Tensile Plasticity

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1 Introduction and the yield functions

Tensile, or Rankine, plasticity is designed to simulate a material that fails when the maximum principal stress exceeds the material's tensile strength. Its yield function is therefore

$$f = \sigma_{III} - T \ , \tag{1.1}$$

where σ_{III} is the maximum[1] principal stress (the largest eigenvalue of the stress tensor) and T is the tensile strength.

One yield function is sufficient because of the definition $\sigma_I \leq \sigma_{II} \leq \sigma_{III}$. For instance, if during the return-map process both σ_{II} and σ_{III} exceed T the corresponding admissible configuration is that both of them are equal to T. While one yield function is sufficient, it is convenient to use three yield functions in total:

$$\begin{aligned} f_0 &= \sigma_{III} - T \ , \\ f_1 &= \sigma_{II} - T \ , \\ f_2 &= \sigma_I - T \ . \end{aligned} \tag{1.2}$$

This is the version used by `TensileStressUpdate`.

The return-map algorithm first rotates σ from the physical frame to the principal-stress frame (where $\sigma = \mathrm{diag}(\sigma_I, \sigma_{II}, \sigma_{III})$). The rotation matrices used are assumed not to change during the return-map process: only σ_I, σ_{II} and σ_{III} change. Therefore, at the end of the return-map process these rotation matrices may be used to find the final stress in the physical frame.

The three yield functions of Eqn (1.2) are smoothed using the method encoded in `MultiParameterPlasticitySt`
An example is shown Figures 1.1 and 1.2. Figure 1.2 shows slices of the yield surface at three values of the mean stress, and the triangular-pyramid nature of the yield surface is evident. The slices are taken near the tip region only, to highlight: (1) the smoothing; (2) that the smoothing is unsymmetric.

The unsymmetric nature of the yield surface only occurs near the tip region where the smoothing mixes the three yield surfaces. For instance, the black line in Figure 1.2 is symmetric, while the blue and red lines are unsymmetric. The amount of asymmetry is small, but it is evident that the red curve is not concentric with the remainder of the curves shown in Figure 1.2. The order of the yield functions in Eqn (1.2) has been chosen so that the curves intersect the $\sigma_{III} = \sigma_{II}$ line at 90°, but not on the $\sigma_{II} = \sigma_I$ line near the pyramid's tip. The asymmetry does not affect MOOSE's convergence, and of course it is physically irrelevant (since there is no one "correct" smoothed yield surface).

[1] Often the maximum principal is denoted by σ_I, but the notation used in this document is motivated by the C++ code. The code uses the `dsymmetricEigenvalues` method of `RankTwoTensor` and this orders the eigenvalues from smallest to greatest.

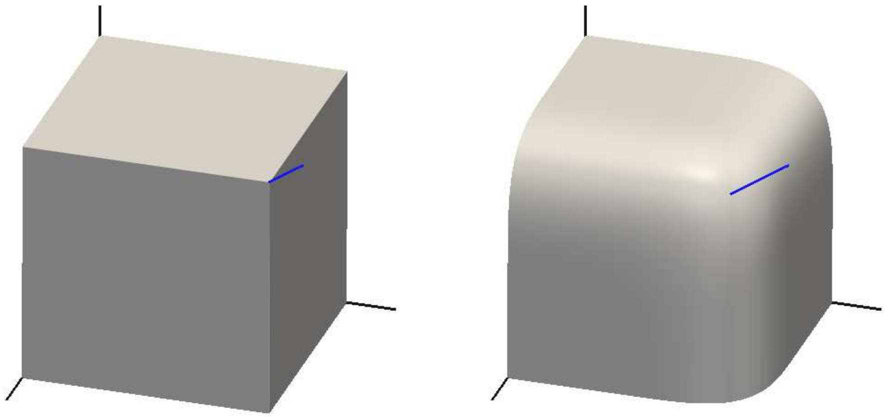

Figure 1.1: Left: the unsmoothed yield surface defined by Eqn (1.2) is a trianglar pyramid. Right: a smoothed version. The principal stress directions are shown with black lines, and the mean stress direction is shown with a blue line. The three planes extend infinitely (there is no base to the triangular pyramid) but these pictures show the tip region only.

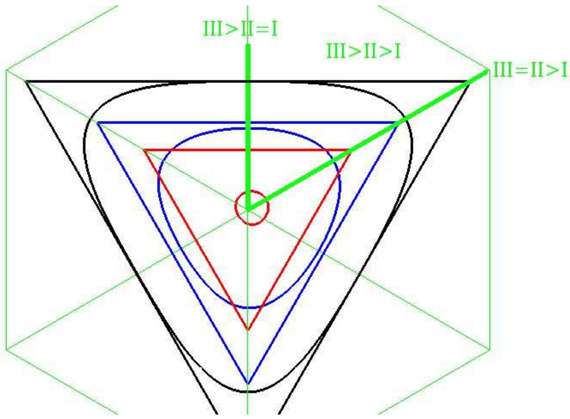

Figure 1.2: Slices of the unsmoothed and smoothed yield functions at various value of mean stress. In this example $T = 1$, the smoothing tolerance is 0.5, and the values of mean stress shown are: black, 0.7; blue, 0.8, red 0.85. An unusually large amount of smoothing has been used here to highlight various features: in real simulations users are encouraged to use smoothing approximately $T/10$. The whole octahedral plane is shown in this figure, but only one sextant is physical, which is indicated by the solid green lines.

2 Flow rules and hardening

This plasticity is associative: the flow potential is

$$g = f = max(f_0, f_1, f_2) + \text{smoothing} . \tag{2.1}$$

Here "smoothing" indicates the smoothing mentioned in Chapter 1.

The flow rules are

$$s_a = s_a^{\text{trial}} - \gamma E_{ab} \frac{\partial g}{\partial s_a} , \tag{2.2}$$

where $s_a = \{\sigma_I, \sigma_{II}, \sigma_{III}\}$ and

$$E_{ab} = \frac{\partial s_a}{\partial \sigma_{ij}} E_{ijkl} \frac{\partial s_b}{\partial \sigma_{kl}} . \tag{2.3}$$

In this equation E_{ijkl} is the elasticity tensor.

An assumption that is made in `MultiParameterPlasticityStressUpdate` is that E_{ab} is independent of the stress parameters, s_a and the internal variables. In this case[1]

$$\frac{\partial s_a}{\partial \sigma_{ij}} = v_i^a v_j^a , \tag{2.4}$$

where v^a is the eigenvector corresponding to the eigenvalue s^a (of the stress tensor) and there is no sum over a on the right-hand side. Recall that the eigenvectors are fixed during the return-map process, so the RHS is fixed, meaning that E_{ab} is indeed independent of the stress parameters. Also recall that the eigenvectors induce a rotation (to the principal-stress frame), so assuming that E_{ijkl} is isotropic

$$E_{ab} = E_{aabb} . \tag{2.5}$$

The assumption of isotropy is appropriate for this type of isotropic plasticity.

It is assumed that there is just one internal parameter, i, and that it is defined by

$$i = i_{\text{old}} + (\sigma_{III}^{\text{trial}} - \sigma_{III})/E_{22} , \tag{2.6}$$

during the return-map process. The tensile strength is a function of this single internal parameter

[1] Special precautions are taken when the eigenvalues are equal, as described in `RankTwoTensor`.

3 Technical discussions

3.1 Unknowns and the convergence criterion

The return-map problem involves solving the four equations: $f = 0$ (smoothed yield function should be zero) and the flow equations (2.2). The unknowns are the 3 stress parameters $s_a = \{\sigma_I, \sigma_{II}, \sigma_{III}\}$ and the plasticity multiplier γ. Actually, to make the units consistent the algorithm uses γE_{22} instead of simply γ. Convergence is deemed to be achieved when the sum of squares of the residuals of these 4 equations is less than a user-defined tolerance.

3.2 Iterative procedure and initial guesses

A Newton-Raphson process is used, along with a cubic line-search. The process may be initialised with the solution that is correct for perfect plasticity (no hardening) and no smoothing, if the user desires. Smoothing adds nonlinearities, so this initial guess will not always be the exact answer. For hardening, it is not always advantageous to initialise the Newton-Raphson process in this way, as the yield surfaces can move dramatically during the return process.

3.3 Substepping the strain increments

Because of the difficulties encountered during the Newton-Raphson process during rapidly hardening/softening moduli, it is possible to subdivide the applied strain increment, $\delta\varepsilon$, into smaller substeps, and do multiple return-map processes. The final returned configuration may then be dependent on the number of substeps. While this is simply illustrating the non-uniqueness of plasticity problems, in my experience it does adversely affect MOOSE's nonlinear convergence as some Residual calculations will take more substeps than other Residual calculations: in effect this is reducing the accuracy of the Jacobian.

3.4 The consistent tangent operator

MOOSE's Jacobian depends on the derivative

$$H_{ijkl} = \frac{\delta\sigma_{ij}}{\delta\varepsilon_{kl}} \; . \tag{3.1}$$

The quantity H is called the consistent tangent operator. For pure elasticity it is simply the elastic tensor, E, but it is more complicated for plasticity. Note that a small $\delta\varepsilon_{kl}$ simply changes $\delta\sigma^{\mathrm{trial}}$, so H is capturing the change of the returned stress ($\delta\sigma$) with respect to a change in the trial stress ($\delta\sigma^{\mathrm{trial}}$). In formulae:

$$H_{ijkl} = \frac{\delta\sigma_{ij}}{\delta\sigma^{\mathrm{trial}}_{mn}}\frac{\delta\sigma^{\mathrm{trial}}_{mn}}{\delta\varepsilon_{kl}} = \frac{\delta\sigma_{ij}}{\delta\sigma^{\mathrm{trial}}_{mn}}E_{mnkl} \ . \tag{3.2}$$

In the case at hand,

$$\sigma_{ij} = \sum_a R_{ia}s_aR^{\mathrm{T}}_{aj} \ . \tag{3.3}$$

In this formula σ_{ij} is the returned stress, s_a are the returned stress parameters (eigenvalues), and R is the rotation matrix, defined through the eigenvectors, v^a ($a = 1,2,3$) of the trial stress:

$$R_{ia} = v^a_i \ . \tag{3.4}$$

The three eigenvectors remain unchanged during the return-map process. However, of course they change under a change in σ^{trial}. The relevant formulae are

$$\frac{\delta s^{\mathrm{trial}}_a}{\delta\sigma^{\mathrm{trial}}_{kl}} = v^a_i v^a_j \ , \tag{3.5}$$

$$\frac{\delta v^a_i}{\delta\sigma^{\mathrm{trial}}_{kl}} = \sum_{b\neq a}\frac{v^b_i(v^b_k v^a_l + v^b_l v^a_k)}{2(s_a - s_b)} \ . \tag{3.6}$$

On the RHS of these equations there is no sum over a.

The final piece of information is

$$\frac{\delta s_b}{\delta s^{\mathrm{trial}}_a} \ . \tag{3.7}$$

`MultiParameterPlasticityStressUpdate` computes this after each Newton step, for any aribtrary plasticity model.

The nontrivial part to the consistent tangent operator is therefore

$$\frac{\delta\sigma_{ij}}{\delta\sigma^{\mathrm{trial}}_{mn}} = \sum_a \frac{\delta R_{ia}}{\delta\sigma^{\mathrm{trial}}_{mn}}s_aR^{\mathrm{T}}_{aj} + \sum_a\sum_b R_{ia}\frac{\delta s_a}{\delta s^{\mathrm{trial}}_b}\frac{\delta s^{\mathrm{trial}}_b}{\delta\sigma^{\mathrm{trial}}_{mn}}R^{\mathrm{T}}_{aj} + \sum_a R_{ia}s_a\frac{\delta R^{\mathrm{T}}_{aj}}{\delta\sigma^{\mathrm{trial}}_{mn}} \ . \tag{3.8}$$

All the components of this equation have been provided above.