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Efficient Causal Dynamical Triangulations

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Abstract

I review constructing piecewise simplicial manifolds using efficient methods for constructing Delaunay triangulations. I then evaluate the use of the Metropolis-Hastings algorithm in the Causal Dynamical triangulations program. I highlight inefficiencies and propose solutions.

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1. Introduction

> Nevertheless, due to the inneratomic (sic) movements of electrons, atoms would have to radiate not only electromagnetic but also gravitational energy, if only in tiny amounts. As this is hardly true in nature, it appears that quantum theory would have to modify not only Maxwellian electrodynamics, but also the new theory of gravitation. [1]
>
> –Einstein, 1916 *Approximative Integration of the Field Equations of Gravitation, p.209*

Quantum gravity is, perhaps, the preeminent hard problem [2] remaining in theoretical physics, and has been worked on for many years [3].

Although difficult to test experimentally, a quantum theory of gravity appears to be the key to resolving several important questions, such as the black hole information paradox. [4]

Causal Dynamical Triangulations (CDT) [5] is a useful approach to quantum gravity. It is based on the Regge action [6], which describes General Relativity on simplicial manifolds similarly to the Einstein-Hilbert action on differentiable manifolds.

Using the Metropolis-Hasting algorithm, in the class of Markov Chain Monte Carlo methods (MCMC), unlike other methods it allows for the analysis of complex distributions in higher dimensions. Better still, in calculating the path integral as a ratio, it allows the factoring out of terms that we would have a hard time of finding out.

Nevetheless, MCMC algorithms suffer from known problems such as exponentially long convergence times to stationary distributions and sensitivity to step size.

Methods such as slice sampling, Hamiltonian Monte Carlo, and Simulated Annealing are other methods that may be used instead of MCMC. But each has respective drawbacks:

Slice sampling requires that the sample is evaluatable, which is not always possible. It also runs into difficulties at higher dimensions.

Hamiltonian Monte Carlo (HMC) computes expectations by exploring a continuous parameter space of probability distributions. [7]. In certain implementations it has been show to be extremely fast and efficient [8], but it's not necessarily clear how to set this up for the Regge action. Additionally, the parameters may be hard to tune, and it does not handle multimodality well, which is an expected output of quantum gravity (”crumpled or polymer” phase and ”other phase of CDT”). Nontheless, I think this is a worthwhile possiblity worth exploring in a future paper.

Like HMC, Simulated Annealing also requires a global parameter space to optimize. [9] Implementing this in the context of CDT has not, to my knowledge, been explored.

2. Background

The Einstein equation describes the curvature of spacetime $R_{\mu\nu}$ in terms of the stress-energy-momentum tensor $T_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{2.1}$$

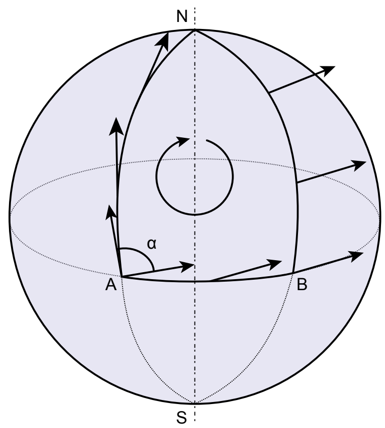

Figure 1: Parallel Transport on a spherical surface by Fred the Oyster, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=35124171

The Reimann tensor is given by:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{2.2}$$

Where the Affine connection $\Gamma^{\lambda}_{\mu\nu}$ is defined by:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tag{2.3}$$

And the (cylindrically symmetric) metric is:

$$g_{\mu\nu} = \begin{pmatrix} e^{2\lambda} & 0 & 0 & 0 \\ 0 & -e^{2(\nu-\lambda)} & 0 & 0 \\ 0 & 0 & -e^{2(\nu-\lambda)} & 0 \\ 0 & 0 & 0 & -\frac{r^2}{e^{2\lambda}} \end{pmatrix} \tag{2.4}$$

$R^{\rho}_{\sigma\mu\nu}$ can be thought of as encapsulating the intrinsic curvature (see Figure 1).

From the Reimann tensor one obtains the Ricci tensor using $R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$, and likewise the Ricci scalar is $R = R^{\mu}_{\mu}$ using the Einstein summation convention.

Given the Ricci scalar the Einstein-Hilbert action is:

$$I_{EH} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}(R - 2\Lambda) \tag{2.5}$$

Where G_N is Newton's Gravitational constant and Λ is the cosmological constant.

Extremizing the Einstein-Hilbert action produces the equations of motion.

Figure 2: Delaunay triangulation (left) Not a Delaunay triangulation (right)

$$\partial I_{EH} = 0 \rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{2.6}$$

In quantum mechanics, one is interested in the transition probability amplitude $\langle B|T|A\rangle$, which is the conditional probability of being in state B given previously being in state A. This is generally computed using the path integral.

$$\langle B|T|A\rangle = \int \mathcal{D}[g] e^{iI_{EH}} \tag{2.7}$$

Such path integrals are typically not directly computable, for a number of reasons. Quantum Field Theory uses perturbative summation techniques such as Feynman diagrams, but these require a notion of renormalizability for various infinite divergences, and gravity has been shown to be definitively non-renormalizable. [10]

In 1961 Regge developed his calculus replacing smooth differentiable manifolds with simplicial manifolds, obeying the following two properties

1. close: every $(n-1)$-dimensional subsimplex of a simplex in the manifold is also in the manifold;
2. connectivity: two connected n-dimensional simplices share one and only one $(n-1)$-dimensional subsimplex;

From here on, simplicial manifolds will be referred to as triangulations. Of special note are Delaunay Triangulations, which are well-behaved simplicial manifolds with a circumsphere property of member simplices which may be seen intuitively in Figure 2.

The discrete version of the Einstein-Hilbert action is the Regge action:

$$I_R = \frac{1}{8\pi G_N}\left(\sum_{hinges} A_h\delta_h - \Lambda \sum_{simplices} V_s\right) \tag{2.8}$$

And the discrete version of the path integral is (after a Wick rotation:

$$\langle B|T|A\rangle = \sum_{triangulations} \frac{1}{C(T)} e^{-I_R(T)} \tag{2.9}$$

Here, we take a sum over all inequivalent triangulations. In 1991 Pachner [11], building on Alexander's work in the 1930s [12] showed that elementary operations, now called Pachner moves, could transform a triangulation T to another manifold T' homeomorphic to T. The set of all inequivalent triangulations could be then be explored via a series of Pachner moves. [13]

Equation (2.9) takes advantage of the distinctly causal nature of Causal Dynamical Triangulations. That is, the triangulations are foliated by hypersurfaces of distinct time. Using this innovation allows an explicit calculation of the CDT action, which has been done for 2, 3, and 4 dimensions. The subject of this paper is the 3D action:

$$\begin{aligned}
I_{CDT}^{(3)} &= 2\pi k\sqrt{\alpha}N_1^{TL} \\
&+ N_3^{(3,1)}\left[-3k\operatorname{arcsinh}\left(\frac{1}{\sqrt{3}\sqrt{4\alpha+1}}\right) - 3k\sqrt{\alpha}\arccos\left(\frac{2\alpha+1}{4\alpha+1}\right) - \frac{\lambda}{12}\sqrt{3\alpha+1}\right] \\
&+ N_3^{(2,2)}\left[2k\operatorname{arcsinh}\left(\frac{2\sqrt{2}\sqrt{2\alpha+1}}{4\alpha+1}\right) - 4k\sqrt{\alpha}\arccos\left(\frac{-1}{4\alpha+1}\right) - \frac{\lambda}{12}\sqrt{4\alpha+2}\right]
\end{aligned}$$

Where α is the length of the timelike edges (spacelike edges are length 1), $k = \frac{1}{8\pi G_N}$, and $\lambda = k * \Lambda$.

To evaluate Equation (2.9), we use the Metropolis-Hastings algorithm as follows:

1. Selection: Pick a Pachner move;
2. Acceptance: Make that move with a probability of $a = a_1 a_2$, where

$$a_1 = \frac{move[i]}{\sum_i move[i]} \tag{2.10}$$

$$a_2 = e^{\Delta I_{CDT}} \tag{2.11}$$

Note that we have divided out the partition function $\frac{1}{C(T)}$ in Equation (2.9), which we didn't know how to evaluate anyway.

After thermalization, the Metropolis-Hastings algorithm gives us the distribution of triangulations for comoputing the path integral. We can then perform measures on these representative ensembles to calculate properties such as spectral dimension. [14, 15]

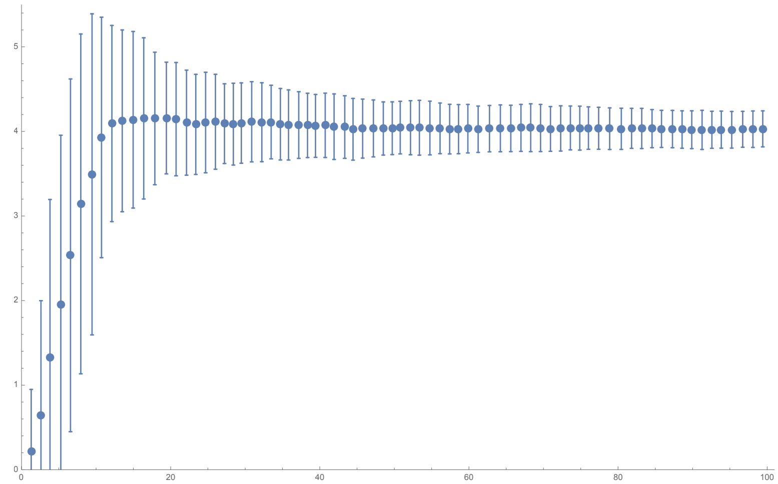

Figure 3: Myrheim-Meyer dimension for subsets of a relatively large causal set

3. Dynamical System

To generate the causal sets used in our analysis, we created a Mathematica notebook that allowed us to select random points uniformly from a causal diamond in Minkowski space, initially in four dimensions. We calculated the Myrheim-Meyer dimension and verified that it agreed with the dimension of of the background manifold in the limit of dense sprinklings. As is evident from figure 3, this limit was already reached with sprinklings of about 20 points, so we used this as a typical size.

To investigate the dependence of dimension on volume, one must choose a way to of select causal sets of "small" volume. Since the volume of a causal set is determined by the number of points in the set, it is tempting to simply average over all subsets containing a given number of points. This can be misleading, though: while such sets are "small" if viewed outside the context of the background spacetime, most of them do not come from a small region of the background spacetime, but include points spread across a large region of the background manifold. In particular, two points with a lightlike separation can be "adjacent" in a causal set even if they are widely separated in spacetime.

As an alternative, for each of our sprinklings we considered successively smaller sub-diamonds in the background spacetime. The points in each sub-diamond constitute a new causal set, whose volume and Myrheim-Meyer dimension we computed. We repeated the process for 10,000 sprinklings, and then averaged the dimension at each volume. We initially applied this analysis to four-dimensional Minkowski space, but subsequently repeated it for $d = 3$ and $d = 5$.

As noted above, the Myrheim-Meyer dimension is not well-defined for single points or causal sets with no edges. While this concern is unimportant for large causal sets, it must be confronted for the very small sets we are interested in. We explored two reasonable possibilities: taking the volume of an isolated point to be zero (they are, after all, single points) or dropping edgeless causal sets from our counting (they are causally disconnected from the rest of spacetime).

4. Results

In each of the background dimensions we studied, we found that dimensional reduction does indeed occur as the volume decreases. As shown in figures 4–6, the process appears to be smooth, but has a rather abrupt onset. The transition to lower dimension starts at a volume of approximately $V = 8$ points in three dimensions, $V = 16$ points in four dimensions, and $V = 22$ points in five dimensions.

At volumes above the transition, the Myrheim-Meyer dimension remains stable and equal to the dimension of the background Minkowski space. Below the transition, the decrease is quite rapid. For each of the background dimensions we considered, the minimum Myrheim-Meyer dimension falls to $d_M \approx 0$ if edgeless causal sets are taken to have dimension zero, and $d_M \approx 2$ if they are omitted. We can understand the latter result by noting that the smallest causal set with an edge—two points with one relation—has a Myrheim-Meyer dimension of two.

Figures 4–6 show $1\,\sigma$ error bars. We believe these are not a result of poor statistics, but are rather a consequence of our definition of volume. A causal diamond of a given volume in a background Minkowski space can contain many different causal sets, which will not all have identical Myrheim-Meyer dimensions. This leads to a genuine statistical fluctuation in dimension, especially at small volumes.

The end point $d_M \approx 2$ is reminiscent of the behavior seen in other investigations of quantum gravity. More precisely, when edgeless causal sets are discarded, we find a minimum dimension of $d_M = 2.08 \pm .26$ in three background dimensions, $d_M = 2.13 \pm .39$ in four background dimensions, and $d_M = 2.19 \pm .40$ in five background dimensions. It would be interesting to understand the fluctuations better, especially since a few other approaches to quantum gravity suggest a minimum dimension of 3/2

We would also like to understand what determines the scale at which dimensional reduction sets in. For three and four background dimensions, the transition seems to occur at a characteristic length of about twice the sprinkling length—that is, $V \sim 2^d$ points—but this pattern appears to break down for background dimension five. We also plan to investigate the behavior of another standard dimensional estimator, midpoint scaling dimension

Ideally, we would like to do more. The results we have presented here have the awkward feature of relying on the background Minkowski space to define the small regions whose dimension we measure. This was necessary to avoid picking out causal subsets that were "small" in the sense of having few points, but "large" in the sense of occupying a highly extended region. Recently, some progress has been made in defining "local" regions entirely in the context of causal sets, without reference to any background . It might be possible to use this work to investigate dimensional reduction more intrinsically.

Fractional volumes appear in the graphs because at a given background volume in Minkowski space, causal sets with varying numbers of points may be present.

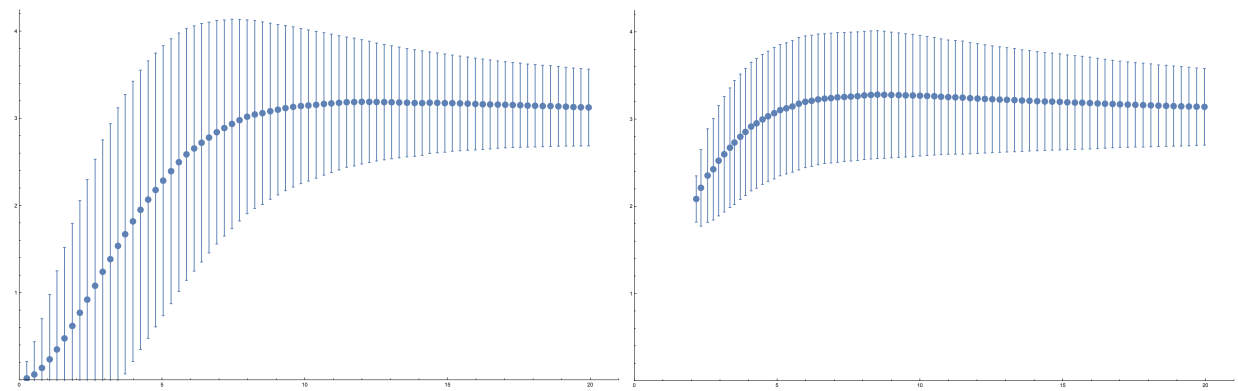

Figure 4: Myrheim-Meyer dimension in a three-dimensional background, with edgeless sets counted as dimension zero (left) or omitted (right)

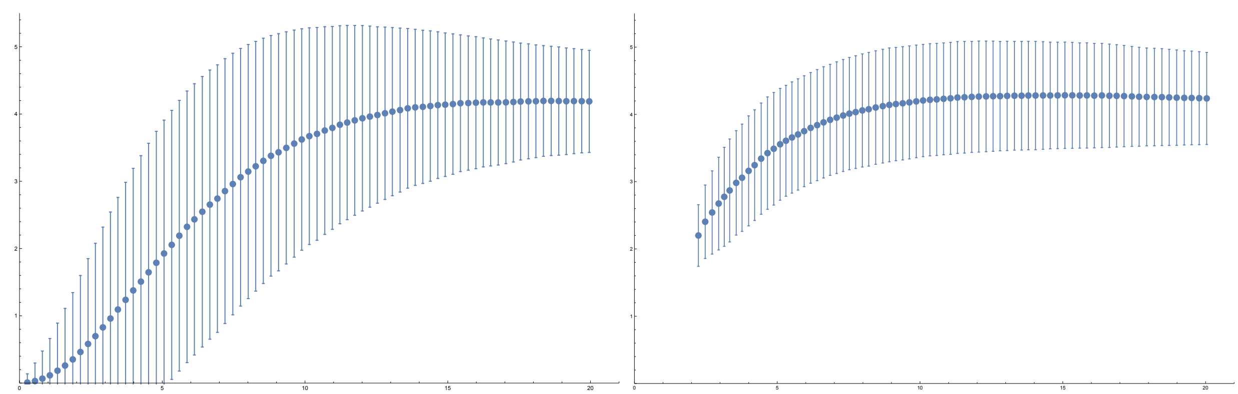

Figure 5: Myrheim-Meyer dimension in a four-dimensional background, with edgeless sets counted as dimension zero (left) or omitted (right)

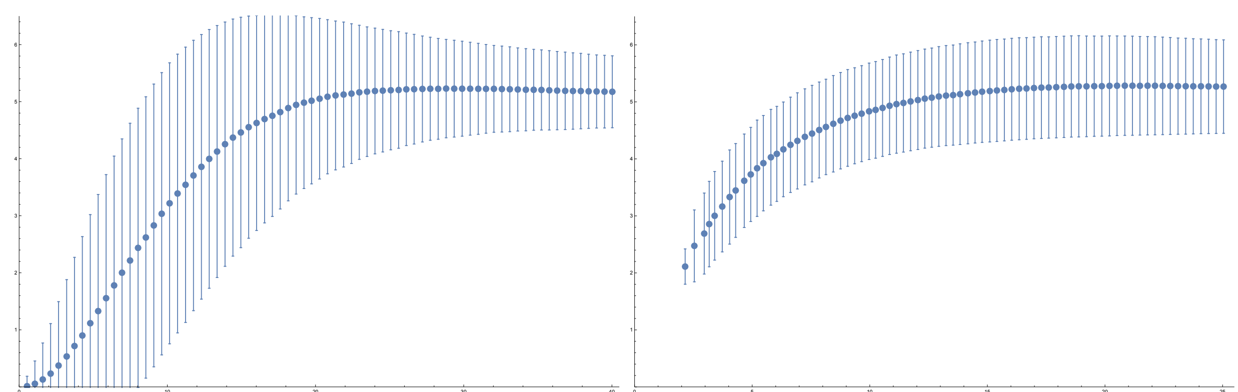

Figure 6: Myrheim-Meyer dimension in a five-dimensional background, with edgeless sets counted as dimension zero (left) or omitted (right)

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