June 2018

Efficient Causal Dynamical Triangulations

A. Getchell*
Department of Physics
University of California
Davis, CA 95616
USA

Abstract

I review constructing piecewise simplicial manifolds using efficient methods for constructing Delaunay triangulations. I then evaluate the use of the Metropolis-Hastings algorithm in the Causal Dynamical triangulations program. I highlight inefficiencies and propose solutions.

* *email: acgetchell@ucdavis.edu*

1. Introduction

> Nevertheless, due to the inneratomic (sic) movements of electrons, atoms would have to radiate not only electromagnetic but also gravitational energy, if only in tiny amounts. As this is hardly true in nature, it appears that quantum theory would have to modify not only Maxwellian electrodynamics, but also the new theory of gravitation. [1]
>
> –Einstein, 1916 *Approximative Integration of the Field Equations of Gravitation, p.209*

Quantum gravity is, perhaps, the preeminent hard problem [2] remaining in theoretical physics, and has been worked on for many years [3].

Although difficult to test experimentally, a quantum theory of gravity appears to be the key to resolving several important questions, such as the black hole information paradox. [4]

Causal Dynamical Triangulations (CDT) [5–9] is a useful approach to quantum gravity. It is based on the Regge action [10], which describes General Relativity on simplicial manifolds similarly to the Einstein-Hilbert action on differentiable manifolds, and has been independently validated in 3 and 4 dimensions. [11]

Using the Metropolis-Hasting algorithm [12], in the class of Markov Chain Monte Carlo methods (MCMC), allows for the analysis of complex distributions in higher dimensions, unlike other methods. [13]

However, and this is the central point of this paper, Metropolis-Hastings algorithms suffer from known problems such as exponentially long convergence times to stationary distributions and sensitivity to step size (from 23% to 70% is given as a suitable acceptance rate [14,15]); both may occur within the context of CDT.

Methods such as slice sampling, Hamiltonian Monte Carlo, and Simulated Annealing are other methods that may be used instead. But each has respective drawbacks:

Slice sampling [16] requires that the sample is evaluatable, which is not always possible. It also runs into difficulties at higher dimensions.

Hamiltonian Monte Carlo (HMC) computes expectations by exploring a continuous parameter space of probability distributions. [17]. In certain implementations it has been show to be extremely fast and efficient [18], but it's not necessarily clear how to set this up for the Regge action. Additionally, the parameters may be hard to tune, and it does not handle multimodality well, which is an expected output of quantum gravity ("crumpled or polymer" phase and "other phase of CDT"). Nontheless, I think this is a worthwhile possiblity worth exploring in a future paper.

Like HMC, Simulated Annealing also requires a global parameter space to optimize. [19] Implementing this in the context of CDT has not, to my knowledge, been explored.

In this paper, I address efficiencies in the Causal Dynamical Triangulations approach by:

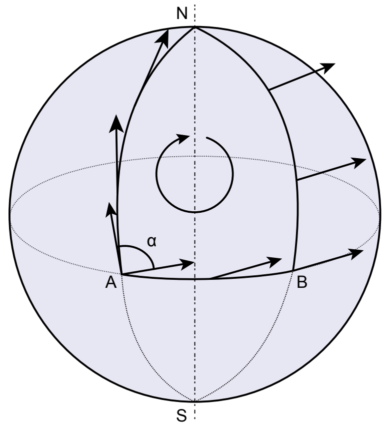

Figure 1: Parallel Transport on a spherical surface by Fred the Oyster, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=35124171

2. Background

The Einstein equation describes the curvature of spacetime $R_{\mu\nu}$ in terms of the stress-energy-momentum tensor $T_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{2.1}$$

The Reimann tensor is given by:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{2.2}$$

Where the Affine connection $\Gamma^{\lambda}_{\mu\nu}$ is defined by:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tag{2.3}$$

And the (cylindrically symmetric) metric is:

$$g_{\mu\nu} = \begin{pmatrix} e^{2\lambda} & 0 & 0 & 0 \\ 0 & -e^{2(\nu-\lambda)} & 0 & 0 \\ 0 & 0 & -e^{2(\nu-\lambda)} & 0 \\ 0 & 0 & 0 & -\frac{r^2}{e^{2\lambda}} \end{pmatrix} \tag{2.4}$$

$R^{\rho}_{\sigma\mu\nu}$ can be thought of as encapsulating the intrinsic curvature (see Figure 1).

From the Reimann tensor one obtains the Ricci tensor using $R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$, and likewise the Ricci scalar is $R = R^{\mu}_{\mu}$ using the Einstein summation convention.

Given the Ricci scalar the Einstein-Hilbert action is:

$$I_{EH} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}(R-2\Lambda) \tag{2.5}$$

Where G_N is Newton's Gravitational constant and Λ is the cosmological constant. Extremizing the Einstein-Hilbert action produces the equations of motion.

$$\partial I_{EH} = 0 \to R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{2.6}$$

In quantum mechanics, one is interested in the transition probability amplitude $\langle B|T|A\rangle$, which is the conditional probability of being in state B given previously being in state A. This is generally computed using the path integral.

$$\langle B|T|A\rangle = \int \mathcal{D}[g]e^{iI_{EH}} \tag{2.7}$$

Such path integrals are typically not directly computable, for a number of reasons. Quantum Field Theory uses perturbative summation techniques such as Feynman diagrams, but these require a notion of renormalizability for various infinite divergences, and gravity has been shown to be definitively non-renormalizable. [20]

In 1961 Regge developed his calculus replacing smooth differentiable manifolds with simplicial manifolds, obeying the following two properties

1. close: every $(n-1)$-dimensional subsimplex of a simplex in the manifold is also in the manifold;
2. connectivity: two connected n-dimensional simplices share one and only one $(n-1)$-dimensional subsimplex;

From here on, simplicial manifolds will be referred to as triangulations. Of special note are Delaunay Triangulations, which are well-behaved simplicial manifolds with a circumsphere property of member simplices which may be seen intuitively in Figure 2.

The discrete version of the Einstein-Hilbert action is the Regge action:

$$I_R = \frac{1}{8\pi G_N}\left(\sum_{hinges} A_h\delta_h - \Lambda\sum_{simplices} V_s\right) \tag{2.8}$$

And the discrete version of the path integral is (after a Wick rotation:

$$\langle B|T|A\rangle = \sum_{triangulations}\frac{1}{C(T)}e^{-I_R(T)} \tag{2.9}$$

Figure 2: Delaunay triangulation (left) Not a Delaunay triangulation (right)

Here, we take a sum over all inequivalent triangulations. In 1991 Pachner [21], building on Alexander's work in the 1930s [22] showed that elementary operations, now called Pachner moves, could transform a triangulation T to another manifold T' homeomorphic to T. The set of all inequivalent triangulations could be then be explored via a series of Pachner moves. [23]

Equation (2.9) takes advantage of the distinctly causal nature of Causal Dynamical Triangulations. That is, the triangulations are foliated by hypersurfaces of distinct time. Using this innovation allows an explicit calculation of the CDT action, which has been done for 2, 3, and 4 dimensions. The subject of this paper is the 3D action (Equation 35 from [6]):

$$\begin{aligned} I^{(3)}_{CDT} &= 2\pi k\sqrt{\alpha}N_1^{TL} \\ &+ N_3^{(3,1)}\left[-3k\text{arcsinh}\left(\frac{1}{\sqrt{3}\sqrt{4\alpha+1}}\right) - 3k\sqrt{\alpha}\text{arccos}\left(\frac{2\alpha+1}{4\alpha+1}\right) - \frac{\lambda}{12}\sqrt{3\alpha+1}\right] \\ &+ N_3^{(2,2)}\left[2k\text{arcsinh}\left(\frac{2\sqrt{2}\sqrt{2\alpha+1}}{4\alpha+1}\right) - 4k\sqrt{\alpha}\text{arccos}\left(\frac{-1}{4\alpha+1}\right) - \frac{\lambda}{12}\sqrt{4\alpha+2}\right] \end{aligned}$$

Where α is the length of the timelike edges (spacelike edges are length 1), $k = \frac{1}{8\pi G_N}$, and $\lambda = k * \Lambda$.

To evaluate Equation (2.9), we use the Metropolis-Hastings algorithm as follows:

1. Selection: Pick a Pachner move;
2. Acceptance: Make that move with a probability of $a = a_1 a_2$, where

$$a_1 = \frac{move[i]}{\sum\limits_i move[i]} \tag{2.10}$$

$$a_2 = e^{\Delta I_{CDT}} \tag{2.11}$$

Note that we have divided out the partition function $\frac{1}{C(T)}$ in Equation (2.9), which we didn't know how to evaluate anyway.

After thermalization, the Metropolis-Hastings algorithm gives us the distribution of triangulations for comoputing the path integral. We can then perform measures on these representative ensembles to calculate properties such as spectral dimension. [24, 25]

3. Dynamical System

4. Methods

5. Results

6. Conclusion

References

[1] A. Einstein, "Volume 6: The Berlin Years: Writings, 1914-1917 (English translation supplement) Page 201 (213 of 462)."

[2] Steve Carlip, "Why Quantum Gravity is Hard," in *Conceptual and Technical Challenges for Quantum Gravity*, (Rome), Sept. 2014.

[3] C. Rovelli, "Notes for a brief history of quantum gravity," *gr-qc/0006061*, June 2000. `http://arxiv.org/abs/gr-qc/0006061`.

[4] J. Preskill, "Do Black Holes Destroy Information?," *arXiv:hep-th/9209058*, Sept. 1992. arXiv: hep-th/9209058.

[5] J. Ambjorn, J. Jurkiewicz, and R. Loll, "Non-perturbative 3d Lorentzian Quantum Gravity," *hep-th/0011276*, Nov. 2000. Phys.Rev.D64:044011,2001.

[6] J. Ambjorn, J. Jurkiewicz, and R. Loll, "Dynamically Triangulating Lorentzian Quantum Gravity," *Nuclear Physics B*, vol. 610, pp. 347–382, May 2001. Nucl.Phys. B610 (2001) 347-382.

[7] R. Loll, "A discrete history of the Lorentzian path integral," *Lecture Notes in Physics*, vol. 631, pp. 137–171, 2003.

[8] J. Ambjorn, A. Goerlich, J. Jurkiewicz, and R. Loll, "Quantum Gravity via Causal Dynamical Triangulations," *arXiv:1302.2173*, Feb. 2013.

[9] J. H. Cooperman, "Making the case for causal dynamical triangulations," *arXiv:1410.0670 [gr-qc, physics:hep-th]*, Oct. 2014. arXiv: 1410.0670.

[10] T. Regge, "General relativity without coordinates," *Il Nuovo Cimento Series 10*, vol. 19, pp. 558–571, Feb. 1961.

[11] R. Kommu, "A Validation of Causal Dynamical Triangulations," *arXiv:1110.6875*, Oct. 2011.

[12] C. P. Robert, "The Metropolis-Hastings algorithm," *arXiv:1504.01896 [stat]*, Apr. 2015. arXiv: 1504.01896.

[13] P. Grišins and I. E. Mazets, "Metropolis–Hastings thermal state sampling for numerical simulations of Bose–Einstein condensates," *Computer Physics Communications*, vol. 185, pp. 1926–1931, July 2014.

[14] M. Bédard, "Optimal acceptance rates for Metropolis algorithms: Moving beyond 0.234," *Stochastic Processes and their Applications*, vol. 118, pp. 2198–2222, Dec. 2008.

[15] E. Xing, "Markov Chain Monte Carlo."

[16] R. M. Neal, "Slice sampling," *The Annals of Statistics*, vol. 31, pp. 705–767, June 2003.

[17] M. Betancourt, "A Conceptual Introduction to Hamiltonian Monte Carlo," *arXiv:1701.02434 [stat]*, Jan. 2017. arXiv: 1701.02434.

[18] M. D. Hoffman and A. Gelman, "The No-U-Turn Sampler: Adaptively Setting Path Lengths in Hamiltonian Monte Carlo," *arXiv:1111.4246 [cs, stat]*, Nov. 2011. arXiv: 1111.4246.

[19] F. Busetti, "Simulated annealing overview," p. 10.

[20] A. Shomer, "A pedagogical explanation for the non-renormalizability of gravity," *arXiv:0709.3555 [gr-qc, physics:hep-ph, physics:hep-th]*, Sept. 2007. arXiv: 0709.3555.

[21] U. Pachner, "P.L. Homeomorphic Manifolds are Equivalent by Elementary Shellings," *European Journal of Combinatorics*, vol. 12, pp. 129–145, Mar. 1991.

[22] J. W. Alexander, "The Combinatorial Theory of Complexes," *Annals of Mathematics*, vol. 31, pp. 292–320, May 1930.

[23] M. Gross and S. Varsted, "Elementary moves and ergodicity in D-dimensional simplicial quantum gravity," *Nuclear Physics B*, vol. 378, pp. 367–380, July 1992.

[24] J. Ambjorn, J. Jurkiewicz, and R. Loll, "Spectral Dimension of the Universe," *Phys*, vol. 95, no. 2005, p. 171301.

[25] T. P. Sotiriou, M. Visser, and S. Weinfurtner, "Spectral dimension as a probe of the ultraviolet continuum regime of causal dynamical triangulations," arXiv e-print 1105.5646, May 2011. Physical Review Letters 107 (2011) 131303.