Detailed calculations for the Weyl metric

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December 5, 2013

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1 Vacuum solution to the Weyl metric

Starting from the cylindrically symmetric (Weyl) vacuum metric [1]:

$$ds^2 = e^{2\lambda}dt^2 - e^{2(\nu-\lambda)}\left(dr^2 + dz^2\right) - r^2 e^{-2\lambda} d\phi^2 \tag{1}$$

$$g_{\mu\nu} = \begin{pmatrix} e^{2\lambda} & 0 & 0 & 0 \\ 0 & -e^{2(\nu-\lambda)} & 0 & 0 \\ 0 & 0 & -e^{2(\nu-\lambda)} & 0 \\ 0 & 0 & 0 & -r^2 e^{-2\lambda} \end{pmatrix} \tag{2}$$

In this coordinate basis, the definition of the Christoffel connection is: [2]

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tag{3}$$

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The non-zero Christoffel connections are:

$$
\begin{aligned}
\Gamma^t_{tr} &= \partial_r \lambda \\
\Gamma^t_{tz} &= \partial_z \lambda \\
\Gamma^r_{tt} &= e^{4\lambda - 2\nu} \partial_r \lambda \\
\Gamma^r_{rr} &= \partial_r \nu - \partial_r \lambda \\
\Gamma^r_{rz} &= \partial_z \nu - \partial_z \lambda \\
\Gamma^r_{zz} &= \partial_r \lambda - \partial_r \nu \\
\Gamma^r_{\phi\phi} &= r e^{-2\nu} \left(r \partial_r \lambda - 1 \right) \\
\Gamma^z_{tt} &= e^{4\lambda - 2\nu} \partial_z \lambda \\
\Gamma^z_{rr} &= \partial_z \lambda - \partial_z \nu \\
\Gamma^z_{rz} &= \partial_r \nu - \partial_r \lambda \\
\Gamma^z_{zz} &= \partial_z \nu - \partial_z \lambda \\
\Gamma^z_{\phi\phi} &= r^2 e^{-2\nu} \partial_z \lambda \\
\Gamma^\phi_{r\phi} &= \frac{1}{r} - \partial_r \lambda \\
\Gamma^\phi_{z\phi} &= -\partial_z \lambda
\end{aligned}
\tag{4}
$$

The components of the Riemann tensor are given by:

$$
R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \tag{5}
$$

Using the properties of the Riemann tensor:

$$
\begin{aligned}
R_{\rho\sigma\mu\nu} &= -R_{\rho\sigma\nu\mu} \\
R_{\rho\sigma\mu\nu} &= -R_{\sigma\rho\mu\nu} \\
R_{\rho\sigma\mu\nu} &= R_{\mu\nu\rho\sigma} \\
R_{\rho[\sigma\mu\nu]} &= 0
\end{aligned}
\tag{6}
$$

The non-zero components of the Riemann tensor are:

$$
\begin{aligned}
R^t_{rtr} &= -\partial_r^2 \lambda + (\partial_z \lambda)^2 - 2 (\partial_r \lambda)^2 + \partial_r \lambda \partial_r \nu - \partial_z \lambda \partial_z \nu \\
R^t_{rtz} &= -\partial_r \partial_z \lambda - 3 \partial_r \lambda \partial_z \lambda + \partial_r \lambda \partial_z \nu + \partial_r \nu \partial_z \lambda \\
R^t_{ztz} &= -\partial_z^2 \lambda - 2 (\partial_z \lambda)^2 + (\partial_r \lambda)^2 - \partial_r \lambda \partial_r \nu + \partial_z \lambda \partial_z \nu \\
R^t_{\phi t\phi} &= r e^{-2\nu} \left(r (\partial_r \lambda)^2 - \partial_r \lambda + r (\partial_z \lambda)^2 \right) \\
R^r_{zrz} &= \partial_r^2 \lambda - \partial_r^2 \nu + \partial_z^2 \lambda - \partial_z^2 \nu \\
R^z_{\phi z\phi} &= r e^{-2\nu} \left(r \partial_z^2 \lambda - r \partial_z \lambda \partial_z \nu + r \partial_r \lambda \partial_r \nu - r (\partial_r \lambda)^2 + \partial_r \lambda - \partial_r \nu \right) \\
R^z_{\phi\phi r} &= r e^{-2\nu} \left(-r \partial_r \partial_z \lambda + r \partial_r \nu \partial_z \lambda - r \partial_r \lambda \partial_z \lambda + r \partial_r \lambda \partial_z \nu - \partial_z \nu \right) \\
R^\phi_{r\phi r} &= \partial_r^2 \lambda + \frac{1}{r} \partial_r \nu - \partial_r \lambda \partial_r \nu - (\partial_z \lambda)^2 + \partial_z \lambda \partial_z \nu + \frac{1}{r} \partial_r \lambda
\end{aligned}
\tag{7}
$$

The Ricci tensor is given by:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \tag{8}$$

The non-zero components of the Ricci tensor are:

$$\begin{aligned}
R_{tt} &= \frac{e^{4\lambda-2\nu}}{r}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right) \\
R_{rr} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2\left(\partial_r\lambda\right)^2 + \frac{1}{r}\partial_r\lambda + \frac{1}{r}\partial_r\nu \\
R_{rz} &= \frac{1}{r}\partial_z\nu - 2\partial_r\lambda\partial_z\lambda \\
R_{zz} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2\left(\partial_z\lambda\right)^2 + \frac{1}{r}\partial_r\lambda - \frac{1}{r}\partial_r\nu \\
R_{\phi\phi} &= re^{-2\nu}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right)
\end{aligned} \tag{9}$$

The Ricci scalar is defined as:

$$R = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu} \tag{10}$$

Which is:

$$R = 2e^{2(\lambda-\nu)}\left(\partial_r^2\nu + \partial_z^2\nu - \partial_r^2\lambda - \partial_z^2\lambda + \left(\partial_r\lambda\right)^2 + \left(\partial_z\lambda\right)^2 - \frac{1}{r}\partial_r\lambda\right) \tag{11}$$

Einstein's equation in a vacuum is:

$$G_{\mu\nu} = 0 \tag{12}$$

Whence Einstein's equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{13}$$

However, we can take a shortcut by using:

$$R_{\mu\nu} = 0 \tag{14}$$

Since the trace of a zero-valued matrix is identically zero, and thus Equation (14) automatically satisfies Equation (12).

Applying Equation (14) to Equation (9) gives the following:

$$\partial_r^2\lambda + \frac{1}{r}\partial_r\lambda + \partial_z^2\lambda = 0 \tag{15}$$

$$\partial_r\nu = r\left(\partial_r^2\nu + \partial_z^2\nu + 2\left(\partial_r\lambda\right)^2\right) \tag{16}$$

$$\partial_z\nu = 2r\partial_r\lambda\partial_z\lambda \tag{17}$$

$$\partial_r^2\nu + \partial_z^2\nu + \left(\partial_r\lambda\right)^2 + \left(\partial_z\lambda\right)^2 = 0 \tag{18}$$

Equation (15) is the two-dimensional Laplace equation in cylindrical coordinates. That is:

$$\nabla^2\lambda(r,z) = 0 \tag{19}$$

Plugging Equation (18) into Equation (16) gives:

$$\partial_r \nu = r\left((\partial_r\lambda)^2 - (\partial_z\lambda)^2\right) \tag{20}$$

Using Equations (17) and (20) we find solutions for ν are given by:

$$\nu = \int r[\left((\partial_r\lambda)^2 - (\partial_z\lambda)^2\right)dr + (2\partial_r\lambda\partial_z\lambda)\,dz] \tag{21}$$

The solutions must satisfy Equations (19) and (21). A particular solution corresponding to two objects (given by Curzon in 1924 [3]) is:

$$\lambda_0(r,z) = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} \tag{22}$$

$$\nu_0(r,z) = -\frac{1}{2}\frac{\mu_1^2 r^2}{r_1^4} - \frac{1}{2}\frac{\mu_2^2 r^2}{r_2^4} + \frac{2\mu_1\mu_2}{(z_1-z_2)^2}\left[\frac{r^2+(z-z_1)(z-z_2)}{r_1 r_2} - 1\right] \tag{23}$$

Where z_1 and z_2 correspond to the positions on the z-axis for the two objects, μ_1 and μ_2 are length parameters, and:

$$r_1 = \sqrt{r^2 + (z-z_1)^2} \tag{24}$$

$$r_2 = \sqrt{r^2 + (z-z_2)^2} \tag{25}$$

Just as a final check, plugging Equations (15) and (18) into Equation (11) gives $R = 0$, which shows that our solutions are consistent with our assumptions.

By construction, these solutions only apply to empty space, and so must exclude the two objects at z_1 and z_2. In addition, as noted by Synge [1], the z axis between the two objects must also be excluded due to violation of elementary flatness. We will examine this in the next section.

2 Curvature from Parallel Transport

Consider parallel transport of a vector V about the z-axis in the $\hat{\phi}$ direction. The equation for parallel transport is generally given by:

$$\tfrac{D}{d\lambda} = \tfrac{dx^\mu}{d\lambda}\nabla_\mu = 0 \quad \text{along} \quad x^\mu(\lambda) \tag{26}$$

That is, the directional covariant derivative is equal to zero along the curve x^μ parameterized by λ. For a vector this can be simply written as:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda}V^\lambda = 0 \tag{27}$$

Starting with parallel transport along $\hat{e}_\phi$, Equation (27) along with the relevant Christoffel symbols $\Gamma^r_{\phi\phi}$, $\Gamma^z_{\phi\phi}$, $\Gamma^\phi_{\phi r}$, and $\Gamma^\phi_{\phi z}$ gives:

$$
\begin{aligned}
\partial_\phi V^r + \Gamma^r_{\phi\phi} V^\phi &= 0 \\
\partial_\phi V^z + \Gamma^z_{\phi\phi} V^\phi &= 0 \\
\partial_\phi V^\phi + \Gamma^\phi_{\phi r} V^r + \Gamma^\phi_{\phi z} V^z &= 0
\end{aligned} \tag{28}
$$

Plugging in the values from Equation (4), our equations are:

$$
\partial_\phi V^r + \left(re^{-2\nu} \left(r\partial_r \lambda - 1 \right) \right) V^\phi = 0 \tag{29}
$$

$$
\partial_\phi V^z + \left(r^2 e^{-2\nu} \partial_z \lambda \right) V^\phi = 0 \tag{30}
$$

$$
\partial_\phi V^\phi + \left(\frac{1}{r} - \partial_r \lambda \right) V^r - \partial_z \lambda V^z = 0 \tag{31}
$$

Differentiating Equation (31) with respect to ϕ and plugging it into Equation (29) gives:

$$
\partial^2_\phi V^\phi - \partial_z \lambda \partial_\phi V^z + r^2 e^{-2\nu} \left(\partial_r \lambda - \frac{1}{r} \right)^2 V^\phi = 0 \tag{32}
$$

Plugging in the expression for $\partial_\phi V^z$ from Equation (30) and letting

$$
\chi = re^{-\nu} \sqrt{(\partial_z \lambda)^2 + \left(\frac{1}{r} - \partial_r \lambda \right)^2} \tag{33}
$$

We have the simple differential equation:

$$
\partial^2_\phi V^\phi + \chi^2 V^\phi = 0 \tag{34}
$$

For which the solution is:

$$
V^\phi = A \sin \chi\phi + B \cos \chi\phi \tag{35}
$$

Therefore, integrating Equation (29) with respect to ϕ we get:

$$
V^r = \frac{r^2 e^{-2\nu} (\partial_r \lambda - \frac{1}{r})}{\chi} \left(A \cos \chi\phi - B \sin \chi\phi \right) \tag{36}
$$

And from Equation (30):

$$
V^z = \frac{r^2 e^{-2\nu} \partial_z \lambda}{\chi} \left(A \cos \chi\phi - B \sin \chi\phi \right) \tag{37}
$$

So our general vector is then:

$$
\begin{aligned}
V = \frac{r^2 e^{-2\nu} (\partial_r \lambda - \frac{1}{r})}{\chi} & \left(A \cos \chi\phi - B \sin \chi\phi \right) \hat{e}_r \\
+ \frac{r^2 e^{-2\nu} \partial_z \lambda}{\chi} & \left(A \cos \chi\phi - B \sin \chi\phi \right) \hat{e}_z \\
& + \left(A \sin \chi\phi + B \cos \chi\phi \right) \hat{e}_\phi
\end{aligned} \tag{38}
$$

Normalizing $V(\phi = 0)$:

$$g_{\mu\nu}V^{\mu}V^{\nu} = 1 \tag{39}$$

We obtain the condition that:

$$A^2 + B^2 = r^{-2}e^{2\lambda} \tag{40}$$

For simplicity, we choose $A^2 = r^{-2}e^{2\lambda}$ and $B^2 = 0$.

Now, when we parallel transport V around to $\phi = 2\pi$ there will be an angle between $V(\phi = 0)$ and $V(\phi = 2\pi)$ given by the definition of the scalar product:

$$\cos(\beta) = \frac{g_{\mu\nu}V^{\mu}(0)V^{\nu}(2\pi)}{g_{\mu\nu}V^{\mu}(0)V^{\nu}(0)} \tag{41}$$

Since we have normalized our vectors, the denominator is equal to 1, and we get the expression that:

$$\cos\beta = \cos(2\pi\chi) \tag{42}$$

Where χ is given by Equation (33). Hence $\beta = 2\pi\chi$. We can now use the definition of the deficit angle:

$$\Delta = 2\pi - \beta = 2\pi(1-\chi) \tag{43}$$

To get the curvature $\mathscr{R}$ via:

$$\mathscr{R} = \lim_{A\to 0}\frac{\Delta}{A} \tag{44}$$

The area A is defined on the reduced metric:

$$ds^2 = e^{2(\nu-\lambda)}dr^2 + r^2e^{-2\lambda}d\phi^2 \tag{45}$$

Via:

$$A = \int\sqrt{|g|}d^nx = \int_{\phi=0}^{\phi=2\pi}\int_{r=0}^{r=R}\sqrt{r^2e^{2\nu-4\lambda}}\,drd\phi = 2\pi\int_{r=0}^{r=R}re^{\nu-2\lambda}dr \tag{46}$$

Plugging Equations (22) and (23) into Equation (46) gives:

$$A = \tag{47}$$

And thus the curvature is:

$$\mathscr{R} = \tag{48}$$

From the definition of the curvature in Equation (11) we can obtain the Ricci tensor, and hence the Einstein tensor. Then reading off the value of G_{zz} we obtain the desired T_{zz}.

References

[1] J. L. Synge, *Relativity: the general theory*. North-Holland Pub. Co., 1960.

[2] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*. Benjamin Cummings, Sept. 2003.

[3] H. E. J. Curzon, "Cylindrical Solutions of Einstein's Gravitational Equations," *Proceedings of the London Mathematical Society*, vol. s2–23, pp. 477–480, 1925.