Expressivity and Autosegmental Structure

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Introduction

- **Main goal:** a restrictive yet sufficient theory of well-formedness in tone
- **Tool:** a theory of simple computations over autosegmental grammars
- **Side benefit:** further understanding of the relationship between expressivity and phonological representation

Introduction

- **Result:** *Graph Strictly Local* (GSL) patterns provide a restrictive, sufficient, and unified characterization of the typology of tone
- GSL is based on *banned subgraphs* in autosegmental structures
- A *sufficient* theory from enriched representation; *restrictive* theory comes from computationally simple nature of banned substructure constraints

Computation and representation

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$$L(R) = \{CV, VC, CVC, VCV, CVCV, VCVC, CVCVC, ...\}$$

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- *CCC, *#bn, *HH, etc.

Computation and representation

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- Capures long-distance dissimilation and harmony with blocking (Heinz et al., 2011; McMullin and Hansson, 2016)

Computation and representation

- **Strictly Piecewise (SP)** grammars: sub*sequence* (precedence), not substrings (Heinz, 2010; Rogers et al., 2010)

$$R = \{s...ʃ, ʃ...s\}$$

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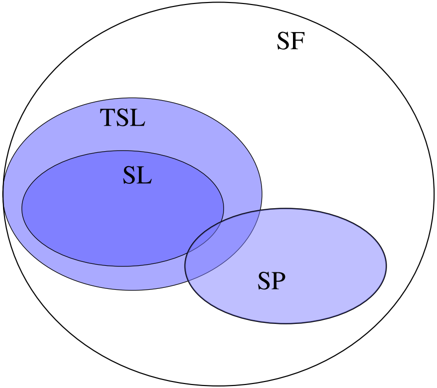
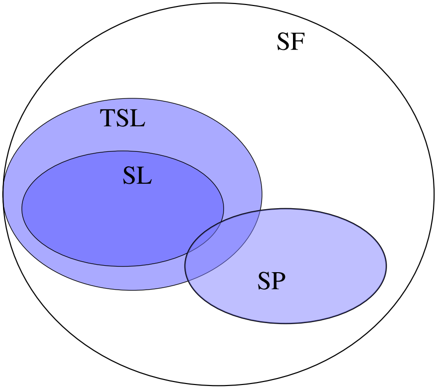
- Good fit to typology of consonant harmony (Heinz, 2010)

Computation and representation



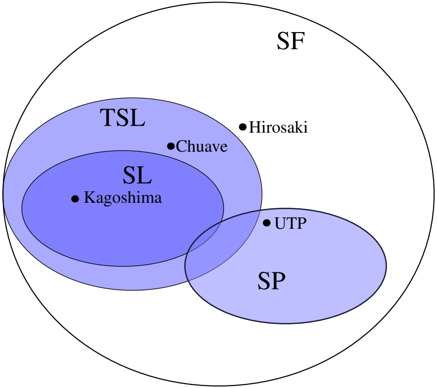
- SL, TSL, and SP provide a robust, yet restrictive, theory of segmental phonotactics
- Computation is based on **banned substructures**; differences are representational

Computation and representation



- Opposed to, ex., **Star Free (SF)** class, which allows for global reasoning about a structure (McNaughton and Papert, 1971; Rogers et al., 2013; Jardine and Heinz, in press)

Well-formedness in tone



- Tone has both local and non-local patterns (Yip, 2002; Hyman, 2011)
- The following sample of *positional*, *obligatoriness*, and *culminativity* generalizations in tone fall in SL, TSL, SP, and SF

Well-formedness in tone

Positional

- Kagoshima Japanese: **Final or penult H**
  (Hirayama, 1951; Haraguchi, 1977; Kubozono, 2012)

a.	hána	‘nose’	HL
b.	sakúra	‘cherry blossom’	LHL
c.	kagaríbi	‘watch fire’	LLHL
d.	kagaribí-ga	‘watch fire’ + NOM	LLLHL
			...
e.	haná	‘flower’	LH
f.	usagí	‘rabbit’	LLH
g.	kakimonó	‘document’	LLLH
h.	kakimono-gá	‘document’ + NOM	LLLLH
			...

Well-formedness in tone

Obligatoriness

- Chuave: **At least one H** (Donohue, 1997)

a.	kán H	‘stick’	e.	gíngódí HHH	‘snore’	*L
b.	gáán HH	‘child’	f.	dénkábu HHL	‘mosquito’	*LL
c.	gáam HL	‘skim’	g.	énugú HLH	‘smoke’	*LLL
d.	kubá LH	‘bamboo’	h.	amámó LHH	‘k.o. yam’	*LLLL
			i.	kóiom HLL	‘wing’	...
			j.	komári LHL	‘before’	
			k.	koiyóm LLH	‘navel’	

Well-formedness in tone

Culminativity

- Unbounded Tone Plateauing (UTP): **At most one *span* of H** (Hyman, 2011; Jardine, 2016)

a.	kitabo	‘book’	LLL
b.	mutéma	‘chopper’	LHL
c.	kisikí	‘log’	LLH
d.	mutémá+bísíkí	‘log chopper’	LHHHHH
e.	*mutéma+bisikí	” ”	*LHLLLH

(Luganda; Hyman, 2011; Hyman and Katamba, 2010)

Well-formedness in tone

Positional + obligatoriness + culminativity

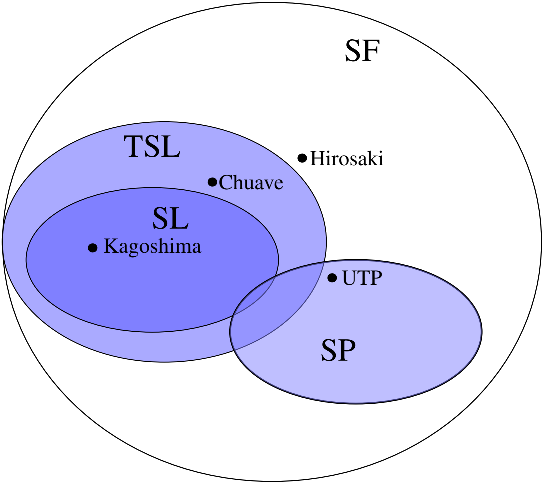
- Hirosaki Japanese: **Exactly one H or F, F only word final** (Haraguchi, 1977)

Noun	Isolation	+Nom	Noun	Isolation
a. ‘handle’	é	e-gá	f. ‘chicken’	niwatorí
	H	LH		LLLH
b. ‘picture’	ê	é-ga	g. ‘lightning’	kaminarî
	F	HL		LLLF
c. ‘candy’	amé	ame-gá	h. ‘fruit’	kudamóno
	LH	LLH		LLHL
d. ‘rain’	amê	amé-ga	i. ‘trunk’	toránku
	LF	LHL		LHLL
e. ‘autumn’	áki	áki-ga	j. ‘bat’	kóomori
	HL	HLL		HLLL
	*LLLL	*HLLH	*HLLF	*FLLL

Well-formedness in tone

- **Kagoshima:** penult or final H *positional*
- **Chuave:** at least one H *obligatoriness*
- **UTP:** At most one plateau of H *culminativity*
- **Hirosaki:** exactly one H or F; F word-final *all 3*

Tone well-formedness and formal language complexity



Tone well-formedness and formal language complexity

Positional constraints are SL

$$\text{Kagoshima pattern:}\quad \left\{ \begin{array}{ll} \rtimes\text{HL}\ltimes, & \rtimes\text{LH}\ltimes, \\ \rtimes\text{LHL}\ltimes, & \rtimes\text{LLH}\ltimes, \\ \rtimes\text{LLHL}\ltimes, & \rtimes\text{LLLH}\ltimes, \\ ... & \end{array} \right\}$$

- $R = \{\text{HLL, HH, HLH, LL}\ltimes, \rtimes\text{L}\ltimes\}$

$$*\rtimes\text{HLLLL}\ltimes, *\rtimes\text{HLLHL}\ltimes,$$
$$*\rtimes\text{LLHHL}\ltimes, *\rtimes\text{HLHL}\ltimes, *\rtimes\text{LLLLL}\ltimes, ...$$

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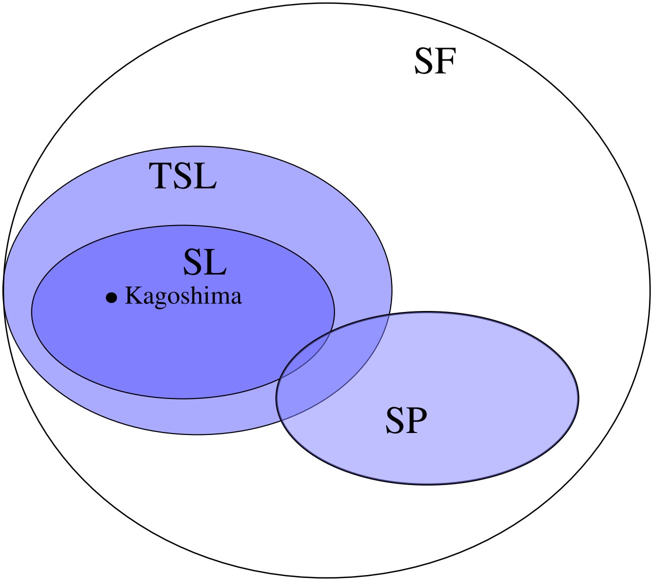

Tone well-formedness and formal language complexity

Obligatoriness constraints are TSL

Chuave pattern: $\{ \rtimes LH \ltimes, \rtimes HL \ltimes, \rtimes HH \ltimes, \rtimes LLH \ltimes, \rtimes LHL \ltimes, \rtimes LHH \ltimes, \rtimes HLL \ltimes, \rtimes HLH \ltimes, \rtimes HHL \ltimes \rtimes HHH \ltimes, \rtimes LLLH \ltimes, \ldots \}$

$* \rtimes L \ltimes$, $* \rtimes LL \ltimes$, $* \rtimes LLL \ltimes$, $* \rtimes LLLL \ltimes$, ...

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- $\langle T = \{\text{H}\}, R = \{ \rtimes\ \ltimes \} \rangle$

$$*\rtimes\text{L}\ltimes,\ *\rtimes\text{LL}\ltimes,\ *\rtimes\text{LLL}\ltimes,\ *\rtimes\text{LLLL}\ltimes,\ ...$$

Tone well-formedness and formal language complexity

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Chuave pattern: $\{$ $\rtimes$LH$\ltimes$, $\rtimes$HL$\ltimes$, $\rtimes$HH$\ltimes$, $\rtimes$LLH$\ltimes$, $\rtimes$LHL$\ltimes$, $\rtimes$LHH$\ltimes$, $\rtimes$HLL$\ltimes$, $\rtimes$HLH$\ltimes$, $\rtimes$HHL$\ltimes$ $\rtimes$HHH$\ltimes$, $\rtimes$LLLH$\ltimes$, ... $\}$

- $\langle T = \{\mathrm{H}\}, R = \{\rtimes\ltimes\}\rangle$

$$\texttt{erase}_T(\rtimes\mathrm{LLH}\ltimes) = \rtimes\mathrm{H}\ltimes$$
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$*\rtimes$L$\ltimes$, $*\rtimes$LL$\ltimes$, $*\rtimes$LLL$\ltimes$, $*\rtimes$LLLL$\ltimes$, ...

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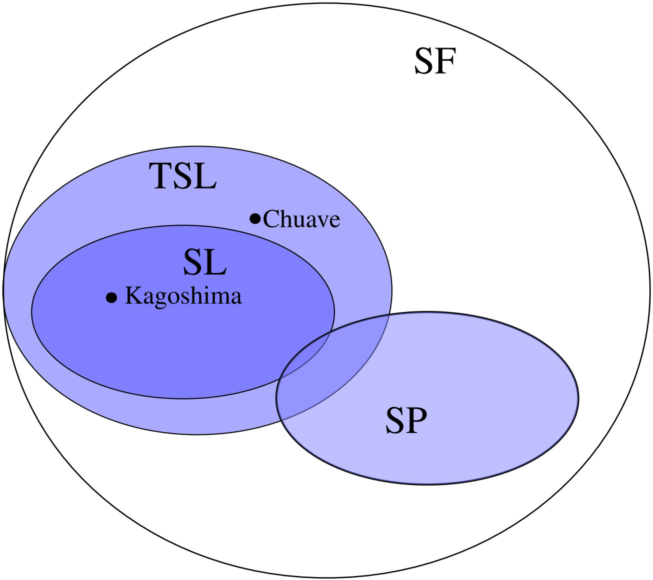
Tone well-formedness and formal language complexity



Tone well-formedness and formal language complexity

Culminativity constraints are SP

UTP pattern: $\{ \rtimes\text{LLL}\ltimes,\ \rtimes\text{LHL}\ltimes,\ \rtimes\text{LLH}\ltimes,\ \rtimes\text{LHHHH}\ltimes,\ \rtimes\text{LLHHHLL}\ltimes,\ \rtimes\text{HHLLLL}\ltimes, \ldots \}$

*HLH, *HLLH, *HLLLH, *HLLLLH, *HLLLLLH,
*LHHLLLLHHHL, *LHHHLLHHHHLLL, ...

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- $R = \{\text{H...L...H}\}$

*HLH, *HLLH, *HLLLH, *HLLLLH, *HLLLLLH,
*LHHLLLLHHHL, *LHHHLLHHHHLLL, ...

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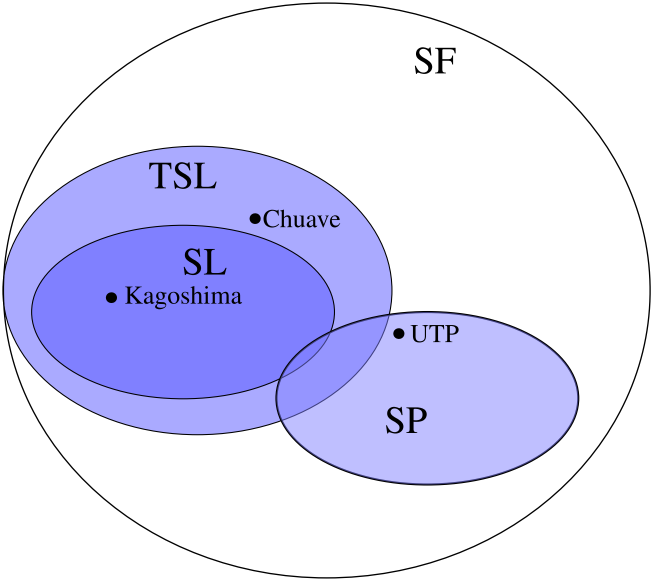

Tone well-formedness and formal language complexity

Combined constraints are not necessarily SL, TSL, or SP

$$\text{Hirosaki pattern:} \quad \left\{ \begin{array}{lll} \rtimes\text{LLH}\ltimes, & \rtimes\text{LLF}\ltimes, & \\ \rtimes\text{LHL}\ltimes, & \rtimes\text{LLLF}\ltimes, & \\ \rtimes\text{HLL}\ltimes, & \rtimes\text{LLLLF}\ltimes, & \\ \rtimes\text{LLLH}\ltimes, & \rtimes\text{LLLLLF}\ltimes, & \\ ... & ... & \end{array} \right\}$$

$$*\rtimes\text{LLL}\ltimes, *\rtimes\text{LLLL}\ltimes, *\rtimes\text{LLLLL}\ltimes, *\rtimes\text{LLLLL}\ltimes, ...$$
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$$*\rtimes\text{LFL}\ltimes, *\rtimes\text{FLL}\ltimes, *\rtimes\text{LLFL}\ltimes, *\rtimes\text{LFLL}\ltimes, *\rtimes\text{FLLL}\ltimes, ...$$

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- TSL: $\langle T = \{\text{H,F}\}, R = \{\rtimes\ltimes, \text{HF}, \text{FH}\}\rangle$

$$*\rtimes\text{LLL}\ltimes, *\rtimes\text{LLLL}\ltimes, *\rtimes\text{LLLLL}\ltimes, *\rtimes\text{LLLLL}\ltimes, ...$$
$$*\rtimes\text{HLF}\ltimes, *\rtimes\text{HLLF}\ltimes, *\rtimes\text{HLLLF}\ltimes, *\rtimes\text{HLLLF}\ltimes, ...$$
$$*\rtimes\text{LFL}\ltimes, *\rtimes\text{FLL}\ltimes, *\rtimes\text{LLFL}\ltimes, *\rtimes\text{LFLL}\ltimes, *\rtimes\text{FLLL}\ltimes, ...$$

Tone well-formedness and formal language complexity

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\text{Hirosaki pattern:} \quad \left\{ \begin{array}{ll} \rtimes\text{LLH}\ltimes, & \rtimes\text{LLF}\ltimes, \\ \rtimes\text{LHL}\ltimes, & \rtimes\text{LLLF}\ltimes, \\ \rtimes\text{HLL}\ltimes, & \rtimes\text{LLLLF}\ltimes, \\ \rtimes\text{LLLH}\ltimes, & \rtimes\text{LLLLLF}\ltimes, \\ \ldots & \ldots \end{array} \right\}
$$

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*\rtimes\text{LLL}\ltimes,\ *\rtimes\text{LLLL}\ltimes,\ *\rtimes\text{LLLLL}\ltimes,\ *\rtimes\text{LLLLL}\ltimes,\ \ldots
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$$

$$
*\rtimes\text{LFL}\ltimes,\ *\rtimes\text{FLL}\ltimes,\ *\rtimes\text{LLFL}\ltimes,\ *\rtimes\text{LFLL}\ltimes,\ *\rtimes\text{FLLL}\ltimes,\ \ldots
$$

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- TSL: $\langle T = \{\text{H,F}\}, R = \{\rtimes\ltimes, \text{HF}, \text{FH}\}\rangle$
- SL: $R = \{\text{FL}\}$

$$*\rtimes\text{LLL}\ltimes, *\rtimes\text{LLLL}\ltimes, *\rtimes\text{LLLLL}\ltimes, *\rtimes\text{LLLLL}\ltimes, ...$$
$$*\rtimes\text{HLF}\ltimes, *\rtimes\text{HLLF}\ltimes, *\rtimes\text{HLLLF}\ltimes, *\rtimes\text{HLLLF}\ltimes, ...$$
$$*\rtimes\text{LFL}\ltimes, *\rtimes\text{FLL}\ltimes, *\rtimes\text{LLFL}\ltimes, *\rtimes\text{LFLL}\ltimes, *\rtimes\text{FLLL}\ltimes, ...$$

Tone well-formedness and formal language complexity

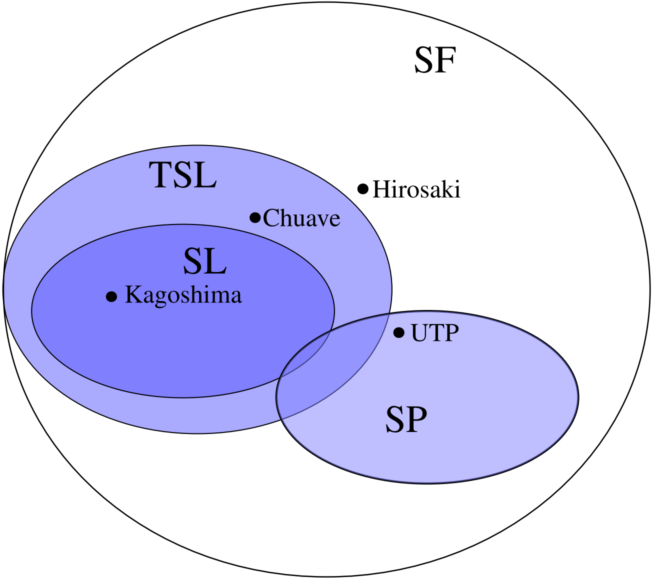
Combined constraints are not necessarily SL, TSL, or SP

$$\text{Hirosaki pattern:} \quad \left\{ \begin{array}{ll} \rtimes\text{LLH}\ltimes, & \rtimes\text{LLF}\ltimes, \\ \rtimes\text{LHL}\ltimes, & \rtimes\text{LLLF}\ltimes, \\ \rtimes\text{HLL}\ltimes, & \rtimes\text{LLLLF}\ltimes, \\ \rtimes\text{LLLH}\ltimes, & \rtimes\text{LLLLLF}\ltimes, \\ \ldots & \ldots \end{array} \right\}$$

- TSL: $\langle T = \{\text{H,F}\}, R = \{\rtimes\ltimes, \text{HF}, \text{FH}\}\rangle$
- SL: $R = \{\text{FL}\}$

$$*\rtimes\text{LLL}\ltimes,\ *\rtimes\text{LLLL}\ltimes,\ *\rtimes\text{LLLLL}\ltimes,\ *\rtimes\text{LLLLL}\ltimes,\ \ldots$$
$$*\rtimes\text{HLF}\ltimes,\ *\rtimes\text{HLLF}\ltimes,\ *\rtimes\text{HLLLF}\ltimes,\ *\rtimes\text{HLLLF}\ltimes,\ \ldots$$
$$*\rtimes\text{LFL}\ltimes,\ *\rtimes\text{FLL}\ltimes,\ *\rtimes\text{LLFL}\ltimes,\ *\rtimes\text{LFLL}\ltimes,\ *\rtimes\text{FLLL}\ltimes,\ \ldots$$

Tone well-formedness and formal language complexity



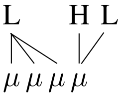
Local graph grammars

- String-based complexity classes provide a *restrictive*, but not entirely *sufficient* nor *unified*, characterization of tone

Local graph grammars

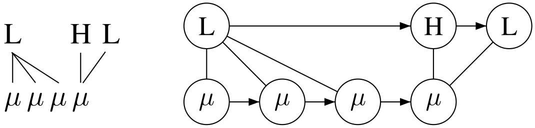
- String-based complexity classes provide a *restrictive*, but not entirely *sufficient* nor *unified*, characterization of tone
- Not unsurprising; tone has been claimed to be fundamentally *autosegmental* (Goldsmith, 1976; Yip, 2002; Hyman, 2011)

kaminarî LLLF 'lightning'
(Hirosaki)



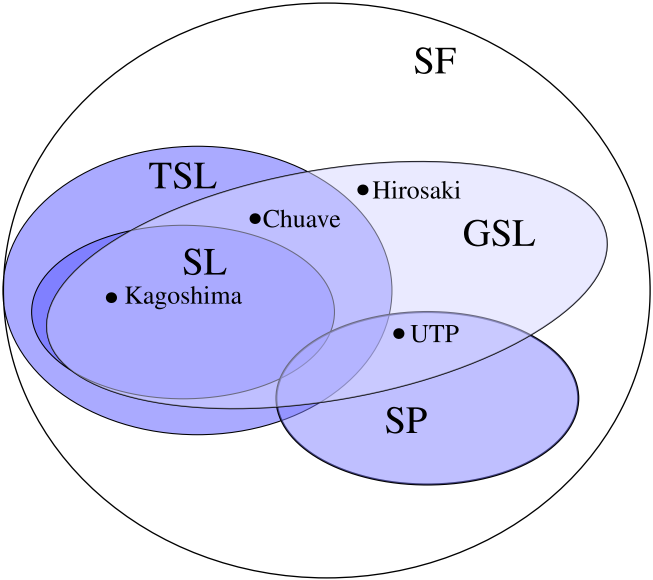
Local graph grammars

- Autosegmental representations are **graphs** (Goldsmith, 1976; Coleman and Local, 1991)



- We can instead consider **Graph Strictly Local** grammars, defined by restricted sub**graphs**

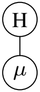
Local graph grammars

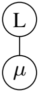


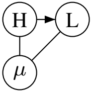
Graph Strictly Local patterns

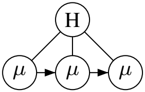
Building structure

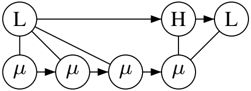
- We can define a function $\texttt{gr}(w)$ that generates an autosegmental representation from strings (Jardine and Heinz, 2015)

$\texttt{gr}(\text{H}) =$ 

$\texttt{gr}(\text{L}) =$ 

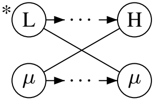
$\texttt{gr}(\text{F}) =$ 

$\texttt{gr}(\text{HHH}) =$ 

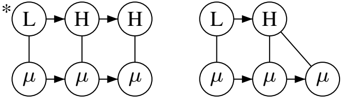
$\texttt{gr}(\text{LLLF}) =$ 

Graph Strictly Local patterns

- Association preserves precedence relations **(the No-Crossing Constraint (NCC))**

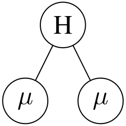


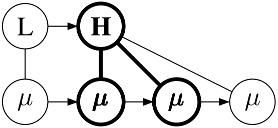
- Adjacent nodes on tonal tier cannot be identical **(the Obligatory Contour Principle (OCP))**



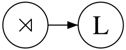
Graph Strictly Local patterns

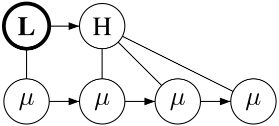
- Let a **subgraph** be some finite, connected piece of a graph





- Subgraphs may refer to boundaries on each tier (not depicted in full graphs)





Graph Strictly Local patterns

- R is some set of restricted subgraphs

$$L(R) = \{\ w \mid \text{no graph in } R \text{ is a subgraph of } \mathtt{gr}(w)\}$$

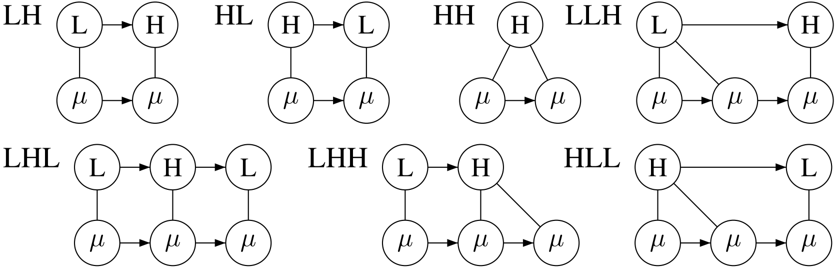
- Let us consider strings over $\{\mathrm{H}, \mathrm{L}, \mathrm{F}\}$

Graph Strictly Local patterns

Chuave: At least one H	{ LH,	HL,	HH,	
	LLH,	LHL,	LHH,	
	HLL,	HLH,	HHL	
	HHH,	LLLH,	...	}

Graph Strictly Local patterns

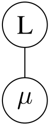
Chuave: At least one H { LH, HL, HH, LLH, LHL, LHH, HLL, HLH, HHL, HHH, LLLH, ... }

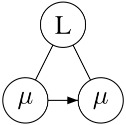


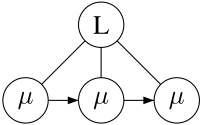
Graph Strictly Local patterns

Chuave: At least one H { LH, HL, HH,
LLH, LHL, LHH,
HLL, HLH, HHL
HHH, LLLH, ... }

- No all L toned words:

*L 

*LL 

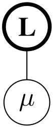
*LLL 

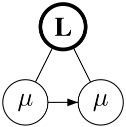
...

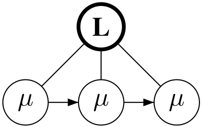
Graph Strictly Local patterns

Chuave: At least one H { LH, HL, HH,
LLH, LHL, LHH,
HLL, HLH, HHL
HHH, LLLH, ... }

- No all L toned words:

*L 

*LL 

*LLL 

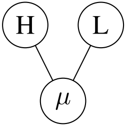
...

- First banned subgraph: $\rtimes \rightarrow \mathrm{L} \rightarrow \ltimes$

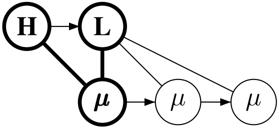
Graph Strictly Local patterns

Chuave: At least one H { LH, HL, HH, LLH, LHL, LHH, HLL, HLH, HHL, HHH, LLLH, ... }

- No contours:

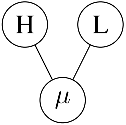


*FLLL



Graph Strictly Local patterns

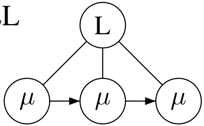
Chuave: At least one H { LH, HL, HH,
LLH, LHL, LHH,
HLL, HLH, HHL
HHH, LLLH, ... }

- $R = \Big\{ \rtimes \rightarrow \mathrm{L} \rightarrow \ltimes$,  $\Big\}$

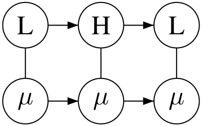
Graph Strictly Local patterns

UTP: At most one span of H { LLL, LHL, LLH, LHHHH, LLHHHLL, HHLLLL, ... }

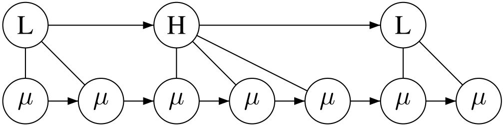
LLL



LHL

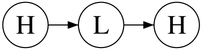


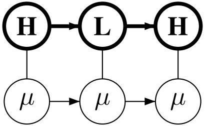
LLHHHLL

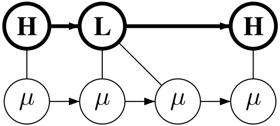


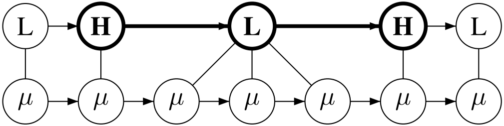
Graph Strictly Local patterns

UTP: At most one span of H { LLL, LHL, LLH, LHHHH, LLHHHLL, HHLLLL, ... }

- Only one H tone per word: 

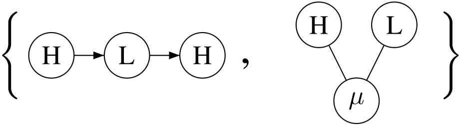
*HLH 

*HLLH 

*LHLLLHL 

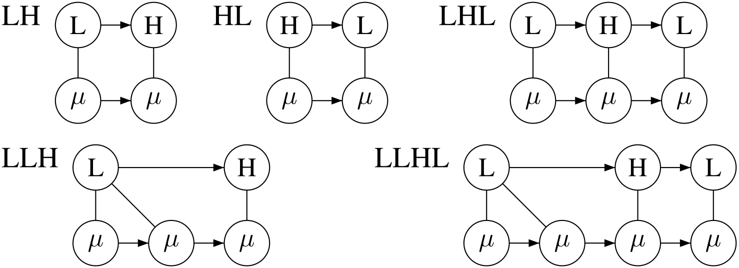
Graph Strictly Local patterns

UTP: At most one span of H { LLL, LHL, LLH, LHHHH, LLHHHLL, HHLLLL, ... }

- $R =$ 

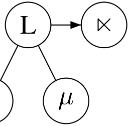
Graph Strictly Local patterns

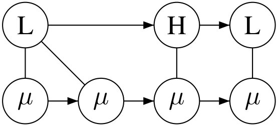
Kagoshima: One penult or final H { HL, LH, LHL, LLH, LLHL, LLLH, ... }

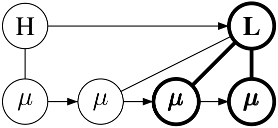


Graph Strictly Local patterns

Kagoshima: One penult or final H { HL, LH, LHL, LLH, LLHL, LLLH, ... }

- A final L tone cannot be multiply associated: 

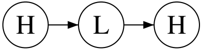
LLHL 

*HLLL 

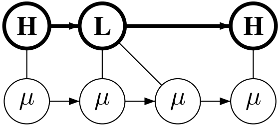
Graph Strictly Local patterns

Kagoshima: One penult or final H { HL, LH, LHL, LLH, LLHL, LLLH, ... }

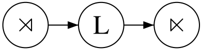
- Only one H tone per word



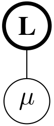
*HLLH



- No all L toned words



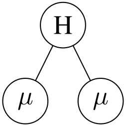
*L



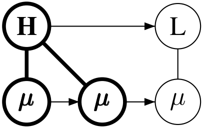
Graph Strictly Local patterns

Kagoshima: One penult or final H { HL, LH, LHL, LLH, LLHL, LLLH, ... }

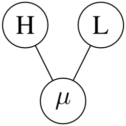
- No spreading of H



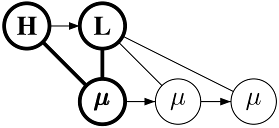
*HHL



- No contours

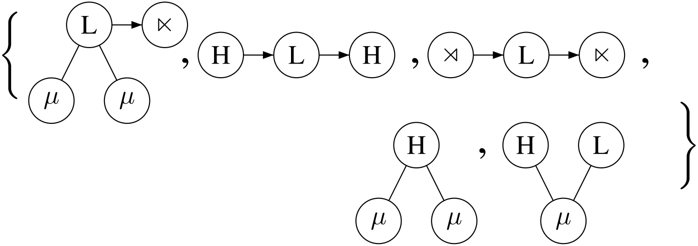


*FLLL



Graph Strictly Local patterns

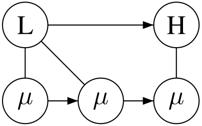
Kagoshima: One penult or final H { HL, LH, LHL, LLH, LLHL, LLLH, ... }

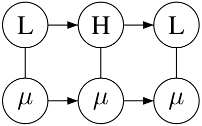
- $R =$ 

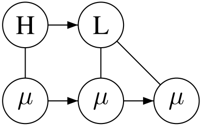
Graph Strictly Local patterns

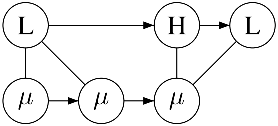
Hirosaki: Exactly one H or F; F always final

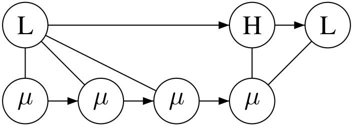
{ LLH, LHL, HLL, LLLH, . . .
LLF, LLLF, LLLLF, LLLLLF, . . . }

LLH


LHL


HLL


LLF


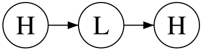
LLLF


Graph Strictly Local patterns

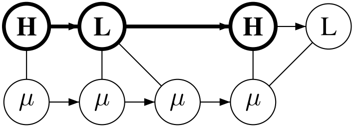
Hirosaki: Exactly one H or F; F always final

{ LLH, LHL, HLL, LLLH, . . .
LLF, LLLF, LLLLF, LLLLLF, . . . }

- No two Hs in the melody:



*HLLF

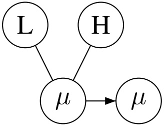


Graph Strictly Local patterns

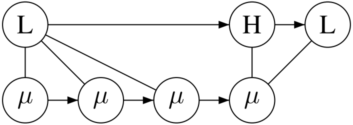
Hirosaki: Exactly one H or F; F always final

{ LLH, LHL, HLL, LLLH, . . .
LLF, LLLF, LLLLF, LLLLLF, . . . }

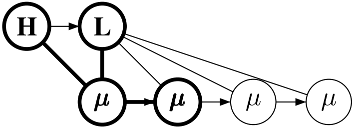
- No nonfinal contours:



LLLF



*FLLL

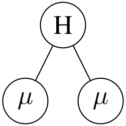


Graph Strictly Local patterns

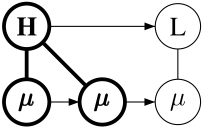
Hirosaki: Exactly one H or F; F always final

{ LLH, LHL, HLL, LLLH, . . .
LLF, LLLF, LLLLF, LLLLLF, . . . }

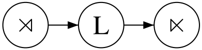
- No spreading of H



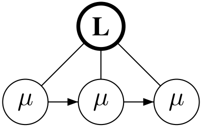
*HHL



- No all L toned words



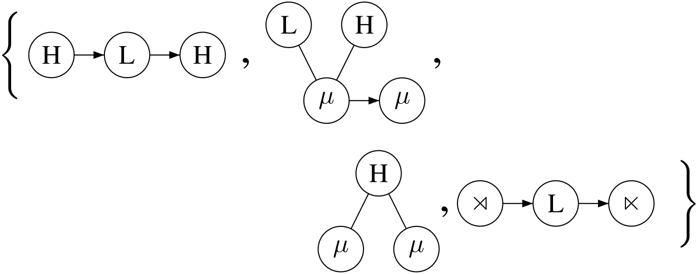
*LLL



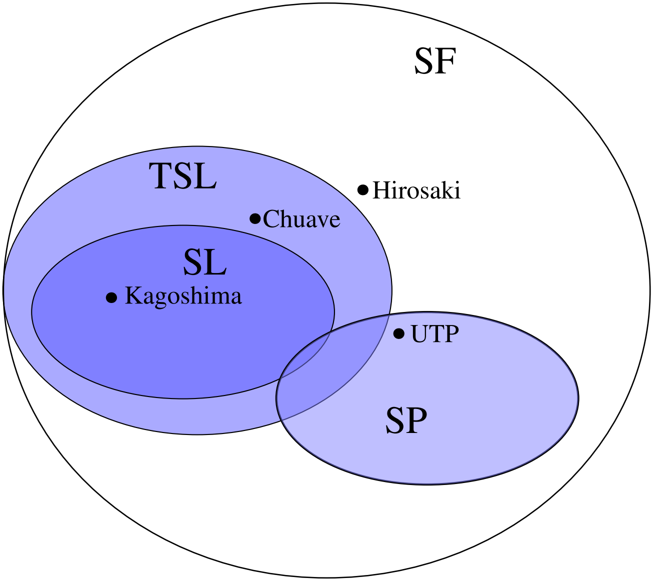
Graph Strictly Local patterns

Hirosaki: Exactly one H or F; F always final

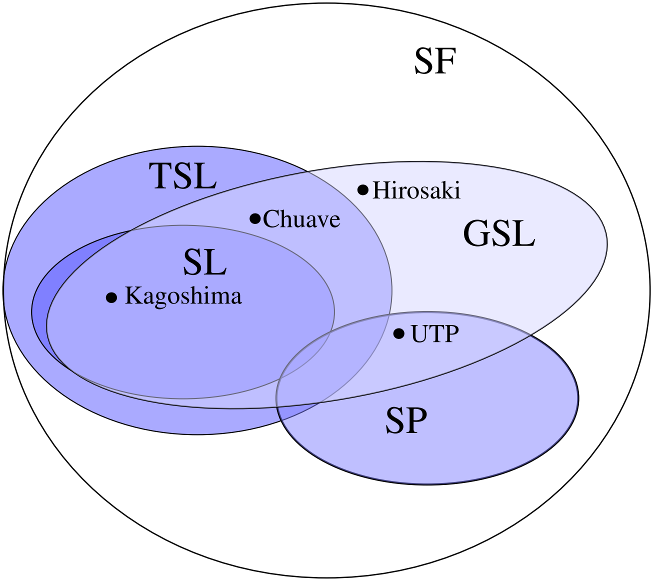
{ LLH, LHL, HLL, LLLH, . . .
LLF, LLLF, LLLLF, LLLLLF, . . . }

- $R =$ 

Graph Strictly Local patterns



Graph Strictly Local patterns



Discussion

- Tonal constraints fall into a number of distinct classes of string grammars
- Banned **subgraph** grammars provide a unified theory of positional, culminativity, and obligatoriness constraints in tone
- They are **restrictive** in that we can only *ban* structures–we can't require them (Jardine and Heinz, in press)

Discussion

- Tonal constraints fall into a number of distinct classes of string grammars
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- They are **restrictive** in that we can only *ban* structures–we can't require them (Jardine and Heinz, in press)
  - Example: 'First last' patterns (Lai, 2012, 2015): $\rtimes\mathrm{H} \leftrightarrow \mathrm{H}\ltimes$

Discussion

- We can define mappings like $\texttt{gr}(w)$ through mathematical logic (Courcelle, 1994; Engelfriet and Hoogeboom, 2001)
- The *structure* is restrictive because $\texttt{gr}(w)$ is **first-order definable** from strings (using the order $<$)
- The structural relationships in an autosegmental structure are thus equivalent to FO-statements in a string
- Thus, using local autosegmental grammars will never take us out of SF
- (This is also true for $\texttt{erase}_T(w)$)

Discussion

- Such structure-creating functions can aid in **learning**
- Banned substructure grammars have established learning techniques (García et al., 1990; Heinz, 2010; Heinz and Rogers, 2010)
- These techniques can learn long-distance patterns with additional structure known *a priori* (Hayes and Wilson, 2008; Heinz et al., 2011; Jardine and Heinz, 2016b)
- Tier structure can be learned (Goldsmith and Riggle, 2012; Jardine and Heinz, 2016a; Jardine and McMullin, to appear), but no work yet on autosegmental structure

Conclusion

- We have characterized tone by extending **banned subgraph** grammars to autosegmental representations
- This provided a sufficient and unified, yet restrictive, characterization of tone
- What about other structure: correspondence, syllables, stress grids, feet?
- How does autosegmental structure interact with the complexity of *transformations*? (Jardine, 2016)

Acknowledgments

Thank you!

This work is indebted to Jeff Heinz, Jim Rogers, Jane Chandlee, Bill Idsardi, the UD Phonetics & Phonology group, the students of my Phonology III course at Rutgers (Eileen Blum, Hazel Mitchley, Luca Iacoponi, and Nick Danis), and audiences at the 2016 LSA annual meeting, NAPhC, UPenn, and Rutgers University. The majority of this research was done under the auspices of a University of Delaware dissertation fellowship.

Defining $\texttt{gr}(w)$ **in FO**

- First order logic for strings over $\{\mathrm{H}, \mathrm{L}\}$
  - Variables x, y, z, ..., ranging over positions in the string

$$\texttt{Input:}\quad \mathrm{H} \rightarrow \mathrm{L} \rightarrow \mathrm{L} \rightarrow \mathrm{L} \rightarrow \mathrm{H}$$

Defining $\texttt{gr}(w)$ in FO

- First order logic for strings over $\{\mathrm{H}, \mathrm{L}\}$
  - Variables x, y, z, ..., ranging over positions in the string
  - Predicates $\mathrm{H}(x)$ and $\mathrm{L}(x)$

`Input:` $\mathrm{H} \to \mathrm{L} \to \mathrm{L} \to \mathrm{L} \to \mathrm{H}$

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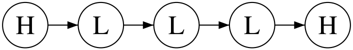
  - Predicates $x \lhd y$ and $x < y$

`Input :`



Appendix

Defining $\texttt{gr}(w)$ in FO

- First order logic for strings over $\{\mathrm{H}, \mathrm{L}\}$
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  - Predicates $\mathrm{H}(x)$ and $\mathrm{L}(x)$
  - Predicates $x \lhd y$ and $x < y$
  - Logical connectives $\neg\phi, \quad \phi \wedge \psi, \quad \phi \vee \psi, \quad \phi \rightarrow \psi$

`Input :` $\mathrm{H} \rightarrow \mathrm{L} \rightarrow \mathrm{L} \rightarrow \mathrm{L} \rightarrow \mathrm{H}$

Appendix

Defining $\texttt{gr}(w)$ in FO

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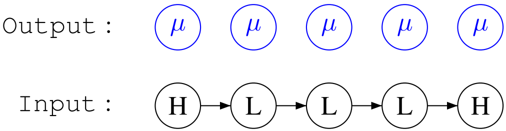
  - Logical connectives $\neg\phi, \quad \phi \wedge \psi, \quad \phi \vee \psi, \quad \phi \rightarrow \psi$
  - Quantifiers $(\forall x)[\phi(x)]$ and $(\exists x)[\phi(x)]$

`Input :` $\mathrm{H} \rightarrow \mathrm{L} \rightarrow \mathrm{L} \rightarrow \mathrm{L} \rightarrow \mathrm{H}$

Defining $\texttt{gr}(w)$ in FO

- Defining autosegmental positions and relationships in terms of the input string
  - $\mu^1_A(x) \stackrel{\text{def}}{=} \mathrm{H}(x) \vee \mathrm{L}(x)$



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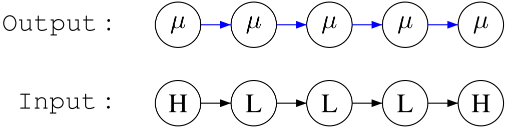
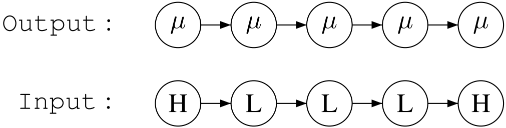


Appendix

Defining $\texttt{gr}(w)$ in FO

- Two useful predicates:
  - $\texttt{LSpanHd}(x) \stackrel{\text{def}}{=} \mathrm{L}(x) \wedge (\forall y)[y \triangleleft x \rightarrow \neg\mathrm{L}(x)]$



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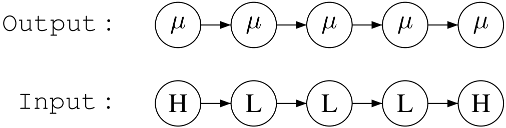

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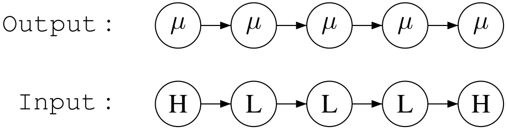
  - $\texttt{HSpanHd}(x) \stackrel{\text{def}}{=} \mathrm{H}(x) \wedge (\forall y)[y \triangleleft x \rightarrow \neg \mathrm{H}(x)]$
  - $\texttt{span}(x, y) \stackrel{\text{def}}{=} \big(\mathrm{H}(x) \wedge \mathrm{H}(y) \wedge (\forall z)[(x < z \wedge z < y) \rightarrow \mathrm{H}(z)]\big)$
    $\vee \big(\mathrm{L}(x) \wedge \mathrm{L}(y) \wedge (\forall z)[(x < z \wedge z < y) \rightarrow \mathrm{L}(z)]\big)$



Appendix

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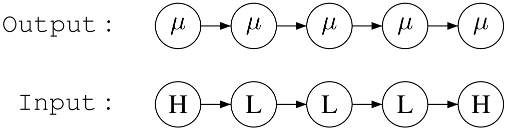
- Defining autosegmental positions and relations in terms of the input string
  - $H^2_A(x) \stackrel{\text{def}}{=} \texttt{HSpanHd}(x)$

"Copy the first H in a sequence of Hs"



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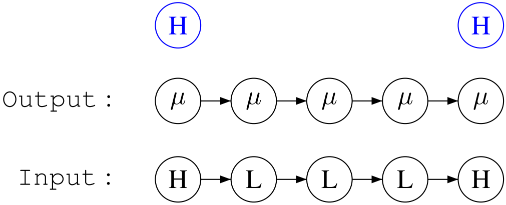

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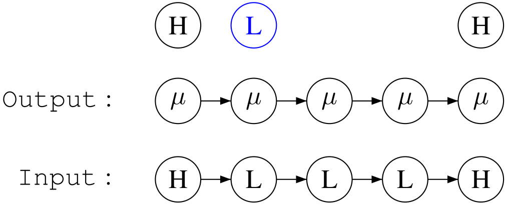
    "Copy the first H in a sequence of Hs"
  - $L^2_A(x) \stackrel{\text{def}}{=} \texttt{LSpanHd}(x)$

    "Copy the first L in a sequence of Ls"

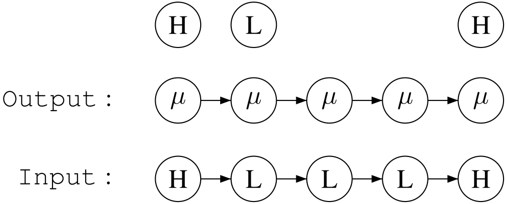


Appendix

Defining $\texttt{gr}(w)$ in FO

- Defining autosegmental positions and relations in terms of the input string
  - $x \triangleleft_A^{2,2} y \overset{\text{def}}{=} x < y \wedge \big(\texttt{HSpanHd}(x) \vee \texttt{LSpanHd}(x)\big) \wedge$

    "x starts a span..."



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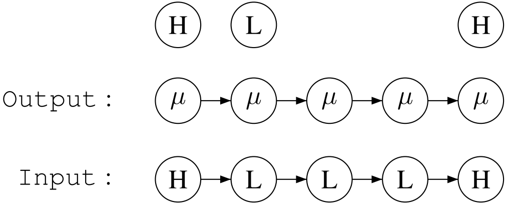
    "x starts a span..."

    $(\forall z)[(x < z \wedge z < y) \rightarrow \texttt{span}(x, z)] \wedge$

    "everything in between x and y is in a span with x"



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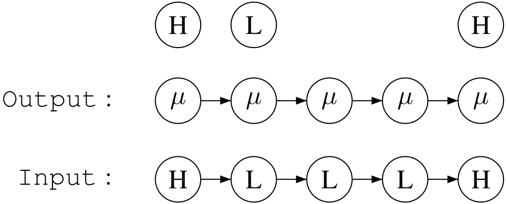
    "everything in between x and y is in a span with x"

    $\neg(\texttt{span}(x, y))$

    "x and y are not in a span"



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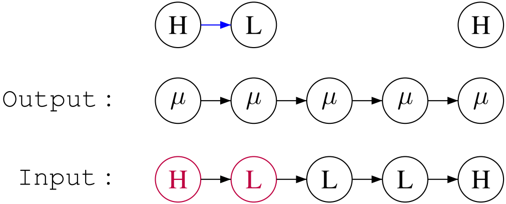

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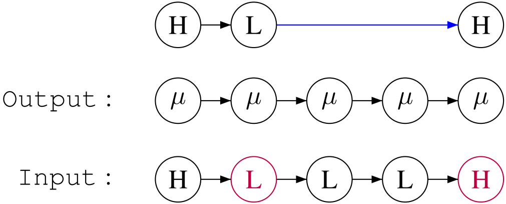

Appendix

Defining $\texttt{gr}(w)$ in FO

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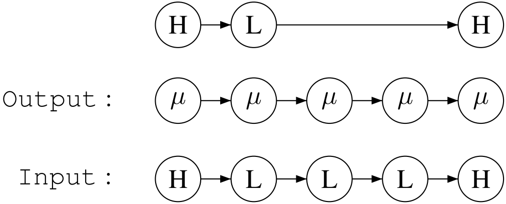
  - $x \circ_A^{2,1} y \stackrel{\text{def}}{=} \big(\texttt{LSpanHd}(x) \vee \texttt{HSpanHd}(x)\big) \wedge \texttt{span}(x, y)$



Appendix

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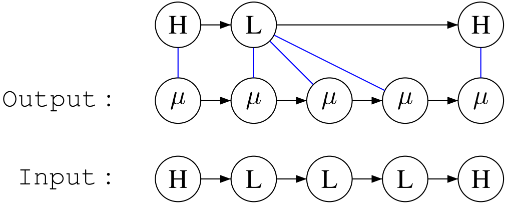


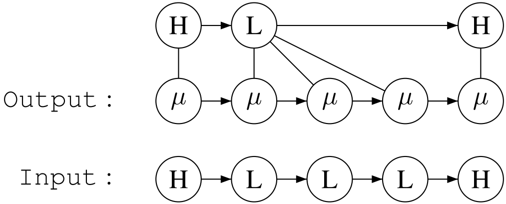
Appendix

Defining $\texttt{gr}(w)$ in FO

- We've defined $\texttt{gr}(w)$ by defining

$$\mu_A(x),\ \ \mathrm{H}_A(x),\ \ \mathrm{L}_A(x),\ \ x \lhd_A^{1,1} y,\ \ x \lhd_A^{2,2} y,\ \ x \circ_A^{2,1} y$$

  in FO terms of the input string



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Defining `gr`(w) **in FO**

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Defining `gr`(w) in FO

- $R = \left\{ \mathrm{H} \to \mathrm{L} \to \mathrm{H} \right\}$
- $\neg(\exists x, y, z)\left[\, x \lhd_A y \ \wedge\ y \lhd_A z \ \wedge\ \mathrm{H}(x) \ \wedge\ \mathrm{L}(y) \ \wedge\ \mathrm{H}(z)\right]$

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Defining $\texttt{gr}(w)$ in FO

- $R = \left\{ \text{H} \to \text{L} \to \text{H} \right\}$
- $\neg(\exists x, y, z)\Big[\underbrace{x \lhd_A y}_{\begin{array}{l}\equiv \boldsymbol{x < y} \wedge \big(\texttt{HSpanHd}(x) \vee \texttt{LSpanHd}(x)\big) \wedge \\ \quad (\forall z)[(x < z \wedge z < y) \to \texttt{span}(x, z)] \wedge \neg(\texttt{span}(x, y))\end{array}} \;\wedge\; y \lhd_A z \;\wedge\; \mathrm{H}(x) \;\wedge\; \mathrm{L}(y) \;\wedge\; \mathrm{H}(z)\Big]$