Prime Symmetric Goldbach

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In this paper I introduce a conjecture in number theory that shines some light on the Goldbach conjecture and the twin prime conjecture, showing that the Goldbach conjecture and the twin prime conjecture are only parts of the whole picture of primes, which is about base six. This new conjecture states that for any non-zero multiple of 6 there is a symmetric 1-or-prime gap to two primes. With other words, multiples of 6 are either located in between twin primes, or they are between two primes that are separated by some prime gap multiplied with 2. If true, then this conjecture makes progress on the Goldback conjecture and the twin prime conjecture by revealing a recursive structure behind primes. I also give a conjecture on the upper bound of the symmetric prime gap, plus a natural sequence of primes according to the theory of Avatar Extensions with an associated prime cover conjecture, and finally a sequence for the first hole of any multiples of natural numbers, with 3 following natural conjectures.

According to Bertrand's postulate, for every integer `n > 1` there is always at least one prime `p` such that `n < p < 2n`. This postulate is implied by the Goldback conjecture, because if it was false, then there would exist some `n` such that `2n` can not be expressed as a sum of two primes:

!Bertrand => !Goldbach

It follows from reverse modus tollens, that requires excluded middle, that:

Goldbach => Bertrand

From Goldbach one has the following, in sense of Bertrand, that for `n > 1`:

$$2n = a + b$$

Where `a, b` are primes. By introducing a 1-or-prime `p`:

$$2n = (a + p) + (b - p)$$

It is natural to examine the following constraint:

$$n = a + p = b - p$$

This is stronger than Goldbach and does not hold for all `n > 1`, but as far as I know, it only holds for `n > 1 & (n % 6) == 0`, which I call "Prime Symmetric Goldbach" conjecture (PSG).

The Dubner conjecture states that all even numbers greater than 4208 are the sum of two prime numbers that have a twin. Now, the Dubner conjecture does not imply that the two primes come from the same twin prime pair. If it did, then the Dubner conjecture would imply the PSG conjecture for numbers greater than 4208, since all mod 6 numbers are even.

So, the PSG conjecture overlaps with the Dubner conjecture and the Goldbach conjecture, since it only holds for mod 6. What makes the PSG conjecture different is that the gap between the two primes is twice the amount of a prime, or `1`.

With other words, every non-zero multiple of 6 is right in the middle between two primes. This generalizes the concept of twin primes to twice 1-or-prime gap.

This implies that there are infinitely number of primes, which has already been proved by Euclid. However, using PSG there is a much simpler proof, because there are infinite multiples of 6. Since there are infinite multiples of 6, it follows that there must be some larger prime for any particular multiple of six larger than any prime associated with lower multiples of 6. Hence, there must be infinite number of primes.

While it is good that the PSG conjecture implies that there is some 1-or-prime `p` such that one can reach two other primes for each non-zero multiple of six `n`, this is not that good because there are many primes, so it just produces an upper bound on `p < n`. In comparison, the computational analysis by T. Oliveira e Silva on the Goldbach conjecture suggests that one of the primes is very small, which allows a relativy simple cover. To improve this upper bound, I did some computational analysis and came up with the following upper bound:

```
upper_bound(n) := min(
    floor(n / 4.8 + 1),
    floor(4.55016035513408 * sqrt(n)),
)
```

The constant `4.8` is exact. The constant `4.55016035513408` is derived by experiment and stops when there is sufficient accuracy. Very likely, this upper bound can be improved.

For example, when `n = 6 000`, there is a prime `p < 352` such that `n – p` and `n + p` are primes.
When `n = 600 000`, there is a prime `p < 3 524` such that `n – p` and `n + p` are primes.
For `n = 60 000 000`, there a a prime `p < 35 245` etc.
Every multiple of 100 produces another digit in the same number for the upper bound of `p`.

This means that there is lower upper limit to how difficult it is to find the next prime. If you have a natural number `x`, then you can calculate `n = x + 6 – (x % 6)` and use this number to do prime search. For large numbers, this is much less than by using Bertrand’s postulate that only stops at `2n`. A prime sieve algorithm can sift out non-prime candidates quickly until there is only one remaining candidate, which according to the PSG conjecture must be a prime.

For example, for `x = 5`, one calculates `n = 5 + 6 – (5 % 6) = 6`. The upper bound gives `p < 2`. This leaves us with just one candidate. Therefore, `n + 1 = 6 + 1 = 7` is a prime.

Now, let us try do the same for `x = 7`. `n = 12` is the next multiple of 6.
The upper bound gives `p < 3`, so `13, 14` are candidates.
The number `14` is divisible by `2`, so `13` must be a prime.

Notice that this algorithm skips `11`. However, since PSG is symmetric, this can be easily fixed.
Since `n = 12` is the next multiple of 6` and `p < 3`, the candidates below `12` are `10, 11`.
The number `10` is divisible by `2`, so `11` must be a prime.

Starting with a list of primes containing `2` and `x = 3`
where `n = x + 6 – (x % 6)` and `g = upper_bound(n)`,
one can use PSG to find all primes by checking any number `> x` in the range `[n + 1 – g, n + g)` and add sift out all non-primes with each prime found in the list, until there are less than or equal to 2 candidates left.

The only known numbers where one gets `<= 2` candidates left are:

> 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25,
> 26, 27, 28, 29, 41, 43, 44, 45, 46, 47, 53

This means, this optimization can not be performed for higher numbers than 53.
Due to this limited property, PSG can not be used to improve prime sieve algorithms a lot.

However, when doing prime sieve mentally, one can use trial division by `2, 3, 5, 7, 11` when filtering the candidates to find the 38 first primes. So, PSG can be used to improve mental math.

Let us try this method with `x = 23`. Here, `n = 24`. The gap is `p < 6`, so the range is `[19, 30)`. However, since we started at `23` the range gets reduced to `[24, 30)`:

> 24, 25, 26, 27, 28, 29

The numbers `24, 26, 28` are divisible by `2`.
The number `27` is divisible by `3`.
The number `25` is divisible by `5`.

The only remaining candidate is `29`, which is less than or equal to 2,
and must therefore be a prime number.

Notice that there was no need to check for divisibility by `7` or `11`.
PSG saves the work needed in mental math up to 151, which is the 38th prime number.

Now, it is time to go back and look at the definition of PSG again:

$$n = a + p = b - p$$

Where `a, b` are primes and `p` is 1-or-prime. This imples that `2n = a + b`.

One can look at this from the perspective of Avatar Extensions. Here, the 0-avatar is `n`, which is the number we focus on. The numbers `a, b` are thought of as extensions of `n`. The first `p` one extends with is 1. The second `p` one extends with is 5. The third `p` one extends with is 7:

1, 5, 7, 13, 17, 11, 19, 43, 41, 59, 23, 29, 67, 31, 53, 47, 71, 101, 89, 113, 37, 73, 109, …

Notice that there is no 2 or 3. It is not surprising that 2 is left out, but lack of 3 is a bit surprising.

So, far I have given two conjectures: PSG and the upper bound on `p`. The third conjecture is about the sequence above, which consists of only 1 or primes. This conjecture states that the sequence of extending `n` with `p` contains all primes, except 2 and 3. I call it “Minus 3”, or M3 for short.

For example, 6 is the first multiple of 6. Here, there are 5 and 7 on each side, extended with 1.
The same goes for 12, where 11 and 13 are also extended with 1.
Again, the same for 18, where 17 and 19 are extended with 1.
The first multiple of 6 that is extended with something else than 1 is 24.
Here, 24 has 19 and 29, extended with 5.

If the M3 conjecture is true, then there is no limit to extensions, because there are infinitely many primes. It is an almost perfect cover of primes, only lacking 2 and 3, but with a different order.

For example, after some multiple of 6 is extended with 17,
the next new extension of multiples of 6 is 11.

This can seem a bit strange, because one might think that since 11 is smaller than 17, it should appear before 17 in the sequence of extensions. Yet, this is not what we observe. The order kind of looks like the prime sequence, but it deviates from it in some places.

Now, consider an even more strange question:
What is the “correct” order, the prime sequence or the M3 sequence?

This question is about mathematical language bias. The prime sequence does not reveal directly the PSG conjecture. This means, when starting with the prime sequence, one has to perform a lot of work to get to the insight of PSG and after even more work get to the M3 conjecture. However, if the starting point is PSG, then the M3 conjecture and hence the associated sequence is easier to find. So, it is possible to take the approach that the M3 sequence is more “correct” according to the theory of Avatar Extensions than the standard prime sequence. Remember, in Avatar Extensions, one does not assume any order such as the number line. Any order argued for in Avatar Extensions takes on the meaning of extending relative to some topic of interest, in this case the PSG conjecture.

With other words, to define “correct” in this sense means to care about the language bias one is using. There are none axioms that says explicitly “this is the correct order”. We have to look for this kind of “correct order” on our own, by taking initiative. This is done in combination with the admitting that we are being biased at the same time, or else there is no particular notion of “correct” that is meaningful here. By acknowledging that we are being subjective some times, one can also agree to some extent that there are some approaches that are “better” than others.

OK? It is important to get the nuance here, or else one can misunderstand the following:

> The PSG conjecture indicates that the standard prime sequence has the “wrong” order

With other words, if one wants to figure out more about primes, one should look into the M3 order.

I am not saying that we can figure out primes using the M3 order. I am saying that there is a mathematical language bias here that suggests the M3 order could be a future approach for research. Nobody knows yet what the M3 order does, but since the theory of Avatar Extensions has turned out to be very powerful before, it would be wise to listen and not just think without reflection that the standard prime sequence is “better” because we are used to think about primes that way.

The Goldbach conjecture is one of the oldest and best-known unsolved problems in number theory. Despite working on it since 1742, this has not yet been proven. However, the PSG conjecture suggests that the Goldbach conjecture is just one side of a symmetrical conjecture about base 6.

Wait a minute! What if there are other bases with the same property?

For any base `b`, one picks `n = b * x` where `x > 1`. If there is no cover of any 1-or-prime `p` such that `n – p` and `n + p` are primes, then the first number `x` where the cover fails is returned. In the case of `b = 6`, one gets an infinite sequence of `x` that has a cover, so one returns `∞` instead.

b	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
	1	1	7	1	∞	1	4	3	8	1	∞	1	2

Notice that primes here fails at `1`, using the same upper bound as with base 6.

Here there are 3 natural conjectures:

1. Prime Base (PB): Every prime fails at 1 using the upper bound
2. Compose Base (CB): Every composite fails at >= 1 using the upper bound
3. Infinity Six Base (ISB): The only infinite values are in bases multiples of 6

So, not only is base 6 telling something recursively about primes using a symmetric prime gap, but by taking a step back and consider every possible base, this also gives a new perspective of primes.

Primes from this perspective are the first numbers that fails that also who do not occur in other tests. Naturally, since the choice of base includes all composites with that base as a factor, the definition of primality in this sense is exactly the same as standard primality. However, here, it has some additional semantics.

For example, `28` might be considered a "relative prime" in base `14`, but without success:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
14	28	42	56	70	84	98	112	126	140	154	168	182
2	1	∞	2	2	∞	1	1	∞	1	2	∞	1

Since `14 = 2 * 7`, the frequency of multiples of 6 doubles in base `14`. Notice that `70 = 5 * 14` fails at `2`, so there is no exact self-similarity between base `14` and the standard number line.

Here is another example, in base `7`:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
7	14	21	28	35	42	49	56	63	70	77	84	91
1	1	3	1	1	∞	1	2	1	2	1	∞	1

Again, `21 = 3 * 7` fails at `3`, so this breaks self-similarity with the standard number line. However, a lot of relative primes fail at `1`, so there is a kind of analogy, if not exact similarity.

Sometimes, mathematicians are caught up in the idea of perfection, that they fail to recognize situations where there is some kind of similarity but not exact match. In Avatar Extensions you have to think about such situations, because there are no axioms that can tell you the "correct" answer.

There is one thing observed from these situations that always seems to map correctly: Base 6.

The base 6 is kind of like the ruler we use to measure similarity. While primes can be an indicator of similarity, they are not guaranteed to behave exactly the same across choices of bases. However, using base 6 one can immediately see whether the chosen base is orthogonal to 6 or not. Orthogonality in this sense means that `∞` occurs regularly in the sequence with 6 steps apart.

This is why the ISB conjecture is so important: If true, then one can tell something about the base by looking at multiples of 6. If false, then it is not possible to know these things about the base.

The sequence of some chosen base is simply called the "PSG Base" sequence. The PSG conjecture states that it contains `∞` of multiples of 6 in every base sequence. The ISB conjecture states that any `∞` in the base sequence must be a multiple of 6. Together, PSG + ISB means that one can tell something about the base from the base sequence, such as whether it is divisible by `6` or `2` or `3`, without knowing the particular choice of base. Hence, the base sequence is kind of like a finger-print of some local number line. Since base 6 is so important, it is a natural choice of base.