Local and Global Flatness

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Given the metric on a cone "induced from the plane":

$$ds^2 = dr^2 + r^2 d\theta^2 \quad \text{with} \quad 0 \leqslant r \leqslant \infty, \quad \beta \leqslant \theta < 2\pi \tag{1}$$

Using the Christoffel connections:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu} \right) \tag{2}$$

We find:

$$\begin{array}{c} \Gamma^{r}_{\theta\theta} = -r \\ \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r} \end{array} \tag{3}$$

To parallel transport around the curve from $\theta = \beta$ to $\theta = 2\pi$ we set the directional covariant derivative equal to zero along the curve:

$$\tfrac{D}{d\lambda} = \tfrac{dx^{\mu}}{d\lambda} \nabla_{\mu} = 0 \quad \text{along} \quad x^{\mu}(\lambda) \tag{4}$$

Since we are parallel transporting a vector, this reduces to:

$$\nabla_{\alpha} V^{\beta} = 0 \tag{5}$$

We are going around the θ axis at $r = r_0$, so we have:

$$\begin{array}{c} \partial_{\theta} V^{r} + \Gamma^{r}_{\theta\theta} V^{\theta} = 0 \\ \partial_{\theta} V^{\theta} + \Gamma^{\theta}_{\theta r} V^{r} = 0 \end{array} \tag{6}$$

Plugging in our connections we have:

$$\partial_{\theta} V^{r} - r V^{\theta} = 0 \tag{7}$$

$$\partial_{\theta} V^{\theta} + \frac{1}{r} V^{r} = 0 \tag{8}$$

Differentiating Equation (7) with respect to θ we get:

$$\partial_{\theta} V^{\theta} = \frac{1}{r} \partial^2_{\theta} V^{r} \tag{9}$$

We can plug Equation (9) into Equation (8) to get:

$$\partial^2_{\theta} V^{r} + V^{r} = 0 \tag{10}$$

For which the solution is:

$$V^r = A\cos(\theta) + B\sin(\theta) \tag{11}$$

Likewise,

$$V^\theta = \frac{1}{r}\left(-A\cos(\theta) + B\sin(\theta)\right) \tag{12}$$

Now, at $\theta = \beta$ then $V = 0\hat{e_r} + 1\hat{e_\theta}$ so we have:

$$V^r = 0 = A\cos(\beta) + B\sin(\beta) \rightarrow A = -B\tan(\beta) \tag{13}$$

$$V^\theta = \frac{1}{r_0}\left(-A\cos(\beta) + B\sin(\beta)\right) \rightarrow B = \frac{r_0}{\tan(\beta)\sin(\beta) + \cos(\beta)} \tag{14}$$

Which gives us (after simplifying trigonometry):

$$\begin{aligned} A &= -r_0\sin(\beta) \\ B &= r_0\cos(\beta) \end{aligned} \tag{15}$$

And the expression for V^θ is:

$$V^\theta = \left(\sin(\theta)\sin(\beta) + \cos(\theta)\cos(\beta)\right)\hat{e}_\theta \tag{16}$$

Which gives, finally, that:

$$V^\theta\left(\theta = 2\pi\right) = \cos(\beta)\hat{e}_\theta \tag{17}$$

As the first sanity check, we note that if $\beta = 0$ then $V^\theta = \hat{e}_\theta$ as expected.

For the second sanity check, we note that $V_\theta V^\theta = 1$ for any value of β or θ, that is, the length of the vector after being parallel transported is unchanged.