Product Form of Inverse of a Non-Singular Matrix for RSM

We wish to compute the inverse of a Basis matrix B_r that is different by only one column (say rth column) from a Basis matrix $\hat{B}$ whose inverse(B^{-1}) is known.

By using the product form of inverse new inverse can be computed in an efficient manner. Steps below:

Let B be the original basis matrix of size n by n.

B_r is the new basis matrix which identical to B except column r.

Let c be the rth column of B_r.

Let $e = B^{-1}C$ = Column vector

$$= \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{r-1} \\ e_r \\ e_{r+1} \\ \vdots \\ e_n \end{bmatrix}, \quad \eta = \begin{bmatrix} -\frac{e_1}{e_r} \\ -\frac{e_2}{e_r} \\ \vdots \\ -\frac{e_{r-1}}{e_r} \\ \frac{1}{e_r} \\ -\frac{e_{r+1}}{e_r} \\ -\frac{e_n}{e_r} \end{bmatrix}$$

where $e_r \neq 0$ (rth element)

$$B_r^{-1} = E_r B^{-1}$$

where

E_r = Identity matrix with its rth column replaced by η vector

B^{-1} = Inverse of the Basis Matrix which is known ($B^{-1} = B = I_n$)

B_r^{-1} = Inverse of B_r.

(This procedure is used in RSM: Revised Simplex Method.)

Example: 1(a)

$$B_r = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}_{3\times 3} \qquad B = B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

3rd column

$r = 3$, $c = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$

$$e = B^{-1}c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$$

Hence

$$\eta = \begin{pmatrix} -\frac{4}{2} \\ \frac{3}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{bmatrix} -2 \\ 3/2 \\ 1/2 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3/2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}_{3\times 3}$$

$$B_r^{-1} = E_3 B^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1/2 \end{bmatrix}$$

Example 1(b)

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Find $B_1^{-1} = E_1 B^{-1} = E_1$

Let $B_2 = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (Note: compare with B_1)

Find $B_2^{-1} = E_2 B_1^{-1} = E_2 E_1$

Let $B_3 = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (Note: compare with B_2)

Find $B_3^{-1} = E_3 B_2^{-1}$

$$= E_3 E_2 E_1 = \begin{bmatrix} 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

where $E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Compute the Inverse of the following matrices:

(i) $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 1 \end{bmatrix}$ (ii) $P = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(iv) $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$

(v) $C = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} = B_4$

(See Page-6)

Example 2:

Let $B = B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$

$B_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ compare B_1 with B

$B_1^{-1} = E_1 B^{-1}$

$B_2 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ compare B_2 with B_1

$B_2^{-1} = E_2 B_1^{-1}$

$B_3 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ compare B_3 with B_2

$B_3^{-1} = E_3 B_2^{-1}$

$B_4 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ compare B_4 with B_3

$B_4^{-1} = E_4 B_3^{-1}$

$$B_4^{-1} = E_4 B_3^{-1} = E_4 E_3 B_2^{-1} = E_4 E_3 E_2 B_1^{-1}$$
$$= E_4 E_3 E_2 E_1$$

(See Page 2 for Computing E_1, E_2, E_3, E_4)