

# Newtonian approximation in Causal Dynamical Triangulations

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## 1 Newton's Law of Gravitation from General Relativity

### 1.1 Vacuum solution to the Weyl metric

Starting from the most general cylindrically symmetric (Weyl) metric [1]:

$$ds^2 = e^{2\lambda} dt^2 - e^{2(\nu-\lambda)} (dr^2 + dz^2) - r^2 e^{-2\lambda} d\phi^2 \quad (1)$$

$$g_{\mu\nu} = \begin{pmatrix} e^{2\lambda} dt^2 & 0 & 0 & 0 \\ 0 & -e^{2(\nu-\lambda)} dr^2 & 0 & 0 \\ 0 & 0 & -e^{2(\nu-\lambda)} dz^2 & 0 \\ 0 & 0 & 0 & -\frac{r^2}{e^{2\lambda}} d\phi^2 \end{pmatrix} \quad (2)$$

The definition of the Christoffel connection is: [2]

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\sigma} (\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu}) \quad (3)$$

The non-zero Christoffel connections are:

$$\begin{aligned}
\Gamma_{tr}^t &= \partial_r \lambda \\
\Gamma_{tz}^t &= \partial_z \lambda \\
\Gamma_{tt}^r &= e^{4\lambda-2v} \partial_r \lambda \\
\Gamma_{rr}^r &= \partial_r v - \partial_r \lambda \\
\Gamma_{rz}^r &= \partial_z v - \partial_z \lambda \\
\Gamma_{zz}^r &= \partial_z \lambda - \partial_z v \\
\Gamma_{\phi\phi}^r &= r e^{-2v} (r \partial_r \lambda - 1) \\
\Gamma_{tt}^z &= e^{4\lambda-2v} \partial_z \lambda \\
\Gamma_{rr}^z &= \partial_z \lambda - \partial_z v \\
\Gamma_{rz}^z &= \partial_r v - \partial_r \lambda \\
\Gamma_{zz}^z &= \partial_r v - \partial_r \lambda \\
\Gamma_{\phi\phi}^z &= r^2 e^{-2v} \partial_z \lambda \\
\Gamma_{r\phi}^\phi &= \frac{1}{r} - \partial_r \lambda \\
\Gamma_{z\phi}^\phi &= -\partial_z \lambda
\end{aligned} \tag{4}$$

The components of the Riemann tensor are given by:

$$R_{\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda \tag{5}$$

Using the properties of the Riemann tensor:

$$\begin{aligned}
R_{\rho\sigma\mu\nu} &= -R_{\rho\sigma\nu\mu} \\
R_{\rho\sigma\mu\nu} &= -R_{\sigma\rho\mu\nu} \\
R_{\rho\sigma\mu\nu} &= R_{\mu\nu\rho\sigma} \\
R_{\rho[\sigma\mu\nu]} &= 0
\end{aligned} \tag{6}$$

The non-zero components of the Riemann tensor are:

$$\begin{aligned}
R_{tr}^t &= -\partial_r^2 \lambda + (\partial_z \lambda)^2 - 2(\partial_r \lambda)^2 + \partial_r \lambda \partial_r v - \partial_z \lambda \partial_z v \\
R_{tz}^t &= -\partial_r \partial_z \lambda - 3\partial_r \lambda \partial_z \lambda + \partial_r \lambda \partial_z v + \partial_r v \partial_z \lambda \\
R_{rz}^t &= -\partial_z^2 \lambda - 2(\partial_z \lambda)^2 + (\partial_r \lambda)^2 - \partial_r \lambda \partial_r v + \partial_z \lambda \partial_z v \\
R_{\phi t}^t &= r e^{-2v} (r(\partial_r \lambda)^2 - \partial_r \lambda + r(\partial_z \lambda)^2) \\
R_{rz}^r &= \partial_r^2 \lambda - \partial_r^2 v + \partial_z^2 \lambda - \partial_z^2 v \\
R_{\phi z}^z &= r e^{-2v} (r \partial_z^2 \lambda - r \partial_z \lambda \partial_z v + r \partial_r \lambda \partial_r v - r(\partial_r \lambda)^2 + \partial_r \lambda - \partial_r v) \\
R_{\phi\phi}^z &= r e^{-2v} (-r \partial_r \partial_z \lambda + r \partial_r v \partial_z \lambda - r \partial_r \lambda \partial_z \lambda + r \partial_r \lambda \partial_z v - \partial_z v) \\
R_{r\phi}^\phi &= \partial_r^2 \lambda + \frac{1}{r} \partial_r v - \partial_r \lambda \partial_r v - (\partial_z \lambda)^2 + \partial_z \lambda \partial_z v + \frac{1}{r} \partial_r \lambda
\end{aligned} \tag{7}$$

The Ricci tensor is given by:

$$R_{\mu\nu} = R_{\mu\lambda\nu}^\lambda \tag{8}$$

The non-zero components of the Ricci tensor are:

$$\begin{aligned}
R_{tt} &= \frac{e^{4\lambda-2\nu}}{r} \left( r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda \right) \\
R_{rr} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2(\partial_r\lambda)^2 + \frac{1}{r}\partial_r\lambda + \frac{1}{r}\partial_r\nu \\
R_{rz} &= \frac{1}{r}\partial_z\nu - 2\partial_r\lambda\partial_z\lambda \\
R_{zz} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2(\partial_z\lambda)^2 + \frac{1}{r}\partial_r\lambda - \frac{1}{r}\partial_r\nu \\
R_{\phi\phi} &= re^{-2\nu} \left( r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda \right)
\end{aligned} \tag{9}$$

Einstein's equation in a vacuum is:

$$R_{\mu\nu} = 0 \tag{10}$$

Applying this complete set of relations to Equation (9) gives the following:

$$\partial_r^2\lambda + \frac{1}{r}\partial_r\lambda + \partial_z^2\lambda = 0 \tag{11}$$

$$\partial_r\nu = r \left( \partial_r^2\nu + \partial_z^2\nu + 2(\partial_r\lambda)^2 \right) \tag{12}$$

$$\partial_z\nu = 2r\partial_r\lambda\partial_z\lambda \tag{13}$$

$$\partial_r^2\nu + \partial_z^2\nu + (\partial_r\lambda)^2 + (\partial_z\lambda)^2 = 0 \tag{14}$$

Equation (11) is the two-dimensional Laplace equation in cylindrical coordinates. Plugging Equation (14) into Equation (12) gives:

$$\partial_r\nu = r \left( (\partial_r\lambda)^2 - (\partial_z\lambda)^2 \right) \tag{15}$$

Using Equations (13) and (15) we find solutions for  $\nu$  given by:

$$\nu = \int r \left[ \left( (\partial_r\lambda)^2 - (\partial_z\lambda)^2 \right) dr + (2\partial_r\lambda\partial_z\lambda) dz \right] \tag{16}$$

## 1.2 Elementary Flatness

In order to be certain that our spacetime is truly flat, we want to check for elementary flatness, that is, that the ratio of the circumference and radius is equal to  $2\pi$ . To do this we will parallel transport a vector  $W$  about the  $\phi$ -axis, and parallel transport a vector  $V$  along the  $r$ -axis for  $r \rightarrow R$ , and check that their ratio is  $2\pi$ . This will give us restrictions on our solutions for  $\lambda(r, z)$  and  $\nu(r, z)$ .

The equation for parallel transport is generally given by:

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu = 0 \quad \text{along } x^\mu(\lambda) \tag{17}$$

That is, the directional covariant derivative is equal to zero along the curve  $x^\mu$  parameterized by  $\lambda$ . For a vector this can be simply written as:

$$\nabla_\alpha X^\beta = 0 \tag{18}$$

Starting with parallel transport along  $\hat{e}_r$  for  $r \rightarrow R$ , Equation (18) along with the non-zero Christoffel connections yields:

$$\begin{aligned}
\partial_r V^t + \Gamma_{rt}^t V^t &= 0 \\
\partial_r V^r + \Gamma_{rr}^r V^r &= 0 \\
\partial_r V^r + \Gamma_{rz}^r V^z &= 0 \\
\partial_r V^z + \Gamma_{rr}^z V^r &= 0 \\
\partial_r V^z + \Gamma_{rz}^z V^z &= 0 \\
\partial_r V^\phi + \Gamma_{r\phi}^\phi V^\phi &= 0
\end{aligned} \tag{19}$$

Plugging in the values from Equation (4) gives:

$$\partial_r V^t + \partial_r \lambda V^t = 0 \tag{20}$$

$$\partial_r V^r + (\partial_r v - \partial_r \lambda) V^r = 0 \tag{21}$$

$$\partial_r V^r + (\partial_z v - \partial_z \lambda) V^z = 0 \tag{22}$$

$$\partial_r V^z + (\partial_z \lambda - \partial_z v) V^r = 0 \tag{23}$$

$$\partial_r V^z + (\partial_r v - \partial_r \lambda) V^z = 0 \tag{24}$$

$$\partial_r V^\phi + \left( \frac{1}{r} - \partial_r \lambda \right) V^\phi = 0 \tag{25}$$

These six equations aren't really too forbidding, however, and can be solved by inspection and what we already know.

The solution for Equation (20) is:

$$V^t = C_1 e^{-\lambda} \tag{26}$$

The solution for Equation (21) is:

$$V^r = C_2 e^{\lambda-v} \tag{27}$$

The solution for Equation (22) also:

$$V^z = C_3 e^{\lambda-v} \tag{28}$$

The solution for Equation (25) can be easily guessed:

$$V^\phi = C_4 \frac{e^\lambda}{r^2} \tag{29}$$

In fact, with a convenient choice of constants we have:

$$V = e^{-2\lambda} \hat{e}_t - e^{2(\lambda-\nu)} \hat{e}_r - e^{2(\lambda-\nu)} \hat{e}_z - \frac{e^{2\lambda}}{r^2} \hat{e}_\phi \quad (30)$$

Now onto  $W$  which is to be parallel transported along the  $\phi$ -axis. The relevant Christoffel symbols are  $\Gamma_{\phi\phi}^r$ ,  $\Gamma_{\phi\phi}^z$ ,  $\Gamma_{\phi r}^\phi$ , and  $\Gamma_{\phi z}^\phi$ , so that our equations are:

$$\partial_\phi W^r + \left( r e^{-2\nu} (r \partial_r \lambda - 1) \right) W^\phi = 0 \quad (31)$$

$$\partial_\phi W^z + \left( r^2 e^{-2\nu} \partial_z \lambda \right) W^\phi = 0 \quad (32)$$

$$\partial_\phi W^\phi + \left( \frac{1}{r} - \partial_r \lambda \right) W^r = 0 \quad (33)$$

$$\partial_\phi W^\phi - \partial_z \lambda W^z = 0 \quad (34)$$

Differentiating Equation (33) with respect to  $\phi$  and plugging it into Equation (31) gives:

$$\partial_\phi^2 W^\phi = r^2 e^{-2\nu} W^\phi \quad (35)$$

For which the solution is:

$$W^\phi = A \cos(\phi r e^{-\nu}) + B \sin(\phi r e^{-\nu}) \quad (36)$$

Therefore:

$$W^r = \frac{r e^{-\nu}}{\partial_r \lambda - \frac{1}{r}} \{ B \cos(\phi r e^{-\nu}) - A \sin(\phi r e^{-\nu}) \} \quad (37)$$

And from Equation (34):

$$W^z = \frac{r e^{-\nu}}{\partial_z \lambda} \{ B \cos(\phi r e^{-\nu}) - A \sin(\phi r e^{-\nu}) \} \quad (38)$$

Now, at  $\phi = 0$  we have  $W = 0\hat{e}_t + 0\hat{e}_r + 0\hat{e}_z + 1\hat{e}_\phi$  which gives us  $A = 1$  and  $B = 0$ . Therefore,

$$W = \frac{r e^{-\nu}}{\partial_r \lambda - \frac{1}{r}} \{ \cos(\phi r e^{-\nu}) \} \hat{e}_r - \frac{r e^{-\nu}}{\partial_z \lambda} \{ \sin(\phi r e^{-\nu}) \} \hat{e}_z + \cos(\phi r e^{-\nu}) \hat{e}_\phi \quad (39)$$

### 1.3 Matter solution to the Weyl metric

In principle, we have solutions for axially symmetric static vacuum spacetimes. We now wish to add matter. If the object is also axially symmetric and static, then we can consider solutions in the form of an external metric  $E$  and an internal metric  $I$ , where  $E$  is given by Equations (1) and (16).

The solution of Laplace's equation for a point particle of mass  $m$  at  $z = z_0$  is well known [3]:

$$\lambda(r, z) = -\frac{m}{\sqrt{r^2 + (z - z_0)^2}} \quad (40)$$

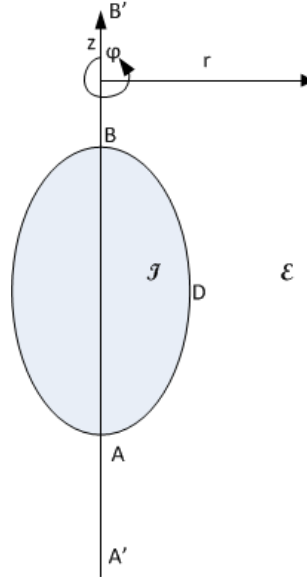
Likewise, we can easily verify for a single particle that the solution for  $v$  from Equations (16) and (40) is (setting integration constants equal to zero):

$$v(r, z) = -\frac{m^2 r^2}{(r^2 + (z - z_0)^2)^2} \quad (41)$$

However, before we can consider this to be a complete solution we must consider elementary flatness. This condition requires that, for any infinitesimal spacelike circle, the ratio of circumference to radius is  $2\pi$ . The most likely place to run into issues is along the  $z$ -axis, for which  $r = 0$ . Looking back at Equation (1) we see that the necessary condition is:

$$\lim_{r \rightarrow 0} v = 0 \quad (42)$$

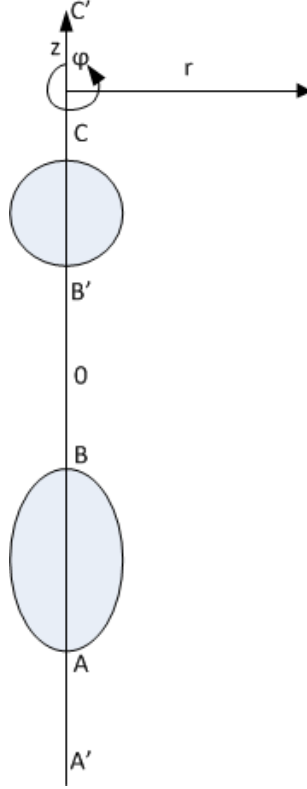
Consider the following diagram<sup>1</sup> :



<sup>1</sup>Adapted from Synge

Equation (16) includes a constant, which we can set by choosing that  $v = 0$  at  $A$ . So,  $v = 0$  along the  $z$ -axis from  $A'$  to  $A$ . The same applies from  $B$  to  $B'$ . Then our path  $ADB$  may be deformed into an infinite semicircle.

Now consider two bodies, as in the following diagram:



Since solutions to Laplace's equation are linear, we have:

$$\lambda(r, z) = -\frac{m_1}{R_1} - \frac{m_2}{R_2} \quad (43)$$

Plugging this into Equation (16) yields:

$$v(r, z) = -\frac{m_1^2 r^2}{R_1^4} - \frac{m_2^2 r^2}{R_2^4} + \frac{4m_1 m_2}{(z_1 - z_2)^2} \frac{r^2 + (z - z_1)(z - z_2)}{R_1 R_2} \quad (44)$$

$$R_i = \sqrt{r^2 + (z - z_i)^2} \quad (45)$$

In this case, we expect  $v = 0$  along  $A'A$  and  $C'C$  as before. But there is no *a priori* reason to think that  $v = 0$  along  $B'B$ . This means that our vacuum solution fails along the  $z$ -axis. Therefore, there must be a strut of matter, i.e. a metric  $I$  such that  $R_{\mu\nu} \neq 0$ , along the  $z$ -axis  $B'B$  separating the two objects. This corresponds with the expectation that two masses will attract each other and not remain at rest.

Indeed, if we apply the condition that  $r = 0$  we get:

$$v(0, z) = \frac{4m_1 m_2}{(z_1 - z_2)^2} \quad (46)$$

Which means that in order for Equation (42) to hold, our strut must have:

$$v = -\frac{4m_1 m_2}{(z_1 - z_2)^2} \quad (47)$$

To get the force on the strut, we can integrate the z-component of the stress-energy tensor over the area:

$$F_z = \int T_{zz} d\sigma \quad (48)$$

We can get the stress-energy tensor from Einstein's equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (49)$$

We have all of the relevant components, except the Ricci scalar:

$$R = R^\mu_\mu = g^{\mu\nu} R_{\mu\nu} \quad (50)$$

Which is:

$$R = 2e^{2(\lambda-\nu)} \left( \partial_r^2 v + \partial_z^2 v - \partial_r^2 \lambda - \partial_z^2 \lambda + (\partial_r \lambda)^2 + (\partial_z \lambda)^2 - \frac{1}{r} \partial_r \lambda \right) \quad (51)$$

Recall that earlier we asserted in Equation (10) that we had a vacuum solution. In order for this to be true, our Ricci scalar better be equal to zero, since a vacuum solution actually corresponds to:

$$G_{\mu\nu} = 0 \quad (52)$$

Using Equations (11) and (14) and plugging them into Equation (51) we see that this is indeed the case.

Continuing, we get:

$$G_{zz} = (\partial_r \lambda)^2 - (\partial_z \lambda)^2 - \frac{1}{r} \partial_r v \quad (53)$$

Again, generally speaking  $G_{zz} = 0$ , which we can see by plugging Equation (15) into Equation (53). So we have this object which is zero everywhere except along the z-axis.

That object is a conical singularity [4].

We can solve this problem by using the Gauss-Bonnet theorem: [5]

$$\iint_M K dA = 2\pi \chi(M) \quad (54)$$



Where  $K$  is the Gaussian curvature and  $\chi(M)$  is the Euler characteristic. As we are looking at  $T_{zz}$ , we are interested in the submanifold of constant  $t$  and  $z$  which has the metric:

$$ds^2 = e^{2(v-\lambda)} dr^2 + r^2 e^{-2\lambda} d\phi^2 \quad (55)$$

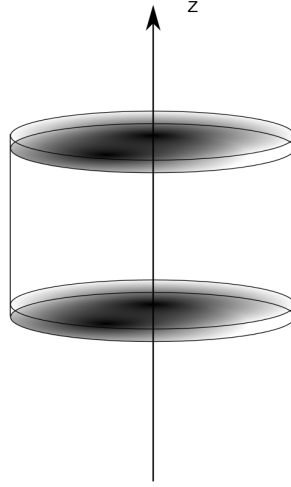
By putting this in the first fundamental form [6]:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \quad (56)$$

We can calculate  $K$  using the orthogonal parameterization [7] (since  $F = 0$ ):

$$K = -\frac{1}{2\sqrt{EG}} \left( \frac{\partial}{\partial u} \frac{\partial_u G}{\sqrt{EG}} + \frac{\partial}{\partial v} \frac{\partial_v E}{\sqrt{EG}} \right) = -\frac{1}{2} \frac{1}{\sqrt{EG}} \frac{d}{dr} \left( \frac{1}{\sqrt{EG}} \frac{dG}{dr} \right) \quad (57)$$

To solve the left side of Equation (54), consider the following diagram:



The top of the cylinder is given by:

$$\int K dA = -\frac{1}{2} \int_0^R \int_0^{2\pi} \frac{d}{dr} \left( \frac{1}{\sqrt{EG}} \frac{dG}{dr} \right) dr d\phi \quad (58)$$

And:

$$\left( \frac{1}{\sqrt{EG}} \frac{dG}{dr} \right) = 2e^{-v} (1 - r\partial_r \lambda) \quad (59)$$

Then:

$$\int K dA = 2\pi [e^{-v} (r\partial_r \lambda - 1)]_0^R \quad (60)$$

Taking the limit as  $R \rightarrow \infty$  and recalling elementary flatness (Equation (42)), the Gauss-Bonnet theorem gives:

$$K = 2\pi \left(1 - e^{-v(r,z)}\right) \delta(r) \quad (61)$$

Where the delta-function  $\delta(r)$  is given by [4]:

$$\delta(r) = \int_0^\infty \int_0^{2\pi} \delta(r) e^{2(\lambda-v)} r dr d\phi = 1 \quad (62)$$

The two-dimensional sub-manifold in  $(r, \phi)$  is diagonal, so that the only non-vanishing components of the Ricci tensor are  $R_{rr} = R_{\phi\phi} = K$ . Then applying Einstein's equation (Equation (49)) we have:

$$T_{zz} = \frac{1}{8\pi G} 2\pi \left(1 - e^{-v(r,z)}\right) \delta(r) \quad (63)$$

Thus:

$$F = \int T_{zz} dA = \frac{1}{4G} \left(1 - e^{-v(r,z)}\right) \quad (64)$$

Where  $G$  is Newton's constant. Looking over our Newtonian potentials  $(\lambda)$  from Equation (43) (which suppressed factors of  $G$ ) and expanding the exponential:

$$e^{-v(0,z)} = 1 + (-v(0,z)) + (-v(0,z))^2 + \dots \quad (65)$$

After applying the solution for  $v(0,z)$  from Equation (46), the first order approximation is:

$$F = \frac{Gm_1 m_2}{(z_1 - z_2)^2} \quad (66)$$

Where higher order terms of  $v(0,z)$  are corrections to Newton's law.

## 2 Application to Causal Dynamical Triangulations

### References

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