

# Detailed calculations for the Weyl metric

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## 1 Vacuum solution to the Weyl metric

Starting from the cylindrically symmetric (Weyl) vacuum metric [1]:

$$ds^2 = e^{2\lambda} dt^2 - e^{2(\nu-\lambda)} (dr^2 + dz^2) - r^2 e^{-2\lambda} d\phi^2 \quad (1)$$

$$g_{\mu\nu} = \begin{pmatrix} e^{2\lambda} & 0 & 0 & 0 \\ 0 & -e^{2(\nu-\lambda)} & 0 & 0 \\ 0 & 0 & -e^{2(\nu-\lambda)} & 0 \\ 0 & 0 & 0 & -r^2 e^{-2\lambda} \end{pmatrix} \quad (2)$$

In this coordinate basis, the definition of the Christoffel connection is: [2]

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\sigma} (\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu}) \quad (3)$$

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The non-zero Christoffel connections are:

$$\begin{aligned}
\Gamma_{tr}^t &= \partial_r \lambda \\
\Gamma_{tz}^t &= \partial_z \lambda \\
\Gamma_{tt}^r &= e^{4\lambda-2\nu} \partial_r \lambda \\
\Gamma_{rr}^r &= \partial_r \nu - \partial_r \lambda \\
\Gamma_{rz}^r &= \partial_z \nu - \partial_z \lambda \\
\Gamma_{zz}^r &= \partial_r \lambda - \partial_r \nu \\
\Gamma_{\phi\phi}^r &= re^{-2\nu} (r\partial_r \lambda - 1) \\
\Gamma_{tt}^z &= e^{4\lambda-2\nu} \partial_z \lambda \\
\Gamma_{rr}^z &= \partial_z \lambda - \partial_z \nu \\
\Gamma_{rz}^z &= \partial_r \nu - \partial_r \lambda \\
\Gamma_{zz}^z &= \partial_z \nu - \partial_z \lambda \\
\Gamma_{\phi\phi}^z &= r^2 e^{-2\nu} \partial_z \lambda \\
\Gamma_{r\phi}^\phi &= \frac{1}{r} - \partial_r \lambda \\
\Gamma_{z\phi}^\phi &= -\partial_z \lambda
\end{aligned} \tag{4}$$

The components of the Riemann tensor are given by:

$$R_{\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda \tag{5}$$

Using the properties of the Riemann tensor:

$$\begin{aligned}
R_{\rho\sigma\mu\nu} &= -R_{\rho\sigma\nu\mu} \\
R_{\rho\sigma\mu\nu} &= -R_{\sigma\rho\mu\nu} \\
R_{\rho\sigma\mu\nu} &= R_{\mu\nu\rho\sigma} \\
R_{\rho[\sigma\mu\nu]} &= 0
\end{aligned} \tag{6}$$

The non-zero components of the Riemann tensor are:

$$\begin{aligned}
R_{trr}^t &= -\partial_r^2 \lambda + (\partial_z \lambda)^2 - 2(\partial_r \lambda)^2 + \partial_r \lambda \partial_r \nu - \partial_z \lambda \partial_z \nu \\
R_{rtz}^t &= -\partial_r \partial_z \lambda - 3\partial_r \lambda \partial_z \lambda + \partial_r \lambda \partial_z \nu + \partial_r \nu \partial_z \lambda \\
R_{ztz}^t &= -\partial_z^2 \lambda - 2(\partial_z \lambda)^2 + (\partial_r \lambda)^2 - \partial_r \lambda \partial_r \nu + \partial_z \lambda \partial_z \nu \\
R_{\phi t\phi}^t &= re^{-2\nu} (r(\partial_r \lambda)^2 - \partial_r \lambda + r(\partial_z \lambda)^2) \\
R_{zrz}^r &= \partial_r^2 \lambda - \partial_r^2 \nu + \partial_z^2 \lambda - \partial_z^2 \nu \\
R_{\phi z\phi}^z &= re^{-2\nu} (r\partial_z^2 \lambda - r\partial_z \lambda \partial_z \nu + r\partial_r \lambda \partial_r \nu - r(\partial_r \lambda)^2 + \partial_r \lambda - \partial_r \nu) \\
R_{\phi\phi r}^z &= re^{-2\nu} (-r\partial_r \partial_z \lambda + r\partial_r \nu \partial_z \lambda - r\partial_r \lambda \partial_z \lambda + r\partial_r \lambda \partial_z \nu - \partial_z \nu) \\
R_{r\phi r}^\phi &= \partial_r^2 \lambda + \frac{1}{r} \partial_r \nu - \partial_r \lambda \partial_r \nu - (\partial_z \lambda)^2 + \partial_z \lambda \partial_z \nu + \frac{1}{r} \partial_r \lambda
\end{aligned} \tag{7}$$

The Ricci tensor is given by:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \quad (8)$$

The non-zero components of the Ricci tensor are:

$$\begin{aligned} R_{tt} &= \frac{e^{4\lambda-2\nu}}{r} \left( r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda \right) \\ R_{rr} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2(\partial_r\lambda)^2 + \frac{1}{r}\partial_r\lambda + \frac{1}{r}\partial_r\nu \\ R_{rz} &= \frac{1}{r}\partial_z\nu - 2\partial_r\lambda\partial_z\lambda \\ R_{zz} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2(\partial_z\lambda)^2 + \frac{1}{r}\partial_r\lambda - \frac{1}{r}\partial_r\nu \\ R_{\phi\phi} &= re^{-2\nu} \left( r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda \right) \end{aligned} \quad (9)$$

The Ricci scalar is defined as:

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu} \quad (10)$$

Which is:

$$R = 2e^{2(\lambda-\nu)} \left( \partial_r^2\nu + \partial_z^2\nu - \partial_r^2\lambda - \partial_z^2\lambda + (\partial_r\lambda)^2 + (\partial_z\lambda)^2 - \frac{1}{r}\partial_r\lambda \right) \quad (11)$$

Einstein's equation in a vacuum is:

$$G_{\mu\nu} = 0 \quad (12)$$

Whence Einstein's equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu} \quad (13)$$

However, we can take a shortcut by using:

$$R_{\mu\nu} = 0 \quad (14)$$

Since the trace of a zero-valued matrix is identically zero, and thus Equation (14) automatically satisfies Equation (12).

Applying Equation (14) to Equation (9) gives the following:

$$\partial_r^2\lambda + \frac{1}{r}\partial_r\lambda + \partial_z^2\lambda = 0 \quad (15)$$

$$\partial_r\nu = r \left( \partial_r^2\nu + \partial_z^2\nu + 2(\partial_r\lambda)^2 \right) \quad (16)$$

$$\partial_z\nu = 2r\partial_r\lambda\partial_z\lambda \quad (17)$$

$$\partial_r^2\nu + \partial_z^2\nu + (\partial_r\lambda)^2 + (\partial_z\lambda)^2 = 0 \quad (18)$$

Equation (15) is the two-dimensional Laplace equation in cylindrical coordinates. That is:

$$\nabla^2 \lambda(r, z) = 0 \quad (19)$$

Plugging Equation (18) into Equation (16) gives:

$$\partial_r v = r \left( (\partial_r \lambda)^2 - (\partial_z \lambda)^2 \right) \quad (20)$$

Using Equations (17) and (20) we find solutions for  $v$  are given by:

$$v = \int r \left[ \left( (\partial_r \lambda)^2 - (\partial_z \lambda)^2 \right) dr + (2\partial_r \lambda \partial_z \lambda) dz \right] \quad (21)$$

The solutions must satisfy Equations (19) and (21). A particular solution corresponding to two objects (given by Curzon in 1924 [3]) is:

$$\lambda_0(r, z) = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} \quad (22)$$

$$v_0(r, z) = -\frac{1}{2} \frac{\mu_1^2 r^2}{r_1^4} - \frac{1}{2} \frac{\mu_2^2 r^2}{r_2^4} + \frac{2\mu_1 \mu_2}{(z_1 - z_2)^2} \left[ \frac{r^2 + (z - z_1)(z - z_2)}{r_1 r_2} - 1 \right] \quad (23)$$

Where  $z_1$  and  $z_2$  correspond to the positions on the  $z$ -axis for the two objects,  $\mu_1$  and  $\mu_2$  are length parameters, and:

$$r_1 = \sqrt{r^2 + (z - z_1)^2} \quad (24)$$

$$r_2 = \sqrt{r^2 + (z - z_2)^2} \quad (25)$$

Just as a final check, plugging Equations (15) and (18) into Equation (11) gives  $R = 0$ , which shows that our solutions are consistent with our assumptions.

By construction, these solutions only apply to empty space, and so must exclude the two objects at  $z_1$  and  $z_2$ . In addition, as noted by Synge [1], the  $z$  axis between the two objects must also be excluded due to violation of elementary flatness. We will examine this in the next section.

## 2 Curvature from Parallel Transport

Consider parallel transport of a vector  $V$  about the  $z$ -axis in the  $\hat{\phi}$  direction. The equation for parallel transport is generally given by:

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu = 0 \quad \text{along } x^\mu(\lambda) \quad (26)$$

That is, the directional covariant derivative is equal to zero along the curve  $x^\mu$  parameterized by  $\lambda$ . For a vector this can be simply written as:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\lambda}^\nu V^\lambda = 0 \quad (27)$$

Starting with parallel transport along  $\hat{\phi}$ , Equation (27) along with the relevant Christoffel symbols  $\Gamma_{\phi\phi}^r$ ,  $\Gamma_{\phi\phi}^z$ ,  $\Gamma_{\phi r}^\phi$ , and  $\Gamma_{\phi z}^\phi$  gives:

$$\begin{aligned}
\partial_\phi V^r + \Gamma_{\phi\phi}^r V^\phi &= 0 \\
\partial_\phi V^z + \Gamma_{\phi\phi}^z V^\phi &= 0 \\
\partial_\phi V^\phi + \Gamma_{\phi r}^\phi V^r + \Gamma_{\phi z}^\phi V^z &= 0
\end{aligned} \tag{28}$$

Plugging in the values from Equation (4), our equations are:

$$\partial_\phi V^r + \left( r e^{-2\nu} (r \partial_r \lambda - 1) \right) V^\phi = 0 \tag{29}$$

$$\partial_\phi V^z + \left( r^2 e^{-2\nu} \partial_z \lambda \right) V^\phi = 0 \tag{30}$$

$$\partial_\phi V^\phi + \left( \frac{1}{r} - \partial_r \lambda \right) V^r - \partial_z \lambda V^z = 0 \tag{31}$$

Differentiating Equation (31) with respect to  $\phi$  and plugging it into Equation (29) gives:

$$\partial_\phi^2 V^\phi - \partial_z \lambda \partial_\phi V^z + r^2 e^{-2\nu} \left( \partial_r \lambda - \frac{1}{r} \right)^2 V^\phi = 0 \tag{32}$$

Plugging in the expression for  $\partial_\phi V^z$  from Equation (30) and letting

$$\chi = r e^{-\nu} \sqrt{(\partial_z \lambda)^2 + \left( \frac{1}{r} - \partial_r \lambda \right)^2} \tag{33}$$

We have the simple differential equation:

$$\partial_\phi^2 V^\phi + \chi^2 V^\phi = 0 \tag{34}$$

For which the solution is:

$$V^\phi = A \sin \chi \phi + B \cos \chi \phi \tag{35}$$

Therefore, integrating Equation (29) with respect to  $\phi$  we get:

$$V^r = \frac{r^2 e^{-2\nu} (\partial_r \lambda - \frac{1}{r})}{\chi} (A \cos \chi \phi - B \sin \chi \phi) \tag{36}$$

And from Equation (30):

$$V^z = \frac{r^2 e^{-2\nu} \partial_z \lambda}{\chi} (A \cos \chi \phi - B \sin \chi \phi) \tag{37}$$

So our general vector is then:

$$\begin{aligned}
V = & \frac{r^2 e^{-2\nu} (\partial_r \lambda - \frac{1}{r})}{\chi} (A \cos \chi \phi - B \sin \chi \phi) \hat{e}_r \\
& + \frac{r^2 e^{-2\nu} \partial_z \lambda}{\chi} (A \cos \chi \phi - B \sin \chi \phi) \hat{e}_z \\
& + (A \sin \chi \phi + B \cos \chi \phi) \hat{e}_\phi
\end{aligned} \tag{38}$$

Normalizing  $V(\phi = 0)$ :

$$g_{\mu\nu}V^\mu V^\nu = 1 \quad (39)$$

We obtain the condition that:

$$A^2 + B^2 = r^{-2}e^{2\lambda} \quad (40)$$

For simplicity, we choose  $A^2 = r^{-2}e^{2\lambda}$  and  $B^2 = 0$ .

Now, when we parallel transport  $V$  around to  $\phi = 2\pi$  there will be an angle between  $V(\phi = 0)$  and  $V(\phi = 2\pi)$  given by the definition of the scalar product:

$$\cos(\beta) = \frac{g_{\mu\nu}V^\mu(0)V^\nu(2\pi)}{g_{\mu\nu}V^\mu(0)V^\nu(0)} \quad (41)$$

Since we have normalized our vectors, the denominator is equal to 1, and we get the expression that:

$$\cos\beta = \cos(2\pi\chi) \quad (42)$$

Where  $\chi$  is given by Equation (33). Hence  $\beta = 2\pi\chi$ . We can now use the definition of the deficit angle:

$$\Delta = 2\pi - \beta = 2\pi(1 - \chi) \quad (43)$$

To get the curvature  $\mathcal{R}$  via:

$$\mathcal{R} = \lim_{A \rightarrow 0} \frac{\Delta}{A} \quad (44)$$

The area  $A$  is defined on the reduced metric:

$$ds^2 = e^{2(v-\lambda)}dr^2 + r^2e^{-2\lambda}d\phi^2 \quad (45)$$

Via:

$$A = \int \sqrt{|g|} d^n x = \int_{\phi=0}^{\phi=2\pi} \int_{r=0}^{r=R} \sqrt{r^2 e^{2v-4\lambda}} dr d\phi = 2\pi \int_{r=0}^{r=R} r e^{v-2\lambda} dr \quad (46)$$

Plugging Equations (22) and (23) into Equation (46) gives:

$$A = \quad (47)$$

And thus the curvature is:

$$\mathcal{R} = \quad (48)$$

## References

- [1] J. L. Synge, *Relativity: the general theory*. North-Holland Pub. Co., 1960.
- [2] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*. Benjamin Cummings, Sept. 2003.
- [3] H. E. J. Curzon, “Cylindrical Solutions of Einstein’s Gravitational Equations,” *Proceedings of the London Mathematical Society*, vol. s2–23, pp. 477–480, 1925.