta^{t} \tag{1.16}$$

where the angles θ are measured from the normal to the interface, the z axis here. Let us notice that c_a being smaller than c_w, k_z^t is a real number and can be written as above, with $-\pi/2 < \theta^t < \pi/2$. The incidence and transmitted angles θ^{inc} and θ^t, are linked through Snell's law

$$\frac{\sin\theta^{inc}}{c_w} = \frac{\sin\theta^t}{c_a} \tag{1.17}$$

If r and t denote the reflection and transmission coefficients, b/a and b'/a respectively, the continuity of δp at $z = 0$ leads to

$$1 + r = t \tag{1.18}$$

The continuity of the vertical component of the velocity is a direct consequence of the concept of interface between two homogeneous media. The two fluids are not supposed to mix, neither to move away from each other, under the small perturbation of an acoustic wave. According to eq.(**??**), the continuity of u_z entails that of $\frac{1}{\rho}\frac{\partial \delta p}{\partial z}$ and leads to

$$\frac{k_z^{inc}}{\rho_w} + r\frac{k_z^r}{\rho_w} = t\frac{k_z^t}{\rho_a} \tag{1.19}$$

Combined with eq.(**??**), it comes

$$1 - r = t\frac{\rho_w c_w}{\rho_a c_a}\frac{\cos\theta^t}{\cos\theta^{inc}} \tag{1.20}$$

Noticing that $\frac{\rho_w c_w}{\rho_a c_a} \simeq 4400$ and that $\cos\theta^t > \cos\theta^{inc}$, we deduce that t is very small and $r \simeq -1$. As a result, the acoustic wave is totally reflected downwards, with a phase shift of π.

1.3 Reflexion at the water-seabed interface

Again, let us assume the interface is flat and that an acoustic wave is propagating in water down to the sea bottom. The main difference with the water-air interface is that the contrast of impedance between seabed and water Z_s/Z_w, where $Z = \frac{1}{\rho c}$, is much smaller here. Therefore, if one applies the results of the previous section and replaces the impedance of air Z_a by that of seabed Z_s, the transmission coefficient writes as

$$t = \frac{2}{1 + \frac{Z_s}{Z_w}\frac{\cos\theta^t}{\cos\theta^{inc}}} \tag{1.21}$$

which in general is not very small.
For instance, for sediment with $c_s = 1700$ m/s and $\rho_s = 1800$ kg/m^3, illuminated under vertical incidence ($\theta^{inc} = 0$, which implies $\theta^t = 0$ according to Snell's law), one gets $t \simeq 1.34$. This results leads us to make two remarks.
First, t greater than 1 does not mean that more energy is going through into the sediment than it was sent from the source in the water. Although the density of energy is proportional to the square modulus of the amplitude of the wave, the expression of the flux of energy through the interface also involves the density of the sediment and the associated wavenumber, such that the principle of energy conservation si not violated. As a result of the energy balance, the part of energy reflected in water in this case, if $c_w = 1500$ m/s, is $R = |r|^2 = |t-1|^2 \simeq 0.117$ and the part of energy going through is $T = 1 - R \simeq 0.883$.
Secondly, since a wave may propagate in seabed, it may hit deeper interfaces separating layers made of various materials (multilayer seabed). In this case, the previous formula (**??**) does not hold. It can only be applied if the transmitted wave does not "see" any other medium than the sediment, because the layers are deeply embedded and the wave is attenuated there. The attenuation depends on the absorbing properties of the seabed material, but also on the incidence angle. When the latter increases, so does θ^t, which, in the previous example, reaches $\pi/2$ when $\sin\theta^t = \frac{c_w}{c_s}\sin\theta^{inc} = 1$, thus $\theta^{inc} = 62°$. In this case, $r = t - 1 = 1$, which entails that 100% of energy is reflected.
It is to be noticed that when $\theta^{inc} > 62°$, the horizontal component of the wavevector κ_0, which is the same for the three waves (Snell's law), has its norm greater than the wavenumber in the sediment $k_s = \frac{\omega}{c_s}$, leading to $q_0^{'2} = k_s^2 - \kappa_0^2 < 0$. Hence, the transmitted wave writes as

$$\delta p^t(\mathbf{r}, t) = \Re\left(b' \exp\left(\sqrt{\kappa_0^2 - k_s^2}\, z\right)\exp(i\kappa_0.\mathbf{r} - i\omega t)\right) \tag{1.22}$$

which represents a so called "surface wave", as it propagates along the horizontal plane and its amplitude exponentially decreases with respect to depth in the sediment ($z < 0$), as in an absorbing medium with $\frac{20}{\ln 10}\sqrt{\kappa_0^2 - k_s^2}$ dB/km attenuation. Therefore, if sound velocity is greater in seabed than in water, and if seabed can be considered as a homogeneous medium, at large incidence angles, the reflection coefficient becomes

$$r = \frac{\frac{q_0}{\rho} - i\frac{g_0'}{\rho'}}{\frac{q_0}{\rho} + i\frac{g_0'}{\rho'}} \tag{1.23}$$

where $q_0 = \sqrt{k_w^2 - \kappa_0^2}$ and $g_0' = \sqrt{\kappa_0^2 - k_s^2}$. Since the denominator is the complex conjugate of the numerator, $|r| = 1$ and the wave is totally reflected back into water, with phase shift $\Phi = 2\arctan\left(\frac{g_0'\rho}{q_0\rho'}\right)$.

1.4 Velocity linear in depth

Let us assume that the sound velocity increases linearly with depth z. Denoting by c_0 the velocity at the surface ($z = 0$), c writes as

$$c(z) = c_0\left(1 + \frac{z}{L}\right) \tag{1.24}$$

where L is the characteristic length of the fluctuations of c, of which order of magnitude is typically tens of kilometers.
Let us consider a ray starting from depth z_s with angle θ_s, measured from the downward direction et let us derive the trajectory of the ray. Snell's law ensures that the horizontal component of the wavevector, denoted by κ here, remains constant. Since the frequency ω is constant along the ray, at any point we have

$$\kappa = \frac{\omega}{c(z)} \sin\theta(z) = \frac{\omega}{c(z_s)} \sin\theta_s \tag{1.25}$$

During the lap of time dt, the horizontal and vertical displacements of the ray are dx and dz, respectively, with $dx/dz = \tan\theta$. Hence

$$dx = \frac{\sin\theta(z)}{\sqrt{1-\sin^2\theta(z)}} dz = \frac{c(z)}{\sqrt{\frac{\omega^2}{\kappa^2} - c^2(z)}} dz \tag{1.26}$$

Since c is a linear function of z, the numerator is proportional to the derivative of the denominator

$$\frac{dc^2}{dz} = 2\frac{c_0}{L} c(z) \tag{1.27}$$

which allows us to integrate analytically

$$x = -\frac{L}{c_0} \left[\sqrt{\frac{\omega^2}{\kappa^2} - c^2(z)} \right]_{z_s}^{z} \tag{1.28}$$

leading to

$$\left(x - \frac{L}{c_0} \sqrt{\frac{\omega^2}{\kappa^2} - c^2(z_s)} \right)^2 = \frac{L^2}{c_0^2} \left(\frac{\omega^2}{\kappa^2} - c^2(z) \right) \tag{1.29}$$

thus

$$\left(x - \frac{L}{c_0} \sqrt{\frac{\omega^2}{\kappa^2} - c^2(z_s)} \right)^2 + (z+L)^2 = \frac{L^2}{c_0^2} \frac{\omega^2}{\kappa^2} \tag{1.30}$$

which is the equation of a circle with radius $R = L\frac{\omega}{|\kappa| c_0}$ and center located at $x_c = R\cos\theta_s$, $z_c = -L$. Noticing that L/c_0 is the inverse of the gradient of velocity, we can write the radius as

$$R = \frac{1}{\sin\theta_s} \frac{c(z_s)}{|\, dc/dz \,|} \tag{1.31}$$

which is the general formula for the radius of curvature of the ray as a function of the slope of the velocity.
Below is an example with $dc/dz = 0.0163 s^{-1}$ and $c_0 = 1450 ms^{-1}$.

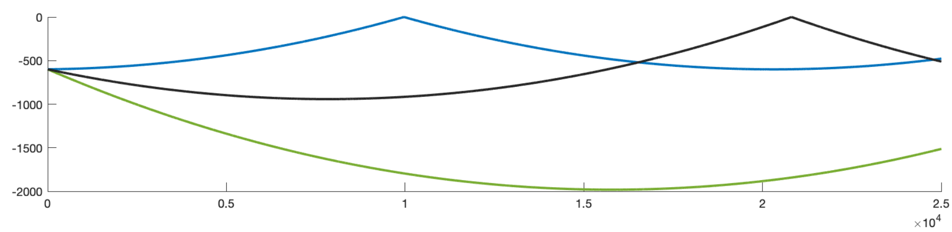

Figure 1.1: Rays starting from 600 m depth, with $\theta_s = 80°$ (green), 85° (black) and 90° (blue)

Bibliography

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