# The Newtonian approximation in Causal Dynamical Triangulations

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May 16, 2013

## Contents



# 1 Newton's Law of Gravitation from General Relativity

This treatment follows that of Katz [1].

## 1.1 Vacuum solution to the Weyl metric

Starting from the cylindrically symmetric (Weyl) vacuum metric [2]:

$$ds^2 = e^{2\lambda}dt^2 - e^{2(\nu-\lambda)}\left(dr^2 + dz^2\right) - r^2 e^{-2\lambda} d\phi^2 \tag{1}$$

$$g_{\mu\nu} = \begin{pmatrix} e^{2\lambda}dt^2 & 0 & 0 & 0 \\ 0 & -e^{2(\nu-\lambda)}dr^2 & 0 & 0 \\ 0 & 0 & -e^{2(\nu-\lambda)}dz^2 & 0 \\ 0 & 0 & 0 & -\frac{r^2}{e^{2\lambda}}d\phi^2 \end{pmatrix} \tag{2}$$

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In this coordinate basis, the definition of the Christoffel connection is: [3]

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left( \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \tag{3}$$

The non-zero Christoffel connections are:

$$\begin{aligned}
\Gamma^{t}_{tr} &= \partial_r \lambda \\
\Gamma^{t}_{tz} &= \partial_z \lambda \\
\Gamma^{r}_{tt} &= e^{4\lambda - 2\nu} \partial_r \lambda \\
\Gamma^{r}_{rr} &= \partial_r \nu - \partial_r \lambda \\
\Gamma^{r}_{rz} &= \partial_z \nu - \partial_z \lambda \\
\Gamma^{r}_{zz} &= \partial_r \lambda - \partial_r \nu \\
\Gamma^{r}_{\phi\phi} &= r e^{-2\nu} \left( r \partial_r \lambda - 1 \right) \\
\Gamma^{z}_{tt} &= e^{4\lambda - 2\nu} \partial_z \lambda \\
\Gamma^{z}_{rr} &= \partial_z \lambda - \partial_z \nu \\
\Gamma^{z}_{rz} &= \partial_r \nu - \partial_r \lambda \\
\Gamma^{z}_{zz} &= \partial_z \nu - \partial_z \lambda \\
\Gamma^{z}_{\phi\phi} &= r^2 e^{-2\nu} \partial_z \lambda \\
\Gamma^{\phi}_{r\phi} &= \frac{1}{r} - \partial_r \lambda \\
\Gamma^{\phi}_{z\phi} &= -\partial_z \lambda
\end{aligned} \tag{4}$$

The components of the Riemann tensor are given by:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma} \tag{5}$$

Using the properties of the Riemann tensor:

$$\begin{aligned}
R_{\rho\sigma\mu\nu} &= -R_{\rho\sigma\nu\mu} \\
R_{\rho\sigma\mu\nu} &= -R_{\sigma\rho\mu\nu} \\
R_{\rho\sigma\mu\nu} &= R_{\mu\nu\rho\sigma} \\
R_{\rho[\sigma\mu\nu]} &= 0
\end{aligned} \tag{6}$$

The non-zero components of the Riemann tensor are:

$$
\begin{aligned}
R^{t}_{rtr} &= -\partial_r^2\lambda + (\partial_z\lambda)^2 - 2(\partial_r\lambda)^2 + \partial_r\lambda\partial_r\nu - \partial_z\lambda\partial_z\nu \\
R^{t}_{rtz} &= -\partial_r\partial_z\lambda - 3\partial_r\lambda\partial_z\lambda + \partial_r\lambda\partial_z\nu + \partial_r\nu\partial_z\lambda \\
R^{t}_{ztz} &= -\partial_z^2\lambda - 2(\partial_z\lambda)^2 + (\partial_r\lambda)^2 - \partial_r\lambda\partial_r\nu + \partial_z\lambda\partial_z\nu \\
R^{t}_{\phi t\phi} &= re^{-2\nu}\left(r(\partial_r\lambda)^2 - \partial_r\lambda + r(\partial_z\lambda)^2\right) \\
R^{r}_{zrz} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu \\
R^{z}_{\phi z\phi} &= re^{-2\nu}\left(r\partial_z^2\lambda - r\partial_z\lambda\partial_z\nu + r\partial_r\lambda\partial_r\nu - r(\partial_r\lambda)^2 + \partial_r\lambda - \partial_r\nu\right) \\
R^{z}_{\phi\phi r} &= re^{-2\nu}\left(-r\partial_r\partial_z\lambda + r\partial_r\nu\partial_z\lambda - r\partial_r\lambda\partial_z\lambda + r\partial_r\lambda\partial_z\nu - \partial_z\nu\right) \\
R^{\phi}_{r\phi r} &= \partial_r^2\lambda + \frac{1}{r}\partial_r\nu - \partial_r\lambda\partial_r\nu - (\partial_z\lambda)^2 + \partial_z\lambda\partial_z\nu + \frac{1}{r}\partial_r\lambda
\end{aligned}
\tag{7}
$$

The Ricci tensor is given by:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \tag{8}$$

The non-zero components of the Ricci tensor are:

$$
\begin{aligned}
R_{tt} &= \frac{e^{4\lambda-2\nu}}{r}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right) \\
R_{rr} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2(\partial_r\lambda)^2 + \frac{1}{r}\partial_r\lambda + \frac{1}{r}\partial_r\nu \\
R_{rz} &= \frac{1}{r}\partial_z\nu - 2\partial_r\lambda\partial_z\lambda \\
R_{zz} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2(\partial_z\lambda)^2 + \frac{1}{r}\partial_r\lambda - \frac{1}{r}\partial_r\nu \\
R_{\phi\phi} &= re^{-2\nu}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right)
\end{aligned}
\tag{9}
$$

Einstein's equation in a vacuum is:

$$R_{\mu\nu} = 0 \tag{10}$$

Applying this complete set of relations to Equation (9) gives the following:

$$\partial_r^2\lambda + \frac{1}{r}\partial_r\lambda + \partial_z^2\lambda = 0 \tag{11}$$

$$\partial_r\nu = r\left(\partial_r^2\nu + \partial_z^2\nu + 2(\partial_r\lambda)^2\right) \tag{12}$$

$$\partial_z\nu = 2r\partial_r\lambda\partial_z\lambda \tag{13}$$

$$\partial_r^2\nu + \partial_z^2\nu + (\partial_r\lambda)^2 + (\partial_z\lambda)^2 = 0 \tag{14}$$

Equation (11) is the two-dimensional Laplace equation in cylindrical coordinates. That is:

$$\nabla^2\lambda(r,z) = 0 \tag{15}$$

Plugging Equation (14) into Equation (12) gives:

$$\partial_r \nu = r\left((\partial_r \lambda)^2 - (\partial_z \lambda)^2\right) \tag{16}$$

Using Equations (13) and (16) we find solutions for $\nu$ are given by:

$$\nu = \int r[\left((\partial_r \lambda)^2 - (\partial_z \lambda)^2\right) dr + (2\partial_r \lambda \partial_z \lambda)\, dz] \tag{17}$$

The solutions must satisfy Equations (15) and (17). A particular solution corresponding to two objects (given by Curzon in 1924 [4] ) is:

$$\lambda_0 = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} \tag{18}$$

$$\nu_0 = \frac{1}{2}\frac{\mu_1^2 r^2}{r_1^4} - \frac{1}{2}\frac{\mu_2^2 r^2}{r_2^4} + \frac{2\mu_1\mu_2}{(z-z_2)^2}\left[\frac{r^2 + (z-z_1)(z-z_2)}{r_1 r_2} - 1\right] \tag{19}$$

Where $z_1$ and $z_2$ correspond to the positions on the z-axis for the two objects, $\mu_1$ and $\mu_2$ are length parameters, and:

$$r_1 = \sqrt{r^2 + (z-z_1)^2} \tag{20}$$

$$r_2 = \sqrt{r^2 + (z-z_2)^2} \tag{21}$$

These solutions, however, only apply to empty space (recall the use of Equation (10)). In addition, as noted by Synge [2], the z axis between $z_1$ and $z_2$ must also be excluded due to violation of elementary flatness. We will examine this in the next section.

### 1.2 Elementary Flatness

In order to be certain that our spacetime is truly flat, we impose the condition of elementary flatness: the ratio of the circumference to the radius is equal to $2\pi$. This gives restrictions on solutions for $\lambda(r,z)$ and $\nu(r,z)$.

To do this we will first integrate in the $\hat{\phi}$ direction at some $r$ and then divide by $r$. This gives:

$$L = \int ds = \int_0^{2\pi} \sqrt{-r^2 e^{-2\lambda} d\phi^2} = \pm\frac{2\pi r}{e^{\lambda}} \tag{22}$$

Then the condition that $\frac{L}{r} = 2\pi$ holds provided that $e^{-\lambda} = 1$. That is,

$$\lambda(0,z) \to 0 \tag{23}$$

But since $\frac{L}{r}$ is not well-defined as $r \to 0$, this is a sign that we need to look more carefully at the $z$-axis.

Consider parallel transport of a vector $V$ about the $z$-axis in the $\hat{\phi}$ direction, demanding that the values for $\phi = 0$ and $\phi = 2\pi$ are equal.

The equation for parallel transport is generally given by:

$$\tfrac{D}{d\lambda} = \tfrac{dx^\mu}{d\lambda}\nabla_\mu = 0 \quad \text{along} \quad x^\mu(\lambda) \tag{24}$$

That is, the directional covariant derivative is equal to zero along the curve $x^\mu$ parameterized by $\lambda$. For a vector this can be simply written as:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda = 0 \tag{25}$$

Starting with parallel transport along $\hat{e}_\phi$, Equation (25) along with the relevant Christoffel symbols $\Gamma^r_{\phi\phi}$, $\Gamma^z_{\phi\phi}$, $\Gamma^\phi_{\phi r}$, and $\Gamma^\phi_{\phi z}$ gives:

$$\begin{aligned} \partial_\phi V^r + \Gamma^r_{\phi\phi} V^\phi &= 0 \\ \partial_\phi V^z + \Gamma^z_{\phi\phi} V^\phi &= 0 \\ \partial_\phi V^\phi + \Gamma^\phi_{\phi r} V^r + \Gamma^\phi_{\phi z} V^z &= 0 \end{aligned} \tag{26}$$

Plugging in the values from Equation (4), our equations are:

$$\partial_\phi V^r + \left( re^{-2\nu} \left( r\partial_r \lambda - 1 \right) \right) V^\phi = 0 \tag{27}$$

$$\partial_\phi V^z + \left( r^2 e^{-2\nu} \partial_z \lambda \right) V^\phi = 0 \tag{28}$$

$$\partial_\phi V^\phi + \left( \frac{1}{r} - \partial_r \lambda \right) V^r - \partial_z \lambda V^z = 0 \tag{29}$$

Differentiating Equation (29) with respect to $\phi$ and plugging it into Equation (27) gives:

$$\partial^2_\phi V^\phi - \partial_z \lambda \partial_\phi V^z + r^2 e^{-2\nu} \left( \partial_r \lambda - \frac{1}{r} \right)^2 V^\phi = 0 \tag{30}$$

Plugging in the expression for $\partial_\phi V^z$ from Equation (28) and letting

$$\chi = re^{-\nu} \sqrt{(\partial_z \lambda)^2 + \left( \partial_r \lambda - \frac{1}{r} \right)^2} \tag{31}$$

We have the simple differential equation:

$$\partial^2_\phi V^\phi + \chi^2 V^\phi = 0 \tag{32}$$

For which the solution is:

$$V^\phi = A \sin \chi\phi + B \cos \chi\phi \tag{33}$$

Therefore, integrating Equation (27) with respect to $\phi$ we get:

$$V^r = \frac{r^2 e^{-2\nu} (\partial_r \lambda - \frac{1}{r})}{\chi} \left( A \cos \chi\phi - B \sin \chi\phi \right) \tag{34}$$

And from Equation (28):

$$V^z = \frac{r^2 e^{-2\nu} \partial_z \lambda}{\chi} \left( A \cos \chi\phi - B \sin \chi\phi \right) \tag{35}$$

At $\phi = 0$ we have $V^\phi = 1$ and $V^r = r_0$ (leaving aside for the moment $V^z$, since we are free to parallel transport about $\phi$ anywhere along the z-axis). Then the condition that $V^\phi = 1$ leads to $B = 1$. Likewise, setting $V^r = r_0$ leads to:

$$\frac{Ae^{-\nu}(\partial_r\lambda - \frac{1}{r_0})}{\sqrt{(\partial_z\lambda)^2 + (\partial_r\lambda - \frac{1}{r})^2}} = 1 \tag{36}$$

We set $A = 1$ for convenience. Then from Equation (36) taking the limit as $r_0 \to 0$ we find:

$$\lim_{r_0\to 0} \frac{e^{-\nu(r_0,z)}(\partial_r\lambda - \frac{1}{r_0})}{\sqrt{(\partial_z\lambda)^2 + (\partial_r\lambda - \frac{1}{r})^2}} = e^{-\nu(r_0,z)} = 1 \tag{37}$$

As $r_0$ is completely arbitrary we can characterize this as:

$$\lim_{r\to 0} \nu(0,z) = 0 \tag{38}$$

The general expression for the vector is then:

$$V = \left(\frac{re^{-\nu}}{\sqrt{(\partial_z\lambda)^2 + (\partial_r\lambda - \frac{1}{r})^2}}\right)(\cos\chi\phi - \sin\chi\phi)\left(\left(\partial_r\lambda - \frac{1}{r}\right)\hat{e}_r + \partial_z\lambda\hat{e}_z\right) + (\sin(\chi\phi) + \cos(\chi\phi))\,\hat{e}_\phi \tag{39}$$

Then,

$$\lim_{r_0\to 0} V(\phi = 0) = \hat{e_\phi} \tag{40}$$

Thus, the requirement that the vector is identical at $\phi = 2\pi$ after being transported around the circle starting at $\phi = 0$ as $r_0 \to 0$ is, from Equation (39):

$$\sin 2\pi\chi + \cos 2\pi\chi = 1 \tag{41}$$

In general, this has a number of solutions: all the integers $\chi = n = 1, 2, \ldots$ and $n = \frac{1}{4}, \frac{9}{4}, \frac{17}{4}$. The equivalent to Equations (40) and (41) for arbitrary $r_0$ from Equation (39) are much more complex.

What this tells us is that we do not, in general, have elementary flatness around the z-axis. We must therefore exclude it from our solution.

## 1.3 Matter solution to the Weyl metric

In principle, we have solutions for axially symmetric static vacuum spacetimes, subject to the conditions of Equations (23) and (38). These solutions, however, exclude the general masses defined by Equations (18) and (19), as well as the $z = 0$ axis ("strut") between them. We now wish to consider these objects.

The most general cylindrically symmetric static metric may be expressed as:

$$ds^2 = e^{2\lambda}dt^2 - e^{2(\nu-\sigma)}\left(dr^2 + dz^2\right) - r^2e^{-2\lambda}d\phi^2 \tag{42}$$

Where $\lambda$, $\nu$, and $\sigma$ are functions of $r$ and $z$. Comparing this to the solutions of the empty-space metric Equation (1), and allowing for deviations from these values due to the strut, we make the identifications:

$$\lambda = \lambda_0 + f(r,z) \tag{43}$$

$$\sigma = \lambda_0 + g(r,z) \tag{44}$$

$$\nu = \nu_0 + h(r,z) \tag{45}$$

In empty space outside the strut Equation (42) reduces to Equation (1) which implies $f(r,z) = g(r,z)$. Evaluating Equation (19) for $r \to 0$ and taking into account the condition of Equation (38) we obtain:

$$\nu_0(0,z) = \begin{cases} \frac{\mu_1\mu_2}{(z_1-z_2)^2} & \text{for} \quad z_1 < z < z_2 \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

From far away, our configuration should have spherical symmetry (i.e. the masses and strut become pointlike). This implies:

$$\lim_{r\to\infty} h(r,z) \to 0 \tag{47}$$

This reasoning will be addressed in the next section.
This in turn implies that:

$$h_0(0,z) = \begin{cases} -\frac{\mu_1\mu_2}{(z_1-z_2)^2} & \text{for} \quad z_1 < z < z_2 \\ 0 & \text{otherwise} \end{cases} \tag{48}$$

$$\nu = -\frac{4\mu_1\mu_2}{(z_1-z_2)^2} \tag{49}$$

To get the force on the strut, we can integrate the z-component of the stress-energy tensor over the area:

$$F_z = \int T_{zz} d\sigma \tag{50}$$

We can get the stress-energy tensor from Einstein's equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{51}$$

We have all of the relevant components, except the Ricci scalar:

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu} \tag{52}$$

Which is:

$$R = 2e^{2(\lambda-\nu)} \left( \partial_r^2 \nu + \partial_z^2 \nu - \partial_r^2 \lambda - \partial_z^2 \lambda + (\partial_r \lambda)^2 + (\partial_z \lambda)^2 - \frac{1}{r}\partial_r \lambda \right) \tag{53}$$

Recall that earlier we asserted in Equation (10) that we had a vacuum solution. In order for this to be true, our Ricci scalar better be equal to zero, since a vacuum solution actually corresponds to:

$$G_{\mu\nu} = 0 \tag{54}$$

Using Equations (11) and (14) and plugging them into Equation (53) we see that this is indeed the case.
Continuing, we get:

$$G_{zz} = (\partial_r \lambda)^2 - (\partial_z \lambda)^2 - \frac{1}{r} \partial_r \nu \tag{55}$$

Again, generally speaking $G_{zz} = 0$, which we can see by plugging Equation (16) into Equation (55). So we have this object which is zero everywhere except along the z-axis.

That object is a conical singularity [5].

(fill in)

Where the delta-function $\delta(r)$ is given by [5]:

$$\delta(r) = \int_0^\infty \int_0^{2\pi} \delta(r) e^{2(\lambda-\nu)} r dr d\phi = 1 \tag{56}$$

The two-dimensional sub-manifold in $(r, \phi)$ is diagonal, so that the only non vanishing components of the Ricci tensor are $R_{rr} = R_{\phi\phi} = K$. Applying Einstein's equation (Equation (51)) we have:

$$T_{zz} = \frac{1}{8\pi G} 2\pi \left(1 - e^{-\nu(r,z)}\right) \delta(r) \tag{57}$$

Thus:

$$F = \int T_{zz} dA = \frac{1}{4G} \left(1 - e^{-\nu(r,z)}\right) \tag{58}$$

Where G is Newton's constant. Looking over our Newtonian potentials ($\lambda$) from Equation (fill in) (which suppressed factors of G) and expanding the exponential:

$$e^{-\nu(0,z)} = 1 + (-\nu(0,z)) + (-\nu(0,z))^2 + \ldots \tag{59}$$

After applying the solution for $\nu(0,z)$ from Equation (46), the first order approximation is (recalling that $\mu_1 = Gm_1$ and $\mu_2 = Gm_2$):

$$F = \frac{Gm_1 m_2}{(z_1 - z_2)^2} \tag{60}$$

Where higher order terms of $\nu(0,z)$ are corrections to Newton's law.

### 1.4 The Schwarszchild solution in cylindrical coordinates

To address if we are justified in applying Equation (47), we should check to see if our solution reduces to the Schwarszchild solution for $r \to \infty$.

### 1.5 Extrinsic Curvature

$$K_{ab} = \frac{1}{2} \mathscr{L}_n h_{ab} \tag{61}$$

## 2 Application to Causal Dynamical Triangulations

Causal Dynamical Triangulations uses a path integral over all possible configurations between boundary conditions. The path integral is given by:

$$Z = \int \mathscr{D}[g] e^{iS_{EH}} \tag{62}$$

Where:

$$S_{EH} = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g} (R - 2\Lambda) \tag{63}$$

Given (62) and [6]

### 2.1 Regge Calculus

## References


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