# The Newtonian approximation in Causal Dynamical Triangulations

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**Abstract**

I review how to derive Newton's law from the Weyl strut between two Chazy-Curzon particles. I then apply this approach in Causal Dynamical Triangulations, modifying the algorithm to keep two simplicial complexes with curvature (i.e. mass) a fixed distance within each other (modulo regularized deviations) across all time slices. I then examine the results to determine if CDT produces an equivalent Weyl strut, which can then be used to obtain the Newtonian limit.

## 1 Introduction

Causal Dynamical Triangulations [1] is a promising approach to the problems of quantum gravity. Since the 1930's [2] attempts have been made to unify quantum mechanics with general relativity. "This is a hard problem, no one agrees on the answers, and perhaps if we knew why it was hard maybe it wouldn't be hard." The underlying difficulties are that observables in general relativity are necessarily non-local, making it difficult to write down a theory that extracts observable results.

Causal Dynamical Triangulations uses the path integral approach, and has had notable successes [3]. However, a difficulty is taking and extracting data that has physical meaning. One cannot identify points in a path integral, nor talk about functions of a point.

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A fundamental question is, does Causal Dynamical Triangulations have physical meaning? Attempts have been made before to relate CDT to the semi-classical limit [4, 5], but not everyone is convinced.

This paper attempts to answer this question by directly finding the Newtonian approximation in Causal Dynamical Triangulations.

## 2 Newton's Law of Gravitation from General Relativity

Starting from the cylindrically symmetric (Weyl) vacuum metric [6]

$$ds^2 = e^{2\lambda}dt^2 - e^{2(\nu-\lambda)}\left(dr^2 + dz^2\right) - r^2 e^{-2\lambda}d\phi^2 \tag{1}$$

where $\lambda$ and $\nu$ are both functions of $r$ and $z$ we find that

$$\partial_r^2\lambda + \frac{1}{r}\partial_r\lambda + \partial_z^2\lambda = \nabla^2\lambda(r,z) = 0 \tag{2}$$

$$\nu = \int r[\left((\partial_r\lambda)^2 - (\partial_z\lambda)^2\right)dr + (2\partial_r\lambda\partial_z\lambda)\,dz]. \tag{3}$$

The solutions must satisfy Equations (2) and (3). A particular solution corresponding to two objects (given by Curzon in 1924 [7] ) is

$$\lambda_0 = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} \tag{4}$$

$$\nu_0 = \frac{1}{2}\frac{\mu_1^2 r^2}{r_1^4} - \frac{1}{2}\frac{\mu_2^2 r^2}{r_2^4} + \frac{2\mu_1\mu_2}{(z-z_2)^2}\left[\frac{r^2 + (z-z_1)(z-z_2)}{r_1 r_2} - 1\right] \tag{5}$$

where $z_1$ and $z_2$ correspond to the positions on the z-axis for the two objects, $\mu_1$ and $\mu_2$ are length parameters, and

$$r_1 = \sqrt{r^2 + (z-z_1)^2} \tag{6}$$

$$r_2 = \sqrt{r^2 + (z-z_2)^2}. \tag{7}$$

Up to this point we have been assuming spacetime is truly flat. We check this assumption via the condition of elementary flatness: the ratio of the circumference to the radius is equal to $2\pi$.

To do this we integrate in the $\hat{\phi}$ direction at some $r$ and then divide by $r$. This gives

$$C = \int ds = \int_0^{2\pi} \sqrt{r^2 e^{-2\lambda} d\phi^2} = 2\pi r e^{-\lambda}. \tag{8}$$

Then the condition that $\frac{C}{r} = 2\pi$ holds provided that

$$\lambda(0, z) \to 0. \tag{9}$$

But Equation (4) contradicts Equation (9) and $\frac{C}{r}$ is not at all well-defined as $r \to 0$. Indeed, Einstein and Rosen [8] first noted that the Weyl metric cannot be a purely vacuum solution, and that there must be a strut on the z-axis.

Now to the salient point: we can use this strut to our advantage by obtaining $T_{zz}$ and thence the Newtonian gravitational interaction via

$$F_z = \int T_{zz} d\sigma \tag{10}$$

as was done by Katz in 1967 [9]. We will use a different approach, however, from either Katz or more recent literature [10].

Taking the parallel transport of a vector around the strut, we obtain

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Using the appropriate connections we obtain

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Now we can just read off the value of $G_{zz}$ and thence $T_{zz}$ to get

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# 3 Geometry

Triangulations in Causal Dynamical Triangulations refers to the use of $d$-simplices to construct a spacetime lattice. In general, a $d$-dimensional simplex has $d + 1$ points, which are also referred to as 0-simplices. For a $d$-dimensional simplex there are $\binom{d+1}{k+1}$ $k$-dimensional faces, or sub-simplices.

Causal refers to the fact that the triangulations generally span two adjacent timeslices. Using a notation $\{k, n\}$ where $k$ is the number of points on the higher timelike slice and $n$ is the number of points in the lower timelike slice, we summarize simplex geometry in Table 1. This will be useful in the discussion of ergodic moves that follows.

| Name | Dimension | 0-faces | 1-faces | 2-faces | 3-faces | Causal Structures |
|---|---|---|---|---|---|---|
| Vertex | 0 | 1 | | | | |
| Edge | 1 | 2 | 1 | | | {1,1} |
| Triangle | 2 | 3 | 3 | 1 | | {2,1} {1,2} |
| Tetrahedron | 3 | 4 | 6 | 4 | 1 | {3,1} {2,2} {1,3} |
| Pentatope | 4 | 5 | 10 | 10 | 5 | {4,1} {3,2} {2,3} {1,4} |

Table 1: Types and causal structures of simplices

# 4 Mass and the Einstein tensor

The general idea behind Causal Dynamical Triangulations is

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To do this, we start from the results of Regge Calculus [11]

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Using Barrett, we can derive the Einstein tensor in Regge Calculus as follows

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In order to introduce mass, we

TODO

# 5 Notes on Implementation

These ideas are implemented using CGAL [12], a time-tested library of geometric algorithms in continuous development since 1995.

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# References


[1] J. Ambjorn, J. Jurkiewicz, and R. Loll, "A non-perturbative lorentzian path integral for gravity," *Physical Review Letters*, vol. 85, pp. 924–7, Feb. 2000. Phys.Rev.Lett. 85 (2000).

[2] C. Rovelli, "Notes for a brief history of quantum gravity," *gr-qc/0006061*, June 2000.

[3] R. Kommu, "A validation of causal dynamical triangulations," *arXiv:1110.6875*, Oct. 2011.

[4] J. Ambjorn, A. Gorlich, J. Jurkiewicz, R. Loll, J. Gizbert-Studnicki, and T. Trzesniewski, "The semiclassical limit of causal dynamical triangulations," *1102.3929*, Feb. 2011.

[5] J. Ambjorn, J. Jurkiewicz, and R. Loll, "Semiclassical universe from first principles," *Physics Letters B*, vol. 607, no. 2005, pp. 205–213.

[6] J. L. Synge, *Relativity: the general theory.* North-Holland Pub. Co., 1960.

[7] H. E. J. Curzon, "Cylindrical Solutions of Einstein's Gravitational Equations," *Proceedings of the London Mathematical Society*, vol. s2–23, pp. 477–480, 1925.

[8] A. Einstein and N. Rosen, "Two-body problem in general relativity theory," *Phys. Rev.*, vol. 49, pp. 404–405, Mar 1936.

[9] A. Katz, "Derivation of Newton's Law of Gravitation from General Relativity," *Journal of Mathematical Physics*, vol. 9, pp. 983–985, Sept. 1967.

[10] P. S. Letelier and S. R. Oliveira, "Superposition of weyl solutions: The equilibrium forces," *arXiv:gr-qc/9710122*, Oct. 1997. Class.Quant.Grav. 15 (1998) 421-433.

[11] T. Regge, "General relativity without coordinates," *Nuovo Cimento A*, vol. 19, pp. 558–571, 1961.

[12] "CGAL, Computational Geometry Algorithms Library." http://www.cgal.org.