Newtonian approximation in Causal Dynamical Triangulations

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1 Motivation

1.1 Newton's Law of Gravitation from General Relativity

Starting from the most general cylindrically symmetric (Weyl) metric [1]:

$$ds^2 = e^{2\lambda}dt^2 - e^{2(\nu-\lambda)}\left(dr^2 + dz^2\right) - r^2 e^{-2\lambda} d\phi^2 \tag{1}$$

$$g_{\mu\nu} = \begin{pmatrix} e^{2\lambda}dt^2 & 0 & 0 & 0 \\ 0 & -e^{2(\nu-\lambda)}dr^2 & 0 & 0 \\ 0 & 0 & -e^{2(\nu-\lambda)}dz^2 & 0 \\ 0 & 0 & 0 & -\frac{r^2}{e^{2\lambda}}d\phi^2 \end{pmatrix} \tag{2}$$

The definition of the Christoffel connection is: [2]

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}\right) \tag{3}$$

With the assumption of zero torsion:

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu} \tag{4}$$

The non-zero Christoffel connections are:

$$
\begin{aligned}
&\Gamma^t_{tr} = \partial_r\lambda \\
&\Gamma^t_{tz} = \partial_z\lambda \\
&\Gamma^r_{tt} = e^{4\lambda-2\nu}\partial_r\lambda \\
&\Gamma^r_{rr} = \partial_r\nu - \partial_r\lambda \\
&\Gamma^r_{rz} = \partial_z\nu - \partial_z\lambda \\
&\Gamma^r_{zz} = \partial_z\lambda - \partial_z\nu \\
&\Gamma^r_{\phi\phi} = re^{-2\nu}\left(r\partial_r\lambda - 1\right) \\
&\Gamma^z_{tt} = e^{4\lambda-2\nu}\partial_z\lambda \\
&\Gamma^z_{rr} = \partial_z\lambda - \partial_z\nu \\
&\Gamma^z_{rz} = \partial_r\nu - \partial_r\lambda \\
&\Gamma^z_{zz} = \partial_r\nu - \partial_r\lambda \\
&\Gamma^z_{\phi\phi} = r^2e^{-2\nu}\partial_z\lambda \\
&\Gamma^\phi_{r\phi} = \tfrac{1}{r} - \partial_r\lambda \\
&\Gamma^\phi_{z\phi} = -\partial_z\lambda
\end{aligned}
\tag{5}
$$

The components of the Riemann tensor are given by:

$$
R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{6}
$$

Using the properties of the Riemann tensor:

$$
\begin{aligned}
&R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} \\
&R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \\
&R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \\
&R_{\rho[\sigma\mu\nu]} = 0
\end{aligned}
\tag{7}
$$

The non-zero components of the Riemann tensor are:

$$
\begin{aligned}
&R^t_{rtr} = -\partial_r^2\lambda + \left(\partial_z\lambda\right)^2 - 2\left(\partial_r\lambda\right)^2 + \partial_r\lambda\partial_r\nu - \partial_z\lambda\partial_z\nu \\
&R^t_{rtz} = -\partial_r\partial_z\lambda - 3\partial_r\lambda\partial_z\lambda + \partial_r\lambda\partial_z\nu + \partial_r\nu\partial_z\lambda \\
&R^t_{ztz} = -\partial_z^2\lambda - 2\left(\partial_z\lambda\right)^2 + \left(\partial_r\lambda\right)^2 - \partial_r\lambda\partial_r\nu + \partial_z\lambda\partial_z\nu \\
&R^t_{\phi t\phi} = re^{-2\nu}\left(r\left(\partial_r\lambda\right)^2 - \partial_r\lambda + r\left(\partial_z\lambda\right)^2\right) \\
&R^r_{zrz} = \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu \\
&R^z_{\phi z\phi} = re^{-2\nu}\left(r\partial_z^2\lambda - r\partial_z\lambda\partial_z\nu + r\partial_r\lambda\partial_r\nu - r\left(\partial_r\lambda\right)^2 + \partial_r\lambda - \partial_r\nu\right) \\
&R^z_{\phi\phi r} = re^{-2\nu}\left(-r\partial_r\partial_z\lambda + r\partial_r\nu\partial_z\lambda - r\partial_r\lambda\partial_z\lambda + r\partial_r\lambda\partial_z\nu - \partial_z\nu\right) \\
&R^\phi_{r\phi r} = \partial_r^2\lambda + \tfrac{1}{r}\partial_r\nu - \partial_r\lambda\partial_r\nu - \left(\partial_z\lambda\right)^2 + \partial_z\lambda\partial_z\nu + \tfrac{1}{r}\partial_r\lambda
\end{aligned}
\tag{8}
$$

The Ricci tensor is given by:

$$
R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \tag{9}
$$

The non-zero components of the Ricci tensor are:

$$
\begin{aligned}
&R_{tt} = \tfrac{e^{4\lambda-2\nu}}{r}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right)\\
&R_{rr} = \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2\left(\partial_r\lambda\right)^2 + \tfrac{1}{r}\partial_r\lambda + \tfrac{1}{r}\partial_r\nu\\
&R_{rz} = \tfrac{1}{r}\partial_z\nu - 2\partial_r\lambda\partial_z\lambda\\
&R_{zz} = \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2\left(\partial_z\lambda\right)^2 + \tfrac{1}{r}\partial_r\lambda - \tfrac{1}{r}\partial_r\nu\\
&R_{\phi\phi} = re^{-2\nu}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right)
\end{aligned}
\tag{10}
$$

Einstein's equation in a vacuum is:

$$R_{\mu\nu} = 0 \tag{11}$$

Applying this complete set of relations to Equation (10) gives the following:

$$\partial_r^2\lambda + \frac{1}{r}\partial_r\lambda + \partial_z^2\lambda = 0 \tag{12}$$

$$\partial_r\nu = r\left(\partial_r^2\nu + \partial_z^2\nu + 2\left(\partial_r\lambda\right)^2\right) \tag{13}$$

$$\partial_z\nu = 2r\partial_r\lambda\partial_z\lambda \tag{14}$$

$$\partial_r^2\nu + \partial_z^2\nu + \left(\partial_r\lambda\right)^2 + \left(\partial_z\lambda\right)^2 = 0 \tag{15}$$

Equation (12) is the two-dimensional Laplace equation in cylindrical coordinates, for which the known solutions are:

$$\lambda(r,z) = \sum_{m=0}^{\infty}\left[A_m J_m(kr) + B_m N_m(kr)\right]\left[C_m \sinh(kr) + D_m \cosh(kr)\right] \tag{16}$$

References

[1] J. L. Synge, *Relativity: the general theory*. North-Holland Pub. Co., 1960.

[2] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*. Benjamin Cummings, Sept. 2003.