Newtonian approximation in Causal Dynamical Triangulations

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Contents

1 Motivation

We would like to address the following issues:

- Can we recover $F = -\frac{Gm_1m_2}{r^2}$ from CDT?
- Do we have a sensible notion of "mass" in CDT?
- Semi-classical approximations not yet completely convincing [1]

In order to do so, we look for a solution with the following properties:

- Separation between two objects $\gg$ Schwarzschild radius
- Self-fields are not excluded
- Object size $\ll$ separation

Following previous work [2], the plan of attack is:

1. Find the most general axisymmetric (Weyl) metric

2. Impose boundary conditions
3. Find an expression for the "strut" of stress energy between two masses
4. Integrate T_{zz} around the "strut"
5. Recover $F = -\frac{Gm_1m_2}{r^2}$

1.1 Newton's Law of Gravitation from General Relativity

We begin by deriving the general axisymmetric Weyl metric.

Following Synge [3], we have a set of components $\{x^0, x^1, x^2, x^3\}$

where x^0 is the time coordinate, x^1,x^2 are any two coordinates in the meridianal (vertical) plane containing the z-axis, and x^3 is the angular component about the z-axis.

Imposing the static condition implies that our metric is independent of x^0, and axisymmetry implies that $x^3 \to x^3 + 2\pi$.

However, this does not sufficiently restrict possible solutions. We also assume time and angular reversibility, i.e. $\phi = -\phi$ and $t = -t$. Then we have:

$$ds^2 = g_{00}\left(dx^0\right)^2 - \Phi - g_{33}\left(dx^3\right)^2 \tag{1}$$

where

$$\Phi = g_{11}\left(dx^1\right)^2 + 2g_{12}dx^1dx^2 + g_{22}\left(dx^2\right)^2 \tag{2}$$

We furthermore assume that $g_{\mu\nu} = f\left(x^1, x^2\right)$

For positive definite quadratic differential forms of two variables such as explicit values of g_{11},g_{12}, and g_{22} from Φ in Equation (1), one can make a real, single-valued, continuous transformation from x^1 and x^2 to u and v by:

$$x^1 = x^1(u,v),\, x^2 = x^2(u,v) \tag{3}$$

where $J = \left[{}^{\partial(x^1,x^2)}/{}_{\partial(u,v)}\right] \neq 0$ such that:

$$g_{11}\left(dx^1\right)^2 + 2g_{12}dx^1dx^2 + g_{22}\left(dx^2\right)^2 = \beta^2\left(du^2 + dv^2\right) \tag{4}$$

Since x^1, x^2, u, and v are all arbitrary, Equation (1) becomes:

$$ds^2 = \alpha^2\left(dx^0\right)^2 - \beta^2\left(\left(dx^1\right)^2 + \left(dx^2\right)^2\right) - \gamma^2\left(dx^3\right)^2 \tag{5}$$

Explicitly, we then have the metric:

$$g_{\mu\nu} = \begin{pmatrix} \alpha^2\left(dx^0\right)^2 & 0 & 0 & 0 \\ 0 & -\beta^2\left(dx^1\right)^2 & 0 & 0 \\ 0 & 0 & -\beta^2\left(dx^2\right)^2 & 0 \\ 0 & 0 & 0 & -\gamma^2\left(dx^3\right)^2 \end{pmatrix} \tag{6}$$

Using [4]:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu} \right) \tag{7}$$

We can get the non-zero Christoffel connections as:

$$\begin{gathered}
\Gamma^{0}_{01} = \frac{\alpha_1}{\alpha} \\
\Gamma^{0}_{02} = \frac{\alpha_2}{\alpha} \\
\Gamma^{1}_{00} = \frac{\alpha\alpha_1}{\beta^2} \\
\Gamma^{1}_{11} = \frac{\beta_1}{\beta} \\
\Gamma^{1}_{12} = \frac{\beta_2}{\beta} \\
\Gamma^{1}_{22} = -\frac{\beta_1}{\beta} \\
\Gamma^{1}_{33} = -\frac{\gamma\gamma_1}{\beta^2} \\
\Gamma^{2}_{00} = \frac{\alpha\alpha_2}{\beta^2} \\
\Gamma^{2}_{11} = -\frac{\beta_2}{\beta} \\
\Gamma^{2}_{12} = \frac{\beta_1}{\beta} \\
\Gamma^{2}_{22} = \partial_{\nu} m \\
\Gamma^{2}_{33} = -\frac{\gamma\gamma_2}{\beta^2} \\
\Gamma^{3}_{13} = \frac{\gamma_1}{\gamma} \\
\Gamma^{3}_{23} = \frac{\gamma_2}{\gamma}
\end{gathered} \tag{8}$$

Where the subscripts denote partial derivatives, i.e. $\alpha_1 = \frac{\partial\alpha}{\partial x^1}$

The components of the Riemann tensor are given by:

$$R^{\rho}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{9}$$

Using Synge's substitutions:

$$\begin{gathered}
01 \longleftrightarrow 1 \\
02 \longleftrightarrow 2 \\
03 \longleftrightarrow 3 \\
12 \longleftrightarrow 4 \\
23 \longleftrightarrow 5 \\
31 \longleftrightarrow 6
\end{gathered} \tag{10}$$

The 20 independent components of the 4D Rieman tensor are given by:

$$\begin{array}{cccccc}
R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\
 & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\
 & & R_{33} & R_{34} & R_{35} & R_{36} \\
 & & & R_{44} & R_{45} & R_{46} \\
 & & & & R_{55} & R_{56} \\
 & & & & & R_{66}
\end{array} \tag{11}$$

Subject to:

$$R_{15}+R_{26}+R_{34}=0 \tag{12}$$

Which is the equivalent of:

$$R_{\rho[\sigma\mu\nu]}=0 \tag{13}$$

The non-zero components of the Riemann tensor are:

$$\begin{gathered}
R_{11}\to R^{0}_{101}=-\left(\tfrac{\alpha_1}{\alpha}\right)_1+\tfrac{\alpha_1\beta_1}{\alpha\beta}-\tfrac{\alpha_2\beta_2}{\alpha\beta}-\left(\tfrac{\alpha_1}{\alpha}\right)^2\\
R_{12}\to R^{0}_{102}=-\left(\tfrac{\alpha_2}{\alpha}\right)_2+\tfrac{\alpha_2\beta_1}{\alpha\beta}-\tfrac{\alpha_2\alpha_1}{\alpha^2}\\
R_{22}\to R^{0}_{202}=-\left(\tfrac{\alpha_2}{\alpha}\right)_2-\tfrac{\alpha_1\beta_1}{\alpha\beta}+\tfrac{\alpha_2\beta_2}{\alpha\beta}-\left(\tfrac{\alpha_2}{\alpha}\right)^2\\
\Gamma^1_{11}=\tfrac{\beta_1}{\beta}\\
\Gamma^1_{12}=\tfrac{\beta_2}{\beta}\\
\Gamma^1_{22}=-\tfrac{\beta_1}{\beta}\\
\Gamma^1_{33}=-\tfrac{\gamma\gamma_1}{\beta^2}\\
\Gamma^2_{00}=\tfrac{\alpha\alpha_2}{\beta^2}\\
\Gamma^2_{11}=-\tfrac{\beta_2}{\beta}\\
\Gamma^2_{12}=\tfrac{\beta_1}{\beta}\\
\Gamma^2_{22}=\partial_\nu m\\
\Gamma^2_{33}=-\tfrac{\gamma\gamma_2}{\beta^2}\\
\Gamma^3_{13}=\tfrac{\gamma_1}{\gamma}\\
\Gamma^3_{23}=\tfrac{\gamma_2}{\gamma}
\end{gathered} \tag{14}$$

The Ricci tensor is given by:

$$R_{\mu\nu}=R^{\lambda}_{\mu\lambda\nu} \tag{15}$$

We get the Ricci tensor components as:

$$
\begin{aligned}
\Gamma^0_{01} &= \frac{\alpha_1}{\alpha} \\
\Gamma^0_{02} &= \frac{\alpha_2}{\alpha} \\
\Gamma^1_{00} &= \frac{\alpha\alpha_1}{\beta^2} \\
\Gamma^1_{11} &= \frac{\beta_1}{\beta} \\
\Gamma^1_{12} &= \frac{\beta_2}{\beta} \\
\Gamma^1_{22} &= -\frac{\beta_1}{\beta} \\
\Gamma^1_{33} &= -\frac{\gamma\gamma_1}{\beta^2} \\
\Gamma^2_{00} &= \frac{\alpha\alpha_2}{\beta^2} \\
\Gamma^2_{11} &= -\frac{\beta_2}{\beta} \\
\Gamma^2_{12} &= \frac{\beta_1}{\beta} \\
\Gamma^2_{22} &= \partial_v m \\
\Gamma^2_{33} &= -\frac{\gamma\gamma_2}{\beta^2} \\
\Gamma^3_{13} &= \frac{\gamma_1}{\gamma} \\
\Gamma^3_{23} &= \frac{\gamma_2}{\gamma}
\end{aligned}
\tag{16}
$$

And the Ricci scalar given by:

$$
R = R^\mu_\mu = g^{\mu\nu} R_{\mu\nu} \tag{17}
$$

Is:

$$
R = R^\mu_\mu = g^{\mu\nu} R_{\mu\nu} \tag{18}
$$

Finally we have the Einstin tensor:

$$
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{19}
$$

And solving for the Einstein Field Equations in a vacuum (i.e. $G_{\mu\nu} = 0$) we get:

$$
G_{00} = -e^{2(\nu - m)} \left((\partial_v n)^2 + \partial_v^2 m + \partial_v^2 n + (\partial_u n)^2 + \partial_u^2 m + \partial_u^2 n \right) = 0 \tag{20}
$$

$$
\begin{aligned}
G_{11} = -\partial_v m \partial_v n + (\partial_v n)^2 - \partial_v m \partial_v \nu + \partial_v n \partial_v \nu + (\partial_v \nu)^2 \\
+ \partial_v^2 n + \partial_v^2 \nu + \partial_u m \partial_u n + \partial_u m \partial_u \nu + \partial_u n \partial_u \nu = 0
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
G_{21} = \partial_v n \partial_u m + \partial_v \nu \partial_u m + \partial_v m \partial_u n - \partial_v n \partial_u n + \partial_v m \partial_u \nu \\
- \partial_v \nu \partial_u \nu - \partial_u \partial_v n - \partial_u \partial_v \nu = 0
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
G_{22} = \partial_v m \partial_v n + \partial_v m \partial_v \nu + \partial_v n \partial_v \nu - \partial_u m \partial_u n + \partial_u m \partial_u \nu \\
+ (\partial_u n)^2 - \partial_u m \partial_u \nu + \partial_u n \partial_u \nu + (\partial_u \nu)^2 + \partial_u^2 n + \partial_u^2 \nu = 0
\end{aligned}
\tag{23}
$$

$$G_{33} = e^{2(n-m)} \left((\partial_v \nu)^2 + \partial_v^2 m + \partial_v^2 \nu + (\partial_u \nu)^2 + \partial_u^2 m + \partial_u^2 \nu \right) = 0 \tag{24}$$

Let:

$$\chi = n + \nu \tag{25}$$

Adding together Eqns. (20) and (24) gives:

$$\partial_u^2 \chi + \partial_v^2 \chi + (\partial_u \chi)^2 + (\partial_v \chi)^2 = 0 \tag{26}$$

Setting:

$$\Phi = e^{\chi} = e^{n+\nu} \tag{27}$$

We recover Laplace's equation in the uv-plane:

$$\partial_u^2 \Phi + \partial_v^2 \Phi = 0 \tag{28}$$

The remaining equations are used for boundary conditions on Laplace's equation. In general, for a metric of the form:

$$ds^2 = e^{2\psi} dt^2 - e^{-2\psi} \left[e^{2\omega} \left(dr^2 + dz^2 \right) + r^2 d\phi^2 \right] \tag{29}$$

We have general solutions:

$$\nabla^2 \psi = \partial_r^2 \psi + \frac{\partial_r \psi}{r} + \partial_z^2 \psi \tag{30}$$

$$d\omega [\psi] = r \left[\left((\partial_r \psi)^2 - (\partial_z \psi)^2 \right) dr + 2 \partial_r \psi \partial_z \psi dz \right] \tag{31}$$

Note that Eq(5) can be recovered from Eq(29) by substituting $\psi = \nu, m = \omega - \psi$, and $e^{2n} = r^2 e^{-2\psi}$.

The solution of Eq(30) and Eq(31)for a point particle of mass m at $z = z_0$ is given by (explain "point" in Schwarzschild solution, check for singularities):

$$\psi = -\frac{m}{R} \tag{32}$$

$$\omega = -\frac{m^2 r^2}{2R^4} \tag{33}$$

$$R = \sqrt{r^2 + (z - z_0)^2} \tag{34}$$

What is meant by "point" particle? To find out, let's transform to the Schwarzschild equation:

$$ds^2 = \left(1 - \frac{2GM}{r} \right) dt^2 - \frac{1}{\left(1 - \frac{2GM}{r}\right)} dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \tag{35}$$

using: (TODO: fill in transforms)

For n point particles we have [5]:

$$\psi = -\sum_{j=1}^{N} \frac{m_j}{R_j} \tag{36}$$

$$\omega = -\frac{r^2}{2}\sum_j \frac{m_j^2}{R_j^4} + \sum_{j\neq k} \frac{m_j m_k}{\left(z_j - z_k\right)^2}\left[\frac{r^2 + \left(z - z_j\right)\left(z - z_k\right)}{R_j R_k} - 1\right] \tag{37}$$

$$R = \sqrt{r^2 + \left(z - z_j\right)^2} \tag{38}$$

1.2 Additional issues

2 Applications to Causal Dynamical Triangulations

2.1 Preliminaries

A simplex is a generalization of a triangle to arbitrary dimension. For example, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and a 4-simplex is a pentachoron. Topologically, an n-simplex is equivalent to an n-ball; that is, an n-dimensional manifold with boundary.

An n-dimensional simplex has $n+1$ points or *vertices*. A convex hull, or minimal convex set of these points is the *m-face* of the *n-simplex*. Thus, a vertex is a *0-face*, and an edge between two vertices is the *1-face*. We can extend this notation to *2-faces* (triangles), *3-faces* (tetrahedrons), *4-faces* (pentachorons). We will not, at present, consider simplices of dimension higher than $n = 4$, but this generalization gives us a useful way to reason about higher dimensional spaces.

The number of *m-faces* on our *n-simplex* is given by the binomial coefficient as:

$$\begin{pmatrix} n+1 \\ m+1 \end{pmatrix} \tag{39}$$

Thus, our pentachoron has 5 vertices, 10 edges, 10 faces (triangles), 5 cells (tetrahedrons), and 1 4-face, itself.

A given face can be shared by another simplex. By requiring that [6]:

- Every face of a simplex K is in K, and
- The intersection of any two simplices of K is a face of each of them

We build up a useful structure called a simplicial complex. Informally, this is a space with a triangulation. Formally, simplicial complexes have only been proven for spaces of dimension $d \leq 3$. A simplicial complex has a well-defined homology (simplicial homology) which is easy to compute.

2.2 Code Correctness

Implementing CDT in computer code is non-trivial. As a first significant step, an independent implementation of CDT has given similar results to the original work [7]. We would like to build on Kommu's implementation using Literate Programming [8] coupled with Test Driven Development specific to the programming language used [9]. This provides for the codebase to be better understood by researchers wishing to replicate results or expand the capabilities of the code, and provides inherent integrity checks apart from "it produced what we expected". Such methodology will be critical to expanding the performance of the code by using such techniques as parallel processing and highly optimized algorithms. The adage of "Make it work, make it right, make it fast" applies.

The first building block of the code are the simplexes themselves. Using the known properties of simplicial complexes, we can provide for a series of checks that validate that simplices are being constructed correctly. Such checks will provide useful test cases when the underlying implementation of simplex data structures and moves are changed.

2.3 Data structures

2.4 Issues

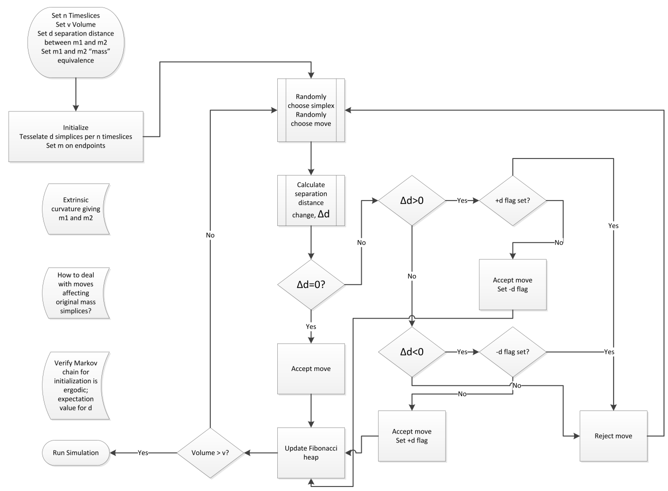
- Extrinsic Curvature *(To Do)*
- Imposing conditions of separation
- Checking that separation >> Schwarzschild radius
- Imposing cylindrical symmetry

2.5 CDT Algorithm

(To Do: insert graphics)

- [(2,8): (1,4) + (4,1) $\rightarrow$8 simplices] + inverse = +2 moves
- [(4,6): ()+()+()+()$\rightarrow$6 simplices] + inverse = +2 moves
- [(2,4): two varieties of ()+()$\rightarrow$4 simplices], self-inverse = +2 moves
- [(3,3): two varieties of ()+()+()$\rightarrow$3 simplices] + inverse = +4 moves

10 moves in all (*Check!)*



Dijkstra’s Algorithm [10]

Solves single-source shortest-path problems on weighted, directed graph G=(V,E) of non-negative edge lengths

- Greedy algorithm
- Proven to be correct
- Complexity
  - $O(V^2)$ naively using adjacency list
  - $O(E \lg V)$ using priority queue iff all vertices reachable from source
  - $O(V \lg V + E)$ using Fibonacci heap (more relaxation calls than extract-min calls)
- Issue: confine edge length algorithm to particular time-slice
- Solution: Store Fibonacci heap of simplices per time-slice
  - Each simplex has 5 neighbors, so more compact than adjacency matrix
  - How to deal with moves affecting original “mass” simplices
  - How to create a 4d cylinder of height z=d
  - Verify Markov chain for initialization is ergodic
  - Calculate <d>

3 Summary

- Insert mass equivalence via extrinsic curvature
- Insert strut by enforcing separation distance

- Filter moves which alter separation distance via Markov chain
- Outlook
  - Write code!
  - Check Extrinsic Curvature
  - Compare results

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