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The expressivity of autosegmental grammars

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Abstract This paper extends a notion of restrictive grammars in formal language theory to autosegmental representations, in order to develop a sufficiently expressive yet computationally restrictive theory of well-formedness in natural language tone patterns. More specifically, it defines a class of *Autosegmental Strictly Local (ASL) grammars* and compares its expressivity to established formal language grammars that have been successfully applied to other areas of phonology.

Keywords formal language theory · graph theory · phonology · autosegmental representations · tone

1 Introduction

An interesting question in formal language theory is to what extent enriching representation increases the expressiveness of a class of grammars. For example, first-order logic describes exactly the Locally Threshhold Testable sets of strings when interpreted over string models with successor, but given string models with the full order over the positions first-order logic describes exactly the Star-Free stringsets, a strict superclass of the Locally Threshhold Testable sets (McNaughton and Papert, 1971). Similarly, Rogers (1997) gives a local logic that, when interpreted over

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strings, describes the exactly Strictly 2-Local stringsets (a strict subclass of the Locally Threshhold Testable sets), but when interpreted over tress describes the Context-Free stringsets.

Motivated by natural language phonology, this paper further explores this question in subclasses of the Star-Free sets. More specifically, it examines the relative expressivity of *banned k-factor grammars* interpreted in different ways in strings. Informally, a k-factor is a connected structure of size k. A banned k-factor grammar is a set of such structures interpreted as describing the set of objects that contain none of the members of the set. For example, $\{bb\}$ can be interpreted as describing the set in (1) of strings over the alphabet $\{a,b\}$ that do not include the *substring bb*.

$$\{\lambda, a, b, aa, ab, ba, aaa, aab, aba, baa, bab, aaaa, ...\} \tag{1}$$

Here, a substring is a k-factor interpreted as a sequence of immediately consecutive symbols in a string—thus (1) is the set of strings in which at no point is a b immediately followed by another b. Because bb is of length 2, the $\{bb\}$ is a banned 2-factor grammar (and (1) is a Strictly 2-Local stringset).

As explained below, banned k-factor grammars as interpreted over strings have been fruitful in positing restrictive computational characterizations of natural language phonology. These characterizations have helped to understand, from a theoretical perspective, how phonological grammars can be learned from positive data, and they have also been shown to be sufficiently expressive for a number of important classes of phonological phenomena. However, this paper shows that for tone patterns—where 'tone' refers to pitch that is phonologically contrastive at the lexical level—no single class of banned k-factor grammars characterizes the full range of attested patterns. It then proposes a new class of grammars, the *Autosegmental Strictly Local* sets, which are defined using banned k-factor grammars over *autosegmental representations*, or the graph-like structures phonologists have previously proposed for tone patterns. As the paper will show, these representations allow the banned k-factor grammars to be expressive enough to capture each of the discussed tone patterns. However, because they are still banned k-factors, they retain their restrictive character: the Autosegmental Strictly Local sets are strictly sub-Star-Free. The novel contribution of this paper is to synthesize previous work relating autosegmental representations to strings (Jardine and Heinz, 2015) to directly compare the expressivity of autosegmental grammars to string-based ones. Before going into more detail about the results of the paper, however, it will be useful to give some more context on the application of formal language theory to phonology.

1.1 Formal language theory and phonology

Formal language-theoretic characterizations of patterns can be used to study the cognitive underpinnings of pattern recognition in human and non-human animals (Gentner et al., 2006; Rogers and Pullum, 2011). In human language, these characterizations are perhaps most directly applied to *well-formedness patterns*, in which speakers of a language can judge whether or not a string of words (in syntax) or sounds (in phonology) is well-formed according to the rules of that language. What is a phonological well-formedness pattern? A classic example is the fact that English speakers, when presented with the two non-words *blick* and *bnick*, will invariably judge *blick* as well-formed but *bnick* as ill-formed (Chomsky and Halle, 1965). This is because in English, *bn* sequences are ill-formed syllable onsets.

It has been well established that phonological well-formedness patterns are at most Regular (Johnson, 1972; Kaplan and Kay, 1994; Heinz and Idsardi, 2011). However, a claim that "phonological well-formedness is Regular" would be an inaccurate characterization. There are many Regular patterns that are not attested as phonological well-formedness patterns; for instance, "the number of *n*s in a word must be even" is Regular, but no such pattern has so far been discovered in natural language. It has been argued, then, that more restrictive *Subregular* (that is, *Star-Free (SF)* and weaker) classes of stringsets are a tighter fit to the range of attested phonological patterns and thus better characterize the nature of the computation of phonological well-formedness (Heinz, 2010a; Heinz et al., 2011; Rogers et al., 2013; McMullin and Hansson, 2016). These classes are, namely, the the *Strictly Local (SL)* stringsets (McNaughton and Papert, 1971; Rogers and Pullum, 2011; Rogers et al., 2013), the *Tier-based Strictly Local (TSL)* stringsets (Heinz et al., 2011), and the *Strictly Piecewise (SP)* stringsets (Rogers et al., 2010; Fu et al., 2011). These classes, which are all sub-SF, are depicted in Fig. 1. All of these classes can be characterized by banned k-factor grammars over strings, and all come with provable learning results (García et al., 1990; Heinz, 2010b; Heinz and Rogers, 2013; Jardine and Heinz, 2016; Jardine and McMullin, 2017). To illustrate, the above restriction on *bn* in English is SL, as it can be (partially) modeled by the banned substring 3-factor grammar $\{\rtimes bn\}$, where $\rtimes$ indicates the beginning of a string (this will be explained in more detail in Sect. 3.2).

1.2 Contributions of this paper

However *tonal* well-formedness patterns have not yet been comprehensively studied from the above perspective. Tone is a particularly interesting object of formal study as it has been claimed to be the upper bound of complexity in natural language phonology; Hyman (2011) states: "Tone can do everything that segmental or metrical phonology can do, but the reverse is not true" (p. 199). This paper demonstrates this from a formal perspective: as depicted in Fig. 1, tone patterns run the gamut of formal language complexity classes previously proposed for various domains in segmental phonology, and even exceed them.

To give an example, all words in Hirosaki Japanese (Haraguchi, 1977) words must carry exactly one high-toned (H) or falling-toned (F) mora, but not both; all other moras must be low-toned (L). Additionally, only final moras can be falling-toned. This pattern can be modeled by the set L_{HJ} of strings over {H, L, F} shown in (2).

$$L_{\text{HJ}} = \{\lambda, \text{H, F, LH, LF, HL, LLH, LLF, LHL, HLL, LLLH, LLLF, ...}\} \quad (2)$$

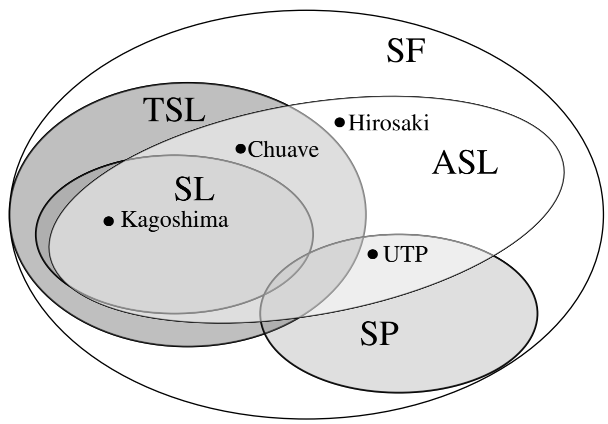


Fig. 1 Tone and the ASL class in the Subregular Hierarchy

As shown in Sec. 4.2, this stringset is properly SF, and thus outside the SL, TSL, and SP classes.

The second contribution of this paper is then to propose a new class of stringsets, the ASL class, and show that it includes each of the tone patterns discussed below. As mentioned above, the ASL class is defined through *autosegmental representations (ARs)*, representational devices that were first proposed to characterize tonal phonology (Leben, 1973; Goldsmith, 1976). ARs are a type of graph that represent tonal units on strings independent of the timing units over which they are realized.[1] Some examples of strings from the Hirosaki Japanese pattern and their corresponding AR graphs are given in Fig. 2.

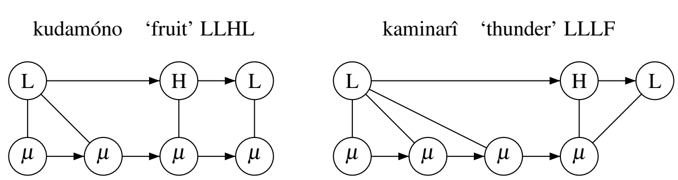


Fig. 2 Example words in Hirosaki Japanese, their corresponding representations as strings of toned moras, and their corresponding representations as ARs. ARs are mixed graphs with both directed edges—representing order on one of multiple strings of units—and undirected edges—representing associations between units on different strings. Nodes are also labeled; here, H and L indicate tonal units, and μ indicate moras.

[1] Tones have been variously been analyzed as being properties of vowels, moras, or syllables, depending on the language. This paper abstracts away from this issue. For discussion, see, e.g., Yip (2002).

For example, the Hirosaki Japanese word [kudamóno] 'fruit', which when represented as a string of moras is LLHL, is represented in an AR as a string of tones LHL associated to four moras (moras denoted with the Greek letter μ). The word [kaminarî] 'thunder' also has a LHL string associated to a string of four moras, but with a different association relation between the strings.

This paper defines the ASL class by extending the notion of banned k-factor grammars to *subgraphs* of ARs. Jardine and Heinz (2015) show how to use graph concatenation from a set of primitives to generate restrictive classes of graphs that correspond to ARs. We can thus describe the set of ARs that obey a particular subgraph grammar; for example, the following banned subgraph grammar in (3) forbids a HLH sequence in the string of tones in an AR.

$$\left\{ \quad \mathrm{H} \rightarrow \mathrm{L} \rightarrow \mathrm{H} \quad \right\} \tag{3}$$

By relating string alphabets to graph primitives, we can use Jardine and Heinz (2015)'s technique to directly relate stringsets and graph sets, and thus we can use banned k-factor grammars over ARs to describe sets of strings. For example, (3) describes the constraint in L_{HJ} that there can be no two H or Fs in a string. The ASL is the class of stringsets that are so describable.

Thus, by considering banned k-factor grammars over a more enriched representation, the ASL class can capture stringsets that banned k-factor grammars over strings cannot. This paper demonstrates the inclusion relationships summarized in Fig. 1: the ASL class includes all of the tone patterns discussed below, but is still sub-SF. To relate this to the above goal of characterizing phonological well-formedness, is that the ASL class forms a tight characterization of well-formedness in tone. Notably, it does this through the additional structural information provided by ARs.

It should be noted that this is by no means the first work on the computation of ARs. There is much work on encoding and finite-state implementation of ARs (Kay, 1987; Wiebe, 1992; Bird and Ellison, 1994; Kornai, 1995; Yli-Jyrä, 2013), and some work on their logical characterizations (Bird and Klein, 1990; Jardine, 2014, to appearb). A valuable lesson from this work is that ARs are within Regular complexity (though see Wiebe 1992 and Jardine to appearb for non-Regular aspects of ARs). However, this paper is the first to explicitly explore the relationship between ARs and *sub*-Regular complexity classes, by positing grammars that operate directly over ARs. A fruitful path for future work is to build on the automata-theoretic work cited above to develop automata-theoretic characterizations of the ASL sets, and compare them to the automata-theoretic characterizations of the SL, TSL, SP, and SF sets.

The idea of using banned k-factor grammars over ARs was first introduced in Jardine (2016a) and Jardine (to appeara), but this is the first paper to explicitly carefully study their expressivity with respect to the Subregular classes of patterns. It also explores tone patterns not discussed in Jardine (2016a) or Jardine (to appeara).

This paper is structured as follows. Sect. 2 outlines the empirical domain by giving examples of commonly attested tone patterns. Sect. 3 gives the formal definitions and relationships among the Subregular stringset classes previously studied with respect to phonological patterns, and Sect. 3 shows that none of these classes capture all of the the tone patterns introduced in Sect. 2. Sect. 5 then introduces the ASL class, shows that it captures all of the tone patterns of Sect. 2, and also discusses in detail its relationship to the other Subregular classes. Sect. 6 summarizes the results and directions for future work, and Sect. 7 concludes.

2 Tone well-formedness patterns

We begin by establishing the empirical domain of interest through a number of key examples of *well-formedness* patterns in tone. Again, these are language-specific patterns in which constraints are placed on the distribution of tones in a word. Tone patterns are common in language, and a full survey of the typology is beyond the scope of this paper (on the range of attested tone patterns see, e.g., Yip, 2002; Hyman, 2011). For the purposes of this paper, it is sufficient to examine a small sample of common tone patterns to show how they range throughout the Subregular Hierarchy. The following looks at three types of tone pattern: *positional* patterns referring to where tones can appear, *obligatoriness* patterns requiring particular tones in a word, and *culminativity* patterns referring to how many times a tone can appear.[2]

2.1 Positional well-formedness patterns

It is common for tones to be restricted to some position in the word. For example, in Kagoshima Japanese, words must have a high tone on either the final or penultimate mora, as in the examples in Table 1 (where high tones are marked with an acute accent on the vowel, [á]).[3] In Table 1, as elsewhere in this paper, schematic representations showing high- and low-toned moras as Hs and Ls are given for each form.

a.	hána	'nose'	HL
b.	sakúra	'cherry blossom'	LHL
c.	kagaríbi	'watch fire'	LLHL
d.	kagaribí-ga	'watch fire' + NOM	LLLHL
e.	haná	'flower'	LH
f.	usagí	'rabbit'	LLH
g.	kakimonó	'document'	LLLH
h.	kakimono-gá	'document' + NOM	LLLLH

Table 1 Tone in Kagoshima Japanese (Hirayama, 1951; Haraguchi, 1977; Kubozono, 2012)

[2] This use of the terms 'obligatoriness' and 'culminativity' is due to Hyman (2009).

[3] The tone patterns in the Japanese dialects are often referred to as 'pitch accent' patterns (see, e.g., Kubozono, 2012), but they fit the definition of tone system in that they use pitch to make lexical contrasts, and it has been argued that there is no reason to treat them as phenomenologically distinct from other tone patterns (Hyman, 2009). This is at least true for the patterns discussed here, which appear both in Japanese dialects and tone systems elsewhere.

Thus, words such as *[kákimono] (HLLL), where a high tone appears before the penult, or a word such as *[kakimono] (LLLL), in which there are no high tones, are ill-formed.

This type of pattern is well-attested in other languages. In Zigula, for example, single high tones must appear on the penultimate mora (Kenstowicz and Kisseberth, 1990); in Copperbelt Bemba, there is a constraint that prefers a high tone to appear on the last mora of a phonological phrase (Bickmore and Kula, 2015).

2.2 Obligatoriness patterns

It is also common for tones to be *obligatory* in a word. This was part of the pattern in Kagoshima Japanese: words with all low tones are banned. A more dramatic example is in Chuave, in which a high tone can appear anywhere in the word, and there is no restriction on the number of high tones. In Chuave, the relevant unit for tone is the syllable, with adjacent vowels appearing in different syllables (the schematic notation of Hs and Ls abstracts away from the question of whether tones are borne by vowels, mora, or syllables).

a.	kán H	'stick'	e.	gíngódí HHH	'snore'	i.	kóiom HLL	'wing'
b.	gáán HH	'child'	f.	dénkábu HHL	'mosquito'	j.	komári LHL	'before'
c.	gáam HL	'skim'	g.	énugú HLH	'smoke'	k.	koiyóm LLH	'navel'
d.	kubá LH	'bamboo'	h.	amámó LHH	'k.o. yam'			

Table 2 Tone in Chuave (Donohue, 1997)

In Chuave, words with all low tones are ill-formed: *L, *LL, *LLL, etc. Other languages with such constraints are Manding, Crow, and Choctaw, among others (Hyman, 2009). Note that unlike primary stress, which is usually considered universally obligatory in stress languages, obligatoriness is not universal for either H or L tones. Tinputz, for example, allows words of the shape L, LL, LLL, etc. (Hostetler and Hostetler, 1975; Hyman, 2009), and many Bantu languages (like Luganda in the section below) contrast H-toned verbs versus all L-toned verbs.

2.3 Culminativity patterns

It is also common for languages to constrain *how many* of a particular tone appear in a word (usually one). We can refer to a tone as *culminative* in such a pattern. This is true for Kagoshima Japanese, as only one high tone appears in any word. This is true also for Tinputz, although the high tone is not restricted to any particular position (Hostetler and Hostetler, 1975; Hyman, 2009).

A perhaps more striking example is a common pattern known as *unbounded tone plateauing* (UTP) (Kisseberth and Odden, 2003; Hyman, 2011; Jardine, 2016a). In

UTP patterns, surface well-formedness dictates that only one *plateau*, or unbroken stretch, of high-toned moras is allowed in a domain. The following examples are from Luganda (Hyman and Katamba, 2010; Hyman, 2011).

a.	kitabo	'book'	LLL
b.	mutéma	'chopper'	LHL
c.	kisikí	'log'	LLH
d.	mutémá+bísíkí	'log chopper'	LHHHHH
e.	*mutéma+bisikí	" "	*LHLLLH

Table 3 Tone in Luganda (Hyman and Katamba, 2010; Hyman, 2011)

In Luganda, high tones are not obligatory, thus forms like Table 3a [kitabo] 'book' (LLL) are licit. However, when we create a compound from two words with high tones, e.g. Table 3b [mutéma] 'chopper' (LHL) and Table 3 [kisikí] 'log' (LLH), the resulting form only has one plateau of high tones, i.e. Table 3d [mutémá+bísíkí] 'log chopper' (LHHHHH). Hypothetical forms with two separate high tones, such as *[mutéma+bisikí] (LHLLLH), are ill-formed.

UTP can be thought of as a *process* that takes two underlying high tones and creates a plateau from them (Hyman, 2011; Jardine, 2016a), but it is generally understood as motivated by a long-distance well-formedness constraint against high-low-high sequences (see, e.g., Yip (2002)'s *TROUGH constraint). Since this effectively limits a word to at most one plateau of high tones in a word, we can also think of UTP as a culminativity pattern (which is borne out from its formal properties, as discussed in Sect. 4.2).

2.4 Composite patterns

There are also tone patterns that are best described as a combination of some or all of the above pattern types. Tone in Hirosaki Japanese (Haraguchi, 1977) simultaneously exhibits positional, obligatoriness, and culminativity well-formedness.[4] Table 4 gives examples of nouns in isolation and suffixed with the nominative suffix. A breve on a vowel [â] denotes a falling tone in the full transcriptions; an F denotes a falling tone in the schematic representations. The relevant unit for tone in Hirosaki Japanese is the mora, including coda nasals (e.g. [ŋ] in Table 4i).

In Hirosaki Japanese, there must be exactly one high- or falling-toned mora in a word, and there never can be both. Thus, forms like *LLLL are attested, as are strings like *HLLH or *HLLF. Thus, there is a combined obligatoriness and culminativity constraint on both H and F tones. Additionally the falling tone can only appear word-finally; strings like *FLLL are also ill-formed. Thus, tonal well-formedness in Hirosaki Japanese is positional, obligatory, and culminative.

[4] For an alternate description of Hirosaki Japanese, see Kobayashi (1970).

Noun	Isolation	+NOM	Noun	Isolation
a. 'handle'	é	e-gá	f. 'chicken'	niwatorí
	H	LH		LLLH
b. 'picture'	ê	é-ga	g. 'thunder'	kaminarî
	F	HL		LLLF
c. 'candy'	amé	ame-gá	h. 'fruit'	kudamóno
	LH	LLH		LLHL
d. 'rain'	amê	amé-ga	i. 'trunk'	toráŋku
	LF	LHL		LHLL
e. 'autumn'	áki	áki-ga	j. 'bat'	kóomori
	HL	HLL		HLLL

Table 4 Tone in Hirosaki Japanese (Haraguchi, 1977)

2.5 Explaining the variation

This concludes our brief survey of positional, obligatoriness, and culminativity well-formedness patterns in tonal phonology. The question then is, how do we best characterize the range of attested tone patterns? Previous work applying formal language theory to segmental (Heinz, 2010a; Heinz et al., 2011; McMullin and Hansson, 2016) and stress (Heinz, 2009; Rogers et al., 2013) well-formedness patterns have shown that sub-Regular classes of patterns offer strong characterizations of the typologies of these phenomena. The following section thus reviews these previously established classes.

3 The Subregular Hierarchy

This section outlines the *Subregular Hierarchy* (Rogers et al., 2013), a hierarchy of formal classes that has been identified in previous research to provide strong characterizations of well-formedness patterns in phonology. We first define the relevant classes and then Sect. 4.1 reviews previous research linking well-formedness patterns in stress and segmental phonology to the Subregular Hierarchy.

3.1 Preliminaries

Let Σ be a finite alphabet of symbols. A string w is a sequence $\sigma_1\sigma_2...\sigma_n$ of symbols $\sigma_i \in \Sigma$ of length n. Let $|w|$ denote the length of w, and let λ denote the *empty string*, or the unique string of length 0. For a symbol $\sigma \in \Sigma$, let σ^n denote the string comprised of σ repeated n times. The set of all strings over Σ, including λ, is denoted Σ^*. For two strings u and v, let uv denote their concatenation.

A *stringset* (or *formal language*) is some (potentially infinite) subset $L \subseteq \Sigma^*$ (the term 'stringset' avoids confusion with natural languages). A *grammar* is a finite representation of a stringset. For a grammar G let $L(G)$ denote the stringset generated by G.

A *class* $\mathscr{C}$ of stringsets is a set of stringsets. A class can be (though not necessarily) discussed in terms of the type of grammar necessary to generate its composite stringsets. For example, the *Regular* stringsets are those generated by Type-3 grammars (Chomsky, 1956), finite-state automata (Kleene, 1956), or sentences of monadic second-order logic (Büchi, 1960). The following reviews *Subregular* classes of stringsets (that is, classes that are properly contained with in the Regular class) that have been successfully applied to the theory of natural language phonology: the *Strictly Local (SL)* stringsets (McNaughton and Papert, 1971; Rogers and Pullum, 2011; Rogers et al., 2013), the *Tier-based Strictly Local (TSL)* stringsets (Heinz et al., 2011), and the *Strictly Piecewise (SP)* stringsets (Rogers et al., 2010; Fu et al., 2011). It is shown that all of these classes share a crucial property: they each can be characterized by *banned k-factor grammars*.[5] These are contrasted with a more complex subclass of the Regular stringsets: the Star Free stringsets, which are characterizable by sentences of first order-logic with precedence (McNaughton and Papert, 1971). For a more detailed review see Rogers et al. (2013).

Banned k-factor grammars describe a stringset by identifying a set of connected structures of size k that cannot appear in any string in the stringset. This property makes these grammars computationally very simple and leads to algorithms for identifying them from positive data. In terms of phonology, the SL, SP, and TSL classes have been previously argued to provide insights into the typology of phonological well-formedness, as discussed in Sect. 4.1.

3.2 The Strictly Local stringsets

The SL stringsets (McNaughton and Papert, 1971; Rogers and Pullum, 2011; Rogers et al., 2013) are defined in terms of *substring k-factors*, so we shall define these first. Let $\rtimes$, $\ltimes$ be special symbols not in Σ marking the beginning and end of a string, and let $\rtimes\Sigma^*\ltimes$ denote the set of strings in Σ^* delineated with these boundary symbols. For any string w, a string u is a *substring* of w if $w = v_1uv_2$ for any two (possibly empty) strings v_1 and v_2. A *substring k-factor* of w is a substring of $\rtimes w\ltimes$ of length k. For example, $\rtimes ab$ is 3-factor of the string $abaa$, because it is a substring of $\rtimes abaa\ltimes$ of length 3. We can define a function $\mathtt{substr}_k$ which takes a string in Σ^* and outputs its set of substring k-factors as follows:

$$\mathtt{substr}_k(w) \stackrel{\text{def}}{=} \begin{array}{ll} \{u \mid u \text{ is a } k\text{-factor of } w\} & \text{if } |\rtimes w \ltimes| > k \\ \{\rtimes w\ltimes\} & \text{otherwise} \end{array}$$

For instance, $\mathtt{substr}_k(abaa) = \{\rtimes ab, aba, baa, aa\ltimes\}$. We extend $\mathtt{substr}_k$ to stringsets in the natural way; i.e. for $L \subseteq \Sigma^*$, $\mathtt{substr}_k(L) = \bigcup_{w\in L} \mathtt{substr}_k(w)$.

A *Strictly k-Local (SL$_k$)* stringset can then be described with a *set of banned substring k-factors* $B \subseteq \mathtt{substr}_k(\Sigma^*)$. The stringset described by B, denoted $L(B)$, is thus the set of strings that contain no substrings in B:

$$L(B) \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid \mathtt{substr}_k(w) \cap B = \emptyset\}$$

[5] This terminology, which distinguishes the notion of factor from the (stronger) model theoretic notion of substructure, is due to James Rogers (p.c.).

For example, if $\Sigma = \{a,b\}$, then the set $B = \{\rtimes b, b\ltimes\}$ is exactly the set of strings of as and bs that neither start nor end with b. For example, $baaaa \notin L(B)$ because $\rtimes b \in B$ and $\rtimes b \in \texttt{substr}_2(baaa)$, whereas $abaa \in L(B)$ because both $\rtimes b \notin \texttt{substr}_2(abaa)$ and $b\ltimes \notin \texttt{substr}_2(abaa)$. Note that $k = 2$ because B is composed of substring 2-factors; we can thus say that $L(B)$ is SL$_2$. More examples of $L(B)$ are given in (4).

$$L(B) = \{\lambda, a, aa, aaa, aba, aaaa, aaba, abaa, abba, aaaaa, aaaba, ...\} \tag{4}$$

For any k, the SL$_k$ class of stringsets is a proper subset of the SL$_{k+1}$ (Rogers et al., 2013, Thm. 2). Thus, these classes form an infinite hierarchy, as given in (5). We say a language is SL if it is SL$_k$ for some k.

$$\text{SL}_1 \subsetneq \text{SL}_2 \subsetneq \text{SL}_3 \subsetneq \ldots \subsetneq \text{SL} \tag{5}$$

An important property of any SL$_k$ language is *suffix substitution closure (SSC)*.

Theorem 1 (Suffix substitution closure (Rogers and Pullum, 2011)) *A stringset L is SL$_k$ iff for any string x of length $k-1$ and any strings w, u, y, and z,*

$$wxu \in L \text{ and } yxz \in L \text{ implies } wxz \in L$$

The intuition here is that the membership problem for an SL$_k$ language only requires paying attention to chunks of a string up to length k. SSC is an abstract characterization that allows us to prove that a language is not SL$_k$. For example, take the following stringset $L_{\text{one-}b}$ of strings over $\Sigma = \{a,b\}$ containing exactly one b:

$$L_{\text{one-}b} = \{b, ab, ba, aab, aba, baa, aaab, aaba, abaa, baaa, aaaab, ...\} \tag{6}$$

This language is not SL$_2$. Consider $x = a$, which is of length $k-1 = 1$. Then set $w = b$, $u = a$, $y = a$, and $z = b$. We then have

$$\underbrace{b}_{w}\underbrace{a}_{x}\underbrace{a}_{u} \in L_{\text{one-}b} \text{ and } \underbrace{a}_{y}\underbrace{a}_{x}\underbrace{b}_{z} \in L_{\text{one-}b} \text{ but } \underbrace{b}_{w}\underbrace{a}_{x}\underbrace{b}_{z} \notin L_{\text{one-}b}$$

In fact, $L_{\text{one-}b}$ is not SL$_k$ for *any* k, as we can always plug in a^{k-1} (a string of $k-1$ as) and still fail the implication in Thm 1:

$$\underbrace{b}_{w}\underbrace{a^{k-1}}_{x}\underbrace{a}_{u} \in L_{\text{one-}b} \text{ and } \underbrace{a}_{y}\underbrace{a^{k-1}}_{x}\underbrace{b}_{z} \in L_{\text{one-}b} \text{ but } \underbrace{b}_{w}\underbrace{a^{k-1}}_{x}\underbrace{b}_{z} \notin L_{\text{one-}b}$$

Thus, $L_{\text{one-}b}$ is not SL.

As to be described in Sect. 4.1, many well-formedness patterns in natural language phonology can be modeled as SL stringsets. The SL also class has a number of attractive properties with respect to natural language: it is closed under intersection (McNaughton and Papert, 1971) and for a given k, the class of SL$_k$ stringsets is learnable in the limit from positive data (García et al., 1990).

To describe SL patterns, we interpret each k-factor $\sigma_1\sigma_2...\sigma_k$ as a contiguous substring; that is, it is *connected* by the successor relation. We can consider different classes by varying the interpretation of the k-factors, but it remains true that they are connected by some relation.

3.3 The Tier-based Strictly Local stringsets

The TSL stringsets (Heinz et al., 2011) are a generalization of SL languages in which banned k-factors in B are evaluated as substrings comprised of symbols on a *tier* $T \subseteq \Sigma$, ignoring all symbols not on the tier. They thus capture a notion of 'relativized locality' that commonly appears in natural language.

The grammars that describe *Tier-based Strictly k-Local (TSL$_k$)* stringsets thus have two parameters $\langle T, B\rangle$, a tier $T \subseteq \Sigma$ and a set $B \subseteq \mathtt{substr}_k(T^*)$ of *banned tier substring k-factors*. From T we define a function $\mathtt{erase}_T(w)$ which takes as input a string $w \in \Sigma^*$ and outputs the string in T^* removing all non-tier symbols from w:

$$\mathtt{erase}_T(w) \stackrel{\text{def}}{=} \begin{array}{ll} \mathtt{erase}_T(u)\sigma & \text{if } w = u\sigma, u \in \Sigma^*, \sigma \in T \\ \mathtt{erase}_T(u) & \text{if } w = u\sigma, u \in \Sigma^*, \sigma \notin T \end{array}$$

For example, if $\Sigma = \{a,b\}$ and $T = \{b\}$, then $\mathtt{erase}_T(abaaaaba) = bb$. The banned tier substring k-factors are then evaluated over these 'erased' strings:

$$L(\langle T,B\rangle) \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid \mathtt{substr}_k(\mathtt{erase}_T(w)) \cap B = \emptyset\}$$

For example, let $\Sigma = \{a,b\}$ and $T = \{b\}$, and $B = \{\rtimes\ltimes, bb\}$. The string $aabaa \in L(\langle T,B\rangle)$, because $\mathtt{erase}_T(aabaa) = b$, and neither $\rtimes\ltimes$ nor bb is a substring 2-factor of b. However, $aaaa \notin L(\langle T,B\rangle)$, because $\mathtt{erase}_T(aaaa) = \lambda$, and $\rtimes\ltimes \in \mathtt{substr}_2(\lambda)$. Similarly, $abaaaaba \notin L(\langle T,B\rangle)$ because $\mathtt{erase}_T(abaaaaba) = bb$ and $bb \in \mathtt{substr}_2(bb)$. In fact, the reader can confirm that $L(\langle T,B\rangle)$ is the set of strings containing exactly one b; in other words, $L(\langle T,B\rangle) = L_{\text{one-}b}$. By using $\mathtt{erase}_T$ to ignore intervening as, we can thus reduce this non-SL stringset to a pattern defined by banned tier substring k-factors.

We say that a stringset is TSL if it is TSL$_k$ for some k. As witnessed by $L_{\text{one-}b}$, a TSL$_k$ class is strictly more expressive than its corresponding SL$_k$ class; note that SL$_k$ is the special case for which $T = \Sigma$. Thus SL is a proper subset of TSL. The TSL class is also a proper subclass of the Star-Free class (Heinz et al., 2011). TSL$_k$ stringsets are also efficiently learnable in the limit from positive data, both when T is known to the learner a priori (Heinz et al., 2011) or not (Jardine and Heinz, 2016; Jardine and McMullin, 2017).

Interestingly, the class of TSL$_k$ stringsets is closed under composition when T is constant but not when T varies, as witnessed by the following. Let $\Sigma = \{a,b,c\}$ and consider the following stringsets $L_{\text{one-}b/c}$, the set of strings with exactly one b or c and $L_{\text{final-}c}$, the set of strings in which c can only appear in final position in the string (but b and a have no such restriction).

$$\begin{array}{rl} L_{\text{one-}b/c} \stackrel{\text{def}}{=} & L(\langle T = \{b,c\}, B = \{\rtimes\ltimes, bc, cb, bb, cc\}\rangle) = \\ & \{b, c, ab, ac, ba, ca, aab, aac, aba, aca, baa, caa, ...\} \end{array} \tag{7}$$

$$\begin{array}{rl} L_{\text{final-}c} \stackrel{\text{def}}{=} & L(\langle T = \{a,b,c\}, B = \{ca, cb, cc\}\rangle) = \\ & \{a, b, c, aa, ab, ac, ba, bb, bc, aaa, aab, aac, aba, abb, abc, ...\} \end{array} \tag{8}$$

Note that T is distinct for each stringset. If we take the intersection of these two stringsets, we get the set of strings with exactly one b or c, and in the case of c, it must appear in the final position:

$$L_{\text{one-}b/c} \cap L_{\text{final-}c} = \{b, c, ab, ac, ba, aab, aac, aba, baa, aaab, aaac, ...\} \quad (9)$$

This stringset is not TSL, as given in Remark 1.

Remark 1 $L_{\text{one-}b/c} \cap L_{\text{final-}c}$ is not TSL for any T or any k.

Proof The only possible T we can consider is $\{a,b,c\}$, because B for the stringset must be $\{\rtimes\ltimes, bc, cb, bb, ca, cc\}$, or the union of the banned k-factors for $L_{\text{one-}b}$ and $L_{\text{final-}c}$, and this set includes a k-factor containing each of a, b, and c. (Recall that B must be made up of k-factors of T^*.) Of course $\{a,b,c\} = \Sigma$ and so we can appeal directly to SSC. The set $L_{\text{one-}b} \cap L_{\text{final-}c}$ fails SSC for any k for the same reason as $L_{\text{one-}b}$. For $w = b$, $u = a$, $y = a$, $z = b$, and $x = a^{k-1}$, $wxv = ba^{k-1}a \in L_{\text{one-}b} \cap L_{\text{final-}c}$, $yxz = aa^{k-1}b \in L_{\text{one-}b} \cap L_{\text{final-}c}$, but $wxz = ba^{k-1}b \notin L_{\text{one-}b} \cap L_{\text{final-}c}$. □

Remark 1 will be relevant when we consider the connection of TSL languages to tone patterns.

3.4 The Strictly Piecewise stringsets

One final class of stringsets describable by banned k-factor grammars is the SP class (Rogers et al., 2010; Fu et al., 2011). Here, instead of banning sub*strings* we ban *subsequences*, where a string $\sigma_1\sigma_2...\sigma_k \in \Sigma^*$ is a k-subsequence of a string w iff

$$w = w_0\sigma_1 w_1 \sigma_2 w_2 ... w_{k-1}\sigma_k w_k$$

for some $w_0, ..., w_k \in \Sigma^*$. In other words, a subsequence is like a substring except that it is interpreted as being connected by precedence and not successor; that is, its composite symbols need not be contiguous in the full string (of course they must be in the same order). For example, bb is a subsequence 2-factor of $abaab$ because we can break $abaab$ down into $w_0 b w_1 b w_2$ where $w_0 = a$, $w_1 = aa$, and $w_2 = \lambda$. The full set of subsequence 2-factors of $abaab$ is $\{aa, ab, ba, bb\}$.

Analagous to $\texttt{substr}_k$ we define a function $\texttt{subseq}_{\leq k}(w)$ which returns all subsequence n-factors in w for $0 \leq n \leq k$:

$$\texttt{subseq}_{\leq k}(w) \stackrel{\text{def}}{=} \{u \mid u \text{ is a } n\text{-subsequence of } w,\ 0 \leq n \leq k\}$$

Note that in the case of subsequence k-factors the boundary symbols $\rtimes$ and $\ltimes$ are irrelevant.

We can then interpret a set B of k-factors as a grammar that describes a stringset thusly:

$$L(B) \stackrel{\text{def}}{=} \left\{w \in \Sigma^* \mid \texttt{subseq}_{\leq k}(w) \cap \texttt{subseq}_{\leq k}(B) = \emptyset\right\}$$

The stringsets that are so describable by some B of factors of length k are the *Strictly k-Piecewise (SP_k)* stringsets. For instance, if $B = \{bb\}$, then $L(B)$ is the set of *at most* one b, as strings such as *abaab* will contain *bb* as a subsequence:

$$L(B) = \{\lambda, a, b, aa, ab, ba, aaa, aab, aba, baa, aaaa, ...\}$$

As in the SL class, The SP_k classes form the following infinite hierarchy, where SP is the class of stringsets that are SP_k for some k.

$$\mathrm{SP}_1 \subsetneq \mathrm{SP}_2 \subsetneq \mathrm{SP}_3 \subsetneq \ldots \subsetneq \mathrm{SP} \tag{10}$$

The SP stringsets have a number of interesting properties (Fu et al., 2011), of particular interest is that they are exactly the stringsets closed under subsequence (that is, for any SP language L and $w \in L$, if u is a subsequence of L then $u \in L$) (Rogers et al., 2010). They are also efficiently learnable in the limit from positive data (Heinz, 2010b; Heinz and Rogers, 2013) and statistical versions of SP stringsets are efficiently estimable from positive data (Heinz and Rogers, 2010).

The SP stringsets are incomparable with both the SL and TSL classes. This is witnessed by the stringsets $L_{\text{final-}c}$ and $L_{\text{one-}b/c}$ mentioned in the previous section; they are SL and TSL, respectively, but neither is closed under sequence and thus not SP (the same goes for their intersection). To show that SP $\not\subseteq$ TSL consider the stringset over $\Sigma = \{a, b\}$ described by the set $B = \{bab\}$ interpreted as a set of banned k-subsequences:

$$\begin{aligned} L_{\text{no-}bab} \stackrel{\text{def}}{=} L(B) = \{&\lambda, a, b, aa, ba, ab, bb, aaa, aab, abb, bba, bbb, \\ &aaaa, aaab, aabb, abba, abbb, bbaa, bbba, ...\} \end{aligned} \tag{11}$$

This is the set of strings with at most one contiguous sequence of bs, as $B = \{bab\}$ bans a subsequence of any two bs separated by an a. As it is relevant to the discussion below of the relation to TSL and tone patterns, Remark 2 highlights the fact that $L_{\text{no-}bab}$ is not TSL.

Remark 2 $L_{\text{no-}bab}$ is not TSL for any T or any k.

Proof It is clear that $T = \{\}$ or $T = \{a\}$ cannot capture $L_{\text{no-}bab}$. For $T = \{b\}$, we cannot ban any $u \in \mathtt{substr}_k(b^*)$ because it is easy to see that $u \in \mathtt{substr}_k(L_{\text{no-}bab})$. The only possible value for T is then $\{a, b\} = \Sigma$, which reduces to the SL case, but $L_{\text{no-}bab}$ fails SSC for any k for the same reason as $L_{\text{one-}b/c} \cap L_{\text{final-}c}$ in the proof of Remark 1. $\square$

This concludes our discussion of Subregular classes of stringsets describable with banned k-factor grammars.

3.5 The Star Free stringsets

Returning to $L_{\text{one-}b/c} \cap L_{\text{final-}c}$, while it is not TSL or SP, it can be shown to be SF. The SF stringsets have a number of converging characterizations, among which are that they are exactly the stringsets describable by counter-free automata and by first-order logic with precedence (FO[$<$]) (McNaughton and Papert, 1971). We use this latter characterization, as it is useful in considering the relationship between string and non-string structures later in the paper. To do this we can consider a string w of length n as a relational model $\mathscr{M}_w = \langle U, <, (P_\sigma)_{\sigma\in\Sigma}\rangle$ where $U = \{1,...,n\}$ is the universe of n positions in the string, $<$ is the usual ordering on these positions, and a set of unary relations P_σ for each $\sigma \in \Sigma$ representing the positions labeled with each $\sigma \in \Sigma$. For example, for the string *abc*,

$$\mathscr{M}_{abc} = \left\langle \{1,2,3\}_U, \{(1,2),(1,3),(2,3)\}_<, \{1\}_{P_a}, \{2\}_{P_b}, \{3\}_{P_c} \right\rangle.$$

We can then define a first-order language FO[$<$] for these structures in which variables $x, y, z, x_1, ...$ range over positions in the string, $x = y$, $x < y$, and $\sigma(x)$ for each $\sigma \in \Sigma$ are *atomic predicates* representing equality and each of the relations in the model. We then define FO[$<$] statements using the logical connectives $\wedge$, $\neg$, $\vee$, and $\rightarrow$ and the quantifiers $\exists$ and $\forall$ in the usual way. We write $\varphi(x_1,...,x_k)$ for a FO[$<$] statement φ for which $x_1,...,x_n$ is exactly the set of *free* variables—i.e., those not bound with any quantifier—in φ. Thus we can define new shorthand predicates, such as

$$\texttt{last}(x) \stackrel{\text{def}}{=} (\forall y)[\neg y < x],$$

which is true when x is interpreted as the last position in a string.

A FO[$<$] statement φ is a *sentence* if it has free variables. Let *satisfaction* of φ for a model $\mathscr{M}_w$, written $\mathscr{M}_w \models \varphi$, be defined in the usual way. For example, for the sentence

$$\varphi_{\text{one-}b/c} \stackrel{\text{def}}{=} (\exists x)[b(x) \vee c(x)] \wedge (\forall x, y)\left[\big((b(x) \vee c(x)) \wedge (b(y) \vee c(y))\big) \rightarrow x = y\right]$$

is satisfied by all and only the strings in which there is exactly one b or c. Thus, for the string *aaab*, $\mathscr{M}_{aaab} \models \varphi_{\text{one-}b/c}$ but for the string *acab*, $\mathscr{M}_{acab} \not\models \varphi_{\text{one-}b/c}$. We can interpret a FO[$<$] sentence φ as a grammar for the set of strings that satisfy it:

$$L(\varphi) \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid \mathscr{M}_w \models \phi\}.$$

The stringsets so definable are the SF stringsets.

If we then define

$$\varphi_{\text{final-}c} \stackrel{\text{def}}{=} (\forall x)[c(x) \rightarrow \texttt{last}(x)]$$

then $L_{\text{one-}b/c} \cap L_{\text{final-}c} = L(\varphi_{\text{one-}b/c} \wedge \varphi_{\text{final-}c})$ and thus is SF. It is simple to show that for any SL, TSL, or SP grammar, we can write an equivalent FO[$<$] sentence. Thus, SL, TSL, and SP are all subclasses of SF (Rogers et al., 2013). That the inclusion is strict is witnessed by $L_{\text{one-}b/c} \cap L_{\text{final-}c}$.

4 The Subregular Hierarchy and phonological well-formedness

Having established the existing complexity classes in the Subregular Hierarchy, this section first (in Sect. 4.1) reviews previous work establishing the relevance of these stringset classes to phonology. Then, in Sect. 4.2, it establishes that the tone patterns introduced in Sect 2 each fall into the different classes shown in Fig. 1. This motivates the ASL class, which is defined in Sect. 5.

4.1 Segmental and stress patterns

A striking result in applying formal language theory to natural language phonology is that many segmental well-formedness patterns fall into either the SL, TSL, or SP classes. Consider the English example from the beginning of Sect. 2 in which *bn* is an ill-formed onset. We can model this as an SL pattern with a grammar B such that $\rtimes bn \in B$.[6]

Many phonological well-formedness patterns are local in this way, such as local agreement constraints against adjacent segments that disagree in voicing (Lombardi, 1999), the famous constraint in Yawelmani Yokuts against three consonants in a row (Kisseberth, 1970) or stress constraints against *clashes* and *lapses*, that is, sequences of adjacent stressed and unstressed syllables, respectively (Hayes, 1995; Kager, 1995).

Long-distance patterns are also possible, but they come in one of two varieties. One is a sort of 'relativized locality' (Nevins, 2010), in which constraints hold over sequences of some class of segments ignoring other segments not in that class. A common example is that of vowel harmony, in which a certain set of vowels agree with each other, ignoring intervening consonants and vowels not of that set (Nevins, 2010; Walker, 2011). Such 'relativized locality' patterns are capturable with TSL constraints (Heinz et al., 2011). The other type of long-distance pattern is one in which constraints hold across segments no matter what intervenes between them. For example, in Inseño Chumash, all sibilants in a word must be either [s] or [ʃ], regardless of what intervenes between sibilants (Applegate, 1972). Heinz (2010a) shows how SP stringsets accurately capture the typology these long-distant consonant harmony patterns; for example, Inseño Chumash can be modeled with a SP grammar $B = \{$sʃ, ʃs$\}$ (though see McMullin and Hansson (2016) for arguments that the TSL class is a better characterization of this type of pattern).

An important generalization we can draw from this is that most segmental and stress well-formedness patterns studied so far can be modeled with a stringset describable with some set B of banned k-factors. The only difference is whether B is interpreted as banned substrings (i.e., as an SL grammar), banned *tier* substrings (TSL), or banned subsequences (SP). Thus, while there are phonological well-formedness patterns that hold over long distances, the full expressive power of SF is largely not necessary.[7] This can be taken to mean that the notion of banning k-factors (as contrasted with, say, statements in FO logic) is fundamental to phonological cognition.

4.2 Tone and the Subregular Hierarchy

As just mentioned, little work exists on locating tone well-formedness patterns in the Subregular Hierarchy (though see, e.g., Bird and Ellison (1994) and Yli-Jyrä (2013)

[6] This pattern is more precisely described by referring to syllable structure and the relative sonority of consonants in the onset. (See, e.g., Strother-Garcia forthcoming.) However, adding syllable structure or featural representations does not change the fundamentally local nature of the pattern, which refers to sequences of adjacent segments.

[7] For discussion of some non-SP or TSL constraints in stress patterns see Rogers et al. (2013).

on finite-state methods in tonal phonology and Jardine (2016a) on the complexity of tonal processes). The following shows that the tone patterns described in Sect. 2 run the gamut from SL to SF, in terms of their complexity in the Subregular Hierarchy.

Recall the positional Kagoshima Japanese pattern from Sect. 2.1, in which a high tone must appear either on the penultimate or final mora. This type of pattern is SL. We can represent this pattern as in (12) as a set of strings over the alphabet $\Sigma = \{\mathrm{H}, \mathrm{L}\}$, where H represents a high-toned mora and L represents a low-toned mora.

$$L_{\mathrm{KJ}} = \{\lambda, \mathrm{H}, \mathrm{HL}, \mathrm{LHL}, \mathrm{LLHL}, \mathrm{LLLHL}, ..., \mathrm{LH}, \mathrm{LLH}, \mathrm{LLLH}, \mathrm{LLLLH}, ...\} \quad (12)$$

In other words, it is the set of strings in which exactly one H appears either in the last or second-to-last position. This can be described with the following set of banned 3-factors.

$$B_{\mathrm{KJ}} = \{\mathrm{HLL}, \mathrm{LL}\ltimes, \mathrm{HLH}, \mathrm{HHL}, \mathrm{HH}\ltimes, \mathrm{LHH}, \rtimes\mathrm{HH}, \mathrm{HHH}\} \quad (13)$$

The reader can confirm that L_{KJ} is the set of all and only the strings that do not contain any 3-factor in B_{KJ}. No string in L_{KJ} will contain HLL, as this represents a high tone that appears anywhere before the penultimate position. Conversely, banning LL$\ltimes$ requires that a high tone appears either in penultimate or final position. The remainder of B_{KJ} is the set of 3-factors that contain more than one H, none of which appear in any string in L_{KJ}. In general, positional well-formedness patterns are SL_k when they refer to some position $k-1$ positions away from the word edge.

Obligatoriness patterns, however, are not SL_k for any k. This follows the logic of the proof for why $L_{\text{one-}b}$ is not SL. Recall the pattern from Chuave, in which every word had to contain at least one H tone. This pattern is recast as a set of strings over $\Sigma = \{\mathrm{H}, \mathrm{L}\}$ in (14).

$$L_{\mathrm{Ch}} = \{\lambda, \mathrm{H}, \mathrm{HL}, \mathrm{LH}, \mathrm{HH}, \mathrm{LLH}, \mathrm{LHL}, \mathrm{LHH}, \mathrm{HLL}, \mathrm{HLH}, \mathrm{HHH}, \mathrm{LLLH}, ...\} \quad (14)$$

No string L^n is in L_{Ch}, for any n. Thus, L_{Ch} fails the SCC of Thm 1 given any k.

$$\underbrace{\mathrm{L}}_{w}\underbrace{\mathrm{L}^{k-1}}_{x}\underbrace{\mathrm{H}}_{u} \in L_{\mathrm{Ch}} \text{ and } \underbrace{\mathrm{H}}_{y}\underbrace{\mathrm{L}^{k-1}}_{x}\underbrace{\mathrm{L}}_{z} \in L_{\mathrm{Ch}} \text{ but } \underbrace{\mathrm{L}}_{w}\underbrace{\mathrm{L}^{k-1}}_{x}\underbrace{\mathrm{L}}_{z} \notin L_{\mathrm{Ch}}$$

Likewise, L_{Ch} is not SP: it is not closed under subsequence. For any k, L^k is a subsequence of a string $\mathrm{L}^k\mathrm{H} \in L_{\mathrm{Ch}}$, but $\mathrm{L}^k \notin L_{\mathrm{Ch}}$.

However, L_{Ch} is TSL_2, as witnessed by the TSL_2 grammar in (15).

$$\langle T_{\mathrm{Ch}} = \{\mathrm{H}\}, B_{\mathrm{Ch}} = \{\rtimes\ltimes\}\rangle \quad (15)$$

Since any string $w \in L_{\mathrm{Ch}}$ contains at least one H, $\mathtt{erase}_{T_{\mathrm{Ch}}}(w) = \mathrm{H}^n$ for some $n \geq 1$, and $\rtimes\ltimes \notin \mathtt{substr}_2(\mathrm{H}^n)$. Conversely, for L^n for any n, $\mathtt{erase}_{T_{\mathrm{Ch}}}(\mathrm{L}^n) = \lambda$, and $\mathtt{substr}_2(\lambda) = \rtimes\ltimes$. So $\langle T_{\mathrm{Ch}} = \{\mathrm{H}\}, B_{\mathrm{Ch}} = \{\rtimes\ltimes\}\rangle$ exactly describes L_{Ch}.

Culminativity constraints, in contrast, are in general SP. Given any pattern in which some element a cannot appear n or more times in a string, we can describe it with a singleton set banning the subsequence $B = \{a^n\}$. Such a pattern is also TSL, but this is not true for all culminativity patterns. The UTP pattern described in Sect.

2.3 is SP but not TSL. Recall that UTP is a pattern in which at most one unbroken sequence of Hs can appear in a string. This pattern can be seen as the set in (16) of strings over $\Sigma = \{\text{H, L}\}$.

$$L_{\text{UTP}} = \{ \lambda, \text{L, H, LL, LH, HL, HH, LLL, LLH, LHL, LHH, HLL, HHL, HHH, LLLL, LLLH, LLHL, LLHH, LHLL, LHHL, LHHH, HLLL, HHLL, HHHL, HHHH, ...} \} \tag{16}$$

This set is described by the SP_3 grammar $B = \{\text{HLH}\}$. This is because L_{UTP} is the set of strings with at most one unbroken sequence of Hs, so no string in L_{UTP} is of the form $w_1\text{H}w_2\text{L}w_3\text{H}w_4$ for any $w_1, w_2, w_3, w_4 \in \Sigma^*$. To see that L_{UTP} is strictly SP, and not TSL for any k, it is sufficient to note that it is the image of a bijective symbol-to-symbol homomorphism of $L_{\text{no-}bab}$ (where L is mapped to a and H is mapped to b) is mamentioned in Sect. 3.4 and thus the proof of Remark 2 that $L_{\text{no-}bab}$ is not TSL applies also to L_{UTP}.

Finally, a pattern which combines any of the above types of patterns is not necessarily going to be SL, TSL, or SP. This is true for the Hirosaki Japanese pattern, which was shown in Sect. 2.4 to exhibit obligatoriness, culminativity, and positional well-formedness. Recall that tonal well-formedness in Hirosaki Japanese dictates that a word must contain exactly one high- or falling-toned mora, but not both, and additionally that final-toned moras can only appear word-finally. This well-formedness pattern is represented in (17) as a set of strings over the alphabet $\Sigma = \{\text{H, L, F}\}$.

$$L_{\text{HJ}} = \{\lambda, \text{H, F, LH, LF, HL, LLH, LLF, LHL, HLL, LLLH, LLLF, ...}\} \tag{17}$$

This is not TSL for any k, nor is it SP. To see why, note that it is the image of a bijective symbol-to-symbol homomorphism of $L_{\text{one-}b/c} \cap L_{\text{final-}c}$ (where L is mapped to a, H to b, and F to c), discussed in Sects. 3.3 and 3.5. Thus, the proof for Remark 1 that $L_{\text{one-}b/c} \cap L_{\text{final-}c}$ is not TSL applies equally to L_{HJ}. Intuitively, this is because L_{HJ} requires two different tiers: one that captures the local, positional constraint that F must appear at the end of the string, and one that captures the non-local requirement that exactly one H or F must appear in the string. It is simple to show that L_{HJ} is not closed under subsequence and thus also not SP.

Like $L_{\text{one-}b/c} \cap L_{\text{final-}c}$, L_{HJ} does admit a FO characterization and so is SF. We simply translate the logical sentences from Sect. 3.5 describing $L_{\text{one-}b/c} \cap L_{\text{final-}c}$ to the FO sentences for strings over $\Sigma = \{\text{H,L,F}\}$ in (18) and (19). The FO description of L_{HJ} is thus the sentence in (20) resulting from the conjunction of these two sentences.

$$\begin{aligned}\varphi_{\text{one-H/F}} \stackrel{\text{def}}{=} \ & (\exists x)[\text{H}(x) \vee \text{F}(x)] \wedge \\ & (\forall x, y)\left[\big((\text{H}(x) \vee \text{F}(x)) \wedge (\text{H}(y) \vee \text{F}(y))\big) \rightarrow x = y\right]\end{aligned} \tag{18}$$

$$\varphi_{\text{final-F}} \stackrel{\text{def}}{=} (\forall x)[\text{F}(x) \rightarrow \texttt{last}(x)] \tag{19}$$

$$L_{\text{HJ}} \equiv L(\varphi_{\text{one-H/F}} \wedge \varphi_{\text{final-F}}) \tag{20}$$

In conclusion, there are tone well-formedness patterns that are SL, TSL, SP, and properly TSL, as depicted in Fig. 1. Thus, none of these classes provide a unifying characterization of tonal well-formedness. The following section proposes the ASL stringsets, a complexity class that does.

5 Local autosegmental grammars

This section introduces a particular type of *autosegmental representations (ARs)*, non-string structures that have previously been proposed to account for tone patterns. It then defines banned *k*-factor grammars for these representations and shows how these correspond to a new class of stringsets we call the *Autosegmental Strictly Local (ASL)* class. As shown in Fig. 1, this class encompasses all of the tone patterns discussed above, yet still falls within the SF class. It thus provides a restrictive, yet unified characterization of tonal well-formedness.

5.1 Autosegmental representations

Early in the program of generative phonology it became clear that tone patterns required mechanisms that allowed tonal units to operate independent of the segmental material over which they are realized (Leben, 1973). The most successful such mechanism has been ARs (Goldsmith, 1976; Clements, 1977; Archangeli and Pulleyblank, 1994). In an AR, tonal information is represented as a separate string from the *tone-bearing units (TBUs)*, or the units on which they are realized—e.g. mora or syllables. Units on these two strings are related via an *association* relation, usually depicted with straight lines. Some example ARs representing words in Hirosaki Japanese are given in Fig. 3.

a. kóomori ‘bat’ (HLLL) (Table 4j)

b. kudamóno ‘fruit’ (LLHL) (Table 4h)

c. kaminarî ‘thunder’ (LLLF) (Table 4f)

Fig. 3 Example ARs from Hirosaki Japanese.

In Fig. 3a, for example, is the four-mora word [kóomori] ‘bat’, which when viewed schematically as a string of high- and low-toned moras is HLLL. As an AR, this word is represented by a string of four moras associated to a string of tones HL such that the H is associated to the first mora and the L is associated to the final three moras. This string of tones is referred to as a *melody*. Similarly, Fig. 3b represents the word [kudamóno] ‘fruit’ as an AR with a LHL melody associated to four moras such that the initial L is associated to the first two moras and the remaining H and L are associated to a single mora each. Finally, Fig. 3c shows the falling tone on the final mora of [kaminarî] ‘thunder’ represented as an H and L associated to the same mora. These examples illustrate how ARs represent plateaus (such as the sequence of L tones in [kóomori] 'bat') and *contour tones* such as the falling tone in [kaminarî] 'thunder' through the *multiple association* of one tone to many mora or one mora to many tones, respectively.

Various authors have posited axioms governing ARs. Perhaps the most robustly regarded as universal is the *no-crossing constraint (NCC)* (Goldsmith, 1976; Hammond, 1988; Sagey, 1988) which states that association lines cannot 'cross', as below in Fig. 4a. More formally, this means that for any position x in one string in the AR that is associated to an element w on another string, there cannot be another element y which follows x in its string and is also associated to some v which precedes w. In other words, association must respect the order in both strings.

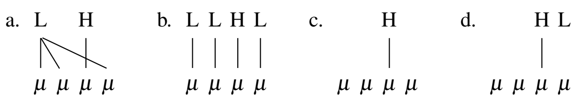


Fig. 4 Ill-formed ARs.

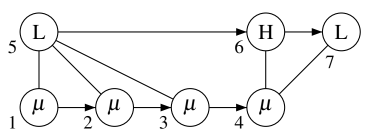
The other axioms are less often posited as universal but will be useful for the purposes of this paper. The first, the *obligatory contour principle (OCP)* (Leben, 1973), states that any adjacent elements in the melody must be distinct. Thus, a melody of LHL is well-formed with respect to the OCP, but a melody like LLHL, as in Fig. 4b, is ill-formed. This has the effect of forcing sequences of mora of the same tone to be associated to the same element in the melody (contrast the first two mora in Fig. 3b with those of Fig. 4b). Finally, it was originally proposed in Goldsmith (1976) that every element in an AR must be associated with some other element—thus ARs like Fig 4c and d are ill-formed.

Goldsmith (1976) or Odden (1986) argue against the universality of the OCP, but Jardine (2016a) defends the OCP as applying to surface well-formedness patterns. Below, it is shown that this allows non-SL patterns to be described with local autosegmental grammars. Similarly, ARs violating the axiom requiring all elements to be associated have been successfully used to capture tonal processes (see, e.g., Pulleyblank, 1986). The consequences of abandoning these axioms as universal are discussed below where appropriate.

Finally, a survey of the arguments for using ARs is beyond the purview of this paper. For arguments in favor of ARs, see, for example, Goldsmith (1976); Clements (1977); Archangeli and Pulleyblank (1994); Hyman (2014). (On alternative representations for tone, see Cassimjee and Kisseberth (1998); Leben (2006); Shih and Inkelas (2015).) However, the following provides a *computational* argument for ARs: they allow for a unified characterization of tone. First, the following defines the set of well-formed ARs explicitly and shows how they can be related to strings over a particular alphabet.

5.2 Explicitly defining autosegmental representations

To define ARs explicitly, and to relate them to strings, we follow the work of Jardine and Heinz (2015) generating the set of well-formed ARs through *graph concatenation* (Engelfriet and Vereijken, 1997). The information in an AR can explicitly as a *labeled mixed graph* (henceforth *graph*) $\langle V, E, A, \ell \rangle$ where V is the set of nodes or units in the representation, E is the set of undirected edges representing association, A is the set of directed edges or *arcs* representing the order on each string in the AR, and $\ell : V \to \Gamma$ is a total function mapping the nodes to an alphabet Γ of labels. For ARs representing high tones, low tones, and moras, we use $\Gamma = \{\text{H}, \text{L}, \mu\}$. An example graph for the AR for the Hirosaki Japanese [kaminarî] 'thunder' (c.f. Fig. 3c) is given in Fig 5.

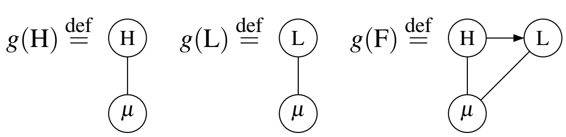


$$\begin{aligned} \langle\, & V = \{1,2,3,4,5,6,7\}, \\ & E = \{\{5,1\}, \{5,2\}, \{5,3\}, \{6,4\}, \{7,4\}\} \\ & A = \{(1,2),(2,3),(3,4),(5,6),(6,7)\} \\ & \ell(1) = \ell(2) = \ell(3) = \ell(4) = \mu, \\ & \ell(5) = \ell(6) = \text{L}, \ell(7) = \text{H} \quad \rangle \end{aligned}$$

Fig. 5 Graph representation of the AR for Hirosaki Japanese word [kaminarî] 'thunder'. Indices are given for each node. Associations are represented with undirected edges and the order (successor relation) on each string is represented with directed edges.

The graph in Fig. 5 is simply an explicit representation of all of the information in the AR in Fig. 3c. (Unless necessary, all following depictions of graphs will not include node indices.)

Let $\text{GR}(\Gamma)$ be the set of all such graphs. Obviously, not every graph represents an AR. How, then, do we define the subset of $\text{GR}(\Gamma)$ that represents the set of ARs obeying the NCC and OCP axioms? Jardine and Heinz (2015) propose deriving the set of well-formed AR graphs through the concatenation of AR *graph primitives* that can be connected to an alphabet of string symbols. For the alphabet $\Sigma = \{\text{H, L, F}\}$ consider the following function $g : \Sigma \to \text{GR}(\Gamma)$.



(21)

They then define for two graphs $G_1, G_2 \in \text{GR}(\Gamma)$ a concatenation operation $G_1 \circ G_2$ that joins the two graphs as follows. For the last member of each string in G_1, draw an arc from that member to the first member of the same string in G_2 (as in Fig. 6a) unless the following holds. If the last member of the melody (the 'upper' string) in G_1 has the same label as the first member of the melody in G_2, then merge them (as in 6b). As the full definition is techincal, we suffice it to illustrate the operation with examples.

In Fig. 6a, concatenation draws an arc from the node labeled H in the first graph (which is the last node in the melody string) to the node labeled L in the second graph. Likewise for the last node on the μ string in the first graph and the first μ node in

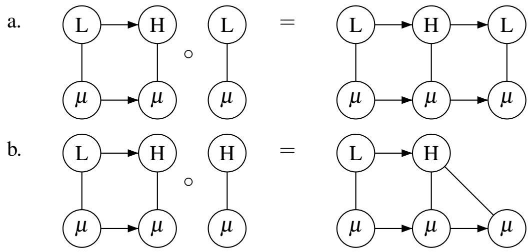


Fig. 6 Examples of graph concatenation

the second graph. In Fig. 6b, the last node in the melody string is labeled H, which is the same as the label for the first node on the melody tier in the second graph. Thus, instead of an arc being drawn between them, they are merged into a single node. As the resulting graph shows, this process of merging melody units creates ARs with multiple association of tones to moras.

We then extend g from symbols in Σ to strings over Σ^* in the following way. Note that G_λ is the *empty graph*; $G_\lambda = \langle V = \{\}, E = \{\}, A = \{\}, \ell = \{\}\rangle$.[8]

Definition 1 For a function g mapping symbols in Σ to graph primitives in $\mathrm{GR}(\Gamma)$, for $w \in \Sigma^*$,

$$g(w) \stackrel{\text{def}}{=} \begin{cases} G_\lambda & \text{if } w = \lambda \\ g(u) \circ g(\sigma) & \text{if } w = u\sigma, u \in \Sigma^*, \sigma \in \Sigma \end{cases}$$

Figure 7 gives some examples.

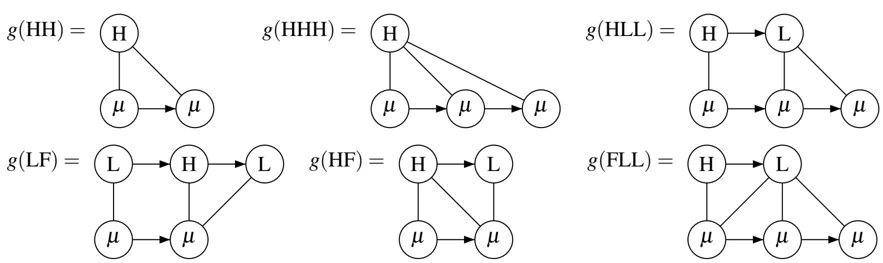


Fig. 7 Examples of $g(w)$ for some strings $w \in \Sigma$

The relationship that g defines between strings in Σ^* and graphs in $\mathrm{GR}(\Gamma)$ can be thought of in intuitive terms as follows. For the symbol H$\in \Sigma$, it represents a high-toned mora; $g(\mathrm{H})$ is thus the explicit autosegmental representation of a mora with a high tone. For some string w of symbols in Σ representing a string of toned moras, the concatenation operation is defined such that $g(w)$ is an AR representing that string. Importantly, note that due to the merging operation of concatenation, there

[8] The use of an empty graph with 0 nodes has sometimes been argued against (Harary and Read, 1974), but for this purpose (and for, e.g., Engelfriet and Vereijken (1997)) it serves as a useful correlate of λ.

is no bound on the number of moras to which a single melody node may be associated. For any n, we can generate a H node associated to n μ nodes by taking $g(\mathrm{H}^n)$:

$g(\mathrm{H}^n) =$

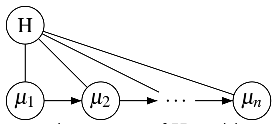


Fig. 8 Representing a span of H positions in the string with a single H node in the graph

This is an important difference between AR graphs and their string representations: a single node on the melody represents, in its corresponding string, a *span* of consecutive positions with the same symbol (in Fig. 8, the sequence of n Hs in the string). In the following section, it will be shown how this can describe non-local dependencies in the string in a local manner. Note that this is dependent on the OCP discussed above: consecutive H positions are not assigned separate, consecutive H melody nodes in the graph. Note also that contours cannot be so created through concatenation. They must appear in the set of graph primitives over which g is originally defined (e.g. as in $g(\mathrm{F})$ in (21)).

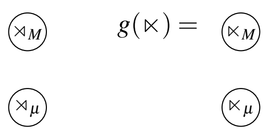
We can then define $g(\Sigma^*)$ as the set of all such AR graphs:

$$g(\Sigma^*) = \{g(w) | w \in \Sigma^*\}$$

Jardine and Heinz (2016) show that, given particular properties of g, the concatenation operation of $\circ$ is associative, and every graph in $g(\Sigma^*)$ obeys both the OCP and the NCC. In other words. $g(\Sigma^*)$ represents the set of well-formed ARs, with the caveat that the contours that appear in $g(\Sigma^*)$ are only those that appear in the graph primitives that define g.

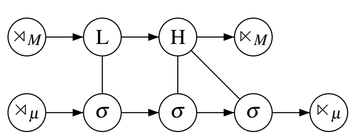
Parallel to how the explicit boundary symbols $\rtimes$ and $\ltimes$ played a crucial role in SL and TSL grammars, we also define special boundary graph primitives as given in (22).

$g(\rtimes) =$ $g(\ltimes) =$ (22)



The subscripts M and μ on the labels of these nodes are to indicate which are added to the melody string ($\rtimes_M$ and $\ltimes_M$) which are added to the string of μ nodes ($\rtimes_\mu$ and $\ltimes_\mu$). Thus, for example, using this extended g, $g(\rtimes\mathrm{LHH}\ltimes)$ is as in (23).

$g(\rtimes\mathrm{LHH}\ltimes) =$ (23)



The subscripts indicating which string a node labeled with a boundary symbol will be concatenated to can be ignored in most contexts and thus will henceforth not be written.

With an explicitly defined universe of ARs, and an method of relating them to strings, we can then define grammars over ARs, and then directly compare their expressivity to the string grammars discussed in the previous sections.

5.3 Banned subgraph grammars

We can extend the notion of a banned k-factor grammar to graphs through the notion of a *subgraph*. This was first proposed for ARs by Jardine (2016b, to appeara). The following defines subgraphs for graphs in GR(Γ).

Definition 2 (Subgraph) For a graph $G = \langle V, E, A, \ell \rangle$ a *subgraph* of G is (isomorphic to) a graph $G' = \langle V', E', A', \ell' \rangle$ for which $V' \subseteq V$, $E' \subseteq E$, $A' \subseteq A$, and $\ell : V' \rightarrow \Sigma$ is a labeling function such that for all $x \in V'$, $\ell(x) = \ell'(x)$.

Essentially, a subgraph of a graph G is (isomorphic to) a subset of the nodes, edges, and arcs of G such that the labeling function is the same. Note that Def 2 does not distinguish among isomorphic graphs. Thus, the graph G' is a subgraph of G in Fig. 9, as shown by the isomorphism indicated by the dotted arrows.

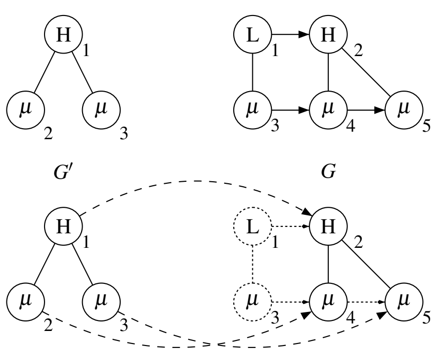


Fig. 9 The graph G' is a subgraph of G

Def. 2 is not quite strong enough: in the banned k-factor grammars of Sec. 3, the factors were all necessarily connected. In order to posit graph grammars that are comparable to the banned k-factor grammars discussed in Sec. 3, we must also ensure that the subgraphs are connected.

For a graph $G = \langle V, E, A, \ell \rangle$, let its *underlying graph* $G^U = \langle V, U \rangle$ be a simple undirected graph where $U = E \cup \{\{v_1, v_2\} \,|\, (v_1, v_2) \in A \text{ or } (v_2, v_1) \in A\}$. We say G is connected iff for any $v, v' \in V$, there is some sequence $v, v_1, v_2, ..., v_n, v'$ such that $\{v, v_1\} \in U, \{v_n, v'\} \in U$, and for all $1 \leq i < n$, $\{v_i, v_{i+1}\} \in U$. In other words, there is some path from any v to any v' over the edges and arcs of G, ignoring the direction of the arcs. Fig. 10 contrasts G_1, whose nodes 1, 2, and 3 are all reachable from each other in the corresponding underlying graph G_1^U, with G_2, whose node 3 cannot be reached from either 1 or 2 in the underlying graph G_2^U.

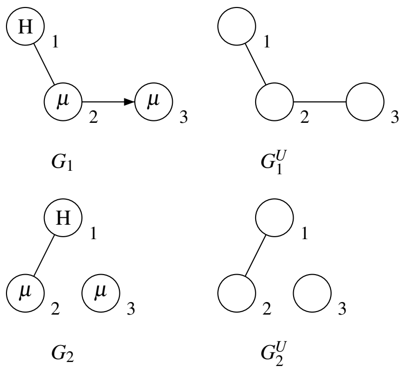


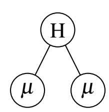
Fig. 10 A connected graph G_1, a graph G_2 that is not connected, and their underlying graphs G_1^U and G_2^U

We thus write $G' \sqsubseteq G$ if G' is a connected subgraph of G. Furthermore, we consider the size $|G|$ of a subgraph G to be equal to the number of its nodes, i.e. $|G| = |V|$. We can then consider a set B of graphs of size at most k as a *banned k-subgraph grammar*. Using g, we can interpret such a grammar as describing a set of strings as follows:

$$L(B) = \{w \in \Sigma^* \mid \text{there is no } b \in B \text{ such that } b \sqsubseteq g(\rtimes w \ltimes)\}$$

Importantly, because these subgraphs are connected, banned k-subgraph grammars are banned k-factor grammars.

For example, consider the following set B containing a single graph of size 3.

$B = \Big\{$  $\Big\}$ (24)

This describes the set of all strings in Σ^* for $\Sigma = \{\text{H, L, F}\}$ containing no HH or HF sequences of adjacent symbols.

$$L(B) = \{\lambda, \text{L, H, F, LL, LH, LF, HL, FH, FF, LLL, LLH, LLF, LHL, LFL, LFF, HLL, HLH, HLF, ...}\} \tag{25}$$

This is because the graph corresponding to any string containing an adjacent HH or HF sequence will contain the subgraph in B in (24), as the examples below in Fig. 11 illustrate. Instances of the offending subgraph are highlighted in the graphs.

What is the class of stringsets described by such grammars? The following section answers this question in depth.

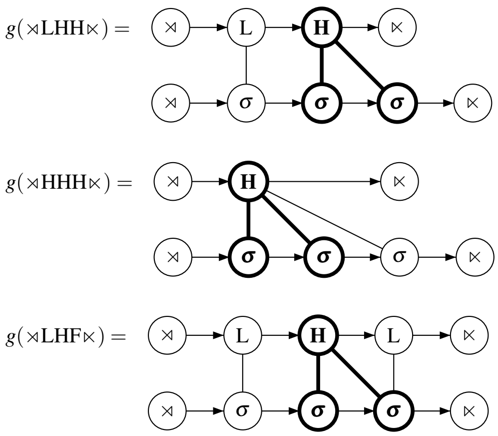


Fig. 11 Strings whose graphs contain instances of the banned subgraph in B in (24)

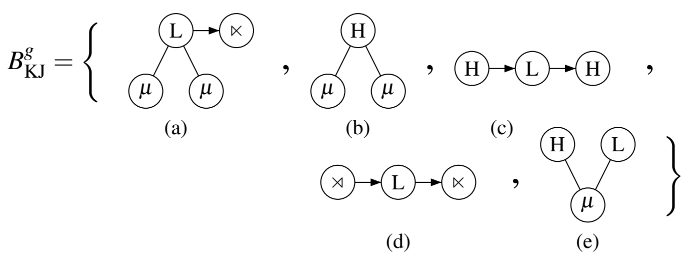
5.4 Tone and Autosegmental Strictly Local sets

Let the sets of strings describable by a banned k-subgraph grammar be the *Autosegmental k-Strictly Local (ASL$_k$)* sets; let ASL refer to the class of stringsets that are ASL$_k$ for some k. This section will show how these sets relate to the aforementioned classes in the Subregular Hierarchy; importantly, as originally shown in Fig. 1, that the ASL class includes all of the tone patterns discussed in Sect. 2. It will also show that, while the ASL class is included in the SF class, it is incomparable to the SL, TSL, and SP classes. Note that these relationships hold for $\Sigma = \{\text{H, L, F}\}$ the g as defined in (21) and (22) in the previous section. How the relationships between classes change, or if they change, depending on the properties of g is a deep topic that will be left for future work. However, this Σ and this g have been shown to have linguistic significance, and thus serve as an appropriate starting point.

We begin with the SL stringset L_{KJ} for the pattern in Kagoshima Japanese, which exemplified a positional tone pattern. This stringset is repeated below in (26). Again, this is the set of strings with a single H either in final or penultimate position.

$$L_{\text{KJ}} = \{\lambda, \text{H, HL, LHL, LLHL, LLLHL, ..., LH, LLH, LLLH, LLLLH, ...}\} \quad (26)$$

This is describable with the banned 4-subgraph grammar B^g_{KJ} given in (27). Each banned subgraph is labeled for later reference.



(27)

For any string w over $\Sigma = \{\text{H, L, F}\}$ that is *not* in L_{KJ}, $g(\rtimes w \ltimes)$ will contain one of these subgraphs. The graph for any string for which the final two symbols are L will contain (27a), as shown in the example below in Fig. 12a. The graph for any string with adjacent Hs will contain (27b), as shown in the example below in Fig. 12b (more examples can be seen above in Fig. 11). The graph for any string containing more than one *non*-adjacent Hs will contain (27b), as shown in the example below in Fig. 12c. Note that this holds for two Hs separated by a sequence of Ls for *any* length, due to the fact that the merging operation in concatenation will assign these to the same L node in the melody of the corresponding AR graph. Similarly, for any string of Ls, its corresponding graph will contain (27d), as exemplified in Fig. 12d. Finally, the graph for any string containing the F symbol will contain (27e) (recall that Kagoshima Japanese has no falling tones); an example is given in Fig. 12e.

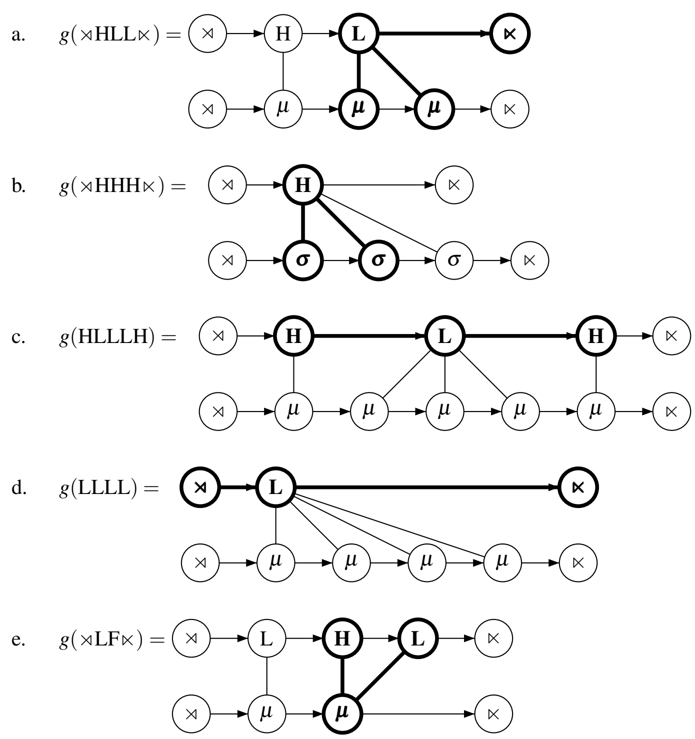


Fig. 12 Subgraphs from B in (27) in $g(w)$ for some examples $w \notin L_{\text{KJ}}$

In contrast, no strings in L_{KJ} contain any of the subgraphs in B^g_{KJ}. Fig. 13 gives examples of the graph versions of strings from L_{KJ}.

Thus, B^g_{KJ} describes exactly the strings in L_{KJ}.

Banned subgraph grammars can also capture non-SL stringsets, again due to the fact that the merging operation in concatenation assigns adjacent like symbols in the string to the same node on the melody of the corresponding graph. First consider L_{Ch},

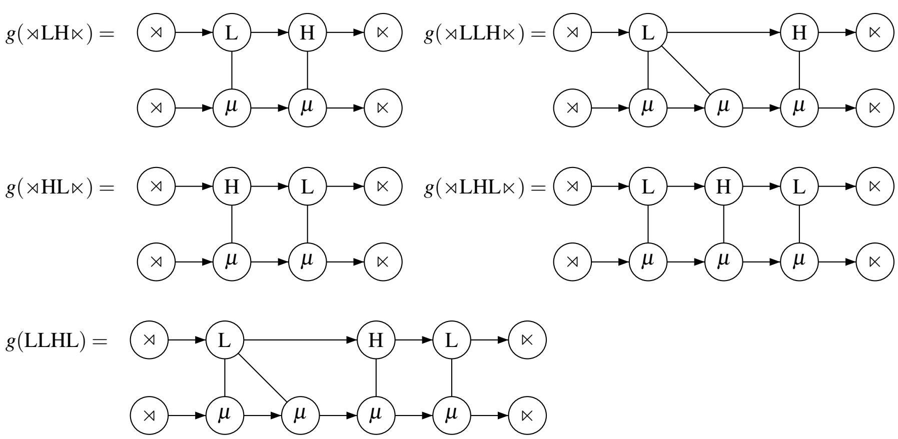
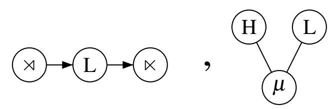


Fig. 13 g applied to strings in L_{KJ}

the stringset corresponding to the obligatoriness constraint in Chuave. Recall that in L_{Ch}, repeated below in (28), at least one H must appear in each string.

$$L_{\mathrm{Ch}} = \{\lambda, \text{H, HL, LH, HH, LLH, LHL, LHH, HLL, HLH, HHH, LLLH, ...}\} \quad (28)$$

This stringset is not SL, but it is TSL. It is also ASL, as witnessed by the following banned subgraph grammar B^g_{Ch}.

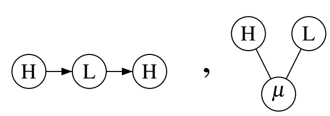
$B^g_{\mathrm{Ch}} = \Big\{$  $\Big\}$ (29)

Note that B^g_{Ch} is just a subset of B_{KJ}. The first banned subgraph discludes strings of all Ls, again because due to the merging in the concatenation operation any such string will contain a single L in the melody (see Fig. 12d). The second banned subgraph again bans any strings that contain F. Any other combinations of L and H are allowed; this is exactly the set of strings in L_{Ch}. It is thus ASL.

The properly SP stringset L_{UTP}, which again modelled the culminative pattern UTP banning more than one plateau of H tones, can also be described with a subset of B^g_{KJ}. This stringset is repeated below in (30).

$$\begin{aligned} L_{\mathrm{UTP}} = \{ & \lambda, \text{L, H, LL, LH, HL, HH, LLL, LLH, LHL, LHH,} \\ & \text{HLL, HHL, HHH, LLLL, LLLH, LLHL, LLHH,} \\ & \text{LHLL, LHHL, LHHH, HLLL, HHLL, HHHL, HHHH, ... } \} \end{aligned} \quad (30)$$

This can be captured with the banned subgraph grammar B^g_{UTP} in (31).

$B^g_{\mathrm{UTP}} = \Big\{$  $\Big\}$ (31)

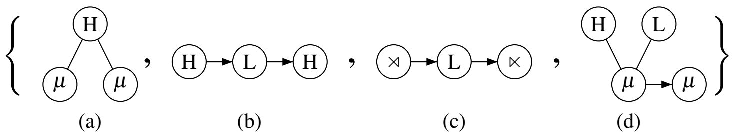
Similar to B^g_{Ch}, B^g_{UTP} bans any string containing a string of Ls surrounded on both sides by Hs (i.e., a substring of the form HL^{n}H). This is because the corresponding AR graph for any such string will contain the sequence of nodes HLH; for an example, see Fig. 12c. Thus, L_{UTP} is describable by B^g_{UTP}, and it is thus ASL.

Finally, the stringset L_{HJ}, which modelled the confluence of positional, obligatoriness, and culminativity constraints in Hirosaki Japanese, is also ASL. Recall that L_{HJ} is the set of strings such that exactly one H or F appears in the string, and Fs must only appear at the end of the string.

$$L_{\mathrm{HJ}} = \{\lambda, \mathrm{H}, \mathrm{F}, \mathrm{LH}, \mathrm{LF}, \mathrm{HL}, \mathrm{LLH}, \mathrm{LLF}, \mathrm{LHL}, \mathrm{HLL}, \mathrm{LLLH}, \mathrm{LLLF}, ...\} \tag{32}$$

This stringset is also describable by a subset of B^g_{KJ} augmented with one additional subgraph banning Fs that appear before the end of the string. This grammar, B^g_{HJ}, is given below in (33).

$B^g_{\mathrm{KJ}} =$



(33)

The subgraphs (33a) through (c) have already appeared in the above banned subgraph grammars, and ban adjacent Hs, nonadjacent Hs, and strings of all Ls, respectively. Note that (33b) bans both strings with two, nonadjacent Hs and strings containing both H and Fs, because $g(\mathrm{F})$ contributes a H node to the melody. An example is given in Fig. 14a. The subgraph (33d) will appear in the graph corresponding to any string of the form Fw, where w is a nonempty string. An example is given in Fig. 14b.

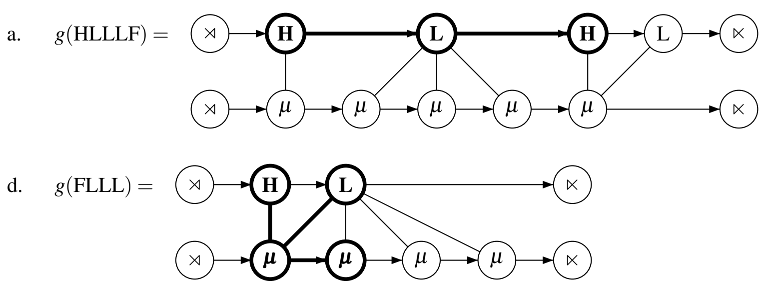


Fig. 14 Subgraphs from B in (33) in $g(w)$ for some examples $w \notin L_{\mathrm{HJ}}$

Thus, B^g_{HJ} captures exactly the strings in L_{HJ}, and so this stringset is ASL.

5.5 Autosegmental Strictly Local sets and the Subregular Hierarchy

The preceding section has shown that the ASL class intersects with the SL, TSL, and SP sets. The following briefly completes the picture of where ASL lies in the

Subregular Hierarchy, as originally depicted in Fig 1: it is properly included within SF, but (at least given the Σ and g we have been using) it is incomparable with SL, TSL, and SP.

First, Jardine (to appearb) shows that g is FO-definable. This means that there is an interpretation of the FO-language of graphs in $g(\Sigma^*)$—call it FO$[E,A,\ell]$—in FO$[<]$, the FO-language of strings introduced in Sect. 3.5.[9] Let us take FO$[<]$ as interpreted over strings in $\Sigma = \{\mathrm{H, L, F}\}$ and let FO$[E,A,\ell]$ be defined as usual with the atomic predicates $E(x,y)$,$A(x,y)$, and $\gamma(x)$ for each $\gamma \in \Gamma \cup \{\rtimes_M, \ltimes_M, \rtimes_\mu, \ltimes_\mu\}$ corresponding to E, A, and ℓ in graphs in $g(\Sigma^*)$ extended with the boundary symbols. Then, for instance,

$$\mathrm{H}(x) \stackrel{\text{def}}{=} \mathrm{H}(x) \wedge (\forall z,y)[(x < z \wedge \neg(z < y \wedge y < x)) \rightarrow \neg\mathrm{H}(z)] \tag{34}$$

In other words, $\mathrm{H}(x)$ is true for a node in $g(w)$ when it corresponds to a position in w that is the first of a sequence of consecutive Hs. This follows from the operation of graph concatenation: when concatenating $g(w)$ and $g(\mathrm{H})$ a new H node is added only when w is empty or ends in L or F; after another H, it is merged with the previous H node. The result in Jardine (to appearb) is thus that there is a similar translation for every other atomic predicate in FO$[E,A,\ell]$ to a formula in FO$[<]$. This allows us to make the following claim that any banned k-subgraph grammar B is expressible in FO$[<]$.

Lemma 1 *For any banned k-subgraph grammar B, there is a* FO$[<]$ *sentence* φ *such that* $\mathscr{M}_w \models \varphi$ *iff* $\mathtt{subg}_k(g(\rtimes w \ltimes)) \cap B = \emptyset$.

Proof It is easy to show that for any $b \in B$ there is an existentially quantified sentence ψ_b in FO$[E,A,\ell]$ that defines b exactly; thus $\neg\psi_b$ is a sentence that true in a graph G iff b is not (isomorphic to) a subgraph of G. The sentence $\psi_B = \bigwedge_{b \in B} \neg\psi_b$ thus describes the set of graphs $G \in g(\Sigma^*)$ extended with the boundary symbols for which $\mathtt{subg}_k(G) \cap B = \emptyset$. From the result in Jardine (to appearb) we know that there is an interpretation $h : \mathrm{FO}[E,A,\ell] \rightarrow \mathrm{FO}[<]$. Thus for $\varphi = h(\psi_B)$, where φ is the translation of ψ_B to FO$[<]$, $\mathscr{M}_w \models \varphi$ iff $\mathtt{subg}_k(g(\rtimes w \ltimes)) \cap B = \emptyset$. □

Theorem 2 *ASL*$\subsetneq$*SF*

Proof That ASL$\subset$SF follows from Lemma 1. That ASL$\neq$FO is witnessed by $L(\varphi)$ for φ is defined as

$$\varphi \stackrel{\text{def}}{=} (\forall x,y)\left[\left(\mathrm{H}(x) \wedge \mathtt{first}(x) \wedge \mathtt{last}(y)\right) \rightarrow \mathrm{H}(y)\right],$$

where $\mathtt{first}(x)$ is defined parallel to $\mathtt{last}(x)$ defined in Sect. 3.5; i.e. it is true when x is the first position in the string. Thus $L(\varphi)$ is exactly the set of strings such that if the first position is H, then the last position must also be H. In other words,

$$L(\varphi) = \Sigma^* - \{\, \mathrm{H}w\mathrm{L} \mid w \in \Sigma^* \,\},$$

that is, the complement of the set of strings that start with H and end with L.

[9] Crucially, this is only defined for graphs in $g(\Sigma^*)$, not GR(Γ) in general.

If $L(\varphi)$ were ASL, there must then be some k such that there is a set B of banned k-subgraphs that ban all and only the strings of the form HwL for some $w \in \Sigma^*$. This requires banning exactly the graphs $g(\rtimes \text{H}w\text{L} \ltimes)$ for all $w \in \Sigma^*$. These are exactly the graphs that share the melody

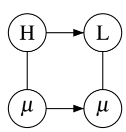
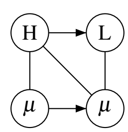
$$\rtimes \to \text{H} \to \cdots \to \text{L} \to \ltimes$$

where '...' indicates the portion of the melody contributed by $g(w)$. The size of $g(w)$ grows arbitarily long as w gets longer (consider $g(\text{HL}), g(\text{HLHL}), g(\text{HLHLHL}), \ldots$). Because B must contain connected graphs, there is thus no finite set B (and thus no finite k) that can ban all such melodies. □

It can also be shown that there are SL (and thus TSL) stringsets that are not ASL.

Theorem 3 *ASL and (T)SL are incomparable.*

Proof That ASL$\not\subset$SL and ASL$\not\subset$TSL has already been shown by the fact that L_{UTP} (which is neither SL or TSL) is ASL. That SL$\not\subset$ASL and TSL$\not\subset$ASL is witnessed by $L(B)$ for the the banned substring 2-factor grammar $B = \{\text{HL}\}$. Thus $L(B)$ contains no string such that HL is a substring, but HF is still a valid substring. However, as shown below, $g(\text{HL})$ is a subgraph of $g(\text{HF})$.

$g(\text{HL}) =$  $g(\text{HF}) =$ 

By the definition of a banned subgraph grammar, there can be no banned subgraph grammar B^G that excludes $g(\text{HL})$ but not $g(\text{HF})$, because any subgraph of $g(\text{HL})$ is also a subgraph of $g(\text{HF})$. □

This is a peculiarity of ARs modeling contour tones: sequences of 'pure'-toned moras are subgraphs of sequences of contour-toned moras with the same melody. For the consequences of this for phonological theory and possible solutions, see Jardine and Heinz (in press). Note that this issue does not arise when g does not include such contour graph primitives; in these cases, ASL may properly include SL and TSL. (Note also that because SL$\subset$SF the above example is an additional proof that SF$\neq$ASL.)

Finally, we show that SP and ASL are incomparable.

Theorem 4 *ASL and SP are incomparable.*

Proof That ASL$\not\subset$SP has already been shown by the fact that L_{HJ}, which is not SP, is ASL. That SP$\not\subset$ASL is witnessed by $L(B)$ for the banned subsequence 2-factor grammar $B = \{\text{FF}\}$. This is exactly the set of strings with at most one F. Note that

$$\text{F}w \in L(B) \text{ for all } w \in \{\text{H,L}\}$$

but

$$\mathrm{F}w\mathrm{F} \notin L(B) \text{ for } \textit{no } w \in \{\mathrm{H}, \mathrm{L}\}\,.$$

For the same reasoning as the proof for Thm. 2, it is impossible to distinguish every $g(\mathrm{F}w)$ from $g(\mathrm{F}w\mathrm{F})$ given arbitary $g(w)$ with a banned k-subgraph grammar B^g for a finite k. □

We have now shown what is summarized visually in Fig. 1: given this Σ and g, ASL is properly included in SF but incomparable to SL, TSL, and SP.

6 Discussion

This thus concludes the main results of the paper. The following summarizes the results and then discusses how future work can build on them.

6.1 Interpreting the results

The previous two sections contain two main results. The first, shown in Sect. 4.2, showed that positional, obligatorniess, and culmanitivity well-formedness generalizations in tone introduced in Sect. 2 appear in each of the subclasses of the SF class of stringsets that have previously been proposed to be relevant to phonology, and Hirosaki Japanese is even properly SF. Thus, neither SL, TSL, nor SP is a sufficiently expressive characterization of tone patterns. The next section, Sect. 5, introduced the ASL class of stringsets by extending the notion of banned k-factor grammar to ARs and by using a function g that established a correspondence between strings and ARs. It was then shown, in 5.4, that this class contains all of the tone patterns of Sect. 2. It was also shown, in Sect. 5.5, that this class is *restrictive* in that it is properly sub-SF.

Thus, as a *theory* of well-formedness in tone, the ASL class is both sufficiently expressive to capture major classes of tone patterns but also restrictive in that it is properly sub-SF.[10] In other words, it is a tight characterization of well-formedness in tone. This increase in expressivity is thanks to the additional AR structure provided by g. That is, we kept the notion of banned k-factor grammars constant, but enriched our representational information. We can thus take this as evidence that this representational information nontrivially captures how tone is computed in phonological cognition.

6.2 Future work

The preceding section established some basic facts about the ASL class, especially with respect to its characterization of well-formedness constraints in tone. However, some interesting questions remain. First, what are the closure properties of ASL? Because the ASL stringsets are defined by sets of banned k-factors, they are closed under intersection: the intersection of two ASL stringsets is simply the stringset described

[10] For other tone patterns captured by banned k-factor grammars over ARs, see Jardine (to appeara).

by the union of their respective banned subgraph grammars. However, like the SL stringsets, they are likely not closed under union, complement or concatenation.

To prove such properties definitively, it will be useful to have an abstract characterization of the ASL class, like SSC for the SL class. Such a characterization would be a generalization of the proofs for Thms. 2 and 3 showing non-membership of SF and SL languages in ASL. Based on these, we can make the following conjecture.

Conjecture 1 (SSC for ASL) A stringset $L \subseteq \Sigma^*$ is ASL_k iff

1. For any two distinct strings $w, v \in \Sigma^*, |w| = |v|$ such that $w \in L$ and $v \notin L$, $g(w) \not\sqsubseteq g(v)$, and
2. for any strings $w_1, w_2, u_1, u_2, x \in \Sigma^*$ such that $\texttt{short}_x(w_1 x w_2) = \texttt{short}_x(u_1 x u_2) = k-1$,
$$w_1 x w_2 \in L \text{ and } u_1 x u_2 \in L \text{ implies } w_1 x u_2 \in L$$

where (informally) $\texttt{short}_x(w_1 x w_2)$ is the length of the shortest path in $g(w_1 x w_2)$ connecting nodes in $g(w_1)$ and $g(w_2)$ through nodes in $g(x)$.

Part 1 of Conj. 1 is based on the proof of Thm. 3, which showed that it is possible to have two distinct strings w and v of the same length such that $g(w)$ is a subgraph of $g(v)$, and so banned subgraph grammars cannot exclude w from a stringset without also excluding v. Part 2 of Conj. 1 is based on the proof of Thm. 2. Like the SSC for SL stringsets, it essentially states that k bounds the size of 'chunks' of a graph that a banned k-subgraph grammar can pay attention to. Here, we use the function $\texttt{short}_x(w_1 x u_2)$ to measure these chunks. As proving both ways of the if and only if will be technical, for concerns of length a full proof of the conjecture is left for future work.

Another interesting question is the consequences of varying Σ and g, as briefly described following Thm. 3 in the previous subsection. For example, we can imagine a g such that there are no strings $w, v \in L$, $w \neq v$ and $|w| = |v|$ such that $g(w) \sqsubseteq g(v)$—in other words, Part 1 of Conj. 1 would never be violated. Given such a g, then any banned substring k-factor grammar could be translated into a banned k-subgraph grammar. In this case, then, SL$\subsetneq$ASL, counter to what was found for the g explored in this paper. Jardine and Heinz (2015) also describe some different properties of the set $g(\Sigma^*)$ when g is varied. For example, they show that particular graph primitives can be used to circumvent the OCP.

Armed with a SSC for ASL, future work can explore systematicities in the variation of expressivity of different ASL classes based on different functions g.

Finally, as noted in Sect. 3, the banned k-factor classes of stringsets all have provable, efficient learning results that take advantage of the restrictive structure of banned k-factor grammars. These ideas can thus be likely extended to banned subgraph grammars as well. One related result is that of Ferreira (2013), who gives an efficient method for listing connected subgraphs of size k. While the size of such a list is potentially very large for graphs in general, for the case of a set of graphs generated by a function like g, it is likely to stay within a reasonable bound. Future work can take advantage of these facts to develop an efficient, provably correct algorithm that, as the results of the current paper show, is relevant for the learning of natural language tone patterns.

7 Conclusion

This paper has offered two main results. One, it has showed that classes of stringsets previously studied in the context of well-formedness in phonology are insufficient to characterize well-formedness in tone patterns. Two, it has defined a new class of stringsets based on grammars that ban factors of autosegmental versions of strings, and it has shown that this class does provide a strong characterization of tonal well-formedness. This is thus a step towards understanding how structure can play a role in the expressivity of grammars, specifically with respect to phonological computation.

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