Rational Symmetry Function

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In this paper I introduce a framework for studying primes that are motivated by symmetries in rational numbers. I give axioms for rational symmetry functions and show that this is a unique and well defined strongly normalization function. I show associated fix-point sequence is square free. Finally, I show that this function can be used to describe equations relevant for prime research.

A rational symmetry function `f` has the following axioms:

```
f(0) = 0    f(1) = 1    f(x) = f(1/x)     f(x·y) = f(x)·f(y)

f : ℚ → ℕ          f{ℕ} . f{ℕ} = f{ℕ}      ava_idem(f)
```

Notice that `f` returns a natural number and is idempotent over natural numbers. Furthermore, it is avatar idemponent which means that for any irreducible representation `n`, `f(n) = n`.

For example:

```
f(x²) = 1                 when `x != 0`
```

Proof:

```
∵     f(x²) = 1
∴     f(x·x) = 1
∴     f(x)·f(x) = 1           using `f(x·y) = f(x)·f(y)`
∴     f(x)·f(1/x) = 1         using `f(x) = f(1/x)`
∴     f(x·1/x) = 1            using `f(x·y) = f(x)·f(y)`
∴     f(x/x) = 1
∴     f(1) = 1                using `x/x = 1` when `x != 0`
∴     1 = 1                   using `f(1) = 1`
∴     true
```

One can use this theorem to prove the following:

```
f(x²·y) = f(y)            when `x != 0`
```

Proof:

```
∵     f(x²·y) = f(y)
∴     f(x²)·f(y) = f(y)
∴     1·f(y) = f(y)           using `f(x²) = 1` when `x != 0`
∴     f(y) = f(y)
∴     true
```

Since `f` is idempotent over natural numbers, it can not map natural numbers to other natural numbers than given by minimal reduction. This follows from this proof:

```
∵     f(f(x)) = f(x)          f(x) = y
∴     f(y) = y                using `f(x) = y`
```

Avatar idempotency implies that the above proof using idempotency applies to all primes, because primes are irreducible representations by the given axioms, except `ava_idem(f)`:

```
f(2) = 2    f(3) = 3    f(5) = 5    f(7) = 7    f(11) = 11  …
```

From the definition of primes, it follows that any composite number with unique primes, which means all prime powers by factorization equals 1, are irreducible:

```
f(x) = x    <=>   uniq_primes(x)
```

This property is the most fundamental property for this framework of studying primes that arises from the rational symmetry function. I name it `rsf` to distinguish it from other contexts where `f` is used to describe some unknown function.

```
rsf(x) = x  <=>   uniq_primes(x)
```

With other words, the fix-points of `rsf` are natural numbers with unique primes.

The `rsf` function has a natural number sequence where for any `n`, `rsf(n) = n`:

0, 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35

Except the first number `0`, this sequence is given by A005117 in The On-line Encyclopedia of Integer Sequences (https://oeis.org/A005117).

Now, the reason the `rsf` function is interesting is because it allows equations using fix-points:

```
∵      rsf(n·(n+1)) = n·(n+1)
∴      rsf(n)·rsf(n+1) = n·(n+1)
```

There is no known solution to this equation where `rsf(n) != n`. Very often, this can simplify problems about primes that are composed together to produce new primes, such as primbixes:

```
∵      rsf(2·r·s·(2·r·s+1)) = 2·r·s·(2·r·s+1)          where `r, s` are primes
∴      rsf(2·r·s)·rsf(2·r·s+1) = 2·r·s·(2·r·s+1)
∴      rsf(2)·rfs(r)·rfs(s)·rsf(2·r·s+1) = 2·r·s·(2·r·s+1)
∴      2·r·s·rsf(2·r·s+1) = 2·r·s·(2·r·s+1)
∴      rsf(2·r·s+1) = 2·r·s+1
```

One research question is whether there is some solution to this equation that is not a fix-point.

Here, I did not use non-existence of the unknown solutions of `rsf(n) != n`, because I already know that `r, s` are primes and hence `rsf(r) = r` and `rsf(s) = s`. However, the approach is similar in both cases. It is about simplifying the equation. This way, the `rsf` function can describe lots of constraints at the same time, packing a lot of information into the same equation.

However, as useful as this sounds, this is far from solving open problems about primbixes. The first equation, `rsf(2·r·s·(2·r·s+1)) = 2·r·s·(2·r·s+1)` where `r, s` are primes, has no known solution without a fix-point and implies the second equation `rsf(2·r·s+1) = 2·r·s+1`. Naturally, the second equation describes a fix-point. So, from `r, s` being prime and the first equation one can infer that `2·r·s+1` contains only unique primes. However, if the first equation does not hold, then one can easily find a counter-example `r = 2, s = 11` where `2·r·s+1 = 2·2·11+1 = $3^2$·5`. The answer has not unique primes since the prime power of `3` is `2`.

In general, a statement `rsf(r·y) = r·y` can be reduced to `r(y) = y` when `r` has unique primes.

For primbix research, the hypothesis `rsf(2·r·s+1) = 2·r·s+1` implying that `r, s` are primes can be disproved by the counter-example `r = 2, s = 4`, since ` 2·r·s+1 =  2·2·4+1 = 17` which is a prime. There is no simple structure of primbixes, shown using the counter-examples in both direction.