The Newtonian approximation in Causal Dynamical Triangulations

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1 Newton's Law of Gravitation from General Relativity

This treatment follows that of Katz [1].

1.1 Vacuum solution to the Weyl metric

Starting from the cylindrically symmetric (Weyl) vacuum metric [2]:

$$ds^2 = e^{2\lambda}dt^2 - e^{2(\nu-\lambda)}\left(dr^2 + dz^2\right) - r^2 e^{-2\lambda}d\phi^2 \tag{1}$$

$$g_{\mu\nu} = \begin{pmatrix} e^{2\lambda} & 0 & 0 & 0 \\ 0 & -e^{2(\nu-\lambda)} & 0 & 0 \\ 0 & 0 & -e^{2(\nu-\lambda)} & 0 \\ 0 & 0 & 0 & -\frac{r^2}{e^{2\lambda}} \end{pmatrix} \tag{2}$$

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In this coordinate basis, the definition of the Christoffel connection is: [3]

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \tag{3}$$

The non-zero Christoffel connections are:

$$\begin{aligned}
\Gamma^{t}_{tr} &= \partial_r \lambda \\
\Gamma^{t}_{tz} &= \partial_z \lambda \\
\Gamma^{r}_{tt} &= e^{4\lambda - 2\nu} \partial_r \lambda \\
\Gamma^{r}_{rr} &= \partial_r \nu - \partial_r \lambda \\
\Gamma^{r}_{rz} &= \partial_z \nu - \partial_z \lambda \\
\Gamma^{r}_{zz} &= \partial_r \lambda - \partial_r \nu \\
\Gamma^{r}_{\phi\phi} &= r e^{-2\nu} \left(r \partial_r \lambda - 1 \right) \\
\Gamma^{z}_{tt} &= e^{4\lambda - 2\nu} \partial_z \lambda \\
\Gamma^{z}_{rr} &= \partial_z \lambda - \partial_z \nu \\
\Gamma^{z}_{rz} &= \partial_r \nu - \partial_r \lambda \\
\Gamma^{z}_{zz} &= \partial_z \nu - \partial_z \lambda \\
\Gamma^{z}_{\phi\phi} &= r^2 e^{-2\nu} \partial_z \lambda \\
\Gamma^{\phi}_{r\phi} &= \frac{1}{r} - \partial_r \lambda \\
\Gamma^{\phi}_{z\phi} &= -\partial_z \lambda
\end{aligned} \tag{4}$$

The components of the Riemann tensor are given by:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma} \tag{5}$$

Using the properties of the Riemann tensor:

$$\begin{aligned}
R_{\rho\sigma\mu\nu} &= -R_{\rho\sigma\nu\mu} \\
R_{\rho\sigma\mu\nu} &= -R_{\sigma\rho\mu\nu} \\
R_{\rho\sigma\mu\nu} &= R_{\mu\nu\rho\sigma} \\
R_{\rho[\sigma\mu\nu]} &= 0
\end{aligned} \tag{6}$$

The non-zero components of the Riemann tensor are:

$$\begin{aligned}
R^t_{rtr} &= -\partial_r^2\lambda + (\partial_z\lambda)^2 - 2(\partial_r\lambda)^2 + \partial_r\lambda\partial_r\nu - \partial_z\lambda\partial_z\nu \\
R^t_{rtz} &= -\partial_r\partial_z\lambda - 3\partial_r\lambda\partial_z\lambda + \partial_r\lambda\partial_z\nu + \partial_r\nu\partial_z\lambda \\
R^t_{ztz} &= -\partial_z^2\lambda - 2(\partial_z\lambda)^2 + (\partial_r\lambda)^2 - \partial_r\lambda\partial_r\nu + \partial_z\lambda\partial_z\nu \\
R^t_{\phi t\phi} &= re^{-2\nu}\left(r(\partial_r\lambda)^2 - \partial_r\lambda + r(\partial_z\lambda)^2\right) \\
R^r_{zrz} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu \\
R^z_{\phi z\phi} &= re^{-2\nu}\left(r\partial_z^2\lambda - r\partial_z\lambda\partial_z\nu + r\partial_r\lambda\partial_r\nu - r(\partial_r\lambda)^2 + \partial_r\lambda - \partial_r\nu\right) \\
R^z_{\phi\phi r} &= re^{-2\nu}\left(-r\partial_r\partial_z\lambda + r\partial_r\nu\partial_z\lambda - r\partial_r\lambda\partial_z\lambda + r\partial_r\lambda\partial_z\nu - \partial_z\nu\right) \\
R^\phi_{r\phi r} &= \partial_r^2\lambda + \frac{1}{r}\partial_r\nu - \partial_r\lambda\partial_r\nu - (\partial_z\lambda)^2 + \partial_z\lambda\partial_z\nu + \frac{1}{r}\partial_r\lambda
\end{aligned} \tag{7}$$

The Ricci tensor is given by:

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \tag{8}$$

The non-zero components of the Ricci tensor are:

$$\begin{aligned}
R_{tt} &= \frac{e^{4\lambda-2\nu}}{r}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right) \\
R_{rr} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2(\partial_r\lambda)^2 + \frac{1}{r}\partial_r\lambda + \frac{1}{r}\partial_r\nu \\
R_{rz} &= \frac{1}{r}\partial_z\nu - 2\partial_r\lambda\partial_z\lambda \\
R_{zz} &= \partial_r^2\lambda - \partial_r^2\nu + \partial_z^2\lambda - \partial_z^2\nu - 2(\partial_z\lambda)^2 + \frac{1}{r}\partial_r\lambda - \frac{1}{r}\partial_r\nu \\
R_{\phi\phi} &= re^{-2\nu}\left(r\partial_r^2\lambda + r\partial_z^2\lambda + \partial_r\lambda\right)
\end{aligned} \tag{9}$$

The Ricci scalar is defined as:

$$R = R^\mu_\mu = g^{\mu\nu}R_{\mu\nu} \tag{10}$$

Which is:

$$R = 2e^{2(\lambda-\nu)}\left(\partial_r^2\nu + \partial_z^2\nu - \partial_r^2\lambda - \partial_z^2\lambda + (\partial_r\lambda)^2 + (\partial_z\lambda)^2 - \frac{1}{r}\partial_r\lambda\right) \tag{11}$$

Einstein's equation in a vacuum is:

$$G_{\mu\nu} = 0 \tag{12}$$

Whence Einstein's equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu} \tag{13}$$

However, we can take a shortcut by using:

$$R_{\mu\nu} = 0 \tag{14}$$

Since the trace of a zero-valued matrix is identically zero, and thus Equation (14) automatically satisfies Equation (12).

Applying Equation (14) to Equation (9) gives the following:

$$\partial_r^2 \lambda + \frac{1}{r}\partial_r \lambda + \partial_z^2 \lambda = 0 \tag{15}$$

$$\partial_r \nu = r\left(\partial_r^2 \nu + \partial_z^2 \nu + 2\left(\partial_r \lambda\right)^2\right) \tag{16}$$

$$\partial_z \nu = 2r\partial_r \lambda \partial_z \lambda \tag{17}$$

$$\partial_r^2 \nu + \partial_z^2 \nu + \left(\partial_r \lambda\right)^2 + \left(\partial_z \lambda\right)^2 = 0 \tag{18}$$

Equation (15) is the two-dimensional Laplace equation in cylindrical coordinates. That is:

$$\nabla^2 \lambda(r,z) = 0 \tag{19}$$

Plugging Equation (18) into Equation (16) gives:

$$\partial_r \nu = r\left(\left(\partial_r \lambda\right)^2 - \left(\partial_z \lambda\right)^2\right) \tag{20}$$

Using Equations (17) and (20) we find solutions for ν are given by:

$$\nu = \int r[\left(\left(\partial_r \lambda\right)^2 - \left(\partial_z \lambda\right)^2\right) dr + \left(2\partial_r \lambda \partial_z \lambda\right) dz] \tag{21}$$

The solutions must satisfy Equations (19) and (21). A particular solution corresponding to two objects (given by Curzon in 1924 [4]) is:

$$\lambda_0 = -\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} \tag{22}$$

$$\nu_0 = \frac{1}{2}\frac{\mu_1^2 r^2}{r_1^4} - \frac{1}{2}\frac{\mu_2^2 r^2}{r_2^4} + \frac{2\mu_1\mu_2}{\left(z-z_2\right)^2}\left[\frac{r^2 + (z-z_1)(z-z_2)}{r_1 r_2} - 1\right] \tag{23}$$

Where z_1 and z_2 correspond to the positions on the z-axis for the two objects, μ_1 and μ_2 are length parameters, and:

$$r_1 = \sqrt{r^2 + \left(z-z_1\right)^2} \tag{24}$$

$$r_2 = \sqrt{r^2 + \left(z-z_2\right)^2} \tag{25}$$

Just as a final check, plugging Equations (15) and (18) into Equation (11) gives $R = 0$, which shows that our solutions are consistent with our assumptions.

By construction, these solutions only apply to empty space, and so must exclude the two objects at z_1 and z_2. In addition, as noted by Synge [2], the z axis between the two objects must also be excluded due to violation of elementary flatness. We will examine this in the next section.

1.2 Elementary Flatness

In order to be certain that our spacetime is truly flat, we impose the condition of elementary flatness: the ratio of the circumference to the radius is equal to 2π. This gives restrictions on solutions for $\lambda(r,z)$ and $\nu(r,z)$.

To do this we will first integrate in the $\hat{\phi}$ direction at some r and then divide by r. This gives:

$$L = \int ds = \int_0^{2\pi} \sqrt{-r^2 e^{-2\lambda} d\phi^2} = \pm \frac{2\pi r}{e^{\lambda}} \tag{26}$$

Then the condition that $\frac{L}{r} = 2\pi$ holds provided that $e^{-\lambda} = 1$. That is,

$$\lambda(0,z) \to 0 \tag{27}$$

But since $\frac{L}{r}$ is not well-defined as $r \to 0$, this is a sign that we need to look more carefully at the z-axis.

Consider parallel transport of a vector V about the z-axis in the $\hat{\phi}$ direction, demanding that the values for $\phi = 0$ and $\phi = 2\pi$ are equal.

The equation for parallel transport is generally given by:

$$\tfrac{D}{d\lambda} = \tfrac{dx^{\mu}}{d\lambda} \nabla_{\mu} = 0 \quad \text{along} \quad x^{\mu}(\lambda) \tag{28}$$

That is, the directional covariant derivative is equal to zero along the curve x^{μ} parameterized by λ. For a vector this can be simply written as:

$$\nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda} = 0 \tag{29}$$

Starting with parallel transport along $\hat{e}_{\phi}$, Equation (29) along with the relevant Christoffel symbols $\Gamma^{r}_{\phi\phi}$, $\Gamma^{z}_{\phi\phi}$, $\Gamma^{\phi}_{\phi r}$, and $\Gamma^{\phi}_{\phi z}$ gives:

$$\begin{aligned} \partial_{\phi} V^{r} + \Gamma^{r}_{\phi\phi} V^{\phi} &= 0 \\ \partial_{\phi} V^{z} + \Gamma^{z}_{\phi\phi} V^{\phi} &= 0 \\ \partial_{\phi} V^{\phi} + \Gamma^{\phi}_{\phi r} V^{r} + \Gamma^{\phi}_{\phi z} V^{z} &= 0 \end{aligned} \tag{30}$$

Plugging in the values from Equation (4), our equations are:

$$\partial_{\phi} V^{r} + \left(r e^{-2\nu} (r \partial_r \lambda - 1) \right) V^{\phi} = 0 \tag{31}$$

$$\partial_{\phi} V^{z} + \left(r^2 e^{-2\nu} \partial_z \lambda \right) V^{\phi} = 0 \tag{32}$$

$$\partial_{\phi} V^{\phi} + \left(\frac{1}{r} - \partial_r \lambda \right) V^{r} - \partial_z \lambda V^{z} = 0 \tag{33}$$

Differentiating Equation (33) with respect to ϕ and plugging it into Equation (31) gives:

$$\partial^2_{\phi} V^{\phi} - \partial_z \lambda \partial_{\phi} V^{z} + r^2 e^{-2\nu} \left(\partial_r \lambda - \frac{1}{r} \right)^2 V^{\phi} = 0 \tag{34}$$

Plugging in the expression for $\partial_{\phi} V^{z}$ from Equation (32) and letting

$$\chi = re^{-\nu}\sqrt{(\partial_z\lambda)^2 + \left(\partial_r\lambda - \frac{1}{r}\right)^2} \tag{35}$$

We have the simple differential equation:

$$\partial_\phi^2 V^\phi + \chi^2 V^\phi = 0 \tag{36}$$

For which the solution is:

$$V^\phi = A\sin\chi\phi + B\cos\chi\phi \tag{37}$$

Therefore, integrating Equation (31) with respect to ϕ we get:

$$V^r = \frac{r^2 e^{-2\nu}(\partial_r\lambda - \frac{1}{r})}{\chi}(A\cos\chi\phi - B\sin\chi\phi) \tag{38}$$

And from Equation (32):

$$V^z = \frac{r^2 e^{-2\nu}\partial_z\lambda}{\chi}(A\cos\chi\phi - B\sin\chi\phi) \tag{39}$$

At $\phi = 0$ we have $V^\phi = 1$ and $V^r = r_0$ (leaving aside for the moment V^z, since we are free to parallel transport about ϕ anywhere along the z-axis). Then the condition that $V^\phi = 1$ leads to $B = 1$. Likewise, setting $V^r = r_0$ leads to:

$$\frac{Ae^{-\nu}(\partial_r\lambda - \frac{1}{r_0})}{\sqrt{(\partial_z\lambda)^2 + (\partial_r\lambda - \frac{1}{r})^2}} = 1 \tag{40}$$

We set $A = 1$ for convenience. Then from Equation (40) taking the limit as $r_0 \to 0$ we find:

$$\lim_{r_0\to 0} \frac{e^{-\nu(r_0,z)}(\partial_r\lambda - \frac{1}{r_0})}{\sqrt{(\partial_z\lambda)^2 + (\partial_r\lambda - \frac{1}{r})^2}} = e^{-\nu(r_0,z)} = 1 \tag{41}$$

As r_0 is completely arbitrary we can characterize this as:

$$\lim_{r\to 0} \nu(0,z) = 0 \tag{42}$$

The general expression for the vector is then:

$$V = \left(\frac{re^{-\nu}}{\sqrt{(\partial_z\lambda)^2 + (\partial_r\lambda - \frac{1}{r})^2}}\right)(\cos\chi\phi - \sin\chi\phi)\left(\left(\partial_r\lambda - \frac{1}{r}\right)\hat{e}_r + \partial_z\lambda\hat{e}_z\right) + (\sin(\chi\phi) + \cos(\chi\phi))\hat{e}_\phi \tag{43}$$

Then,

$$\lim_{r_0\to 0} V(\phi = 0) = \hat{e_\phi} \tag{44}$$

Thus, the requirement that the vector is identical at $\phi = 2\pi$ after being transported around the circle starting at $\phi = 0$ as $r_0 \to 0$ is, from Equation (43):

$$\sin 2\pi\chi + \cos 2\pi\chi = 1 \tag{45}$$

In general, this has a number of solutions: all the integers $\chi = n = 1, 2, \ldots$ and $n = \frac{1}{4}, \frac{9}{4}, \frac{17}{4}$. The equivalent to Equations (44) and (45) for arbitrary r_0 from Equation (43) are much more complex.

What this tells us is that we do not, in general, have elementary flatness around the z-axis. We must therefore exclude it from our solution.

1.3 Matter solution to the Weyl metric

In principle, we have solutions for axially symmetric static vacuum spacetimes, subject to the conditions of Equations (27) and (42). These solutions, however, exclude the general masses defined by Equations (22) and (23), as well as the $z = 0$ axis (the Weyl strut) between them. We now wish to consider these objects.

The most general cylindrically symmetric static metric may be expressed as:

$$ds^2 = e^{2\lambda} dt^2 - e^{2(\nu-\sigma)} \left(dr^2 + dz^2 \right) - r^2 e^{-2\lambda} d\phi^2 \tag{46}$$

Where λ, ν, and σ are functions of r and z. Comparing this to the solutions of the empty-space metric Equation (1), and allowing for deviations from these values due to the strut, we make the identifications:

$$\lambda = \lambda_0 + f(r,z) \tag{47}$$

$$\sigma = \lambda_0 + g(r,z) \tag{48}$$

$$\nu = \nu_0 + h(r,z) \tag{49}$$

In empty space outside the strut the metric of Equation (46) reduces to that of Equation (1), which implies:

$$f(r,z) = g(r,z) \tag{50}$$

Evaluating Equation (23) for $r \to 0$ (n.b. you must take the one negative and one positive root of the two square root terms to get a non-zero answer) and taking into account the condition of Equation (42) we obtain:

$$\nu_0(0,z) = \begin{cases} \frac{-4\mu_1\mu_2}{(z_1-z_2)^2} & \text{for} \quad z_1 < z < z_2 \\ 0 & \text{otherwise} \end{cases} \tag{51}$$

From far away, our configuration should have spherical symmetry (i.e. the masses and strut become pointlike). This implies:

$$\lim_{r\to\infty} h(r,z) \to 0 \tag{52}$$

This reasoning will be addressed in the next section.

Combining Equation (49) with Equation (51) and the condition of Equation (42) yields:

$$h_0(0,z) = \begin{cases} \frac{4\mu_1\mu_2}{(z_1-z_2)^2} & \text{for} \quad z_1 < z < z_2 \\ 0 & \text{otherwise} \end{cases} \tag{53}$$

To get the force on the strut, we can integrate the z-component of the stress-energy tensor over the area:

$$F_z = \int T_{zz} d\sigma \tag{54}$$

We can get this by processing the metric of Equation (46) through Einstein's equation (13) resulting in:

$$G_{zz} = -(\partial_r \lambda)^2 + (\partial_z \lambda)^2 + \frac{1}{r}\partial_r \nu - \frac{1}{r}\partial_r \sigma + \frac{1}{r}\partial_r \lambda \tag{55}$$

Taking a first order approximation to Equation (55) eliminates the first two squared terms. Applying Equations (47) to (50) reduces to:

$$G_{zz} = -\frac{1}{r}\left(\partial_r \left(\nu_0 + h(r,z)\right)\right) \tag{56}$$

Now, recalling Equation (52) $h(r,z)$ can be taken as being defined at $r = 0$. Recalling Equation (42) cancels out the ν_0 term, and we are left with:

$$T_{zz} = \frac{1}{8\pi G} G_{zz} = -\frac{1}{8\pi G r}\partial_r h_0(0,z) \tag{57}$$

The integration measure for Equation (54) $d\sigma = r dr d\phi$.
Substituting:

$$F = \int r d\phi \int dr T_{zz} = \frac{1}{8\pi G r} 2\pi r \int dr \partial_r h_0(0,z) = \frac{1}{4G}\frac{-4\mu_1\mu_2}{(z_1 - z_2)^2} \tag{58}$$

Recall that Equations (22) and (23) as solutions to Equation (12) contain the length parameters μ_1 and μ_2 in order to make them dimensionless overall. Dimensional analysis (recall we are working in units of $c = 1$) shows that μ_1 can be taken as Gm_1 and $\mu_2 = Gm_2$.
Thus, the first order approximation is:

$$F = -\frac{Gm_1 m_2}{(z_1 - z_2)^2} \tag{59}$$

Where the negative sign indicates that gravity is attractive [3], and the squared derivative terms we excluded from Equation (55) are corrections to Newton's law.

Therefore, integrating the stress-energy of the Weyl strut between two stationary masses gives Newton's law (plus higher order corrections).

1.4 The Schwarszchild solution in cylindrical coordinates

We have a loose end in Section 1.3. To address if we are justified in applying Equation (52), we should check to see if our solution reduces to the Schwarszchild solution for $r \to \infty$.

1.5 Extrinsic Curvature

2 Application to Causal Dynamical Triangulations

Causal Dynamical Triangulations uses a path integral over all possible configurations between boundary conditions. The path integral is given by:

$$Z = \int \mathscr{D}[g] e^{iS_{EH}} \tag{60}$$

Where:

$$S_{EH} = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g}\,(R - 2\Lambda) \tag{61}$$

Given (60) and [5]

2.1 Regge Calculus

References

[1] A. Katz, "Derivation of Newton's Law of Gravitation from General Relativity," *Journal of Mathematical Physics*, vol. 9, pp. 983–985, Sept. 1967.

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[5] R. Kommu, "A Validation of Causal Dynamical Triangulations," *arXiv:1110.6875*, Oct. 2011.