Inverse kinematics for a Humanoid Robot : a mix between closed form and geometric solutions

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1 author:

Fabrice Noreils
birdconnected

Inverse kinematics for a Humanoid Robot : a mix between closed form and geometric solutions

Fabrice R. Noreils
fnoreils@gmail.com

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1 Introduction

In this paper we present a derivation of the forward kinematics (FK) and inverse kinematics (IK) of a humanoid robot with 32 degrees of freedom, specifically the ODOI platform. A picture of ODOI is shown in Fig. 1. The FK and IK are not solved for the entire 32 joints but instead divided into six parts:the two arms(seven joints each), the two legs (seven joints each), the torso (5 joints), and the head (2 joints).

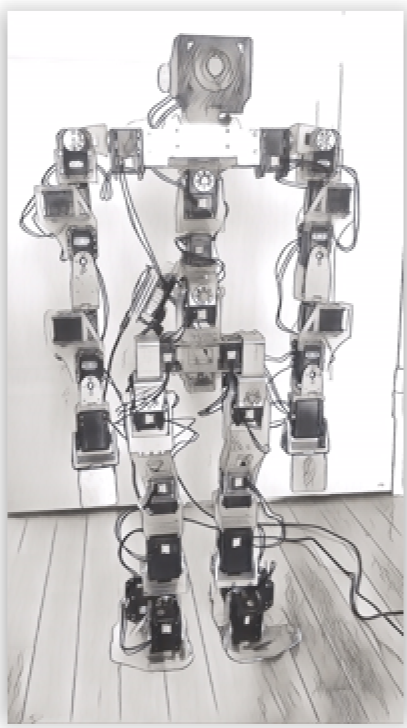


Figure 1: ODOI humanoid robot.

Forward kinematics (FK) and inverse kinematics (IK) are mandatory in any robot manipulator especially for a humanoid robot, because usually it is desirable to control the end-effector of the robot (e.g. the hand of one arm) in its workspace, not in joint space. In other words, tell the hand of the humanoid to go to point (x, y, z) not tell the joints to go to values θ_i. However, the joint values are what can be directly controlled, therefore, one needs to know how to convert from workspace to joint space and vice versa.

Pieper [6] outlined two conditions for finding a closed-form joint solution of a robot manipulator in which either three adjacent joint axes are parallel to one another or they intersect at a single point. A joint solution is said to be "closed-form" if the unknown joint angles can be solved for symbolically in terms of the arc-tangent function.

Although a robot manipulator may satisfy one of these two conditions for finding the closed-form joint solution, it is difficult to develop a consistent procedure for finding a closed-form joint solution for a humanoid robot and selecting one desirable solution from multiple solutions.

However Ali et. al. [5] [1] presented a closed-form solution for the inverse kinematics (IK) of the limbs of the HUBO2+ robot platform . They used a reverse decoupling mechanism method by viewing the kinematic chain of a limb in reverse order and decoupling the position and orientation. The authors then used the inverse transform method to compute eight possible solutions for each limb. The correct solution is selected based on joint limits and constraints. Later, O'Flaherty et al. [4] improved the method.

In this paper a mix approach is introduced: a closed-form solution is proposed for the arms and the torso based on the aforementioned works and a geometric solution is proposed for the legs.

2 Forward and Inverse kinematics for the Arm

2.1 Forward kinematics

The forward kinematics problem is that of solving for the end-effector orientation and position given the joint angles. This is easily solved using the robot geometry and coordinate frames, which are specified in the Denavit-Hartenberg (DH) parameters [3]. The general homogeneous transformation from one link to the next given the DH parameters is represented in matrix form as:

$$ {}^{i-1}\mathbf{T}_i = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (1) $$

where ${}^{i-1}\mathbf{T}_i$ is the transformation from coordinate frame $i-1$ to frame i.

Create a DH representation of a mechanical structure composed of motorized joints can be a tedious task and it is really helpful to have the possibility to visualize it to be sure that the parameters of each joint corresponds to what you have in mind. Please refer to appendix A for some examples of software available for that purpose.

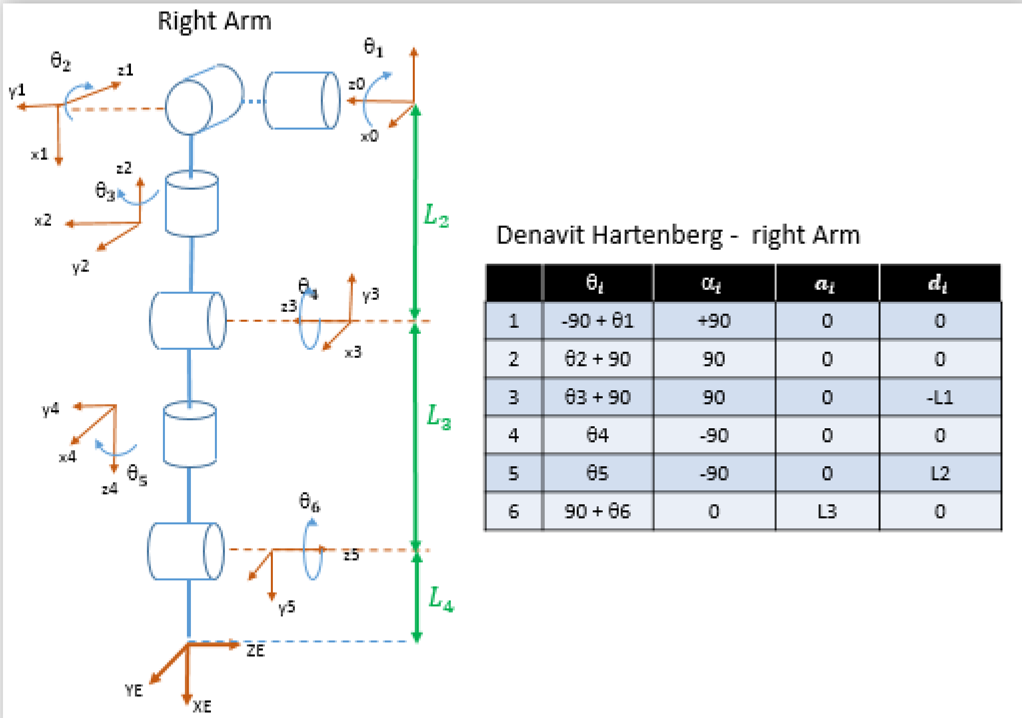
The six ${}^{i-1}\mathbf{T}_i$ transformation matrices for the right arm of the ODOI robot can be found on the basis of the coordinate systems established in Fig.2. These ${}^{i-1}\mathbf{T}_i$ matrices are listed below.

$$ {}^{0}\mathbf{T}_1 = \begin{pmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^{1}\mathbf{T}_2 = \begin{pmatrix} C_2 & 0 & -S_2 & 0 \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^{2}\mathbf{T}_3 = \begin{pmatrix} C_3 & 0 & S_3 & 0 \\ S_3 & 0 & -C_3 & 0 \\ 0 & 1 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

$$ {}^{3}\mathbf{T}_4 = \begin{pmatrix} C_4 & 0 & S_4 & 0 \\ S_4 & 0 & -C_4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^{4}\mathbf{T}_5 = \begin{pmatrix} C_5 & 0 & -S_5 & 0 \\ S_5 & 0 & C_5 & 0 \\ 0 & 1 & 0 & L_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^{5}\mathbf{T}_6 = \begin{pmatrix} C_6 & -S_6 & 0 & L_4C_6 \\ S_6 & C_6 & 0 & L_4S_6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

The ${}^{i}\mathbf{T}_{i-1}$ matrices that will be needed to compute the joints are listed below.

$$ {}^{1}\mathbf{T}_0 = \begin{pmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ S_1 & -C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^{2}\mathbf{T}_1 = \begin{pmatrix} C_2 & -S_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^{3}\mathbf{T}_2 = \begin{pmatrix} C_3 & S_3 & 0 & 0 \\ 0 & 0 & 0 & L_2 \\ S_3 & -C_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$



	θ_i	α_i	a_i	d_i
1	-90 + θ1	+90	0	0
2	θ2 + 90	90	0	0
3	θ3 + 90	90	0	-L1
4	θ4	-90	0	0
5	θ5	-90	0	L2
6	90 + θ6	0	L3	0

Figure 2: Denavit Hartenberg parameters for the right arm

$$^{4}\mathbf{T}_3 = \begin{pmatrix} C_4 & S_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ S_4 & -C_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad , {}^{5}\mathbf{T}_4 = \begin{pmatrix} C_5 & S_5 & 0 & 0 \\ 0 & 0 & 1 & -L_3 \\ S_5 & -C_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $S_i \equiv \sin\theta_i$ and $C_i \equiv \cos\theta_i$

In order to calculate the forward kinematics (FK), the six transformation matrices from each joint are pre-multiplied to obtain the position and orientation of the end-effector **relative to the shoulder**. We define the transformation from the shoulder to the hand as:

$$^{0}\mathbf{T}_6 = \prod_{i=1}^{6} {}^{i-1}\mathbf{T}_i = {}^{0}\mathbf{T}_1 {}^{1}\mathbf{T}_2 {}^{2}\mathbf{T}_3 {}^{3}\mathbf{T}_4 {}^{4}\mathbf{T}_5 {}^{5}\mathbf{T}_6 \tag{2}$$

3 Inverse kinematic Joint Position solution for the Arm

The inverse kinematics is the problem of solving for the joint angles given the end-effector orientation and position. This is a much harder problem because there are multiple solutions. As mentioned before, when solving the inverse kinematics of a manipulator, Pieper [6] indicates that a closed-form solution exists if three consecutive joint axes of the manipulator are parallel to one another, or intersect at a single point. The three shoulder joint axes on the ODOI intersect at a single point for the arms therefore a closed-form kinematic solution exists for the arms.

Let us write $^{0}\mathbf{T}_6$ as:

$$^{0}\mathbf{T}_6 = \begin{pmatrix} x_6 & y_6 & z_6 & p_6 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} n & s & a & p \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

where x_6, y_6, and z_6 are the unit vectors along the principal axes of the hand frame and p_6 is the position vector describing the location of the hand relative to the shoulder. These three unit vectors describe the orientation of the hand coordinate frame relative to the shoulder coordinate frame. The vectors n, s, a, and p represent the normal vector, sliding vector, approach vector, and position vector of the hand, respectively [2].

Using this knowledge, the arm can be viewed in reverse so that the last three joints make up the shoulder, thus the position and orientation of the shoulder frame can be described relative to the hand frame. This new position vector, $p^{'}$, is only a function of θ_4, θ_5 and θ_6, and thus decouples the arm into position and orientation components. The IK problem is solved in this reverse method by taking the inverse of both sides of Eq. (3). A detailed discussion (and argumentation) about this method can be found in [5].

$$
{}^{0}\mathbf{T}_6{}^{'} = \begin{pmatrix} n & s & a & p \\ 0 & 0 & 0 & 1 \end{pmatrix}^{'} = \begin{pmatrix} n^{'} & s^{'} & a^{'} & p^{'} \\ 0 & 0 & 0 & 1 \end{pmatrix} = {}^{6}\mathbf{T}_0 \tag{4}
$$

We can use the inverse transform method to solve for the last three joint angles by multiplying both sides of Eq. (4) by ${}^{5}\mathbf{T}_6$. This results in an equation where the left side of the equation is:

$$
\mathbf{G}_{5-arm}^{(LHS)} == {}^{5}\mathbf{T}_6 \begin{pmatrix} n^{'} & s^{'} & a^{'} & p^{'} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} C_6 n_x{}' - S_6 n_y^{'} & C_6 s_x^{'} - S_6 s_y^{'} & C_6 a_x^{'} - S_6 a_y^{'} & C_6 p_x^{'} - S_6 p_y^{'} + C_6 L_4 \\ S_6 n_x{}' + C_6 n_y^{'} & S_6 s_x^{'} + C_6 s_y^{'} & S_6 a_x^{'} + C_6 a_y^{'} & S_6 p_x^{'} + C_6 p_y^{'} + S_6 L_4 \\ n_z^{'} & s_z^{'} & a_z^{'} & p_z^{'} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}
$$

The right side of the equation is:

$$
\mathbf{G}_{5-arm}^{(RHS)} = {}^{5}\mathbf{T}_0 = {}^{5}\mathbf{T}_4{}^{4}\mathbf{T}_3{}^{3}\mathbf{T}_2{}^{2}\mathbf{T}_1{}^{1}\mathbf{T}_0 = {}^{5}\mathbf{T}_6 \begin{pmatrix} g_{511} & g_{521} & g_{531} & S_4 C_5 L_2 \\ g_{512} & g_{522} & g_{532} & -C_4 L_2 - L_3 \\ g_{513} & g_{523} & g_{533} & S_4 S_5 L_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}
$$

with:

$$
\begin{aligned}
g_{511} &= C_1(C_2(C_3C_4C_5 + S_3S_5) - S_2S_3C_5) - S_1(S_3C_4C_5 - C_3S_5) \\
g_{512} &= C_1(C_2C_3S_4 + S_2C_4) + S_1(-S_3S_4) \\
g_{513} &= C_1(C_2(C_3C_4S_5 - S_3C_5) - S_2S_4C_5) - S_1(S_3C_4S_5 + C_3C_5) \\
g_{521} &= S_1(C_2(C_3C_4C_5 + S_3S_5) - S_2S_4C_5) + C_1(S_3C_4C_5 - C_3S_5) \\
g_{522} &= S_1(C_2C_3S_4 + S_2C_4) + C_1(S_3S_4) \\
g_{523} &= S_1(C_2(C_3C_4S_5 - C_3C_5) - S_2S_4C_5) + C_1(S_3C_4S_5 + C_3S_5) \\
g_{531} &= S_2(C_3C_4C_5 + S_3S_5) + C_2S_4C_5 \\
g_{532} &= S_2C_3S_4 - C_2C_4 \\
g_{533} &= S_2(C_3C_4S_5 - S_3C_5) + C_2S_4S_5
\end{aligned}
$$

We can solve for the last three joint angles by equating the position terms (last column) of Eq. (5) and Eq. (6) to get:

$$
C_6(p_x^{'} + L_4) - S_6 p_y^{'} = S_4 C_5 L_2 \tag{7}
$$

$$
S_6(p_x^{'} + L_4) + C_6 p_y^{'} = -C_4 L_2 - L_3 \tag{8}
$$

$$
p_z^{'} = S_4 S_5 L_2 \tag{9}
$$

Let $p_x^{'} + L_4 = rC_\psi$ and $p_y^{'} = rS_\psi$, and substituting them into Eqs. (7) and (8), we get:

$$
\begin{aligned}
rC_{6\psi} &= S_4 C_5 L_2 && (10)\\
rS_{6\psi} &= -C_4 L_2 - L_3 && (11)\\
p_z^{'} &= S_4 S_5 L_2 && (12)
\end{aligned}
$$

where $r = \sqrt{(p_x^{'} + L_4)^2 + (p_y^{'})^2}$ and $\psi = atan2(p_y^{'}, p_x^{'} + L_4)$. By squaring Eqs. (10), (11) and (12) and adding them, we can obtain an equation for C_4:

$$
\begin{aligned}
C_4 &= \frac{(p_x^{'} + L_4)^2 + (p_y^{'})^2 + (p_z^{'})^2 - L_2^2 - L_3^2}{2L_2L_3} && (13)\\
\theta_4 &= atan2(\pm\sqrt{1 - C_4^2}, C_4) && (14)
\end{aligned}
$$

From Eq. (12), we can find S_5:

$$
S_5 = \frac{p_z^{'}}{S_5 L_2}; \qquad \theta_5 = atan2(S_5, \pm\sqrt{1 - S_5^2}) \tag{15}
$$

By dividing Eqs (10) and (11), we get:

$$
\begin{aligned}
\tan_{6\psi} &= \frac{S_{6\psi}}{C_{6\psi}} = \frac{-C_4 L_2 - L_3}{S_4 C_5 L_2} && (16)\\
\theta_6 &= \mathbf{wrapToPI(atan2(-C_4L_2 - L_3, S_4C_5L_2)} - \psi)^1 && (17)
\end{aligned}
$$

For the solution of joint angles θ_1, θ_2 and θ_3, multiplying both sides of Eq. (4) by ${}^3\mathbf{T}_4$, ${}^4\mathbf{T}_5$ and ${}^5\mathbf{T}_6$. This results in an equation where the left side of the equation is:

$$
\mathbf{G}_{4-arm}^{(LHS)} == {}^3\mathbf{T}_4{}^4\mathbf{T}_5{}^5\mathbf{T}_6 \begin{pmatrix} n^{'} & s^{'} & a^{'} & p^{'} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} g_{411} & g_{421} & g_{431} & g_{441} \\ g_{412} & g_{422} & g_{432} & g_{442} \\ g_{413} & g_{423} & g_{433} & g_{443} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{18}
$$

with:

$$
\begin{aligned}
g_{411} &= n_x^{'}(S_4S_6 + C_4C_5C_6) + n_y^{'}(S_4C_6 - C_4C_5S_6) + n_z^{'}C_4S_5\\
g_{412} &= n_x^{'}(S_4C_5C_6 - C_4S_6) - n_y^{'}(C_4C_6 + S_4C_56S_6) + n_z^{'}S_4S_5\\
g_{413} &= n_x^{'}S_5C_6 - n_y^{'}S_5S_6 - n_z^{'}C_5\\
g_{421} &= s_x^{'}(S_4S_6 + C_4C_5C_6) + s_y^{'}(S_4C_6 - C_4C_5S_6) + s_z^{'}C_4S_5\\
g_{422} &= s_x^{'}(S_4C_5C_6 - C_4S_6) - s_y^{'}(C_4C_6 + S_4C_56S_6) + s_z^{'}S_4S_5\\
g_{423} &= s_x^{'}S_5C_6 - s_y^{'}S_5S_6 - s_z^{'}C_5\\
g_{431} &= a_x^{'}(S_4S_6 + C_4C_5C_6) + a_y^{'}(S_4C_6 - C_4C_5S_6) + a_z^{'}C_4S_5\\
g_{432} &= a_x^{'}(S_4C_5C_6 - C_4S_6) - a_y^{'}(C_4C_6 + S_4C_56S_6) + a_z^{'}S_4S_5\\
g_{433} &= a_x^{'}S_5C_6 - a_y^{'}S_5S_6 - a_z^{'}C_5\\
g_{441} &= p_x^{'}(C_4C_5C_6 + S_4S_6) + p_y^{'}(C_6S_4 - C_4C_5S_6) + p_z^{'}(C_4S_5 + S_4L_3) + L_4(C_4C_6 + S_4S_6)\\
g_{442} &= p_x^{'}(S_4C_5C_6 - C_4S_6) - p_y^{'}(C_6C_4 + S_4C_5S_6) + p_z^{'}(S_4S_5 - C_4L_3) + L_4(S_4C_6 - C_4S_6)\\
g_{443} &= p_x^{'}S_5C_6 - p_y^{'}S_5S_6 - p_z^{'}C_5 + L_4S_5C_6
\end{aligned}
$$

and the right side of the equation is:

$$
\mathbf{G}_{4-arm}^{(RHS)} = {}^3\mathbf{T}_2{}^2\mathbf{T}_1{}^1\mathbf{T}_0 = {}^5\mathbf{T}_6 \begin{pmatrix} C_1C_2C_3 - S_1S_3 & C_1S_3 + S_1C_2C_3 & S_2C_3 & 0\\ -C_1S_2 & -S_1S_2 & C_2 & L_2\\ S_1C_3 + C_1C_2S_3 & S_1C_2S_3 - C_1C_3 & S_2S_3 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}
$$

By comparing the element (2,3)[2] of LHS and RHS of G_{4-arm}, we can obtain C_2:

$$
C_2 = g_{432} = a_x^{'}(S_4C_5C_6 - C_4S_6) - a_y^{'}(C_4C_6 + S_4C_56S_6) + a_z^{'}S_4S_5 \tag{20}
$$

$$
\theta_2 = atan2(\pm\sqrt{1 - C_2^2}, C_2) \tag{21}
$$

By comparing the elements (1,3) and (3,3) of LHS and RHS of G_{4-arm}, we obtain two equations. By dividing these two equations, we can determine the joint solution θ_3:

$$
\frac{S_2S_3}{S_2C_3} = \frac{g_{433}}{g_{431}} \tag{22}
$$

$$
\frac{S_3}{C_3} = \frac{a_x^{'}S_5C_6 - a_y^{'}S_5S_6 - a_z^{'}C_5}{a_x^{'}(S_4S_6 + C_4C_5C_6) + a_y^{'}(S_4C_6 - C_4C_5S_6) + a_z^{'}C_4S_5} \tag{23}
$$

$$
\theta_3 = atan2(g_{433}, g_{431}) \tag{24}
$$

by comparing the elements (2,1) and (2,2) of LHS and RHS of G_{4-arm}, we obtain two equations relating to θ_1. By dividing these two equations, we can determine the joint solution θ_1:

$$
\frac{-S_1S_2}{-C_1S_2} = \frac{g_{422}}{g_{412}} \tag{25}
$$

$$
\frac{-S_1}{-C_1} = \frac{s_x^{'}(S_4C_5C_6 - C_4S_6) - s_y^{'}(C_4C_6 + S_4C_56S_6) + s_z^{'}S_4S_5}{n_x^{'}(S_4C_5C_6 - C_4S_6) - n_y^{'}(C_4C_6 + S_4C_56S_6) + n_z^{'}S_4S_5} \tag{26}
$$

$$
\theta_1 = atan2(-g_{422}, -g_{412}) \tag{27}
$$

[2] The notation (row, column) is used in this document.

θ_1															
Input	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
Computed	140	...	175	...	-175	...	-110	...	-70	...	70	...	5	...	40
$sin(\theta_2) < 0$	-40	...	-5	...	5	...	70	...	110	...	175	...	-175	...	-140
Objective	-220	...	-185	...	-175	...	-110	...	-70	...	-5	...	5	...	40
θ_2															
Input	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
Computed	-40	...	-5	...	5	...	70	...	110	...	175	...	-175	...	-120
Objective	-40	...	-5	...	5	...	70	...	110	...	175	...	185	...	220
θ_3															
Input	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
Computed	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
$sin(\theta_2) < 0$	50	...	85	...	95	...	160	...	-160	...	-95	...	-85	...	-50
Objective	-130	...	-95	...	-85	...	-110	...	20	...	85	...	95	...	130

Table 1: (Right arm) Angle values of θ_1 and θ_3 depend on the sign of $\sin(\theta_2)$

At this point, we have a solution for all joints, but the value of θ_1 and θ_3 depend on the sign of $\sin(\theta_2)$. Table 1 shows three tables, one for each joint, namely θ_1 , θ_2 and θ_3. Each table contain four rows:

- Input: which is the angle provided as a parameter;
- Computed: which is the angle computed based on the above equations;
- $\sin(\theta_2) < 0$: which contains the angle computed when $\sin(\theta_2) < 0$;
- Objective: which is the expected angle value (remember that for θ_1 and θ_2 there is $\pm 90 + \theta$ in the DH parameters).

In some cases, one can see that in order to match the objective it is necessary to shift the computed angles by $\pm \pi or 2\pi$.

3.1 Singularities

Three singularities exist for the arm: Elbow, Shoulder and Shoulder-Elbow. Let us detail each of them.

3.1.1 Case 1: Elbow singularity or $\theta_4 = 0$

It means that θ_3 and θ_5 are collinear. In this case we need to re-calculate all the angles because the equations introduced in Section 3 were based on the value of C_4. It is necessary to go back to Eqs. (5) and (6) with $\theta_4 = 0$ or $C_4 = 1 and S_4 = 0$.

Using (3,1) from LHS and RHS of $\mathbf{G}_{5-arm}$ gives:

$$S_2(C_3C_5 + S_2S_5) = S_2(C_{3,5}) = C_6a_x^{'} - S_6a_y^{'}$$

Using (3,3) from LHS and RHS of $\mathbf{G}_{5-arm}$ gives:

$$S_2(C_3S_5 - S_2S_5) = -S_2(S_{3,5}) = a_z^{'}$$

Thus:

$$\frac{-S_{3,5}}{C_{3,5}} = \frac{a_z^{'}}{C_6a_x^{'} - S_6a_y^{'}} \tag{28}$$

$$\theta_{T(3-5)} = atan2(-a_z^{'}, C_6a_x^{'} - S_6a_y^{'} \tag{29}$$

Using (2,1) from LHS and RHS of $\mathbf{G}_{5-arm}$ gives:

$$C_1 S_2 C_4 = S_6 n_x{}' + C_6 n_y^{'}$$

Using (2,3) from LHS and RHS of $\mathbf{G}_{5-arm}$ gives:

$$-C_2 C_4 = S_6 a_x^{'} + C_6 a_y^{'}$$

thus:

$$\frac{S_2}{-C_2} = \frac{S_6 n_x{}' + C_6 n_y^{'}}{C_1 (S_6 a_x^{'} + C_6 a_y^{'})} \tag{30}$$

$$\theta_2 = atan2(S_6 n_x{}' + C_6 n_y^{'}, -C_1 (S_6 a_x^{'} + C_6 a_y^{'})) \tag{31}$$

Please note that in this case the value of θ_2 will be affected by the sign of C_1 and the values of both θ_1 and θ_T are also affected by the sign of S_2.

Using (1,4) from LHS and RHS of $\mathbf{G}_{5-arm}$ gives:

$$0 = C_6 p_x^{'} - S_6 p_y^{'} + C_6 L_4 \tag{32}$$

$$S_6 p_y^{'} = C_6 (p_x^{'} + L_4) \tag{33}$$

$$\frac{S_6}{C_6} = \frac{p_x^{'} + L_4}{p_y^{'}} \tag{34}$$

$$\theta_6 = atan2(p_x^{'} + L_4, p_y^{'}); \tag{35}$$

Using (2,1) and (2,2) from LHS and RHS of $\mathbf{G}_{5-arm}$ gives:

$$\theta_1 = atan2(S_6 s_x^{'} + C_6 s_y^{'}, S_6 n_x{}' + C_6 n_y^{'}) \tag{36}$$

θ_1															
Input	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
Computed	140	...	175	...	-175	...	-110	...	-70	...	70	...	5	...	40
$sin(\theta_2) < 0$	-40	...	-5	...	5	...	70	...	110	...	175	...	-175	...	-140
Objective	-220	...	-185	...	-175	...	-110	...	-70	...	-5	...	5	...	40
θ_2															
Input	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
Computed	-140	...	-175	...	5	...	70	...	110	...	175	...	-5	...	-40
$theta_1 < 0$	40	...	5	...	-175	...	-110	...	-70	...	-5	...	175	...	140
Objective	-40	...	-5	...	5	...	70	...	110	...	175	...	185	...	220
θ_T															
Computed	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
$sin(\theta_2) < 0$	50	...	85	...	95	...	160	...	-160	...	-95	...	-85	...	-50
Objective	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130

Table 2: (Right arm) $\theta_4 = 0$: Angles are shifted according to both the sign of θ_1 and the sign of $\sin(\theta_2)$

The value of θ_3 will be fixed to 0 and thus $\theta_5 = -\theta_T$. Again angles are shifted according to both the sign of θ_1 and the sign of $\sin(\theta_2)$ as it is shown on Table 2.

3.1.2 Case 2: Shoulder singularity or $\theta_2 = 0$

It means that θ_1 and θ_3 are collinear. In this case, it is necessary to go back to Eqs. (5) and (6) with $\theta_2 = 0$ or $C_2 = 1$ and $S_2 = 0$.

In this specific case, we need to re-calculate θ_1 and θ_3 only. As C_4 is not equal to 1 the values of θ_4, θ_5 and θ_6 with equations described in Section 3 still hold.

Using (2,1) from LHS and RHS of $\mathbf{G}_{5-arm}$ gives:

$$S_4(C_1C_3 - S_1S_3) = S_6{n_x}' + C_6n_y^{'}$$

Using (2,2) from LHS and RHS of $\mathbf{G}_{5-arm}$ gives:

$$S_4(S_1C_3 + C_1S_3) = S_6s_x^{'} + C_6s_y^{'}$$

thus:

$$\frac{S_{3+1}}{C_{3+1}} = \frac{S_6s_x^{'} + C_6s_y^{'}}{S_6{n_x}' + C_6n_y^{'}} \tag{37}$$

$$\theta_T = atan2(S_6s_x^{'} + C_6s_y^{'}, S_6{n_x}' + C_6n_y^{'}) \tag{38}$$

By fixing $\theta_1 = 0$ then $\theta_3 = \mathbf{wrapToPI}(\frac{\pi}{\mathbf{2}} - \theta_{\mathbf{T}})$. Moreover, in this case there is no need to shift angles.

3.1.3 Case 3 Elbow-shoulder singularity or θ_4 and $\theta_2 = 0$

It means that θ_1, θ_2 and θ_3 are collinear. Using (2,1) from LHS and RHS of $\mathbf{G}_{5-arm}$ gives:

$$C_1(C_3S_5 - S_3C_5) - S_1(S_3S_5 + C_3C_5) = n_z^{'}$$

Using (2,2) from LHS and RHS of $\mathbf{G}_{5-arm}$ gives:

$$S_1(C_3S_5 - S_3C_5) + S_1(S_3S_5 + C_3C_5) = s_z^{'}$$

thus:

$$\theta_{T(1+5-3)} = atan2(-n_z^{'}, s_z^{'}) \tag{39}$$

By fixing $\theta_1 = 0$ and $\theta_3 = 0$, then $\theta_5 = \mathbf{wrapToPI}(\theta_{\mathbf{T(1+5-3)}} + \frac{\pi}{\mathbf{2}})$

θ_6 is processed as in Section 3.1.1:

$$\theta_6 = atan2(p_x^{'} + L_4, p_y^{'}); \tag{40}$$

3.2 Decision table and Metric

The above equations give us 8 solutions, as it is shown in Fig. 3, each one define a **Pose**: $\theta_1 ... \theta_6$ that brings the arm to the right position and orientation of the end-effector.

The question is: how to select the right one? This is here that we can select the pose that will meet specific constraints such as torque limitation or the arm must move along a specific geometric constraint for instance.

For our purpose, I choose to limit the variation of the angles between the previous pose and the next one:

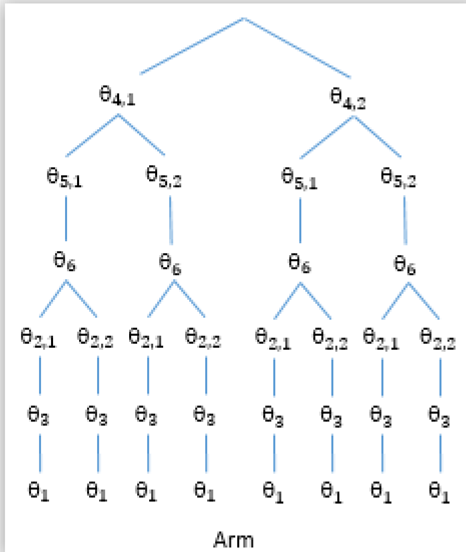


Figure 3: Decision Table for the right/left arm

$$\begin{aligned} \underset{j}{\text{minimize}} \quad & cost(j) \\ \text{subject to} \quad & cost(j) = \sum_{i=1}^{6} (\theta_i^{current} - \theta_i^j)^2. \end{aligned} \tag{41}$$

where $\theta_i^{current}$ represents the the current value of the i^{th} joint and θ_i^j is the computed value of the i^{th} joint in the j^{th} row in the decision table.

3.3 Algorithm

Once we have identified all the cases, the algorithm is the following:

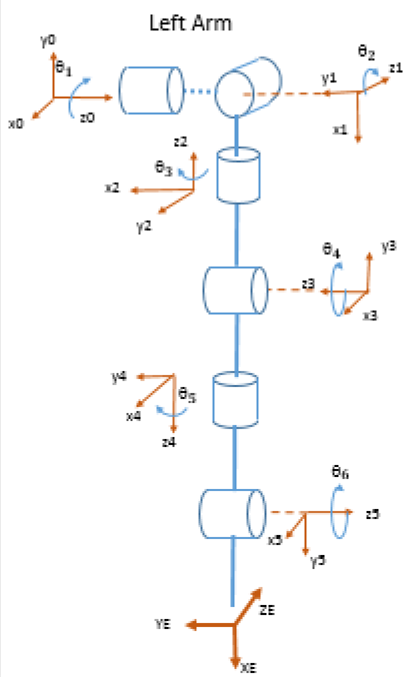
Algorithm 1 Calculate Inverse kinematic for the Arm

$C_4 = \frac{(p_x^{'}+L_4)^2+(p_y^{'})^2+(p_z^{'})^2-L_2^2-L_3^2}{2L_2L_3}$
if $C_4 = 1$ **then**
  Elbow Singularity
  $\theta_4 = 0$
  $\theta_{T(3-5)} = atan2(-a_z^{'}, C_6 a_x^{'} - S_6 a_y^{'}$
  $\theta_1 = atan2(S_6 s_x^{'} + C_6 s_y^{'}, S_6 n_x{'} + C_6 n_y^{'})$
  $\theta_3 = 0$
  $\theta_5 = -\theta_{T(3-5)}$
  $\theta_6 = atan2(p_x^{'} + L_4, p_y^{'})$
  $\theta_2 = atan2(S_6 n_x{'} + C_6 n_y^{'}, -C_1(S_6 a_x^{'} + C_6 a_y^{'})$
  if $\theta_2 = \pi$ **then**
    Elbow Shoulder" Singularity
    $\theta_{T(1+5-3)} = atan2(-n_z^{'}, s_z^{'})$
    $\theta_1 = 0$, $\theta_3 = 0$
    if right arm **then**
      $\theta_5 = \mathbf{wrapToPI}(\theta_{\mathbf{T(1+5-3)}} + \frac{\pi}{\mathbf{2}})$
    else
      $\theta_5 = \mathbf{wrapToPI}(\frac{\pi}{\mathbf{2}} - \theta_{\mathbf{T}})$
    end if
  end if
else
  (End of $C_4 = 1$) Shift angle according to Fig. **??** (right arm) or Fig. **??** (left arm)
else
  $\theta_4 = atan2(\pm\sqrt{1 - C_4^2}, C_4)$
  $\theta_6 = wrapToPI(atan2(-C_4 L_2 - L_3, S_4 C_5 L_2) - \psi)$
  $\theta_5 = atan2(S_5, \pm\sqrt{1 - S_5^2})$
  $\theta_2 = atan2(\pm\sqrt{1 - C_2^2}, C_2)$
  if $\theta_2 = \pi$ **then**
    Shoulder singularity
    $\theta_T = atan2(S_6 s_x^{'} + C_6 s_y^{'}, S_6 n_x{'} + C_6 n_y^{'})$
    $\theta_3 = -\mathbf{wrapToPI}(\frac{\pi}{\mathbf{2}} - \theta_{\mathbf{T}})$
    $\theta_1 = 0$
    if right arm **then**
      $\theta_3 = \mathbf{wrapToPI}(\frac{\pi}{\mathbf{2}} - \theta_{\mathbf{T}})$
    else
      $\theta_3 = -\mathbf{wrapToPI}(\frac{\pi}{\mathbf{2}} - \theta_{\mathbf{T}})$
    end if
  else
    $\theta_3 = atan2(-g_{422}, -g_{412})$
    $\theta_1 = atan2(-g_{422}, -g_{412})$
    shift angles according to Fig. **??**(right arm) or Fig.**??** (left arm)
    Create the decision table and apply the cost function to find the relevant **Pose**
  end if
end if

4 Forward and Inverse kinematic for the left arm

4.1 Inverse kinematic joint position for the left arm

The DH representation for the left arm is given on Fig. 4. All the equations to process $\theta_1 ... \theta_6$ associated with the right arms, which have been defined in Section 3 are still valid.



Denavit Hartenberg - left Arm

	θ_i	α_i	a_i	d_i
1	90 + θ1	-90	0	0
2	θ2 - 90	90	0	0
3	θ3 - 90	90	0	-L1
4	θ4	-90	0	0
5	θ5	-90	0	L2
6	90 + θ6	0	L3	0

Figure 4: DH representation for the left arm

However, the shifting of angles for the left arm is different from the right arm and the values are given on Table 3. This table is used to shift θ_1, θ_2 and θ_3 angles.

θ_1															
Input	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
Computed	140	...	175	...	-175	...	-110	...	-70	...	70	...	5	...	40
$sin(\theta_2) < 0$	-40	...	-5	...	5	...	70	...	110	...	175	...	-175	...	-140
Objective	-220	...	-185	...	-175	...	-110	...	-70	...	-5	...	5	...	40
θ_2															
Input	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
Computed	-140	...	-175	...	175	...	110	...	70	...	5	...	-5	...	-40
Objective	-40	...	-5	...	5	...	70	...	110	...	175	...	185	...	220
θ_3															
Input	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
Computed	50	...	85	...	95	...	160	...	-160	...	-95	...	-85	...	-50
$sin(\theta_2) < 0$	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
Objective	-130	...	-95	...	-85	...	-110	...	20	...	85	...	95	...	130

Table 3: (Left arm) Angle values of θ_1 and θ_3 depend on the sign of $\sin(\theta_2)$

4.2 Singularities

4.2.1 Case 1: Elbow singularity or $\theta_4 = 0$

Equations which have been defined for the right arm in Section 3.1.1 are valid for the left arm. However the shifting of angles, as expected, is different and is detailed on Table 4.

It is interesting to notice that the correct values for θ_T are computed when $sin(\theta_1) < 0$

θ_1															
Input	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
Computed	140	...	175	...	-175	...	-110	...	-70	...	70	...	5	...	40
$sin(\theta_2) < 0$	-40	...	-5	...	5	...	70	...	110	...	175	...	-175	...	-140
Objective	-220	...	-185	...	-175	...	-110	...	-70	...	-5	...	5	...	40
θ_2															
Input	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
Computed	-40	...	-5	...	175	...	110	...	70	...	5	...	-5	...	-40
$theta_1 < 0$	140	...	175	...	-5	...	-70	...	-110	...	-175	...	5	...	40
Objective	-40	...	-5	...	5	...	70	...	110	...	175	...	185	...	220
θ_T															
Computed	50	...	85	...	95	...	160	...	-160	...	-95	...	-85	...	-50
$sin(\theta_2) < 0$	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130
Objective	-130	...	-95	...	-85	...	-20	...	20	...	85	...	95	...	130

Table 4: (Left arm) $\theta_4 = 0$: Angles are shifted according to both the sign of θ_1 and the sign of $\sin(\theta_2)$

.

4.2.2 Case 2: Shoulder singularity or $\theta_2 = 0$

Equations which have been defined for the right arm in Section 3.1.2 are valid for the left arm. However the value for θ_3 is:

$$\theta_3 = -\mathbf{wrapToPI}(\frac{\pi}{\mathbf{2}} - \theta_\mathbf{T}) \tag{42}$$

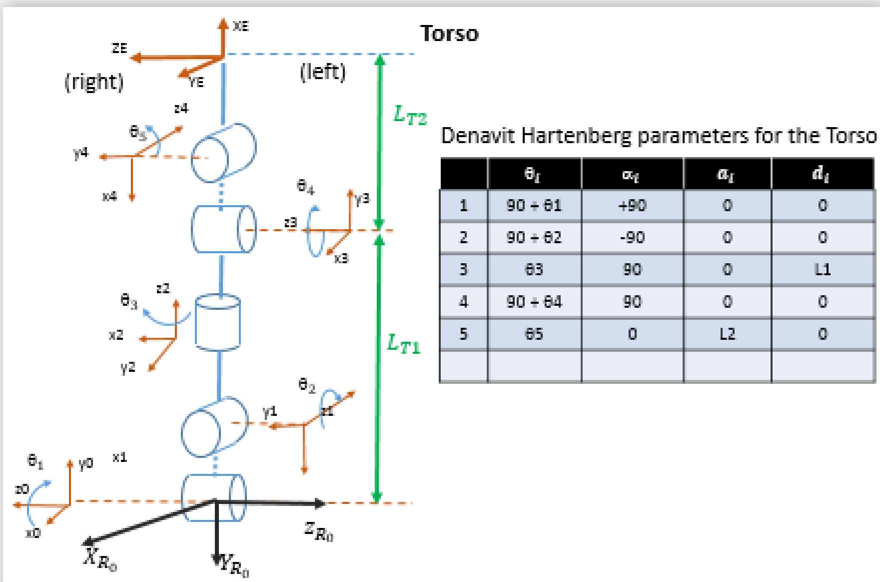
4.2.3 Case 3 Elbow-shoulder singularity or θ_4 and $\theta_2 = 0$

Equations which have been defined for the right arm in Section 3.1.3 are valid for the left arm. However the value for θ_5 is:

$$\theta_5 = \mathbf{wrapToPI}(\frac{\pi}{\mathbf{2}} - \theta_{\mathbf{T}}) \tag{43}$$

5 Forward and Inverse kinematic for the Torso

The five ${}^{i-1}\mathbf{T}_i$ transformation matrices for the Torso of the ODOI robot can be found on the basis of the coordinate systems established in Fig.5. These ${}^{i-1}\mathbf{T}_i$ matrices are listed below.



	θ_i	α_i	a_i	d_i
1	90 + θ1	+90	0	0
2	90 + θ2	-90	0	0
3	θ3	90	0	L1
4	90 + θ4	90	0	0
5	θ5	0	L2	0

Figure 5: Denavit Hartenberg parameters for the Torso

$$ {}^{0}\mathbf{T}_1 = \begin{pmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^{1}\mathbf{T}_2 = \begin{pmatrix} C_2 & 0 & -S_2 & 0 \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^{2}\mathbf{T}_3 = \begin{pmatrix} C_3 & 0 & S_3 & 0 \\ S_3 & 0 & -C_3 & 0 \\ 0 & 1 & 0 & L_{T1} \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

$$ {}^{3}\mathbf{T}_4 = \begin{pmatrix} C_4 & 0 & S_4 & 0 \\ S_4 & 0 & -C_4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^{4}\mathbf{T}_5 = \begin{pmatrix} C_5 & -S_5 & 0 & L_{T2}C_5 \\ S_5 & C_5 & 0 & L_{T2}S_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

The ${}^{i}\mathbf{T}_{i-1}$ matrices that will be needed to compute the joints are listed below.

$$ {}^{1}\mathbf{T}_0 = \begin{pmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ S_1 & -C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^{2}\mathbf{T}_1 = \begin{pmatrix} C_2 & -S_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^{3}\mathbf{T}_2 = \begin{pmatrix} C_3 & S_3 & 0 & 0 \\ 0 & 0 & 0 & -L_{T1} \\ S_3 & -C_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

$$
{}^{4}\mathbf{T}_{3} = \begin{pmatrix} C_4 & S_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ S_4 & -C_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

In order calculate the forward kinematics (FK), the five transformation matrices from each joint are pre-multiplied to obtain the position and orientation of the end-effector **relative to main frame**:

$$
{}^{0}\mathbf{T}_5 = \prod_{i=1}^{5} {}^{i-1}\mathbf{T}_i = {}^{0}\mathbf{T}_1 {}^{1}\mathbf{T}_2 {}^{2}\mathbf{T}_3 {}^{3}\mathbf{T}_4 {}^{4}\mathbf{T}_5 \tag{44}
$$

5.1 Inverse kinematics

The same method as for the right arm is applied. Let us write ${}^{0}\mathbf{T}_5$ as:

$$
{}^{0}\mathbf{T}_5 = \begin{pmatrix} x_5 & y_5 & z_5 & p_5 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} n & s & a & p \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{45}
$$

where x_5, y_5, and z_5 are the unit vectors along the principal axes of the hand frame and p_5 is the position vector describing the location of the hand relative to the shoulder. These three unit vectors describe the orientation of the hand coordinate frame relative to the shoulder coordinate frame. The vectors n, s, a, and p represent the normal vector, sliding vector, approach vector, and position vector of the hand, respectively [2].

$$
{}^{0}\mathbf{T}_5{}^{'} = \begin{pmatrix} n & s & a & p \\ 0 & 0 & 0 & 1 \end{pmatrix}^{'} = \begin{pmatrix} n^{'} & s^{'} & a^{'} & p^{'} \\ 0 & 0 & 0 & 1 \end{pmatrix} = {}^{5}\mathbf{T}_0 \tag{46}
$$

We use the inverse transform method to solve for the last two joint angles by multiplying both sides of Eq. (4) by ${}^{4}\mathbf{T}_5$. This results in an equation where the left side of the equation is:

$$
\mathbf{G}_{4-torso}^{(LHS)} == {}^{4}\mathbf{T}_5 \begin{pmatrix} n^{'} & s^{'} & a^{'} & p^{'} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} C_5 n_x{}' - S_5 n_y^{'} & C_5 s_x^{'} - S_5 s_y^{'} & C_5 a_x^{'} - S_5 a_y^{'} & C_5 p_x^{'} - S_5 p_y^{'} + C_5 L_{T2} \\ S_5 n_x{}' + C_5 n_y^{'} & S_5 s_x^{'} + C_5 s_y^{'} & S_5 a_x^{'} + C_5 a_y^{'} & S_5 p_x^{'} + C_5 p_y^{'} + S_5 L_{T2} \\ n_z^{'} & s_z^{'} & a_z^{'} & p_z^{'} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{47}
$$

The right side of the equation is:

$$
\mathbf{G}_{4-torso}^{(RHS)} = {}^{4}\mathbf{T}_0 = {}^{4}\mathbf{T}_3 {}^{3}\mathbf{T}_2 {}^{2}\mathbf{T}_1 {}^{1}\mathbf{T}_0 = \begin{pmatrix} g_{411} & g_{421} & g_{431} & -L_{T1} S_4 \\ g_{412} & g_{422} & g_{432} & 0 \\ g_{413} & g_{423} & g_{433} & L_{T1} C_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{48}
$$

with:

$g_{411} = C_1(C_2C3C4 - S_2S_4) - S_1S_3C_4$
$g_{412} = C_1C_2S_3 + S_1C_3$
$g_{413} = C_1(C_2C3C4 + S_2C_4) + S_1S_3S_4$
$g_{421} = S_1(C_2C3C4 - S_2S_4) + C_1S_3C_4$
$g_{422} = S_1C_2S_3 - C_1C_3$
$g_{433} = S_1(C_2C_3S_4 + S_2C_4) - C_1S_3S_4$
$g_{431} = S_2C_3C_4 + C_2S_4$
$g_{432} = S_2S_3$
$g_{433} = C_3$

We can solve for the last two (for the torso) joint angles by equating the position terms (last column) of Eq.(47) and Eq.(48) to get:

$$C_5(p_x^{'} + L_{T2}) - S_5p_y^{'} = -L_{T1}S_4 \tag{49}$$

$$p_z^{'} = L_{T1}C_4 \tag{50}$$

$$\tag{51}$$

From Eq. (50) we can find the value of C_4 and thus θ_4:

$$C_4 = \frac{(p_z^{'}}{L_{T1}} \tag{52}$$

$$\theta_4 = atan2(\pm\sqrt{1 - C_4^2}, C_4) \tag{53}$$

Let $p_x^{'} + L_{T2} = rC_\psi$ and $p_y^{'} = rS_\psi$, and substituting them into Eq. (49), we get:

$$rC_5C_\psi - rS_5S_\psi = -L_{T1}S_4 \tag{54}$$

$$C_{5\psi} = \frac{-L_{T1}S_4}{r} \tag{55}$$

$$\theta_5 = wrapToPI(atan2(\pm\sqrt{1 - \frac{-L_{T1}S_4}{r}^2}, \frac{-L_{T1}S_4}{r}) - \psi) \tag{56}$$

where $r = \sqrt{(p_x^{'} + L_{T2})^2 + (p_y^{'})^2}$ and $\psi = atan2(p_y^{'}, p_x^{'} + L_{T2})$

For the solution of joint angles θ_1, θ_2 and θ_3, multiplying both sides of Eq. (46) by ${}^3\mathbf{T}_4$ and ${}^4\mathbf{T}_5$. This results in an equation where the left side of the equation is:

$$\mathbf{G}_{3-torso}^{(LHS)} == {}^3\mathbf{T}_4{}^4\mathbf{T}_5 \begin{pmatrix} n^{'} & s^{'} & a^{'} & p^{'} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} g_{311} & g_{321} & g_{331} & g_{341} \\ g_{312} & g_{322} & g_{332} & g_{342} \\ g_{313} & g_{323} & g_{333} & g_{343} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{57}$$

with:

$g_{311} = n_x^{'} C_4 C_5 - n_y^{'} C_4 S_5 + n_z^{'} S_4$
$g_{312} = n_x^{'} S_4 C_5 - n_y^{'} S_4 S_5 - n_z^{'} C_4$
$g_{313} = n_x^{'} S_5 + n_y^{'} C_5$
$g_{321} = s_x^{'} C_5 C_5 - s_y^{'} C_4 S_5 + s_z^{'} S_4$
$g_{322} = s_x^{'} S_S C_5 - s_y^{'} S_4 S_5 - s_z^{'} C_4$
$g_{323} = s_x^{'} S_5 + s_y^{'} C_5$
$g_{331} = a_x^{'} C_4 C_5 - a_y^{'} C_4 S_5 + a_z^{'} S_4$
$g_{332} = a_x^{'} S_4 C_5 - a_y^{'} S_4 S_5 - a_z^{'} C_4$
$g_{333} = a_x^{'} S_5 + a_y^{'} C_5$
$g_{341} = p_x^{'} C_4 C_5 - p_y^{'} C_4 S_5 + p_z^{'} S_4 + L_{T2} C_4 C_5$
$g_{342} = p_x^{'} S_4 C_5 - p_y^{'} S_4 S_5 - p_z^{'} C_4 + L_{T2} S4 C_5$
$g_{343} = p_x^{'} S_5 + p_y^{'} C_5 + L_{T2} S_5$

and the right side of the equation is:

$$\mathbf{G}_{3-torso}^{(RHS)} = {}^3\mathbf{T}_2{}^2\mathbf{T}_1{}^1\mathbf{T}_0 = \begin{pmatrix} C_1C_2C_3 - S_1S_3 & C_1S_3 + S_1C_2C_3 & S_2C_3 & 0 \\ -C_1S_2 & -S_1S_2 & C_2 & -L_{T1} \\ S_1C_3 + C_1C_2S_3 & S_1C_2S_3 - C_1C_3 & S_2S_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{58}$$

By comparing the element (2,3) of LHS and RHS of $G_{3-torso}$, we can obtain C_2:

$$C_2 = g_{332} = a_x^{'} S_4 C_5 - a_y^{'} S_4 S_5 - a_z^{'} C_4 \tag{59}$$
$$\theta_2 = atan2(\pm\sqrt{1 - C_2^2}, C_2) \tag{60}$$

By comparing the elements (1,3) and (3,3) of LHS and RHS of $G_{3-torso}$, we obtain two equations. By dividing these two equations, we can determine the joint solution θ_3:

$$\frac{S_2S_3}{S_2C_3} = \frac{g_{333}}{g_{331}} \tag{61}$$
$$\frac{S_3}{C_3} = \frac{a_x^{'} S_5 + a_y^{'} C_5}{a_x^{'} C_4 C_5 - a_y^{'} C_4 S_5 + a_z^{'} S_4} \tag{62}$$
$$\theta_3 = atan2(g_{333}, g_{331}) \tag{63}$$

by comparing the elements (2,1) and (2,2) of LHS and RHS of $G_{3-torso}$, we obtain two equations relating to θ_1. By dividing these two equations, we can determine the joint solution θ_1:

$$\frac{-S_1S_2}{-C_1S_2} = \frac{g_{322}}{g_{312}} \tag{64}$$
$$\frac{-S_1}{-C_1} = \frac{s_x^{'} S_S C_5 - s_y^{'} S_4 S_5 - s_z^{'} C_4}{n_x^{'} S_4 C_5 - n_y^{'} S_4 S_5 - n_z^{'} C_4} \tag{65}$$
$$\theta_3 = atan2(-g_{322}, -g_{312}) \tag{66}$$

In the case of the Torso, there is no need to shift angle values and there is no singularity as well.

5.2 Decision table and Metrics

The above equations give us 8 solutions, as it is shown in Fig. 6, each one define a **Pose** that brings the torso to the right position and orientation .

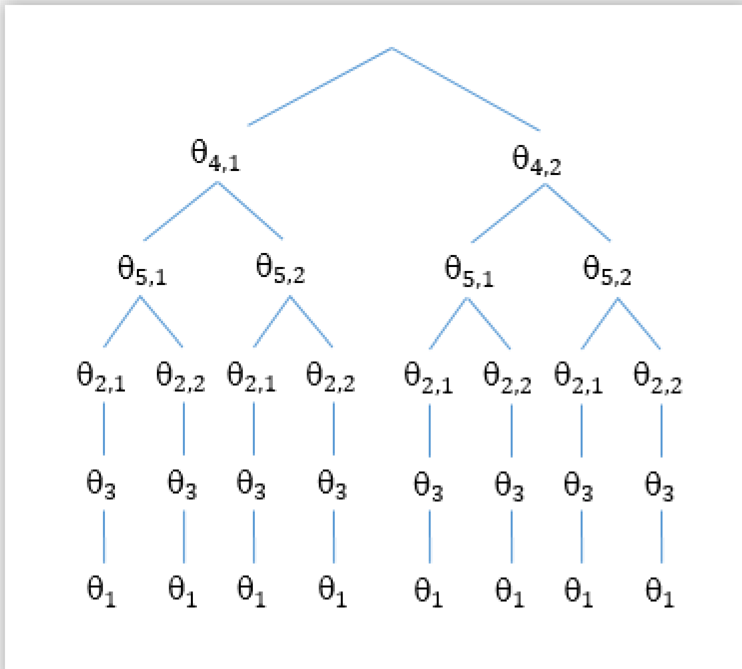


Figure 6: Decision Table for the torso

The question is: how to select the right one? This is here that we can select the pose that will meet specific constraints such as torque limitation or the arm must move along a specific geometric constraint for instance.

For our purpose, like for the arms, I choose to limit the variation of the angles between the previous pose and the next one:

$$\begin{aligned} & \underset{j}{\text{minimize}} & & cost(j) \\ & \text{subject to} & & cost(j) = \sum_{i=1}^{5}(\theta_i^{current} - \theta_i^j)^2. \end{aligned} \tag{67}$$

where $\theta_i^{current}$ represents the the current value of the i^{th} joint and θ_i^j is the computed value of the i^{th} joint in the j^{th} row in the decision table.

5.3 Algorithm

In order to compute all the joints the process is quite straightforward because on one hand there is no singularity and on the other hand not need to shift angle:

- Apply the above equations;

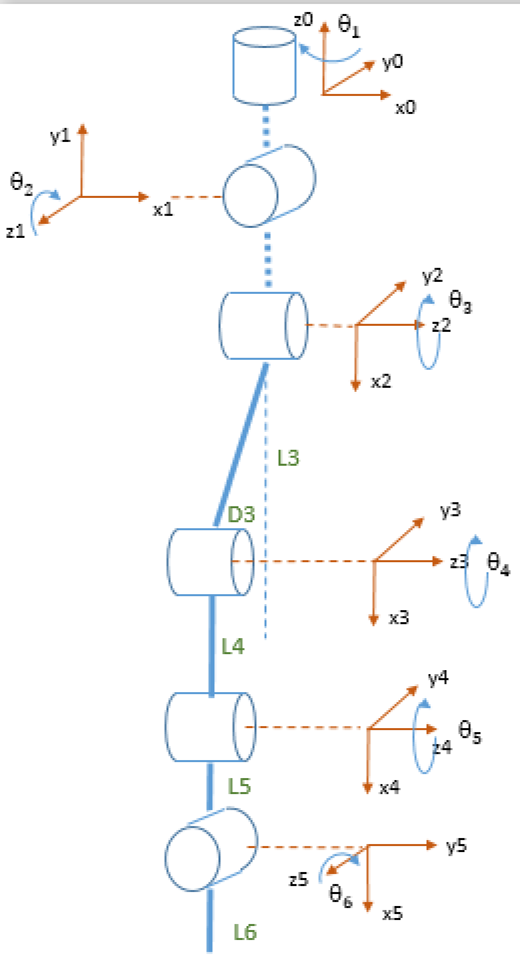
- Create the decision table;
- Apply a given metrics to choose the appropriate pose for the torso.

6 Inverse kinematic Joint Position solution for the legs

For information the DH representation of the right leg is pictured on Fig. 7. However, because of the fact that:

- The leg and thigh are not aligned (parameters L3 and D3);
- There is an additional segment (L5) between the rotation of the foot in the frontal plane (θ_5 and the rotation of the leg in the saggital plane (θ_4).

The methodology we applied successfully for the Arms and the Torso leads us to complex equations which cannot be solved. This is why for the leg a solution based on geometry is proposed.



Denavit Hartenberg - right leg

0	θ_i	α_i	a_i	d_i
1	θ1	+90	0	0
2	θ2 - 90	-90	0	0
3	θ3	0	L3	D3
4	θ4	0	L4	0
5	θ5	90	L5	0
6	θ6	0	L6	0

Figure 7: DH parameters for the right leg

6.1 let us start with a simple observation

Let us start with an observation based on the mechanical structure of the leg and the thigh. As mentioned before, the thigh is not aligned with the leg but it makes an angle which is referred to as α.

As it is shown on Fig. 8 if the thigh is rotating along the point **K**, the trajectory of the perpendicular projection (represented by a blue dotted line) along the knee point of the point **H**, which is referred to as $\mathbf{H}_P$, is making a circle, like **H**.

Moreover:

- The distance **D** between **H** and $\mathbf{H}_P$ is constant and equal to $L_T \sin(\alpha)$;
- The rotation angle θ between $\mathbf{H}(\theta_1)$ and $\mathbf{H}(\theta_2)$ is the same as the one between $\mathbf{H}_P(\theta_1)$ and $\mathbf{H}_P(\theta_2)$.

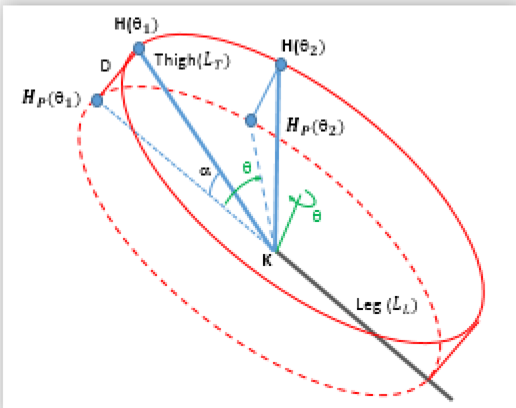


Figure 8: Decision Table for the torso

6.2 Construction of the tangent plane - resolve θ_6

The input is the position (given by **v** and orientation of the foot $(n_{\vec{foot}}, s_{\vec{foot}} and a_{\vec{foot}})$, as the foot is articulated (active articulation of the forefoot and passive articulation of the heel), we may have an angle between the ground and the foot.

In order to compute θ_6, which is referred to as the lateral angle of the ankle, we will compute the plane, referred to as $Plane_P$ which is (see Fig. 9.(A)):

- Tangent to a sphere whose radius is D (as defined in **??**) and the center is one extremity of the Pelvic;
- which is passing through the foot articulation F.

In order to compute this plane, we need to compute the points of tangency between the sphere and $Plane_P$ The math to compute these points is discussed in Appendix B.

Actually, two points are computed: T_1 and T_2. In order to select to correct one, the dot products $\vec{P} \bullet \vec{T}1$ and $\vec{P} \bullet \vec{T}2$ are computed and the selection will be based on the sign value which must be positive (see Fig. 9.(C)).

Once the point T is known, we are able to compute the vector normal to $Plane_P$, referred to as $\vec{n_P}$:

$$\vec{n_P} = \vec{FT} \times n_{\vec{foot}} \tag{68}$$

where $\times$ represents the cross product and $n_{\vec{foot}}$ represents the orientation of the foot with respect of the ground - remember that the foot is articulated!

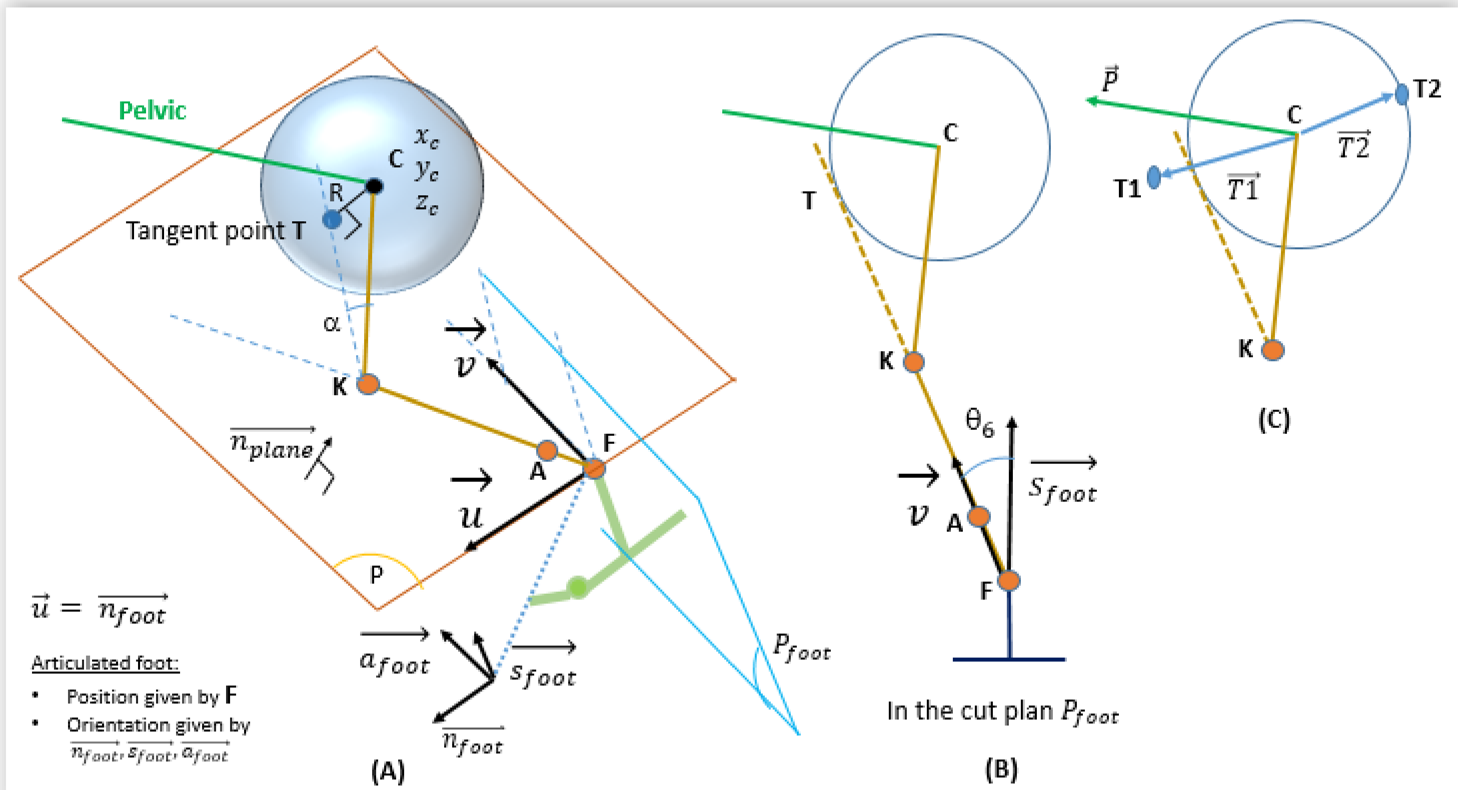


Figure 9: Computation of the tangent plane

Then we compute the intersection of $Plane_P$ and the plane which represents the orientation of the foot with respect to the vertical. The intersection is a line which is defined by its vector director $\vec{v}$.

Now we know all the information to compute θ_6 (see Fig. 9.(B)):

$$\theta_6 = arccos(\frac{\vec{\mathbf{v}} \bullet \mathbf{s}_{\vec{foot}}}{\|\mathbf{v}\| \|\mathbf{s_{foot}}\|}) \tag{69}$$

6.3 Projection in the tangent plane: resolution of ankle θ_5 and knee θ_4 angles

In order to compute θ_5 and θ_4, we compute the point (x_H, y_H) which is the projection of the vector $\vec{FT}$ on $Plane_P$ (see Fig. 10):

$$x_H = \vec{FT} \bullet \vec{u} y_H = \vec{FT} \bullet \vec{v} \tag{70}$$

where $\vec{u} = n_{\vec{foot}}$.

Then we compute the intersection of two circles, one with a radius R which is equal to the length of the leg L_L and one with radius Ra which is equal to $L_T \sin(\alpha)$ (refer to 6.1.

There are two solutions K_1 and K_2 and the sign of the cross product $\vec{OH} \times K\vec{1,}2 = x_k y_{K1,2} - y_k x_{K1,2}$ allow us to select the correct point, referred to as **K**.

Once **K** is known we are able to compute the ankle (θ_4) and knee (θ_5) saggital angles:

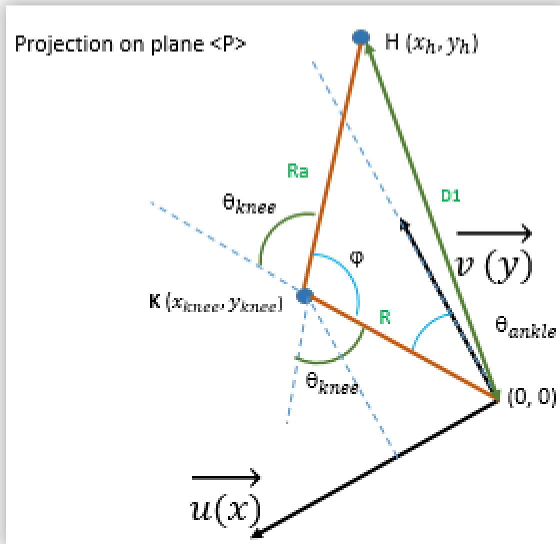


Figure 10: Computation of θ_5 and θ_4.

$$
\begin{aligned}
\phi &= \arccos(\frac{D1^2 - R^2 - Ra^2}{-2RRa}) \\
\theta_4 &= 180 - \phi \\
\theta_5 &= \arctan(\frac{x_K}{y_K})
\end{aligned}
\tag{71}
$$

This is here also where we take case of the singularity regarding the alignment of the leg and the thigh, i.e.$\theta_5 = 0$

6.4 And finally the last three angles θ_1, θ_2 and θ_3

Let us go back to the sphere and the tangent point. If we take a closer look at the sphere, as it is shown on Fig. 11, the tangent point allows us two to compute θ_1 and θ_2.

However, we need to know $\mathbf{T}$ in the Pelvic frame, which is computed as follow:

$$
\begin{pmatrix} T_x^p \\ T_y^p \\ T_z^p \\ 1 \end{pmatrix} = (\mathbf{T}_{pelvic})^{-1} \begin{pmatrix} T_x \\ T_y \\ T_z \\ 1 \end{pmatrix}
$$

Where $\mathbf{T}_{pelvic}$ is the transformation matrix associated with the Pelvic. Knowing $\mathbf{T}$ in the pelvic frame, θ_1 and θ_2 are computed as follow:

$$
\begin{aligned}
\theta_1 &= arctan(\frac{T_x}{T_z}) \\
\theta_2 &= arcsin(\frac{T_y}{r}) \\
r &= \sqrt{{T_x}^2 + {T_y}^2 + {T_z}^2}
\end{aligned}
\tag{72}
$$

Once θ_1 and θ_2 are known, it is possible to get θ_3. We have all the information to set the Thigh at a position where $\theta_3 = 0$. This is done by applying ROT_x and ROT_y to the Pelvic transformation:

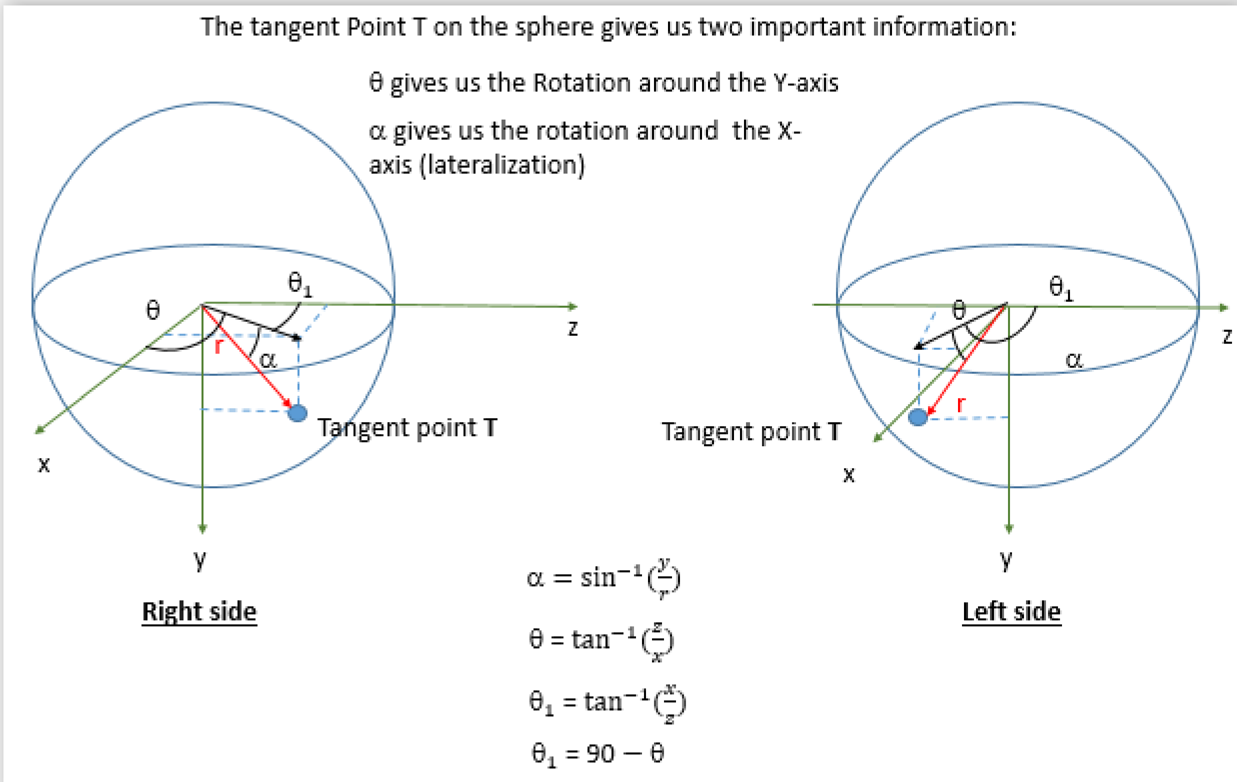


Figure 11: Computation of θ_1 and θ_2.

$$\mathbf{T}_{thigh}(\theta_3 = 0) = \mathbf{T}_{pelvic} ROT_y(\theta_2) ROT_x(\theta_1)$$

By providing this transformation to the thigh, we are able to know the position of the knee $\mathbf{K_0}$ when $\theta_3 = 0$ (and $\mathbf{K_0}$ belongs to $Plane_P$).

$$\theta_6 = \arccos(\frac{\vec{\mathbf{CK}} \bullet \vec{\mathbf{CK}}_0}{\|\mathbf{CK}\|\|\mathbf{CK_0}\|}) \tag{73}$$

7 Put things together

In this Section, we will bring together the torso and the arms. Transformations related to the torso will be represented by a $\mathbf{T}$ and transformations related to the arms with $\mathbf{A}$

We define two new "end points" for the torso, the right shoulder (rs) and the left shoulder (ls), these two end points are computed based on the transformations below:

$$^{5}\mathbf{T}_{RS} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -L_{T3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , ^{5}\mathbf{T}_{LS} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_{T3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The right shoulder is defined by:

$$^{0}\mathbf{T}_{RS} = {}^{0}\mathbf{T}_1 {}^{1}\mathbf{T}_2 {}^{2}\mathbf{T}_3 {}^{3}\mathbf{T}_4 {}^{4}\mathbf{T}_5 {}^{5}\mathbf{T}_{RS} \tag{74}$$

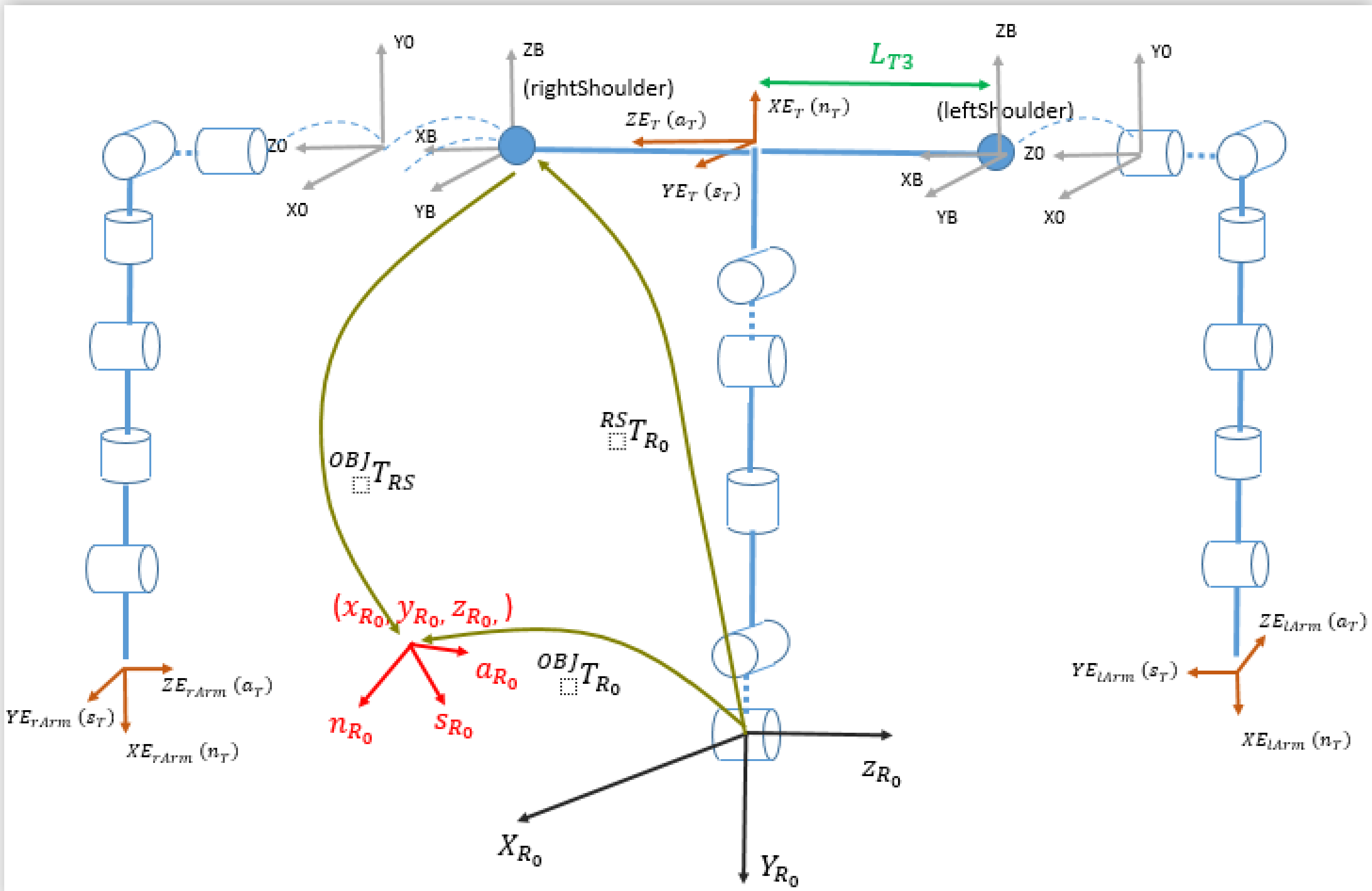


Figure 12: Decision Table for the torso

and the left shoulder by:

$$^{0}\mathbf{T}_{LS} = {}^{0}\mathbf{T}_1{}^{1}\mathbf{T}_2{}^{2}\mathbf{T}_3{}^{3}\mathbf{T}_4{}^{4}\mathbf{T}_5{}^{5}\mathbf{T}_{LS} \tag{75}$$

In order to connect the arms, two other matrices are required: ${}^{S}\mathbf{T}_B$ which rotates the frame (XE_T, YE_T, ZE_T) related to Torso and ${}^{B}\mathbf{T}_{0_{arm}}$which rotates the frame (X_B, Y_B, Z_B) to match the root frame (X_0, Y_0, Z_0) or the arm.

$${}^{RS}\mathbf{T}_B = {}^{LS}\mathbf{T}_B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , {}^{B}\mathbf{T}_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Finally, we defined two transformations :

$$\begin{aligned} {}^{0}\mathbf{A}_{0_{rarm}} &= {}^{0_T}\mathbf{T}_{RS}{}^{RS}\mathbf{T}_B{}^{B}\mathbf{T}_{0_{arm}} \\ {}^{0}\mathbf{A}_{0_{larm}} &= {}^{0_T}\mathbf{T}_{LS}{}^{LS}\mathbf{T}_B{}^{B}\mathbf{T}_{0_{arm}} \end{aligned} \tag{76}$$

The arms will then be connected to the torso by pre-multiplying ${}^{0}\mathbf{A}_{6_{rarm}}$ (as defined by Eq.2) by ${}^{0}\mathbf{T}_{0_{rarm}}$ and ${}^{0}\mathbf{A}_{6_{larm}}$ by ${}^{0}\mathbf{T}_{0_{larm}}$.

The complete 3D representation of the robot is displayed on Fig. 13

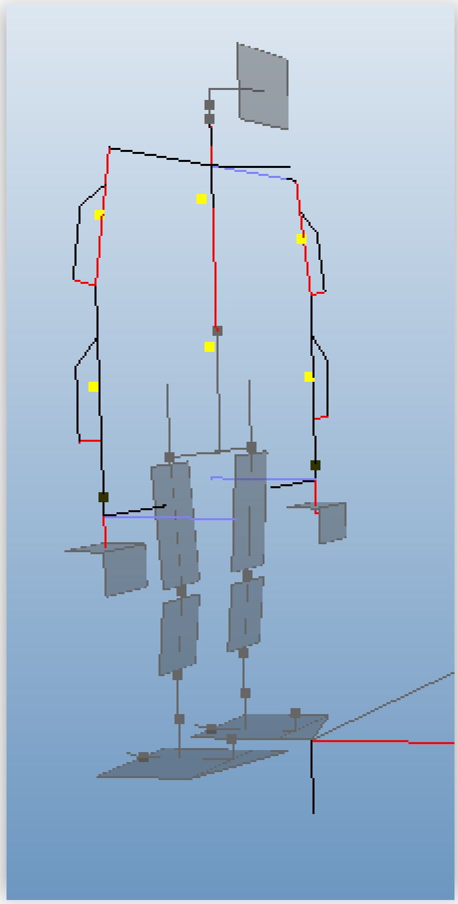


Figure 13: 3D representation of the robot once all the "pieces" are assembled

The Inverse kinematics equations work if the objective is defined in the shoulder frame. As the objective is defined in the main frame, we need to write the equation to compute it in the right/left shoulder frame. If we look at Fig. 12, we can see in red color an objective defined by its coordinates ($X_{R_0}, Y_{R_0}, Z_{R_0}$ and its orientation ($\vec{n}_{R_0}, \vec{s}_{R_0}, \vec{a}_{R_0}$) in the main frame. If we consider the transformations which link the objective in the main frame to the right shoulder we have:

$$^{Obj}\mathbf{T}_{R_0} = {}^{Obj}\mathbf{T}_{RS}{}^{RS}\mathbf{T}_{R_0} \tag{77}$$

Therefore the transformation which brings the objective in the right shoulder frame is:

$$\begin{aligned} ^{Obj}\mathbf{T}_{RS} &= (^{Obj}\mathbf{T}_{R_0})^{-1}{}^{RS}\mathbf{T}_{R_0} \\ ^{Obj}\mathbf{T}_{RS} &= (^{0}\mathbf{T}_{RS}{}^{rS}\mathbf{T}_{B}{}^{B}\mathbf{T}_{0})^{-1}{}^{RS}\mathbf{T}_{R_0} \end{aligned} \tag{78}$$

The objective in the Shoulder frame is defined as:.

$$\begin{pmatrix} n_{RS} & s_{RS} & a_{RS} & p_{RS} \\ 0 & 0 & 0 & 1 \end{pmatrix} = {}^{Obj}\mathbf{T}_{RS} \begin{pmatrix} n_{R_0} & s_{R_0} & a_{R_0} & p_{R_0} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{79}$$

A Useful package to visualize DH-based representation of an articulated body

I found one very useful package developed in Python by Rufus Fraanje from the Hague University of Applied Sciences. It is straightforward to install and easy to use. It is based on Python-visual and can be downloaded from github: `https://github.com/prfraanje/python-robotics`

The picture below gives you an example of how a manipulator is displayed.

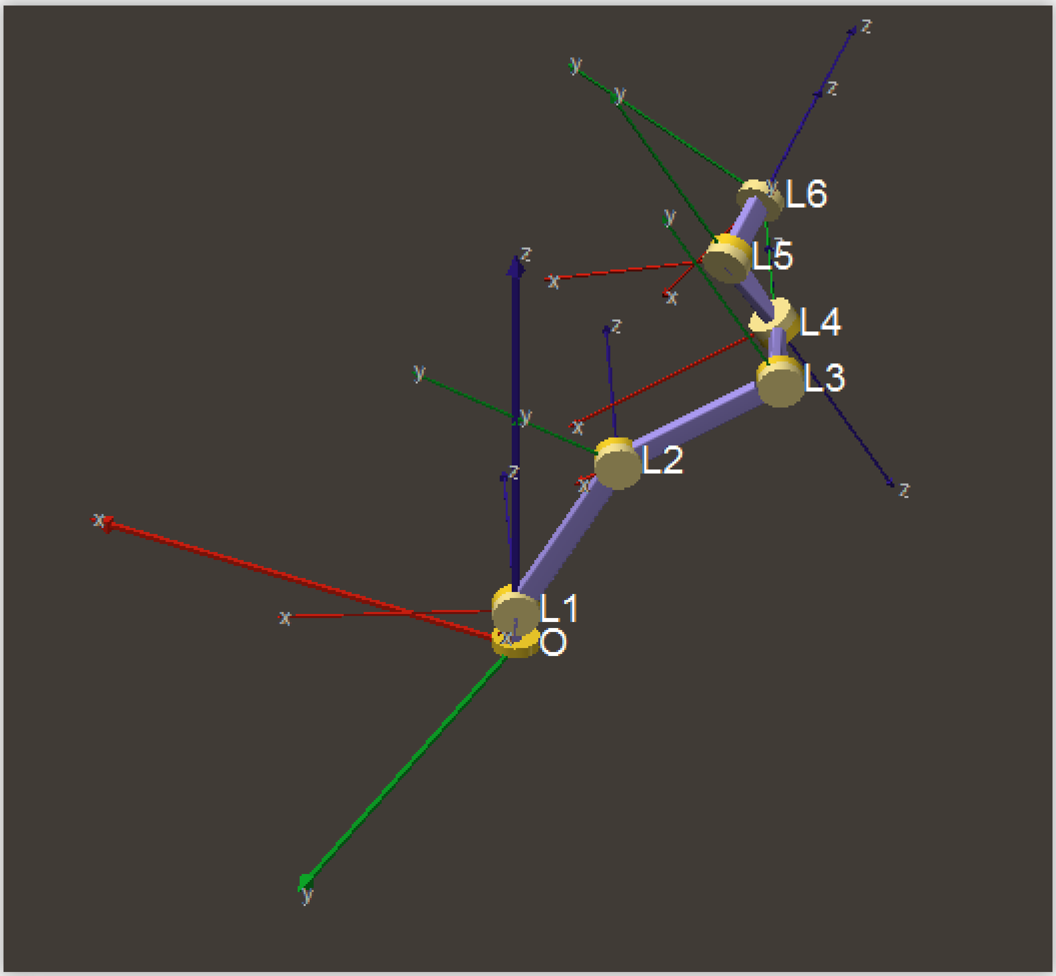


Figure 14: 3D graphic display of a manipulator based on a DH-based representation

The code is pretty straighworward as one can see:

```
from links import *

scene = display( title ='UR 5',center=(0,0,0),forward=(−1,−1,−1),up=(0,0,1),background=visual.color.background,range=5)
scene.fov = 0.01 # mimic orthographic projection

scale = 1;
F0 = frame([0,0,0], label='O',scale=scale)

# Denavit−Hartenberg parameters of the UR5 robot, see e.g.
# http://www.universal−robots.com/how−tos−and−faqs/faq/ur−faq/actual−center−of−mass−for−robot−17264/
# note that the a's and the d's here are in meters, and the
```

```
# alpha's and the theta's are in degrees
# you may use radians as well, but then change the unit in links below to
# unit='rad'
a1     = 0 # link length
alpha1 = 90 # link twist
d1     = .089159 # link offset (controllable)
theta1 = 0 # joint angle (controllable)
joint1 = 'r'

a2     = −0.42500 # link length
alpha2 = 0 # link twist
d2     = 0 # link offset (controllable)
theta2 = −90 # joint angle (controllable)
joint2 = 'r'

a3     = −0.39225 # link length
alpha3 = 0 # link twist
d3     = 0.0 # link offset (controllable)
theta3 = 0 # joint angle (controllable)
joint3 = 'r'

a4     = 0 # link length
alpha4 = 90 # link twist
d4     = 0.20915 # link offset (controllable)
theta4 = −90 # joint angle (controllable)
joint4 = 'r'

a5     = 0 # link length
alpha5 = −90 # link twist
d5     = −0.19465 # link offset (controllable)
theta5 = 0 # joint angle (controllable)
joint5 = 'r'

a6     = 0 # link length
alpha6 = 0 # link twist
d6     = 0.1823 # link offset (controllable)
theta6 = 0 # joint angle (controllable)
joint6 = 'r'

# these are the joint variables
# in these case they are all angles (in degrees)
# that come on top of the offsets defined by the
# theta angles above, so use q as the joint variable
# and theta just as the offset
q1 = 30
q2 = 30
q3 = 30
q4 = 90
q5 = 30
q6 = −30

scale = 0.5

L1 = link(F0,a=a1,alpha=alpha1,d=d1,theta=theta1,q=q1,joint=joint1,label='L1',scale=scale,unit='deg')
L2 = link(L1.frame,a=a2,alpha=alpha2,d=d2,theta=theta2,q=q2,joint=joint2,label='L2',scale=scale,unit='deg')
L3 = link(L2.frame,a=a3,alpha=alpha3,d=d3,theta=theta3,q=q3,joint=joint3,label='L3',scale=scale,unit='deg')
L4 = link(L3.frame,a=a4,alpha=alpha4,d=d4,theta=theta4,q=q4,joint=joint4,label='L4',scale=scale,unit='deg')
L5 = link(L4.frame,a=a5,alpha=alpha5,d=d5,theta=theta5,q=q5,joint=joint5,label='L5',scale=scale,unit='deg')
L6 = link(L5.frame,a=a6,alpha=alpha6,d=d6,theta=theta6,q=q6,joint=joint6,label='L6',scale=scale,unit='deg')

scene.camera(F0,forward=[−1,−1,−1],up=[0,0,1])
```

B Find a plan tangent to a sphere

So we want the plane that passes both P and Q and touches the sphere (radius r). Let that point, where the sphere and the plane meet be $X(\alpha, \beta, \gamma)$.

Since X is on the sphere, it satisfies $\alpha^2 + \beta^2 + \gamma^2 = r^2$.

Also, the vector $\vec{OX}$ will be the normal vector to the tangent plane. The plane passes X so, the equation of the plane is

$$\alpha(x - \alpha) + \beta(y - \beta) + \gamma(z - \gamma) = 0$$

Thus,

$$\alpha x + \beta y + \gamma z = r^2$$

This plane passes through P and Q, so plugging the values in gives us,

$$\alpha x_1 + \beta y_1 + \gamma z_1 = r^2$$

$$\alpha x_2 + \beta y_2 + \gamma z = r_2^2$$

Overall, solve these three equations,

$$\alpha x_1 + \beta y_1 + \gamma z_1 = r^2 \tag{80}$$

$$\alpha x_2 + \beta y_2 + \gamma z_2 = r^2 \tag{81}$$

$$\alpha^2 + \beta^2 + \gamma^2 = r^2 \tag{82}$$

to get the values of α, β, γ.

From Eq. (80) we have:

$$\alpha = \frac{r^2 - \beta y_1 - \gamma z_1}{x_1} \tag{83}$$

input α in Eq. (81):

$$\frac{x_2}{x_1}(r^2 - \beta y_1 - \gamma z_1) + \beta y_2 + \gamma z_2 = r^2$$

thus

$$\beta(y_2 - \frac{y_1 x_2}{x_1}) - \gamma(\frac{z_1 x_2}{x_1} - z_2) = r^2(1 - \frac{x_2}{x_1})$$

or

$$\beta = \mathbf{A}\gamma + \mathbf{B} \tag{84}$$

with:

$$\mathbf{A} = \frac{\frac{z_1 x_2}{x_1} - z_2}{y_2 - \frac{y_1 x_2}{x_1}}$$

$$\mathbf{B} = \frac{r^2(1 - \frac{x_2}{x_1})}{y_2 - \frac{y_1 x_2}{x_1}}$$

Reintroducing Eq. (84) in Eq. (83):

$$\alpha = \frac{r^2 - (\mathbf{A}\gamma + \mathbf{B})y_1 - \gamma z_1}{x_1}$$

$$\alpha = \frac{\gamma(-\mathbf{A}y_1 - z_1) + r^2 - \mathbf{B}y_1}{x_1}$$

$$\alpha = \mathbf{C}\gamma + \mathbf{D} \tag{85}$$

with:

$$\mathbf{C} = \frac{-\mathbf{A}y_1 - z_1}{x_1}$$

$$\mathbf{D} = \frac{r^2 - \mathbf{B}y_1}{x_1}$$

Input Eqs (84) and (85) in Eq. (82):

$$(\mathbf{C}\gamma + \mathbf{D})^2 + (\mathbf{A}\gamma + \mathbf{B}^2 + \gamma^2 = r^2$$

$$\gamma^2(\mathbf{C}^2 + \mathbf{A}^2 + 1) + \gamma(2\mathbf{CD} + 2\mathbf{AB}) + \mathbf{D}^2 + \mathbf{B}^2 - \mathbf{r}^2 = 0$$

by solving this second order equation we have two values for γ and we can compute two values for β with Eq. (84) and two values for α with Eq. (85).

Contents

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