# Geometric lower bounds for stochastic processing networks with limited connectivity

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# Outline

Stochastic processing networks

Flexibility metrics

Overview of results

Monotone transformations of networks

Proof sketches

Final remarks

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# Introduction

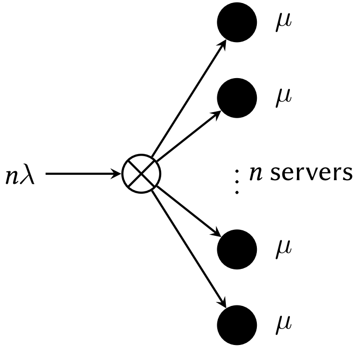
- Load balancing plays a central role in parallel-processing systems.
- Balance incoming tasks across distributed servers.
- A lot of attention in recent decades, see e.g. [van der Boor et al., 2022].

Main goal: complete resource pooling, i.e. the system performs efficiently.

Problem: we also need scalable policies.

# Supermarket model

[Vvedenskaya et al., 1996; Mitzenmacher, 2001]




$$\rho := \frac{\lambda}{\mu}$$

- Arrivals at rate $\lambda$, $n$ parallel servers.
- Tasks are allowed to run in any server.
- Tasks are queued at the servers.
- Exponential assumptions.

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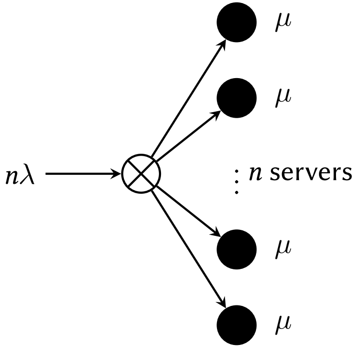


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## Join-the-shortest-queue (JSQ)

- Upon arrival, choose the shortest queue.
- Optimal policy. Achieves pooling for large $n$.
- Large communication overhead.

# Supermarket model

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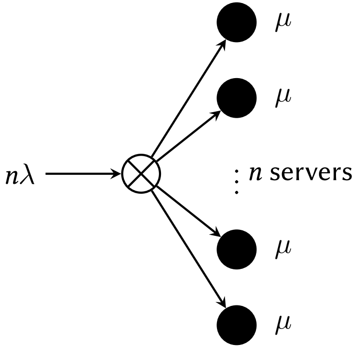


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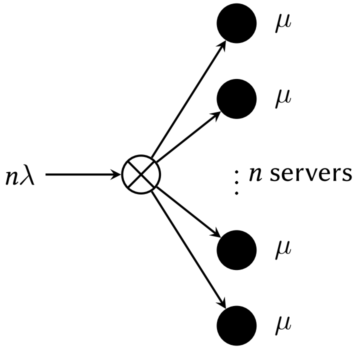
## Power-of-$d$ (PoD)

- Upon arrival, sample $d$ queues at random.
- Choose the shortest among them.
- Doubly exponential decay with minimal overhead.

# Supermarket model

[Vvedenskaya et al., 1996; Mitzenmacher, 2001]




$$\rho := \frac{\lambda}{\mu}$$

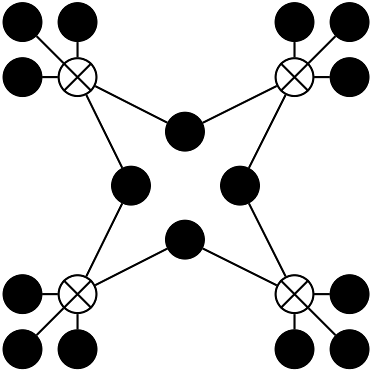
- Arrivals at rate $\lambda$, $n$ parallel servers.
- Tasks are allowed to run in any server.
- Tasks are queued at the servers.
- Exponential assumptions.

## Join-the-idle-queue (JIQ)

- Upon arrival, choose an idle queue (you'll find one, trust me)
- Minimal communication overhead.
- Good if $\rho/n$ away from 1.

# Networks are not that simple...

- Multiple entry points, types of tasks...
- Heterogeneous servers.
- Compatibility constraints: not every dispatcher is connected to every server.



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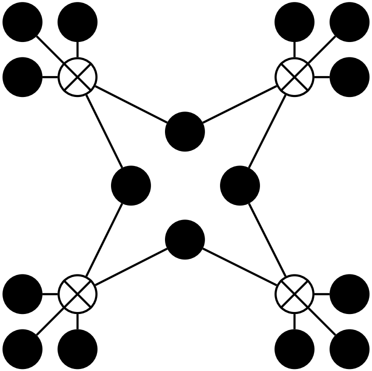


What's the right abstraction for this system?

# Stochastic processing networks

Consider a bipartite graph $G = (D, S, E)$:

- $d \in D$ is a dispatcher, receives tasks at rate $\lambda(d)$.
- $s \in S$ is a server. Executes tasks sequentially at rate $\mu(s)$ and maintains a queue.
- $(d, s) \in E$ encodes compatibility constraints.
- We assume JSQ is used...

## Definition

$\mathbf{X}(t, u)$, the number of tasks in server $u$ at time $t$ is the *load balancing process* associated with the bipartite graph $G = (D, S, E)$ and the rate functions $\lambda : D \to (0, \infty)$ and $\mu : S \to (0, \infty)$.

# Stability conditions

[Foss and Chernova, 1998, Theorem 2.5]

For $\mathbf{X}$ to be stable (ergodic), it is sufficient that:

$$\sum_{\mathcal{N}(d)\subset U} \lambda(d) < \sum_{u\in U} \mu(u) \quad \text{for all nonempty } U \subset S$$

where:

$$\mathcal{N}(d) := \{u \in S : (d,u) \in E\}$$

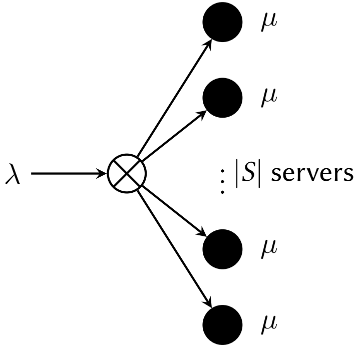
denotes the set of servers that are compatible with some dispatcher $d$.

In what follows we always assume stability, and define $X(u)$ to be the steady-state of the load balancing process.

# Simple processing network

## Definition

Consider a network $G = (D, S, E)$ with $D$ a singleton, $E = D \times S$, and constant $\lambda$ and $\mu$. We call the load balancing process of this network *simple* with load $\rho := \lambda/\mu$.




- The general definition contains the supermarket model using JSQ.
- Ergodic if $\rho < |S|$

Simple load balancing processes would be key to our proofs.

# Performance measure

## Definition (Queue occupancy measure)

Suppose that $\mathbf{X}$ is an ergodic load balancing process and let $X$ denote its stationary distribution. The steady-state occupancy is the random sequence:

$$q(i) := \frac{1}{|S|} \sum_{u \in S} \mathbf{1}_{\{X(u) \geqslant i\}} \quad \text{for all} \quad i \in \mathbb{N}.$$

- Lower values of $q$ imply better performance.
- $|S| \sum_i q(i) =$ total number of tasks, and thus delay by Little's law.

# Queue occupancy in simple networks

Consider a sequence of simple networks of growing size $|S| = n \to \infty$, and $\lambda^{(n)} = n\lambda$, then $\rho^{(n)} = \rho$. In the mean field limit:

- $q(i) = \rho^i$ under random routing (geometric decay).
- $q(i) = \rho^{\frac{d^i - 1}{d-1}}$ under PoD (double exponential decay).
- $q(1) = \rho$ and $q(i) = 0$ for $i > 1$ under JSQ (and JIQ).

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But the simple network is fully flexible...

Question: Can we have the same performance in the limit with less flexibility?

# Asymptotic results under partial connectivity

[Rutten and Mukherjee, 2024] and others

Consider a sequence of graphs $G^{(n)}$ where $n = |S|$, $|D| = M(n)$ and $\lambda^{(n)} \equiv \frac{\lambda n}{M(n)}$ (so the total arrival rate is $\lambda n$).

Introduce the regularity and diversity metrics:

$$\phi(G) = \max_u \left| \frac{|S|}{|D|} \sum_{d \in \mathcal{N}(u)} \frac{1}{\deg(d)} - 1 \right| \quad \text{and} \quad \gamma(G) = \frac{1}{|D|} \sum_d \frac{1}{\deg(d)}$$

- $\phi$ is a measure of (ir)regularity. How diverse are the degrees of the dispatchers.
- $\gamma$ is an (in)flexibility metric. When $\gamma \to 0$, the average degree (options) grow.

# Asymptotic results under partial connectivity

[Rutten and Mukherjee, 2024] and others

### Theorem

*If* $\phi(G^{(n)}) \to 0$ *and* $\gamma(G^{(n)}) \to 0$*, then the load balancing process associated to the bipartite graph under PoD is "close" to the solution for a fully connected bipartite graph.*

# Asymptotic results under partial connectivity

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*If* $\phi(G^{(n)}) \to 0$ *and* $\gamma(G^{(n)}) \to 0$, *then the load balancing process associated to the bipartite graph under PoD is "close" to the solution for a fully connected bipartite graph.*

- In plain terms, if the flexibility is large, and there are no dispatchers with few choices, then we recover the PoD behavior of the fully connected network.
- Similar results hold for other policies as well.
- Note $\gamma(G^{(n)}) \to 0$ implies that the average degree of dispatchers must go to $\infty$.

# In this talk...

- We look for converse results!
- In particular, we provide geometric lower bounds for the network behavior when connectivity is limited.
- We do so by introducing a novel bottleneck measure, as well as the average degree.
- We show that, unless these metrics diverge, there will always be a geometric tail, even under JSQ.

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# Bottleneck metric

Given a bipartite graph $G = (D, S, E)$, we define:

## Definition (Bottleneck metric)

$$\alpha_G := \frac{1}{|S|} \sum_{u \in S} \min \{\deg(d) : d \in \mathcal{N}(u)\}$$

## Interpretation:

- For a server $u$, $\min \{\deg(d) : d \in \mathcal{N}(u)\}$ is a measure of how important is this server for some nodes.
- Now pick a server at random, what is the average "importance".

# Bottleneck metric

## Further interpretation

- Assume that some dispatchers have only few options.
- Then the subset of servers that serve them are clearly a bottleneck for the network.
- Congestion will occur in these servers.
- If the size of this subset grows with $|S|$, then you're in trouble.

# Average degree metric

Given a bipartite graph $G = (D, S, E)$, we define:

## Definition (Average degree metric)

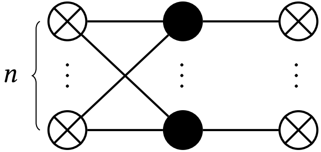
$$\beta_G := \frac{1}{|D|} \sum_{d \in D} \deg(d).$$

- Simpler than $\gamma(G)$, pick a dispatcher at random, what are they options on average.
- By Jensen's inequality:

$$\gamma(G) = \frac{1}{|D|} \sum_{d} \frac{1}{\deg(d)} \geqslant \frac{1}{\frac{1}{|D|} \sum_d \deg(d)} = \frac{1}{\beta_G},$$

so again $\gamma(G) \to 0$ implies $\beta_G \to \infty$.

Why both metrics are necessary for the results? Let's look at the following examples:



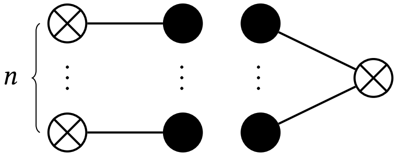

$$\alpha_{G_n^1} = 1,$$

$$\beta_{G_n^1} = \frac{n+1}{2}.$$

- In this case, $\alpha_G$ remains bounded, because at least half of the dispatchers crucially depend on a single server.
- However, the average degree of the network grows without bound.

# Examples

## A network with very flexible nodes hiding others




$$\alpha_{G_n^2} = \frac{n+1}{2},$$

$$\beta_{G_n^2} = \frac{2n}{n+1}.$$

- In this second case, half of the network is clearly disconnected, and thus has bounded $\min(d)$.
- However, the flexible dispatcher on the right makes the average $\alpha_G \to \infty$.
- Crucially in this case the average degree remains bounded.

# Outline

# A useful lemma

## Lemma

*Let $\mathbf{X}$ be a simple and ergodic load balancing process with load $\rho$. Then its steady-state occupancy satisfied:*

$$E\left[q(i)\right] \geqslant \frac{\left[r(\rho, |S|)\right]^i}{|S|} \quad \textit{for all} \quad i \in \mathbb{N},$$

*where:*

$$r(\rho, x) := \left(\frac{\rho}{x}\right)^x.$$

# A useful lemma

## Lemma

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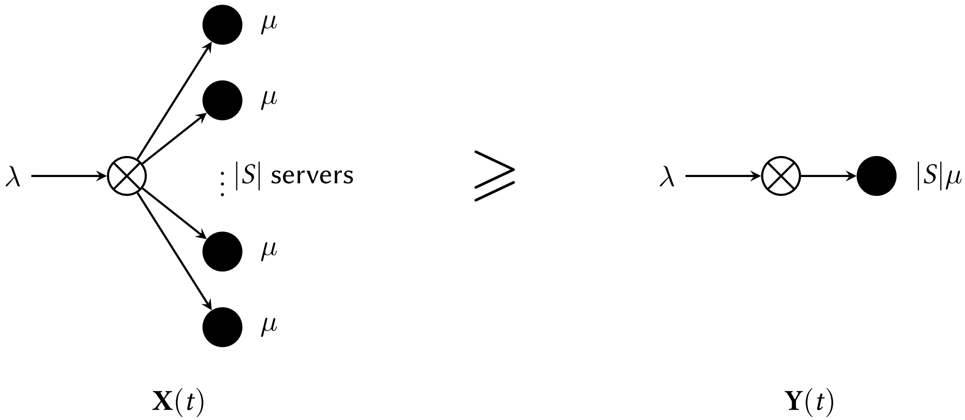
So every simple network has a geometric tail for finite $|S|$.

# Proof sketch I

## Coupling with a single server queue



# Proof sketch II

## Coupling with a single server queue

- By construction, if both systems start empty:

$$\sum_{u \in S} \mathbf{X}(t, u) \geqslant \mathbf{Y}(t) \quad \text{for all} t$$

- $\mathbf{X}$ ergodic $\Rightarrow \lambda < \mu|S|$, therefore $\mathbf{Y}$ ergodic.

- Then in steady-state:

$$P\left(\sum_{u \in S} X(u) \geqslant i\right) \geqslant P\left(Y \geqslant i\right) \quad \text{for all} \quad i \in \mathbb{N}.$$

# Proof sketch III

## Coupling with a single server queue

- The above inequality implies that

$$P\left(Y \geqslant |S|i\right) \leqslant P\left(\sum_{u\in S} X(u) \geqslant |S|i\right) \leqslant P\left(\bigcup_{u\in S}\{X(u) \geqslant i\}\right) \leqslant |S| P\left(X(v) \geqslant i\right).$$

where $v$ is any server.

- We conclude by:

$$P\left(X(u) \geqslant i\right) \geqslant \frac{1}{|S|} P\left(Y \geqslant |S|i\right) = \frac{1}{|S|}\left(\frac{\rho}{|S|}\right)^{|S|i} = \frac{\left[r(\rho,|S|)\right]^i}{|S|} \quad \text{for all } u \in S,$$

and summing over $u \in S$.

# Simple network flexibility

For a simple network, note that:

$$\alpha_G = \frac{1}{|S|} \sum_{u \in S} \min \{\deg(d) : d \in \mathcal{N}(u)\} = |S|,$$

$$\beta_G = \frac{1}{|D|} \sum_{d \in D} \deg(d) = |S|,$$

# Simple network flexibility

For a simple network, note that:

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So we can deduce from the Lemma that:

$$E[q(i)] \geqslant \frac{[r(\rho, |S|)]^i}{|S|} = \frac{[r(\rho, \alpha_G)]^i}{\alpha_G} = \frac{[r(\rho, \beta_G)]^i}{\beta_G}$$

where again:

$$r(\rho, x) := \left(\frac{\rho}{x}\right)^x.$$

# Bounds for general networks

The above geometric tail generalizes to any network:

## Theorem ($\alpha_G$ bound)

*Suppose that* $\mathbf{X}$ *is the load balancing process of an ergodic stochastic processing network with* $\lambda(u)$ *and* $\mu(s)$*.*
*Assume that:*

$$0 < \lambda_0 \leqslant \min_{d \in D} \lambda(d) \quad \text{and} \quad \max_{u \in S} \mu(u) \leqslant \mu_0 < \infty,$$

*and let* $\rho_0 := \frac{\lambda_0}{\mu_0}$*. If* $q$ *is the steady state occupancy then:*

$$E\left[q(i)\right] \geqslant \frac{\left[r(\rho_0, \alpha_G)\right]^i}{\alpha_G} \quad \text{for all} \quad i \geqslant \frac{1}{\rho_0},$$

# Bounds for general networks

## Theorem ($\beta_G$ bound)

*Suppose that* $\mathbf{X}$ *is the load balancing process of an ergodic stochastic processing network with* $\lambda(u)$ *and* $\mu(s)$.
*Assume that:*

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*and let* $\rho_0 := \frac{\lambda_0}{\mu_0}$. *If* $q$ *is the steady state occupancy then:*

$$E\left[q(i)\right] \geqslant C(\beta_G, \rho_0) \frac{\left[r(\rho_0, \beta_G + 1)\right]^i}{\beta_G + 1} \quad \textit{for all} \quad i \geqslant \frac{1}{\rho_0},$$

# Main result

## Theorem

*Consider a sequence of bipartite graphs* $G_n = (D_n, S_n, E_n)$ *with*

$$\alpha := \liminf_{n\to\infty} \alpha_{G_n} \quad \textit{and} \quad \beta := \liminf_{n\to\infty} \beta_{G_n}.$$

*In addition, fix rate functions* $\lambda_n : D_n \to (0,\infty)$ *and* $\mu_n : S_n \to (0,\infty)$ *such that*

$$\liminf_{n\to\infty} \min_{d\in D_n} \lambda_n(d) > \lambda_0 > 0 \quad \textit{and} \quad \limsup_{n\to\infty} \max_{u\in S_n} \mu_n(u) < \mu_0 < \infty.$$

*If the associated* $\mathbf{X}_n$ *are ergodic and* $\rho_0 := \lambda_0/\mu_0$*, then the steady-state occupancies satisfy:*

$$\liminf_{n\to\infty} E\left[q_n(i)\right] \geqslant \max\left\{\frac{\left[r(\rho_0,\alpha)\right]^i}{\alpha}, C(\beta,\rho_0)\frac{\left[r(\rho_0,\beta+1)\right]^i}{\beta+1}\right\} \quad \textit{for all} \quad i \geq \frac{1}{\rho_0}.$$

# Proof sketch

- By assumption, there exists $n_0 \geq 1$ such that

$$\min_{d \in D_n} \lambda_n(d) > \lambda_0 \quad \text{and} \quad \max_{u \in S_n} \mu_n(u) < \mu_0 \quad \text{for all} \quad n \geqslant n_0.$$

- As a result, the bound Theorems imply that, for $n \geqslant n_0$,

$$E\left[q_n(i)\right] \geqslant \max\left\{\frac{\left[r(\rho_0, \alpha_{G_n})\right]^i}{\alpha_{G_n}}, C(\beta_{G_n}, \rho_0)\frac{\left[r(\rho_0, \beta_{G_n}+1)\right]^i}{\beta_{G_n}+1}\right\} \quad \text{for all} \quad i \geqslant \frac{1}{\rho_0}.$$

- Since $x \mapsto \left[r(\rho_0, x)\right]^i / x$ is continuous in $(0, \infty)$ and $\min\{\alpha_{G_n}, \beta_{G_n}\} \geqslant 1$, we get

$$\liminf_{n \to \infty} E\left[q_n(i)\right] \geqslant \max\left\{\frac{\left[r(\rho_0, \alpha)\right]^i}{\alpha}, C(\beta, \rho_0)\frac{\left[r(\rho_0, \beta+1)\right]^i}{\beta+1}\right\} \quad \text{for all} \quad i \geqslant \frac{1}{\rho_0}.$$

# Unavoidable geometric tails

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- This gives a partial converse to the results in [Mukherjee et al., 2020; Rutten and Mukherjee, 2024, 2023; Budhiraja et al., 2019; Weng et al., 2020; Zhao et al., 2022; Zhao and Mukherjee, 2023].
- They prove that, under suitable connectivity assumptions, the mean-field limit behaves as if the graph were complete, and thus decays faster that geometric.
- All of their connectivity assumptions imply $\beta = \infty$.

# Unavoidable geometric tails

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- They prove that, under suitable connectivity assumptions, the mean-field limit behaves as if the graph were complete, and thus decays faster that geometric.
- All of their connectivity assumptions imply $\beta = \infty$.

The previous Theorem implies that such mean-field limits are not possible if $\beta < \infty$.

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# Proof technique

We now move into the details of the proofs of the bounds.

- Key idea: make a sequence of *monotone transformations* to the original network, that preserve order (i.e. each step has better performance)

- These networks are coupled to preserve the laws of the original network.

- In the end, we end up with a series of $|D|$ (coupled) simple networks, where we can apply the previous bounds (and thus, geometric tails).

- Since these networks have better performance, the original network must also have a geometric tail.

We consider the following transformations $G_1 \mapsto G_2$, $\lambda_1 \mapsto \lambda_2$, $\mu_1 \mapsto \mu_2$:

## Arrival rate decrease

The arrival rate of tasks is decreased for some dispatchers. Specifically:

$$\lambda_1(d) \geqslant \lambda_2(d) \quad \text{for all } d \in D_1$$

while $G_2 := G_1$ and $\mu_2 := \mu_1$.

## Service rate increase

The service rate of tasks is increased for some servers. Specifically:

$$\mu_1(u) \leqslant \mu_2(u) \quad \text{for all } u \in S_1$$

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It is clear that both transformations will produce a less congested network, i.e.:

$$P\left(X(u) \geqslant i\right) \geqslant P\left(X(u) \geqslant i\right) \quad \text{for all} \quad u \in S_1 \text{ and } i \in \mathbb{N}$$

in steady-state.

# Monotone transformations

## Edge simplification

- Our third transformation requires *coupling* some servers, so we need to keep track of these couplings.
- Consider a stochastic processing network $G = (D, S, E)$ with rates $\lambda$, $\mu$.
- Associate with $\mathbf{X}$ a partition $\mathcal{S}$ of $S$ such that all the servers in $U \in \mathcal{S}$ have the same *potential* departure process.
- Clearly, we must have $\mu(u) = \mu(v)$ if $u, v \in U$ and $U \in \mathcal{S}$.
- Initially, $\mathcal{S} = \{\{u\} : u \in S\}$.

Our third transformation is thus the following:

## Edge simplificatioon

A compatibility relation $(d, u) \in E_1$ is removed while a server $v \notin S_1$ and the edge $(d, v)$ are incorporated. Specifically,

$$D_2 := D_1, \quad S_2 := S_1 \cup \{v\} \quad \text{and} \quad E_2 := (E_1 \setminus \{(d, u)\}) \cup \{(d, v)\}.$$

The potential departure process of $v$ is the same as for $u$. Namely, suppose that $U$ is the element of the partition $\mathcal{S}_1$ such that $u \in U$. Then we let

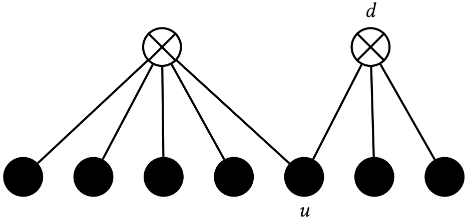
$$\mathcal{S}_2 := (\mathcal{S}_1 \setminus \{U\}) \cup \{U \cup \{v\}\} \quad \text{and} \quad \mu_2(v) := \mu_1(u).$$

Further, $\lambda_2(d) := \lambda_1(d)$ for all $d \in D_2$ and $\mu_2(w) := \mu_1(w)$ for all $w \in S_1$.

# Monotone Transformations

## Edge simplification example

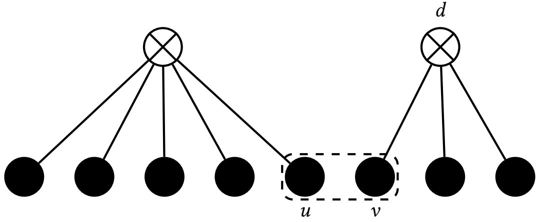
Assume that you start with the following network:



# Monotone Transformations

## Edge simplification example

And apply edge simplification:




- Edge simplification that removes the compatibility relation $(d, u)$ and incorporates server $v$ and the compatibility relation $(d, v)$.
- The servers $u$ and $v$ have the same potential departure process (coupling).

# Edge simplification improves performance

## Proposition

*Suppose now that $\mathbf{X}_2$ is obtained from $\mathbf{X}_1$ by edge simplification, removing $(d, u)$ and incorporating server $v$. Assume $\mathbf{X}_1$ is ergodic, then the following inequalities hold in steady-state:*

$$P\left(X_1(u) \geqslant i\right) \geqslant P\left(X_2(v) \geqslant i\right) \quad \textit{and} \quad P\left(X_1(w) \geqslant i\right) \geqslant P\left(X_2(w) \geqslant i\right)$$

*for all $w \in S_1$ and $i \in \mathbb{N}$.*

# Outline

# Proof of the $\alpha$ bound I

First let's recall the Theorem:

### Theorem ($\alpha_G$ bound)

*Suppose that* $\mathbf{X}$ *is the load balancing process of an ergodic stochastic processing network with* $\lambda(u)$ *and* $\mu(s)$*.*
*Assume that:*

$$0 < \lambda_0 \leqslant \min_{d \in D} \lambda(d) \quad \textit{and} \quad \max_{u \in S} \mu(u) \leqslant \mu_0 < \infty,$$

*and let* $\rho_0 := \frac{\lambda_0}{\mu_0}$*. If* $q$ *is the steady state occupancy then:*

$$E\left[q(i)\right] \geqslant \frac{\left[r(\rho_0, \alpha_G)\right]^i}{\alpha_G} \quad \textit{for all} \quad i \geqslant \frac{1}{\rho_0},$$

# Proof of the $\alpha$ bound II

Given a graph $G = (D, S, E)$ and rates $\lambda, \mu$:

- We perform an edge simplification at all the edges sequentially.
- From these transformations we get the bipartite graph $G_0 = (D_0, S_0, E_0)$ given by

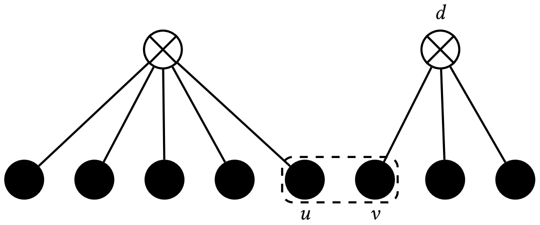
$$D_0 := D, \quad S_0 := \{u_d : (d, u) \in E\} \quad \text{and} \quad E_0 := \{(d, u_d) : (d, u) \in E\},$$

- The sets of coupled servers are given by:

$$\mathcal{S}_0 := \{\{u_d : u \in \mathcal{N}(d)\} : d \in D\}.$$

# Proof of the $\alpha$ bound III

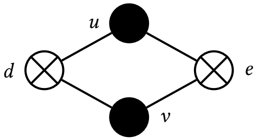
In other words we do this...

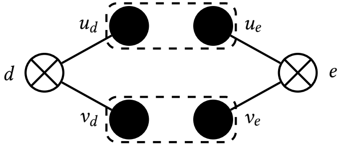



...until all dispatchers get a simple network of their own (with coupled departure processes).

# Proof of the $\alpha$ bound IV

Moreover, after decomposition, all coupled servers end up in different neighborhoods!



# Proof of the $\alpha$ bound V

- Now apply the arrival rate decrease and departure rate increase transformation...
- ...until all the dispatchers have the same arrival rate $\lambda_0$,
- ...and all servers have the same rate $\mu_0$.

Call $\mathbf{X}_0$ the resulting load balancing process, it follows that $\mathbf{X}_0$ is ergodic and in steady-state:

$$P\left(X(u) \geqslant i\right) \geqslant P\left(X_0(u_d) \geqslant i\right) \quad \text{for all} \quad (d,u) \in E \quad \text{and} \quad i \in \mathbb{N},$$

# Proof of the $\alpha$ bound VI

- Choose now weights for each edge $\theta : D \times S \to [0,1]$ such that

$$\sum_{d \in \mathcal{N}(u)} \theta(d,u) = 1 \quad \text{for all} \quad u \in S.$$

- Then we have the following for the steady-state occupancy:

$$E\left[q(i)\right] = \frac{1}{|S|} \sum_{u \in S} P\left(X(u) \geqslant i\right) \geqslant \frac{1}{|S|} \sum_{u \in S} \sum_{d \in \mathcal{N}(u)} \theta(d,u) P\left(X_0(u_d) \geqslant i\right) \text{ for all } i \in \mathbb{N}.$$

- Now for each dispatcher, we have a simple load-balancing process, with the same degree as in the original graph.

# Proof of the $\alpha$ bound VII

- We can apply the proposition to each component to get:

$$E\left[q(i)\right] \geqslant \frac{1}{|S|} \sum_{u \in S} \sum_{d \in \mathcal{N}(u)} \theta(d,u) \frac{\left[r\left(\rho_0, \deg(d)\right)\right]^i}{\deg(d)},$$

- Observe that the function:

$$f(x) := \frac{\left[r(\rho,x)\right]^k}{x} = \frac{1}{x}\left(\frac{\rho}{x}\right)^{kx} \quad \text{for all} \quad x > 0$$

is strictly decreasing and convex in $[\rho, \infty)$.

- Observe that:

$$\sum_{u \in S} \sum_{d \in \mathcal{N}(u)} \frac{\theta(d,u)}{|S|} = 1,$$

# Proof of the $\alpha$ bound VIII

- Therefore, using Jensen's inequality:

$$E\left[q(i)\right] \geqslant \frac{\left[r(\rho_0, \theta_G)\right]^i}{\theta_G} \quad \text{for all } i \geqslant \frac{1}{\rho_0} \quad \text{with } \theta_G := \frac{1}{|S|} \sum_{u \in S} \sum_{d \in \mathcal{N}(u)} \theta(d,u) \deg(d).$$

- Finally ecall that:

$$\alpha_G := \frac{1}{|S|} \sum_{u \in S} \min\left\{\deg(d) : d \in \mathcal{N}(u)\right\}$$

is just one possible $\theta$ (in fact is the one that achieves the $\sup$ over all $\theta$, and thus the better bound).

# Outline

# Final remarks

- Simple load-balancing networks always have geometric tails for finite number of servers.

- We defined two flexibility measures $\alpha_G$ and $\beta_G$ that describe dispatcher-serer connectivity.

- We showed that, in a sequence of growing size networks, unless $\alpha_G \to \infty$, $\beta_G \to \infty$, the geometric tails do not disappear in the limit.

- Future work: characterize the size of the bottleneck set, by defining:

$$\alpha_G(U) = \frac{1}{|U|} \sum_{u \in U} \min \{\deg(d) : d \in \mathcal{N}(u)\} \qquad U \subset S$$

and study its properties.

# Merci beaucoup!

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Arxiv version

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