IEOR 4106 : Optional Project 1

1.1 PERPETUAL AMERICAN OPTIONS

Background : In contrast to the case for an European option, whose discounted process is a martingale under the risk-neutral measure, the discounted price process of an American option is super-martingale under this measure. The holder of this option may fail to exercise at the optimal exercise date, and thus the discounted option price has a tendency to fall.

The Perpetual American option with payoff h allows its owner to exercise his rights at any moment in $[0, \infty]$.

We consider Black-Scholes model for which the underlying Stock Process S_t is given by:

$$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t \quad - (1.1)$$

Under risk-neutral measure, we can rewrite (1.1) as

$$dS_t = r S_t\,dt + \sigma S_t\,dB_t^{Q} \quad - (1.2)$$

where r is the interest rate and B_t^{Q} is the Brownian Motion under risk-neutral measure Q.

The Perpetual American put pays $[K - S_t]^+$ if it is exercised at time t. This is its intrinsic value.

Let $\mathcal{T}$ be the set of all stopping times. The price of the perpetual American put is defined as

$$V_*(x) = \max_{\tau \in \mathcal{T}} \tilde{E}\left[e^{-r\tau}(K - S(\tau))^+\right] \quad - (1.3)$$

where $x = S_0$, and $h(S_\tau) = [K - S_\tau]^+$

A stopping time τ is a random variable taking values in $[0, \infty]$ and satisfying

$$\{\tau \leq t\} \in F(t) \qquad \forall t \geq 0$$

τ has the property that the decision to stop at time t must be based on information available at time t.

The idea behind eq. (1.3) is that the owner of the perpetual American put option can chose to exercise at τ, subject to the condition that he/she may not look ahead to determine when to exercise.

The owner of the option should choose the exercise strategy that maximizes the expected payoff, discounted back to time zero, and hence the reasoning behind (1.3) to define the price of the option to be maximum over $\tau \in \mathcal{T}$ of the discounted expected payoffs.

In particular, there is no expiration date after which the put can no longer be exercised. Hence, it is reasonable to assume that the optimal exercise policy must depend on value of S_t and not on t.

(1.) → In the event that $\tau = \infty$, $\lim_{\tau \to \infty} e^{-r\tau}(K - S_\tau)$ tends to zero, and hence we must exercise ~~that~~ to receive non-zero payoff.

→ The owner must exercise when S_t falls ~~fa~~ ~~fall~~ far below K to maximize his payoff.

→ Since the payoff $h(x) = (K - x)^+$, for any $x < K$, we have an non-zero payoff.

Hence, we can say that the ~~~~ owner exercises the put when S_t falls to the level L_*.

§ Laplace Transform for the first passage time of drifted Brownian Motion,

Let $X_t = \mu t + B_t$ $\qquad X_0 = 0$

$\tau_m = \min\{t \geq 0 : X_t = m\}$

Define: $\lambda = \sigma\mu + \frac{1}{2}\sigma^2 \implies \sigma = -\mu + \sqrt{\mu^2 + 2\lambda}$

Define: $Z_t = \exp(\sigma X_t - \lambda t)$

We have $Z_t = \exp(\sigma \cancel{\mu t} + \sigma B_t - \cancel{\sigma\mu t} - \frac{1}{2}\sigma^2 t)$

$\implies Z_t = \exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$

Applying Ito on Z_t, we have

$dZ_t = \cancel{-\frac{\sigma^2}{2} Z_t\, dt} + \sigma Z_t\, dB_t + \cancel{\frac{\sigma^2}{2} Z_t\, dt}$

$\therefore dZ_t = \sigma Z_t\, dB_t$.

Hence Z_t is a Martingale since it only involves a stochastic integral without any drift term.

Using Optional Stopping Theorem:

$\mathbb{E}[Z_{\tau_m}] = \mathbb{E}[Z_0]$

$\implies \mathbb{E}[\exp(\sigma X_{\tau_m} - \lambda\tau_m)] = 1$

$\implies \mathbb{E}[\exp(-\lambda\tau_m)] = \exp(-\sigma m) \quad \because X_{\tau_m} = m$

$\therefore \boxed{\mathbb{E}[\exp(-\lambda\tau_m)] = \exp(-m(-\mu + \sqrt{\mu^2 + 2\lambda})) \quad \lambda > 0}$

Applying Ito on $\log(S_t)$ from 1.2, we get

$$S_t = x\exp\left\{\sigma B_t^{Q} + \left(r - \frac{\sigma^2}{2}\right)t\right\}$$

Since the owner of American perpetual put exercises the option when S_t reaches a level say $L < K$, if the initial stock price S_t is at or below L, she exercise immediately (at $t = 0$).

Then $V_L(x) = [K - x]^+$ where $S_0 = x$, $0 \le x \le L$

If the initial $x > L$, option is exercised at stopping time $\tau_L = \min\{t \ge 0 : S_t = L\}$

Thus we have: $S_t = L \Rightarrow x\exp\left(\sigma B_t^{Q} + \left(r - \frac{\sigma^2}{2}\right)t\right) = L$

$$\Rightarrow \frac{1}{\sigma}\log\frac{x}{L} = -B_t^{Q} - \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)$$

With $\lambda = r$ and $H = -\frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)$

$$\mathbb{E}[e^{-r\tau_L}] = \exp\left\{\left(-\frac{1}{\sigma}\log\frac{x}{L}\ \frac{2r}{\sigma}\right)\right\} = \left(\frac{x}{L}\right)^{-2r/\sigma^2}$$

Thus we have,

$$V_L(x) = \begin{cases} [K - x] & 0 \le x \le L \\ (K-L)\left(\frac{x}{L}\right)^{-2r/\sigma^2} & x \ge L \end{cases}$$

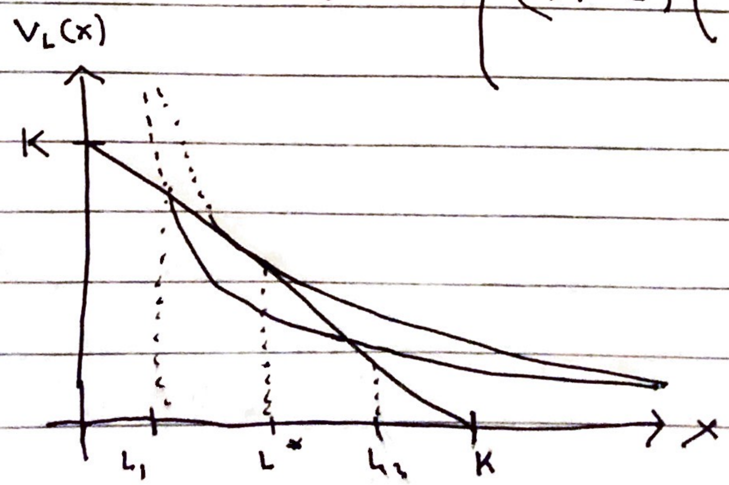


Fig. 1.1

$$V_L(x) = (K-L)L^{\frac{2r}{\sigma^2}} x^{-\frac{2r}{\sigma^2}} \qquad x \geq L$$

Let $g(L) = (K-L) L^{2r/\sigma^2}$

$\therefore g'(L) = 0 \Rightarrow \boxed{L_* = \frac{2r}{2r+\sigma^2} K}$

$$\therefore g(L_*) = \frac{\sigma^2}{2r+\sigma^2} \left(\frac{2r}{2r+\sigma^2} \right)^{\frac{2r}{\sigma^2}} K^{\frac{2r+\sigma^2}{\sigma^2}}$$

From the figure 1.1 and with all the computations, we conclude that we expect an exercise region for which it is optimal to exercise the American perpetual put option.
Also, using optional Stopping Theorem, we computed the optimal payoff ~~where~~ when the owner exercises the option at stopping time to maximize the expected payoff. Also, we have an explicit closed form solution for the price of such an option by varying L.

2. According to the Hamilton-Jacobi-Bellman equation,

$$\min\left\{ r v(x) - \mu x v'(x) - \frac{1}{2}\sigma^2 x^2 v''(x),\ v(x) - h(x) \right\} = 0 \quad \text{— (1.4)}$$

Since $v(x)$ is the maximum payoff we can get by exercising the option at optimal stopping time τ, and also from the previous question and reasoning we have concluded that where $S_t = x^*$, it is optimal to exercise to maximize $h(x)$.

Hence, if we have $x > x^*$, $h(x) < h(x^*)$ [$\because h(x) = [K - x]^+$], thus we have $v(x) > h(x)$ when $x > x^*$.

From (1.4) and since $v(x) > h(x)$, we have

$$r v(x) - \mu x v'(x) - \tfrac{1}{2}\sigma^2 x^2 v''(x) = 0, \quad x > x^* \quad \text{— (1.5)}$$

Since we need a general solution of the above ODE in the form $v(x) = Ax^m + Bx^n$, $m \leq 0$ and $n \geq 1$

$$v'(x) = Amx^{m-1} + Bnx^{n-1}$$
$$v''(x) = Am(m-1)x^{m-2} + Bn(n-1)x^{n-2}$$

Substituting the values of $v(x)$, $v'(x)$ and $v''(x)$ in 1.5, we have

$$Ax^m\left[r - m\mu - \frac{\sigma^2}{2}m(m-1)\right] + Bx^n\left[r - n\mu - \frac{\sigma^2}{2}n(n-1)\right] = 0 \quad \forall x > x^*$$

Since $m \le 0 \Rightarrow x^m > 0$ and $n \ge 1 \Rightarrow x^n > 0$,

We have,

$$A\left[r - m\mu - \frac{\sigma^2}{2}m(m-1)\right] = 0$$

$$B\left[r - n\mu - \frac{\sigma^2}{2}n(n-1)\right] = 0$$

If A and B are non-zero, we will have

$$r - m\mu - \frac{\sigma^2}{2}m(m-1) = 0$$

and $$r - n\mu - \frac{\sigma^2}{2}n(n-1) = 0$$

Solving the quadratic equations above, we have

$$\bar{m} = \frac{\left(\frac{\sigma^2}{2} - \mu\right) - \sqrt{\left(\frac{\sigma^2}{2} - \mu\right)^2 + 2r\sigma^2}}{\sigma^2} \quad (\text{since } m \le 0)$$

$$\bar{n} = \frac{\left(\frac{\sigma^2}{2} - \mu\right) + \sqrt{\left(\frac{\sigma^2}{2} - \mu\right)^2 + 2r\sigma^2}}{\sigma^2} \quad (\text{since } n \ge 1)$$

Thus $v(x) = Ax^m + Bx^n$

with $A = 0$, $n = \bar{n}$

$B = 0$, $m = \bar{m}$

$A = B = 0$, $n = \bar{n}$, $m = \bar{m}$.

3. We proved in ① that,

$$V_L(x) = (K-L)\left(\frac{x}{L}\right)^{-2r/\sigma^2} \qquad x \geq L$$

Since $\frac{2r}{\sigma^2} > 0$, we have $V_L(L) = K-L$

$$V_L(\infty) = 0$$

The $V_L(x) \in [0, K-L] \quad \forall x \geq L$

Hence $V_L(x)$ is <u>bounded</u>.

Since, $V(x) = Ax^m + Bx^n \qquad m \leq 0$ and $n \geq 1$

Hence, $V(x) \to \infty$ and $x \to \infty$

But since $V(x)$ is bounded, we must have $B = 0$.

4. Notice that $L = x^*$

Since $V(x)$ is C^1 at $x = x^*$, we can write

$$V(x) = \begin{cases} Ax^m, & x > x^* \\ K-x, & x \leq x^* \end{cases}$$

Since $\lim_{x \to x^*_+} V(x) = \lim_{x \to x^*_-} V(x)$ ($V(x)$ is continuous)

$$\Rightarrow \lim_{h \to 0^+} \frac{V(x+h) - V(x)}{h} = \lim_{h \to 0^-} \frac{V(x+h) - V(x)}{h} \quad (V(x) \text{ is differentiable})$$

Thus we can write $Ax_*^m = K - x_*$

And, we can also write, $Amx_*^{m-1} = -1$

$$\Rightarrow A = -\frac{1}{mx_*^{m-1}}$$

Plugging in $A = -\frac{1}{mx_*^{m-1}}$ in $Ax_*^m = K - x_*$, we have

$$-\frac{1}{m x_*^{m-1}} x_*^m = K - x^* \Rightarrow \boxed{x^* = \frac{K}{1 - 1/m}}$$

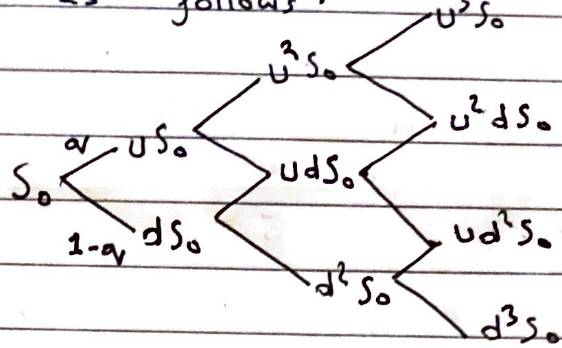
where, $$m = \frac{\left(\frac{\sigma^2}{2} - \mu\right) - \sqrt{\left(\frac{\sigma^2}{2} - \mu\right)^2 + 2r\sigma^2}}{\sigma^2} \leq 0$$

<u>Case $r \to 0$:</u> $2r\sigma^2 \to 0 \Rightarrow m \to 0 \Rightarrow x^* \to 0$

Since $x^* \to 0$ corresponds to maximize payoff of K at maturity, this implies that it is not optimal to exercise the option early when $r \to 0$.

[Jupyter Notebook attached for this problem]

1.2 ① The binomial tree of Stock Process S_t is given as follows:



The lower triangular matrix SS is given by,

$$
\begin{array}{cc}
n+1 & \begin{bmatrix} S_0 & & & 0 \\ dS_0 & uS_0 & & \\ d^2S_0 & udS_0 & u^2S_0 & \\ \vdots & & & \\ d^nS_0 & d^{n-1}uS_0 & \cdots & u^nS_0 \end{bmatrix}
\end{array}
$$

② The recursion formula is given as:

$$p^n = h(S_{t_n}) = [S_{t_n} - K]^+ \quad \text{(For European call)}$$

$$p^i = \mathbb{E}^{Q}[e^{-rT/n} p^{i+1} \mid \mathcal{F}_{t_i}]$$

$$\Rightarrow \quad p^i = e^{-rT/n}\left[q\, p^{i+1}(u) + (1-q)\, p^{i+1}(d)\right]$$

$$= e^{-rT/n}\left[\frac{e^{rT/n} - d}{u-d}\, p^{i+1}(u) + \frac{u - e^{rT/n}}{u-d}\, p^{i+1}(d)\right]$$

Codes are attached for this problem where the above formula is used to compute the price of European option on every node.

③ The recursion formula is given as follows:

<u>American Call</u>: $CA^{i} = \max\left\{ \max(S_{t_i}-K,0),\; e^{-rT/n}\left(\frac{e^{rT/n}-d}{u-d} CA^{i+1}(u) + \frac{u-e^{rT/n}}{u-d} CA^{i+1}(d) \right) \right\}$

<u>American Put</u>: $PA^{i} = \max\left\{ \max(K-S_{t_i},0),\; e^{-rT/n}\left(\frac{e^{rT/n}-d}{u-d} PA^{i+1}(u) + \frac{u-e^{rT/n}}{u-d} PA^{i+1}(d) \right) \right\}$

When $r=0$, the price of the American call is same as that of the European call for every node in the binomial tree. This implies that when $r=0$, it is never optimal to exercise the American put early (similar to American call where we know that it is never optimal to exercise early before maturity)

④ By changing the value of r in the code, it is emperically derived that the efficient frontier strongly depends on r. With increase in r, the no. of early exercise counts in the nodes of binomial tree increase while decreasing r causes the no. of early exercise in the nodes of binomial tree to decrease.
Hence, area of the exercise region increases/decreases with increase/decrease in r.

Note: Codes ~~are~~ is attached for this problem.

Note: Codes are attached in the Jupyter notebook

1.3 Finite differences for American Options

① Since we are given AP, by matrix algebra we can compute $A_{(M+1)\times(M+1)}$ as follows :-

$$A_{jk} = \begin{cases} -\dfrac{\sigma^2 x_j^2}{2h^2} - \dfrac{r x_j}{h}, & k = j+1 \\ \dfrac{\sigma^2 x_j^2}{h^2} + \dfrac{r x_j}{h} + r, & k = j \\ -\dfrac{\sigma^2 x_j^2}{2h^2}, & k = j-1 \\ 0, & \text{otherwise} \end{cases}$$

The code for computions is attached.

② We know that,

$$\min\left\{ \frac{P_j^{n+1} - P_j^n}{\Delta t} + \frac{\sigma^2 x_j^2}{2}\left(\frac{-P_{j-1}^n + 2P_j^n - P_{j+1}^n}{h^2}\right) - r x_j \frac{(P_{j+1}^n - P_j^n)}{h} + rP_j^n,\ P_j^{n+1} - g(x_j) \right\} = 0 \quad \text{— (1.3.1)}$$

$$P_{M+1}^{n+1} = 0$$

and

$$(AP)_j = \frac{\sigma^2 x_j^2}{2}\left(\frac{-P_{j-1}^n + 2P_j^n - P_{j+1}^n}{h^2}\right) - r x_j \frac{P_{j+1}^n - P_j^n}{h} + rP_j^n, \quad \text{— (1.3.2)}$$

$$0 \leq j \leq M$$

Plugging (1.3.2) in (1.3.1) gives

$$\min\left\{ \frac{P_j^{n+1} - P_j^n}{\Delta t} + (AP)_j^n,\ P_j^{n+1} - g(x_j) \right\} = 0$$

$$0 \leq j \leq M+1$$

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$\because$ $\phi = (g(x_j))_{0 \le j \le M+1}$, we can write in matrix form:

$$\min\left\{ \frac{p^{n+1} - p^n}{\Delta t} + Ap^n ,\ p^{n+1} - \phi \right\} = 0$$

Case I: $\dfrac{p^{n+1} - p^n}{\Delta t} + Ap^n > p^{n+1} - \phi \Rightarrow p^{n+1} - \phi = 0$

$$\Rightarrow \underline{p^{n+1} = \phi} \quad (1.3.3)$$

Case II: $\dfrac{p^{n+1} - p^n}{\Delta t} + Ap^n < p^{n+1} - \phi$

$$\Rightarrow \frac{p^{n+1} - p^n}{\Delta t} + Ap^n = 0$$

$$\Rightarrow p^{n+1} - p^n + \Delta t\, Ap^n = 0$$

$$\Rightarrow \underline{p^{n+1} = p^n - \Delta t\, Ap^n} \quad (1.3.4)$$

Consider Case I: $\dfrac{p^{n+1} - p^n}{\Delta t} + Ap^n > p^{n+1} - \phi$

From (1.3.3), $\dfrac{p^{n+1} - p^n}{\Delta t} > -Ap^n$

$$\Rightarrow \underline{p^{n+1} > p^n - \Delta t\, Ap^n, \text{ when } p^{n+1} = \phi} \quad - (1.3.5)$$

Consider case II: $\dfrac{p^{n+1} - p^n}{\Delta t} + Ap^n < p^{n+1} - \phi$

From (1.3.4), $0 < p^{n+1} - \phi$

$$\Rightarrow \underline{p^{n+1} > \phi, \text{ when } p^{n+1} = p^n - \Delta t\, Ap^n} \quad - (1.3.6)$$

From (1.3.5) and (1.3.6), it is easy to convince ourself that indeed $\boxed{p^{n+1} = \max\left\{ p^n - \Delta t\, Ap^n, \phi \right\}}$

③ After playing with the code by changing the parameter's M and N's values, it is observed that the graph is well behaved when $M = 20$ but increasing the value of M to 50 causes the graph to be unstable. This is due to the unstability of the explicit Euler scheme.

Implicit Euler-Scheme

④ $F(x) = \min(Bx - b,\ x - \phi) = 0$

Case I: $Bx - b < x - \phi$

$\Rightarrow F(x) = Bx - b$

$\Rightarrow F'(x) = B$

There $F'(x)_{i,j} = B_{i,j}$

Case II: $Bx - b > x - \phi$

$\Rightarrow F(x) = x - \phi$

$\Rightarrow F'(x) = I$, $I \to$ Identity Matrix

Thus we can write,

$$F'(x)_{i,j} = \begin{cases} B_{i,j} & \text{if } (Bx-b)_i \leq (x-\phi)_i \\ \delta_{i,j}, & \text{otherwise} \end{cases}$$

where $\delta_{i,j} = \begin{matrix} 1 & \text{if } i=j \\ 0 & \text{, otherwise} \end{matrix}$

Note: ~~Computating~~ Computations are demonstrated on Jupyter notebook.