Newtonian approximation in Causal Dynamical Triangulations

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Contents

1 Motivation

1.1 Newton's Law of Gravitation from General Relativity

Starting from the most general cylindrically symmetric (Weyl) metric [1]:

$$ds^2 = e^{2\lambda} dt^2 - e^{2(\nu-\lambda)} \left(dr^2 + dz^2\right) - r^2 e^{-2\lambda} d\phi^2 \tag{1}$$

$$g_{\mu\nu} = \begin{pmatrix} e^{2\lambda} dt^2 & 0 & 0 & 0 \\ 0 & -e^{2(\nu-\lambda)} dr^2 & 0 & 0 \\ 0 & 0 & -e^{2(\nu-\lambda)} dz^2 & 0 \\ 0 & 0 & 0 & -\frac{r^2}{e^{2\lambda}} d\phi^2 \end{pmatrix} \tag{2}$$

The definition of the Christoffel connection is: [2]

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tag{3}$$

The non-zero Christoffel connections are:

$$
\begin{aligned}
&\Gamma^t_{tr} = \partial_r\lambda \\
&\Gamma^t_{tz} = \partial_z\lambda \\
&\Gamma^r_{tt} = e^{4\lambda-2\nu}\partial_r\lambda \\
&\Gamma^r_{rr} = \partial_r\nu - \partial_r\lambda \\
&\Gamma^r_{rz} = \partial_z\nu - \partial_z\lambda \\
&\Gamma^r_{zz} = \partial_z\lambda - \partial_z\nu \\
&\Gamma^r_{\phi\phi} = re^{-2\nu}\left(r\partial_r\lambda - 1\right) \\
&\Gamma^z_{tt} = e^{4\lambda-2\nu}\partial_z\lambda \\
&\Gamma^z_{rr} = \partial_z\lambda - \partial_z\nu \\
&\Gamma^z_{rz} = \partial_r\nu - \partial_r\lambda \\
&\Gamma^z_{zz} = \partial_r\nu - \partial_r\lambda \\
&\Gamma^z_{\phi\phi} = r^2e^{-2\nu}\partial_z\lambda \\
&\Gamma^\phi_{r\phi} = \tfrac{1}{r} - \partial_r\lambda \\
&\Gamma^\phi_{z\phi} = -\partial_z\lambda
\end{aligned}
\tag{4}
$$

The components of the Riemann tensor are given by:

$$
R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}
\tag{5}
$$

Using the properties of the Riemann tensor:

$$
\begin{aligned}
&R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} \\
&R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \\
&R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \\
&R_{\rho[\sigma\mu\nu]} = 0
\end{aligned}
\tag{6}
$$

The non-zero components of the Riemann tensor are:

$$
\begin{aligned}
&R^t_{rtr} = -\partial_r^2\lambda + \left(\partial_z\lambda\right)^2 - 2\left(\partial_r\lambda\right)^2 + \partial_r\lambda\partial_r\nu - \partial_z\lambda\partial_z\nu \\
&R^t_{rtz} = -\partial_r\partial_z\lambda - 3\partial_r\lambda\partial_z\lambda + \partial_r\lambda\partial_z\nu + \partial_r\nu\partial_z\lambda \\
&R^t_{ztz} = -\partial_z^2\lambda - 2\left(\partial_z\lambda\right)^2 + \left(\partial_r\lambda\right)^2 - \partial_r\lambda\partial_r\nu + \partial_z\lambda\partial_z\nu \\
&R^t_{\phi t\phi} = re^{-2\nu}\left(r\left(\partial_r\lambda\right)^2 - \partial_r\lambda + r\left(\partial_z\lambda\right)^2\right) \\
&R^r_{zrz} = \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu \\
&R^z_{\phi z\phi} = re^{-2\nu}\left(r\partial_z^2\lambda - r\partial_z\lambda\partial_z\nu + r\partial_r\lambda\partial_r\nu - r\left(\partial_r\lambda\right)^2 + \partial_r\lambda - \partial_r\nu\right) \\
&R^z_{\phi\phi r} = re^{-2\nu}\left(-r\partial_r\partial_z\lambda + r\partial_r\nu\partial_z\lambda - r\partial_r\lambda\partial_z\lambda + r\partial_r\lambda\partial_z\nu - \partial_z\nu\right) \\
&R^\phi_{r\phi r} = \partial_r^2\lambda + \tfrac{1}{r}\partial_r\nu - \partial_r\lambda\partial_r\nu - \left(\partial_z\lambda\right)^2 + \partial_z\lambda\partial_z\nu + \tfrac{1}{r}\partial_r\lambda
\end{aligned}
\tag{7}
$$

The Ricci tensor is given by:

$$
R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}
\tag{8}
$$

The non-zero components of the Ricci tensor are:

$$\begin{aligned}
&R_{tt} = \tfrac{e^{4\lambda-2\nu}}{r}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right)\\
&R_{rr} = \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2\left(\partial_r\lambda\right)^2 + \tfrac{1}{r}\partial_r\lambda + \tfrac{1}{r}\partial_r\nu\\
&R_{rz} = \tfrac{1}{r}\partial_z\nu - 2\partial_r\lambda\partial_z\lambda\\
&R_{zz} = \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2\left(\partial_z\lambda\right)^2 + \tfrac{1}{r}\partial_r\lambda - \tfrac{1}{r}\partial_r\nu\\
&R_{\phi\phi} = re^{-2\nu}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right)
\end{aligned} \tag{9}$$

Einstein's equation in a vacuum is:

$$R_{\mu\nu} = 0 \tag{10}$$

Applying this complete set of relations to Equation (9) gives the following:

$$\partial_r^2\lambda + \frac{1}{r}\partial_r\lambda + \partial_z^2\lambda = 0 \tag{11}$$

$$\partial_r\nu = r\left(\partial_r^2\nu + \partial_z^2\nu + 2\left(\partial_r\lambda\right)^2\right) \tag{12}$$

$$\partial_z\nu = 2r\partial_r\lambda\partial_z\lambda \tag{13}$$

$$\partial_r^2\nu + \partial_z^2\nu + \left(\partial_r\lambda\right)^2 + \left(\partial_z\lambda\right)^2 = 0 \tag{14}$$

Equation (11) is the two-dimensional Laplace equation in cylindrical coordinates, for which the known general solutions are:

$$\lambda(r,z) = \sum_{n=0}^{\infty}\left[A_n J_n(kr) + B_n Y_n(kr)\right]\left[C_n \sinh(kz) + D_n \cosh(kz)\right] \tag{15}$$

Plugging Equation (14) into Equation (12) gives:

$$\partial_r\nu = r\left(\left(\partial_r\lambda\right)^2 - \left(\partial_z\lambda\right)^2\right) \tag{16}$$

Using Equations (13), (15) and (16) we find solutions for ν given by:

$$\nu = \int r[\left(\left(\partial_r\lambda\right)^2 - \left(\partial_z\lambda\right)^2\right)dr + \left(2\partial_r\lambda\partial_z\lambda\right)dz] \tag{17}$$

In principle, we have solutions for axially symmetric static vacuum spacetimes. We now wish to add matter. If the object is also axially symmetric and static, then we can consider solutions in the form of an external metric E and an internal metric I, where E is given by Equations (1), (15), and (17).

The solution of Laplace's equation for a point particle of mass m at $z = z_0$ is well known [3]:

$$\lambda(r,z) = -\frac{m}{\sqrt{r^2 + (z-z_0)^2}} \tag{18}$$

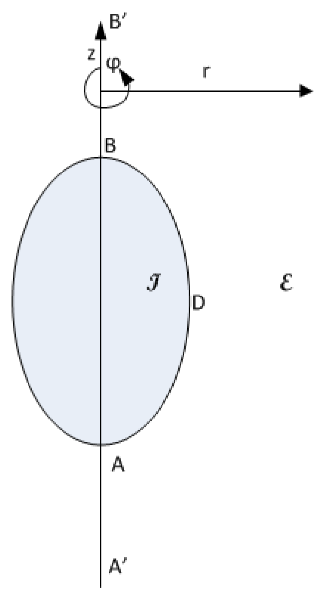
Likewise, we can easily verify for a single particle that the solution for ν from Equations (17) and (18) is (setting integration constants equal to zero):

$$\nu(r,z) = -\frac{m^2 r^2}{\left(r^2 + (z - z_0)^2\right)^2} \tag{19}$$

However, before we can consider this to be a complete solution we must consider elementary flatness. This condition requires that, for any infinitesimal spacelike circle, the ratio of circumference to radius is 2π. The most likely place to run into issues is along the z-axis, for which $r = 0$. Looking back at Equation (1) we see that the necessary condition is:

$$\lim_{r\to 0} \nu = 0 \tag{20}$$

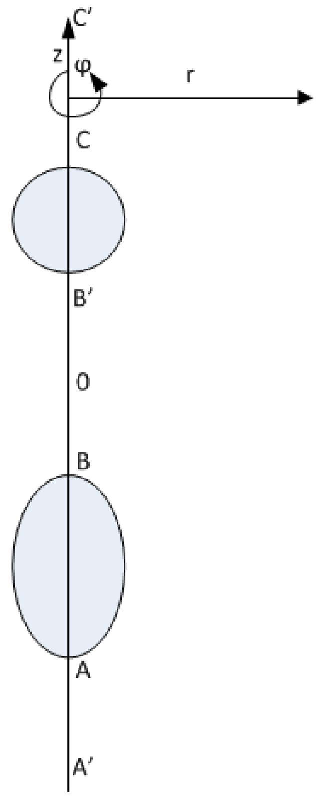
Consider the following diagram[1] :



Equation (17) includes a constant, which we can set by choosing that $\nu = 0$ at A. So, $\nu = 0$ along the z-axis from A' to A. The same applies from B to B'. Then our path ADB may be deformed into an infinite semicircle.

Now consider two bodies, as in the following diagram:

[1] Adapted from Synge



Since solutions to Laplace's equation are linear, we have:

$$\lambda(r,z) = -\frac{m_1}{R_1} - \frac{m_2}{R_2} \tag{21}$$

Plugging this into Equation (17) yields:

$$\nu(r,z) = -\frac{m_1^2 r^2}{R_1^4} - \frac{m_2^2 r^2}{R_2^4} + \frac{4 m_1 m_2}{(z_1 - z_2)^2} \frac{r^2 + (z - z_1)(z - z_2)}{R_1 R_2} \tag{22}$$

$$R_i = \sqrt{r^2 + (z - z_i)^2} \tag{23}$$

In this case, we expect $\nu = 0$ along *A'A* and *C'C* as before. But there is no *a priori* reason to think that $\nu = 0$ along *B'B*. This means that our vacuum solution fails along the z-axis. Therefore, there must be a strut of matter, i.e. a metric I such that $R_{\mu\nu} \neq 0$, along the z-axis *B'B* separating the two objects. This corresponds with the expectation that two masses will attract each other and not remain at rest.

Indeed, if we apply the condition that $r = 0$ we get:

$$\nu(0,z) = \frac{4 m_1 m_2}{(z_1 - z_2)^2} \tag{24}$$

Which means that in order for Equation (20) to hold, our strut must have:

$$\nu = -\frac{4m_1 m_2}{(z_1 - z_2)^2} \tag{25}$$

To get the force on the strut, we can integrate the z-component of the stress-energy tensor over the area:

$$F_z = \int T_{zz} d\sigma \tag{26}$$

We can get the stress-energy tensor from Einstein's equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{27}$$

We have all of the relevant components, except the Ricci scalar:

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu} \tag{28}$$

Which is:

$$R = 2e^{2(\lambda-\nu)} \left(\partial_r^2 \nu + \partial_z^2 \nu - \partial_r^2 \lambda - \partial_z^2 \lambda + (\partial_r \lambda)^2 + (\partial_z \lambda)^2 - \frac{1}{r} \partial_r \lambda \right) \tag{29}$$

Recall that earlier we asserted in Equation (10) that we had a vacuum solution. In order for this to be true, our Ricci scalar better be equal to zero, since a vacuum solution actually corresponds to:

$$G_{\mu\nu} = 0 \tag{30}$$

Using Equations (11) and (14) and plugging them into Equation (29) we see that this is indeed the case.

Continuing, we get:

$$G_{zz} = (\partial_r \lambda)^2 - (\partial_z \lambda)^2 - \frac{1}{r} \partial_r \nu \tag{31}$$

Again, generally speaking $G_{zz} = 0$, which we can see by plugging Equation (16) into Equation (31). So we have this object which is zero everywhere except along the z-axis.

That object is a conical singularity [4].

We can solve this problem by using the Gauss-Bonnet theorem: [5]

$$\iint_M K dA = 2\pi \chi(M) \tag{32}$$

Where K is the Gaussian curvature and $\chi(M)$ is the Euler characteristic. As we are looking at T_{zz}, we are interested in the submanifold of constant t and z which has the metric:

$$ds^2 = e^{2(\nu-\lambda)} dr^2 + r^2 e^{-2\lambda} d\phi^2 \tag{33}$$

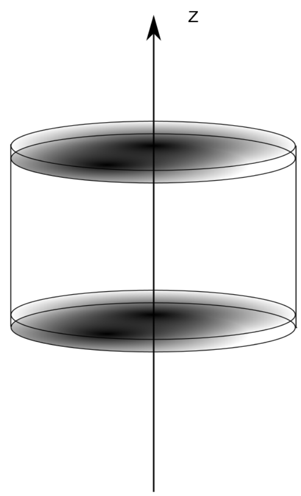
By putting this in the first fundamental form [6]:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{34}$$

We can calculate K using the orthogonal parameterization [7] (since $F = 0$):

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{\partial_u G}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{\partial_v E}{\sqrt{EG}}\right) = -\frac{1}{2}\frac{1}{\sqrt{EG}}\frac{d}{dr}\left(\frac{1}{\sqrt{EG}}\frac{dG}{dr}\right) \tag{35}$$

To solve the left side of Equation (32), consider the following diagram:



The top of the cylinder is given by:

$$\int KdA = -\frac{1}{2}\int_0^R \int_0^{2\pi} \frac{d}{dr}\left(\frac{1}{\sqrt{EG}}\frac{dG}{dr}\right) drd\phi \tag{36}$$

Thus, the Gauss-Bonnet theorem gives us:

$$K_{top} = K_{bottom} = 2\pi\left(1 - e^{-\nu(0,z)}\right) \tag{37}$$

The "edge" of the top disk contributes nothing to K, since as $R \to \infty$ this surface approaches that of a plane, for which $K = 0$.

The Gaussian curvature of the cylinder wall can also be calculated by letting $R \to \infty$. For this case, the metric of the top and bottom are Minkowski spaces, thus flat planes ($K = 0$). Then the entire cylinder is homeomorphic to the 2-sphere, for which the Euler characteristic $\chi = 2$ and thus by the Gauss-Bonnet theorem $K = 4\pi$. Since the other components (disks and disk edges) have $K = 0$ the cylnder wall must contain all of the curvature, that is, $K_{cylinderwall} = K = 4\pi$.

Thus,

$$\int KdA = K_{top} + K_{bottom} + K_{cylinderwall} = 4\pi \left(1 - e^{-\nu(r,z)}\right) + 4\pi = 2\pi * (2) \tag{38}$$

It has been shown that:

$$K = 2\pi \left(1 - e^{-\nu(r,z)}\right) \delta(r) \tag{39}$$

Where the delta-function $\delta(r)$ is given by:

$$\delta(r) = \int_0^\infty \int_0^{2\pi} \delta(r) e^{2(\lambda - \nu)} rdrd\phi = 1 \tag{40}$$

The two-dimensional sub-manifold in (r, ϕ) is diagonal, Thus:

$$T_{zz} = \frac{1}{8\pi G}(stuffhopefully\nu) \tag{41}$$

References

[1] J. L. Synge, *Relativity: the general theory*. North-Holland Pub. Co., 1960.

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[3] P. S. Letelier and S. R. Oliveira, "Superposition of weyl solutions: The equilibrium forces," *arXiv:gr-qc/9710122*, Oct. 1997. Class.Quant.Grav. 15 (1998) 421-433.

[4] M. E. AraÃºjo and S. R. Oliveira, "Static axisymmetric approach for the head-on collision of two black holes," *Physical Review D*, vol. 52, pp. 816–820, July 1995.

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[6] Wikipedia contributors, "First fundamental form," Mar. 2012. Page Version ID: 479957456.

[7] N. Schleifer, "Condition of elementary flatness and the two-particle curzon solution," *Physics Letters A*, vol. 112, pp. 204–207, Oct. 1985.