Lane-Emden Equation

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Contents

Lane-Emden equation

- Aim: use the numerical methods to solve Lane-Emden equation
- Benchmark of performance for the different numerical methods
- Applications: Lotka-Volterra, Nuclear decay chains, World population
- Particular relevance in Astrophysics
- Lane-Emden equation is a "dimensionless form of Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid"

Description

- Basic assumption is to consider a self-gravitating, spherically symmetric fluid in hydrostatic equilibrium
- The star's structure is static and thus none of the physical variables are time-dependent
- Spherical symmetry:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

- ρ is the density and it is a function of r

Description

- Assume there is also a polytopic relation between pressure and density:

$$P = K\rho^{\frac{n+1}{n}}$$

- where K and n are positive constants, n is called polytropic index
- Hydrostatic equilibrium leads to:

$$\frac{dP}{dr} = \frac{-Gm}{r^2}\rho$$

Description

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\theta^n$$

- $\rho = \rho_c \theta^n$ and $r = \alpha\xi$
- where $\alpha^2 = \frac{K(n+1)\rho_c^{\frac{1-n}{n}}}{4\pi G}$
- Boundary conditions:

$$\theta(\xi = 0) = 1$$

,

$$\frac{d\theta(\xi = 0)}{d\xi} = 0$$

Analytical solutions

There are analytical solutions for only three values of polytropic index:

- n=0 $\theta(\xi) = 1 - \frac{1}{6}\xi^2$
- n=1 $\theta(\xi) = \frac{sin\xi}{\xi}$
- n=5 $\theta(\xi) = \frac{1}{\sqrt{1+\frac{\xi^2}{3}}}$

Notation

$$\frac{dy}{dt} = f(t, y)$$

$$y(t_0) = y_0$$

$$f(y_0) = f_0$$

$$y(t) = y(t_0 + nh) = y_n$$

$$f(y_n) = f_n$$

$$y(t + h) = y(t_0 + (n + 1)h) = y_{n+1}$$

$$f(y_{n+1}) = f_{n+1}$$

Euler method -Forward

- Simplest algorithm for solving ODE
- Taylor expansion: $f(t+h) = f(t) + hy'(t) + \mathcal{O}(h^2)$ where $\mathcal{O}(h^2)$ is the order of the local truncation error for the step y_n to y_{n+1}
- Example: Starting at y_0 we calculate $y_1 = y_0 + hf(y_0)$
- Global truncation error $= N\mathcal{O}(h^2) = \frac{L\mathcal{O}(h^2)}{h} = L\mathcal{O}(h)$ The smaller the step size the smaller the global error. not quite effective, double accuracy means double the number of points

Multistep

Also known as Adams-Bashford's method

- Multistep method because it requires information from two previous steps
- $y_{n+1} = y_n + h(\frac{3}{2}f(x_n, y_n) - \frac{1}{2}f(x_{n-1}, y_{n-1})) + \mathcal{O}(h^3)$

Runge-Kutta methods

Used to avoid calculating the derivative

- Gives the best balance between speed and accuracy

$$k_1 = hf(t_i, y_i)$$

$$k_2 = hf(t_i + \frac{h}{2}, y_i + \frac{k_1}{2})$$

$$k_3 = hf(t_i + \frac{h}{2}, y_i + \frac{k_2}{2})$$

$$k_4 = hf(t_{i+1}, y_i + k_3)$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)) + \mathcal{O}(h^5)$$

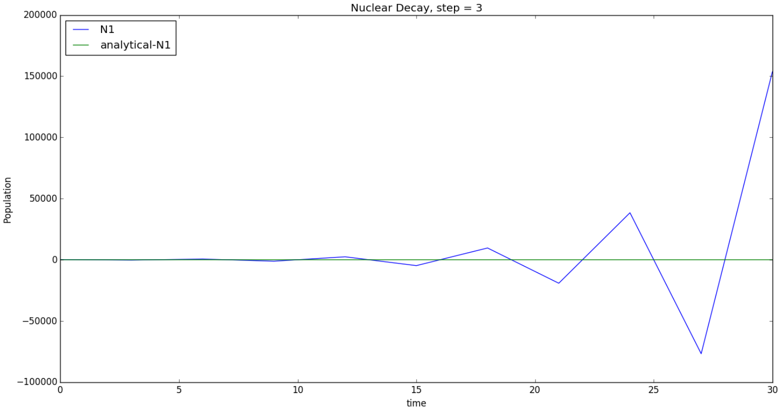
Instability

- Given the following ODE $y' = -\lambda y$ with $y(0) = \beta$ and λ positive constant
- Numerically this is solved with the following: $y(x+h) = y(x) + hy'(x)$
- $y(x+h) = y(x) + h(-\lambda y) = (1 - \lambda h)y(x)$
- Therefore, the numerical method is stable if:

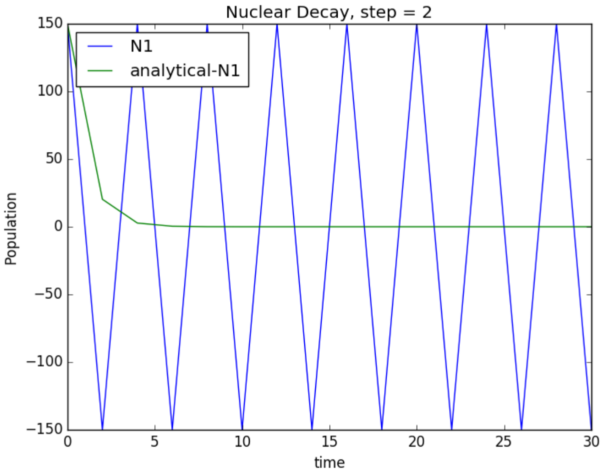
$$|1 - \lambda h| \leq 1$$

- In our case: $\lambda = 1$

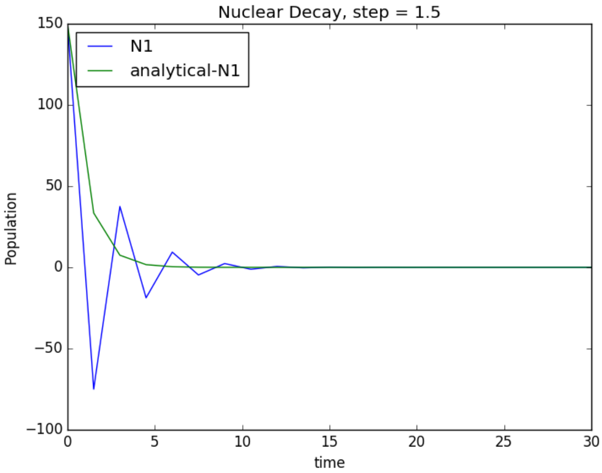
Examples


Nuclear Decay, step = 3
N1
analytical-N1
Population
time
200000
150000
100000
50000
0
−50000
−100000
0
5
10
15
20
25
30

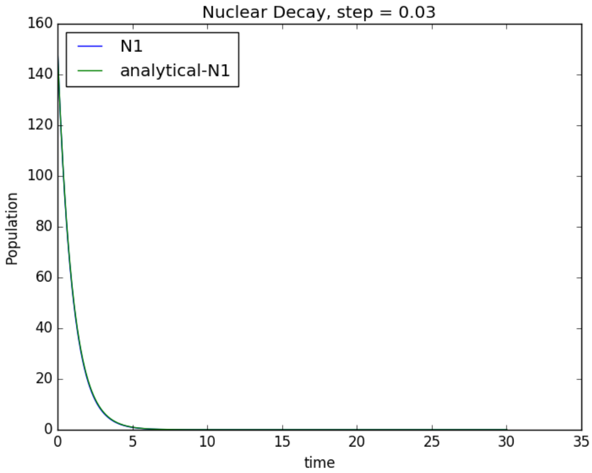
Examples


Nuclear Decay, step = 2
N1
analytical-N1
150
100
50
0
−50
−100
−150
Population
0
5
10
15
20
25
30
time

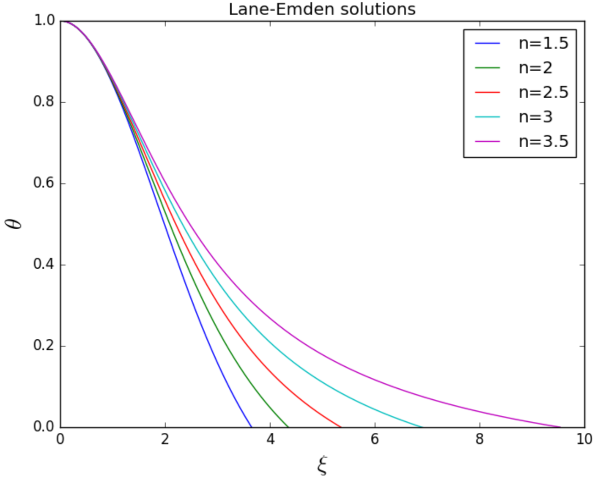
Examples


Nuclear Decay, step = 1.5
N1
analytical-N1
Population
time
150
100
50
0
−50
−100
0
5
10
15
20
25
30

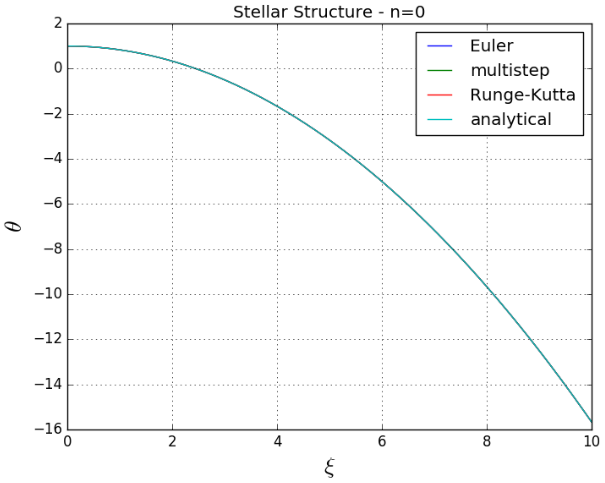
Examples


Nuclear Decay, step = 0.03
N1
analytical-N1
Population
time
160
140
120
100
80
60
40
20
0
0
5
10
15
20
25
30
35

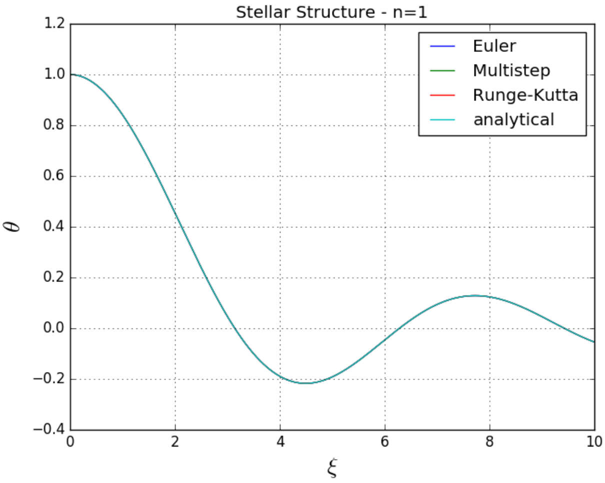
Lane-Emden solutions



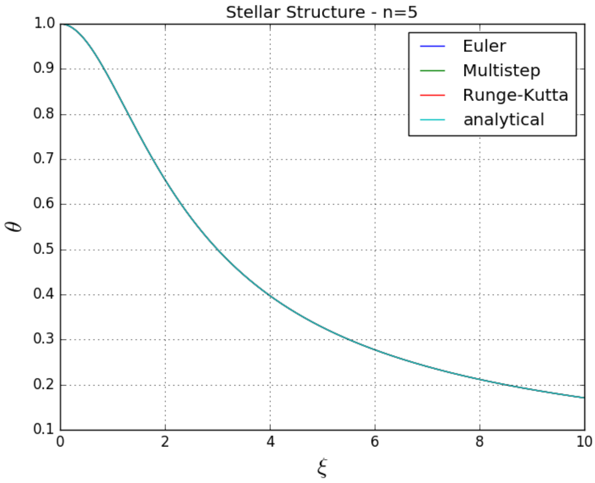
Lane-Emden solutions n=0



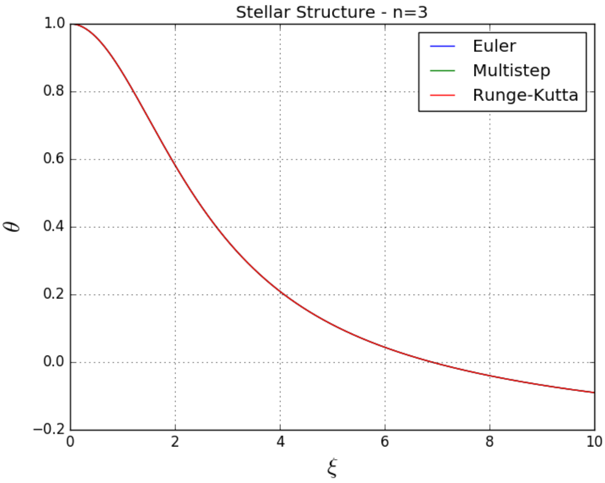
Lane-Emden solutions n=1



Lane-Emden solutions n=5



Lane-Emden solutions n=3

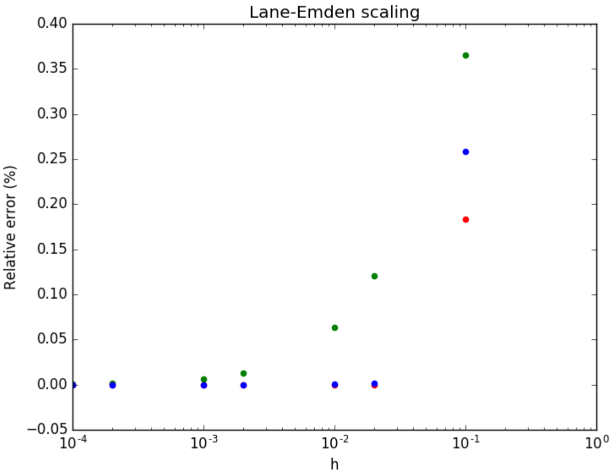


Benchmark

Execution time for each method and for each step size
Relative error for different step sizes: 0.1,0.01,0.001,0.0001

Green: Euler
Blue: Multistep
Red: Runge-Kutta

Benchmark n=0



Orders of magnitude - benchmark of performance n=0

Euler		
h	Execution time	Relative Error (%)
0.1	0.001	0.35
0.01	0.01	0.06
0.001	0.100	0.005
0.0001	0.98	0.0005

Orders of magnitude - benchmark of performance

Multistep		
h	Execution time	Relative Error (%)
0.1	0.007	0.27
0.01	0.048	0.001
0.001	0.431	10^{-5}
0.0001	4.176	10^{-7}

Orders of magnitude - benchmark of performance

Runge-Kutta		
h	Execution time	Relative Error (%)
0.1	0.059	0.17
0.01	0.162	10^{-5}
0.001	0.598	10^{-9}
0.0001	6.08	10^{-13}

Mass:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

Hydrostatic equilibrium:

$$\frac{dP}{dr} = \frac{-Gm}{r^2}\rho$$

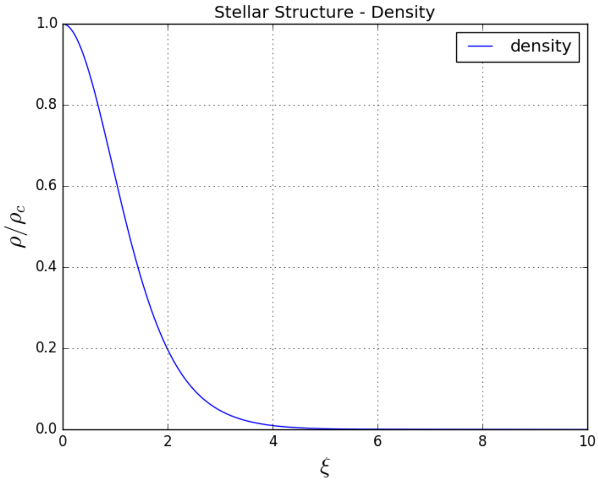
Energy Transport due to Radiation:

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{k\rho}{T^3}\frac{L}{4\pi r^2}$$

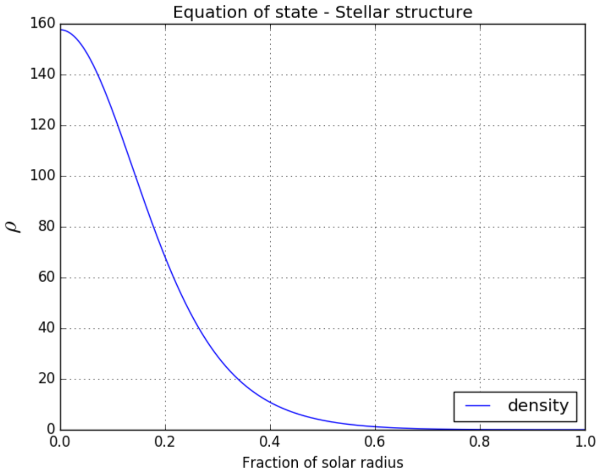
Energy generation

$$\frac{dL}{dr} = 4\pi r^2 \rho q$$

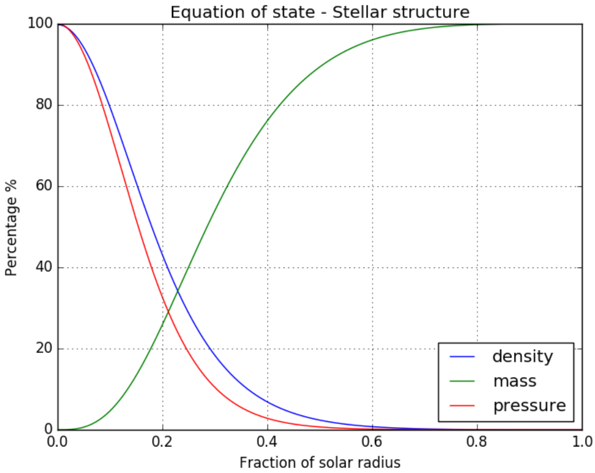
Sun - Density


Stellar Structure - Density
density
ρ/ρ_c
ξ
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10

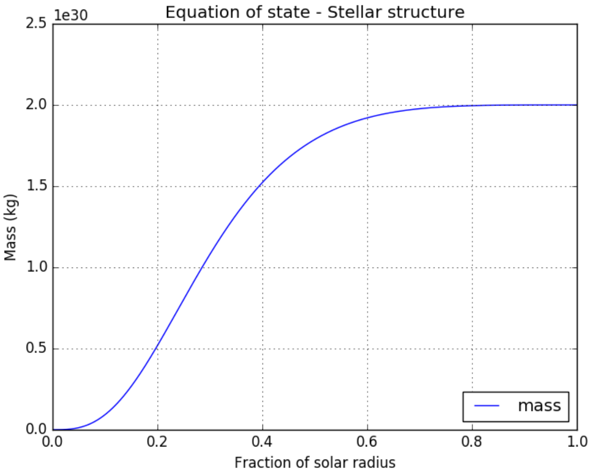
Sun - Density of the sun



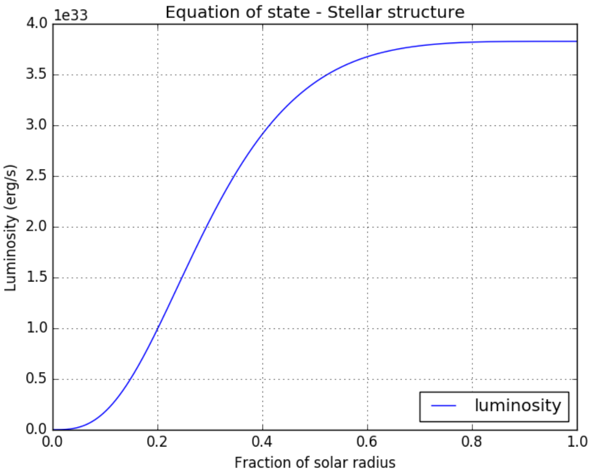
Sun - Density-Pressure-Mass



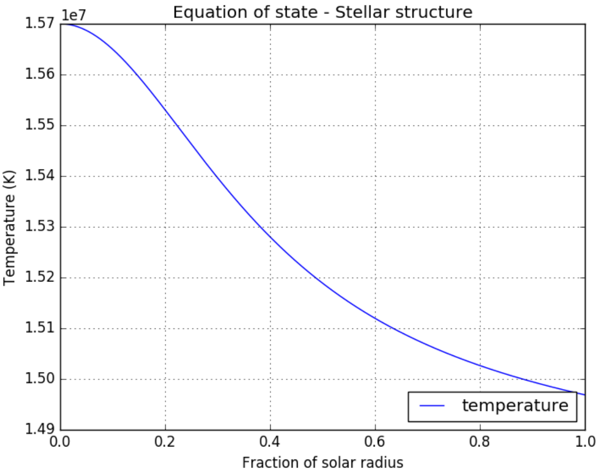
Sun - Mass of the Sun



Sun - Luminosity of the Sun



Sun - Temperature of the Sun



White dwarf

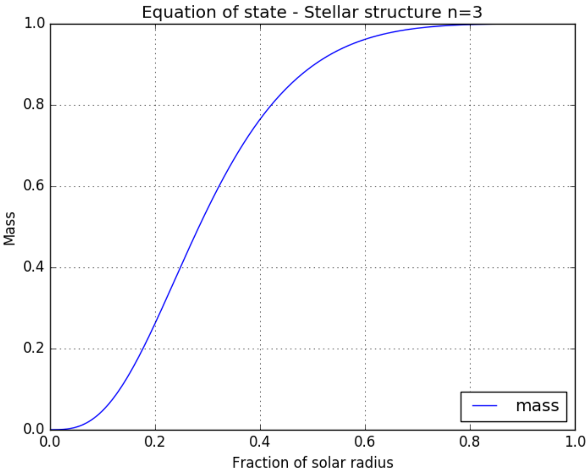
For polytropic index n=1.5 we have a good description for an object of mass $M << M_{\odot}$ such as a white dwarf.
In fact, for n=1.5 $P = K\rho^{\frac{5}{3}}$ This is in agreement with the theoretical behaviour of the pressure:

$$P = \frac{(3\pi^2)^{\frac{2}{3}}}{5} \frac{\hbar^2}{m_e} (\frac{Z\rho}{Am_H})^{\frac{5}{3}}$$

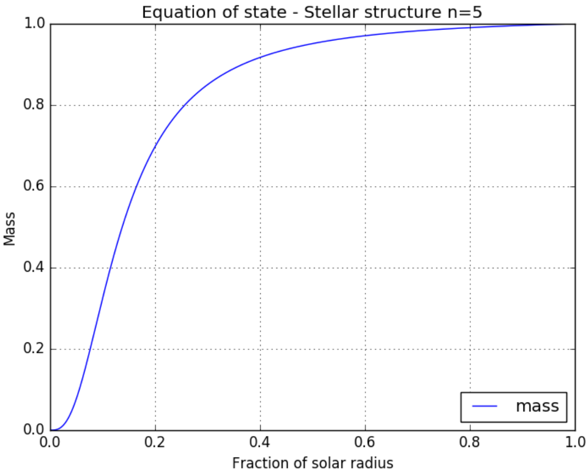
Different polytropic index

For polytropic index n=3 we have a good approximation of a gas in radiative equilibrium such as the sun.
For n=5 the mass is finite. For $n > 5$ the mass is infinite.

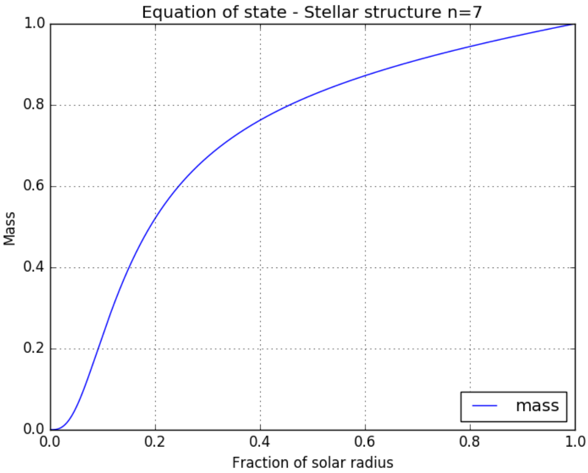
Mass for n=3


Equation of state - Stellar structure n=3
1.0
0.8
0.6
0.4
0.2
0.0
Mass
0.0
0.2
0.4
0.6
0.8
1.0
Fraction of solar radius
mass

Mass for n=5


Equation of state - Stellar structure n=5
1.0
0.8
0.6
0.4
0.2
0.0
Mass
0.0
0.2
0.4
0.6
0.8
1.0
Fraction of solar radius
mass

Mass for n=7


Equation of state - Stellar structure n=7
1.0
0.8
0.6
0.4
0.2
0.0
Mass
0.0
0.2
0.4
0.6
0.8
1.0
Fraction of solar radius
mass