Newtonian approximation in Causal Dynamical Triangulations

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1 Motivation

1.1 Newton's Law of Gravitation from General Relativity

Starting from the most general cylindrically symmetric (Weyl) metric [1]:

$$ds^2 = e^{2\lambda}dt^2 - e^{2(\nu-\lambda)}\left(dr^2 + dz^2\right) - r^2 e^{-2\lambda} d\phi^2 \tag{1}$$

$$g_{\mu\nu} = \begin{pmatrix} e^{2\lambda}dt^2 & 0 & 0 & 0 \\ 0 & -e^{2(\nu-\lambda)}dr^2 & 0 & 0 \\ 0 & 0 & -e^{2(\nu-\lambda)}dz^2 & 0 \\ 0 & 0 & 0 & -\frac{r^2}{e^{2\lambda}}d\phi^2 \end{pmatrix} \tag{2}$$

The definition of the Christoffel connection is: [2]

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tag{3}$$

With the assumption of zero torsion:

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu} \tag{4}$$

The non-zero Christoffel connections are:

$$
\begin{aligned}
&\Gamma^t_{tr} = \partial_r\lambda \\
&\Gamma^t_{tz} = \partial_z\lambda \\
&\Gamma^r_{tt} = e^{4\lambda-2\nu}\partial_r\lambda \\
&\Gamma^r_{rr} = \partial_r\nu - \partial_r\lambda \\
&\Gamma^r_{rz} = \partial_z\nu - \partial_z\lambda \\
&\Gamma^r_{zz} = \partial_z\lambda - \partial_z\nu \\
&\Gamma^r_{\phi\phi} = re^{-2\nu}\left(r\partial_r\lambda - 1\right) \\
&\Gamma^z_{tt} = e^{4\lambda-2\nu}\partial_z\lambda \\
&\Gamma^z_{rr} = \partial_z\lambda - \partial_z\nu \\
&\Gamma^z_{rz} = \partial_r\nu - \partial_r\lambda \\
&\Gamma^z_{zz} = \partial_r\nu - \partial_r\lambda \\
&\Gamma^z_{\phi\phi} = r^2e^{-2\nu}\partial_z\lambda \\
&\Gamma^\phi_{r\phi} = \tfrac{1}{r} - \partial_r\lambda \\
&\Gamma^\phi_{z\phi} = -\partial_z\lambda
\end{aligned}
\tag{5}
$$

The components of the Riemann tensor are given by:

$$
R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{6}
$$

Using the properties of the Riemann tensor:

$$
\begin{aligned}
&R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} \\
&R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \\
&R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \\
&R_{\rho[\sigma\mu\nu]} = 0
\end{aligned}
\tag{7}
$$

The non-zero components of the Riemann tensor are:

$$
\begin{aligned}
&R^t_{rtr} = -\partial_r^2\lambda + \left(\partial_z\lambda\right)^2 - 2\left(\partial_r\lambda\right)^2 + \partial_r\lambda\partial_r\nu - \partial_z\lambda\partial_z\nu \\
&R^t_{rtz} = -\partial_r\partial_z\lambda - 3\partial_r\lambda\partial_z\lambda + \partial_r\lambda\partial_z\nu + \partial_r\nu\partial_z\lambda \\
&R^t_{ztz} = -\partial_z^2\lambda - 2\left(\partial_z\lambda\right)^2 + \left(\partial_r\lambda\right)^2 - \partial_r\lambda\partial_r\nu + \partial_z\lambda\partial_z\nu \\
&R^t_{\phi t\phi} = re^{-2\nu}\left(r\left(\partial_r\lambda\right)^2 - \partial_r\lambda + r\left(\partial_z\lambda\right)^2\right) \\
&R^r_{zrz} = \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu \\
&R^z_{\phi z\phi} = re^{-2\nu}\left(r\partial_z^2\lambda - r\partial_z\lambda\partial_z\nu + r\partial_r\lambda\partial_r\nu - r\left(\partial_r\lambda\right)^2 + \partial_r\lambda - \partial_r\nu\right) \\
&R^z_{\phi\phi r} = re^{-2\nu}\left(-r\partial_r\partial_z\lambda + r\partial_r\nu\partial_z\lambda - r\partial_r\lambda\partial_z\lambda + r\partial_r\lambda\partial_z\nu - \partial_z\nu\right) \\
&R^\phi_{r\phi r} = \partial_r^2\lambda + \tfrac{1}{r}\partial_r\nu - \partial_r\lambda\partial_r\nu - \left(\partial_z\lambda\right)^2 + \partial_z\lambda\partial_z\nu + \tfrac{1}{r}\partial_r\lambda
\end{aligned}
\tag{8}
$$

The Ricci tensor is given by:

$$
R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \tag{9}
$$

The non-zero components of the Ricci tensor are:

$$\begin{aligned}
R_{tt} &= \tfrac{e^{4\lambda-2\nu}}{r}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right) \\
R_{rr} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2\left(\partial_r\lambda\right)^2 + \tfrac{1}{r}\partial_r\lambda + \tfrac{1}{r}\partial_r\nu \\
R_{rz} &= \tfrac{1}{r}\partial_z\nu - 2\partial_r\lambda\partial_z\lambda \\
R_{zz} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2\left(\partial_z\lambda\right)^2 + \tfrac{1}{r}\partial_r\lambda - \tfrac{1}{r}\partial_r\nu \\
R_{\phi\phi} &= re^{-2\nu}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right)
\end{aligned} \tag{10}$$

Einstein's equation in a vacuum is:

$$R_{\mu\nu} = 0 \tag{11}$$

Applying this complete set of relations to Equation (10) gives the following:

$$\partial_r^2\lambda + \frac{1}{r}\partial_r\lambda + \partial_z^2\lambda = 0 \tag{12}$$

$$\partial_r\nu = r\left(\partial_r^2\nu + \partial_z^2\nu + 2\left(\partial_r\lambda\right)^2\right) \tag{13}$$

$$\partial_z\nu = 2r\partial_r\lambda\partial_z\lambda \tag{14}$$

$$\partial_r^2\nu + \partial_z^2\nu + \left(\partial_r\lambda\right)^2 + \left(\partial_z\lambda\right)^2 = 0 \tag{15}$$

Equation (12) is the two-dimensional Laplace equation in cylindrical coordinates, for which the known solutions are:

$$\lambda(r,z) = \sum_{m=0}^{\infty}\left[A_m J_m(kr) + B_m N_m(kr)\right]\left[C_m \sinh(kr) + D_m \cosh(kr)\right] \tag{16}$$

Plugging Equation (15) into Equation (13) gives:

$$\partial_r\nu = r\left(\left(\partial_r\lambda\right)^2 - \left(\partial_z\lambda\right)^2\right) \tag{17}$$

Using Equations (14), (16) and (17) we find solutions for ν given by:

$$\nu = \int r[\left(\left(\partial_r\lambda\right)^2 - \left(\partial_z\lambda\right)^2\right)dr + \left(2\partial_r\lambda\partial_z\lambda\right)dz] \tag{18}$$

However, before we can consider this to be a complete solution we must consider elementary flatness. This condition requires that, for any infinitesimal spacelike circle, the ratio of circumference to radius is 2π. The most likely place to run into issues is along the z-axis, for which $r = 0$. Looking back at Equation (1) we see that the necessary condition is:

$$\nu = 0 \quad \text{for} \quad r = 0 \tag{19}$$

References

[1] J. L. Synge, *Relativity: the general theory*. North-Holland Pub. Co., 1960.

[2] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*. Benjamin Cummings, Sept. 2003.