Detailed calculations for the Weyl metric

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1   Vacuum solution to the Weyl metric

Starting from the cylindrically symmetric (Weyl) vacuum metric [1]:

$$ds^2 = e^{2\lambda}dt^2 - e^{2(\nu-\lambda)}\left(dr^2 + dz^2\right) - r^2 e^{-2\lambda}d\phi^2 \tag{1}$$

$$g_{\mu\nu} = \begin{pmatrix} e^{2\lambda} & 0 & 0 & 0 \\ 0 & -e^{2(\nu-\lambda)} & 0 & 0 \\ 0 & 0 & -e^{2(\nu-\lambda)} & 0 \\ 0 & 0 & 0 & -r^2 e^{-2\lambda} \end{pmatrix} \tag{2}$$

In this coordinate basis, the definition of the Christoffel connection is: [2]

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}\right) \tag{3}$$

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The non-zero Christoffel connections are:

$$\Gamma^t_{tr} = \partial_r \lambda$$
$$\Gamma^t_{tz} = \partial_z \lambda$$
$$\Gamma^r_{tt} = e^{4\lambda - 2\nu} \partial_r \lambda$$
$$\Gamma^r_{rr} = \partial_r \nu - \partial_r \lambda$$
$$\Gamma^r_{rz} = \partial_z \nu - \partial_z \lambda$$
$$\Gamma^r_{zz} = \partial_r \lambda - \partial_r \nu$$
$$\Gamma^r_{\phi\phi} = re^{-2\nu} \left(r\partial_r \lambda - 1 \right)$$
$$\Gamma^z_{tt} = e^{4\lambda - 2\nu} \partial_z \lambda \tag{4}$$
$$\Gamma^z_{rr} = \partial_z \lambda - \partial_z \nu$$
$$\Gamma^z_{rz} = \partial_r \nu - \partial_r \lambda$$
$$\Gamma^z_{zz} = \partial_z \nu - \partial_z \lambda$$
$$\Gamma^z_{\phi\phi} = r^2 e^{-2\nu} \partial_z \lambda$$
$$\Gamma^\phi_{r\phi} = \frac{1}{r} - \partial_r \lambda$$
$$\Gamma^\phi_{z\phi} = -\partial_z \lambda$$

The components of the Riemann tensor are given by:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \tag{5}$$

Using the properties of the Riemann tensor:

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$$
$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$$
$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \tag{6}$$
$$R_{\rho[\sigma\mu\nu]} = 0$$

The non-zero components of the Riemann tensor are:

$$R^t_{rtr} = -\partial_r^2 \lambda + (\partial_z \lambda)^2 - 2(\partial_r \lambda)^2 + \partial_r \lambda \partial_r \nu - \partial_z \lambda \partial_z \nu$$
$$R^t_{rtz} = -\partial_r \partial_z \lambda - 3\partial_r \lambda \partial_z \lambda + \partial_r \lambda \partial_z \nu + \partial_r \nu \partial_z \lambda$$
$$R^t_{ztz} = -\partial_z^2 \lambda - 2(\partial_z \lambda)^2 + (\partial_r \lambda)^2 - \partial_r \lambda \partial_r \nu + \partial_z \lambda \partial_z \nu$$
$$R^t_{\phi t\phi} = re^{-2\nu} \left(r(\partial_r \lambda)^2 - \partial_r \lambda + r(\partial_z \lambda)^2 \right)$$
$$R^r_{zrz} = \partial_r^2 \lambda - \partial_r^2 \nu + \partial_z^2 \lambda - \partial_z^2 \nu \tag{7}$$
$$R^z_{\phi z\phi} = re^{-2\nu} \left(r\partial_z^2 \lambda - r\partial_z \lambda \partial_z \nu + r\partial_r \lambda \partial_r \nu - r(\partial_r \lambda)^2 + \partial_r \lambda - \partial_r \nu \right)$$
$$R^z_{\phi\phi r} = re^{-2\nu} \left(-r\partial_r \partial_z \lambda + r\partial_r \nu \partial_z \lambda - r\partial_r \lambda \partial_z \lambda + r\partial_r \lambda \partial_z \nu - \partial_z \nu \right)$$
$$R^\phi_{r\phi r} = \partial_r^2 \lambda + \frac{1}{r}\partial_r \nu - \partial_r \lambda \partial_r \nu - (\partial_z \lambda)^2 + \partial_z \lambda \partial_z \nu + \frac{1}{r}\partial_r \lambda$$

The Ricci tensor is given by:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \tag{8}$$

The non-zero components of the Ricci tensor are:

$$R_{tt} = \frac{e^{4\lambda-2\nu}}{r}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right)$$

$$R_{rr} = \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2\left(\partial_r\lambda\right)^2 + \frac{1}{r}\partial_r\lambda + \frac{1}{r}\partial_r\nu$$

$$R_{rz} = \frac{1}{r}\partial_z\nu - 2\partial_r\lambda\partial_z\lambda \tag{9}$$

$$R_{zz} = \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2\left(\partial_z\lambda\right)^2 + \frac{1}{r}\partial_r\lambda - \frac{1}{r}\partial_r\nu$$

$$R_{\phi\phi} = re^{-2\nu}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right)$$

The Ricci scalar is defined as:

$$R = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu} \tag{10}$$

Which is:

$$R = 2e^{2(\lambda-\nu)}\left(\partial_r^2\nu + \partial_z^2\nu - \partial_r^2\lambda - \partial_z^2\lambda + (\partial_r\lambda)^2 + (\partial_z\lambda)^2 - \frac{1}{r}\partial_r\lambda\right) \tag{11}$$

Einstein's equation in a vacuum is:

$$G_{\mu\nu} = 0 \tag{12}$$

Whence Einstein's equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu} \tag{13}$$

However, we can take a shortcut by using:

$$R_{\mu\nu} = 0 \tag{14}$$

Since the trace of a zero-valued matrix is identically zero, and thus Equation (14) automatically satisfies Equation (12).

Applying Equation (14) to Equation (9) gives the following:

$$\partial_r^2\lambda + \frac{1}{r}\partial_r\lambda + \partial_z^2\lambda = 0 \tag{15}$$

$$\partial_r\nu = r\left(\partial_r^2\nu + \partial_z^2\nu + 2(\partial_r\lambda)^2\right) \tag{16}$$

$$\partial_z\nu = 2r\partial_r\lambda\partial_z\lambda \tag{17}$$

$$\partial_r^2\nu + \partial_z^2\nu + (\partial_r\lambda)^2 + (\partial_z\lambda)^2 = 0 \tag{18}$$

Equation (15) is the two-dimensional Laplace equation in cylindrical coordinates. That is:

$$\nabla^2 \lambda(r,z) = 0 \tag{19}$$

Plugging Equation (18) into Equation (16) gives:

$$\partial_r \nu = r\left((\partial_r \lambda)^2 - (\partial_z \lambda)^2\right) \tag{20}$$

Using Equations (17) and (20) we find solutions for ν are given by:

$$\nu = \int r[\left((\partial_r \lambda)^2 - (\partial_z \lambda)^2\right) dr + (2\partial_r \lambda \partial_z \lambda) dz] \tag{21}$$

The solutions must satisfy Equations (19) and (21). A particular solution corresponding to two objects (given by Curzon in 1924 [3]) is:

$$\lambda_0(r,z) = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} \tag{22}$$

$$\nu_0(r,z) = \frac{1}{2}\frac{\mu_1^2 r^2}{r_1^4} - \frac{1}{2}\frac{\mu_2^2 r^2}{r_2^4} + \frac{2\mu_1\mu_2}{(z-z_2)^2}\left[\frac{r^2 + (z-z_1)(z-z_2)}{r_1 r_2} - 1\right] \tag{23}$$

Where z_1 and z_2 correspond to the positions on the z-axis for the two objects, μ_1 and μ_2 are length parameters, and:

$$r_1 = \sqrt{r^2 + (z-z_1)^2} \tag{24}$$

$$r_2 = \sqrt{r^2 + (z-z_2)^2} \tag{25}$$

Just as a final check, plugging Equations (15) and (18) into Equation (11) gives $R = 0$, which shows that our solutions are consistent with our assumptions.

By construction, these solutions only apply to empty space, and so must exclude the two objects at z_1 and z_2. In addition, as noted by Synge [1], the z axis between the two objects must also be excluded due to violation of elementary flatness. We will examine this in the next section.

2   Curvature from Parallel Transport

Consider parallel transport of a vector V about the z-axis in the $\hat{\phi}$ direction. The equation for parallel transport is generally given by:

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu = 0 \quad \text{along} \quad x^\mu(\lambda) \tag{26}$$

That is, the directional covariant derivative is equal to zero along the curve x^μ parameterized by λ. For a vector this can be simply written as:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda = 0 \tag{27}$$

Starting with parallel transport along $\hat{e}_\phi$, Equation (27) along with the relevant Christoffel symbols $\Gamma^r_{\phi\phi}$, $\Gamma^z_{\phi\phi}$, $\Gamma^\phi_{\phi r}$, and $\Gamma^\phi_{\phi z}$ gives:

$$\partial_\phi V^r + \Gamma^r_{\phi\phi} V^\phi = 0$$
$$\partial_\phi V^z + \Gamma^z_{\phi\phi} V^\phi = 0 \tag{28}$$
$$\partial_\phi V^\phi + \Gamma^\phi_{\phi r} V^r + \Gamma^\phi_{\phi z} V^z = 0$$

Plugging in the values from Equation (4), our equations are:

$$\partial_\phi V^r + \left(re^{-2\nu} \left(r\partial_r \lambda - 1 \right) \right) V^\phi = 0 \tag{29}$$

$$\partial_\phi V^z + \left(r^2 e^{-2\nu} \partial_z \lambda \right) V^\phi = 0 \tag{30}$$

$$\partial_\phi V^\phi + \left(\frac{1}{r} - \partial_r \lambda \right) V^r - \partial_z \lambda V^z = 0 \tag{31}$$

Differentiating Equation (31) with respect to ϕ and plugging it into Equation (29) gives:

$$\partial_\phi^2 V^\phi - \partial_z \lambda \partial_\phi V^z + r^2 e^{-2\nu} \left(\partial_r \lambda - \frac{1}{r} \right)^2 V^\phi = 0 \tag{32}$$

Plugging in the expression for $\partial_\phi V^z$ from Equation (30) and letting

$$\chi = re^{-\nu} \sqrt{(\partial_z \lambda)^2 + \left(\frac{1}{r} - \partial_r \lambda \right)^2} \tag{33}$$

We have the simple differential equation:

$$\partial_\phi^2 V^\phi + \chi^2 V^\phi = 0 \tag{34}$$

For which the solution is:

$$V^\phi = A \sin \chi \phi + B \cos \chi \phi \tag{35}$$

Therefore, integrating Equation (29) with respect to ϕ we get:

$$V^r = \frac{r^2 e^{-2\nu} (\partial_r \lambda - \frac{1}{r})}{\chi} (A \cos \chi \phi - B \sin \chi \phi) \tag{36}$$

And from Equation (30):

$$V^z = \frac{r^2 e^{-2\nu} \partial_z \lambda}{\chi} (A \cos \chi \phi - B \sin \chi \phi) \tag{37}$$

So our general vector is then:

$$V = \frac{r^2 e^{-2\nu} (\partial_r \lambda - \frac{1}{r})}{\chi} (A \cos \chi \phi - B \sin \chi \phi) \hat{e}_r + \frac{r^2 e^{-2\nu} \partial_z \lambda}{\chi} (A \cos \chi \phi - B \sin \chi \phi) \hat{e}_z + (A \sin \chi \phi + B \cos \chi \phi) \hat{e}_\phi \tag{38}$$

Normalizing $V(\phi = 0)$:

$$g_{\mu\nu}V^\mu V^\nu = 1 \tag{39}$$

We obtain the condition that:

$$A^2 + B^2 = r^{-2}e^{2\lambda} \tag{40}$$

For simplicity, we choose $A^2 = r^{-2}e^{2\lambda}$ and $B^2 = 0$.

Now, when we parallel transport V around to $\phi = 2\pi$ there will be an angle between $V(\phi = 0)$ and $V(\phi = 2\pi)$ given by the definition of the scalar product:

$$\cos(\beta) = \frac{g_{\mu\nu}V^\mu(0)V^\nu(2\pi)}{g_{\mu\nu}V^\mu(0)V^\nu(0)} \tag{41}$$

Since we have normalized our vectors, the denominator is equal to 1, and we get the expression that:

$$\cos\beta = \cos(2\pi\chi) \tag{42}$$

Where χ is given by Equation (33). Hence $\beta = 2\pi\chi$. We can now use the definition of the deficit angle:

$$\Delta = 2\pi - \beta = 2\pi(1 - \chi) \tag{43}$$

To get the curvature $\mathcal{R}$ via:

$$\mathcal{R} = \lim_{A\to 0} \frac{\Delta}{A} \tag{44}$$

The area A is defined on the reduced metric:

$$ds^2 = e^{2(\nu-\lambda)}dr^2 + r^2 e^{-2\lambda}d\phi^2 \tag{45}$$

Via:

$$A = \int \sqrt{|g|}d^n x = \int_{\phi=0}^{\phi=2\pi}\int_{r=0}^{r=R}\sqrt{r^2 e^{2\nu-4\lambda}}\,dr d\phi = 2\pi\int_{r=0}^{r=R} r e^{\nu-2\lambda}\,dr \tag{46}$$

Plugging Equations (22) and (23) into Equation (46) gives:

$$A = \tag{47}$$

And thus the curvature is:

$$\mathcal{R} = \tag{48}$$

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3   Matter solution to the Weyl metric

In principle, we have solutions for axially symmetric static vacuum spacetimes, subject to the conditions of **????**. These solutions, however, exclude the general masses defined by Equations (22) and (23), as well as the $z = 0$ axis (the Weyl strut) between them. We now wish to consider these objects using the method of Katz [4].

The most general cylindrically symmetric static metric may be expressed as:

$$ds^2 = e^{2\lambda} dt^2 - e^{2(\nu - \sigma)} \left(dr^2 + dz^2 \right) - r^2 e^{-2\lambda} d\phi^2 \tag{49}$$

Where λ, ν, and σ are functions of r and z. Comparing this to the solutions of the empty-space metric Equation (1), and allowing for deviations from these values due to the strut, we make the identifications:

$$\lambda = \lambda_0 + f(r, z) \tag{50}$$

$$\sigma = \lambda_0 + g(r, z) \tag{51}$$

$$\nu = \nu_0 + h(r, z) \tag{52}$$

In empty space outside the strut the metric of Equation (49) reduces to that of Equation (1), which implies:

$$f(r, z) = g(r, z) \tag{53}$$

Evaluating Equation (23) for $r \to 0$ (n.b. you must take the one negative and one positive root of the two square root terms to get a non-zero answer) and taking into account the condition of **??** we obtain:

$$\nu_0(0, z) = \begin{cases} \frac{-4\mu_1\mu_2}{(z_1 - z_2)^2} & \text{for} \quad z_1 < z < z_2 \\ 0 & \text{otherwise} \end{cases} \tag{54}$$

From far away, our configuration should have spherical symmetry (i.e. the masses and strut become pointlike). This implies:

$$\lim_{r \to \infty} h(r, z) \to 0 \tag{55}$$

This reasoning will be addressed in the Section 4.

Combining Equation (52) with Equation (54) and the condition of **??** yields:

$$h_0(0, z) = \begin{cases} \frac{4\mu_1\mu_2}{(z_1 - z_2)^2} & \text{for} \quad z_1 < z < z_2 \\ 0 & \text{otherwise} \end{cases} \tag{56}$$

To get the force on the strut, we can integrate the z-component of the stress-energy tensor over the area:

$$F_z = \int T_{zz} d\sigma \tag{57}$$

We can get this by processing the metric of Equation (49) through Einstein's equation (13) resulting in:

$$G_{zz} = - (\partial_r \lambda)^2 + (\partial_z \lambda)^2 + \frac{1}{r} \partial_r \nu - \frac{1}{r} \partial_r \sigma + \frac{1}{r} \partial_r \lambda \tag{58}$$

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Taking a first order approximation to Equation (58) eliminates the first two squared terms. Applying Equations (50) to (53) reduces to:

$$G_{zz} = -\frac{1}{r} \left(\partial_r \left(v_0 + h(r,z) \right) \right) \tag{59}$$

Now, recalling Equation (55) $h(r,z)$ can be taken as being defined at $r = 0$. Recalling **??** cancels out the v_0 term, and we are left with:

$$T_{zz} = \frac{1}{8\pi G} G_{zz} = -\frac{1}{8\pi Gr} \partial_r h_0(0,z) \tag{60}$$

The integration measure for Equation (57) $d\sigma = r dr d\phi$.
Substituting:

$$F = \int r d\phi \int dr T_{zz} = \frac{1}{8\pi Gr} 2\pi r \int dr \partial_r h_0(0,z) = \frac{1}{4G} \frac{-4\mu_1 \mu_2}{(z_1 - z_2)^2} \tag{61}$$

Recall that Equations (22) and (23) as solutions to Equation (12) contain the length parameters μ_1 and μ_2 in order to make them dimensionless overall. Dimensional analysis (recall we are working in units of $c = 1$) shows that μ_1 can be taken as Gm_1 and $\mu_2 = Gm_2$.

Thus, the first order approximation is:

$$F = -\frac{Gm_1 m_2}{(z_1 - z_2)^2} \tag{62}$$

Where the negative sign indicates that gravity is attractive [2], and the squared derivative terms we excluded from Equation (58) are corrections to Newton's law.

Therefore, integrating the stress-energy of the Weyl strut between two stationary masses gives Newton's law (plus higher order corrections).

4   The Schwarszchild solution in cylindrical coordinates

We have a loose end in Section 3. To address if we are justified in applying Equation (55), we should check to see if our solution reduces to the Schwarszchild solution for $r \to \infty$.

5   Extrinsic Curvature

We should find the extrinsic curvature of the induced metric.

6   Application to Causal Dynamical Triangulations

Causal Dynamical Triangulations uses a path integral over all possible configurations between boundary conditions. The path integral is given by:

$$Z = \int \mathscr{D}[g] e^{iS_{EH}} \tag{63}$$

Where:

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$$S_{EH} = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g}\,(R - 2\Lambda) \tag{64}$$

References

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[3] H. E. J. Curzon, "Cylindrical Solutions of Einstein's Gravitational Equations," *Proceedings of the London Mathematical Society*, vol. s2–23, pp. 477–480, 1925.

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