Basic Feasible Solution Procedure to solve an LP Problem.

Let $AX = b$ be system of m linear equations with n variables ($n > m$) where A is a real matrix of size m by n, X is a column vector having n elements i.e. $X = (x_1, x_2, x_3, \ldots, x_n)^T$, and b is a non-zero column vector having m elements i.e. $b = (b_1, b_2, b_3, \ldots, b_m)^T$.

The system is consistent and has infinite number of solutions

if $r(A) = r(A|b) = m < n$

i.e. Rank of A and Rank of augmented matrix $(A|b)$ are equal

and less than n.

If the system $AX = b$ is consistent, we may select any m variables out of n variables $(x_1, x_2, x_3, \ldots, x_n)$ and set the remaining $(n - m)$ variables to zero.

Remarks:

The variables which are set as zero are called non-basic variables and rest of the variables are called basic variables.

After setting (n - m) variables to zero the system $AX = b$ becomes $BX_B = b$ where B is a non-singular matrix of order m (i.e. $|B| \neq 0$) and X_B is a column vector with m elements. If it has a solution then

$$X_B = B^{-1}b.$$

X_B is called a Basic Solution of the system $AX = b$.

Maximum number of possible Basic Solutions is equal to $^nC_m = {}^nC_{n-m}$.

If all the basic variables are non-negative then a Basic Solution is called a Basic Feasible Solution (B.F.S.).

Example:

Let us consider the LPP:

$$\text{Max} : Z = x_1 + 4x_2$$

Subject to

$$x_1 + x_2 \leq 10$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

After introducing slack variables x_3, x_4 now we have two equations and four variables. Let us find the B.F.S.

$$x_1 + x_2 + x_3 = 10$$

$$x_1 + 4x_2 \quad + x_4 = 16$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Table for Basic Solution and Basic Feasible Solution

Sl. No.	Non-Basic Variables	Basic Variables
1.	$x_1=0, x_2=0$	$x_3=10, x_4=16$
2.	$x_1=0, x_3=0$	$x_2=10, x_4=-24$
3.*	$x_1=0, x_4=0$	$x_2=4, x_3=6$

Table (Contd.)

Sl. No.	Non-Basic Variables	Basic Variables
4.	$x_2=0,\ x_3=0$	$x_1=10,\ x_4=6$
5.	$x_2=0,\ x_4=0$	$x_1=16,\ x_3=-6$
6.*	$x_3=0,\ x_4=0$	$x_1=8,\ x_2=2$

There are six Basic Solutions. Only four of them are Basic Feasible Solutions. Sl. No. (2) and (5) are not Basic Feasible Solutions.

Optimal Solution (Max.) is obtained at Sl. No. (3) i.e. $x_1=0, x_2=4$ and Sl. No. (6) i.e. $x_1=8, x_2=2$

Maximum value of $Z=16$.

We obtain two Optimal Solutions.

$(x, y)=(0, 4)$ and $(x, y)=(8, 2)$

<u>Remark:</u>

This LP problem has infinite number of Optimal Solutions.

General Remarks:

- In general a system $AX = b$ has either unique Solution or no solution or infinite number of solutions.
- In general a LP problem has either one optimal Solution or no optimal solution or infinite number of optimal solutions.

Example – 2(a) & 2(b)

(a) max : $Z_1 = 4x_1 + 6x_2$

(b) min : $Z_2 = 2x_1 + 3x_2$

s.to

$$2x_1 + x_2 - x_3 = 20$$
$$3x_1 + 4x_2 - x_4 = 50$$
$$x_1 + 2x_2 - x_5 = 20$$
$$x_1, x_2 \geq 0$$
$$x_3, x_4, x_5 \geq 0$$

(i) Find all the Basic solutions.

(ii) Find all the Basic Feasible Solutions.

(iii) Find the optimal solution(s)

Example 3 (a) & 3(b)

$$\max: Z_1 = x_1 + 9x_2 + x_3 + x_4$$

$$\min: Z_2 = 9x_1 + x_2 + x_3 + 9x_4$$

s.to

$$x_1 + 2x_2 + 3x_3 + x_4 = 90$$

$$3x_1 + 2x_2 + 2x_3 + x_4 = 150$$

$$4x_1 + 4x_2 + 5x_3 + 2x_4 = 240$$

$$x_1, x_2, x_3, x_4 \geq 0$$

(i) Find all the Basic solutions,

(ii) State all the Basic Feasible solution,

(iii) Find the max. value of Z_1

(iv) Find the min. value of Z_2.