ARCOS Group

uc3m | Universidad Carlos III de Madrid

L3: Representation of information
Computer Structure

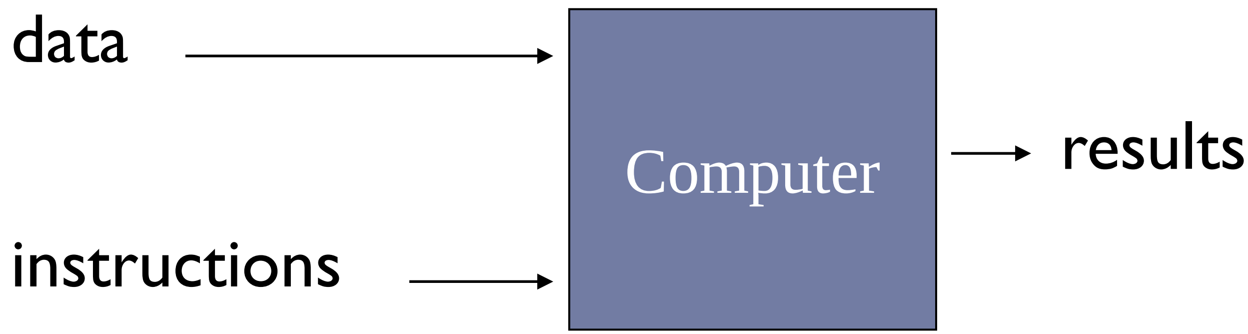
Bachelor in Computer Science and Engineering
Bachelor in Applied Mathematics and Computing
Dual Bachelor in Computer Science and Engineering and Business Administration

Contents

Contents

Introduction
Computer

- A computer is a machine designed to process data.



- Instructions are applied and results are obtained.

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Computer

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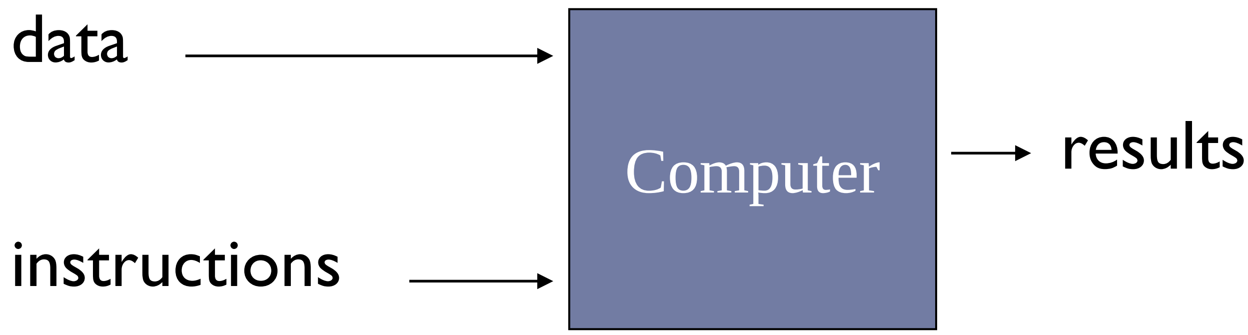

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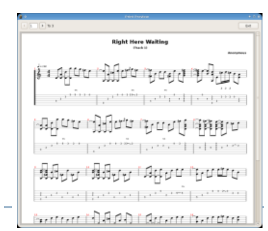

https://www.pixar.com/feature-films/up

Introduction
Computer

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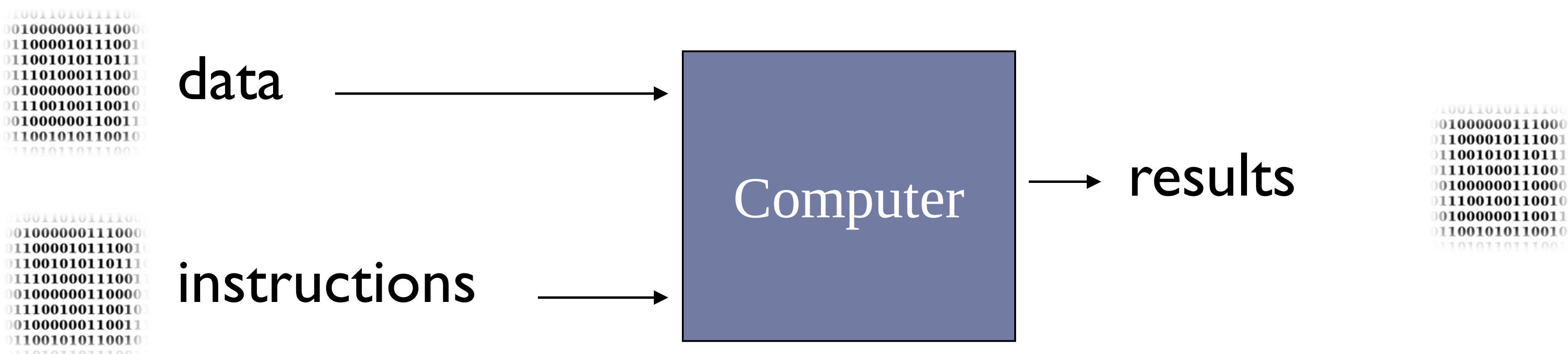
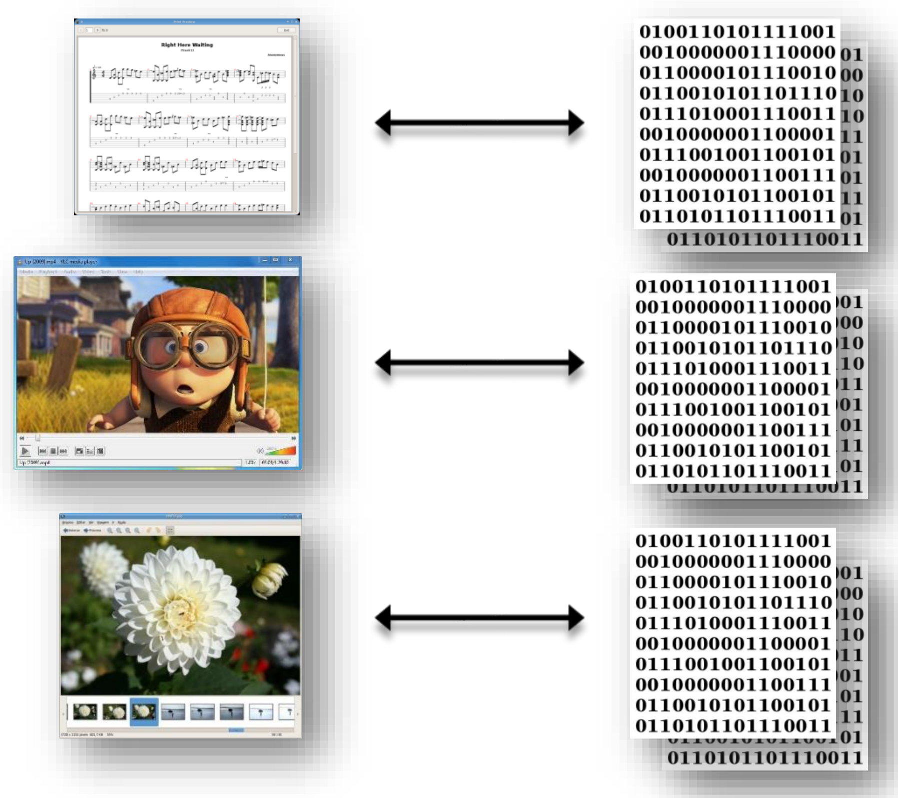


- Instructions are applied and results are obtained.
- The data/information can be of different types.
- A computer uses only one representation: binary.

Introduction
Information representation

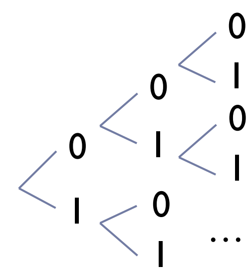
- The use of a representation allows the transformation of different types of information into binary (and vice versa).



https://www.pixar.com/feature-films/up

Introduction
Characteristics of the information representation

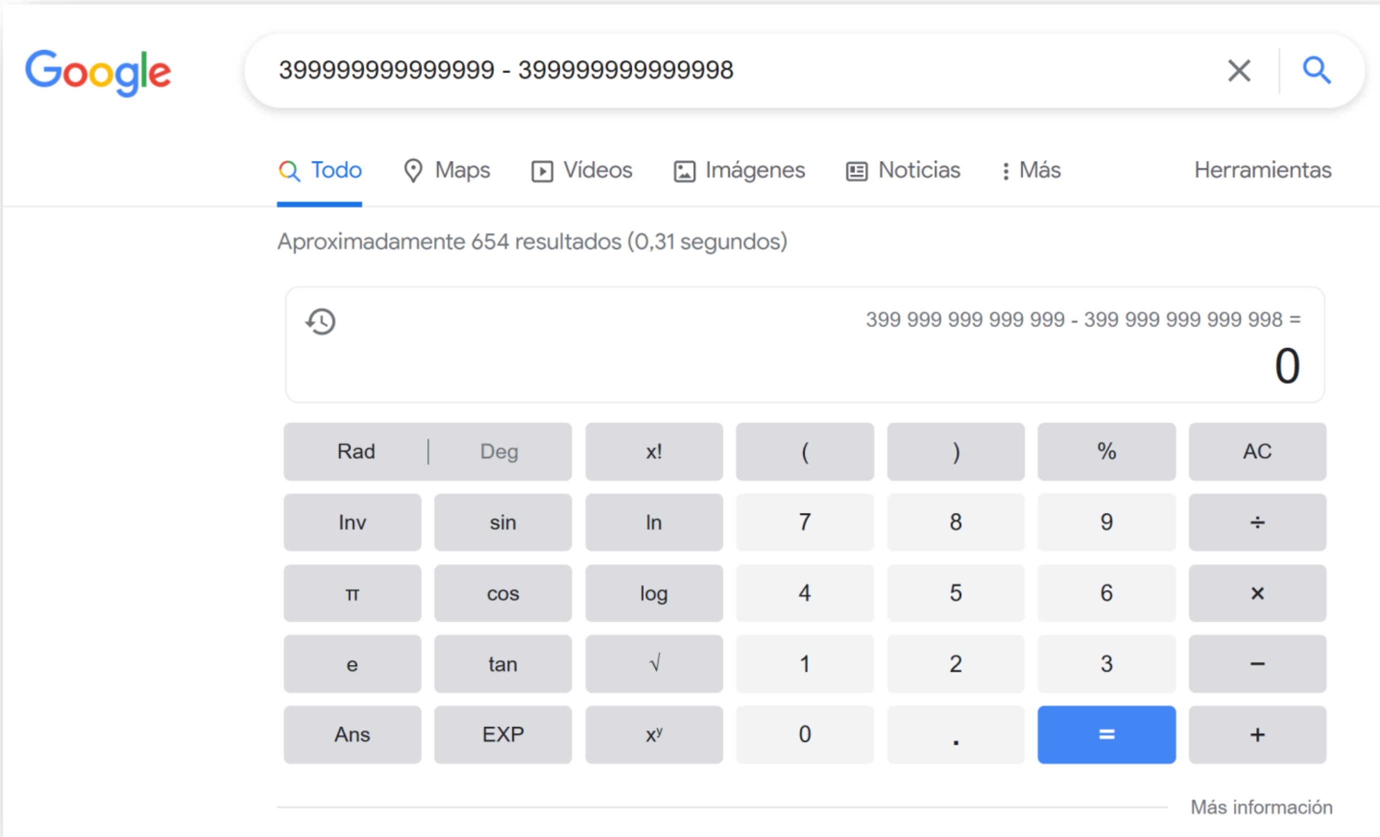
- A computer handles a finite set of values
  - Binary type (two states)
  - Finite (bounded representation)
    - Number of bits of the computer word (32/64) or bit (1), nibble (4), byte (8), half w., double w., …
    - With **n** bits, **2^n** different values can be encoded



- There are some types of information that are infinite
  - Impossible to represent all values of natural numbers, real numbers, etc.

- The chosen representation has limitations.

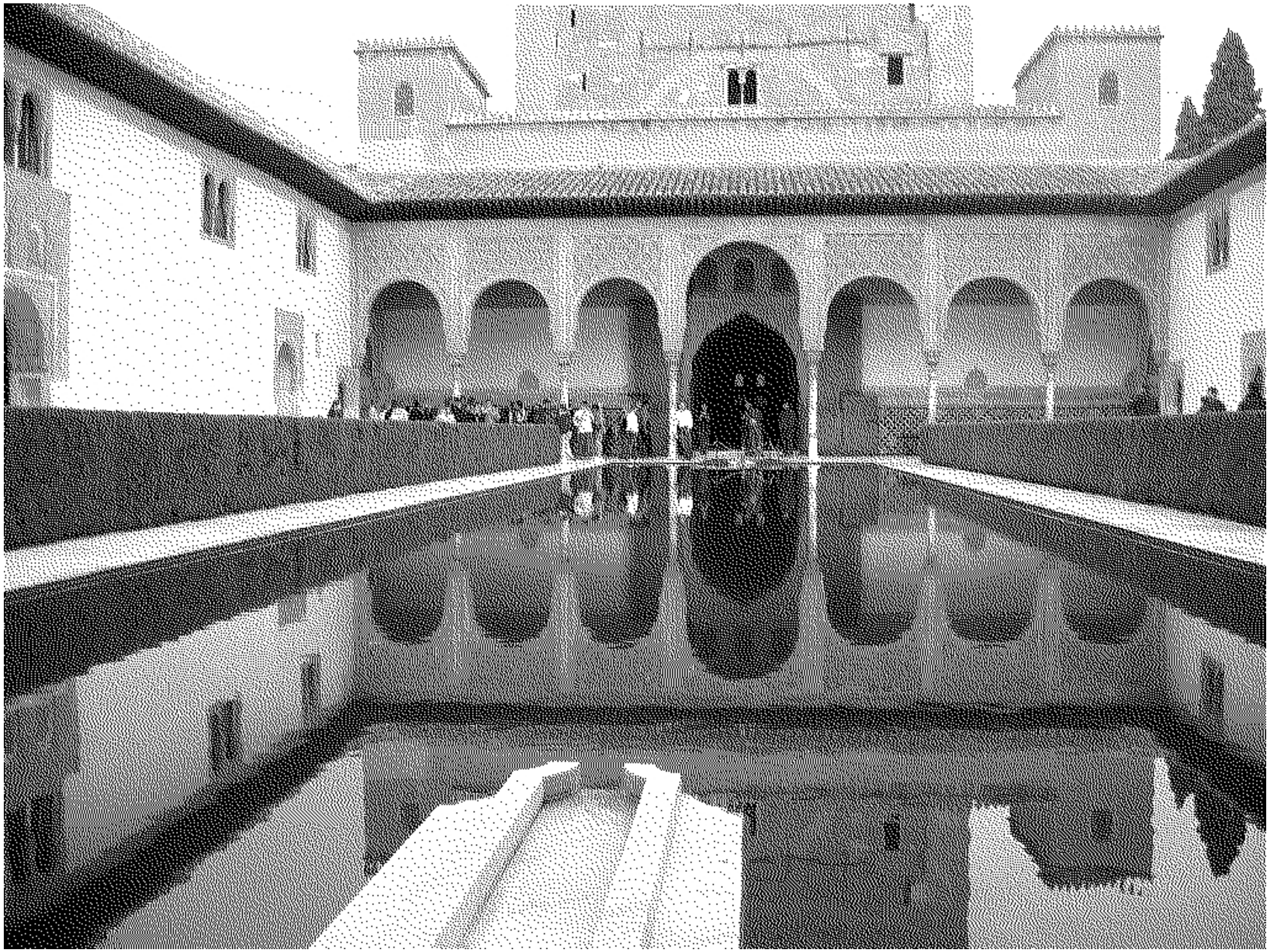
Example 1:
the Google calculator with 15 digits…



http://www.20minutos.es/noticia/415383/0/google/restar/error/

Example 2:
color depth…

1 bit	2 colors
4 bits	16 colors
8 bits	256 colors



http://platea.pntic.mec.es/~lgonzale/tic/imagen/conceptos.html

Example 2:
color depth…

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Example 2:
color depth…

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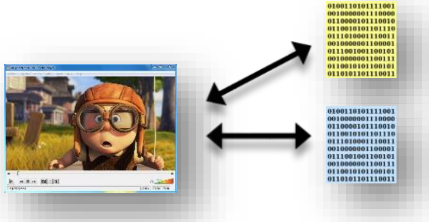
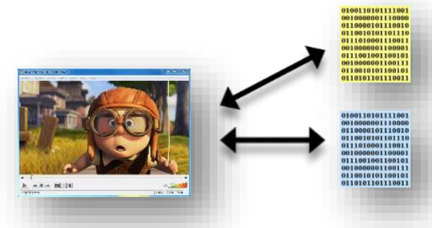
We need…

- To know possible representations:

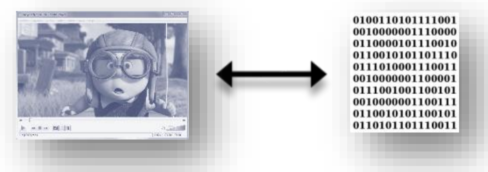


We need…

- To know possible representations:



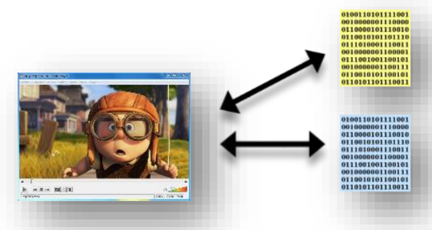
- To know the characteristics of theses representations:
  - Limitations



https://www.pixar.com/feature-films/up

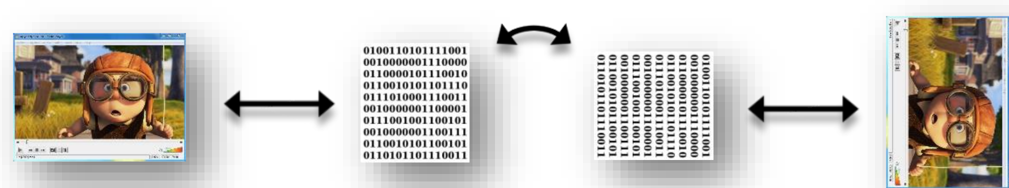
We need…

- To know possible representations:



- To know the characteristics of theses representations:
  - Limitations

- To know how work with the selected representation:



https://www.pixar.com/feature-films/up

Contents

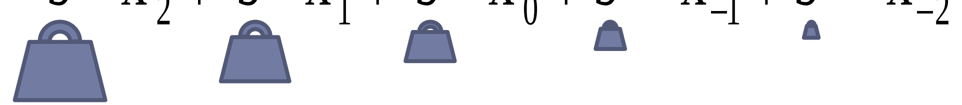
Positional representation systems

- A number is defined by a ordered list of digits, each of which is affected by a scaling factor that depends on the position it occupies in the list.

- Given a numbering base b,
  a number X is defined as the list of digits:
  $X = (\ldots\ \ x_2\ \ x_1\ \ x_0\ ,\ \ x_{-1}\ \ x_{-2}\ \ \ldots)_b$ Con $0 \leq x_i < b$
  with a list of associated weights:
  $P\ = (\ldots\ \ b^2\ \ b^1\ \ b^0\ \ \ \ b^{-1}\ \ b^{-2}\ \ \ldots)_b$

- Its value is:

$$V(X) = \sum_{i=-\infty}^{+\infty} b^i \cdot x_i = \cdots b^2 \cdot x_2 + b^1 \cdot x_1 + b^0 \cdot x_0 + b^{-1} \cdot x_{-1} + b^{-2} \cdot x_{-2} \cdots$$



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Positional representation systems

- Decimal

  X = 9 7 3 1

  … 10^3 10^2 10^1 10^0

- Binary

  X = 0 1 0 1

  … 2^3 2^2 2^1 2^0

- Hexadecimal

  X = 1 F A 8

  … 16^3 16^2 16^1 16^0

Positional representation systems

- Decimal

  X = 9 7 3 1

  … 10^3 10^2 10^1 10^0

- Binary

  X = 0 1 0 1

  … 2^3 2^2 2^1 2^0

- Hexadecimal

  X = 1 F A 8

  … 16^3 16^2 16^1 16^0

From binary to hexadecimal:

- Group by 4 bits, right to left
- Each 4 bits is the value of a hexadecimal digit
- E.g.: $\underbrace{1\ 0\ 1\ 0}_{A}\ \underbrace{0\ 1\ 0\ 1}_{5}$

  0x A 5

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Positional representation systems

- Decimal

  X = 9 7 3 1

  … 10^3 10^2 10^1 10^0

¿?

- Binary

  X = 0 1 0 1

  … 2^3 2^2 2^1 2^0

- Hexadecimal

  X = 1 F A 8

  … 16^3 16^2 16^1 16^0

Exercise

- To represent 342 in binary:

256	128	64	32	16	8	4	2	1
?	?	?	?	?	?	?	?	?

Exercise (solution)

- To represent 342 in binary:

256	128	64	32	16	8	4	2	1
1	0	1	0	1	0	1	1	0
342-256=86		86-64=22		22-16=6		6-4=2	2-2=0	

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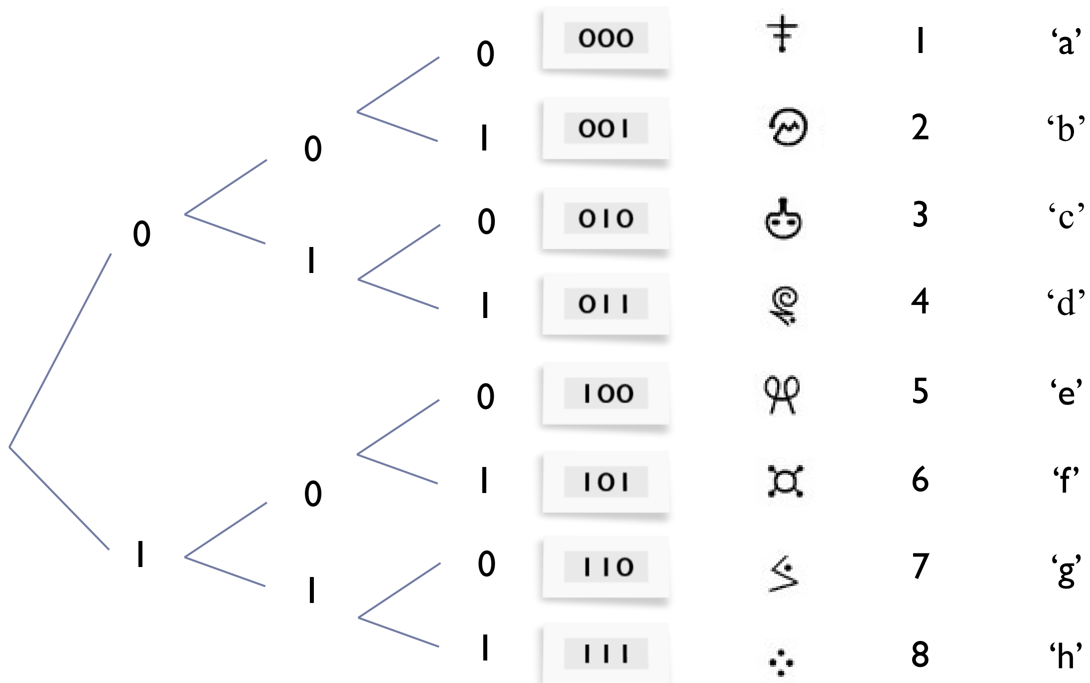
Positional representation systems

- With 3 binary digits, up to 8 symbols can be represented:



Positional representation systems

- How many values can be represented with n bits?

- How many bits are needed to represent m 'values'?

- With n bits, if the minimum representable value corresponds to the number 0, what is the maximum representable numerical value?

Positional representation systems

- How many values can be represented with n bits?
  - **2^n**
  - E.g.: with 4 bits up to 16 values can be represented

- How many bits are needed to represent m 'values'?
  - **$\lceil$Log2(n)$\rceil$** (Log2(n) round up)
  - E.g.: 6 bits are required to represent 35 values

- With n bits, if the minimum representable value corresponds to the number 0,
  what is the maximum representable numerical value?
  - **2^n-1**

Exercise

- To compute the value of (23 ones):

$$11111111111111111111111_2$$

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Exercise (solution)

- To compute the value of (23 ones):

$$11111111111111111111111_2$$

$$X = 2^{23} - 1$$

Tip:

$$
\begin{array}{rll}
 & 11111111111111111111111_2 & = X \\
+ & 00000000000000000000001_2 & = 1 \\
\hline
 & 100000000000000000000000_2 & = 2^{23}
\end{array}
$$

$$X = 2^{23} - 1$$

Example: operations

- Add in binary:

	1		1	1		
		1	0	1	0	0
+		1	1	1	1	0
	1	1	0	0	1	0

Example: operations

- Add in binary:

$$\begin{array}{ccccccc} & 1 & 1 & 1 & & & \\ & & 1 & 0 & 1 & 0 & 0 \\ + & & 1 & 1 & 1 & 1 & 0 \\ \hline & 1 & 1 & 0 & 0 & 1 & 0 \end{array}$$

- Subtract in binary:

$$\begin{array}{cccccc} & & & 1\rightarrow & 1\rightarrow & \\ & 0 & 1 & 1 & 0 & 0 \\ - & 0 & 1 & 0 & 1 & 1 \\ \hline & 0 & 0 & 0 & 0 & 1 \end{array}$$

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Exercise

2 minutes máx.

You have a 5-liter bottle and a 3-liter bottle.
How can you get 4 liters just right?

Exercise (solution)

2 minutes máx.

You have a 5-liter bottle and a 3-liter bottle.
How can you get 4 liters just right?

- Fill the 5-liter jar
- Empty it into the 3-liter jar
  - There are 2 left in the 5-liter jar (-3 a 5).
- Throw away what is in the 3-liter jar
- Transfer the 2 from the 5-liter jar to the 3-liter jar
  - There are 1 left in the 3-liter jar (-1 to 3).
- Refill the 5-liter jar
- Fill the 3-liter jar to the top,
  what is left in the 5-liter jar is 4 liters

http://www.educaplanet.com/educaplanet/2014/11/acertijo-logica-las-jarras-de-agua/

Exercise

2 minutes máx.

- Using the numbers 112 and -71 in decimal base, perform addition in 10's complement.

Exercise (solution)

2 minutes máx.

- 10's complement, 112 is: 112
- 10's complement, -71 is:

1000
-0071

929

- Adding both:

112
929

~~1~~041

112
-071

041

Contents

Alphanumeric representation

- Each character is encoded as one byte.
- With n bits ⇒ up to 2^n characters can be encoded:

# bits	# characters	Includes…	Example
6	64	• 26 letter: a...z • 10 number: 0...9 • punctuation: . , ; : ... • specials: + - [...	***BCDIC***
7	128	• adds uppercases and control characters	***ASCII***
8	256	• adds accented letters, ñ, semigraphic characters	***EBCDIC*** ***ASCII extended***
16	34.168	• add support for Chinese, Arabic, ...	***UNICODE***

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Example: ASCII table (7 bits)

ASCII value	Character	Control character	ASCII value	Character	ASCII value	Character	ASCII value	Character
000	(null)	NUL	032	(space)	064	@	096	
001	☺	SOH	033	!	065	A	097	a
002	☻	STX	034	"	066	B	098	b
003	♥	ETX	035	#	067	C	099	c
004	♦	EOT	036	$	068	D	100	d
005	♣	ENQ	037	%	069	E	101	e
006	♠	ACK	038	&	070	F	102	f
007	(beep)	BEL	039	'	071	G	103	g
008	◘	BS	040	(	072	H	104	h
009	(tab)	HT	041	)	073	I	105	i
010	(line feed)	LF	042	*	074	J	106	j
011	(home)	VT	043	+	075	K	107	k
012	(form feed)	FF	044	,	076	L	108	l
013	(carriage return)	CR	045	-	077	M	109	m
014	♫	SO	046	.	078	N	110	n
015	☼	SI	047	/	079	O	111	o
016	►	DLE	048	0	080	P	112	p
017	◄	DC1	049	1	081	Q	113	q
018	↕	DC2	050	2	082	R	114	r
019	‼	DC3	051	3	083	S	115	s
020	π	DC4	052	4	084	T	116	t
021	§	NAK	053	5	085	U	117	u
022	▬	SYN	054	6	086	V	118	v
023	↨	ETB	055	7	087	W	119	w
024	↑	CAN	056	8	088	X	120	x
025	↓	EM	057	9	089	Y	121	y
026	→	SUB	058	:	090	Z	122	z
027	←	ESC	059	;	091	[	123	{
028	(cursor right)	FS	060	<	092	\	124	¦
029	(cursor left)	GS	061	=	093	]	125	}
030	(cursor up)	RS	062	>	094	^	126	~
031	(cursor down)	US	063	?	095	_	127	⌂

Example: ASCII table (7 bits)

control characters

ASCII value	Character	Control character	ASCII value	Character	ASCII value	Character	ASCII value	Character
000	(null)	NUL	032	(space)	064	@	096	
001	☺	SOH	033	!	065	A	097	a
002	☻	STX	034	"	066	B	098	b
003	♥	ETX	035	#	067	C	099	c
004	♦	EOT	036	$	068	D	100	d
005	♣	ENQ	037	%	069	E	101	e
006	♠	ACK	038	&	070	F	102	f
007	(beep)	BEL	039	'	071	G	103	g
008	◘	BS	040	(	072	H	104	h
009	(tab)	HT	041	)	073	I	105	i
010	(line feed)	LF	042	*	074	J	106	j
011	(home)	VT	043	+	075	K	107	k
012	(form feed)	FF	044	,	076	L	108	l
013	(carriage return)	CR	045	-	077	M	109	m
014	♫	SO	046	.	078	N	110	n
015	☼	SI	047	/	079	O	111	o
016	►	DLE	048	0	080	P	112	p
017	◄	DC1	049	1	081	Q	113	q
018	↕	DC2	050	2	082	R	114	r
019	‼	DC3	051	3	083	S	115	s
020	π	DC4	052	4	084	T	116	t
021	§	NAK	053	5	085	U	117	u
022	▬	SYN	054	6	086	V	118	v
023	↨	ETB	055	7	087	W	119	w
024	↑	CAN	056	8	088	X	120	x
025	↓	EM	057	9	089	Y	121	y
026	→	SUB	058	:	090	Z	122	z
027	←	ESC	059	;	091	[	123	{
028	(cursor right)	FS	060	<	092	\	124	¦
029	(cursor left)	GS	061	=	093	]	125	}
030	(cursor up)	RS	062	>	094	∧	126	~
031	(cursor down)	US	063	?	095	—	127	⌂

< 32

Example: ASCII table (7 bits)

distance between uppercase and lowercase letters

ASCII value	Character	Control character	ASCII value	Character	ASCII value	Character	ASCII value	Character
000	(null)	NUL	032	(space)	064	@	096	
001	☺	SOH	033	!	065	A	097	a
002	☻	STX	034	"	066	B	098	b
003	♥	ETX	035	#	067	C	099	c
004	♦	EOT	036	$	068	D	100	d
005	♣	ENQ	037	%	069	E	101	e
006	♠	ACK	038	&	070	F	102	f
007	(beep)	BEL	039	'	071	G	103	g
008	◘	BS	040	(	072	H	104	h
009	(tab)	HT	041	)	073	I	105	i
010	(line feed)	LF	042	*	074	J	106	j
011	(home)	VT	043	+	075	K	107	k
012	(form feed)	FF	044	,	076	L	108	l
013	(carriage return)	CR	045	-	077	M	109	m
014	♫	SO	046	.	078	N	110	n
015	☼	SI	047	/	079	O	111	o
016	►	DLE	048	0	080	P	112	p
017	◄	DC1	049	1	081	Q	113	q
018	↕	DC2	050	2	082	R	114	r
019	‼	DC3	051	3	083	S	115	s
020	π	DC4	052	4	084	T	116	t
021	§	NAK	053	5	085	U	117	u
022	▬	SYN	054	6	086	V	118	v
023	↨	ETB	055	7	087	W	119	w
024	↑	CAN	056	8	088	X	120	x
025	↓	EM	057	9	089	Y	121	y
026	→	SUB	058	:	090	Z	122	z
027	←	ESC	059	;	091	[	123	{
028	(cursor right)	FS	060	<	092	\	124	¦
029	(cursor left)	GS	061	=	093	]	125	}
030	(cursor up)	RS	062	>	094	^	126	~
031	(cursor down)	US	063	?	095	_	127	⌂

97-65=32

Example: ASCII table (7 bits)

conversion of a number to a character

ASCII value	Character	Control character	ASCII value	Character	ASCII value	Character	ASCII value	Character
000	(null)	NUL	032	(space)	064	@	096	
001	☺	SOH	033	!	065	A	097	a
002	☻	STX	034	"	066	B	098	b
003	♥	ETX	035	#	067	C	099	c
004	♦	EOT	036	$	068	D	100	d
005	♣	ENQ	037	%	069	E	101	e
006	♠	ACK	038	&	070	F	102	f
007	(beep)	BEL	039	'	071	G	103	g
008	◘	BS	040	(	072	H	104	h
009	(tab)	HT	041	)	073	I	105	i
010	(line feed)	LF	042	*	074	J	106	j
011	(home)	VT	043	+	075	K	107	k
012	(form feed)	FF	044	,	076	L	108	l
013	(carriage return)	CR	045	-	077	M	109	m
014	♫	SO	046	.	078	N	110	n
015	☼	SI	047	/	079	O	111	o
016	►	DLE	048	0	080	P	112	p
017	◄	DC1	049	1	081	Q	113	q
018	↕	DC2	050	2	082	R	114	r
019	‼	DC3	051	3	083	S	115	s
020	π	DC4	052	4	084	T	116	t
021	§	NAK	053	5	085	U	117	u
022	▬	SYN	054	6	086	V	118	v
023	↨	ETB	055	7	087	W	119	w
024	↑	CAN	056	8	088	X	120	x
025	↓	EM	057	9	089	Y	121	y
026	→	SUB	058	:	090	Z	122	z
027	←	ESC	059	;	091	[	123	{
028	(cursor right)	FS	060	<	092	\	124	¦
029	(cursor left)	GS	061	=	093	]	125	}
030	(cursor up)	RS	062	>	094	^	126	~
031	(cursor down)	US	063	?	095	_	127	⌂

6+48=54

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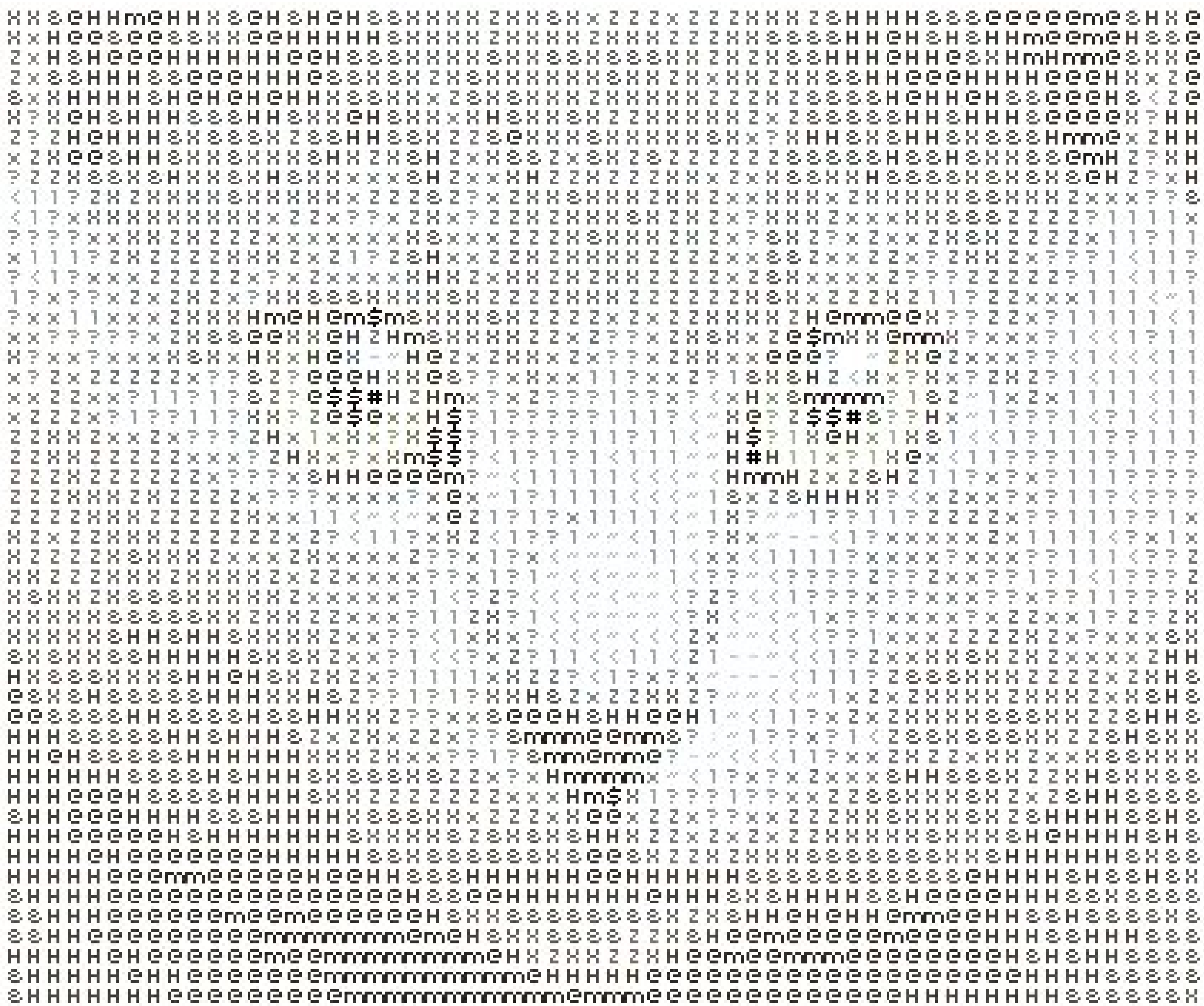
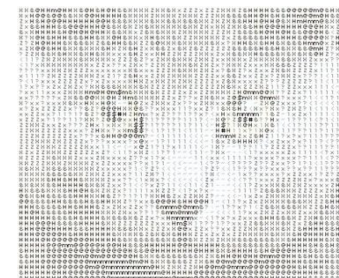
Curiosity:
Display “image” with characters





http://www.typorganism.com/asciiomatic/

1000	00110011
1001	01101100
	…
1008	10100011

Character strings

1. Fixed-length string:

h	e	l	l	o			
1	2	3	4	5	6	7	8

2. Variable-length string with delimiter:

h	e	l	l	o	\0		
1	2	3	4	5	6	7	8

3. Variable-length strings with length in header:

5	h	e	l	l	o		
1	2	3	4	5	6	7	8

Contents

Numerical representation

- Classification of real numbers:
  - Naturals: 0, 1, 2, 3, ...
  - Integers: ... -3, -2, -1, 0, 1, 2, 3, ...
  - Rational: fractions (5/2 = 2,5)
  - Irrational: $2^{1/2}$, π, e, ...

- Infinite sets but finite representation space:
  - Impossible to represent all

- Characteristics of the representation used:
  - Represented element:
    Natural, integer, …
  - Representation range:
    Interval between minor and major not representable
  - Resolution of representation:
    Difference between a representable number and the following one.
    It represents the maximum error committed. It can be cte. or variable.

Most used binary representation systems

A. (Pure) binary — natural

B. Sign-Magnitude

C. One's complement (Ca 1)

D. Two's complement (Ca 2) — integer

E. Biased $2^{n-1}-1$

F. Floating point: IEEE 754 standard — real

(Pure) binary or unsigned binary [natural numbers]

- Positioning system with base 2 and without fractional part.

n-1 0

n bits

$$V(X) = \sum_{i=0}^{n-1} 2^i \cdot x_i$$

- Representation range: $[0, 2^n - 1]$
- Resolution: 1 unit

Comparative example (3 bits)

Decimal	Pure Binary
+7	111
+6	110
+5	101
+4	100
+3	011
+2	010
+1	001
+0	000
-0	N.A.
-1	N.A.
-2	N.A.
-3	N.A.
-4	N.A.
-5	N.A.
-6	N.A.
-7	N.A.

Signed binary number or Sign-Magnitude [integer numbers]

- One bit (S) is reserved for the sign ($0 \Rightarrow +$; $1 \Rightarrow -$)

n-1	n-2 ... 0
S	Magnitude (n-1 bits)

$$\left.\begin{array}{ll} \text{Si } x_{n-1} = 0 & \mathbf{V}(X) = \sum_{i=0}^{n-2} 2^i \cdot \mathbf{x}_i \\ \text{Si } x_{n-1} = 1 & \mathbf{V}(X) = -\sum_{i=0}^{n-2} 2^i \cdot \mathbf{x}_i \end{array}\right| \Rightarrow V(X) = (1 - 2 \cdot x_{n-1}) \cdot \sum_{i=0}^{n-2} 2^i \cdot x_i$$

- Representation range: $[-2^{n-1}+1,\ 2^{n-1}-1]$
- Resolution: 1 unit
- Ambiguity of zero + complex hw. for subtraction

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Comparative example (3 bits)

Decimal	Pure Binary	Sign-Magnitude
+7	111	N.A.
+6	110	N.A.
+5	101	N.A.
+4	100	N.A.
+3	011	011
+2	010	010
+1	001	001
+0	000	000
-0	N.A.	100
-1	N.A.	101
-2	N.A.	110
-3	N.A.	111
-4	N.A.	N.A.
-5	N.A.	N.A.
-6	N.A.	N.A.
-7	N.A.	N.A.

Example

- Can we represent 745_{10} in sign-magnitude with 10 bits?

Example (solution)

- Can we represent 745_{10} in sign-magnitude with 10 bits?

- With 10 bits the range in sign-magnitude is:
  $[-2^9+1,\ldots,-0,+0,\ldots 2^9-1] \Rightarrow [-511, 511]$
  then, **we cannot represent** 745

One's complement (to the base minus one) [integer] (1/3)

- **Positive number**:
  is represented in pure binary with n-1 bits

n-1	n-2 … 0
0	Magnitude

$$V(X) = \sum_{i=0}^{n-1} 2^{i} \cdot x_{i} = \sum_{i=0}^{n-2} 2^{i} \cdot x_{i}$$

- Representation range (+): $[0, 2^{n-1} - 1]$
- Resolution: 1 unit

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One's complement (to the base minus one) [integer] (2/3)

- **Negative number**:
  - Complemented to the base minus one.
  - The number $X < 0$ is represented as $2^n - X - 1$ with n bits

n-1	n-2 … 0
1	1'C of magnitude

$$V(X) = -2^n + \sum_{i=0}^{n-1} 2^i \cdot y_i + 1$$

- Representation range (-): $[-(2^{n-1}-1), -0]$
- Resolution: 1 unit

One's complement (to the base minus one) [integer] (3/3)

Tip: Ca1 (X) = X
Ca1 (-X) = change the 1's to 0's and the 0's to 1's

- Example: For n=4 $\Rightarrow$ the value $+3_{10} = 0011_2$
- Example: For n=4 $\Rightarrow$ the value $-3_{10} = 1100_2$
  - \- $\Rightarrow$ 1 (sign bit and also part of magnitude)
  - Ca1(3) $\Rightarrow 2^4 - 0011_2 - 1 = 2^4 - 3 - 1 = 12 \Rightarrow 1100_2$

- Representation range: $[-2^{n-1}+1, 2^{n-1}-1]$
- Resolution: 1 unit
- Zero has a double representation (+0 y -0)
- Symmetrical range

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Comparative example (3 bits)

Decimal	Pure Binary	Sign-Magnitude	One's complement
+7	111	N.A.	N.A.
+6	110	N.A.	N.A.
+5	101	N.A.	N.A.
+4	100	N.A.	N.A.
+3	011	011	011
+2	010	010	010
+1	001	001	001
+0	000	000	000
-0	N.A.	100	111
-1	N.A.	101	110
-2	N.A.	110	101
-3	N.A.	111	100
-4	N.A.	N.A.	N.A.
-5	N.A.	N.A.	N.A.
-6	N.A.	N.A.	N.A.
-7	N.A.	N.A.	N.A.

Example

With n = 5 bits and using one's complement:

- How is represented X = 5?
- How is represented X = -5?
- What is the value of 00111 in 1's complement?
- What is the value of 11000 in 1's complement?

Example (solution)

With n = 5 bits and using one's complement:

- How is represented X = 5?
  - Because is positive then is like (pure) binary
    - 00101
- How is represented X = -5?
  - Because is negative, then 5 is complemented to one (00101)
    - 11010
- What is the value of 00111 in 1's complement?
  - Because is positive then its value is 7
- What is the value of 11000 in 1's complement?
  - Because is negative, then is complemented and is 00111 (7)
    - The value is -7

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Two's complement (complement to the base) [integer] (1/3)

- **Positive number**:
  is represented in pure binary with n-1 bits

n-1	n-2 ... 0
0	Magnitude

$$V(X) = \sum_{i=0}^{n-1} 2^i \cdot x_i = \sum_{i=0}^{n-2} 2^i \cdot x_i$$

- Representation range (+): $[0, 2^{n-1} - 1]$
- Resolution: 1 unit

Two's complement (complement to the base) [integer] (2/3)

- **Negative number**:
  - Complemented to the base.
  - The number $X < 0$ is represented as $2^n - X$ with n bits

n-1	n-2 … 0
1	2'C of the magnitude

$$V(X) = -2^n + \sum_{i=0}^{n-1} 2^i \cdot y_i$$

- Representation range (-): $[-2^{n-1}, -1]$
- Resolution: 1 unit

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Two's complement (complement to the base) [integer] (3/3)

Tip: $C\,a\,2\,(X) = X$

$C\,a\,2\,(-X) = C\,a\,1\,(X) + 1$

- Example: For n=4 $\Rightarrow$ $+3 = 0011_2$
- Example: For n=4 $\Rightarrow$ $-3 = 1101_2$
  - 1 $\Rightarrow$ - (sign bit and also part of magnitude)
  - $C\,a\,2\,(3) = C\,a\,2(0011_2) = 2^4 - 3 = 13 \Rightarrow 1101_2$

- Representation range: $[-2^{n-1}, 2^{n-1}-1]$
- Resolution: 1 unit
- 0 has only one representation ($\nexists$ -0)
- Asymmetric range

Comparative example (3 bits)

Decimal	Pure Binary	Sign-Magnitude	One's complement	Two's complement
+7	111	N.A.	N.A.	N.A.
+6	110	N.A.	N.A.	N.A.
+5	101	N.A.	N.A.	N.A.
+4	100	N.A.	N.A.	N.A.
+3	011	011	011	011
+2	010	010	010	010
+1	001	001	001	001
+0	000	000	000	000
-0	N.A.	100	111	N.A.
-1	N.A.	101	110	111
-2	N.A.	110	101	110
-3	N.A.	111	100	101
-4	N.A.	N.A.	N.A.	100
-5	N.A.	N.A.	N.A.	N.A.
-6	N.A.	N.A.	N.A.	N.A.
-7	N.A.	N.A.	N.A.	N.A.

Two's complement with 32-bits

$0000 \ldots 0000\ 0000\ 0000\ 0000_{2c} = 0_{(10}$
$0000 \ldots 0000\ 0000\ 0000\ 0001_{2c} = 1_{(10}$
$0000 \ldots 0000\ 0000\ 0000\ 0010_{2c} = 2_{(10}$
. . .
$0111 \ldots 1111\ 1111\ 1111\ 1101_{2c} = 2,147,483,645_{(10}$
$0111 \ldots 1111\ 1111\ 1111\ 1110_{2c} = 2,147,483,646_{(10}$
$0111 \ldots 1111\ 1111\ 1111\ 1111_{2c} = 2,147,483,647_{(10}$

$1000 \ldots 0000\ 0000\ 0000\ 0000_{2c} = -2,147,483,648_{(10}$
$1000 \ldots 0000\ 0000\ 0000\ 0001_{2c} = -2,147,483,647_{(10}$
$1000 \ldots 0000\ 0000\ 0000\ 0010_{2c} = -2,147,483,646_{(10}$
. . .
$1111 \ldots 1111\ 1111\ 1111\ 1101_{2c} = -3_{(10}$
$1111 \ldots 1111\ 1111\ 1111\ 1110_{2c} = -2_{(10}$
$1111 \ldots 1111\ 1111\ 1111\ 1111_{2c} = -1_{(10}$

Biased $2^{n-1}-1$ representation [integer]

- X value with n bits is represented as $X + 2^{n-1}-1$
- Bias refers to the value $2^{n-1}-1$

n-1 0

n bits

$$V(X) = \sum_{i=0}^{n-1} 2^{i} \cdot x_{i} - (2^{n-1}-1)$$

- Representation range: $[-(2^{n-1}-1), 2^{n-1}]$
- Resolution: 1 unit
- There is no ambiguity with 0

Comparative example (3 bits)

Decimal	Pure Binary	Sign-Magnitude	One's complement	Two's complement	Biased-3
+7	111	N.A.	N.A.	N.A.	N.A.
+6	110	N.A.	N.A.	N.A.	N.A.
+5	101	N.A.	N.A.	N.A.	N.A.
+4	100	N.A.	N.A.	N.A.	111
+3	011	011	011	011	110
+2	010	010	010	010	101
+1	001	001	001	001	100
+0	000	000	000	000	011
-0	N.A.	100	111	N.A.	N.A.
-1	N.A.	101	110	111	010
-2	N.A.	110	101	110	001
-3	N.A.	111	100	101	000
-4	N.A.	N.A.	N.A.	100	N.A.
-5	N.A.	N.A.	N.A.	N.A.	N.A.
-6	N.A.	N.A.	N.A.	N.A.	N.A.
-7	N.A.	N.A.	N.A.	N.A.	N.A.

Representations
summary

Name	Pure binary	Sign-magnitude	Ca1	Ca2	Bias 2^{n-1}-1
Represent	Natural	Integer	Integer	Integer	Integer
Sign	All bits for magnitude, no sign	MSB is sign (0 ⇒ + y 1 ⇒ -)	MSB is sign **and magnitude** (0 ⇒ + y 1 ⇒ -)	MSB is sign **and magnitude** (0 ⇒ + y 1 ⇒ -)	
Range	$[0, 2^n-1]$	$[-2^{n-1}+1, 2^{n-1}-1]$	$[-2^{n-1}+1, 2^{n-1}-1]$	$[-2^{n-1}, 2^{n-1}-1]$	$[-(2^{n-1}-1), 2^{n-1}]$
Resolution	1 unit	1 unit	1 unit	1 unit	1 unit
Disadvantage	No negative	+0 y -0	+0 y -0	**A**symmetric range	**A**symmetric range
Advantage		Symmetric range	Symmetric range	(No ∃ -0)	(No ∃ -0)
Tip		Remove first bit and compute pure binary value	+: = pure binary -: switch 1 by 0 and 0 by 1	+: = pure binary -: Ca1 + 1	Subtract bias (2^{n-1}-1)
Value		$V(X) = (1 - 2 \cdot x_{n-1}) \cdot \sum_{i=0}^{n-2} 2^i \cdot x_i$	+: $V(X) = \sum_{i=0}^{n-2} 2^i \cdot x_i$ -: $V(X) = -2^n + \sum_{i=0}^{n-1} 2^i \cdot X_i + 1$	+: $V(X) = \sum_{i=0}^{n-2} 2^i \cdot x_i$ -: $V(X) = -2^n + \sum_{i=0}^{n-1} 2^i \cdot X_i$	$V(X) = \sum_{i=0}^{n-1} 2^i \cdot x_i - (2^{n-1} - 1)$

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Comparative example (3 bits)
summary

Decimal	Pure Binary	Sign-Magnitude	One's complement	Two's complement	Biased-3
+7	111	N.A.	N.A.	N.A.	N.A.
+6	110	N.A.	N.A.	N.A.	N.A.
+5	101	N.A.	N.A.	N.A.	N.A.
+4	100	N.A.	N.A.	N.A.	111
+3	011	011	011	011	110
+2	010	010	010	010	101
+1	001	001	001	001	100
+0	000	000	000	000	011
-0	N.A.	100	111	N.A.	N.A.
-1	N.A.	101	110	111	010
-2	N.A.	110	101	110	001
-3	N.A.	111	100	101	000
-4	N.A.	N.A.	N.A.	100	N.A.
-5	N.A.	N.A.	N.A.	N.A.	N.A.
-6	N.A.	N.A.	N.A.	N.A.	N.A.
-7	N.A.	N.A.	N.A.	N.A.	N.A.

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Example

Indicate the representation of the following numbers, giving a brief justification of your answer:

1. **-32** in one's complement with **6 bits**
2. **-32** in two's complement with **6 bits**
3. **-10** in sign-magnitude with **5 bits**
4. **+14** in two's complement with **5 bits**

Example (solution)

1. With 6 bits **is not representable** in 1C: $[-2^{6-1}+1,\ldots,-0,+0,\ldots\ 2^{6-1}-1]$
2. 1C + 1 -> **100000**
3. Sign=1, magnitude=1010 -> **11010**
4. Positive -> 1C=2C=SM -> **01110**

Contents

Comparison of arithmetic in B, 1C and 2C

	Binary	One's complement	Two' complement
Add	10110 01100 ---------- 100010	same as binary	same as binary
Subtract	10110 01100 ---------- 01010	add and if there is Cn-1 then add Cn-1 to total	add and if there is Cn-1 then discard it

In hardware, it is easier to operate with complement

Comparison of arithmetic in B, 1C and 2C

why add the carry to the result in 1C

- -X is represented as $2^n - X - 1$
- -Y is represented as $2^n - Y - 1$
- -(X + Y) is represented as $2^n - (X+Y) - 1$

- -(X + Y) the operation gives $2^n + 2^n - (X + Y) - 2$ + 1

	Bi		ment
Add			
Subtract	10110 01100 ---------- 01010	add and if there is Cn-1 then add Cn-1 to total	add and if there is Cn-1 then discard it

Correction of the result by adding the carry...

Comparison of arithmetic in B, 1C and 2C

why discard the carry in 2C

	Bi...		...ent
Add			
Subtract	10110 01100 ---------- 01010	add and if there is Cn-1 then add Cn-1 to total	add and if there is Cn-1 then discard it

- -X is represented as $2^n - X$
- -Y is represented as $2^n - Y$
- -(X + Y) is represented as $2^n - (X+Y)$

- -(X + Y) the operation gives $2^n + 2^n - (X + Y)$

Correction of the result by discarding the carry...

Comparison of arithmetic in B, 1C and 2C

	Binary	One's complement	Two' complement
Detect overflow	The result needs 1 bit more There are Cn	Adding ++ is –, Adding – – is + Cn <> Cn-1	Adding ++ is –, Adding – – is + Cn <> Cn-1
Sign extension	0…0 10110	1…1 10110 0…0 00110	1…1 10110 0…0 00110
…	…	…	…

Example

- Using 5 bits, compute the followingg additions in 1's complement:
  a) 4 +12
  b) 4 -12
  c) -4 -12

Example (solution)

By using 5 bits in 1's complement

a) 4 +12

```
00100
01100
--------
10000 ⇒ -15 ⇒ negative! ⇒ overflow
```

b) 4 - 12

```
00100
10011
--------
10111 ⇒ -8
```

c) -4 - 12

```
 11011
 10011
 --------
101110 ⇒ 6 bits are needed ⇒ overflow
```

Contents

More representation necessities…

- How to represent?
  - Very large numbers: $30.556.926.000_{(10}$
  - Very small numbers: $0.0000000000529177_{(10}$
  - Fractional numbers: 1.58567

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Reminder
Example of failure…

- **Ariane 5** explosion (first flight)
  - Sent by ESA in June 1996
  - Cost of development:
    10 years and 7 billion dollars
  - Exploded 40 seconds after launch, at 3700 meters altitude.
  - Failure due to total loss of altitude information:
    - The inertial reference system software performed the conversion of a 64-bit floating point real value to a 16-bit integer value.
    - The number to be stored was greater than 32767 (the largest 16-bit signed integer) and a conversion failure and exception occurred.

http://www.rvs.uni-bielefeld.de/lectures/TechInf/TI2/download/ariane_pics.html

Fixed point [racionals]

- The position of the binary point is fixed and the weights associated with the decimal places are used.

- Example:

$$1001.1010 = 2^4 + 2^0 + 2^{-1} + 2^{-3} = 9.625$$

Fractional values in binary with fixed point

- Example with 6 bits:

xx.yyyy

2^{1} 2^{0} 2^{-1} 2^{-2} 2^{-3} 2^{-4}

- Example:
  $10{,}1010_{(2} = 1\text{x}2^{1} + 1\text{x}2^{-1} + 1\text{x}2^{-3} = 2.625_{10}$

- Using this fixed point, the range is:
  - [0 a 3.9375 (almost 4)]

Fractional powers of 2

i	2^{-i}	
0	1.0	1
1	0.5	1/2
2	0.251/4	
3	0.125	1/8
4	0.0625	1/16
5	0.03125	1/32
6	0.015625	
7	0.0078125	
8	0.00390625	
9	0.001953125	
10	0.0009765625	

Contents

Floating-point numbers

$$\underline{9.12}_{10} \times 10^{\underline{25}}$$

mantissa

radix (base)

exponent

- Each number has a mantissa and an $^{\text{exponent}}$
- Scientific notation (in decimal): normalized form
  - Only one digit different to 0 on the left of decimal point

- The number is adapted to the order of magnitude of the value to be represented, by translating the *decimal point* by using the exponent

Scientific notation in binary

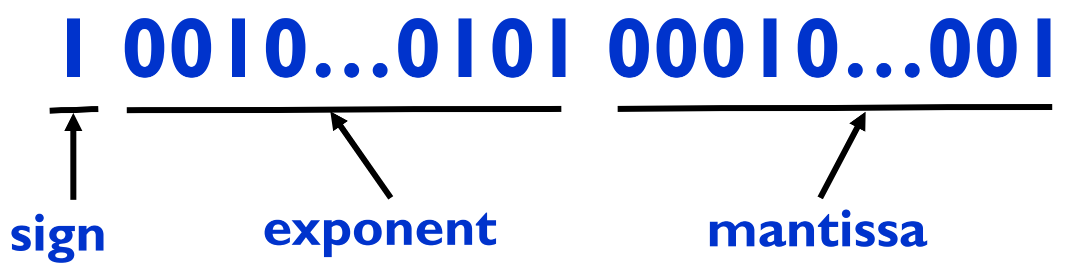
mantissa　　　　　　　　　　exponent

$$1.0_{(2} \times 2^{-1}$$

binary point　　　　　　　　radix (base)

- Normalized form:
  One 1 (only one digit) in the left of the binary point
  - Normalized: 1.0001×2^{-9},
  - Not normalized: 0.0011×2^{-8}, 10.0×2^{-10}

IEEE 754 Floating Point Standard [rationals]



- Floating point standard used in most computers.
- **Characteristics** (unless special cases):
  - Exponent: excess-k with bias $k = 2^{\text{num\_bits\_in\_exponent - 1}} - 1$
  - Mantissa: sign-magnitude, <u>normalized</u>, with <u>implicit bit</u>
- Different **formats**:
  - **Single precision**: 32 bits (sign: 1, exponent: 8, mantissa: 23 and bias: 127)
  - **Double precision**: 64 bits (sign: 1, exponent: 11, mantissa: 52 and bias: 1023)
  - **Quad-precision**: 128 bits (sign: 1, exponent: 15, mantissa: 112 and bias: 16383)

Normalization and implicit bit

- **Normalization**

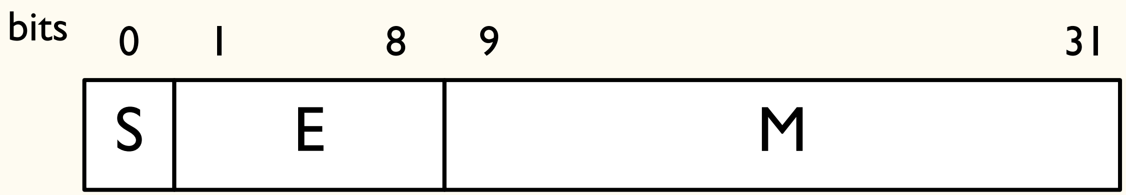
  In order to normalize the mantissa, the exponent is adjusted to have a most significant bit of value 1
  - Example: 100100000000000000000000 x 2^3 (already normalized)
  - Example: 00010000000000010101 x 2^3 (is not)

    10000000000010101000 x 2^0 (now it is)

- **Implicit bit**

  Once normalized, since the most significant bit is 1, it is **not** stored to leave space for one more bit (increases accuracy).
  - This makes it possible to represent mantissa with one bit more

IEEE Standard 754 (single precision)



S: sign (1 bit)
E: exponent (8 bits)
M: mantissa (23 bits)

- The value is computed (unless special cases) as:

$$\mathbf{N = (-1)^{S} \times 2^{E-127} \times 1.M}$$

where:

S = 0 for positive numbers, S =1 for negative numbers
0 < E < 255 (E=0 y E=255 are special cases)
$00000000000000000000000 \leq M \leq 11111111111111111111111$

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IEEE Standard 754 (single precision) [rationals]

- Special cases:

$(-1)^s \times 0.\text{mantissa} \times 2^{-126}$

Exponent	Mantissa	Special value
0 (0000 0000)	0	+/- 0 (depends on sign)
0 (0000 0000)	≠ 0	Number NOT normalized
255 (1111 1111)	≠ 0	NaN (0/0, sqrt(-4), ….)
255 (1111 1111)	0	+/- infinite (depends on sign)
1-254	Any	Normalized number (no special)

$(-1)^s \times 1.\text{mantissa} \times 2^{\text{exponent}-127}$

Examples

S	E	M	N
1	00000000	00000000000000000000000	-0 (Exception 0) E=0 y M=0.
1	01111111	00000000000000000000000	$-2^0 \times 1.0_2 = -1$
0	10000001	11100000000000000000000	$+2^2 \times 1.111_2 = +2^2 \times (2^0+2^{-1}+2^{-2}+2^{-3}) = +7.5$
0	11111111	00000000000000000000000	∞ (Exception ∞) E=255 y M=0
0	11111111	10000000000000000000001	NaN (Not a Number) E=255 y $M \neq 0$.

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Example

a) Calculate the value in decimal associated to this number
0 10000011 11000000000000000000000
represented in IEEE 754 single precision

Example (solution)

a) Calculate the value in decimal associated to this number
0 10000011 11000000000000000000000
represented in IEEE 754 single precision

a) Sign bit: 0 $\Rightarrow (-1)^0 = +1$
b) Exponent: $10000011_2 = 131_{10} \Rightarrow E - 127 = 131 - 127 = 4$
c) Mantissa: 11000000000000000000000 $\Rightarrow 1 \times 2^{-1} + 1 \times 2^{-2} = 0.75$

The decimal value is $+1 \times 2^4 \times 1.75 = +28$

Exercise

b) Represent the number -9 using IEEE 754 single precision

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Exercise (Solution)

b) Represent the number -9 using IEEE 754 single precision

$-9_{10} = -1001_2 = -1001_2 \times 2^0 = -1.001_2 \times 2^3$ (normalized mantissa)

a) Sign: negative ⇨ S=1
b) Exponent: 3+127 (bias) = 130 ⇨ 10000010
c) Mantissa: 1,001 (impl. bit) ⇨ 00100000000000000000000

-9 is represented by 1 10000010 00100000000000000000000

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IEEE Standard 754 (single precision) [rationals]

- Range of representable magnitudes (regardless of sign):
  - Smallest normalized:
    $2^{1-127} \times 1.00000000000000000000000_2$
  - Largest normalized:
    $2^{254-127} \times 1.11111111111111111111111_2$
  - Smallest not normalized :
    $2^{-126} \times 0.00000000000000000000001_2$
  - Largest not normalized :
    $2^{-126} \times 0.11111111111111111111111_2$

$(-1)^s * 0.\text{mantisa} * 2^{-126}$

Exponent	Mantissa	Special value
0	$\neq 0$	Not normalized
1-254	any	Normalized

$(-1)^s * 1.\text{mantisa} * 2^{\text{exponente}-127}$

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IEEE Standard 754 (single precision) [rationals]

- Range of representable magnitudes (regardless of sign):
  - Smallest normalized:
    $2^{1-127} \times 1.00000000000000000000000_2 = 2^{-126}$
  - Largest normalized:
    $2^{254-127} \times 1.11111111111111111111111_2 = 2^{127} \times (2 - 2^{-23}) = 2^{128} \times (1 - 2^{-24})$
  - Smallest not normalized :
    $2^{-126} \times 0.00000000000000000000001_2 = 2^{-149}$
  - Largest not normalized :
    $2^{-126} \times 0.11111111111111111111111_2 = 2^{-126} \times (1 - 2^{-23})$

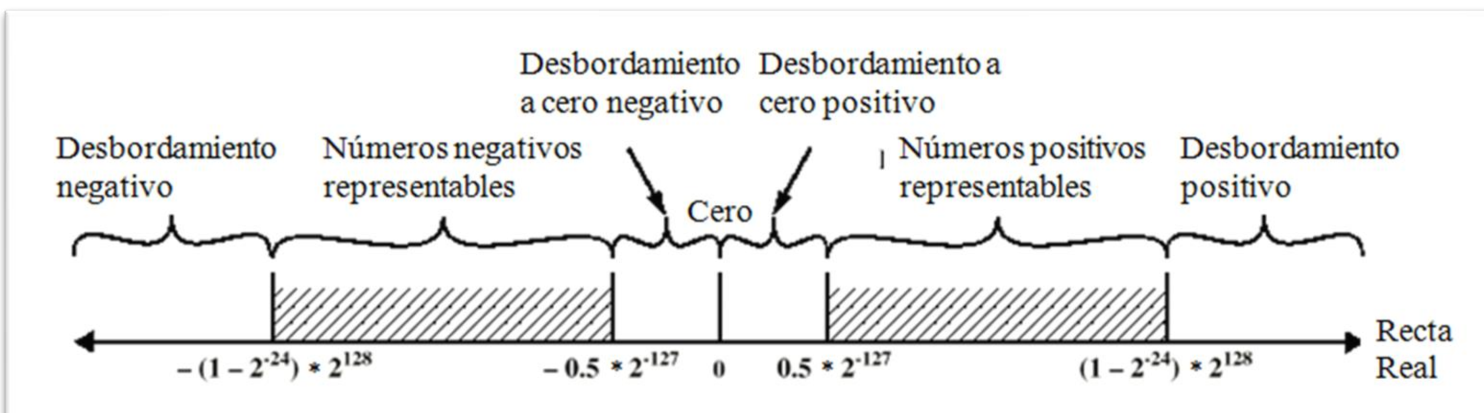
Tip:

$$
\begin{array}{rrcl}
 & 1.11111111111111111111111_2 & = & X \\
+ & 0.00000000000000000000001_2 & = & 2^{-23} \\
\hline
 & 10.00000000000000000000000_2 & = & 2
\end{array}
$$

$$X = 2 - 2^{-23}$$

IEEE Standard 754 (single precision) [rationals]

- Range of representable magnitudes (regardless of sign):
  - Smallest normalized:
    $2^{1-127} \times 1.00000000000000000000000_2 = 2^{-126} = 2^{-127} \times 0.5$
  - Largest normalized:
    $2^{254-127} \times 1.11111111111111111111111_2 = 2^{127} \times (2 - 2^{-23}) = 2^{128} \times (1 - 2^{-24})$
  - Smallest not normalized :
    $2^{-126} \times 0.00000000000000000000001_2 = 2^{-149}$
  - Largest not normalized :
    $2^{-126} \times 0.11111111111111111111111_2 = 2^{-126} \times (1 - 2^{-23})$



Exercise

- How many *floats* (single precision floating point numbers) are between 1 and 2 (not included)?

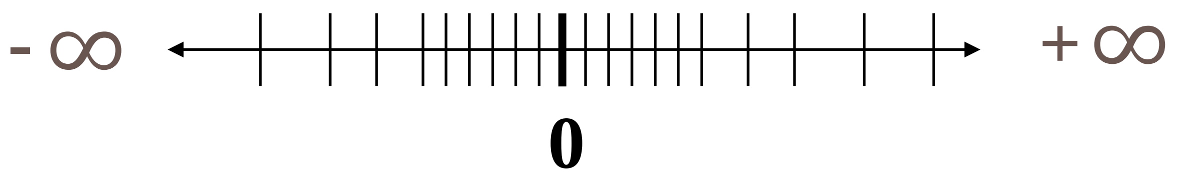
- How many *float* (single precision floating point numbers) are between 2 and 3 (not included)?

Exercise (Solution)

- How many *floats* (single precision floating point numbers) are between 1 and 2 (not included)?
  - $1 = 1.00000000000000000000000 \times 2^0$
  - $2 = 1.00000000000000000000000 \times 2^1$
  - Between 1 and 2 there are 2^{23} numbers
- How many *float* (single precision floating point numbers) are between 2 and 3 (not included)?
  - $2 = 1.00000000000000000000000 \times 2^1$
  - $3 = 1.10000000000000000000000 \times 2^1$
  - Between 2 and 3 there are 2^{22} numbers

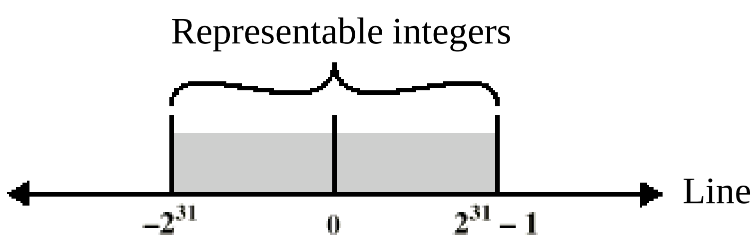
Discrete representation

- Variable resolution:
  denser near zero, less towards infinity

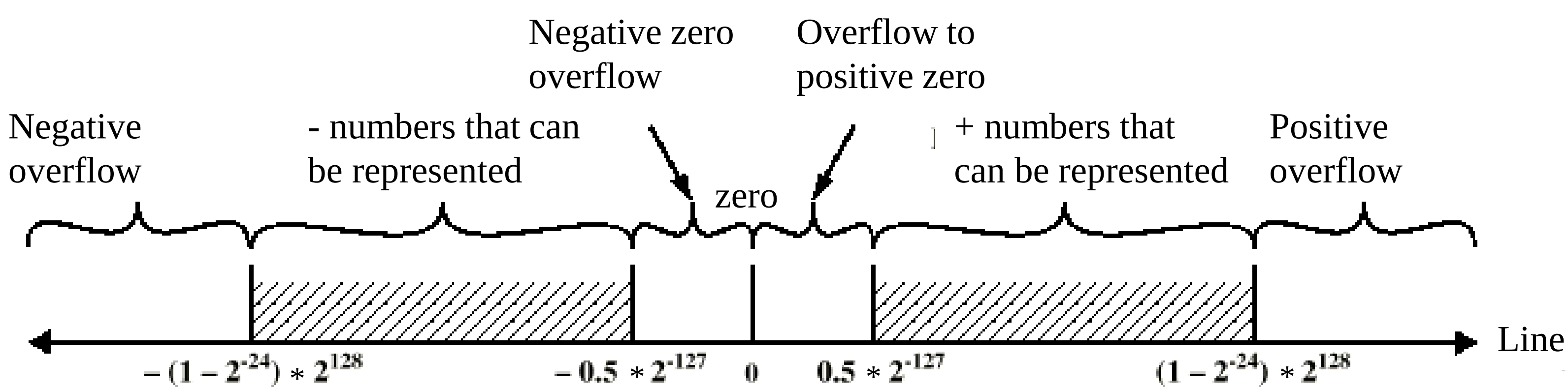


https://www.hobbyconsolas.com/reportajes/toy-story-espacio-disney-infinity-58154

Representable numbers



(a) Two's complement integers



(b) Floating point numbers

www.math.unipd.it/~frossi/cap9-2.pdf

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Example 1
inaccuracy

0.4 →

0	01111101	10011001100110011001101

3.9999998 e-1

0.1 →

0	01111011	10011001100110011001100

9.9999994 e-2

Example 2
inaccuracy

- How does C performs a division?

t2.c

```
#include <stdio.h>

int main ( )
{
   float a ;

   a = 3.0/7.0 ;
   if (a == 3.0/7.0)
          printf("Equal\n") ;
   else printf("Not equal\n") ;
   return (0) ;
}
```


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Example 2
inaccuracy

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   else printf("Not equal\n") ;
   return (0) ;
}
```

```
$ gcc -o t2 t2.c
$ ./t2
Not equal
```


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Example 2
inaccuracy

- How does C performs a division?

t2.c

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#include <stdio.h>

int main ( )
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   float a ;

   a = 3.0/7.0 ;
   if (a == 3.0/7.0)
          printf(“Equal\n") ;
   else printf("Not equal\n") ;
   return (0) ;
}
```

float

double

```
$ gcc -o t2 t2.c
$ ./t2
Not equal
```

Example 3
inaccuracy

- The associative property is not always satisfied

¿ a + (b + c) = (a + b) + c ?

t1.c

```
#include <stdio.h>

int main ( )
{
      float x, y, z ;

      x = 10e30;   y = -10e30;   z = 1;
      printf("(x+y)+z = %f\n",(x+y)+z) ;
      printf("x+(y+z) = %f\n",x+(y+z)) ;

      return (0) ;
}
```

Example 3
inaccuracy

- The associative property is not always satisfied

  ¿ a + (b + c) = (a + b) + c ?

t1.c

```
#include <stdio.h>

int main ( )
{
      float x, y, z ;

      x = 10e30;   y = -10e30;   z = 1;
      printf("(x+y)+z = %f\n",(x+y)+z) ;
      printf("x+(y+z) = %f\n",x+(y+z)) ;

      return (0) ;
}
```

```
$ gcc -o t1 t1.c
$ ./t1
(x+y)+z = 1.000000
x+(y+z) = 0.000000
```

Floating-point is not associative

- Floating-point is not associative
  - $x = -1.5 \times 10^{38}$, $y = 1.5 \times 10^{38}$, $z = 1.0$
  - $x + (y + z) = -1.5 \times 10^{38} + (1.5 \times 10^{38} + 1.0)$
    $= -1.5 \times 10^{38} + (1.5 \times 10^{38}) = 0.0$
  - $(x + y) + z = (-1.5 \times 10^{38} + 1.5 \times 10^{38}) + 1.0$
    $= (0.0) + 1.0 = 1.0$
- Floating point operations are not associatives
  - Results are approximated
  - 1.5×10^{38} is so much larger than 1.0
  - $1.5 \times 10^{38} + 1.0$ in floating point representation is still 1.5×10^{38}

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Example

int → float → int

```
if (i == (int)((float) i)) {
      printf("true");
}
```

- Not always prints `"true"`
- Most integer values (specially larger ones) don't have an exact floating point representation
- What about `double`?

Example

- The number 133000405 in binary is:
  - 111111011010110110011010101 (27 bits)
- $111111011010110110011010101 \times 2^{0}$
- When is normalized:
  - $1.11111011010110110011010101 \times 2^{26}$
  - S = 0 (positive)
  - $e = 26 \rightarrow \quad E = 26 + 127 = 153$
  - M = 11111011010110110011010 (last 3 bits are lost)
- The normalized number stored is:
  - $1.11111011010110110011010 \times 2^{26} =$
  - $111111011010110110011010 \times 2^{3} = 133000400$

Example

float → int → float

```
if (f == (float)((int) f)) {
    printf("true");
}
```

- Not always true
- Numbers with decimals do not have integer representation

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Rounding

- Rounding removes less significant digits from a number to obtain an approximate value.

- Types of rounding:
  - Round to + ∞
    - Round it "up": 2.001 → 3, -2.001 → -2
  - Round to - ∞
    - Round it "down": 1.999 → 1, -1.999 → -2
  - Truncate
    - Discard last bits: 1.299 → 1.2
  - Round to nearest (ties to even)
    - 2.4 → 2, 2.6 → 3, -1.4 → -1
    - If number falls midway then it is rounded to the nearest value with an even least significant digit (+23.5 → +24 ← +24.5; −23.5 → −24 ← −24.5)

Rounding

- Rounding means losing accuracy.
- Rounding occurs:
  - When moving to a representation with fewer representables:
    - E.g.: A value from double to single precision
    - E.g.: A floating point value to integer
  - When performing arithmetic operations:
    - E.g.: After adding two floating-point numbers (using guard bits)

Guard bits

- Guard digits are used to improve accuracy:
  - FP hardware internally includes additional bits for operations
  - After operation, guard bits are eliminated: rounding

- Example: $2.65 \times 10^0 + 2.34 \times 10^2$

	WITHOUT guard bits	WITH guard bits
1.- equalize exponents	0.02×10^2 $+ 2.34 \times 10^2$	0.0265×10^2 $+ 2.3400 \times 10^2$
2.- add	2.36×10^2	2.3665×10^2
3.- round	2.36×10^2	2.37×10^2

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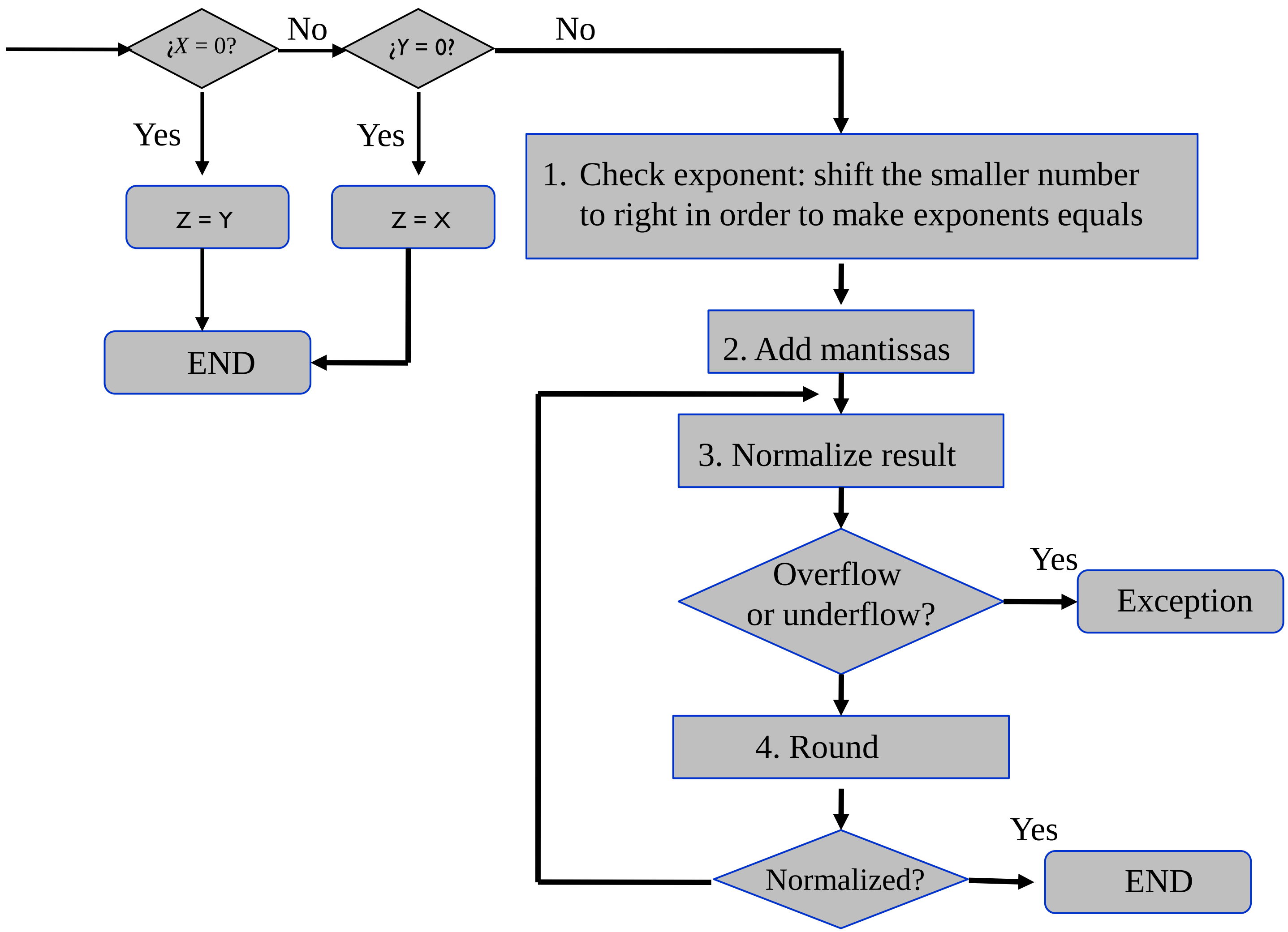
Floating point operations

- Add
- Subtract
  1. Check zero values.
  2. Equalize exponents (shift smaller number to the right).
  3. Add/subtract mantissa.
  4. Normalize the result.

- Multiply
- Divide
  1. Check zero values.
  2. Add/subtract exponents.
  3. Multiply/divide mantissa (taking into account the sign).
  4. Normalize the result.
  5. Rounding the result.

Additions and subtractions: Z=X+Y y Z=X-Y



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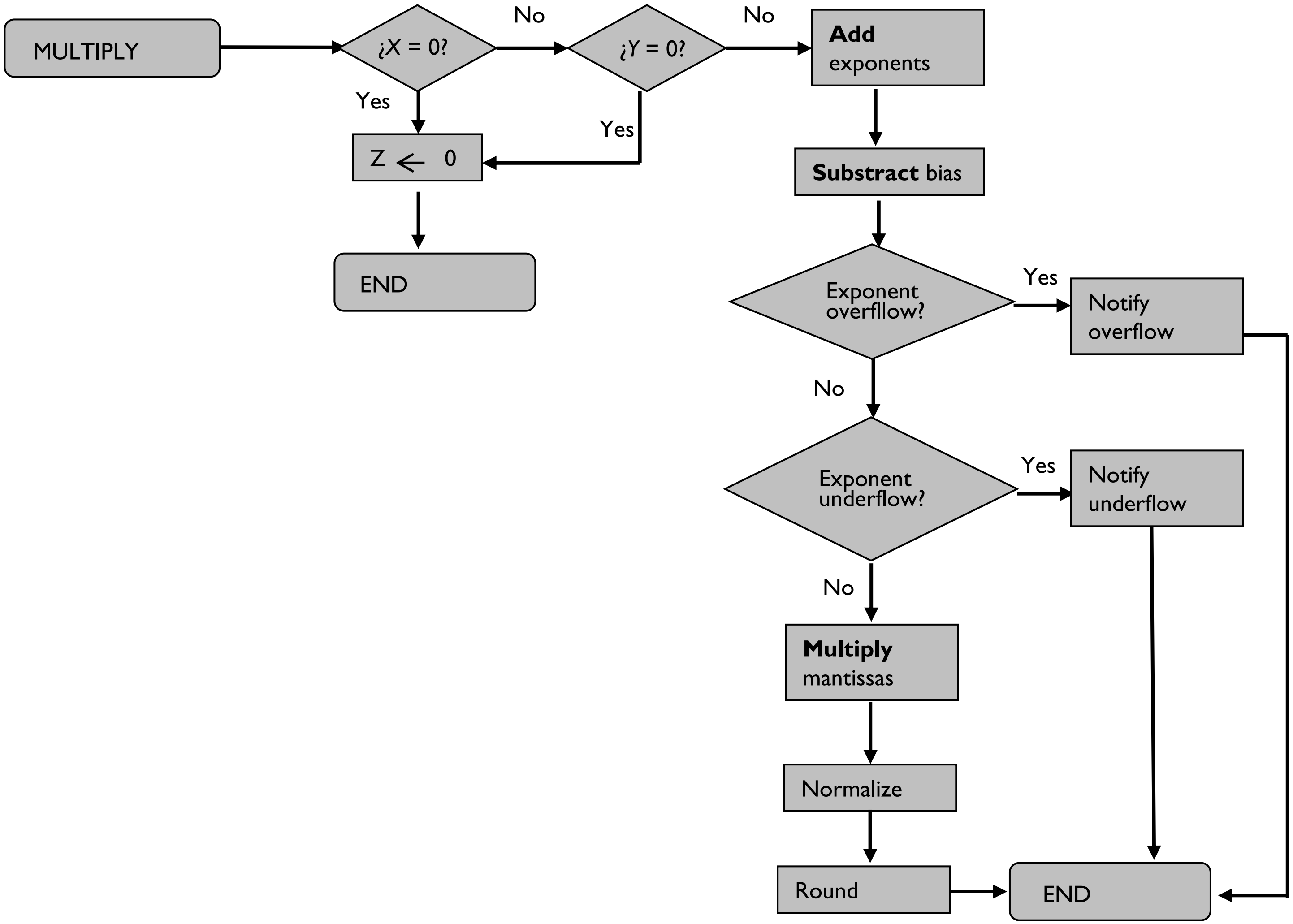
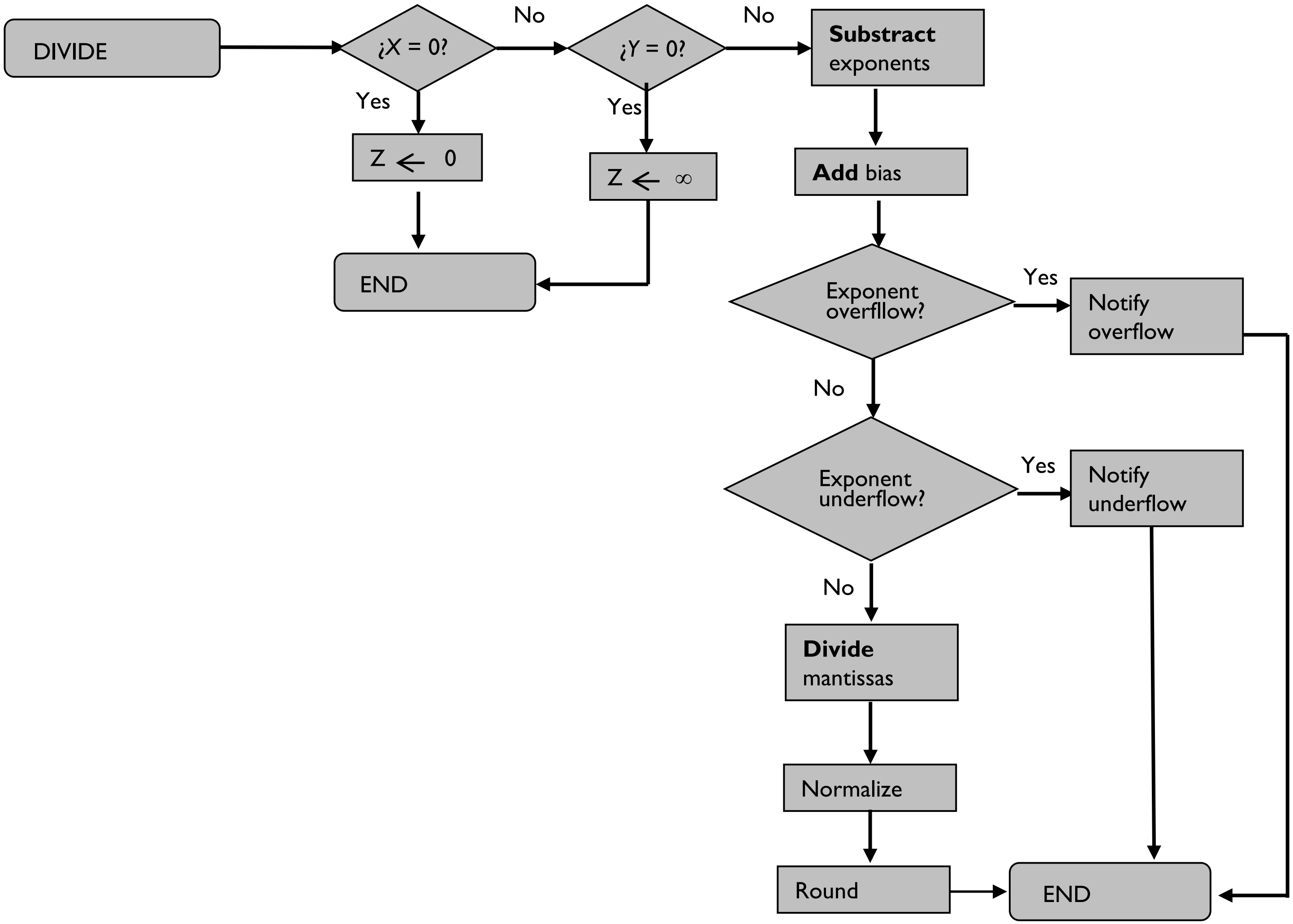
Multiplication: Z=X*Y



Division: Z=X/Y



Exercise

- Using the IEEE 754 format, add 7.5 and 1.5 step by step.

Solution

1) $7.5 + 1.5 =$
2) $1.111*2^2 + 1.1*2^0 =$
3) $1.111*2^2 + 0.011*2^2 =$
4) $10.010*2^2 =$
5) $1.0010*2^3$

To binary

Equalize exponents

Add

Adjust exponents

Solution

- Representation of the numbers

7.5 → `0 10000001 11100000000000000000000`

\+ **1.5** → `0 01111111 10000000000000000000000`

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Solution

- Splitting exponents and mantissas, and adding implicit bit

7.5 →	0	10000001	1.11100000000000000000000
+ **1.5** →	0	01111111	1.10000000000000000000000

Solution

- Equalize exponents

7.5 →	0	10000001	1.11100000000000000000000
+ **1.5** →	0	01111111	1.10000000000000000000000

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Solution

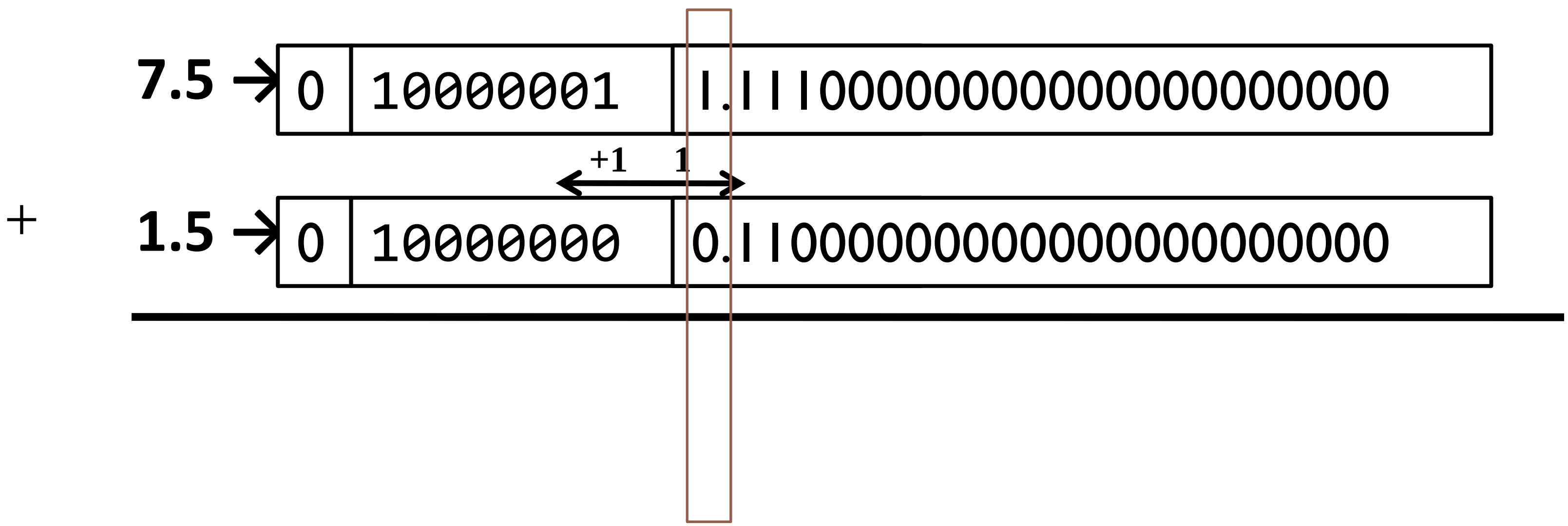
- Equalize exponents

	Sign	Exponent	Mantissa
7.5 →	0	10000001	1.11100000000000000000000
+ **1.5** →	0	01111111	1.10000000000000000000000

+1 1

Solution

- Equalize exponents



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Solution

- Equalize exponents

	Sign	Exponent	Mantissa
7.5 →	0	10000001	1.11100000000000000000000
		+1	1
+ **1.5** →	0	10000001	0.01100000000000000000000

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Solution

- Add mantissas

7.5 →	0	10000001	1.11100000000000000000000
+ **1.5** →	0	10000001	0.01100000000000000000000

Solution

- Normalize result…

7.5 →	0	10000001	1.11100000000000000000000
+ 1.5 →	0	10000001	0.01100000000000000000000
9 →	0	10000001	10.01000000000000000000000

There is carry,
non-normalized mantissa

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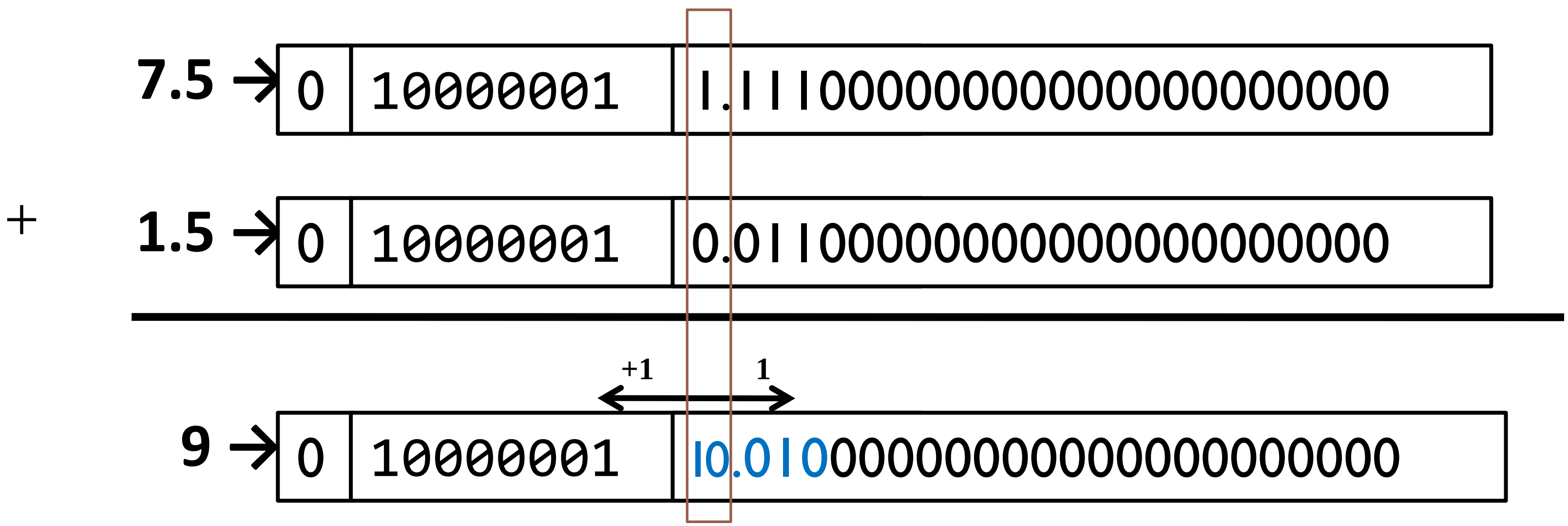
Solution

- Normalize result…



There is carry,
non-normalized mantissa

Solution

	Sign	Exponent	Mantissa
7.5 →	0	10000001	1.11100000000000000000000
+ 1.5 →	0	10000001	0.01100000000000000000000
9 →	0	10000010	1.00100000000000000000000

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Solution

- Eliminate the implicit bit and store the result

9 → `0 10000010 00100000000000000000000`

Exercise

- Using the IEEE 754 format, multiply 7.5 and 1.5 step by step.

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Solution

- Representation of the numbers

7.5 → 0 10000001 11100000000000000000000

1.5 → 0 01111111 11000000000000000000000

Solution

- Splitting exponents and mantissas, and adding implicit bit

7.5 →	0	10000001	1.11100000000000000000000
X **1.5** →	0	01111111	1.10000000000000000000000

The implicit bit is included

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Solution

- Multiply: add exponents and multiply mantissas

7.5 →	0	10000001	1.11100000000000000000000
X **1.5** →	0	01111111	1.10000000000000000000000
		+	×
	0	100000000	10.11010000000000000000000

Solution

- Multiply: remove one bias from exponent (there are two)

	7.5 →	0	10000001	1.11100000000000000000000
X	1.5 →	0	01111111	1.10000000000000000000000
		0	100000000	10.11010000000000000000000
			- 01111111	
		0	10000001	10.11010000000000000000000

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Solution

- Multiply: normalize result…

7.5 →	0	10000001	1.11100000000000000000000
X 1.5 →	0	01111111	1.10000000000000000000000
		+1	1
11.25	0	10000001	10.11010000000000000000000

Solution

- Multiply: normalize result…

		Sign	Exponent	Mantissa
	7.5 →	0	10000001	1.11100000000000000000000
X	1.5 →	0	01111111	1.10000000000000000000000
	11.25	0	10000010	1.011010000000000000000000

Solution

- Eliminate the implicit bit and store the result

11.25 0 10000010 01101000000000000000000

IEEE 754 Evolution

- 1985 – IEEE 754
- 2008 – IEEE 754-2008 (754+854)
- 2011 – ISO/IEC/IEEE 60559:2011 (754-2008)

Name	Common name	Base	Digits	E min	E max	Notes	Decimal digits	Decimal E max
binary16	Half precision	2	10+1	−14	+15	storage, not basic	3.31	4.51
binary32	Single precision	2	23+1	−126	+127		7.22	38.23
binary64	Double precision	2	52+1	−1022	+1023		15.95	307.95
binary128	Quadruple precision	2	112+1	−16382	+16383		34.02	4931.77
decimal32		10	7	−95	+96	storage, not basic	7	96
decimal64		10	16	−383	+384		16	384
decimal128		10	34	−6143	+6144		34	6144

http://en.wikipedia.org/wiki/IEEE_floating_point

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L3: Representation of information

Computer Structure

Bachelor in Computer Science and Engineering
Bachelor in Applied Mathematics and Computing
Dual Bachelor in Computer Science and Engineering and Business Administration