Locality and non-linear representations in phonology

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The Big Picture

- ▶ What is the character of phonological generalizations?
 - ► Well-formedness blick vs. *bnick (Chomsky and Halle, 1965)
 - **▶** Transformations

```
write /raɪt/ \rightarrow[raɪt]
writer /raɪt+ər/ \rightarrow[raɪrər]
```

- ► How do we best characterize cross-linguistic variation in well-formedness patterns and transformations?
 - ► That is, what is a possible/impossible well-formedness pattern/transformation?
- ► How might they be learned?

The Big Picture

- Criteria for evaluating a phonological theory
 - a. Captures the attested generalizations
 - b. Predicted typology is restricted in a principled way
 - c. Has some learning model

This talk

- Criteria for evaluating a phonological theory
 - a. Captures the attested generalizations
 - b. Predicted typology is restricted in a principled way
 - c. Has some learning model
- ➤ This talk demonstrates that **tone well-formedness patterns** are fundamentally **local**, in a computational sense, over autosegmental structures
- ► A theory of well-formedness using language-specific, inviolable constraints which forbid substructures provides a unified explanation of cross-linguistic variation in tone and meets (a)–(c)

This talk

- Criteria for evaluating a phonological theory
 - a. Captures the attested generalizations
 - b. Predicted typology is restricted in a principled way
 - c. Has some learning model
- A local theory compares favorably to previous approaches to tone
- ▶ Directional association and rules in derivational frameworks (e.g., Goldsmith, 1976; Archangeli and Pulleyblank, 1994)
 - ► satisfy (a)
 - unclear with respect to (b)
 - have no current solution for (c)

This talk

- Criteria for evaluating a phonological theory
 - a. Captures the attested generalizations
 - b. Predicted typology is restricted in a principled way
 - c. Has some learning model
- Optimal satisfaction of violable constraints (Meyers, 1997; Yip, 2002; Zoll, 2003)
 - ► satisfies (c)
 - does not satisfy (a) or (b)
- Both derivational and OT explanations invoke a globally evaluated notion of directionality, miss the local nature of tone patterns

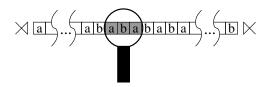
- ▶ blick vs. *bnick
- ► Constraint: *#bn
- ► This is a forbidden substructure constraint







 Well-formedness determined solely by well-formedness of local substructures



Rogers and Pullum (2011); Rogers et al. (2013)

- ► Classes of patterns thus described are extremely restrictive in their expressivity (McNaughton and Papert, 1971; Rogers and Pullum, 2011; Rogers et al., 2013; Heinz, 2010; Heinz et al., 2011)
- ► Such patterns can be efficiently learned (García et al., 1990; Heinz, 2010, 2011; Jardine and Heinz, accepted)

How do we characterize the following variation?

- Attested
 - ► Fixed melodies realized over words of different lengths (all)
 - Contours and plateaus of tone may be restricted to left or right (Mende vs. Hausa)
 - ▶ Plateaus of a particular tone may be restricted (Kukuya)
 - ► Tones may appear on both edges (N. Karanga)
 - ► There must be exactly one of a particular tone (Hirosaki J.)
- Unattested
 - ▶ Patterns calculating over entire representation

Mende word tone (Leben, 1973; Goldsmith, 1976)

```
H a. kó
            Η
                 'war'
                         b. pélé
                                 HH 'house'
                                             c. háwámá HHH
                                                             'waist'
  L d. kpà
               'debt'
                        e. bèlè LL 'pants' f. kpàkàlì
                                                       LLL
                                                             'stool'
 HL g. mbû F
               'owl' h. ngílà HL 'dog' i. félàmà
                                                       HLL
                                                             'junction'
LH j. mbă
           R
                 'rice'
                        k. nìká
                                 LH 'cow' l. ndàvúlá
                                                       LHH
                                                             'sling'
LHL m. mbà R-F 'comp.'
                                                             'nut'
                        n. nyàhâ LF
                                     'woman'
                                             o. nìkílì
                                                       LHL
```

- ► Words choose between 5 melodies (*HLH)
- Plateaus of tone appear at the right edge of the word HHH, HLL
 *LLH, *HHL
- Contours appear at the right edge of the word R, LF, *RL

Hausa tone-integrating suffixes (Newman, 1986, 2000)

```
b. jíráa
                                HH
                                      'wait for'
                                                 c. béebíyáa
                                                               HHH
                                                                      'deaf mute'
a. jáa
            'pull'
c. wàa L
            'who?'
                      d. màcè
                                LL
                                      'woman'
                                                 e zàmfàrà
                                                              LLL
                                                                      'Zamfara'
                      g. jìmìnúu HHL 'ostriches'
                                                 h. bàbbàbbàkú LLLH 'roasted'
f. jàakíi LH 'donkey'
i fáadì
        HL 'fall'
                     j. hántúnàa LLH
                                      'noses'
                                                 k. búhúnhúnàa HHHL 'sacks'
                      m. káràntá
                                HLH
                                      'read'
l. mântá FH 'forget'
                                                 n. kákkáràntá
                                                               HHLH 'reread'
```

- ► HLH allowed (LHL not depicted)
- Plateaus of tone appear at the *left* edge of the word LLH, HHL
 *LHH. *HLL
- Contours appear at the *left* edge of the word FH, *HR

Kukuya word tone (Hyman, 1987; Zoll, 2003)

```
a. kâ 'to pick' F b. sámà 'conversation' HL c. káràgà 'entangled' HLL d. să 'knot' R e. kàrá 'paralytic' LH f. m<sup>w</sup>àrègí 'brother' LLH g. bá 'palms' H h. bágá 'show knives' HH i. bálágá 'fence' HHH j. bví falls' R-F k. pàlí 'goes out' LF l. kàlágí 'turns' LHL
```

- Contours on right edge of word
- ▶ No plateaus of H in the presence of L

N. Karanga Shona non-assertive tense (Odden, 1986; Hewitt and Prince, 1989)

hàndákà-p-á	'I didn't give'	H
hàndákà- tór-à	'I didn't take'	HL
hàndákà-tór-ès-á	'I didn't make take'	HLH
hàndákà-tór-és-èr-á	'I didn't make take for'	HHLH
hàndákà-tór-és-ér-àn-á	'I didn't make take for e.o.'	HHHLH
hàndákà-tór-és-ér-ès-àn-á	'I didn't make take a lot for e.o.'	HHHLLH
hàndákà-tór-és-ér-ès-ès-àn-á	<i>''</i>	HHHLLLH

► Two Hs on either end, first spreads to three syllables maximally

Hirosaki Japanese (Haraguchi, 1977)

Noun	Isolation	+NOM	Noun	Isolation
a. 'handle'	é	e-gá	f. 'chicken'	niwatorí
	Н	LH		LLLH
b. 'picture'	ê	é-ga	g. 'lightening'	kaminarî
	F	HL		LLLF
c. 'candy'	amé	ame-gá	h. 'fruit'	kudamóno
	LH	LLH		LLHL
d.	amê	amé-ga	i. 'trunk'	toránku
	LF	LHL		LHLL
e. 'autumn'	áki	áki-ga	j. 'bat'	kóomori
	HL	HLL		HLLL

► Exactly one H or F; F can only be word-final *Lⁿ, *H...H, *H...F, *FL...

Unattested generalizatons

- ▶ The number of Hs in the word must be in the Fibonacci sequence
- ► The number of Hs and Ls must be equal
- ► A single H appears as close to the center of the word as possible

Summary: Tonal well-formedness patterns

How do we characterize the following variation?

- Attested
 - ► Fixed melodies realized over words of different lengths (all)
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Autosegmental representations

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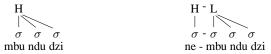


Autosegmental representations

► Tones behave independently of TBUs; Tiv (Pulleyblank, 1986):

sớn è sớn HLH 'bird of bird'
$$\begin{bmatrix} -- \end{bmatrix}$$
 $\begin{bmatrix} H & L & H \\ | & | & | \\ \sigma & \sigma & \sigma \end{bmatrix}$ sớn !sớn $\begin{bmatrix} H'H & (same) \end{bmatrix}$ $\begin{bmatrix} -- \end{bmatrix}$ $\begin{bmatrix} H & L & H \\ | & | & | \\ \sigma & \sigma & \sigma \end{bmatrix}$

Plateaus of tone behave as single units; Shona (Odden, 1980): mbúndúdzí HHH 'worm' né-mbùndùdzì H-LLL 'with worm'



► Thus, I do not use alternatives (Cassimjee and Kisseberth, 2001; Shih and Inkelas, 2014)

Summary: Tonal well-formedness patterns

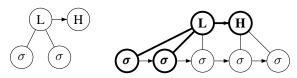
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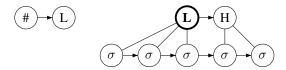
- ▶ What is a substructure in an autosegmental representation?
- ► Autosegmental representations are **graphs** (Goldsmith, 1976; Coleman and Local, 1991)



► Let a **subgraph** be some finite, connected piece of a graph



► Subgraphs may refer to boundaries on each tier (not depicted in full graphs)

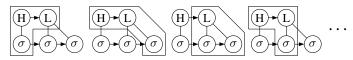


▶ We specify lists of **forbidden subgraph constraints** as in the following, where each ϕ_i is a subgraph:

$$\neg \phi_1 \wedge \neg \phi_2 \wedge \neg \phi_3 \wedge \ldots \wedge \neg \phi_n$$

- $\blacktriangleright \neg \phi$ means " ϕ is a forbidden substructure"
- ► These constraints are **inviolable** and **language-specific**

► Evaluation procedure now 'crawls' through graph



- Substructure constraints **provably** cannot specify patterns in which:
 - ▶ The number of Hs in the word must be in the Fibonacci sequence
 - ▶ The number of Hs and Ls must be equal
 - ▶ A single H appears as close to the center of the word as possible
 - **.**..

Summary: Tonal well-formedness patterns

How do we characterize the following variation?

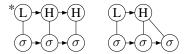
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Some assumptions

 Association preserves precedence relations (the No-Crossing Constraint (NCC))

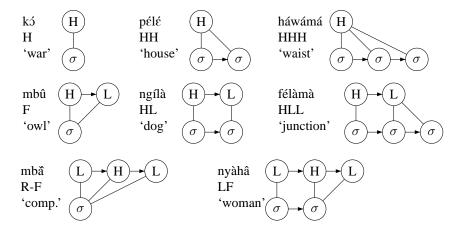


► Adjacent nodes on tonal tier cannot be identical (the Obligatory Contour Principle (OCP)

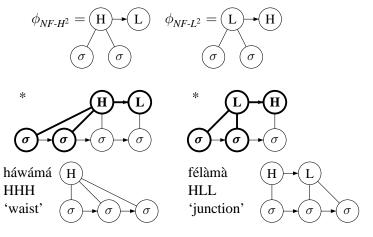


▶ Both of these arise as natural properties when we look at autosegmental representations as concatenation of primitives (Jardine and Heinz, 2015)

Representations

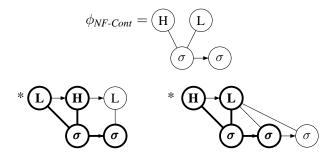


Plateaus



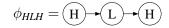
• Kukuya will use ϕ_{NF-H^2} but not ϕ_{NF-L^2}

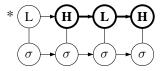
Contours



► c.f. Zhang (2000)

Melody constraint



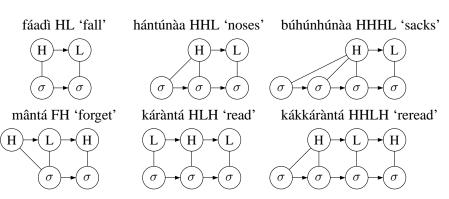


► Mende summary:

$$\neg \phi_{HLH} \wedge \neg \phi_{NF\text{-}Cont} \wedge \neg \phi_{NF\text{-}H^2} \wedge \neg \phi_{NF\text{-}L^2}$$

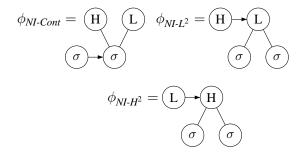
Case study: Hausa

Representations



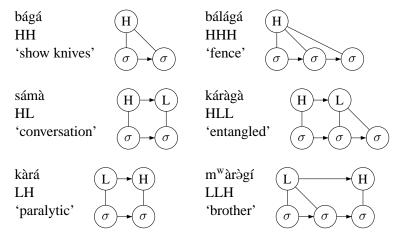
Case study: Hausa

Constraints



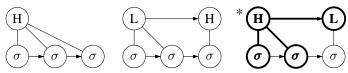
Case study: Kukuya

Representations

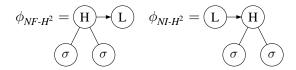


Case study: Kukuya

▶ H cannot spread in the presence of another tone

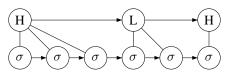


► In terms of forbidden subgraphs, this is just conjunction of $\neg \phi_{NF-H^2}$ (Mende) and $\neg \phi_{NI-H^2}$ (Hausa)

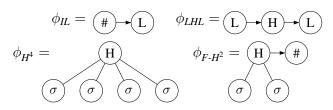


Case study: N. Karanga

▶ H on both edges; first spreads maximally to three syllables

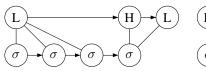


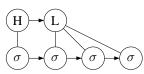
hàndákà-tór-és-ér-ès-àn-á HHHLLH 'I didn't make take a lot for e.o.'



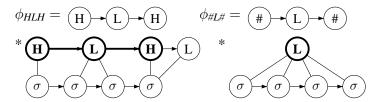
Case study: Hirosaki Japanese

► Exactly one H or F: kaminarî LLLF 'lightening' kóomori HLLL 'bat'





► Unattested: *HLLF, *LLLL



Discussion

- We have characterized the following variation in tone patterns with local constraints:
 - ► Fixed melodies realized over words of different lengths (all)
 - Contours and plateaus of tone may be restricted to left or right (Mende vs. Hausa)
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 - ► There must be exactly one of a particular tone (Hirosaki J.)

 Rule-based employ a plethora of association paradigms (Goldsmith, 1976; Leben, 1978; Yip, 1988; Hewitt and Prince, 1989, inter alia) and language-specific rules (e.g., Hyman, 1987)

- ► They capture the attested patterns
- ▶ No clear constraints on what a possible rule/association paradigm
- ► It is unclear what typology is, or how association paradigms/rules can be learned

- ► Zoll (2003) employs violable constraints in OT; *CLASH for Kukuya, ALIGN for directionality
- ► Clear picture of typology (Prince and Smolensky, 1993, 2004) and learnability (Tesar and Smolensky, 1998; Tesar, 2013)
- ► ALIGN constraints cannot capture N. Karanga Shona (as Zoll 2003 points out)

► ALIGN constraints can generate 'H closest to center' pattern (Eisner, 1997)

		$ALIGN(\sigma,R,H,R)$							
$\sigma\sigma\sigma\sigma\sigma\sigma\sigma\sigma$ +H		σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	(total)
Œ	a. <i>όσσσσσσ</i>	0	*	**	***	****	****	*****	21
	b. <i>σόσσσσσ</i>	*	0	*	**	***	****	****	16
	c. σσόσσσσ	**	*	0	*	**	***	****	13
	d. σσσόσσσ	***	**	*	0	*	**	***	12
	e. <i>σσσσόσσ</i>	****	***	**	*	0	*	**	13
	f. σσσσσόσ	****	****	***	**	*	0	*	16
	g. σσσσσσό	*****	****	****	***	**	*	0	22

▶ Optimization gets complex patterns with simple constraints (Gerdemann and Hulden, 2012)

▶ *Both* derivational and optimization-based approaches miss generalization that well-formedness is local, which is a central feature of the current proposal

Discussion: learning

- Language-specific forbidden substructure constraints are learnable
- ► The main idea: learner only pays attention to substructures of a particular size (García et al., 1990; Heinz, 2010, 2011; Jardine and Heinz, accepted)
- ► This idea can be extended straightforwardly to local autosegmental grammars (Jardine and Heinz, LSA)
- ▶ With some assumptions, autosegmental constraints can be learned directly from strings (Jardine and Heinz, 2015)

Conclusions

- ► The character of tonal well-formedness is fundamentally **local** over autosegmental structures
- ► Forbidden substructure constraints over autosegmental representations are an attractive theory of tonal well-formedness
- ► These constraints are inviolable, language-specific, and local
- ► They capture the full range of patterns, while not overgenerating in the same way as previous theories
- We know how they can be learned

Future work

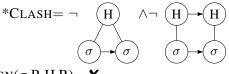
- ► So far, only considered *computational* restrictions
- ▶ We can incorporate other (e.g., perceptual, articulatory) information explicitly in graph and consider restrictions on *type* of subgraphs we can ban
- ► How can this be applied to **segmental** representations? The question of complex segments versus clusters (Jardine et al., 2015)?
- ► How does representation relate to the idea that tone is different from segmental phonology (Hyman, 2011; Jardine, to appear)?

Future work

- ► The locality of the generalizations discussed in this talk is a fact. Incorporating it into **transformations** is an interesting, yet approachable problem
- ► Chandlee (2014) shows how this can be done for strings
- ► For autosegmental representations, we can encode URs and correspondence in graph (Jardine, dissertation; Potts and Pullum, 2002)

Future work

- By varying logical grammars and representation, we get a space of constraint definition languages (de Lacy, 2011) whose relative expressive power is well-defined
- ► This space is not yet well understood!
- ► We can use this to study both the nature of MARKEDNESS and FAITHFULNESS



$$ALIGN(\sigma,R,H,R) = X$$

▶ We can explicitly study and compare other theories of representation (Browman and Goldstein, 1986; Cassimjee and Kisseberth, 2001; Rose and Walker, 2004; Shih and Inkelas, 2014)

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