Magic Square of Perfect Squares

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1 The maths behind it

Each component s_i, $i = 1, 2, 3, \ldots, 9$ of the magic square has to be a perfect square number. Say $s_i = n_i^2$ for $n \in \mathbb{Z}$. From [1] we know that the components need to be such that

$$n_i^2 \equiv 1 \pmod{24} \tag{1}$$

Looking at the smallest perfect square numbers and their modulo 24 (see table 1) we can see some sort of a pattern every 12-th integer. Indeed, as

$$(x+12)^2 \mod 24 = (x^2 + 24x + 144) \mod 24 \tag{2}$$

for any integer x and as

$$(24x + 144) \mod 24 = 0 \tag{3}$$

we can see that

$$x^2 \mod 24 = (x+12)^2 \mod 24 \tag{4}$$

For $x \in [0, 11]$ there are exactly four perfect squares fulfilling the condition (1). Therefore we can build four infinite series $\mathbb{S}_j$, $j = 1, 2, 3, 4$ for which all members fulfill the condition. Let there be t any integer, then

$$\mathbb{S}_1 = 1 + 12t = \{1, 13, 25, \ldots\} \tag{5}$$
$$\mathbb{S}_2 = 5 + 12t = \{5, 17, 29, \ldots\} \tag{6}$$
$$\mathbb{S}_3 = 7 + 12t = \{7, 19, 31, \ldots\} \tag{7}$$
$$\mathbb{S}_4 = 11 + 12t = \{11, 23, 35, \ldots\} \tag{8}$$

The members of these for sets are all and the only perfect square numbers that fulfill the condition (1).

x	x^2	$x^2 \mod 24$	x	x^2	$x^2 \mod 24$
0	0	0	12	144	0
1	1	1	13	169	1
2	4	4	14	196	4
3	9	9	15	225	9
4	16	16	16	256	16
5	25	1	17	289	1
6	36	12	18	324	12
7	49	1	19	361	1
8	64	16	20	400	16
9	81	9	21	441	9
10	100	4	22	484	4
11	121	1	23	529	1

Table 1: Some perfect square numbers and their modulo 24.

2 The generator functions

The above four series can be generated using two generator functions $f_+(g)$ and $f_-(g)$ for any integer g.

$$f_+ = 6 \cdot g + 1 \tag{9}$$

$$f_- = 6 \cdot g - 1 \tag{10}$$

References

[1] P. Zimmermann, P. Pierrat, and F. Thiriet, "Magic squares of squares." http://www.loria.fr/~zimmerma/papers/squares.pdf.