Additional examples for QUINOPT

1 `AdditionalEx1.m`: Poincaré inequality

We compute the smallest constant ν for which the Poincaré inequality

$$\int_{-1}^{1} \nu u_{xx}^2 - u_x^2 \, dx \geq 0$$

holds for all functions u satisfying the Dirichlet boundary conditions $u(-1) = 0$, $u(1) = 0$. This problem can be solved analytically to find $\nu_{\rm opt} = 0.25\pi^{-2}$. We use the basic commands to set up and solve the problem using the default settings of QUINOPT. We also show how to override the default options to refine the relaxation of the integral inequality and improve the solution.

2 `AdditionalEx2.m`: Second order Poincaré-type inequality

We compute the smallest constant ν for which the inequality

$$\int_{-1}^{1} \nu u_{xx}^2 - u_x^2 \, dx \geq 0$$

holds for all functions u satisfying the boundary conditions $u(-1) = 0$, $u(1) = 0$ and $u_x(-1) - u_x(1) = 0$. This problem can be solved analytically to find $\nu_{\rm opt} = \pi^{-2}$. Similarly to `AdditionalEx1.m`, we use the basic commands to set up and solve the problem using the default settings of QUINOPT. We also show how to override the default options to refine the relaxation of the integral inequality and improve the solution.

3 `AdditionalEx3.m`: Energy dissipation bounds for 2D stress-driven shear flow

An upper bound on the energy dissipation coefficient for a 2D shear flow in a periodic box with period Γ driven by a non-dimensional surface stress G, can be found by solving the optimization problem

$$\begin{aligned}
&\min_{\eta,\hat{\phi}_1,\dots,\hat{\phi}_P} \quad 2\eta - 4\hat{\phi}_1 \\
\text{s.t.} \quad &\int_{-1}^{1} \left[\frac{16}{k_n^2}(u_{xx}^2 + v_{xx}^2) + 8(u_x^2 + v_x^2) + k_n^2(u^2+v^2) + \frac{8}{k_n}\phi_x(u_x v - u v_x) \right] dx \geq 0 \text{ for } n = 1, 2, \dots, n_{\max} \\
&\sum_{i=1}^{P} \hat{\phi}_i = \frac{G}{2}, \\
&\begin{bmatrix} \frac{1}{2} & & & & \hat{\phi}_1 \\ & \frac{3}{2} & & & \hat{\phi}_2 \\ & & \ddots & & \vdots \\ & & & \frac{2P-1}{2} & \hat{\phi}_P \\ \hat{\phi}_1 & \hat{\phi}_2 & \dots & \hat{\phi}_P & G\eta \end{bmatrix} \succeq 0.
\end{aligned}$$

where $k_n = 2\pi n/\Gamma$ and $\hat{\phi}_1, \dots \hat{\phi}_P$ are the Legendre coefficients of the derivative ϕ_x of $\phi(x)$, a polynomial of degree P to be determined. The integral constraint should hold for all functions u, v satisfying the boundary conditions

$$
\begin{gathered}
u(-1)=0, \quad u(1)=0, \quad u_x(-1)=0, \quad u_{xx}(1)=0, \\
v(-1)=0, \quad v(1)=0, \quad v_x(-1)=0, \quad v_{xx}(1)=0.
\end{gathered}
$$

We solve this optimisation problem when $G = 1000$, $P = 20$, $\Gamma = 2$ and $n_{\max} = 1$. In this example we introduce the use of polynomials defined in the Legendre basis using the `legpoly` class, and demonstrate how to use QUINOPT when the problem has constraints in addition to the integral inequality constraint.

4 `AdditionalEx4.m`: Energy dissipation bounds for 3D stress-driven shear flow

In this example we compute upper bounds on the energy dissipation for 3D stress-driven shear flows. The problem to be solved is similar to that in `Demo4.m`, with the integral inequality replaced by

$$
\int_{-1}^{1}\left[\frac{16}{k_n^2}v_{xx}^2 + 4u_x^2 + 8v_x^2 + k_n^2(u^2+v^2) + \frac{4}{k_n}\phi_x uv\right]dx \geq 0 \text{ for } n = 1, 2, ..., n_{\max}
$$

with boundary conditions

$$
\begin{gathered}
u(-1)=0, \quad u_x(1)=0, \\
v(-1)=0, \quad v(1)=0, \quad v_x(-1)=0, \quad v_{xx}(1)=0.
\end{gathered}
$$

We solve the problem with $n_{\max} = 3$ to illustrate how multiple integral inequalities can be specified.

5 `AdditionalEx5.m`: Energy stability of 2D Bénard-Marangoni convection

A sinusoidal perturbation with wave number $k > 0$ of the conduction state of the Bénard-Marangoni problem with nondimensional forcing Ma is stable if

$$
\int_{-1}^{1}\left[T_x^2 + k^2T^2 + \mathrm{Ma} f_k(x) T T(1)\right]dx \geq 0
$$

for all functions T satisfying the boundary conditions $T(-1) = 0$ and $T_x(1) = 0$, where the function f_k is given by

$$
f_k(x) = \frac{k\sinh k}{2(\sinh k\cosh k - k)}[kz\cosh(kz) - \sinh(kz) + (1-k\coth k)z\sinh(kz)].
$$

Note that the unknown boundary value $T(1)$ appears explicitly in the integrand of the inequality. In this example we compute an approximation to the maximum Ma for which a perturbation of given wave number k is stable. In particular, we show how to approximate problems with non-polynomial coefficients, and how to solve problems in which the boundary values of the dependent variables appear explicitly in the integral inequality.

6 `AdditionalEx6.m`: Stability of a 1D linear PDE

In this example, we determine the maximum value of the parameter k such that the trivial solution $u = 0$ of the linear PDE

$$
u_t = u_{xx} + (k - 24x + 24x^2)u, \quad u(0) = u(1) = 0,
$$

is stable. We do so by constructing a Lyapunov function of the form

$$
V(u) = \frac{1}{2}\int_0^1 p(x)u^2 dx
$$

where $p(x)$ is a polynomial to be determined. This requires choosing $p(x)$ such that

$$V(u) \geq \int_0^1 cu^2 dx,$$
$$\frac{dV}{dt} = \int_0^1 p(x) u u_t dx \leq 0,$$

for some $c > 0$. Since the problem is homogeneous in p, we may take $c = 1$ without loss of generality. In this example we consider two cases:

1. $p(x) = 1$. With this choice, the only parameter in the problem is k and the integral constraints are convex in k, so the optimal value k_{opt} can be determined using QUINOPT. It may be shown that the optimal value obtained with $p(x) = 1$ is sharp, meaning that the PDE can be shown to unstable for $k > k_{\text{opt}}$.

2. Nonconstant $p(x)$. In this case, the problem is nonconvex in the problem variables (k and the coefficients of p), so we cannot optimize k. Instead, we check that a nonconstant $p(x)$ can be found when $k = k_{\text{opt}}$. This is not the case if one applies the sum-of-squares method proposed by Valmorbida, Ahmadi & Papachristodoulou in *Semi-definite programming and functional inequalities for Distributed Parameter Systems*, 53rd IEEE Conference on Decision and Control, 2014.