Survey of Causal Dynamical Triangulations

Adam Getchell

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Abstract

I give a general overview of approaches to quantum gravity, then specifically detail the method of Causal Dynamical Triangulations.

Contents

1 History

The need for the development of a quantum theory of gravity was recognized almost at the beginning, starting with Einstein in 1916 who noted that quantum effects would lead to modifications in General Relativity. Rosenfeld applied the Pauli method for quantization of fields with gauge groups to the linearized Einstein field equation in 1930, Blokhintsev and Gal'perin describe the spin-two quantum of the gravitational field, the graviton, in a 1934 paper. In 1936 M.P. Bronstein rederived the Rosenfeld-Pauli quantization of linear quantum gravity, but realized that gravity had to be quantized in such a way as to remain background independent (unlike QED) and that non-Riemannian geometry would be needed. In 1938 Heisenberg points out that a dimensionful gravitational coupling constant would cause problems with a quantum field theory of gravity, and Fierz and Pauli discussed linear spin two quantum fields in a 1939 paper. [1]

In short, a sophisticated understanding of many of the issues of quantum gravity developed quickly.

2 Approaches to Quantum Gravity

It is generally agreed today that the goal of quantum gravity is to provide a consistent theoretical framework describing space-time geometry (with or without matter) which reduces in the classical limit to general relativity.

In 1957 Charles Misner introduced the Feynman quantization of general relativity in the following form:

$$\int \exp\left\{({}^{i}\!/\!{}_{\hbar})(S_{Einstein})\right\} d(f_{history}) \tag{1}$$

In his paper [2], Misner describes several modern concerns for a theory of quantum gravity:

- Why the quantum hamiltonian must be zero
- Why individual spacetime points are not defined in the quantum theory
- Why the integral must be gauge invariant

In addition, Misner lays out the three main approaches used to quantize gravity today.

2.1 Covariant

"The covariant line of research is the attempt to build the theory as a quantum field theory of the fluctuations of the metric over a flat Minkowski space, or some other background metric space. The program was started by Rosenfeld, Fierz and Pauli in the thirties. The Feynman rules of general relativity (GR, from now on) were laboriously found by DeWitt and Feynman in the sixties. t'Hooft and Veltman, Deser and Van Nieuwenhuizen, and others, found firm evidence of non-renormalizability at the beginning of the seventies. Then, a search for an extension of GR giving a renormalizable or finite perturbation expansion started. Through high derivative theory and supergravity, the search converged successfully to string theory in the late eighties."[1]

2.2 Canonical

"The canonical line of research is the attempt to construct a quantum theory in which the Hilbert space carries a representation of the operators corresponding to the full metric, or some functions of the metric, without background metric to be fixed. The program was set by Bergmann and Dirac in the fifties. Unraveling the canonical structure of GR turned out to be laborious. Bergmann and his group, Dirac, Peres, Arnowit Deser and Misner completed the task in the late fifties and early sixties. The formal equations of the quantum theory were then written down by Wheeler and DeWitt in the middle sixties, but turned out to be too ill-defined. A well defined version of the same equations was successfully found only in the late eighties, with loop quantum gravity." [1]

2.3 Sum over histories

This uses a version of Feynman's functional integral quantization (see equation 1) to define the theory. Instead of using perturbative approaches to generate effective theories and then dealing with the mechanics of renormalization (the material of this class), a non-perturbative path integral is taken as the starting point. Discrete approaches such as Causal Dynamical Triangulations use this method by exploiting certain relations to generate the summation over all paths, as will be described on this page. [3]

3 Causal Dynamical Triangulations

3.1 What's in a name?

Causal Dynamical Triangulations is a relatively new approach that uses several important technical advances[4]:

- It uses a causal, or Lorentzian, signature for the metric (which avoids problems of degenerate quantum geometries which occur with Euclidean metrics)
- A non-perturbative approach path integral is used, removing the necessity for initial input of a fixed background metric. This allows space-time to be determined dynamically as part of the theory.
- Spacetime is approximated by tiling flat simplicial building blocks. Tiling means that simplexes are connected according to particular rules, the most important being preservation of causality. Flat means that the building

blocks are Minkowski sub-spaces, which allow for genuine manifolds which are locally $\mathbb{R}^d$ where a simplex is a d-dimensional generalized triangle.

3.2 The quantum path integral

In classical General Relativity, we obtain a solution metric spacetime (M, $g_{\mu\nu}$) from the Einstein equations:

$$R_{\mu\nu}[g] - \frac{1}{2} g_{\mu\nu} R[g] + \Lambda g_{\mu\nu} = -8\pi G_N T_{\mu\nu}[\Phi] \tag{2}$$

which are the equations of motion on the space $\mathcal{M}(M)$, the space of all metrics on the differentiable manifold M.

Similarly, we desire to obtain a "quantum space-time" which arises as a solution to a quantum equations of motion on the quantum analog of $\mathcal{M}(M)$. Along the way (and in contrast with the covariant approach), we must give up the flat Minkowski background space-time on which quantization happens, which in turn abandons the Poincare group and all the standard quantum field theory tools for regularization and renormalization.

To begin with (and following Loll, 2008) we consider the path integral representation for a 1-d non-relativistic particle.

$$\psi\left(x'', t''\right) = \int_R G\left(x'', x'; t'', t'\right) \psi\left(x', t'\right) \tag{3}$$

where the propagator or Feynman kernel G is:

$$G\left(x'', x'; t'', t'\right) = lim_{\epsilon \to 0} A^{-N} \prod_{k=1}^{N-1} \int dx_k e^{i \sum_{j=0}^{N-1} \epsilon L(x_{j+1}, (x_{j+1} - x_j)/\epsilon)} \tag{4}$$

and the time interval t"-t' is discretized into N steps of length $\epsilon = {}^{(t''-t')}/_N$ and the right hand side of equation 4 is an integral over all piecewise linear paths propagating from x' to x" as shown in Figure 1.

This gives us the path integral:

$$G(x', t'; x'', t'') = \int \mathcal{D}x(t) \exp^{iS[x(t)]} \tag{5}$$

where $\mathcal{D}x(t)$ is the functional measure on the space of all paths weighted exponentially by the classical action $S[x(t)]$. The usual trick of Wick-rotation $t \to \tau = it$ converts the integrals into real ones.

If we know the propagator G we know the quantum dynamics of the system.

For Einstein gravity, then, we wish to define a gravitational propagator

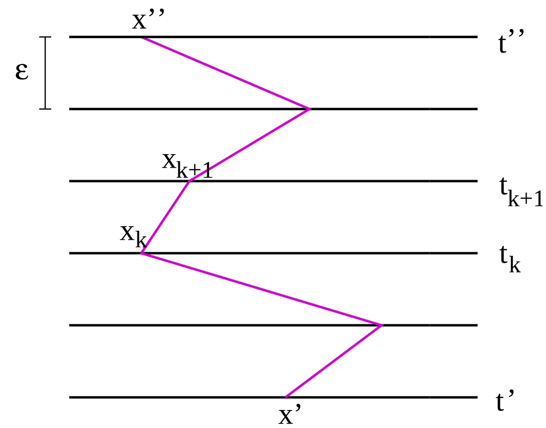

Figure 1: Piecewise linear particle path contributing to the discrete Feynman propagator.

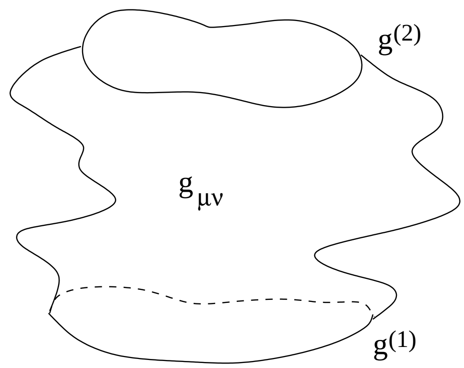

Figure 2: The time-honored way [5] of illustrating the gravitational path integral as the propagator from an initial to a final spatial boundary geometry

$$G\left(\left[g'_{ij}\right], \left[g''_{ij}\right]\right) = \int_{Geom(M)} \mathcal{D}\left[g_{\mu\nu}\right] e^{iS_{Einstein}\left[g_{\mu\nu}\right]} \tag{6}$$

from initial geometry [g'] to final geometry [g"] as the continuum limit of the discrete propagator given by equation 4.

We have several problems with Equation 6:

1. How do we parametrize geometries? (The usual prescription starts with gauge covariant fields and then gauge fixes. This leads to the usual Faddeev-Popov ghosts which are difficult to evaluate non-perturbatively. [6]

2. No matter how we choose the fields, $S_{Einstein}$ is non-quadratic which leaves us with no clear-cut way to evaluate $\int \exp(iS_{Einstein})$
3. No obvious choice of Wick rotation
4. How to non-perturbatively regularize without violating diffeomorphism-invariance [7][1]

3.3 Constructing spacetime a piece at a time

To start with, we will assume that Geom(M) (Loll's notation) is a regularized space of all geometries with a regularized path integral $G(a)$ which contains an ultraviolet cutoff a and converges in a non-trivial region of the coupling constant space. The continuum limit corresponds to $a \to 0$ and if Geom(M) exists it can be investigated with respect to geometric properties and the semiclassical limit. This takes care of diffeomorphism-invariant non-perturbative regularization.

Next, we use the general ideas of Regge calculus [8] and earlier dynamical triangulations by using locally-flat simplices (d-dimensional generalizations of triangles) to carefully build up our manifold, which is also locally-flat (locally $\mathcal{R}^d$). This gives us several crucial items:

- By selecting a proper-time orientation, that is, edges/vertices connected only to time τ or $\tau+1$ we preserve the causal structure and avoid particular problems engendered in Euclidean dynamical triangulations.
- Because we have preferred notion of discrete time, via a long chain of careful reasoning [9] we can define an analytic continuation in time of $\Lambda \to -i\Lambda$and thus unambiguously defining a Wick rotation.
- The simplicial manifolds are completely described by the discrete set $\{l_i^2\}$ of the squared lengths of their edges. This gives us a completely regularized parameterization of Geom(M), and incidentally results in a coordinate-free description of geometry.[2] It also gives a straightforward prescription for evaluating $e^{-iS_{Einstein}}$, since (as we shall see) $S_{Einstein}$ is given simply in terms of explicit countings of geometries.

The crucial technical advance is the preservation of the Lorentzian signature. To see why, let us examine some of the issues that arise from the use of Euclidean simplices.

The phase diagram of Euclidean dynamical triangulations in Figure 3 shows how the model possesses infinite-volume limits everwhere along a critical line $k_3^{crit}(k_0)$, which fixes the bare cosmological constant as a function of $k_0 \sim G_N^{-1}$, the inverse Newton constant.

[1] In particular, although we are acquainted with perturbative dimensional regularization from class it has been argued that this can fail in a nonperturbative setting.

[2] We can recover coordinates by introducing coordinate patches on each individual simplex and define suitable transistion functions between patches. This gives a definite form for the metric tensor, but it is unnecessary in the formulation of the path integral.

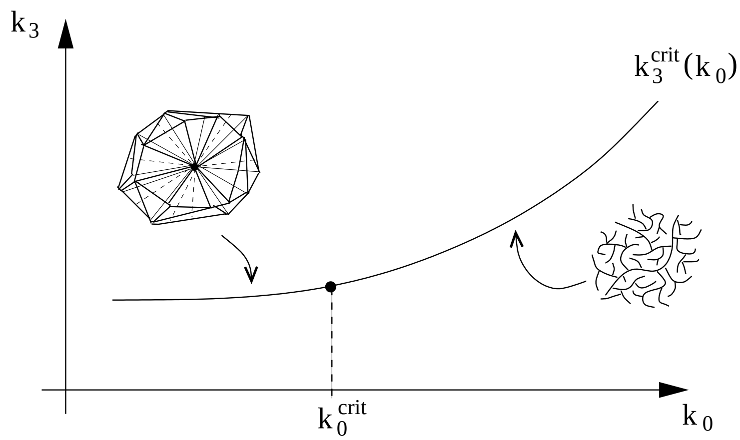

Figure 3: The phase diagram of 3- and 4-d Euclidean dynamical triangulations.

Along this line, there is the critical point k_0^{crit} below which geometries have been shown in simulations to have very large Hausdorff dimension. This corresponds to the condensation of spacetime around a few well-connected vertices such that every other vertex is at most one link-distance away.

Along $k_3^{crit}(k_0)$ and above k_0^{crit} simulations show polymerization: a typical element of the state sum is a thin branched polymer with dimensions curled up such that the effective dimension is ~2.[3]

Now we return to the assertion that piecewise linear manifolds have their intrinsic geometry completely specified by the edge lengths of their constituent simplices.

In two dimensions the linear space is triangulated, and the scalar curvature $R(x)$ corresponds with the Gaussian curvature. If we parallel-transport a vector around a closed curve we will measure the curvature. For the lattice vertices v where the surrounding angles added up to some $\sum_{i \supset v} \alpha_i = 2/pi - \delta$ for $\delta \neq 0$ then this deficit angle δ is precisely the rotation picked up by the vector as seen in Figure 4.

Then the Regge calculus gives the curvature part of the Einstein action in terms of (d-2) dimensional building blocks, which are dual to the localized 2d submanifolds used in the tilings. Equations (7)and (8) give the continuum curvature and volume terms of the action.

[3]As noted by Loll in her paper, this has an interesting correspondence with string theory.

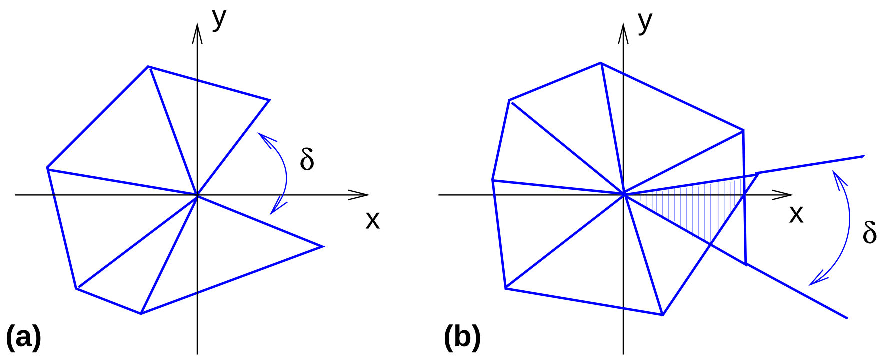

Figure 4: Positive (a) and negative (b) space-like deficit angles δ

$$\frac{1}{2}\int_{\mathcal{R}} d^d x \sqrt{|\det g|}^{(d)} R \longrightarrow \sum_{i\in\mathcal{R}} Vol(i^{th}(d-2)-simplex)\delta_i \tag{7}$$

$$\int_{\mathcal{R}} d^d x \sqrt{|\det g|} \longrightarrow \sum_{i\in\mathcal{R}} Vol(i^{th} d-simplex) \tag{8}$$

3.4 3D Simplices

Now for Lorentzian (causal) triangulations we allow for both spacelike and time-like links such that $l_i^2 \in \left\{-a^2, a^2\right\}$ for the geodesic distance a. In three dimensions we have the following relations: [10]

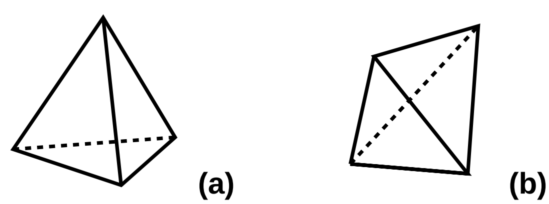

Figure 5: A (3,1)- and a (2,2)-tetrahedron in three dimensions

To distinguish space-like and time-like edges, where space-like edges lie entirely in the plane t=constant and time-like edges connect t and t+1, we take $l_{space}^2 = a^2$ and $l_{time}^2 = -\alpha a^2$.

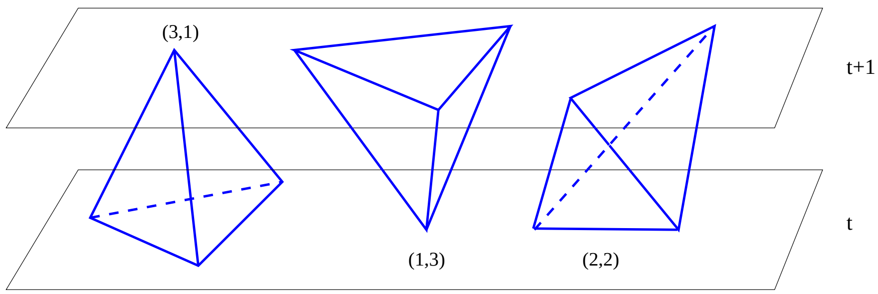

Figure 6: The three types of tetrahedral building blocks used in 3d Lorentzian gravity

Then we have the volume of the (3,1)-tetrahedron given by:

$$Vol(3,1) = \frac{1}{12}\sqrt{3\alpha+1} \tag{9}$$

And the dihedral angle around the space-like vertices are (using complex trigonometric functions):

$$\cos\Theta_{(3,1)} = -\frac{i}{\sqrt{3}\sqrt{4\alpha+1}} \tag{10}$$

$$\sin\Theta_{(3,1)} = \frac{2\sqrt{3\alpha+1}}{\sqrt{3}\sqrt{4\alpha+1}} \tag{11}$$

Around the time-like vertices the relations are:

$$\cos\Theta_{(3,1)} = \frac{2\alpha+1}{4\alpha+1} \tag{12}$$

$$sin\Theta_{(3,1)} = \frac{2\sqrt{\alpha}\sqrt{3\alpha+1}}{4\alpha+1} \tag{13}$$

For the (2,2)-tetrahedra:

$$Vol(2,2) = \frac{1}{6\sqrt{2}}\sqrt{2\alpha+1} \tag{14}$$

The space-like dihedral angle is:

$$\cos\Theta_{(2,2)} = \frac{4\alpha+3}{4\alpha+1} \tag{15}$$

$$\sin\Theta_{(2,2)} = -i\frac{2\sqrt{2}\sqrt{2\alpha+1}}{4\alpha+1} \tag{16}$$

And the time-like dihedral angle is:

$$\cos\Theta_{(2,2)} = -\frac{1}{4\alpha+1} \tag{17}$$

$$\sin\Theta_{(2,2)} = \frac{2\sqrt{2}\sqrt{2\alpha+1}}{4\alpha+1} \tag{18}$$

Remember, by specifying the dihedral angle for each simplex we specify the curvature. We thus have the curvature of the simplex in terms of the edge lengths only, and by tiling the 2+1 space with 3D simplices we now have a well-defined, Minkowski, spacetime lattice with which to perform calculations. The sum over all histories approach then becomes a simple matter of counting all possible tilings.

3.5 4D Simplices

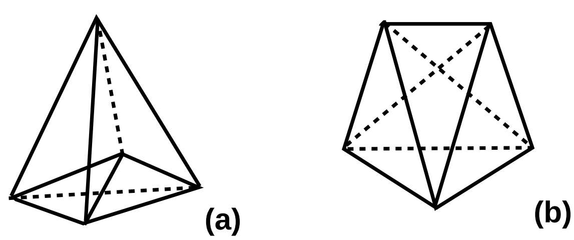

Figure 7: A (4,1)- and a (3,2)-tetrahedron in four dimensions

For d=4 there are reflection symmetries of two types of four-simplices, (4,1) and (3,2). It is also necessary to use equilateral tetrahedra that lie entirely on slices of constant t, given by:

$$Vol_{space-like} = \frac{1}{6\sqrt{2}} \tag{19}$$

$$Vol(4,1) = \frac{1}{96}\sqrt{8\alpha+3} \tag{20}$$

The (4,1)-simplex has 10 dihedral angles of the usual two types, space-like and time-like. Space-like dihedral angles are given by:

$$\cos\Theta_{(4,1)} = -\frac{i}{2\sqrt{2}\sqrt{3\alpha+1}} \tag{21}$$

$$\sin\Theta_{(4,1)} = \sqrt{\frac{3(8\alpha+3)}{8(3\alpha+1)}} \tag{22}$$

Timelike dihedral angles are determined by:

$$\cos\Theta_{(4,1)} = \frac{2\alpha+1}{2(3\alpha+1)} \tag{23}$$

$$\sin\Theta_{(4,1)} = \frac{\sqrt{4\alpha+1}\sqrt{8\alpha+3}}{2(3\alpha+1)} \tag{24}$$

As before, we are using trigonometric functions that are complex. We now generate relations for the (3,2)-simplex, starting with the volume:

$$Vol(3,2) = \frac{1}{96}\sqrt{12\alpha+7} \tag{25}$$

The dihedral angles are a bit trickier, because for the ten angles per four-simplex, there is exactly one spacelike triangle given by:

$$\cos\Theta_{(3,2)spacelike} = \frac{6\alpha+5}{2(3\alpha+1)} \tag{26}$$

$$\sin\Theta_{(3,2)spacelike} = -i\frac{\sqrt{3}\sqrt{12\alpha+7}}{2(3\alpha+1)} \tag{27}$$

There are two types of time-like triangles. First, if Θ is formed by two (2,2)-tetrahedra then:

$$\cos\Theta_{(3,2)\to(2,2)} = \frac{4\alpha+3}{4(2\alpha+1)} \tag{28}$$

$$\sin\Theta_{(3,2)\to(2,2)} = \frac{\sqrt{(4\alpha+1)(12\alpha+7)}}{4(2\alpha+1)} \tag{29}$$

If the two facing tetra hedra are a (3,1) and a (2,2) then the angle is:

$$\cos\Theta_{(3,2)} = \frac{-1}{2\sqrt{2}\sqrt{2\alpha+1}\sqrt{3\alpha+1}} \tag{30}$$

$$\sin\Theta_{(3,2)} = \frac{\sqrt{(4\alpha+1)(12\alpha+7)}}{2\sqrt{2}\sqrt{2\alpha+1}\sqrt{3\alpha+1}} \tag{31}$$

3.6 Topological Identities

The best known is the Euler identity which states:

$$\chi = N_0 - N_1 + N_2 - N_3 + \ldots \tag{32}$$

Where χ is the Euler characteristic of the manifold and N_0 is the number of vertices (points), N_1 is the number of lines, N_2 is the number of triangles, N_3 is the number of tetrahedra, and so forth. By working with the particular topology we can derive relations between these "bulk" variables which we will use to express our action. For details see [10, pp. 9-11].

3.7 Wick Rotation

With the above definitions, it can be shown that for the complex α-plane subject to the constraints:

$$\alpha \mapsto -\alpha \tag{33}$$

$$\alpha > \frac{1}{2} \tag{34}$$

in the 3d case and

$$\alpha > \frac{7}{12} \tag{35}$$

for the 4d case then we can define a non-perturbative Wick roation from Lorentzian to Euclidean discrete geometries according to:

$$e^{iS} \mapsto e^{-S_{Euclidean}} \tag{36}$$

The special case $\alpha = -1$ just reproduces the Euclidean triangulations.

3.8 Gravitational Action

Finally, with a well defined Wick rotation and the bulk variables from our Toplogical Identities, we can define the Gravitation action:

In 3D:

$$S^{(3)} = k\left(\frac{2\pi}{i}N_1^{SL} - \frac{2}{i}N_3^{(2,2)}\arcsin\left(\frac{-i2\sqrt{2}\sqrt{2\alpha+1}}{4\alpha+1}\right) - \frac{3}{i}N_3^{(3,1)}\arccos\left(\frac{-i}{\sqrt{3}\sqrt{4\alpha+1}}\right)\right)$$

$$+k\sqrt{\alpha}\left(2\pi N_1^{TL} - 4N_3^{(2,2)}\arccos\left(\frac{-1}{4\alpha+1}\right) - 3N_3^{(3,1)}\arccos\left(\frac{2\alpha+1}{4\alpha+1}\right)\right)$$

$$-\lambda\left(N_3^{(2,2)}\frac{1}{12}\sqrt{4\alpha+2} + N_3^{(3,1)}\frac{1}{12}\sqrt{3\alpha+1}\right)$$

In 4D:

$$S^{(4)} = k\left(\frac{2\pi}{i}\frac{\sqrt{3}}{4}N_2^{SL} - \frac{\sqrt{3}}{4i}N_4^{(3,2)}\arcsin\left(\frac{-i\sqrt{3}\sqrt{12\alpha+7}}{2(3\alpha+1)}\right)\right)$$

$$-\frac{\sqrt{3}}{i}N_4^{(4,1)}\arccos\left(\frac{-i}{2\sqrt{2}\sqrt{3\alpha+1}}\right) + \frac{k}{4}\sqrt{4\alpha+1}\{2\pi N_2^{TL} -$$

$$-N_4^{(3,2)}\left(6\arccos\left(\frac{-1}{2\sqrt{2}\sqrt{3\alpha+1}}\right) + 3\arccos\left(\frac{4\alpha+3}{4(2\alpha+1)}\right)\right)$$

$$-6N_4^{(4,1)}\arccos\left(\frac{2\alpha+1}{2(3\alpha+1)}\right)\} - \lambda\left(N_4^{(4,1)}\frac{\sqrt{8\alpha+3}}{96} + N_4^{(3,2)}\frac{\sqrt{12\alpha+7}}{96}\right)$$

References

[1] Rovelli, Carlo. "[gr-qc/0006061v3] Notes for a brief history of quantum gravity." Roma, 2000. http://arxiv.org/abs/gr-qc/0006061v3.

[2] C Misner, "Feynman quantization of general relativity", Rev Mod Phys 29 (1957) 497.

[3] R. Loll. "Discrete Approaches to Quantum Gravity in Four Dimensions." Living Reviews in relativity, November 19, 1998. http://relativity.livingreviews.org/Articles/lrr-1998-13/.

[4] Loll, R. "A discrete history of the Lorentzian path integral." Lecture Notes in Physics 631 (2003): 137-171. http://arxiv.org/abs/hep-th/0212340v2.

[5] S.W. Hawking: in General relativity: an Einstein centenary survey, ed. S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979) 746-789.

[6] Zwanziger, Daniel. "Non-perturbative modification of the Faddeev-Popov formula." Physics Letters B 114, no. 5 (August 5, 1982): 337-9. http://www.sciencedirect.com/science?_ob=ArticleURL&_udi=B6TVN-472JTDW-1JT&_user=4421&_rdoc=1&_fmt=&_orig=search&_sort=d&view=c&_acct=C000059598

[7] Philips, Daniel, Silas Beane, and Thomas Cohen. "Nonperturbative Regularization and Renormalization: Simple Examples from Nonrelativistic Quantum Mechanics." Annals of Physics 263, no. 2: 255-75. http://adsabs.harvard.edu/abs/1998AnPhy.263..255P.

[8] T. Regge, Nuovo Cimento 19 (1961) 558

[9] Ambjorn, J., and R. Loll. "Non-perturbative Lorentzian Quantum Gravity, Causality and Topology Change." Nuclear Physics B 536, no. 1998 (May 18, 1998). http://arxiv.org/abs/hep-th/9805108.

[10] J. Ambjorn, J. Jurkiewicz, and R. Loll. "Dynamically Triangulating Lorentzian Quantum Gravity." Nuclear Physics B 610, no. 2001 (May 27, 2001): 347-382. http://arxiv.org/abs/hep-th/0105267.