The Vectorial Parameterization of Rotation.*

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Abstract

The parameterization of rotation is the subject of continuous research and development in many theoretical and applied fields of mechanics, such as rigid body, structural, and multibody dynamics, robotics, spacecraft attitude dynamics, navigation, image processing, and so on. This paper introduces the vectorial parameterization of rotation, a class of parameterization techniques encompassing many formulations independently developed to date for the analysis of rotational motion. The exponential map of rotation, the Rodrigues, Cayley, Gibbs, Wiener, and Milenkovic parameterization all are special cases of the vectorial parameterization. This generalization parameterization sheds additional light on the fundamental properties of these techniques, pointing out the similarities in their formal structure and showing their inter-relationships. Although presented in a compact manner, all of the formulæ needed for a complete implementation of the vectorial parameterization of rotation are included in this paper.

keywords: Finite rotations, Parameterization of rotations.

1 Introduction and Motivation

The effective description of rotational motion has led over the years to the development of numerous techniques, presenting various properties and advantages. Reviews of these parameterization techniques may be found in refs. [1, 2, 3, 4]. Whether originating from geometric, algebraic, or matrix approaches, parameterization of rotations are most naturally categorized into two classes: *vectorial* and *non-vectorial* parameterizations. The former refers to parameterization in which a set of parameters (sometimes called rotational "quasi-coordinates") define a geometric vector, whereas the latter cannot be cast in the form of

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a vector. These two types of parameterization are sometimes denoted as *invariant* and *non-invariant* parameterization, respectively, see ref. [2].

The rotation vector, as well as Cayley, Gibbs, Rodrigues, Wiener, and Milenkovic parameters all are examples of vectorial parameterizations. These are all characterized by a set of three parameters (*i.e.*, a "minimal" set) which behave as the Cartesian components of a geometric vector in 3-D space. Non-vectorial parameterizations, on the other hand, may be either minimal, as in the case of Euler (or Euler-type, or Eulerian) angles, or "redundant", as for Euler-Rodrigues parameters (unit quaternions), Cayley-Klein parameters, and for the matrix of direction cosines. Redundancy arises when more than three parameters are employed: four in the case of Euler-Rodrigues and Cayley-Klein parameters, nine in the case of direction cosines. In fact, rotation may be described as the motion of a point on a 3-D non-linear manifold, the Lie group of special orthogonal transformations of the 3-D space. The various parameterizations of rotation are, in differential geometry terminology, different *charts* available for this particular manifold.

The various parameterization techniques detailed in the literature present distinct advantages and drawbacks. Advantages can be of a theoretical nature (ease of geometric interpretation, or convenience in algebraic manipulations, for instance) or of a computational nature (low cost function evaluations, wide range of singularity free behavior, etc.) These features provide guidelines for selecting parameterizations that are best suited for specific applications. However, a survey of the literature reveals that for both theoretical and numerical applications, the choice of parameterization is often based on personal taste and traditions rather than cost/benefit considerations.

This choice is further complicated by the fact that specific parameterizations are sometimes used to present novel computational algorithms. The desirable properties of such algorithms then seem intimately linked to the specific parameterization used in the derivation. For instance, momentum preserving and/or energy preserving/decaying schemes for time integration of multibody systems have appeared in numerous publications [5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and were presented using either the exponential map, or the Cayley transform representation (*i.e.* Rodrigues parameters). Consequently, these parameterizations seem to be endowed with special properties. However, as pointed out in ref. [9], this reasoning is often incorrect, as in many cases any vectorial parameterization can lead to the desired result.

The vectorial parameterization presented in this paper is a natural consequence of Euler's theorem on rotation. The quaternion formulation is also closely related to the proposed vectorial parameterization. On the other hand, minimal non-vectorial parameterizations such as Euler and Euler-type angles are not easily related to vectorial techniques. Rather, they may be investigated in terms of exponential coordinates of the second kind, whereas the exponential parameterization is an application of exponential coordinates of the first kind [15]. In this work, a complete description of rotations is presented for an arbitrary vectorial parameterization. The particular formulæ for any specific parameterization of this class can then be easily obtained. It is even possible to devise new parameterization techniques subjected to given requirements.

In section 2, Euler's theorem on rotations is reviewed. Although this material is 'classical', a synthetic presentation is given as a preliminary step to the development of a general *vectorial parameterization theory for rotation*, presented in section 3.1. In this section, all the formulæ for the rotation tensor, its derivative, and related quantities are presented without specifying the choice of the parameterizing function. In section 3.2, special choices of the generating function are shown to yield many widely used parameterizations such as the Cayley-Gibbs-Rodrigues parameters, Wiener-Milenkovic parameters, Euler-Rodrigues reduced parameters, linear parameters, and others. Furthermore, these techniques are recovered as members of two different families: the sine and the tangent family. An example of a new parameterization belonging to the sine family is discussed as an example. The occurrence of singularities in the proposed vectorial parameterization is the focus of section 3.4. The Appendix includes various complements to the vectorial parameterization, including the relationships with other approaches to the representation of rotation, such as the use of unit quaternions, the exponential map of rotations, and the Cayley transform.

1.1 Preliminaries on Notation

We denote with the symbol $\mathbb{N}$ the set of natural numbers, *i.e.* the (strictly) positive integers. The symbol $\mathbb{E}^3$ indicates the 3-D Euclidean vector space, equipped with the standard bilinear operations $\{\cdot, \times, \otimes\}$, *i.e.* the scalar product, vector product, and tensor product, respectively. The set of all linear transformations, or tensors, on $\mathbb{E}^3$ is denoted by $\mathrm{Lin}(\mathbb{E}^3)$. Both vectors and tensors (or first and second order tensors, respectively) will be denoted by bold letters. The null vector in $\mathbb{E}^3$ is denoted as $\mathbf{0}$ and the identity in $\mathrm{Lin}(\mathbb{E}^3)$ as $\mathbf{I}$. The operations $\mathrm{tr}(\mathbf{A})$ and $\det(\mathbf{A})$ denote the evaluation of the *trace* (first invariant) and *determinant* (third invariant) of tensor $\mathbf{A}$, respectively. The operations $\mathrm{sym}(\mathbf{A}) = (\mathbf{A} + \mathbf{A}^T)/2$ and $\mathrm{skw}(\mathbf{A}) = (\mathbf{A} - \mathbf{A}^T)/2$ denote the symmetric and skew parts of tensor $\mathbf{A}$, respectively, where the superscript ^{T} denotes transposition operation. Two subsets of $\mathrm{Lin}(\mathbb{E}^3)$ are of importance in the following: the *rotation group* $\mathrm{SO}(\mathbb{E}^3)$, *i.e.* the Lie group composed of all the special orthogonal tensors, and the space $\mathrm{so}(\mathbb{E}^3)$, composed of all the skew-symmetric tensors and corresponding to the Lie algebra of $\mathrm{SO}(\mathbb{E}^3)$. The skew-symmetric tensor obtained from the cross product operator applied to vector $\mathbf{a} \in \mathbb{E}^3$ is denoted $(\mathbf{a}\times) \in \mathrm{so}(\mathbb{E}^3)$, *i.e.*, $(\mathbf{a}\times)\,\mathbf{b} = \mathbf{a} \times \mathbf{b}, \quad \forall \mathbf{a}, \mathbf{b} \in \mathbb{E}^3$. The inverse operation $\mathrm{axial}(\mathbf{A})$ represents the 'extraction' of the *axial vector* from tensor $\mathbf{A} \in \mathrm{Lin}(\mathbb{E}^3)$. The axial vector is defined as vector $\mathbf{a}$ such that $(\mathbf{a}\times) = \mathrm{skw}(\mathbf{A})$. Hence, spaces $\mathbb{E}^3$ and $\mathrm{so}(\mathbb{E}^3)$ are completely identified. Finally, $[\mathbf{A}]_{\mathcal{B}}$ denotes the matrix of components of tensor $\mathbf{A}$ with respect to the basis $\mathcal{B}$. A comprehensive treatment of 3-D vector algebra can be found in numerous textbooks such as refs. [16, 17].

2 Euler's theorem on rotations

The fundamental theorem on finite rotations due to Leonhard Euler states: *"any rigid motion leaving a point fixed may be represented by a rotation about a suitable axis passing through that point"*. The rotation is fully defined by the unit vector of the axis of rotation $\mathbf{u} \in \mathbb{E}^3$ and the angle of rotation φ with respect to a reference configuration. From a purely geometrical standpoint, two rotations about the same axis but through angles differing by $2k\pi$, where $k \in \mathbb{N}$, are clearly indistinguishable. Note that when $\varphi = 2k\pi$ the axis is no longer uniquely determined. Hence, the range $|\varphi| \leq \pi$ covers all possible rotations.

A general rotation, viewed as a transformation of three-dimensional vectors, may be represented by a proper orthogonal tensor $\mathbf{R} \in \mathrm{SO}(\mathbb{E}^3)$, *i.e.* $\mathbf{R}^T = \mathbf{R}^{-1}$ and $\det(\mathbf{R}) = 1$. Elementary geometrical arguments (see, *e.g.*, ref. [18]), can be used to give an explicit expression of the rotation tensor in terms of $(\varphi, \mathbf{u})$

$$\boxed{\mathbf{R} = \mathbf{I} + \sin\varphi\,(\mathbf{u}\times) + (1-\cos\varphi)\,(\mathbf{u}\times)^2.} \tag{1}$$

This equation is known as *Euler-Rodrigues formula*. Note that the rotation corresponding to $(-\varphi, \mathbf{u})$ is equivalent to that corresponding to $(\varphi, -\mathbf{u})$ and is represented by tensor $\mathbf{R}^{-1}$. Euler-Rodrigues formula fails whenever $\varphi = 0$, since $\mathbf{u}$ is then undetermined. Consequently, the use of equation (1) is not recommended in numerical applications.

Inspection of equation (1) gives immediately

$$\mathrm{sym}(\mathbf{R}) = \cos\varphi\,\mathbf{I} + (1-\cos\varphi)\,(\mathbf{u}\otimes\mathbf{u}), \tag{2}$$

$$\mathrm{skw}(\mathbf{R}) = \sin\varphi\,(\mathbf{u}\times) \tag{3}$$

so that

$$\mathrm{tr}(\mathbf{R}) = 1 + 2\,\cos\varphi, \tag{4}$$

$$\mathrm{axial}(\mathbf{R}) = \sin\varphi\,\mathbf{u}. \tag{5}$$

The rotation tensor leaves any vector parallel to the rotation axis unchanged,

$$\mathbf{R}\,\mathbf{u} = \mathbf{u}, \tag{6}$$

as can be verified from equation (1). In other words, the axis unit vector is an eigenvector corresponding to the eigenvalue $\lambda_1(\mathbf{R}) = 1$. The other eigenvalues are $\lambda_{2,3}(\mathbf{R}) = \exp(\pm i\varphi) = \cos\varphi \pm \sin\varphi$, and the corresponding eigenvectors define a plane normal to the rotation axis.

An important issue in computations is the composition of successive rotations. Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be two rotation tensors associated with rotation angles and axes $(\varphi_1, \mathbf{u}_1)$ and $(\varphi_2, \mathbf{u}_2)$, respectively. Rotation $\mathbf{R}_3$ is said to compose rotations $\mathbf{R}_1$ and $\mathbf{R}_2$ if $\mathbf{R}_3 = \mathbf{R}_2\mathbf{R}_1$. The parameters $(\varphi_3, \mathbf{u}_3)$ of rotation $\mathbf{R}_3$ are then related to those of rotations $\mathbf{R}_1$ and $\mathbf{R}_2$ as

$$\cos\frac{\varphi_3}{2} = \cos\frac{\varphi_1}{2}\,\cos\frac{\varphi_2}{2} - \sin\frac{\varphi_1}{2}\,\sin\frac{\varphi_2}{2}\,\mathbf{u}_1\cdot\mathbf{u}_2, \tag{7}$$

$$\sin\frac{\varphi_3}{2}\,\mathbf{u}_3 = \cos\frac{\varphi_1}{2}\,\sin\frac{\varphi_2}{2}\,\mathbf{u}_2 + \cos\frac{\varphi_2}{2}\,\sin\frac{\varphi_1}{2}\,\mathbf{u}_1 - \sin\frac{\varphi_1}{2}\,\sin\frac{\varphi_2}{2}\,\mathbf{u}_1\times\mathbf{u}_2. \tag{8}$$

These formulæ can be derived from purely geometric arguments, or from algebraic considerations, see Appendix A.

The time derivative of tensor $\mathbf{R}$ may be cast in the well-known form

$$\dot{\mathbf{R}} = \boldsymbol{\omega}\times\mathbf{R}, \tag{9}$$

where vector $\boldsymbol{\omega} := \mathrm{axial}(\dot{\mathbf{R}}\,\mathbf{R}^{-1}) \in \mathbb{E}^3$ is called the angular velocity, or spin. With the help of equation (1), an explicit form of the angular velocity vector can be obtained in terms of $(\varphi, \mathbf{u})$ and their time derivatives

$$\boldsymbol{\omega} = \dot{\varphi}\,\mathbf{u} + \big(\sin\varphi\,\mathbf{I} + (1-\cos\varphi)\,(\mathbf{u}\times)\big)\,\dot{\mathbf{u}}. \tag{10}$$

It is well known that the above operations on rotations can be expressed in terms of the Euler-Rodrigues parameterization. The basic relationships associated with this approach are given in Appendix A

3 The vectorial parameterization

3.1 General Formulation

The proposed vectorial parameterization of rotation consists of a minimal set of parameters, *i.e.* 3 scalars, defining the components of a *rotation parameter vector*, $\mathbf{p} \in \mathbb{E}^3$. All parameterizations in the class feature parameter vectors parallel to the rotation axis

$$\boxed{\mathbf{p} = p(\varphi)\,\mathbf{u}.} \tag{11}$$

Thus, rotation parameter vectors are eigenvectors of the rotation tensor corresponding to the positive unit eigenvalue, with magnitude $p := \|\mathbf{p}\|$.

Clearly, a specific vectorial parameterization is completely defined by the choice of a *generating function* $p(\varphi)$. Generating functions must be odd functions of the rotation angle φ and present the following limit behavior

$$\lim_{\varphi \to 0} \frac{p(\varphi)}{\varphi} = \kappa, \tag{12}$$

where κ is a real normalization factor. It will be shown that many widely used parameterization techniques are vectorial parameterizations, for suitable values of κ. In many cases, the normalizing factor is unity, and hence

$$\lim_{\varphi \to 0} p(\varphi) = \varphi, \tag{13}$$

which implies $\lim_{\varphi \to 0} \mathbf{p} = \varphi\,\mathbf{u}$. The following developments will focus on the unit normalizing factor.

The explicit expression of the rotation tensor in term of the vectorial parameterization is easily obtained from equation (1),

$$\mathbf{R} = \mathbf{I} + \frac{\nu^2}{\varepsilon}\,(\mathbf{p}\times) + \frac{\nu^2}{2}\,(\mathbf{p}\times)^2, \tag{14}$$

where ν and ε are even functions of φ defined as

$$\nu(\varphi) = \frac{2\sin(\varphi/2)}{p(\varphi)}, \tag{15}$$

$$\varepsilon(\varphi) = \frac{2\tan(\varphi/2)}{p(\varphi)}. \tag{16}$$

In view of equation (13), $\lim_{\varphi \to 0} \nu = \lim_{\varphi \to 0} \varepsilon = 1$.

Consider an orthonormal basis $\mathcal{B} := \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, where $\mathbf{u}$ is the eigenvector of $\mathbf{R}$ associated with eigenvalue $\lambda_1(\mathbf{R}) = +1$ and $\mathbf{v}$, $\mathbf{w}$ define a plane normal to $\mathbf{u}$. Such basis is called a *canonical basis* for $\mathbf{R}$. The components of tensor $\mathbf{R}$ measured in canonical basis $\mathcal{B}$, denoted $[\mathbf{R}]_{\mathcal{B}}$, are

$$[\mathbf{R}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - p^2\nu^2/2 & -p\,\nu^2/\varepsilon \\ 0 & p\,\nu^2/\varepsilon & 1 - p^2\nu^2/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}. \tag{17}$$

The last two eigenvalues of $\mathbf{R}$ are readily obtained from equation (17) as $\lambda_{2,3}(\mathbf{R}) = (1 - p^2\nu^2/2) \pm i\, p\, \nu^2/\varepsilon = \cos\varphi \pm i \sin\varphi$. Furthermore, $\mathrm{tr}(\mathbf{R}) = 3 - p^2\nu^2 = 1 + 2\cos\varphi$ and $\det(\mathbf{R}) = (1 - p^2\nu^2/2)^2 + (p\,\nu^2/\varepsilon)^2 = \cos^2\varphi + \sin^2\varphi = 1$.

The kinematic inversion, *i.e.* computing $\mathbf{p}$ from $\mathbf{R}$, is an important issue. In practice, it is most expeditious to first compute Euler-Rodrigues parameters using the procedure detailed in Appendix A, then obtain the vectorial parameterization vector using equation (75).

Composition of rotations can be expressed in terms of the vectorial parameterization as follows. Let $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ with rotation angles φ_1, φ_2, and φ_3, respectively, correspond to rotation tensors $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$, respectively. If $\mathbf{R}_3 := \mathbf{R}_2\,\mathbf{R}_1$, the relationship between the various parameters then follows from equations (7) and (8)

$$\frac{\nu_3}{\varepsilon_3} = \frac{\nu_1}{\varepsilon_1}\frac{\nu_2}{\varepsilon_2} - \frac{1}{4}\,(\nu_1\mathbf{p}_1)\cdot(\nu_2\mathbf{p}_2), \tag{18}$$

$$\nu_3\,\mathbf{p}_3 = \frac{\nu_1}{\varepsilon_1}\,(\nu_2\,\mathbf{p}_2) + \frac{\nu_2}{\varepsilon_2}\,(\nu_1\,\mathbf{p}_1) - \frac{1}{2}\,(\nu_1\mathbf{p}_1)\times(\nu_2\mathbf{p}_2). \tag{19}$$

The first equation is used to compute φ_3 since $\nu_3/\varepsilon_3 = \cos(\varphi_3/2)$. The second equation then yields $\mathbf{p}_3$.

Consider a change of basis characterized by tensor $\mathbf{F} \in \mathrm{SO}(\mathbb{E}^3)$. The components of the rotation tensor $\mathbf{R}$ in this basis are $\mathbf{FRF}^T$, since $\mathbf{R}$ is a second order tensor. Equation (14) then implies $\mathbf{FRF}^T = \mathbf{I} + (\nu^2/\varepsilon)\,((\mathbf{Fp})\times) + (\nu^2/2)\,((\mathbf{Fp})\times)^2$, regardless of the particular choice of generating function $p(\varphi)$. This means that the components of $\mathbf{p}$ transform as $\mathbf{Fp}$, *i.e.* $\mathbf{p}$ is a first order tensor. This observation justifies the adjective "vectorial" used to characterize the proposed parameterization.

Next, the relationship between the angular velocity vector and the vectorial parameterization vector and its time derivative is sought:

$$\boldsymbol{\omega} = \mathbf{H}(\mathbf{p})\,\dot{\mathbf{p}}. \tag{20}$$

An explicit expression for tensor $\mathbf{H}$ is found from equation (10), written as $\boldsymbol{\omega} = \mu\,\dot{p}\,\mathbf{u} + p\left((\nu^2/\varepsilon)\,\mathbf{I} + (\nu^2/2)\,(\mathbf{p}\times)\right)$. Next, elementary vector identities imply $\dot{\varphi}\,\mathbf{u} = \mu\left(\mathbf{I} + (\mathbf{u}\times)^2\right)\dot{\mathbf{p}}$, where

$$\mu(\varphi) := \frac{1}{p'(\varphi)}, \tag{21}$$

with $p' := \mathrm{d}p/\mathrm{d}\varphi$. Tensor $\mathbf{H}$ is then given by

$$\mathbf{H} = \mu\,\mathbf{I} + \frac{\nu^2}{2}\,(\mathbf{p}\times) + \frac{1}{p^2}\left(\mu - \frac{\nu^2}{\varepsilon}\right)(\mathbf{p}\times)^2. \tag{22}$$

For equation (13), $\lim_{\varphi\to 0}\mu = 1$, and, hence, $\lim_{\varphi\to 0}\mathbf{H} = \mathbf{I}$.

The components of $\mathbf{H}$ in a canonical basis $\mathcal{B}$ are

$$[\mathbf{H}]_{\mathcal{B}} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & (\mu - \nu^2/\varepsilon)/p^2 & -p\,\nu^2/2 \\ 0 & p\,\nu^2/2 & (\mu - \nu^2/\varepsilon)/p^2 \end{bmatrix} = \nu \begin{bmatrix} \mu/\nu & 0 & 0 \\ 0 & \cos(\varphi/2) & -\sin(\varphi/2) \\ 0 & \sin(\varphi/2) & \cos(\varphi/2) \end{bmatrix}. \tag{23}$$

It follows that the eigenvalues of $\mathbf{H}$ are $\lambda_1(\mathbf{H}) = \mu$, and $\lambda_{2,3}(\mathbf{H}) = \nu^2 (1/\varepsilon \pm i\, p/2) = \nu \big(\cos(\varphi/2) \pm i \sin(\varphi/2) \big)$. It is readily verified that the eigenvector associated with λ_1 is $\mathbf{u}$, since $\mathbf{H}\,\mathbf{u} = \mu\,\mathbf{u}$. The determinant of $\mathbf{H}$ is readily obtained as

$$\det(\mathbf{H}) = \mu\,\nu^2. \tag{24}$$

From equation (20), the relationship between the time derivative of the vectorial parameterization vector and the angular velocity vector is

$$\dot{\mathbf{p}} = \mathbf{H}(\mathbf{p})^{-1}\boldsymbol{\omega}. \tag{25}$$

The inverse of tensor $\mathbf{H}$ is

$$\mathbf{H}^{-1} = \frac{1}{\mu}\mathbf{I} - \frac{1}{2}(\mathbf{p}\times) - \frac{1}{p^2}\left(\frac{1}{\varepsilon} - \frac{1}{\mu}\right)(\mathbf{p}\times)^2. \tag{26}$$

Its components in the canonical basis $\mathcal{B}$ are

$$[\mathbf{H}^{-1}]_{\mathcal{B}} = \begin{bmatrix} 1/\mu & 0 & 0 \\ 0 & 1/\varepsilon & p/2 \\ 0 & -p/2 & 1/\varepsilon \end{bmatrix} = \frac{p}{2}\begin{bmatrix} 2\,p'/p & 0 & 0 \\ 0 & 1/\tan(\varphi/2) & 1 \\ 0 & -1 & 1/\tan(\varphi/2) \end{bmatrix}. \tag{27}$$

It can be readily shown that tensors $\mathbf{R}$ and $\mathbf{H}$ are closely related through the following properties

$$\mathbf{R} = \mathbf{H}\,\mathbf{H}^{-T} = \mathbf{H}^{-T}\,\mathbf{H}; \tag{28}$$

$$\mathbf{R} - \mathbf{I} = (\mathbf{p}\times)\mathbf{H} = \mathbf{H}(\mathbf{p}\times). \tag{29}$$

3.2 Special choices of generating function

The formulation presented thus far is very general, but in practice, a specific choice of the generating function, $p(\varphi)$, must be made. A natural strategy is to select a generating function that will simplify some of the operators involved in rotation manipulations. For instance, the simplest choice of generating function is

$$p(\varphi) = \varphi. \tag{30}$$

Various names are used in the literature for the resulting parameter vector $\mathbf{p} = \varphi\,\mathbf{u}$, such as the 'Euler' vector, the 'principal rotation' vector, or the 'equivalent axis representation' vector. In the following, it will be referred to as the *rotation vector* and will be further discussed in Appendix C.

The expression for the rotation tensor, equation (14), will simplify if $\nu^2 = c\,\varepsilon$, where c is a constant to be determined by imposing the limit condition, equation (13). This yields

$$p(\varphi) = \sin\varphi \tag{31}$$

and $c = 1$. This choice is often called the *linear parameterization*. A similar simplification is obtained if $\nu^2 = 2\,c$, leading to

$$p(\varphi) = 2\,\sin\frac{\varphi}{2} \tag{32}$$

and $c = 2$. This choice is called the *reduced Euler-Rodrigues parameterization*, since it is closely related to the Euler-Rodrigues parameterization (see Appendix A). Some authors define the generating function as $p(\varphi) = \sin(\varphi/2)$, in which case the normalization factor, equation (32), is $\kappa = 1/2$. An equivalent approach consists in requiring the last term in the expression of the tensor $\mathbf{H}^{-1}$, equation (26), to vanish. This implies the nonlinear differential equation $2p' \tan(\varphi/2) = p$ whose solution is $p(\varphi) = 2 \sin \varphi/2$, *i.e.* the generating function of the reduced Euler-Rodrigues parameters is recovered.

An alternate approach is to require the last term in the expression of tensor $\mathbf{H}$, equation (22), to vanish, *i.e.* $\mu = \nu^2/\varepsilon$. This leads to the nonlinear differential equation $p' \sin \varphi = p$, the solution of which is $p(\varphi) = c \tan(\varphi/2)$, where c is an integration constant. The limiting condition, equation (13), then implies $c = 2$, and hence

$$p(\varphi) = 2 \tan \frac{\varphi}{2}. \tag{33}$$

This parameterization is called the Rodrigues [2], Gibbs [19], or Cayley [20] parameterization. In the following, this choice will be referred to as the *Cayley-Gibbs-Rodrigues parameterization*. The generating function defined by equation (33) also implies $\varepsilon = 1$. In some cases, the generating function $p(\varphi) = \tan \varphi/2$ (corresponding to a normalization factor $\kappa = 1/2$) is used, see section D.

In yet another approach, tensor $\mathbf{H}$ is required to be the 'square root', except for a multiplicative scalar factor $\alpha(\varphi)$, of tensor $\mathbf{R}$, *i.e.* $\mathbf{H}^2 = \alpha^2 \mathbf{R}$. This implies two conditions: $p' = 1/\alpha$ and $p\,\alpha = 2 \sin \varphi/2$. Hence, the following differential equation must hold: $2\,p' \sin \varphi/2 = p$. With the help of the limiting condition, the solution becomes

$$p(\varphi) = 4 \tan \frac{\varphi}{4}, \tag{34}$$

and $\alpha(\varphi) = \cos^2(\varphi/4)$. This parameterization also bears various names in the literature: Wiener [21], Milenkovic [22], conformal rotation vector (CRV) [2, 23], or modified Rodrigues parameterization [24]. It shall be referred to as the *Wiener-Milenkovic parameterization* in the following.

The last choice leads to a novel parameterization, demonstrating the versatility of the proposed framework. In order to avoid the appearance of singularities when manipulating operator $\mathbf{H}$, it might be desirable to have $\det(\mathbf{H}) = c$, where c is a constant. In view of equation (24), this requirement implies $c\,p' = \nu^2$. The solution of this nonlinear differential equation is

$$\boxed{p(\varphi) = \sqrt[3]{6(\varphi - \sin \varphi)}.} \tag{35}$$

where constant c was found to be unity with the help of the limiting condition. Hence, this particular parameterization is such that $\det(\mathbf{H}) = 1$ for all values of φ, and $\mathbf{H}$ is always invertible.

3.3 The sine and tangent parameterization families

The preceding discussion clearly indicates that two subclasses of vectorial parameterization enjoy interesting properties:

$$p(\varphi) = m \sin \frac{\varphi}{m}, \qquad \text{and} \qquad p(\varphi) = m \tan \frac{\varphi}{m}, \tag{36}$$

where $m \in \mathbb{N}$. The corresponding functions $\mu(\varphi)$ are

$$\mu(\varphi) = \frac{1}{\cos \dfrac{\varphi}{m}} = \frac{1}{\sqrt{1 - \dfrac{p(\varphi)^2}{m^2}}}, \qquad \text{and} \qquad \mu(\varphi) = \cos^2 \frac{\varphi}{m} = \frac{1}{1 + \dfrac{p(\varphi)^2}{m^2}}. \tag{37}$$

These subclasses are called the *sine* and *tangent* families. The linear parameterization thus coincides with the $m = 1$ member of the sine family, while the reduced Euler-Rodrigues parameterization with the $m = 2$ member. The Cayley-Gibbs-Rodrigues and the Wiener-Milenkovic parameterizations coincide with the $m = 2$ and 4 members of the tangent family, respectively.

Table 1 lists the names of various parameterizations, the corresponding generating function, and the relevant scalar functions $p'(\varphi)$, $\nu(\varphi)$, and $\det(\mathbf{H})(\varphi)$. The range of validity of each parameterization is also indicated. The generating functions for the first five members of the sine and tangent families are depicted in figures 1 and 2, respectively. The generating function for the rotation vector also appears on each figures, for reference. Note that $p(\varphi) \rightarrow \varphi$ for $|\varphi| < \pi$, by lower bound for the sine family, and by upper bound for the tangent family. For rotation angles $|\varphi| \leq \pi$, the generating function becomes a nearly linear function of φ, as m increases. This nearly linear behavior is a desirable characteristic that makes the Wiener-Milenkovic parameterization a popular choice. More recently, the tangent family with higher values of m were presented by Tsiotras et al [24], in relation to higher-order Cayley transforms, see section D.

The sine family and its higher order members have apparently not been investigated in the literature. Since the case $m = 4$ is of particular interest, the formulæ relevant to this parameterization will be readily derived from the general expressions presented in section 3.1, as an example of application of the vectorial parameterization theory. At first, the parameter vector is denoted $\mathbf{p} = 4 \sin \varphi/4\, \mathbf{u}$, and the scalar $p_0 = \cos \varphi/4 = \sqrt{(}1 - p^2/4)$ is defined. Note that $p_0 \in [1, 1/\sqrt{2}]$ for $|\varphi| \in [0, \pi]$. The rotation tensor is then given by equation (14), where

$$\nu \equiv p_0; \quad \varepsilon = \frac{p_0^3}{2p_0^2 - 1}. \tag{38}$$

Tensor $\mathbf{H}$ is then given by equation (22), where

$$\mu = 1/p_0. \tag{39}$$

Interesting properties of this parameterization will be presented in the next section.

3.4 Extending the vectorial parameterization

The vectorial parameterization as presented in the previous sections exhibits desirable features, but also suffers serious drawbacks. In particular, for all generating functions, singularities will occur for specific values of the rotation angle φ. Singular points will appear with any minimal parameterization of rotation, *i.e.*, any technique based on a three-parameter set, including vectorial and non-vectorial parameterizations, such as Euler-type angles [25].

More specifically, singularities can first occur in the definition of the generating function, as $p \rightarrow \infty$. For instance, the Cayley-Gibbs-Rodrigues parameterization is singular when $\varphi =$

$\pm\pi$. Since the representation of arbitrary rotations requires a well defined parameterization for all $|\varphi| \leq \pi$, the Cayley-Gibbs-Rodrigues parameterization can not be used when dealing with rotations of arbitrary magnitude. Next, problems can occur in the kinematic inversion, *i.e.* when determining the rotation parameter vector $\mathbf{p}$ from the rotation tensor $\mathbf{R}$. For example, one may use the inversion procedure for Euler-Rodrigues parameters (which is singularity-free, see Appendix A) and then recover the vectorial parameterization vector. In this case, singularities are encountered when $\nu \to 0$ or ∞. Linear parameters, for instance, experience such singularity when $\nu \to \infty$, *i.e.* when $\varphi = \pm\pi$. A third source of singularities are the operator $\mathbf{H}$ and its inverse. Inspection of equations (22) and (26) reveals that singularities will appear when $p' \to 0$ or ∞ and $\nu \to 0$ or ∞. In summary, singularities will appear when $p \to \infty$, $\nu \to 0$ or ∞, and $\mu \to 0$ or ∞. Figures 3 and 4 depict these relevant function, $\nu(\varphi)$, $p'(\varphi)$, and $\det(\mathbf{H})(\varphi)$ for the sine and tangent families, respectively.

All parameterizations with a validity range of $|\varphi| > \pi$ are able to handle all finite rotations. However, such parameterizations are not necessarily "worry free." Indeed, finite rotation are often used in incremental procedures where a small incremental rotation is added to a finite rotation at each time step, for instance. In this case, angles φ of arbitrary magnitude are routinely encountered; consider, for instance, a rotating shaft, or a satellite tumbling in space. In these cases, singularities will always appear as φ increases to large values.

The validity of the sine and tangent parameterizations for $m = 4$ can be extended by using a *rescaling* operation. This operation is based on the observation that rotations of magnitudes differing by 2π about the same axis $\mathbf{u}$ correspond to the same final configuration, and can hence be represented by identical parameter vectors.

The tangent parameterization for $m = 4$ is considered at first. In this case $\|\mathbf{p}\| = p \leq 4$ when $|\varphi| \leq \pi$. Let $\mathbf{p}$ and $\hat{\mathbf{p}}$ be associated with the rotations φ and $\hat{\varphi} := \varphi \pm 2\pi$, respectively. The relationship between these two parameter vectors is

$$\hat{\mathbf{p}} = 4 \tan \frac{\hat{\varphi}}{4} \mathbf{u} = 4 \tan \left(\frac{\varphi}{4} \pm \frac{\pi}{2} \right) \mathbf{u} = -\frac{4}{\tan \frac{\varphi}{4}} \mathbf{u} = -\frac{1}{\tan^2 \frac{\varphi}{4}} \mathbf{p}, \tag{40}$$

which writes

$$\hat{\mathbf{p}} = -\frac{\nu}{1-\nu} \mathbf{p}. \tag{41}$$

Equation (40) implies

$$p\,\hat{p} = 16. \tag{42}$$

If $\pi < |\varphi| < 2\pi$, $p > 4$, and hence $\hat{p} < 4$; in other words, the rescaling operation represented by adopting $\hat{\mathbf{p}}$ in place of $\mathbf{p}$ decreases the norm of the parameter vector, and can be used to avoid falling into badly conditioned neighborhoods of the singularity at $\varphi = 2\pi$. These ideas were exposed in ref. [23] for the present parameterization. Later, in refs. [26, 24] the subject was discussed in terms of the *shadow* parameter set, based on the interpretation of the generating functions as resulting from different stereographic projections of the unit circle. In these references, the relation between some elements of the tangent family and Euler-Rodrigues parameters (see Appendix A) in connection to stereographic projections is exploited.

Consider the successive composition of incremental rotations starting from an initial orientation described by $\mathbf{p}_1$. The composition formula yielding the parameter vector $\mathbf{p}_3$ of the final orientation resulting from performing an incremental rotation described by $\mathbf{p}_2$, eq.(19), gives the following update relationship

$$\mathbf{p}_3 = \frac{\nu_1\nu_2}{\nu_3}\left(\frac{1}{\varepsilon_2}\mathbf{p}_1 + \frac{1}{\varepsilon_1}\mathbf{p}_2 - \frac{1}{2}\mathbf{p}_1 \times \mathbf{p}_2\right), \tag{43}$$

where, in view of equation (18), $2\nu_3 - 1 = \cos\varphi_3/2 = \nu_1\nu_2\,(1/\varepsilon_1\varepsilon_2 - \mathbf{p}_1 \cdot \mathbf{p}_2/4)$. As incremental rotations are added to the initial orientation, p_3 increases and when $|\varphi_3|$ becomes larger than π, $p_3 > 4$ and the rescaling operation, equation (41), becomes necessary. The two operations, update and rescaling, are conveniently combined into a single operation as follows

$$\mathbf{p}_3 = \begin{cases} \dfrac{\nu_1\nu_2}{\nu_3}\left(\dfrac{1}{\varepsilon_2}\mathbf{p}_1 + \dfrac{1}{\varepsilon_1}\mathbf{p}_2 - \dfrac{1}{2}\mathbf{p}_1 \times \mathbf{p}_2\right) & \text{if} \quad \nu_3 \geq \dfrac{1}{2} \\ -\dfrac{\nu_1\nu_2}{1-\nu_3}\left(\dfrac{1}{\varepsilon_2}\mathbf{p}_1 + \dfrac{1}{\varepsilon_1}\mathbf{p}_2 - \dfrac{1}{2}\mathbf{p}_1 \times \mathbf{p}_2\right) & \text{if} \quad \nu_3 \leq \dfrac{1}{2} \end{cases} \tag{44}$$

Similar developments hold for the sine parameterization with $m = 4$. In this case $p^2 \leq 8$ when $|\varphi| \leq \pi$. The rescaling operation now writes

$$\hat{\mathbf{p}} = \frac{\nu}{\sqrt{1-\nu^2}}\,\mathbf{p}. \tag{45}$$

and implies

$$p^2 + \hat{p}^2 = 16. \tag{46}$$

Here again, the rescaling operation decreases the norm of the parameter vector. Finally, the update and rescaling operations are conveniently combined as

$$\mathbf{p}_3 = \begin{cases} \dfrac{\nu_1\nu_2}{\nu_3}\left(\dfrac{1}{\varepsilon_2}\mathbf{p}_1 + \dfrac{1}{\varepsilon_1}\mathbf{p}_2 - \dfrac{1}{2}\mathbf{p}_1 \times \mathbf{p}_2\right) & \text{if} \quad \nu_3 \geq \dfrac{1}{\sqrt{2}} \\ \dfrac{\nu_1\nu_2}{\sqrt{1-\nu_3^2}}\left(\dfrac{1}{\varepsilon_2}\mathbf{p}_1 + \dfrac{1}{\varepsilon_1}\mathbf{p}_2 - \dfrac{1}{2}\mathbf{p}_1 \times \mathbf{p}_2\right) & \text{if} \quad \nu_3 \leq \dfrac{1}{\sqrt{2}} \end{cases} \tag{47}$$

where, in view of equation (18), $2\nu_3^2 - 1 = \cos\varphi_3/2 = \nu_1\nu_2\,(1/\varepsilon_1\varepsilon_2 - \mathbf{p}_1 \cdot \mathbf{p}_2/4)$.

In summary, the two parameterizations considered here are able to handle rotations of truly arbitrary magnitude provided that any update operation is combined with a possible rescale, as indicated in equations (47) and (44), respectively.

4 Conclusions

A new framework for the derivation and interpretation of minimal parameterizations of the rotation group has been proposed. A general class of vectorial parameterizations was defined and a complete set of relevant formulæ was presented. Several well known techniques adopted in the analysis of rotational motion were shown to be particular cases of this representation, such as the exponential parameterization and the techniques based on sine and tangent families of generating functions.

Within the proposed framework, the characteristics of a specific parameterization can be readily assessed in terms of three scalar function $p(\varphi)$, $\mu(\varphi)$, and $\nu(\varphi)$. These scalar functions were shown to fully define all the operations associated with the manipulation of rotations, and reveal any possible singularity of the chosen representation. Expression for the rotation tensor $\mathbf{R}$, its associated differential tensor $\mathbf{H}$, and their inverse were derived in terms of these quantities.

Since the proposed vectorial parameterization forms a minimal set, singularities always occur for specific values of the rotation angle. However, for the parameterizations based on the generating functions $p(\varphi) = 4\sin\varphi/4$ and $4\tan\varphi/4$, we detailed a procedure that combines rotation composition with a suitable rescaling operation that results in a "worry free" parameterization for rotations of arbitrary magnitude.

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A Euler-Rodrigues Parameterization

Euler-Rodrigues parameters [27, 28, 29] lead to a simple, purely algebraic representation of finite rotations and are are defined as

$$e_0 = \cos\frac{\varphi}{2}, \tag{48}$$

$$\mathbf{e} = \sin\frac{\varphi}{2}\,\mathbf{u}, \tag{49}$$

where e_0 is termed the scalar part, and $\mathbf{e}$ the vector part. These parameters are clearly related by a normality condition,

$$e_0^2 + e^2 = 1, \tag{50}$$

where $e := \|\mathbf{e}\|$. They are thus not independent, forming a one-redundant set of rotation parameters. Introducing Euler-Rodrigues parameters into the expression for the rotation tensor, equation (1), yields

$$\mathbf{R} = \mathbf{I} + 2e_0\,(\mathbf{e}\times) + 2\,(\mathbf{e}\times)^2. \tag{51}$$

Clearly, the rotation tensor component matrix with respect to an arbitrary basis $\mathcal{B}$ of $\mathbb{E}^3$ is a quadratic expression of e_0, e_1, e_2, e_3, where $\{e_k\}_{k=1,2,3}$ indicate the components of $\mathbf{e}$ with respect to that same basis, $[\mathbf{e}]_{\mathcal{B}} = (e_1, e_2, e_3)^T$. Euler-Rodrigues parameters as said to form a *quaternion* [27], *i.e.* a four component array

$$\underline{e} := \begin{bmatrix} e_0 \\ \mathbf{e} \end{bmatrix} \tag{52}$$

that, besides standard linear algebra operations in $\mathbb{R}^4$, supports the well-known composition rule

$$\underline{e}_2 \circ \underline{e}_1 = \begin{bmatrix} e_{10}\,e_{20} - \mathbf{e}_1 \cdot \mathbf{e}_2 \\ e_{10}\,\mathbf{e}_2 + e_{20}\,\mathbf{e}_1 - \mathbf{e}_1 \times \mathbf{e}_2 \end{bmatrix}. \tag{53}$$

The underbar is used here to denote a four-dimensional vector or tensor. Given the normality condition, equation (50), Euler-Rodrigues parameters form a *unit* quaternion.

Compact matrix expressions can be found by introducing the following operators [2, 30]

$$\mathcal{M}_l(\underline{e}) = \begin{bmatrix} e_0 & -\mathbf{e}^T \\ \mathbf{e} & e_0\mathbf{I} + (\mathbf{e}\times) \end{bmatrix}, \tag{54}$$

$$\mathcal{M}_r(\underline{e}) = \begin{bmatrix} e_0 & -\mathbf{e}^T \\ \mathbf{e} & e_0\mathbf{I} - (\mathbf{e}\times) \end{bmatrix}. \tag{55}$$

Operators $\mathcal{M}_l$ and $\mathcal{M}_r$ allow to perform the composition of two quaternions, equation (53), as a standard matrix-vector multiplication by either a 'left' or 'right' multiplication pattern, respectively, as

$$\underline{e}_2 \circ \underline{e}_1 = \mathcal{M}_l(\underline{e}_2)\,\underline{e}_1 = \mathcal{M}_r(\underline{e}_1)\,\underline{e}_2. \tag{56}$$

These operators are orthogonal, as can be readily verified with the help of the normality condition, equation (50),

$$\mathcal{M}_l(\underline{e})\,\mathcal{M}_l(\underline{e})^T = \mathcal{M}_r(\underline{e})\,\mathcal{M}_r(\underline{e})^T = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{57}$$

The rotation tensor is easily expressed in terms of these operators. In fact, by defining

$$\mathcal{R}(\underline{e}) := \mathcal{M}_l(\underline{e})\, \mathcal{M}_r(\underline{e})^T = \mathcal{M}_r(\underline{e})^T \mathcal{M}_l(\underline{e}), \tag{58}$$

it is easily verified that

$$\mathcal{R}(\underline{e}) = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R} \end{bmatrix}. \tag{59}$$

The conjugate quaternion, $\underline{e}^*$, is defined as

$$\underline{e}^* := \begin{bmatrix} e_0 \\ -\mathbf{e} \end{bmatrix}. \tag{60}$$

Consequently, $\mathcal{M}_l(\underline{e}^*) = \mathcal{M}_l(\underline{e})^T$ and $\mathcal{M}_r(\underline{e}^*) = \mathcal{M}_r(\underline{e})^T$, and hence

$$\mathcal{M}_l(\underline{e}^*)\, \mathcal{M}_r(\underline{e}^*)^T = \mathcal{M}_r(\underline{e}^*)^T \mathcal{M}_l(\underline{e}^*) = \mathcal{M}_l(\underline{e})^T \mathcal{M}_r(\underline{e}) = \mathcal{M}_r(\underline{e})\, \mathcal{M}_l(\underline{e})^T = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R}^T \end{bmatrix}. \tag{61}$$

In other words, the rotation tensor corresponding to $\underline{e}^*$ is $\mathbf{R}^T = \mathbf{R}^{-1}$, as can be immediately verified by equation (51).

The determination of the Euler-Rodrigues parameters from the rotation tensor [31, 32], *i.e.* the inverse of the operation defined by equation (51) can be performed by retrieving the following four-dimensional projector

$$\underline{\underline{E}} = \frac{1}{4} \begin{bmatrix} 1 + \mathrm{tr}(\mathbf{R}) & 2\,\mathrm{axial}(\mathbf{R})^T \\ 2\,\mathrm{axial}(\mathbf{R}) & (1 - \mathrm{tr}(\mathbf{R}))\mathbf{I} + \mathbf{R} + \mathbf{R}^T \end{bmatrix} = \underline{e} \otimes \underline{e}. \tag{62}$$

The components E_{ij} of this tensor in basis $\mathcal{B}$ are

$$[\underline{\underline{E}}]_{\mathcal{B}} = \begin{bmatrix} e_0^2 & e_0e_1 & e_0e_2 & e_0e_3 \\ e_0e_1 & e_1^2 & e_1e_2 & e_1e_3 \\ e_0e_2 & e_1e_2 & e_2^2 & e_2e_3 \\ e_0e_3 & e_1e_3 & e_2e_3 & e_3^2 \end{bmatrix}. \tag{63}$$

Euler-Rodrigues parameters can readily be computed from any column of this matrix as

$$e_i = \frac{1}{\Delta_k} E_{ik}, \quad i = 0, 1, 2, 3; \tag{64}$$

where

$$\Delta_k = \sqrt{E_{kk}}. \tag{65}$$

This expression clearly shows the problem associated with the inverse relationships: the results become inaccurate when a Δ_k becomes very small, or zero. Accurate results are obtained by extracting Euler-Rodrigues parameters from the column of $[\underline{\underline{E}}]_{\mathcal{B}}$ which presents the largest diagonal term. In fact, $\max(E_{00}, E_{11}, E_{22}, E_{33}) = \max(\mathrm{tr}(\mathbf{R}), R_{11}, R_{22}, R_{33})$.

The action of a rotation tensor $\mathbf{R}$ on a vector $\mathbf{v}$ to produce $\mathbf{w} = \mathbf{R}\,\mathbf{v}$ can be expressed in a remarkably simple quaternion form as

$$\underline{w} = \mathcal{R}(\underline{e})\, \underline{v} = \underline{e} \circ \underline{v} \circ \underline{e}^*, \tag{66}$$

where $\underline{v}$, $\underline{w}$ are quaternions with a vanishing scalar part $v_0 = w_0 = 0$ and $\mathbf{v}$ and $\mathbf{w}$ as their vector part, respectively. Furthermore, Euler-Rodrigues parameters are particularly convenient for expressing rotation compositions. It can be easily shown that the tensorial composition $\mathbf{R}_3 = \mathbf{R}_2\mathbf{R}_1$ translates into the quaternion composition $\underline{e}_3 = \underline{e}_2 \circ \underline{e}_1$ and vice-versa. In fact,

$$\mathcal{R}(\underline{e}_3)\,\underline{v} = \mathcal{R}(\underline{e}_2)\,\mathcal{R}(\underline{e}_1)\,\underline{v} = \underline{e}_2 \circ (\underline{e}_1 \circ \underline{v} \circ \underline{e}_1^*) \circ \underline{e}_2^* = (\underline{e}_2 \circ \underline{e}_1) \circ \underline{v} \circ (\underline{e}_2 \circ \underline{e}_1)^* = \mathcal{R}(\underline{e}_2 \circ \underline{e}_1)\,\underline{v}, \quad (67)$$

$\forall \underline{v} \in \mathbb{R}^4$, and hence $\underline{e}_3 = \underline{e}_2 \circ \underline{e}_1$.

Introducing Euler-Rodrigues parameters into the expression for the angular velocity vector, equation (10), yields

$$\boldsymbol{\omega} = 2e_0\,\dot{\mathbf{e}} - 2\dot{e}_0\,\mathbf{e} + 2\,\mathbf{e}\times\,\dot{\mathbf{e}}. \quad (68)$$

The following four-dimensional vector

$$\underline{\omega} = \begin{bmatrix} \omega_0 \\ \boldsymbol{\omega} \end{bmatrix}, \quad (69)$$

is defined, where the scalar part ω_0 is defined as $\omega_0 = 2\,(e_0\,\dot{e}_0 + \mathbf{e}\cdot\dot{\mathbf{e}}) = 0$, in view of the normality condition, equation (50). The angular velocity $\boldsymbol{\omega}$ is now related to the derivative of the Euler-Rodrigues quaternion, $\dot{\underline{e}}$, through

$$\underline{\omega} = \mathcal{H}(\underline{e})\,\dot{\underline{e}}. \quad (70)$$

The operator $\mathcal{H}(\underline{e})$ plays the same role as tensor $\mathbf{H}$ in the vectorial parameterization, and it is easily determined from equation (68) as

$$\mathcal{H}(\underline{e}) = 2\,\mathcal{M}_r(\underline{e})^T = 2\,\mathcal{M}_r(\underline{e}^*). \quad (71)$$

Therefore,

$$\underline{\omega} = 2\,\dot{\underline{e}} \circ \underline{e}^*, \quad (72)$$

and the inverse operation simply writes

$$\dot{\underline{e}} = \frac{1}{2}\,\underline{\omega} \circ \underline{e}, \quad (73)$$

as can be verified immediately, since operator $\mathcal{M}_r$ is orthogonal.

The above formulæ demonstrate the power of Euler parameters: all operations associated with finite rotations become purely algebraic operations. In fact the rotation tensor is a quadratic expression of $\underline{e}$, and the angular velocity a bilinear expression in terms of $\underline{e}$ and $\dot{\underline{e}}$. Trigonometric functions have been eliminated, an obvious computational advantage. There are no possible singularities in any of the above relationships. However, these advantages come at a considerable cost: four parameters must be used instead of three, *i.e.* Euler-Rodrigues parameters do not form a minimal set. Furthermore, the normality condition, equation (50), must be enforced as an external constraint.

In the vectorial parameterization, the parameter vector $\mathbf{p}$ is related to the Euler-Rodrigues parameters $\underline{e}$ by

$$e_0 = \frac{\nu}{\varepsilon}. \quad (74)$$

$$\mathbf{e} = \frac{\nu}{2}\,\mathbf{p}, \quad (75)$$

for any choice of the generating function $p(\varphi)$. In other words, the vectorial parameterization $\mathbf{p} = \mathbf{e}$ corresponds to the generating function $p(\varphi) = \sin\varphi/2$ with a normalization factor $\kappa = 1/2$. It coincides with the reduced Euler-Rodrigues discussed in sections 3.2 and 3.3. In this case, $\mu = 2/e_0$, $\nu = 2$, and $\varepsilon = \mu$.

Equations (74) and (75) do not supply an analytical relationship for solving the inverse kinematic problem, *i.e.* obtaining $\mathbf{p}$ as a function of $\mathbf{e}$. Indeed, $\varphi = 2\cos^{-1} e_0$, $\nu = 2\sin(\varphi/2)/p(\varphi)$, and finally $\mathbf{p}$ is obtained from equation (75). However, a possible strategy for solving the inverse kinematic problem can make use of these equations, since the extraction of $\underline{e}$ from $\mathbf{R}$ is computationally simple and does not suffer from singularity problems.

Note that the composition formulæ (18) and (19) can be readily demonstrated by inserting the previous equations in equation (53).

B Rotation Powers and Roots

Since any power of a rotation tensor is a rotation tensor, we may define the m-th root of $\mathbf{R}$, where $m \in \mathbb{N}$ as

$$\mathbf{G}_{(m)} := \mathbf{I} + \sin\frac{\varphi}{m}\,(\mathbf{u}\times) + \left(1 - \cos\frac{\varphi}{m}\right)\,(\mathbf{u}\times)^2, \tag{76}$$

so that

$$\mathbf{R} = {\mathbf{G}_{(m)}}^m. \tag{77}$$

Clearly, following Euler's theorem, $\mathbf{G}_{(m)}$ represents a rotation corresponding to $(\varphi/m, \mathbf{u})$, and the previous equation plainly expresses a decomposition of the total rotation into the subsequent application of m partial rotations by an angle φ/m about the same axis $\mathbf{u}$. If we denote with $\boldsymbol{\gamma}_{(m)}$ the vector characterizing the time derivative of $\mathbf{G}_{(m)}$,

$$\dot{\mathbf{G}}_{(m)} = \boldsymbol{\gamma}_{(m)} \times \mathbf{G}_{(m)}, \tag{78}$$

the following expression in terms of $(\varphi, \mathbf{u})$ is obtained

$$\boldsymbol{\gamma}_{(m)} = \frac{\dot{\varphi}}{m}\,\mathbf{u} + \left(\sin\frac{\varphi}{m}\,\mathbf{I} + \left(1 - \cos\frac{\varphi}{m}\right)\,(\mathbf{u}\times)\right)\,\dot{\mathbf{u}}. \tag{79}$$

Tensor $\mathbf{K}_{(m)}$ can be defined such that

$$\boldsymbol{\gamma}_{(m)} = \mathbf{K}_{(m)}\dot{\mathbf{p}}. \tag{80}$$

By analogy with $\boldsymbol{\omega} = \mathbf{H}\,\dot{\mathbf{p}}$, $\boldsymbol{\omega} \equiv \boldsymbol{\gamma}_{(1)}$ and $\mathbf{H} \equiv \mathbf{K}_{(1)}$.

Within the vectorial parameterization, the square root of $\mathbf{R}$, $\mathbf{G} \equiv \mathbf{G}_{(2)}$, is given by

$$\mathbf{G} = \mathbf{I} + \frac{\nu}{2}\,(\mathbf{p}\times) + \frac{1}{p^2}\left(1 - \frac{\nu}{\varepsilon}\right)\,(\mathbf{p}\times)^2. \tag{81}$$

The relevant angular velocity $\boldsymbol{\gamma} \equiv \boldsymbol{\gamma}_{(2)}$,

$$\boldsymbol{\gamma} = \mathbf{K}\,\dot{\mathbf{p}}, \tag{82}$$

is obtained through tensor $\mathbf{K} \equiv \mathbf{K}_{(2)}$, given by

$$\mathbf{K} = \frac{\mu}{2}\,\mathbf{I} + \frac{1}{p^2}\left(1 - \frac{\nu}{\varepsilon}\right)(\mathbf{p}\times) + \frac{1}{p^2}\,\frac{\mu - \nu}{2}\,(\mathbf{p}\times)^2. \tag{83}$$

In terms of Euler-Rodrigues parameters these relationships become

$$\mathbf{G} = \mathbf{I} + (\mathbf{e}\times) + \frac{1}{1 + e_0}\,(\mathbf{e}\times)^2, \tag{84}$$

and

$$\mathbf{K} = \frac{1}{e_0}\,\mathbf{I} + \frac{1}{1 + e_0}\,\left(\mathbf{I} + \frac{1}{e_0}\,(\mathbf{e}\times)\right)(\mathbf{e}\times). \tag{85}$$

C The rotation vector and exponential map

In the following, we adopt the notation used in refs. [8, 20], where a more detailed exposition of the exponential map of rotation is found. The rotation vector parameterization corresponds to a simple choice of the generating function $p(\varphi) = \varphi$ and will be denoted as

$$\boldsymbol{\varphi} = \varphi\,\mathbf{u}. \tag{86}$$

Clearly, $\|\boldsymbol{\varphi}\| = \varphi$, $\mu = 1$, $\nu = 2\,\sin\varphi/\varphi$,and $\varepsilon = 2\tan\varphi/\varphi$. The expression for the rotation tensor is readily found by inserting $p(\varphi) = \varphi$ into equation (14). Expanding the trigonometric functions into infinite power series then yields

$$\mathbf{R} = \exp(\boldsymbol{\varphi}\times) := \sum_{k=0}^{\infty} \frac{(\boldsymbol{\varphi}\times)^k}{k!}. \tag{87}$$

Given the rotation tensor $\mathbf{R}$, the corresponding rotation vector $\boldsymbol{\varphi}$ is recovered by the inverse formula

$$\boldsymbol{\varphi} = \mathrm{axial}\big(\log(\mathbf{R})\big), \tag{88}$$

where

$$\log(\mathbf{R}) := -\sum_{k=1}^{\infty} \frac{1}{k}\,(\mathbf{I} - \mathbf{R})^k. \tag{89}$$

Note that both the $\exp(\cdot)$ and $\log(\cdot)$ maps are not one-to-one, so that a restriction over all possible (infinite) determinations of $\boldsymbol{\varphi}$ for a given $\mathbf{R}$ must be imposed. This is accomplished selecting the *principal value*, or simply requiring that the magnitude of such vector be in $(-\pi, \pi]$.

Although mathematically attractive, equation (87) should not be used to evaluate $\mathbf{R}$. The finite form representation, equation (14), is preferable. Similarly, equation (89) should not be used for the kinematic inversion. Rather, the procedure outlined in Appendix A should be used to determine the Euler parameters first, then the rotation vector as $\boldsymbol{\varphi} = \mathbf{e}/\nu$.

The expression for tensor $\mathbf{H}$ is readily found by inserting $p(\varphi) = \varphi$ into equation (22). Expanding the trigonometric functions into infinite power series then yields $\mathbf{H} = \mathbf{S}$, where tensor $\mathbf{S}$ is defined as the 'associated differential tensor'

$$\mathbf{S} = \mathrm{dexp}(\boldsymbol{\varphi}\times) := \sum_{k=0}^{\infty} \frac{(\boldsymbol{\varphi}\times)^k}{(k+1)!}. \tag{90}$$

This is sometimes named *coexponential* map [33]. This map is also obtained when looking at the solution of the initial value problem for a general constant coefficient linear ordinary differential equation and its perturbation, as shown in ref. [20]. The exponential map can be interpreted as the *evolution* operator of the constant angular velocity problem, while the associated differential map has the meaning of the corresponding *convolution* operator.

Finally, the inverse of the associated differential tensor is considered

$$\mathbf{S}^{-1} = \mathrm{dexp}^{-1}(\boldsymbol{\varphi}\times) := \mathbf{I} - \frac{1}{2}\,(\boldsymbol{\varphi}\times) - \sum_{k=1}^{\infty} \frac{B_k}{(2\,k)!}\,(\boldsymbol{\varphi}\times)^k, \tag{91}$$

where the scalar coefficients $\{B_k\}$ denote the Bernoulli numbers [34]. Here again, a more practical expression is provided by the finite form, equation (26).

D The Cayley transform

In the following, we adopt the notation used in ref. [20], where a more detailed exposition of the Cayley transform of rotation is found. The *Cayley transform* can be used as a parameterization technique for rotation

$$\mathbf{R} = \mathrm{cay}(\boldsymbol{\zeta}\times) := \big(\mathbf{I} + (\boldsymbol{\zeta}\times)\big)\,\big(\mathbf{I} + (\boldsymbol{\zeta}\times)\big)^{-T} \equiv \big(\mathbf{I} + (\boldsymbol{\zeta}\times)\big)^{-T}\big(\mathbf{I} + (\boldsymbol{\zeta}\times)\big). \tag{92}$$

where $\boldsymbol{\zeta} \in \mathbb{E}^3$ is termed the Cayley rotation vector. It can be shown that

$$\boldsymbol{\zeta} = \tan\frac{\varphi}{2}\,\mathbf{u}, \tag{93}$$

and that the expression for the rotation tensor simplifies as

$$\mathbf{R} = \mathbf{I} + \frac{1}{1+\zeta^2}\,\big(\mathbf{I} + (\boldsymbol{\zeta}\times)\big)\,(\boldsymbol{\zeta}\times). \tag{94}$$

where $\zeta := \|\boldsymbol{\zeta}\|$. Note that $\big(\mathbf{I} + (\boldsymbol{\zeta}\times)\big)^{-T} = (\mathbf{I} + \mathbf{R})/2$, and hence, this represents an operator averaging the initial and rotated orientations. On the other hand, $\big(\mathbf{I} + (\boldsymbol{\zeta}\times)\big)$ represents a linearized rotation as $\varphi \to 0$. In view of these interpretations, equation (92) represents a multiplicative decomposition of the rotation tensor into average and incremental tensors. Taking into account equation (92), a multiplicative decomposition of the vectorial parameterization map is obtained

$$\mathbf{R} = \left(\mathbf{I} + \frac{\varepsilon}{2}\,(\mathbf{p}\times)\right)\,\left(\mathbf{I} + \frac{\varepsilon}{2}\,(\mathbf{p}\times)\right)^{-T} = \left(\mathbf{I} + \frac{\varepsilon}{2}\,(\mathbf{p}\times)\right)^{-T}\,\left(\mathbf{I} + \frac{\varepsilon}{2}\,(\mathbf{p}\times)\right), \tag{95}$$

as $\boldsymbol{\zeta} = (\varepsilon/2)\,\mathbf{p}$ for any choice of the generating function $p(\varphi)$. The corresponding formula using Euler-Rodrigues parameters is

$$\mathbf{R} = \left(\mathbf{I} + \frac{1}{e_0}\,(\mathbf{e}\times)\right)\,\left(\mathbf{I} + \frac{1}{e_0}\,(\mathbf{e}\times)\right)^{-T} = \left(\mathbf{I} + \frac{1}{e_0}\,(\mathbf{e}\times)\right)^{-T}\,\left(\mathbf{I} + \frac{1}{e_0}\,(\mathbf{e}\times)\right), \tag{96}$$

which is closely related to equation (58).

As can be verified from equations (18) and (19), composition of rotations takes a particularly simple form within the Cayley parameterization:

$$\boldsymbol{\zeta}_3 = \boldsymbol{\zeta}_1 + \boldsymbol{\zeta}_2 - \boldsymbol{\zeta}_1 \times \boldsymbol{\zeta}_2, \tag{97}$$

where $\boldsymbol{\zeta}_1$ $\boldsymbol{\zeta}_2$, and $\boldsymbol{\zeta}_3$ are associated with $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$, respectively, and $\mathbf{R}_3 := \mathbf{R}_2\,\mathbf{R}_1$. Following similar reasoning to those already carried out for the vectorial and exponential parameterizations, the associated differential tensor is found as $\mathbf{H} = \mathbf{Y}$, where tensor $\mathbf{Y}$ is defined as the 'associated differential tensor'

$$\mathbf{Y} := \mathrm{dcay}(\boldsymbol{\zeta}\times) = \frac{1}{1+\zeta^2}\big(\mathbf{I} + (\boldsymbol{\zeta}\times)\big). \tag{98}$$

This parameterization clearly falls into the proposed vectorial parameterization class, since $\mathbf{p} = \boldsymbol{\zeta}$ corresponds to the generating function $p(\varphi) = \tan\varphi/2$, with a normalization factor $\kappa = 1/2$. Apart from this normalization, it clearly coincides with the Cayley-Gibbs-Rodrigues parameterization described with in sections 3.2 and 3.3. In this case, $\mu = 2/(1+\zeta^2)$, $\nu = \sqrt{2\,\mu}$, and $\varepsilon = 2$.

The Cayley formalism can be generalized by introducing the *m-th order Cayley transform*,

$$\mathrm{cay}_{(m)}(\mathbf{A}) := (\mathbf{I} + \mathbf{A})^m\,(\mathbf{I} - \mathbf{A})^{-m}, \tag{99}$$

where $m \in \mathbb{N}$. It is readily shown that the domain and codomain of $\mathrm{cay}_{(m)}$ are $\mathrm{so}(\mathbb{E}^3)$ and $\mathrm{SO}(\mathbb{E}^3)$, respectively. Consequently, the m-th order Cayley rotation vector $\boldsymbol{\zeta}_{(m)} \in \mathbb{E}^3$ can be adopted as a vectorial parameterization class

$$\mathbf{R} = \mathrm{cay}_{(m)}(\boldsymbol{\zeta}_{(m)}\times), \tag{100}$$

where it can be demonstrated that

$$\boldsymbol{\zeta}_{(m)} = \tan\frac{\varphi}{2m}\,\mathbf{u}. \tag{101}$$

This extension was carried out in ref. [24]. Clearly, this parameterization class coincides with the even integer divisor subclass of the tangent family with normalization factors $\kappa = 1/(2m)$. For this subclass, $\mathbf{G}_{(m)}$, the m-th root of $\mathbf{R}$ defined in equation (76), becomes

$$\mathbf{G}_{(m)} = \mathrm{cay}(\boldsymbol{\zeta}_{(m)}\times). \tag{102}$$

This is apparent when considering that $\mathrm{cay}_{(m)}(\mathbf{A}) = \mathrm{cay}(\mathbf{A})^m$. In other words, the m-th parameterization map of the even integer divisor subclass of the tangent family is given by the Cayley parameterization map raised to the m-th power.

Name	$p(\varphi)$	p'	ν	ε	**Validity range**
Exponential map	φ	1	$\dfrac{\sin\varphi/2}{\varphi/2}$	$\dfrac{\tan\varphi/2}{\varphi/2}$	$< \pm 2\pi$
Cayley-Gibbs-Rodrigues	$2\tan\varphi/2$	$\dfrac{1}{\cos^2\varphi/2}$	$\cos\varphi/2$	1	$< \pm\pi$
Wiener-Milenkovic	$4\tan\varphi/4$	$\dfrac{1}{\cos^2\varphi/4}$	$\cos^2\varphi/4$	$\dfrac{1}{1-\tan^2\varphi/4}$	$< \pm 2\pi$
Linear Parameters	$\sin\varphi$	$\cos\varphi$	$\dfrac{1}{\cos\varphi/2}$	$\dfrac{1}{\cos^2\varphi/2}$	$< \pm\pi$
Euler Rodrigues	$2\sin\varphi/2$	$\cos\varphi/2$	1	$\dfrac{1}{\cos\varphi/2}$	$< \pm\pi$
	$4\sin\varphi/4$	$\cos\varphi/4$	$\cos\varphi/4$	$\dfrac{\cos\varphi/4}{\cos\varphi/2}$	$< \pm 2\pi$

Table 1: Various choices of the generating function.

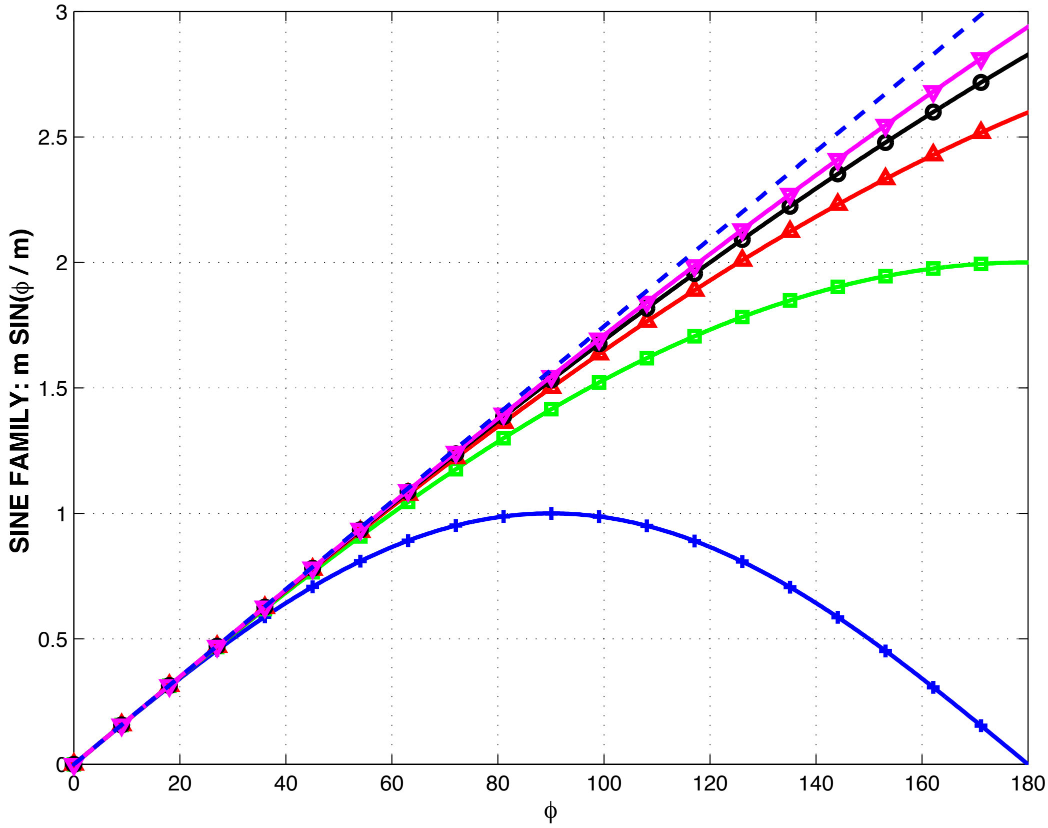

Figure 1: Generating function versus φ for the sine family. m = 1 (+), 2 (□), 3 (△), 4 (○), 5 (▽). The dotted line represents the generating function for the rotation vector $p(\phi) = \phi$.

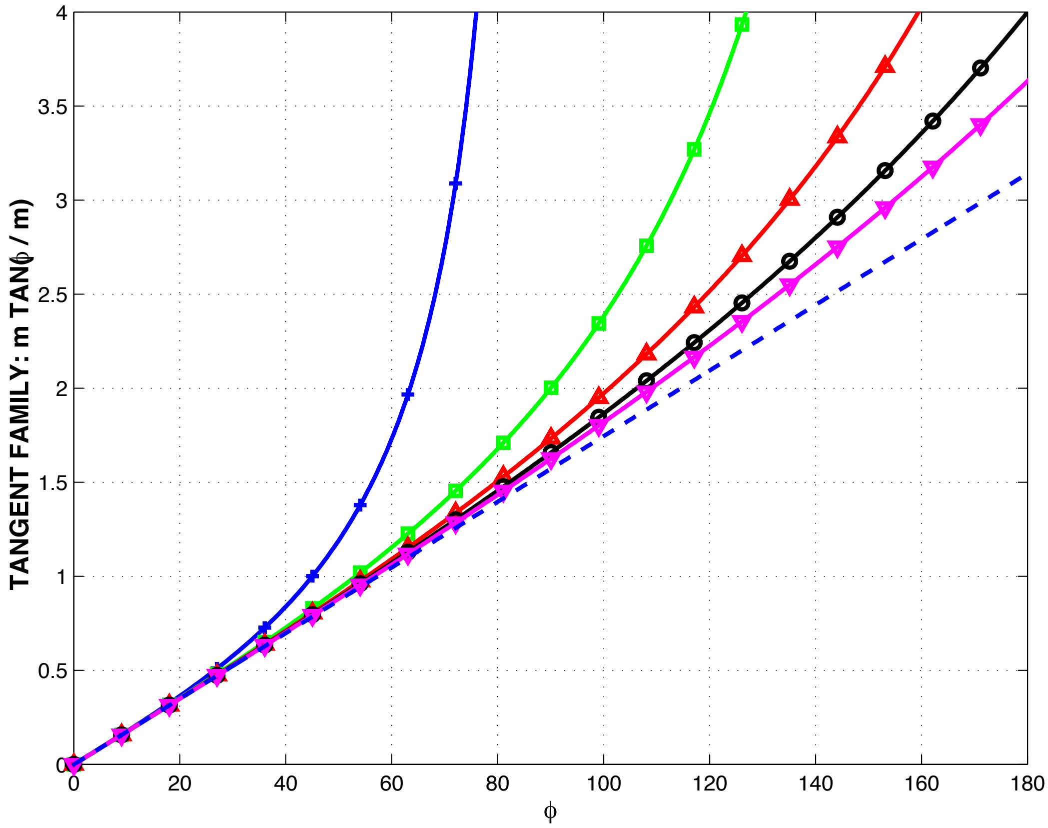

Figure 2: Generating function versus φ for the tangent family. m = 1 (+), 2 (□), 3 (△), 4 (∘), 5 (▽). The dotted line represents the generating function for the rotation vector $p(\phi) = \phi$.

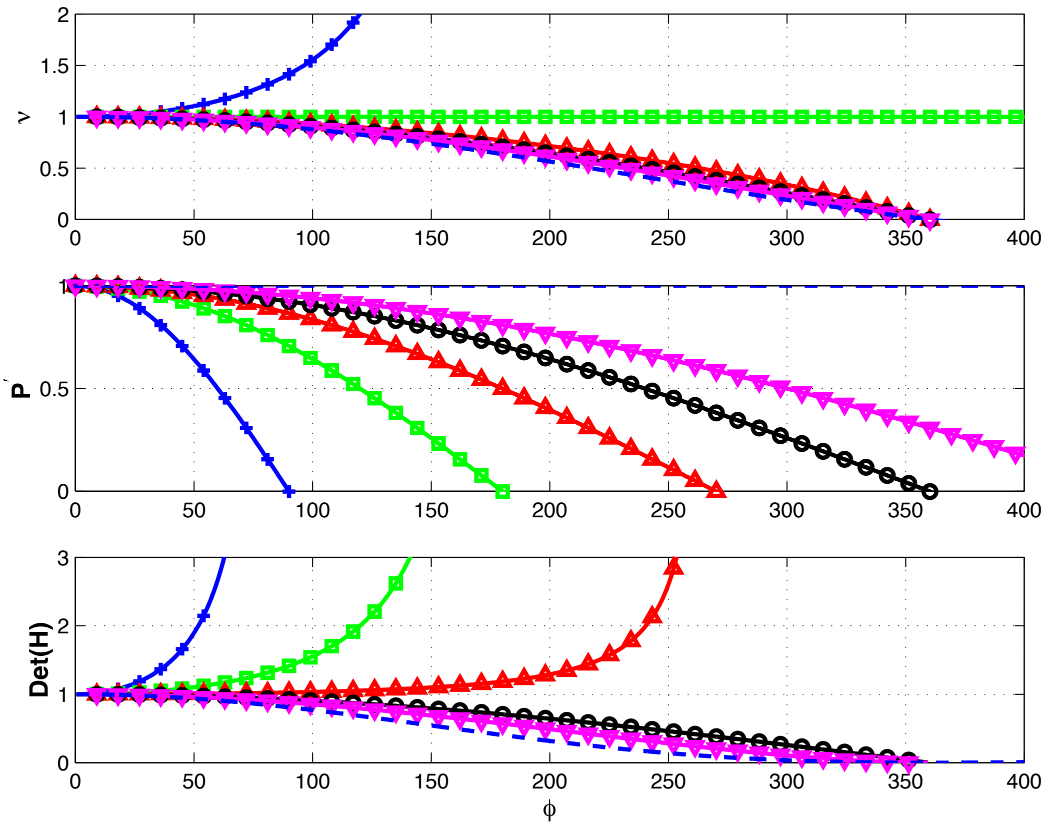

Figure 3: Functions ν (top figure), p' (middle figure), and $\det(\mathbf{H})$ (bottom figure), versus φ for the sine family. m = 1 (+), 2 (□), 3 (△), 4 (○), 5 (▽). The dotted line gives the corresponding quantities for the rotation vector $p(\phi) = \phi$.

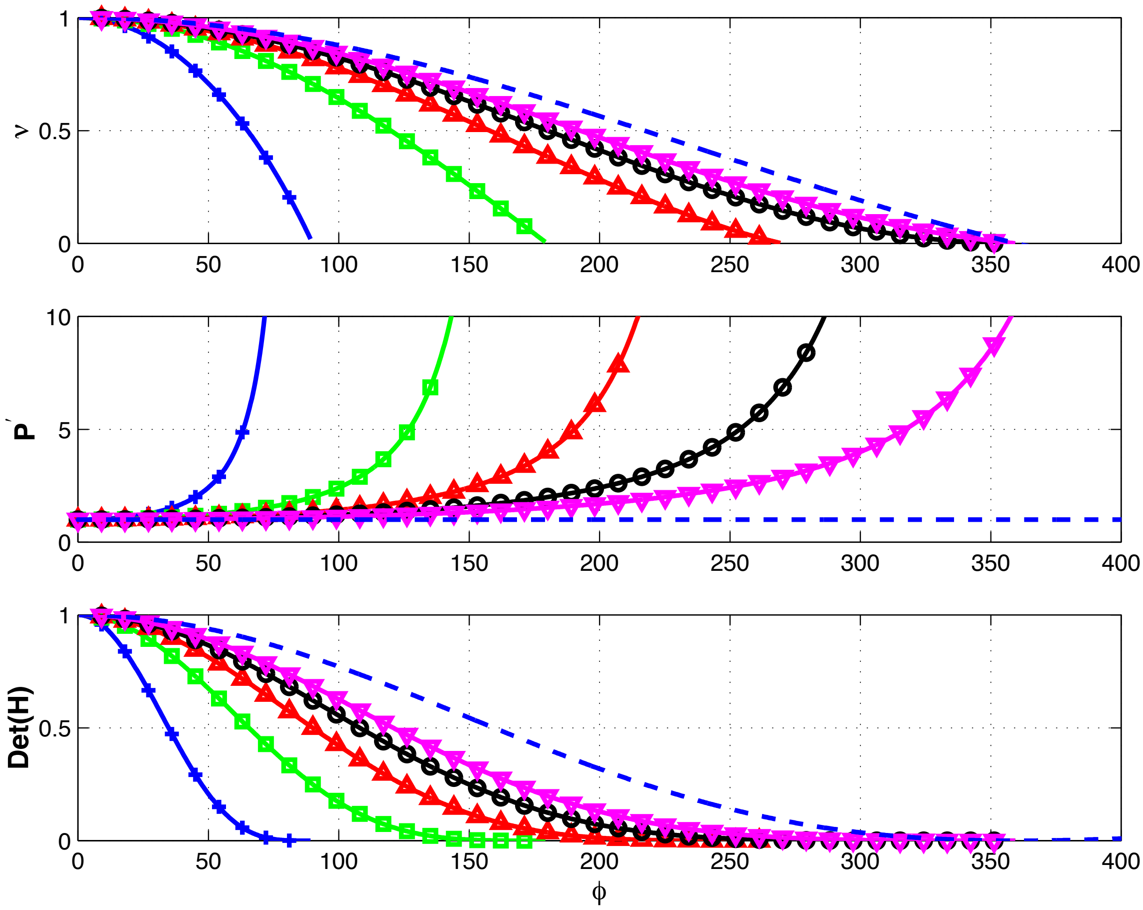

Figure 4: Functions ν (top figure), p' (middle figure), and $\det(\mathbf{H})$ (bottom figure), versus φ for the tangent family. m = 1 (+), 2 (□), 3 (△), 4 (○), 5 (▽). The dotted line gives the corresponding quantities for the rotation vector $p(\phi) = \phi$.

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