Galilean Anomalies and Their Effect on Hydrodynamics

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ABSTRACT: We extend the null background construction of [1, 2] to include torsion and a conserved spin current, and use it to study gauge and gravitational anomalies in Galilean theories coupled to torsional Newton-Cartan backgrounds. We establish that the relativistic anomaly inflow mechanism with an appropriately modified anomaly polynomial, can be used to generate these anomalies. Similar to relativistic case, we find that Galilean anomalies also survive only in even dimensions. Further, these anomalies only effect the gauge and rotational symmetries of a Galilean theory; in particular the Milne boost symmetry remains non-anomalous. We also extend the transgression machinery used in relativistic fluids to fluids on null backgrounds, and use it to determine how these anomalies affect the constitutive relations of a Galilean fluid.

Unrelated to Galilean fluids, we propose an analogue of the off-shell second law of thermodynamics for relativistic fluids introduced by [3], to include torsion and a conserved spin current in Vielbein formalism. Interestingly, we find that even in absense of spin and torsion the entropy currents in two formalisms are different; while the usual entropy current gets a contribution from gravitational anomaly, the entropy current in Vielbein formalism does not have any anomaly induced part.

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1 | Null Reduction and Anomalies

The world around us, for most practical purposes, can be regarded non-relativistic. So it is important to ask how various exotic results in relativistic theories are to be interpreted in non-relativistic limit. Taking this limit however turns out to be a non-trivial task; except in few special cases, non-relativistic limit is either not well defined or is not unique[1], which forces the analysis to resort on approximate methods. It is generally accepted that non-relativistic theories can be very well approximated by Galilean theories. So rather than taking a limit of relativistic theories, one can take a more axiomatic approach of defining the Galilean theories in their own right – as has been historically done – and say something useful about the non-relativistic theories. About a decade after the inception on general relativity it was realized that Galilean spacetimes can also be packaged into a nice covariant language – Newton-Cartan geometries [6, 7]. Since then there has been a huge amount of development in understanding

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[1]For example, Maxwell's electromagnetism is known to have more than one non-relativistic limits [5].

how Galilean theories couple to Newton-Cartan backgrounds [4, 8–21][2]. We recommend looking at §2.1 of [18] for a short and self contained review of Newton-Cartan geometries, which will be extensively used throughout this work. Refer [22–25] for some more recent work on Galilean physics which will not be touched upon here.

There is also a relatively recent way to approach non-relativistic physics – *null reduction* [26–28]. It is known for a long time that Galilean group can be embedded into one dimensional higher Poincaré group. Correspondingly, one can constrain the Poincaré algebra in a certain way, and reduce it to a Galilean algebra. To be more precise, consider generators of 5 dimensional Poincaré algebra written in null coordinates[3] ($A, B = -, +, 1, 2, 3$),

$$\text{Spacetime Translations: } P_A, \qquad \text{Lorentz Transformations: } M_{AB}. \tag{1.1}$$

A subset of these – generators which commute with null momenta P_- ($a, b = 1, 2, 3$),

$$P_-, \quad P_+, \quad P_a, \quad M_{a+}, \quad M_{ab}, \tag{1.2}$$

form a Galilean algebra, with P_- acting as a new Casimir. P_- can be interpreted as continuity operator (with mass as its conserved charge), P_a as translations, P_+ as time translation, M_{a+} as Galilean boosts and finally M_{ab} as rotations (look at [29] for an extensive review)[4]. It has far reaching implications – one has an entire new way to take 'non-relativistic limit'. Rather than starting from a 4 dimensional relativistic theory and taking 'c' to infinity to get a non-relativistic theory, one can start with a 5 dimensional relativistic theory and reduce it over a light cone (introduce a null Killing vector) to get a Galilean theory, which is as good as a non-relativistic theory. This idea (and its generalizations to higher and lower dimensions) have been used readily in the literature to reproduce known results and to get new insights into non-relativistic physics. Probably most important of these was to reproduce (torsional) Newton-Cartan geometries starting from a *Bargmann structure* (relativistic manifold carrying a covariantly constant null Killing vector) in one higher dimension [13, 16, 30, 31]. Also authors in [32] and many following them (e.g. [33, 34]) established that reducing a relativistic fluid on light cone indeed gives the expected constitutive relations of a Galilean fluid, discussed e.g. in [35].

In [34] authors realized that although this mechanism gives 'a' Galilean fluid, it is not the most generic one. Especially thermodynamics which the reduced fluid follows is in some sense more restrictive than the most generic Galilean theories[5]. Also parity-violating sector of the reduced fluid is highly restrictive and survives only in a very special case of 'incompressible fluids kept in a constant magnetic field'. Same authors in [1] provided a resolution to this issue, which however is little different from the usual spirit of null reduction. Rather than doing null reduction of a relativistic fluid, authors suggested to construct a theory of fluids coupled to Bargmann structures from scratch, henceforth referred as *Bargmann fluid* or *null fluid*[6]. In the process

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[2]It is far from the reach of a mortal being to compile an exhaustive list of work on non-relativistic physics; please refer to mentioned works and references therein.

[3]We define the transformation to null coordinates as $x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^4)$.

[4]To be more precise, what we call Galilean group is generally known as the *Bargmann group* which is the *central extension of Galilean group* with central generator P_-. Galilean group sits inside as a special case with $P_- = 0$.

[5]See eqn. IV.121 of [34] and footnote (7) of [2] for more details on this issue.

[6]Why 'null' fluid? A fluid is generally called 'null' if the corresponding fluid velocity is a null vector. Unlike usual relativistic fluids, one can show that on a Bargmann structure (with null Killing vector V^M), a null fluid ($u^M u_M = 0$) and a unit normalized fluid ($w^M w_M = -1$) are related by merely a field redefinition: $u^M = w^M + \frac{1}{2 w^N V_N} V^M$. [1] found that writing a Bargmann fluid in terms of 'null fluid velocity' is more natural from the point of view of a Galilean fluid.

it was realized that there are certain aspects of null fluids which arise just by the introduction of null isometry and have no analogue in usual relativistic fluids. Upon null reduction[7], this null fluid gives rise to the most generic Galilean fluid. In a sense null fluids can be seen as a particular embedding of Galilean fluid into a spacetime of one higher dimension. This approach is more in lines with the axiomatic approach to study Galilean theories, but has the benefit that we have all the well-developed machinery of relativistic physics at our disposal.

The aim of this paper is to address a similar issue, but in a different setting – *anomalies*. Gauge and gravitational anomalies for a non-relativistic quantum field theory (Lifshitz fermions) was discussed in [36] using path integral methods. [37] on the other hand took the conventional null reduction approach to this problem, where author started with an anomalous relativistic theory and figured out its fate upon reduction. There is however an issue with this approach – relativistic anomalies[8] are known to exist only in even dimensions, hence this approach will essentially give anomalies *only in odd dimensional* Galilean theories. This is slightly unpleasant, because if one is to look at Galilean theories as a makeshift for non-relativistic theories which in turn are 'low velocity' limit of relativistic theories in same number of dimensions, one would expect them to be anomalous *only in even dimensions*. Half of this problem can be solved by noting that all the anomalies found by [37] crucially depend on the components of higher dimensional gauge field and affine connection along the Killing direction ($A_\sim$, $\Gamma^M_{\sim N}$ where A_M is the gauge field, Γ^R_{MN} is the affine connection, and the null Killing vector is chosen to be $\partial_\sim$). It was noted by [34] that these components act as sources in the mass conservation Ward identity (look at discussion around eqn. (2.42)). Since we do not know of any such mass sources appearing in nature, it would be better to switch these off (one can check that these mass sources $A_\sim$, $\Gamma^M_{\sim N}$ are well defined gauge covariant tensors). Doing so will eliminate all the anomalies in odd dimensional Galilean theories. We call the Bargmann structures with these mass sources set to zero as *compatible Bargmann structures* or *null backgrounds* following [2]. The other half of the problem is however more challenging – we need to find a consistent mechanism to introduce anomalies in theories coupled to odd dimensional null backgrounds.

The basic idea to do this was illustrated in [2] using abelian gauge anomalies. To motivate this lets consider the simplest case of 4 dimensional flat relativistic theory with a U(1) anomaly. Conservation of corresponding (covariant) current J^μ is given as,

$$\partial_\mu J^\mu = \frac{3}{4} C^{(4)} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{1.3}$$

where $F_{\mu\nu}$ is the field strength tensor and $C^{(4)}$ is the anomaly constant. Upon taking a non-relativistic limit, one would qualitatively expect the conservation law to look like (for appropriately large or small $C^{(4)}$),

$$\partial_t q + \partial_i j_q^i = 3 C^{(4)} \varepsilon^{ijk} F_{0i} F_{jk} = -6 C^{(4)} e_i b^i, \tag{1.4}$$

where e, b are the electric and magnetic fields respectively. This effect can be reproduced after null reduction of a 5 dimensional conservation law,

$$\partial_M J^M = \frac{3}{4} C^{(4)} \epsilon^{MNRST} \bar{V}_M F_{NR} F_{ST}, \tag{1.5}$$

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[7]Since the theory already has a null Killing field, null reduction is defined as choosing a foliation transverse to the Killing field and compactifying the null direction. As we shall discuss in § 2.3, doing this requires introducing a Galilean frame of reference, or in other words a preferred notion of time.

[8]Author in [37] considered both gauge/gravitational as well as Weyl anomalies; however in this work we will only be concerned about the former.

where $\bar{V}^M$ is an arbitrary null vector with $\bar{V}_\sim = -1$. Note that $F_{\sim M} = \partial_\sim A_M - \partial_M A_\sim = 0$ when $A_\sim = 0$. Since one index on ϵ must be '$\sim$', that responsibility lands on $\bar{V}_M$ implying that the mentioned expression doesn't depend on what $\bar{V}^M$ is chosen (these statements will be made more rigorous in § 3.2). It was observed by [2] that this anomaly can indeed be generated by anomaly inflow mechanism exactly in the same way as it works for usual relativistic anomalies, but with a tweaked anomaly polynomial. Authors there were interested in abelian anomalies and how they affect the hydrodynamics at the level of constitutive relations. This work will generalize these arguments to non-abelian and gravitational anomalies, and will give a more rigorous and transparent mechanism to compute their contribution to Galilean hydrodynamics using transgression machinery of relativistic fluids [38].

However unlike [2] we would need to introduce torsion in the game for a clearer analysis of gravitational sector. In Newton-Cartan geometries it is known (cref. [18]) that torsionlessness imposes a dynamic constraint $d\boldsymbol{n} = 0$ on the time metric $\boldsymbol{n} = n_\mu dx^\mu$. It has been noted in [16, 21, 39] that lifting this constraint off-shell is necessary to study energy transport in Galilean theories. Similar issue also showed up in the context of Galilean hydrodynamics discussed in [1] where authors noted that on torsionless Galilean backgrounds second law of thermodynamics fails to capture all the constraints obeyed by transport coefficients of a Galilean fluid. Since we would be interested in off-shell physics to understand anomalies, imposing torsionlessness would only make matters less clear. Nevertheless, on cost of some added technicality, it would allow us to explore null reduction for theories with non-zero spin current, which as far as we can tell hasn't been attempted[9]. [4] has considered the most generic Galilean theories on torsional Newton-Cartan background (without a conserved spin current), which follows very nicely via null reduction. Notably authors in [4] presented their results in a 'frame-independent' manner using an 'extended space representation' of the Galilean group; we show in appendix (B) that this representation is nothing but the theory on a null background seen prior to null reduction.

It is worth noting here that the essence of null reduction, usual or axiomatic, lies in the fact that the sophisticated machinery of relativistic theories can be used to say something useful about non-relativistic theories. This method however has its limitations; one needs to be acquainted with the relativistic side of the story to appreciate the construction. Although we review whatever is required for this work, readers might find it helpful to consult the relativistic results first, or from time to time during the reading. The respective relativistic references will be mentioned on the go.

Unrelated to Galilean fluids, we also make some observations regarding entropy current for a relativistic fluid. Recently an off-shell generalization of the second law of thermodynamics was considered by [3] in context of torsionless relativistic hydrodynamics. The authors in [40, 41] also proposed a new abelian $U(1)_T$ symmetry in hydrodynamics associated with this off-shell statement, with entropy as its conserved charge. We propose a natural generalization of this off-shell statement of second law in Vielbein formalism, in presence of torsion and a conserved spin current. More interestingly, even in absence of torsion we find that the entropy current defined by off-shell second law in Vielbein formalism, is different from what defined in usual formalism (we call the latter as Belinfante entropy current). Vielbein entropy current does not have any anomaly induced part, while the Belinfante entropy current has been shown to get contributions from gravitational anomaly [41]. A similar distinction between two formalisms has been known for energy-momentum tensor as well: while the Vielbein formalism deals with an asymmetric canonical EM tensor (which is Noether current of translations), the usual formalism

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[9]Some authors including [16] have considered null reduction in presence of torsion, but have not included a spin connection as an independent background source.

deals with a symmetric Belinfante EM tensor (which couples to the metric in general relativity) (see footnote 11 for related comments). Motivated from this, and the fact that Vielbein entropy current does not get contributions from anomaly, we guess that it should be in some sense more naturally related to the fundamental $U(1)_T$ symmetry of [40, 41]. In the passing we would also like to note that the two entropy currents are found to differ only off-shell, and boil down to the same on imposing equations of motion. Further, for a spinless fluid the difference only survives in anomalous sector, and is precisely what accounts for the Vielbein entropy current being independent of anomalies. Interested readers can jump directly to appendix (D).

This work is broadly categorized in 5 sections. The remaining of introduction contains a summary of our main results in § 1.1. § 2 starts off by extending null background construction of [2] to include torsion, which is further used to derive Ward identities of a Galilean theory with non-trivial spin current in § 2.3. A review of the relativistic anomaly inflow mechanism has been provided in § 3, which we modify in § 3.2 to account for anomalies in null/Galilean backgrounds and derive corresponding anomalous Ward identities. Later in § 4 we discuss how these anomalies affect the constitutive relations of null/Galilean hydrodynamics. Keeping in mind the technicality of this work, a detailed walkthrough example for the simplest case of 3 dimensional null theories (2 dimensional Galilean theories) has been given in § 5.1, results of which are generalized to arbitrary higher dimensions in § 5.2. In appendix (A) we present some of our results in conventional non-covariant basis for the benefit of readers not acquainted with Newton-Cartan language. Appendix (B) is devoted to a comparison of null backgrounds to the extended space representation of [4]. In appendix (C) we give some notations and conventions for differential forms used throughout this work. Finally in appendix (D) we comment on the entropy current in relativistic hydrodynamics in Vielbein formalism.

1.1 Overview and Results

Skipping all the technicalities we start directly with the results, keeping in mind that these results have been obtained by null reduction of anomalies on null backgrounds. In the following we denote indices on Newton-Cartan (NC) manifold $\mathcal{M}^{\mathrm{NC}}_{(d+1)}$ by $\mu\nu\ldots$, and on a flat spatial manifold by $\mathbb{R}^{(d)}$ by $a, b\ldots$. NC structure is defined by a time-metric n_μ, a degenerate Vielbein $e_a{}^\mu$, and a flat metric δ_{ab}. Further we define a NC frame velocity v^μ, and using it an 'inverse' Vielbein $e^a{}_\mu$ by $v^\mu n_\nu + e_a{}^\mu e^a{}_\nu = \delta^\mu_\nu$ and $e^a{}_\mu e_b{}^\mu = \delta^a_b$. Indices on $\mathcal{M}^{\mathrm{NC}}_{(d+1)}$ cannot be raised/lowered, while on $\mathbb{R}^{(d)}$ can be raised/lowered by δ^{ab}, δ_{ab}. $\mathcal{M}^{\mathrm{NC}}_{(d+1)}$ indices can be projected down to $\mathbb{R}^{(d)}$ using $e^a{}_\mu, e_a{}^\mu$. NC manifold is also equipped with a connection $\Gamma^\lambda{}_{\mu\nu}$, a spin connection $C^a{}_{\mu b}$, a non-abelian gauge field A_μ and a covariant derivative ∇_μ associated with all of these. Differential forms are denoted by bold symbols.

Similar to the relativistic case, we find that (gauge and gravitational) anomalies on an even dimensional NC background $\mathcal{M}^{\mathrm{NC}}_{(2n)}$, are governed by a $(2n+2)$ dimensional anomaly polynomial $\mathfrak{p}^{(2n+2)}$. However here the anomaly polynomial is written in terms of Chern classes of gauge field strength $\boldsymbol{F} = \mathrm{d}\boldsymbol{A} + \boldsymbol{A} \wedge \boldsymbol{A} = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ and Pontryagin classes of NC spatial curvature $\boldsymbol{R}^a{}_b = \mathrm{d}\boldsymbol{C}^a{}_b + \boldsymbol{C}^a{}_c \wedge \boldsymbol{C}^c{}_b = \frac{1}{2}R_{\mu\nu}{}^a{}_b\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. The odd dimensional Galilean theories on the other hand are non-anomalous (in absence of any extra mass source). In presence of anomalies,

the conservation laws of the theory are given as,

$$
\begin{aligned}
\text{Mass Cons. (Continuity):} \quad & \underline{\nabla}_\mu \rho^\mu = 0, \\
\text{Energy Cons. (Time Translation):} \quad & \underline{\nabla}_\mu \epsilon^\mu = [\text{power}] - p^{\mu a} c_{\mu a}, \\
\text{Momentum Cons. (Translations):} \quad & \underline{\nabla}_\mu p^\mu{}_a = [\text{force}]_a - \rho^\mu c_{\mu a}, \\
\text{Temporal Spin Cons. (Galilean Boosts):} \quad & \underline{\nabla}_\mu \tau^{\mu a} = \frac{1}{2} \left(\rho^a - p^a \right), \\
\text{Spatial Spin Cons. (Rotations):} \quad & \underline{\nabla}_\mu \sigma^{\mu ab} = p^{[ba]} + 2\tau^{\mu[a} c_\mu{}^{b]} + \textcolor{red}{\sigma_{\mathrm{H}}^{\perp ab}}, \\
\text{Charge Cons. (Gauge Transformations):} \quad & \underline{\nabla}_\mu j^\mu = \textcolor{red}{\mathrm{j}_{\mathrm{H}}^\perp},
\end{aligned}
\tag{1.6}
$$

where $\underline{\nabla}_\mu = \nabla_\mu + v^\nu H_{\nu\mu} - e_a{}^\nu \mathrm{T}^a{}_{\nu\mu}$. Here $H_{\mu\nu}$ is the temporal torsion and $\mathrm{T}^a{}_{\mu\nu}$ is the spatial torsion. Along with the conservation laws, the associated symmetries and conserved quantities have been specified above. We see that mass is exactly conserved. Energy/momentum is sourced by power/force densities (expressions can be found in § 2.3) and pseudo-power/force due to spacetime dependence of the frame velocity $c_\mu{}^a = e^a{}_\nu \nabla_\mu v^\nu$. Temporal spin is sourced by difference in spatial mass current and momentum density; for spinless theories it implies equality of the two. Barring anomalies, spatial spin is sourced by antisymmetric part of momentum density (causing torque) and pseudo-torque, while charge is exactly conserved. In addition to these, spatial spin and charge are also sourced by gravitational $\sigma_{\mathrm{H}}^{\perp ab}$ and gauge $\mathrm{j}_{\mathrm{H}}^\perp$ anomalies respectively. These anomaly sources can be determined from the anomaly polynomial $\mathbf{p}^{(2n+2)}$ as,

$$
\sigma_{\mathrm{H}}^{\perp ab} = - *_\uparrow \left[\frac{\partial \mathbf{p}^{(2n+2)}}{\partial \mathbf{R}_{ba}} \right], \qquad \mathrm{j}_{\mathrm{H}}^\perp = - *_\uparrow \left[\frac{\partial \mathbf{p}^{(2n+2)}}{\partial \mathbf{F}} \right].
\tag{1.7}
$$

In the study of Galilean hydrodynamics, we can construct the sector of constitutive relations completely determined by mentioned anomalies. For doing this, we first need to define the hydrodynamic shadow gauge field, $\hat{\mathbf{A}} = \mathbf{A} - \mu \mathbf{n}$ and spin connection $\hat{\mathbf{C}}^a{}_b = \mathbf{C}^a{}_b - [\mu_\sigma]^a{}_b \mathbf{n}$, where μ is the gauge chemical potential and $[\mu_\sigma]^a{}_b$ is the spatial spin chemical potential. We call the corresponding field strengths $\hat{\mathbf{F}}$ and $\hat{\mathbf{R}}^a{}_b$, and the anomaly polynomial made out of these to be $\hat{\mathbf{p}}^{(2n+2)}$. Using these we define the transgression form, $\mathcal{V}_{\mathfrak{p}}^{(2n+1)} = -\frac{\mathbf{n}}{H} \wedge \left(\mathbf{p}^{(2n+2)} - \hat{\mathbf{p}}^{(2n+2)} \right)$, where $\mathbf{H} = -d\mathbf{n}$. It can be used to generate the anomalous sector of constitutive relations; only non-zero contributions are given as,

$$
(\epsilon^\mu)_{\mathrm{A}} = *_\uparrow \left[\frac{\partial \mathcal{V}_{\mathfrak{p}}^{(2n+1)}}{\partial \mathbf{H}} \right]^\mu, \qquad (\sigma^{\mu ab})_{\mathrm{A}} = *_\uparrow \left[\frac{\partial \mathcal{V}_{\mathfrak{p}}^{(2n+1)}}{\partial \mathbf{R}_{ba}} \right]^\mu, \qquad (j^\mu)_{\mathrm{A}} = *_\uparrow \left[\frac{\partial \mathcal{V}_{\mathfrak{p}}^{(2n+1)}}{\partial \mathbf{F}} \right]^\mu.
$$

We leave it for the readers to convince themselves that these formulas are well defined. These constitutive relations follow the second law of thermodynamics and off-shell adiabaticity with a trivially zero entropy current. We would like to caution the reader that these are merely the contribution from anomalies to the constitutive relations, there will be further contributions which are independent of anomalies and have not been discussed here.

Explicit examples of the above results in case of U(1) and gravitational anomalies, for 2 dimensions and a generalization to $2n$ dimensions has been given in § 5. But probably the most important take home message of this work is that one can perform a consistent analysis of gauge and gravitational anomalies for Galilean theories using guidelines laid out by relativistic construction. This should be taken as a yet another point in the favor of, or rather an advertisement for, the axiomatic approach to null reduction – null backgrounds [1].

2 | Galilean Theories with Spin and Torsion

The aim of this section is to extend the null background construction of [1, 2] to torsional backgrounds, and derive non-anomalous Ward identities for a Galilean theory with non-zero spin current. We will later introduce anomalies in § 3. The construction is mainly based on the work of [16, 31] on torsional null reductions, with certain modifications. We will be working in Vielbein formalism, which is most natural choice for a spin system. Hence the language and expressions will be slightly different from what seen in the earlier work on null backgrounds [2] where authors focus on torsionless and spinless case.

2.1 Einstein-Cartan Backgrounds

We start with a short review of Einstein-Cartan backgrounds, mostly to setup notation for our later discussion on torsional null backgrounds. A more comprehensive introduction to this formalism can be found in e.g. [42]. Consider a manifold $\mathcal{M}_{(d+2)}$ theories on which are invariant under diffeomorphisms and possibly non-abelian gauge group $\mathcal{G}$. We denote the infinitesimal diffeomorphism and gauge variation parameters by,

$$\psi_\xi = \left\{ \xi = \xi^M \partial_M, \Lambda_{(\xi)} \right\} \in \mathrm{T}\mathcal{M}_{(d+2)} \times \mathfrak{g}. \tag{2.1}$$

We have denoted tangent bundle of $\mathcal{M}_{(d+2)}$ as $\mathrm{T}\mathcal{M}_{(d+2)}$, and Lie group corresponding to $\mathcal{G}$ as $\mathfrak{g}$. Indices on $\mathcal{M}_{(d+2)}$ are denoted by $M, N, R, S \ldots$. $\mathcal{M}_{(d+2)}$ is endowed with a metric $\mathrm{d}s^2 = \mathrm{G}_{MN}\mathrm{d}x^M\mathrm{d}x^N$, a $\mathfrak{g}$ valued gauge field $\boldsymbol{A} = A_M\mathrm{d}x^M$ and a metric compatible affine connection Γ^R_{MS} which is not necessarily symmetric in its last two indices. In the case of torsional geometries it is more natural to shift to Vielbein formalism, which we describe in the following. The condition of local flatness of a manifold allows us to define a map between $\mathrm{T}\mathcal{M}_{(d+2)}$ and (pseudo-Riemannian) flat space $\mathbb{R}^{(d+1,1)}$, realized in terms of a Vielbein E^A_M and its inverse E_A^M, restricted by,

$$\mathrm{G}_{MN} = \mathrm{E}^A_M\mathrm{E}^B_N\eta_{AB}, \qquad \mathrm{G}^{MN} = \mathrm{E}_A^M\mathrm{E}_B^N\eta^{AB}, \tag{2.2}$$

where η_{AB} is the flat metric, and $A, B, C, D \ldots$ denote indices on $\mathbb{R}^{(d+1,1)}$. Indices on $\mathcal{M}_{(d+2)}$ can be raised and lowered by G_{MN}, and on $\mathbb{R}^{(d+1,1)}$ by η_{AB}. Indices on $\mathcal{M}_{(d+2)}$ and $\mathbb{R}^{(d+1,1)}$ can also be interchanged using the E^A_M. Vielbein has $(d+2)^2$ components out of which $\frac{1}{2}(d+2)(d+3)$ are taken away by eqn. (2.2). Remaining $\frac{1}{2}(d+1)(d+2)$ components can be fixed by introducing an additional $\mathrm{SO}(d+1,1)$ symmetry in Vielbein $\mathrm{E}^A_M \sim O^A_B\mathrm{E}^B_M$. Hence E^A_M modded by diffeomorphisms and $\mathrm{SO}(d+1,1)$ has same physical information as G_{MN} modded with only diffeomorphisms. We also define a spin connection for fields living in $\mathbb{R}^{(d+1,1)}$,

$$\boldsymbol{C}^A_B = C^A_{MB}\mathrm{d}x^M = \mathrm{E}_B^S \left(\mathrm{E}^A_R\Gamma^R_{MS} - \partial_M\mathrm{E}^A_S \right)\mathrm{d}x^M, \tag{2.3}$$

which has same information as Γ^R_{MS}. So finally our system can be described by the trio $\{\mathrm{E}^A_M, C^A_{MB}, A_M\}$ modded by diffeomorphisms, gauge transformations, and $\mathrm{SO}(d+1,1)$ rotations denoted by infinitesimal parameters,

$$\psi_\xi = \left\{ \xi^M \partial_M, [\Lambda_{\Sigma(\xi)}]^A_B, \Lambda_{(\xi)} \right\} \in \mathrm{T}\mathcal{M}_{(d+2)} \times \mathfrak{so}(d+1,1) \times \mathfrak{g}. \tag{2.4}$$

Here $\mathfrak{so}(d+1,1)$ denotes the Lie algebra of $\mathrm{SO}(d+1,1)$. ψ_ξ is given a Lie algebra structure by defining a commutator on it,

$$\psi_{[\xi_1,\xi_2]} = [\psi_{\xi_1}, \psi_{\xi_2}] = \delta_{\xi_1}\psi_{\xi_2} = -\delta_{\xi_2}\psi_{\xi_1}, \tag{2.5}$$

where,

$$\delta_{\xi_1}\xi_2 = \pounds_{\xi_1}\xi_2 = -\pounds_{\xi_2}\xi_1 = -\delta_{\xi_2}\xi_1,$$

$$\delta_{\xi_1}[\Lambda_{\Sigma(\xi_2)}]^A{}_B = \pounds_{\xi_1}[\Lambda_{\Sigma(\xi_2)}]^A{}_B + [\Lambda_{\Sigma(\xi_2)}]^A{}_C[\Lambda_{\Sigma(\xi_1)}]^C{}_B - [\Lambda_{\Sigma(\xi_1)}]^A{}_C[\Lambda_{\Sigma(\xi_2)}]^C{}_B - \pounds_{\xi_2}[\Lambda_{\Sigma(\xi_1)}]^A{}_B$$

$$= -\delta_{\xi_2}[\Lambda_{\Sigma(\xi_1)}]^A{}_B,$$

$$\delta_{\xi_1}\Lambda_{(\xi_2)} = \pounds_{\xi_1}\Lambda_{(\xi_2)} + [\Lambda_{(\xi_2)}, \Lambda_{(\xi_1)}] - \pounds_{\xi_2}\Lambda_{(\xi_1)} = -\delta_{\xi_2}\Lambda_{(\xi_1)}. \tag{2.6}$$

Similarly the action of ψ_ξ (denoted by δ_ξ) on an arbitrary field φ (all indices suppressed) obeys an algebra: $[\delta_{\xi_1}, \delta_{\xi_2}]\varphi = \delta_{[\xi_1,\xi_2]}\varphi$. Under the action of ψ_ξ constituent fields vary as,

$$\delta_\xi E^A{}_M = \pounds_\xi E^A{}_M - [\Lambda_{\Sigma(\xi)}]^A{}_B E^B{}_M = \nabla_M \xi^A + \xi^N T^A{}_{NM} - [\nu_{\Sigma(\xi)}]^A{}_B E^B{}_M,$$

$$\delta_\xi C^A{}_{MB} = \pounds_\xi C^A{}_{MB} + \nabla_M[\Lambda_{\Sigma(\xi)}]^A{}_B = \nabla_M[\nu_{\Sigma(\xi)}]^A{}_B + \xi^N R_{NM}{}^A{}_B,$$

$$\delta_\xi A_M = \pounds_\xi A_M + \nabla_M\Lambda_{(\xi)} = \nabla_M\nu_{(\xi)} + \xi^N F_{NM}, \tag{2.7}$$

where $\xi^A = E^A{}_M \xi^M$ and $\pounds_\xi$ denotes Lie derivative along ξ^M. The covariant derivative ∇_M is associated with all the connections $\Gamma^R{}_{MS}, C^A{}_{MB}, A_M$, which acts on a general field $\varphi^R{}_S{}^A{}_B$ transforming in adjoint representation of the gauge group as,

$$\nabla_M\varphi^R{}_S{}^A{}_B = \partial_M\varphi^R{}_S{}^A{}_B + \Gamma^R{}_{MN}\varphi^N{}_S{}^A{}_B - \Gamma^N{}_{MS}\varphi^R{}_N{}^A{}_B + C^A{}_{MC}\varphi^R{}_S{}^C{}_B - C^C{}_{MB}\varphi^R{}_S{}^A{}_C + [A_M, \varphi^R{}_S{}^A{}_B], \tag{2.8}$$

and similarly on higher rank objects. In eqn. (2.7) we have defined[10]

$$\text{Scaled gauge chemical potential:} \qquad \nu_{(\xi)} = \Lambda_{(\xi)} + \xi^N A_N,$$

$$\text{Scaled spin chemical potential:} \qquad [\nu_{\Sigma(\xi)}]^A{}_B = [\Lambda_{\Sigma(\xi)}]^A{}_B + \xi^N C^A{}_{NB}, \tag{2.9}$$

associated with ψ_ξ. We have also defined curvatures of all the constituent fields,

$$\text{Gauge Field Strength:} \qquad \boldsymbol{F} = d\boldsymbol{A} + \boldsymbol{A} \wedge \boldsymbol{A} = \frac{1}{2}F_{MN}dx^M \wedge dx^N,$$

$$\text{Spacetime Curvature:} \qquad \boldsymbol{R}^A{}_B = d\boldsymbol{C}^A{}_B + \boldsymbol{C}^A{}_C \wedge \boldsymbol{C}^C{}_B = \frac{1}{2}R_{MN}{}^A{}_B dx^M \wedge dx^N,$$

$$\text{Spacetime Torsion:} \qquad \boldsymbol{T}^A = d\boldsymbol{E}^A + \boldsymbol{C}^A{}_B \wedge \boldsymbol{E}^B = \frac{1}{2}T^A{}_{MN}dx^M \wedge dx^N. \tag{2.10}$$

On can check that all these quantities $\nu_{(\xi)}, [\nu_{\Sigma(\xi)}]^A{}_B, F_{MN}, R_{MN}{}^A{}_B, T^A{}_{MN}$ transform covariantly under the action of ψ_ξ. It is interesting to note that $C^A{}_{MB}$ transforms as a $\mathfrak{so}(d+1,1)$ valued gauge field. In terms of torsion it is possible to give an exact expression for connections which we note for completeness,

$$\Gamma^R{}_{MS} = \frac{1}{2}G^{RN}\left(\partial_M G_{NS} + \partial_S G_{NM} - \partial_N G_{MS} + T_{NMS} - T_{MSN} - T_{SMN}\right),$$

$$C^A{}_{MB} = \frac{1}{2}\eta_{BD}E^{[D|S}\left[2\left(2\partial_{[S}E^{A]}{}_{M]} - T^{A]}{}_{SM}\right) + E_{CM}E^{A]N}\left(2\partial_{[S}E^C{}_{N]} - T^C{}_{SN}\right)\right]. \tag{2.11}$$

A physical theory on $\mathcal{M}_{(d+2)}$ can be described by a partition function $W[E^A{}_M, C^A{}_{MB}, A_M]$ which is a functional of Vielbein and connections. Under an infinitesimal variation of the sources its response is captured by,

$$\delta W = \int \{dx^M\} \sqrt{|G|}\left(T^M{}_A \delta E^A{}_M + \Sigma^{MA}{}_B \delta C^B{}_{MA} + J^M \cdot \delta A_M\right), \tag{2.12}$$

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[10] By scaled we mean scaled with temperature: $\nu = \mu/\vartheta$, where μ is the chemical potential and ϑ is the temperature. Note that at this point these quantities are just introduced for computational convenience; and will get a physical meaning only in presence of a preferred symmetry data, e.g. when spacetime admits an isometry.

where $X \cdot Y = \text{Tr} [XY]$ for $X, Y \in \mathfrak{g}$ is the inner product on $\mathfrak{g}$. T^{MA} is the *canonical energy-momentum tensor*, Σ^{MAB} is the *spin current* (antisymmetric in its last two indices) and J^M is the *charge current*. Demanding the partition function to be invariant under the action of ψ_ξ we can find the Ward identities[11] related to these currents,

$$\underline{\nabla}_M T^M_{\ N} = T^B_{\ NM} T^M_{\ B} + R_{NM}^{\ \ A}_{\ \ B} \Sigma^{MB}_{\ \ A} + F_{NM} \cdot J^M,$$
$$\underline{\nabla}_M \Sigma^{MAB} = T^{[BA]},$$
$$\underline{\nabla}_M J^M = 0, \tag{2.13}$$

where $\underline{\nabla}_M = \nabla_M - T^N_{\ NM}$.

2.2 Null Backgrounds

We are now ready to define null backgrounds. These kind of backgrounds and their Galilean interpretation goes back to [13, 16, 31, 44]. The idea of null backgrounds is to somewhat tweak the procedure, so that we not only get the correct symmetries, but also reproduce the required background field content after reduction. As we shall show, this even allows us to add anomalies in odd dimensional null backgrounds which naively doesn't look possible.

We will call ψ_ξ a *compatible symmetry data* if the scaled chemical potentials associated with it defined in eqn. (2.9) are identically zero. Now, a manifold $\mathcal{M}_{(d+2)}$ along with fields $\{E^A_M, C^A_{\ MB}, A_M\}$ will be called a *null background* (or more formally a *compatible Bargmann structure*) if it admits a *covariantly constant __compatible__ null isometry* generated by $\psi_V = \{V^M \partial_M, [\Lambda_{\Sigma(V)}]^A_{\ B}, \Lambda_{(V)}\}$ i.e.,

1. Action of ψ_V is an isometry, $\delta_V E^A_M = \delta_V C^A_{\ MB} = \delta_V A_M = 0$,

2. V is null, $V^M V_M = 0$,

3. V is covariantly constant, $\nabla_M V^N = 0$, and

4. ψ_V is compatible, $\nu_{(V)} = V^M A_M + \Lambda_{(V)} = 0$, $[\nu_{\Sigma(V)}]^A_{\ B} = V^M C^A_{\ MB} + [\Lambda_{\Sigma(V)}]^A_{\ B} = 0$.

Although this definition of null backgrounds is little different from [2], one can check that it boils down to the same in torsionless limit. If we drop condition (4), i.e. compatibility, we would be left with the definition of *Bargmann structures* [13] extended to Vielbein formalism. They have some nice properties,

$$T^A_{\ MN} V_A = H_{MN} \equiv 2\partial_{[M} V_{N]}, \qquad R_{MN}^{\ \ A}_{\ \ B} V_A = 0. \tag{2.14}$$

Hence if we are interested in a torsionless theory, we would have to apply a dynamic constraint on V, which can be violated off-shell. Requirement of compatibility further imposes,

$$V^M T^A_{\ MN} = V^M F_{MN} = V^M R_{MN}^{\ \ A}_{\ \ B} = 0, \qquad V^M \nabla_M \varphi = \delta_V \varphi, \tag{2.15}$$

for any tensor φ transforming in an appropriate representation of $\mathfrak{g}$ and $\mathfrak{so}(d+1,1)$ (all indices suppressed). These restrictions are in some sense backbone of null backgrounds. First and

---

[11] Note that we can use the spin Ward identity to eliminate antisymmetric part of canonical EM tensor in the EM conservation equation. Doing this is particularly helpful in torsionless theories where the new EM conservation becomes, $\nabla_M T^{MN}_{(b)} = F^{NM} \cdot J_M$. Here we have defined the symmetric *Belinfante energy-momentum tensor*, $T^{MN}_{(b)} = T^{(MN)} + 2\nabla_R \Sigma^{(MN)R}$. In this work however, we will mostly talk in terms of canonical EM tensor as this is the Noether charge corresponding to translations. Also it is well known that gravitational anomalies do not affect the canonical EM conservation [43].

foremost, they eliminate unphysical mass sources that would otherwise appear in the mass conservation law after reduction. Hints of it were originally found by [34] in an attempt of naive null reduction of charged fluids. We would have more to say about it later. As we shall see, these restrictions also allow for anomalies on odd dimensional null backgrounds and forbid them in even dimensional ones. This is an important feature, if we are to reproduce physically realizable anomalies in Galilean theories in one lower dimension.

We demand that physical theories on null backgrounds (referred as null theories) are not invariant under action of any arbitrary ψ_ξ but only those which leave ψ_V invariant i.e. $[\psi_V, \psi_\xi] = 0$. This requirement ensures that there is no dynamics along the isometry even off-shell. The new partition function variation can be written following eqn. (2.12) as,

$$\delta W = \int \{\mathrm{d}x^M\} \sqrt{|\mathrm{G}|} \left(T^M_{\ A}\delta\mathrm{E}^A_{\ M} + \Sigma^{MA}_{\ \ B}\delta C^B_{\ MA} + J^M \cdot \delta A_M + \#_A \delta V^A\right). \tag{2.16}$$

Note the last term in this expression, which is valid since our restriction does not forbid us from varying V^A. Astute reader might note that we could have absorbed that term into T^{MA} owing to the fact that $\delta V^M = 0$, but we have a better setup in mind. The conditions of null background along with the restrictions we have imposed, imply that null theories are invariant under the following set of current redefinitions,

$$T^{MA} \to T^{MA} + V^M\theta_1^A, \qquad \Sigma^{MAB} \to \Sigma^{MAB} + V^M\theta_2^{AB}, \qquad J^M \to J^M + \theta_3 V^M, \tag{2.17}$$

$$\#^A \to \#^A - \theta_1^A + \theta_4 V^A, \tag{2.18}$$

where θ's are arbitrary scalars transforming in appropriate representations of $\mathfrak{g}$ and $\mathfrak{so}(d+1,1)$. The Ward identities on null backgrounds will also be slightly modified compared to eqn. (2.13)[12],

$$\underline{\nabla}_M T^M_{\ N} = \mathrm{T}^B_{\ NM}T^M_{\ B} + R_{NM}^{\ \ A}_{\ \ B}\Sigma^{MB}_{\ \ A} + F_{NM} \cdot J^M,$$
$$\underline{\nabla}_M \Sigma^{MAB} = T^{[BA]} + \#^{[A}V^{B]},$$
$$\underline{\nabla}_M J^M = 0, \tag{2.19}$$

One can check that these equations are invariant under redefinitions eqn. (2.17). To interpret the new $\#^A$ term, note that spin (angular momentum) conservation are $\frac{1}{2}(d+1)(d+2)$ equations. However as was pointed out in the introduction, after reduction system only respects $\frac{1}{2}d(d-1)$ equations corresponding to rotations and d equations corresponding to Galilean boosts. The job of $\#^A$ is then to eliminate the remaining $(d+1)$ conservation equations, as it does. Practically it is best to fix an 'off-shell gauge' $\delta V^A = 0$, which renders a new invariance in the spin current,

$$\Sigma^{MAB} \to \Sigma^{MAB} + \theta_5^{M[A}V^{B]}, \tag{2.20}$$

and omits the remaining $(d+1)$ components of the spin conservation. Note that it will further restrict ψ_ξ to obey $[\nu_{\Sigma(\xi)}]^A_{\ B}V^B = 0$. This point onward we would assume every symmetry data ψ_ξ to satisfy these requirements, and would term them ψ_V *compatible symmetry data*. From this viewpoint, spin conservation in eqn. (2.19) must be true for some $\#^M$, hence ruling out components involving $\#^M$ as they carry no information.

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[12] Following footnote 11 one might wonder how the respective Belinfante EM conservation law looks like for null theories. Similar to non-null case one can use the spin conservation in EM conservation law, which will give $\nabla_M \left(T^{MN}_{(b)} - \#^{[N}V^{M]}\right) = F^{NM} \cdot J_M$. One can show that the $\#^M$ dependence can be removed by using T^{MA} redefinition eqn. (2.17), after which one recovers the standard Belinfante conservation law (given in footnote 11) even for null theories. Belinfante EM tensor on the other hand is left with redefinition freedom, $T^{MN}_{(b)} \to T^{MN}_{(b)} + \theta_1 V^M V^N$. These were derived directly for a spinless null theory in [1].

On null backgrounds using ψ_V we can also define some more 'thermodynamic' variables associated with ψ_ξ similar to eqn. (2.9),

$$\text{Temperature:}\quad \vartheta_{(\xi)} = -\frac{1}{\xi^N V_N}, \qquad \text{Scaled mass chemical potential:}\quad \varpi_{(\xi)} = -\frac{\xi^M \xi_M}{2\xi^N V_N}, \quad (2.21)$$

and using it we can define chemical potentials from scaled chemical potentials,

$$\mu_{(\xi)} = \vartheta_{(\xi)}\nu_{(\xi)}, \qquad [\mu_{\Sigma(\xi)}]^A{}_B = \vartheta_{(\xi)}[\nu_{\Sigma(\xi)}]^A{}_B, \qquad \mu_{\varpi(\xi)} = \vartheta_{(\xi)}\varpi_{(\xi)}. \qquad (2.22)$$

These abstract definitions will find use later.

2.3 Null Reduction – Newton-Cartan Backgrounds

Having obtained the Ward identities in the null background language, it is now time to see what does they imply for the Galilean theories. To do this we need to pick up a foliation $\mathcal{M}_{(d+2)} = S^1_V \times \mathcal{M}^{\text{NC}}_{(d+1)}$ and compactify along the isometry direction V. Following [2] we note that since V is null, it is not possible to find a unique such foliation without choosing a set of ψ_V compatible *time data*, $\psi_T = \{T^M\partial_M, [\Lambda_{\Sigma(T)}]^A{}_B, \Lambda_{(T)}\}$. This is tantamount to choosing a preferred Galilean frame of reference[13]. Having chosen ψ_T we can define such a foliation as $\mathcal{M}_{(d+2)} = S^1_V \times \mathbb{R}_T \times \mathcal{M}^T_{(d)}$, where we identify $\mathcal{M}^{\text{NC}}_{(d+1)} = \mathbb{R}_T \times \mathcal{M}^T_{(d)}$ as degenerate *Newton-Cartan* (NC) manifold. We define *null reduction* as this choice of foliation and subsequent compactification.

Newton-Cartan Structure: Using ψ_T we can define a null field orthonormal to V as,

$$\bar{V}^M_{(T)} = \vartheta_{(T)}T^M + \mu_{\varpi(T)}V^M, \qquad (2.23)$$

such that $\bar{V}^M_{(T)}\bar{V}_{(T)M} = 0$, and $\bar{V}^M_{(T)}V_M = -1$. Here $\vartheta_{(T)}$, $\mu_{\varpi(T)}$ have been defined in eqns. (2.21) and (2.22). Without loss of generality we choose a basis on $\mathcal{M}_{(d+2)}$, $x^M = \{x^\sim, x^\mu\}$ such that $\psi_V = \{\partial_\sim, 0, 0\}$. On the other hand on $\mathbb{R}^{(d+1,1)}$ we choose a basis $x^A = \{x^-, x^+, x^a\}$ such that $V = \partial_-$ and $\bar{V}_{(T)} = \partial_+$. At this stage we choose a specific representation of η_{AB}, $\mathrm{E}^A{}_M$ and $\mathrm{E}_A{}^M$ compatible with the mentioned basis,

$$\eta_{AB} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \delta_{ab} \end{pmatrix}, \qquad \mathrm{E}^A{}_M = \begin{pmatrix} 1 & -B_\mu \\ 0 & n_\mu \\ 0 & e^a{}_\mu \end{pmatrix}, \qquad \mathrm{E}_A{}^M = \begin{pmatrix} 1 & 0 \\ B_\nu v^\nu & v^\mu \\ B_\nu e_a{}^\nu & e_a{}^\mu \end{pmatrix}, \qquad (2.24)$$

such that,

$$n_\mu v^\mu = 1, \qquad e_a{}^\mu n_\mu = 0, \qquad e^a{}_\mu v^\mu = 0, \qquad e^a{}_\mu e_b{}^\mu = \delta^a_b, \qquad v^\mu n_\nu + e_a{}^\mu e^a{}_\nu = \delta^\mu_\nu. \qquad (2.25)$$

This can be identified as *Newton-Cartan* (NC) structure. We can also define the NC degenerate metric by,

$$h_{\mu\nu} = e^a{}_\mu e^b{}_\nu \delta_{ab}, \qquad h^{\mu\nu} = e_a{}^\mu e_b{}^\nu \delta^{ab}. \qquad (2.26)$$

Since there is no non-degenerate metric on $\mathcal{M}^{\text{NC}}_{(d+1)}$, raising/lowering of $\mu, \nu \ldots$ indices is not permitted. However $a, b \ldots$ indices can be raised/lowered using δ_{ab}. NC Vielbein $e^a{}_\mu$ is not a

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[13][4] proposed a formalism for Galilean theories independent of the choice of frame. But on a closer look it would be clear that they have just discovered null backgrounds from a different perspective. The Ward identities as described by [4] are just the null background Ward identities with slight rearrangement; we give a comparison in appendix (B).

'square matrix' and hence does not furnish an invertible map between tensors on $\mathcal{M}^{\text{NC}}_{(d+1)}$ and $\mathbb{R}^{(d)}$. It however can be used to project tensors on $\mathcal{M}^{\text{NC}}_{(d+1)}$ to tensors on $\mathbb{R}^{(d)}$, and tensors on $\mathbb{R}^{(d)}$ to 'spatial tensors' on $\mathcal{M}^{\text{NC}}_{(d+1)}$,

$$e^a{}_\mu X^\mu = X^a, \qquad e_a{}^\mu Y_\mu = Y_a, \qquad X^a e_a{}^\mu = h^\mu{}_\nu X^\nu, \qquad Y_a e^a{}_\mu = h^\nu{}_\mu Y_\nu, \qquad (2.27)$$

where $h^\mu{}_\nu = h^{\mu\rho} h_{\sigma\nu}$. The compatibility of null isometry switches off many components of the connections $\Gamma^M{}_{\sim N}$, $\Gamma^M{}_{\mu\sim}$, $C^A{}_{\sim B}$, $C^A{}_{\mu-}$, $C^+{}_{\mu B}$ and $A_\sim$. Remaining non-zero components can be determined to be,

$$C^-{}_{\mu a} = c_{\mu a}, \qquad C^a{}_{\mu+} = c_\mu{}^a, \qquad \Gamma^\sim{}_{\mu\nu} = c_{\mu\nu} - \nabla_\mu B_\nu,$$

$$\Gamma^\lambda{}_{\mu\nu} = v^\lambda \partial_\mu n_\nu + \frac{1}{2} h^{\lambda\sigma} \left(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu} \right) + n_{(\mu} \Omega_{\nu)\sigma} h^{\lambda\sigma} + \frac{1}{2} \left(e_a{}^\lambda \mathrm{T}^a{}_{\mu\nu} - 2 e_{a(\nu} \mathrm{T}^a{}_{\mu)\sigma} h^{\lambda\sigma} \right),$$

$$C^a{}_{\mu b} = \frac{1}{2} n_\mu \Omega_b{}^a + \frac{1}{2} \eta_{bd} e^{[d|\nu} \left[2 \left(2\partial_{[\nu} e^{a]}{}_{\mu]} - \mathrm{T}^{a]}{}_{\nu\mu} \right) + e_{c\mu} e^{a]\sigma} \left(2\partial_{[\nu} e^c{}_{\sigma]} - \mathrm{T}^c{}_{\nu\sigma} \right) \right]. \qquad (2.28)$$

Here we have defined spacetime dependence of frame velocity $c_{\mu\nu} = h_{\sigma\nu} \nabla_\mu v^\sigma$ in terms of which frame vorticity is given by $\Omega_{\mu\nu} = 2c_{[\mu\nu]}$. We call a time data (reference frame) ψ_T to be *globally inertial* if $c_{\mu\nu} = 0$. We choose the connections on $\mathcal{M}^{\text{NC}}_{(d+1)}$ to be $\Gamma^\lambda{}_{\mu\nu}$, $C^a{}_{\mu b}$ and A_μ, and denote associated covariant derivative by ∇_μ, acting on a general field $\varphi^\rho{}_\sigma{}^a{}_b$ transforming in adjoint representation of the gauge group as,

$$\nabla_\mu \varphi^\rho{}_\sigma{}^a{}_b = \partial_\mu \varphi^\rho{}_\sigma{}^a{}_b + \Gamma^\rho{}_{\mu\nu} \varphi^\nu{}_\sigma{}^a{}_b - \Gamma^\nu{}_{\mu\sigma} \varphi^\rho{}_\nu{}^a{}_b + C^a{}_{\mu c} \varphi^\rho{}_\sigma{}^c{}_b - C^c{}_{\mu b} \varphi^\rho{}_\sigma{}^a{}_c + \left[A_\mu, \varphi^\rho{}_\sigma{}^a{}_b \right], \quad (2.29)$$

and similarly on higher rank objects. Action of ∇_μ on NC structure can be found to be,

$$\nabla_\mu n_\nu = 0, \quad \nabla_\mu e_a{}^\nu = 0, \quad \nabla_\mu h^{\rho\sigma} = 0, \quad \nabla_\mu h_{\nu\rho} = -2c_{\mu(\nu} n_{\rho)}, \quad \nabla_\mu e^a{}_\nu = -n_\nu c_\mu{}^a. \qquad (2.30)$$

One can check that $\nabla_\mu, \Gamma^\lambda{}_{\mu\nu}$ agrees with the most generic NC covariant derivative and connection written down in [4]. One can also perform the reduction of curvatures. Surviving components of gauge field strength are $F_{\mu\nu}$ which act as NC gauge field strength. Similarly surviving components of torsion are spatial torsion $\mathrm{T}^a{}_{\mu\nu}$, 'mass torsion' $\mathrm{T}_{+\mu\nu} = -\mathrm{T}^-{}_{\mu\nu}$ and temporal torsion $H_{\mu\nu} = -\mathrm{T}^+{}_{\mu\nu}$. Finally we have the surviving components of curvature,

$$R_{\mu\nu}{}^a{}_+ = 2\partial_{[\mu} c_{\nu]}{}^a + 2C^a{}_{[\mu|b} c_{\nu]}{}^b, \qquad R_{\mu\nu}{}^a{}_b = 2\partial_{[\mu} C^a{}_{\nu]b} + 2C^a{}_{[\mu|c} C^c{}_{\nu]b}, \qquad (2.31)$$

which act as NC temporal and spatial curvatures respectively. Both curvatures can also be combined into a full NC curvature,

$$R_{\mu\nu}{}^\rho{}_\sigma = e_a{}^\rho \left(R_{\mu\nu}{}^a{}_+ n_\sigma + R_{\mu\nu}{}^a{}_b e^b{}_\sigma \right). \qquad (2.32)$$

We define raised NC volume element,

$$\varepsilon_\uparrow^{\mu\nu\cdots} = \bar{V}_M \epsilon^{M\mu\nu\cdots} = -\epsilon^{-\mu\nu\cdots}. \qquad (2.33)$$

Again since volume element is defined with all indices up, and there is no lowering operation, the corresponding Hodge dual $*_\uparrow$ gives a map from differential forms to completely antisymmetric contravariant tensor fields. It is also possible to define a lowered volume element, but we would not require it for our purposes. More details on NC volume forms and Hodge duals can be found in appendix (C).

Conserved Currents and Ward Identities: Now we need to decompose the currents in this basis,

$$T^M{}_A = \begin{pmatrix} \times & \times & \times \\ -\rho^\mu & -\epsilon^\mu & p^\mu{}_a \end{pmatrix}, \quad J^M = \begin{pmatrix} \times \\ j^\mu \end{pmatrix}, \quad \Sigma^{\sim AB} = \times, \quad \Sigma^{\mu AB} = \begin{pmatrix} 0 & \times & \times \\ \times & 0 & \tau^{\mu b} \\ \times & -\tau^{\mu a} & \sigma^{\mu ab} \end{pmatrix}.$$

(2.34)

Here we have denoted unphysical components by $\times$ which can be eliminated by redefinitions eqns. (2.17) and (2.20). We identify ρ^μ as mass current, ϵ^μ as energy current, $p^{\mu a}$ as momentum current, $\tau^{\mu a}$ as temporal spin current, $\sigma^{\mu ab}$ as spatial spin current, and finally j^μ as charge current. We can also project the μ index in these currents onto $\mathbb{R}^{(d)}$ to get corresponding 'spatial currents'. On the other hand we define various densities as the projection of these currents along n_μ,

$$\rho = n_\mu \rho^\mu, \quad \epsilon = n_\mu \epsilon^\mu, \quad p^a = n_\mu p^{\mu a}, \quad \tau^a = n_\mu \tau^{\mu a}, \quad \sigma^{ab} = n_\mu \sigma^{\mu ab}, \quad q = n_\mu j^\mu. \quad (2.35)$$

In terms of these, physical components of Ward identities eqn. (2.19) can be expressed as,

$$
\begin{aligned}
\text{Mass Cons. (Continuity):} \quad & \underline{\nabla}_\mu \rho^\mu = 0, \\
\text{Energy Cons. (Time Translation):} \quad & \underline{\nabla}_\mu \epsilon^\mu = [\text{power}] - p^{\mu a} c_{\mu a}, \\
\text{Momentum Cons. (Translations):} \quad & \underline{\nabla}_\mu p^\mu{}_a = [\text{force}]_a - \rho^\mu c_{\mu a}, \\
\text{Temporal Spin Cons. (Galilean Boosts):} \quad & \underline{\nabla}_\mu \tau^\mu{}_a = \frac{1}{2}(\rho_a - p_a), \\
\text{Spatial Spin Cons. (Rotations):} \quad & \underline{\nabla}_\mu \sigma^{\mu ab} = p^{[ba]} + 2\tau^{\mu[a} c_\mu{}^{b]}, \\
\text{Charge Cons. (Gauge Transformations):} \quad & \underline{\nabla}_\mu j^\mu = 0,
\end{aligned}
$$

(2.36)

where $\underline{\nabla}_\mu = \nabla_\mu + v^\nu H_{\nu\mu} - e_a{}^\nu T^a{}_{\nu\mu}$. These are the (non-anomalous) conservation laws of a Galilean theory with spin current. We have mentioned above what are the conserved quantities (and what is the underlying symmetry). The temporal conservation equation, which is slightly less familiar, is akin to the Milne boost Ward identity of torsionless case, which states that spatial mass current must equal the momentum density (look e.g. [18] and follow references therein). Here [power] and [force]$_a$ are power and force densities due to background fields,

$$[\text{power}] = -v^\nu \left(H_{\nu\mu} \epsilon^\mu + T_{+\nu\mu} \rho^\mu + T^a{}_{\nu\mu} p^\mu{}_a + R_{\mu\nu}{}^a{}_+ \tau^\mu{}_a + R_{\nu\mu ab} \sigma^{\mu ba} + F_{\nu\mu} \cdot j^\mu \right),$$

$$[\text{force}]_a = e_a{}^\nu \left(H_{\nu\mu} \epsilon^\mu + T_{+\nu\mu} \rho^\mu + T^a{}_{\nu\mu} p^\mu{}_a + R_{\mu\nu}{}^a{}_+ \tau^\mu{}_a + R_{\nu\mu ab} \sigma^{\mu ba} + F_{\nu\mu} \cdot j^\mu \right), \quad (2.37)$$

which act as energy and momentum sources respectively. The terms coupling to $c_{\mu a}$ in eqn. (2.36) are due to the chosen Galilean frame (time data) not being globally inertial and hence causes pseudo-power, pseudo-force and pseudo-torque.

One could have taken a slightly different approach to get these Ward identities and performed null reduction at the level of partition function eqn. (2.16) itself,

$$\delta W = \int \{dx^\mu\} \sqrt{|\gamma|} \left(\rho^\mu \delta B_\mu - \epsilon^\mu \delta n_\mu + p^\mu{}_a \delta e^a{}_\mu + 2\tau^{\mu a} \delta c_{\mu a} + \sigma^{\mu a}{}_b \delta C^b{}_{\mu a} + j^\mu \cdot \delta A_\mu \right), \quad (2.38)$$

where $\gamma_{\mu\nu} = h_{\mu\nu} + n_\mu n_\nu$ and $\gamma = \det \gamma_{\mu\nu} = -G$. Symmetry data ψ_ξ breaks up in NC basis as,

$$\psi_\xi^{\text{NC}} = \left\{ \quad \Lambda_{M(\xi)} \equiv -\xi^\sim, \quad \xi^\mu, \quad [\Lambda_{\tau(\xi)}]_a \equiv [\Lambda_{\Sigma(\xi)}]^-{}_a, \quad [\Lambda_{\sigma(\xi)}]^a{}_b \equiv [\Lambda_{\Sigma(\xi)}]^a{}_b, \quad \Lambda_{(\xi)} \quad \right\}, \quad (2.39)$$

Variation of various constituent fields under the action of ψ_ξ^{NC} (also denoted as δ_ξ) can be obtained via null reduction[14],

$$\delta_\xi B_\mu = \pounds_\xi B_\mu + \partial_\mu \Lambda_{\text{M}(\xi)} + [\Lambda_{\tau(\xi)}]_a e^a{}_\mu = \partial_\mu \nu_{\text{M}(\xi)} + \xi^\nu T_{+\nu\mu} - \xi^\nu c_{\mu\nu} - [\nu_{\tau(\xi)}]_a e^a{}_\mu$$

$$\delta_\xi n_\mu = \pounds_\xi n_\mu = \partial_\mu \xi^+ - \xi^\nu H_{\nu\mu}$$

$$\delta_\xi e^a{}_\mu = \pounds_\xi e^a{}_\mu - [\Lambda_{\sigma(\xi)}]^a{}_b e^b{}_\mu - [\Lambda_{\tau(\xi)}]^a n_\mu = \nabla_\mu \xi^a + \xi^+ c_\mu{}^a + \xi^\nu T^a{}_{\nu\mu} - [\nu_{\sigma(\xi)}]^a{}_b e^b{}_\mu - [\nu_{\tau(\xi)}]^a n_\mu$$

$$\delta_\xi c_{\mu a} = \pounds_\xi c_{\mu a} + \left(\partial_\mu [\Lambda_{\tau(\xi)}]_a - C^b{}_{\mu a}[\Lambda_{\tau(\xi)}]_b \right) + [\Lambda_{\sigma(\xi)}]^b{}_a c_{\mu b} = \nabla_\mu [\nu_{\tau(\xi)}]_a + [\nu_{\sigma(\xi)}]^b{}_a c_{\mu b} - \xi^\nu R_{\nu\mu+a}$$

$$\delta_\xi C^a{}_{\mu b} = \pounds_\xi C^a{}_{\mu b} + \left(\partial_\mu [\Lambda_{\sigma(\xi)}]^a{}_b + C^a{}_{\mu c}[\Lambda_{\sigma(\xi)}]^c{}_b - C^c{}_{\mu b}[\Lambda_{\sigma(\xi)}]^a{}_c \right) = \nabla_\mu [\nu_{\sigma(\xi)}]^a{}_b + \xi^\nu R_{\nu\mu}{}^a{}_b,$$

$$\delta_\xi A_\mu = \pounds_\xi A_\mu + \partial_\mu \Lambda_{(\xi)} + [A_\mu, \Lambda_{(\xi)}] = \nabla_\mu \nu_{(\xi)} + \xi^\nu F_{\nu\mu}. \tag{2.40}$$

Looking at these expressions we can identify $\Lambda_{\text{M}(\xi)}$ as continuity parameter, ξ^μ the spacetime translation parameter, $[\Lambda_{\tau(\xi)}]_a$ as Galilean boost parameter, $[\Lambda_{\sigma(\xi)}]^a{}_b$ as rotation parameter, and $\Lambda_{(\xi)}$ as gauge parameter. It is further noteworthy that $\xi^+ = n_\mu \xi^\mu$ and $\xi^a = e^a{}_\mu \xi^\mu$ serve as time translation and space translation parameters respectively. Demanding invariance of eqn. (2.38) under all these parameters one can recover Ward identities eqn. (2.36). One can compare these results to those of [4].

In the first equation of (2.40) we have defined *scaled* *total* *mass chemical potential* associated with ψ_ξ as $\nu_{\text{M}(\xi)} = \Lambda_{\text{M}(\xi)} + \xi^\mu B_\mu = \xi^M \bar{V}_{(T)M}$. It is differs from the scaled mass chemical potential $\varpi_{(\xi)}$ defined in eqn. (2.21), by a 'kinetic' part, $\nu_{\text{M}(\xi)} = \varpi_{(\xi)} - \frac{1}{2\vartheta_{(\xi)}} \bar{V}^a_{(\xi)} \bar{V}_{(\xi)a}$. Following eqn. (2.22) we can also define total mass chemical potential as, $\mu_{\text{M}(\xi)} = \vartheta_{(\xi)} \nu_{\text{M}(\xi)} = \mu_{\varpi(\xi)} - \frac{1}{2} \bar{V}^a_{(\xi)} \bar{V}_{(\xi)a}$.[15]

Finally we would like to note that mass being exactly conserved is a consequence of compatibility. Otherwise the respective conservation equation would look something like,

$$-\underline{\nabla}_\mu \rho^\mu = T^A{}_{\sim M} T^M{}_A + R_{\sim M}{}^A{}_B \Sigma^{MB}{}_A + F_{\sim M} \cdot J^M$$
$$= T^M{}_A \left(E^B{}_M [\nu_{\Sigma(V)}]^A{}_B - E^A{}_N \nabla_M V^N \right) - \Sigma^{MB}{}_A \nabla_M [\nu_{\Sigma(V)}]^A{}_B - J^M \cdot \nabla_M \nu_{(V)}. \tag{2.42}$$

One can clearly see that $\nabla_M V^N$, $[\nu_{\Sigma(V)}]^A{}_B$ and $\nu_{(V)}$ source this conservation. One of the prime reasons for imposing compatibility is to get rid of these mass sources.

Comparing our analysis to the torsionless case of [2] one would note that authors there also imposed a 'T-redefinition' invariance in the theory, which leads to Galilean boost transformation upon reduction. Note that on defining $\bar{\psi}^\mu = [\Lambda_{\tau(\xi)}]^a e_a{}^\mu$, our Galilean boost transformation,

$$\delta_\xi\big|_{\bar{\psi}} B_\mu = \bar{\psi}_\mu, \qquad \delta_\xi\big|_{\bar{\psi}} v^\mu = \bar{\psi}^\mu, \qquad \delta_\xi\big|_{\bar{\psi}} h_{\mu\nu} = -2\bar{\psi}_{(\mu} n_{\nu)}, \tag{2.43}$$

boils down to (infinitesimal) T-redefinition transformation of [2]. Hence for us imposing T-redefinition is redundant. Actually even for [2], imposing T-redefinition was redundant, as the authors noted that the corresponding Ward identity is trivially satisfied for theories obtained

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[14] Note that fixing V^M or V^A is not a 'gauge fixing', as transformations shifting these are not part of our symmetries on null backgrounds. On the other hand fixing $\bar{V}^A_{(T)}$ is a gauge fixing which can be violated off-shell. If we fix this gauge even off-shell we would miss the corresponding temporal spin conservation equation.

[15] Although we will not be using it in this work, it is interesting to differentiate in two types of mass chemical potentials. Consider that our system has a preferrend symmetry data ψ_U. Naively μ_ϖ corresponds to the first law of thermodynamics written in terms of internal energy E, while μ_{M} corresponds to the first law in terms of total energy $E_{\text{tot}} = E + \frac{1}{2} R u^a u_a$ (where $u^a = \bar{V}^a_{(U)}$; subscripts (U) have been dropped),

$$dE = \vartheta dS + \mu_\varpi dR + [\mu_\Sigma]^B{}_A d[Q_\Sigma]^A{}_B + \mu \cdot dQ, \quad dE_{\text{tot}} = \vartheta dS + \mu_{\text{M}} dR + u^a d(R u_a) + [\mu_\Sigma]^B{}_A d[Q_\Sigma]^A{}_B + \mu \cdot dQ. \tag{2.41}$$

When working with total energy as a thermodynamic variable, the thermodynamics becomes frame dependent and first law has a term corresponding to work done due to momentum density $R u_a$ as well. Notation used here can be found in [2].

by null reduction. It was helpful however to have this transformation there, because Galilean currents are not boost invariant and there was no non-trivial inherent symmetry of the partition function to keep track of these transformations.

3 | Galilean Gauge and Spin Anomalies

In the previous section we used null reduction to obtain Ward identities for a Galilean theory with non-trivial spin current. Now we would like to take this a step ahead and ask – how these identities modify in presence of a gauge and gravitational anomalies. We would give away the suspense right away, because the following story is quite technical. As one would expect, the gauge anomaly in null theory translates to gauge anomaly in Galilean theory as well, while the gravitational anomaly manifests itself purely through spatial spin conservation. Other 4 out of 6 conservation laws in eqn. (2.36) remain non-anomalous. In formulation of anomalies in Cartan language it is not surprising; it is known that gravitational anomaly acts as Lorentz anomaly in this formalism and only violates spin conservation [43]. What is surprising is that we did not find any anomalies in temporal spin conservation (or correspondingly Milne boost Ward identity). We do not claim that this anomaly cannot be introduced by other means or that we are not missing anything, but the fact that number of anomaly coefficients in our treatment and that of a relativistic theory match exactly (in fact they both are determined by the same anomaly polynomial), gives us some confidence in our results.

3.1   Anomaly Inflow on Einstein-Cartan Backgrounds

In relativistic theories anomaly inflow has been by far the best way to understand gauge and gravitational anomalies [45]. We would like to take a step back and first describe the anomaly inflow mechanism for generic Einstein-Cartan theories. The extension to null theories will then be more transparent. A good discussion on anomaly inflow for torsionless relativistic theories can be found in §2 of [46]. We consider that our manifold of interest $\mathcal{M}_{(d+2)}$ lives on the boundary of a *bulk* manifold $\mathcal{B}_{(d+3)}$. Bulk coordinates are denoted with a bar, and we choose a basis $x^{\bar{M}} = \{x^{\perp}, x^M\}$, where $x^{\perp}$ corresponds to depth into the bulk. All the field content $\mathrm{E}^{\bar{A}}_{\bar{M}}$, $A_{\bar{M}}$, $C^{\bar{A}}_{\bar{M}\bar{B}}$ is extended down into the bulk with the requirement that all $\perp$ components vanish at the boundary.

Now we keep our theory of interest on $\mathcal{M}_{(d+2)}$, whose generating functional $W_{\mathcal{M}}$ is not necessarily invariant under symmetries of the theory – i.e. is anomalous. In the bulk we keep some theory with generating functional $W_{\mathcal{B}}$, which is invariant under all symmetries up to some non-trivial boundary terms. The full theory described by $W = W_{\mathcal{M}} + W_{\mathcal{B}}$ is assumed to be invariant under all symmetries. It is actually this non-trivial boundary term in $W_{\mathcal{B}}$ which induces anomaly in the boundary theory, hence the name *anomaly inflow*. Note that in absence of anomalies $W_{\mathcal{B}} = 0 \Rightarrow W = W_{\mathcal{M}}$ which was discussed in last section. Let us assume for now that we have figured out such a $W_{\mathcal{B}}$, and parametrize its infinitesimal variation as[16],

$$\delta W_{\mathcal{B}} = \int \{\mathrm{d}x^{\bar{M}}\} \sqrt{|\mathrm{G}_{(d+3)}|} \left(\mathrm{T}_{\mathrm{H}}^{\bar{M}\bar{A}} \delta \mathrm{E}_{\bar{A}\bar{M}} + \Sigma_{\mathrm{H}}^{\bar{M}\bar{A}\bar{B}} \delta C_{\bar{B}\bar{M}\bar{A}} + \mathrm{J}_{\mathrm{H}}^{\bar{M}} \cdot \delta A_{\bar{M}}\right)$$
$$+ \int \{\mathrm{d}x^M\} \sqrt{|\mathrm{G}|} \left(\mathrm{T}_{\mathrm{BZ}}^{MA} \delta \mathrm{E}_{AM} + \Sigma_{\mathrm{BZ}}^{MAB} \delta C_{BMA} + \mathrm{J}_{\mathrm{BZ}}^{M} \cdot \delta A_M\right). \quad (3.1)$$

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[16]Note that $SO(d+1,1)$ transformations leave the flat metric η_{AB} invariant, hence it can commute freely through variations.

It is generally known that $W_{\mathcal{B}}$ is topological and hence does not depend on the metric/Vielbein, but we keep it here just for the sense of generality; we will see the respective terms vanishing when we put in the allowed expression for $W_{\mathcal{B}}$. The *Hall currents* in the bulk must be manifestly symmetry covariant by definition of $W_{\mathcal{B}}$. The boundary *Bardeen-Zumino currents* on the other hand are symmetry non-covariant. Variation of $W_{\mathcal{M}}$ will generate the *consistent currents* which due to anomaly are not symmetry covariant either,

$$\delta W_{\mathcal{M}} = \int \{\mathrm{d}x^M\} \sqrt{|\mathrm{G}|} \left(T_{\mathrm{cons}}^{MA} \delta \mathrm{E}_{AM} + \Sigma_{\mathrm{cons}}^{MAB} \delta C_{BMA} + J_{\mathrm{cons}}^M \cdot \delta A_M \right). \tag{3.2}$$

Since the full partition function W should be symmetry invariant, we can read out the symmetry covariant, *covariant currents* in the boundary,

$$T^{MA} = T_{\mathrm{cons}}^{MA} + \mathrm{T}_{\mathrm{BZ}}^{MA}, \qquad \Sigma^{MAB} = \Sigma_{\mathrm{cons}}^{MAB} + \Sigma_{\mathrm{BZ}}^{MAB}, \qquad J^M = J_{\mathrm{cons}}^M + \mathrm{J}_{\mathrm{BZ}}^M. \tag{3.3}$$

Demanding W to be invariant under all symmetries of the theory, we will get the anomalous Ward identities for these currents,

$$\begin{aligned}
\underline{\nabla}_M T^M{}_N &= \mathrm{T}^A{}_{NM} T^M{}_A + R_{NM}{}^A{}_B \Sigma^{MB}{}_A + F_{NM} \cdot J^M + \mathrm{T}_{\mathrm{H}\,N}^\perp, \\
\underline{\nabla}_M \Sigma^{MAB} &= T^{[BA]} + \Sigma_{\mathrm{H}}^{\perp AB}, \\
\underline{\nabla}_M J^M &= \mathrm{J}_{\mathrm{H}}^\perp.
\end{aligned} \tag{3.4}$$

We verify that the bulk Hall currents source anomaly in the boundary theory. On the other hand Hall currents themselves must satisfy the non-anomalous Ward identities eqn. (2.13) in the bulk, which would be trivial if $W_{\mathcal{B}}$ is chosen properly. Now depending on the field content of the theory one would have to construct the most generic allowed $W_{\mathcal{B}}$ and read out from there the Hall currents. This would determine the most generic anomalies that can occur in the respective theory which can be modeled using anomaly inflow. In notation of differential forms $W_{\mathcal{B}}$ is given by integration of a full rank form $\boldsymbol{I}^{(d+3)}$,

$$W_{\mathcal{B}} = \int_{\mathcal{B}_{(d+3)}} \boldsymbol{I}^{(d+3)}. \tag{3.5}$$

Requirement that its variation should be symmetry invariant up to a boundary term can be recast into the requirement that $\boldsymbol{\mathcal{P}}^{(d+4)} = \mathrm{d}\boldsymbol{I}^{(d+3)}$ should be symmetry invariant. $\boldsymbol{\mathcal{P}}^{(d+4)}$ is called the anomaly polynomial, which encodes all the non-trivial information about anomaly. It is evident that $\boldsymbol{\mathcal{P}}^{(d+4)}$ needs to be closed, symmetry invariant, and should not be expressible as exterior derivative of a symmetry invariant form. For example on usual backgrounds (not null), $\boldsymbol{\mathcal{P}}^{(2n+4)}$ is given by the Chern-Simons anomaly polynomial $\boldsymbol{\mathcal{P}}_{\mathrm{CS}}^{(2n+4)}$ for even dimensional boundary theories, and no such term is possible in odd dimensions. $\boldsymbol{\mathcal{P}}_{\mathrm{CS}}^{(2n+4)}$ is a 'polynomial' made out of Chern classes of $\boldsymbol{F}$ and Pontryagin classes of $\boldsymbol{R}$. Look e.g. [46] for more details.

3.2   Anomaly Inflow on Null/Newton-Cartan Backgrounds

Now we come back to our case of interest – null backgrounds. Above procedure goes more or less through, except that bulk $\mathcal{B}_{(d+3)}$ is now required to possess a compatible null isometry ψ_V, which translates itself to a compatible null isometry on the boundary $\mathcal{M}_{(d+2)}$ since all the $\perp$ components vanish. Variation of $W_{\mathcal{B}}$ in eqn. (3.1) remains unchanged under a ψ_V compatible variation, except all the currents now follow redefinitions specified in eqns. (2.17) and (2.20).

Consequently we can find the anomalous Ward identities for null backgrounds,

$$\underline{\nabla}_M T^M{}_N = \mathrm{T}^A{}_{NM} T^M{}_A + R_{NM}{}^A{}_B \Sigma^{MB}{}_A + F_{NM} \cdot J^M + \mathrm{T}^\perp_{\mathrm{H}N},$$
$$\underline{\nabla}_M \Sigma^{MAB} = T^{[BA]} + \Sigma^{\perp AB}_{\mathrm{H}} + \#^{[A} V^{B]},$$
$$\underline{\nabla}_M J^M = \mathrm{J}^\perp_{\mathrm{H}}, \tag{3.6}$$

for some $\#^M$. These are same as non-null identities except that just like non-anomalous case some components of spin current conservation have been discarded using spin current redefinition eqn. (2.20). Physical components of these laws can be expressed after reduction as anomalous Galilean conservation laws,

$$
\begin{aligned}
\text{Mass Cons. (Continuity):} \quad & \underline{\nabla}_\mu \rho^\mu = \rho^\perp_{\mathrm{H}}, \\
\text{Energy Cons. (Time Translation):} \quad & \underline{\nabla}_\mu \epsilon^\mu = [\text{power}] - p^{\mu a} c_{\mu a} + \varepsilon^\perp_{\mathrm{H}}, \\
\text{Momentum Cons. (Translations):} \quad & \underline{\nabla}_\mu p^\mu{}_a = [\text{force}]_a - \rho^\mu c_{\mu a} + \mathrm{p}^\perp_{\mathrm{H}a}, \\
\text{Temporal Spin Cons. (Galilean Boosts):} \quad & \underline{\nabla}_\mu \tau^{\mu a} = \frac{1}{2} (\rho^a - p^a) + \tau^{\perp a}_{\mathrm{H}}, \\
\text{Spatial Spin Cons. (Rotations):} \quad & \underline{\nabla}_\mu \sigma^{\mu ab} = p^{[ba]} + 2\tau^{\mu[a} c_\mu{}^{b]} + \sigma^{\perp ab}_{\mathrm{H}}, \\
\text{Charge Cons. (Gauge Transformations):} \quad & \underline{\nabla}_\mu j^\mu = \mathrm{j}^\perp_{\mathrm{H}}, \tag{3.7}
\end{aligned}
$$

where we have decomposed the Hall currents as,

$$\mathrm{T}^\perp_{\mathrm{H}A} = \begin{pmatrix} -\rho^\perp_{\mathrm{H}} & -\varepsilon^\perp_{\mathrm{H}} & \mathrm{p}^\perp_{\mathrm{H}a} \end{pmatrix}, \qquad \mathrm{J}^\perp_{\mathrm{H}} = \mathrm{j}^\perp_{\mathrm{H}}, \qquad \Sigma^{\perp AB}_{\mathrm{H}} = \begin{pmatrix} 0 & \times & \times \\ \times & 0 & \tau^{\perp b}_{\mathrm{H}} \\ \times & -\tau^{\perp a}_{\mathrm{H}} & \sigma^{\perp ab}_{\mathrm{H}} \end{pmatrix}. \tag{3.8}$$

We hence see that in principle anomaly inflow can destroy all the conservation laws. It is now the form of $\boldsymbol{\mathcal{P}}^{(d+4)}$ which will determine how many of these anomalies are permissible and in what dimensions.

On even dimensional ($d = 2n$) null backgrounds the allowed anomaly polynomial takes the usual Chern-Simons structure of relativistic theories $\boldsymbol{\mathcal{P}}^{(2n+4)} = \boldsymbol{\mathcal{P}}^{(2n+4)}_{\mathrm{CS}}$ which is made up of Chern classes of $\boldsymbol{F}$ and Pontryagin classes of $\boldsymbol{R}$. Note however that neither of $\boldsymbol{F}, \boldsymbol{R}$ have a leg along V, hence $\boldsymbol{\mathcal{P}}^{(2n+4)}_{\mathrm{CS}}$ is identically zero. The corresponding $\boldsymbol{I}_{\mathrm{CS}}$ might still have a leg along V since $\iota_V \boldsymbol{A}, \iota_V \boldsymbol{C}^A{}_B \neq 0$ for a general null theory. But one can check that the corresponding $\perp$ components of (dual) Hall currents again have no leg along V and hence the Ward identities become non-anomalous. This suggests that we cannot get anomalies in an even dimensional null theory, and hence odd dimensional Galilean theories are anomaly free.

At this point we would like to point out some subtle differences from the analysis of [37]. In the cited reference author does not impose compatibility of the isometry, and hence $\boldsymbol{F}, \boldsymbol{R}$ does have a leg along V. This results in anomalous conservation laws that crucially depend on $\nu_{(V)}$, $[\nu_{\Sigma(V)}]^A{}_B$ – additional fields which are otherwise switched off by compatibility. As we mentioned in the introduction, we have chosen to switch off these fields as they serve as a 'mass source' in the Galilean theory, and we do not see these sources in non-relativistic theories that occur in nature.

Now we shift our attention to the more interesting case of odd dimensional ($d = 2n - 1$) null backgrounds. One can check that with the field content at hand, it is not possible to naively construct an anomaly polynomial. Following [2] however, we note that we can remedy this problem by introducing auxiliary *time data* ψ_T that was used to perform null reduction in § 2.3.

Using the corresponding $\bar{V}_{(T)}$ defined in eqn. (2.23), we can write the only allowed anomaly polynomial,

$$\boldsymbol{\mathcal{P}}^{(2n+3)} = \bar{\boldsymbol{V}}_{(T)} \wedge \boldsymbol{\mathcal{P}}^{(2n+2)}_{\mathrm{CS}}, \tag{3.9}$$

where $\bar{\boldsymbol{V}}_{(T)} = \bar{V}_{(T)M} \mathrm{d}x^M$. Although this expression has an explicit dependence on ψ_T, one can show that it is invariant under any arbitrary redefinition of ψ_T. This follows from the fact that change in $\bar{V}_{(T)M}$ does not have any leg along V, due to normalization property $\delta(\bar{V}_{(T)M}V^M) = V^M \delta\bar{V}_{(T)M} = 0$. For this reason we drop the subscript (T) from $\bar{V}_{(T)}$ this point onward. Readers should convince themselves that there are no more terms which can be written in the anomaly polynomial. However, we have a problem; anomaly polynomial eqn. (3.9) is not exact,

$$\boldsymbol{\mathcal{P}}^{(2n+3)} = -\mathrm{d}\left(\bar{\boldsymbol{V}} \wedge \boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}\right) + \mathrm{d}\bar{\boldsymbol{V}} \wedge \boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}. \tag{3.10}$$

Hence for $\boldsymbol{I}^{(2n+2)}$ to be well defined, the second term must vanish. In general however it does not as $\boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}$ does have a leg along V. It hence forces us to choose *transverse gauge* for the null isometry generated ψ_V, i.e.,

$$\Lambda_{(V)} = [\Lambda_{\Sigma(V)}]^A{}_B = 0, \tag{3.11}$$

which ensures $\boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}$ does not have any leg along V. Some comments are due; different choices of ψ_V represent different null theories, as we are not allowed to perform transformations which alter these (we demanded the partition function to be invariant under ψ_V preserving transformations). Hence the statement is that not all such null theories can be anomalous; only null theories with transverse null isometry can exhibit anomalies. Note that in conventional null reduction, one generally chooses $\psi_V = \{\partial_{\sim}, 0, 0\}$ which by definition satisfies the transversality requirement. Modulo this subtlety we can find,

$$\boldsymbol{I}^{(2n+2)} = -\bar{\boldsymbol{V}} \wedge \boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}. \tag{3.12}$$

Computing its variation one can find *Hall* and *Bardeen-Zumino* currents defined in eqn. (3.1),

$$\mathrm{T}_{\mathrm{H}}^{\bar{M}\bar{A}} = 0, \qquad \star_{(2n+2)}\boldsymbol{\Sigma}_{\mathrm{H}}^{\bar{A}\bar{B}} = \bar{\boldsymbol{V}} \wedge \frac{\partial \boldsymbol{\mathcal{P}}^{(2n+2)}_{\mathrm{CS}}}{\partial \boldsymbol{R}_{\bar{B}\bar{A}}}, \qquad \star_{(2n+2)}\mathbf{J}_{\mathrm{H}} = \bar{\boldsymbol{V}} \wedge \frac{\partial \boldsymbol{\mathcal{P}}^{(2n+2)}_{\mathrm{CS}}}{\partial \boldsymbol{F}},$$

$$\mathrm{T}_{\mathrm{BZ}}^{MA} = 0, \qquad \star\boldsymbol{\Sigma}_{\mathrm{BZ}}^{AB} = \bar{\boldsymbol{V}} \wedge \frac{\partial \boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}}{\partial \boldsymbol{R}_{BA}}, \qquad \star\mathbf{J}_{\mathrm{BZ}} = -\bar{\boldsymbol{V}} \wedge \frac{\partial \boldsymbol{I}^{(2n+1)}_{\mathrm{CS}}}{\partial \boldsymbol{F}}. \tag{3.13}$$

We verify that $T_{\mathrm{H}}^{\bar{M}\bar{A}}, T_{\mathrm{BZ}}^{MA}$ vanish. It immediately follows that mass, energy and momentum conservation are non-anomalous. Also the (Milne) boost Ward identity stays non-anomalous as matrix indices of $\Sigma_{\mathrm{H}}^{MAB}$ comes from $\boldsymbol{R}^A{}_B$ which have a zero contraction along V. Again this follows from compatibility of isometry, and is not true for considerations of [37], which is why they find a Milne anomaly. These statements can be recast as,

$$\rho_{\mathrm{H}}^{\perp} = \varepsilon_{\mathrm{H}}^{\perp} = \mathrm{p}_{\mathrm{H}}^{\perp a} = \tau_{\mathrm{H}}^{\perp a} = 0, \tag{3.14}$$

which follows directly from null reduction. The only laws that get anomalous are hence spin and charge conservation. Explicit expressions for their Hall currents follow from reduction,

$$\mathrm{j}_{\mathrm{H}}^{\perp} = -*_{\uparrow}\left[\frac{\partial \mathfrak{p}^{(2n+2)}}{\partial \boldsymbol{F}}\right], \qquad \sigma_{\mathrm{H}}^{\perp ab} = -*_{\uparrow}\left[\frac{\partial \mathfrak{p}^{(2n+2)}}{\partial \boldsymbol{R}_{ba}}\right]. \tag{3.15}$$

Here we have formally denoted $\boldsymbol{\mathcal{P}}^{(2n+2)}_{\mathrm{CS}}$ as $\mathfrak{p}^{(2n+2)}$ after reduction; the distinction is purely notational. $*_{\uparrow}$ is the Hodge dual associated with raised Newton-Cartan volume element $\varepsilon_{\uparrow}$; look

appendix (C) for more details. Putting back eqns. (3.14) and (3.15) into eqn. (2.36) we can get the anomalous Ward identities for Galilean theories.

Before closing this section, we would like to make some due comments on even dimensional case. One might worry that we can use ψ_T to define anomalies in even dimensions as well. However one can check that only possible symmetry-covariant anomaly polynomial we can write involving ψ_T is,

$$\boldsymbol{\mathcal{P}}^{(2n+4)} = \boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{\mathcal{P}}_{\mathrm{CS}}^{(2n+2)}, \tag{3.16}$$

where $\boldsymbol{V} = V_M \mathrm{d}x^M$. This anomaly polynomial is however not an exact form,

$$\boldsymbol{\mathcal{P}}^{(2n+4)} = \mathrm{d}\left(\boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{I}_{\mathrm{CS}}^{(2n+1)}\right) - \boldsymbol{H} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{I}_{\mathrm{CS}}^{(2n+1)} + \boldsymbol{V} \wedge \mathrm{d}\bar{\boldsymbol{V}} \wedge \boldsymbol{I}_{\mathrm{CS}}^{(2n+1)}. \tag{3.17}$$

The last term can be removed just like before by going to transverse gauge, but the second last term cannot. We hence see that the current formalism does not allow for anomalous even dimensional null theories. From this point onward we will assume our null background to be odd dimensional, and hence set $d = 2n - 1$.

With this we conclude our discussion on generic anomalous Galilean theories. Using the construction of null backgrounds, we have found a set of conservation laws which determine the dynamics of these theories in terms of a set of currents. These laws have already been well explored in literature, but the fact that they follow by trivially choosing a basis in a higher dimensional null theory is to be appreciated. Going along lines of [4], it appears to us that null backgrounds are the true 'covariant' and 'frame independent' formalism of Galilean physics, which appear pretty natural from a 5 dimensional perspective. We refer the reader to appendix (B) for more comments on these issues.

All of the results presented here are in Newton-Cartan notation, which is the natural covariant prescription for Galilean physics. In appendix (A) we present some of our results in conventional non-covariant notation, for the benefit of readers who are not comfortable with the Newton-Cartan language. Even otherwise, seeing the results in non-covariant form might help us relate it better to day to day physics, where we are used to viewing time and space separately.

4 | Anomalous Galilean Hydrodynamics

In previous sections we have obtained the anomalous conservation laws for a null/Galilean theory with non-zero spin current. Here we want to study these theories in hydrodynamic limit – near equilibrium effective description of any quantum system. Before going to that let us make some general comments about hydrodynamics on Einstein-Cartan backgrounds. We start by picking up a collection of *hydrodynamic fields* which can be exactly solved for using the equations of motion of the theory. Since there is an equation of motion for each symmetry data, we choose hydrodynamic fields to be a set of symmetry data[17] $\psi_U = \{U^M, [\Lambda_\Sigma]^A{}_B, \Lambda\}$. The *fluid* (hydrodynamic system) is characterized by conserved currents T^{MA}, Σ^{MAB}, J^M written as the most generic tensors made out of hydrodynamic fields ψ_U and background sources $\mathrm{E}^A{}_M$, $C^A{}_{MB}$, A_M, arranged in a derivative expansion. These are known as *constitutive relations* of the fluid. Near equilibrium assumption of hydrodynamics implies that derivatives of quantities are small compared to quantities themselves, which allows for proper truncation of the derivative expansion as dictated by the cause. Dynamics of these constitutive relations in turn is governed by the

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[17]We drop the subscript (U) for ψ_U and hope that it will be clear by context.

conservation laws eqn. (3.4). These constitutive relations are further subjected to the second law of thermodynamics, i.e. the requirement of an entropy current S^M such that $\underline{\nabla}_M S^M \geq 0$, whenever equations of motion are satisfied. This requirement imposes various constraints on the constitutive relations, and the job of hydrodynamics is to monitor these constraints. Having done so, one can in principle plug these constitutive relations back into the equations of motion and solve for exact 'configurations' of hydrodynamic fields, which is not in the scope of hydrodynamics. A nice and modern review of relativistic hydrodynamics can be found in §1 of [41].

Another notion which is inherent to any statistical system is *equilibrium*. Equilibrium is the steady state of hydrodynamics, when the fluid has come in terms with the background and has aligned itself accordingly. In this state, the fluid can be described by a partition function W^{eqb} written purely in terms of background data, and equations of motion are trivially satisfied. Equilibrium is generally defined by a collection of symmetry data $\psi_K = \{K^M, [\Lambda_{\Sigma(K)}]^A{}_B, \Lambda_{(K)}\}$ which acts as an isometry on the background. For our constitutive relations to be physical, we will need to ensure that on introducing ψ_K they trivially satisfy the equations of motion eqn. (3.4).

Please note that ψ_U is a set of variables we have picked up to solve the system; like in any field theory we could do an arbitrary field redefinition of ψ_U without changing the physics. This is known as *hydrodynamic redefinition freedom*. By convention ψ_U is defined to agree with ψ_K in equilibrium at zero derivative order (this goes into definition of fluid velocity, temperature and chemical potential in equilibrium), which fixes a huge amount of this freedom. Further fixing of this freedom can be dealt in various different ways, which takes the name of hydrodynamic frames (find more thorough discussion on these frames for null fluids in [2]). Here we would work in the so called *equilibrium frame* where $\psi_U = \psi_K$ exactly in equilibrium, not just at zero derivative order. Note that this does not fix the freedom completely, we can still perturb this relation with anything that vanishes in equilibrium. For now we conclude that on setting $\psi_U = \psi_K$ i.e. on promoting ψ_U to an isometry, the constitutive relations should identically satisfy the equations of motion.

It was noted in [3] for relativistic fluids that it is helpful to remove the clause 'whenever equations of motion are satisfied' from the second law requirement and upgrade it to an off-shell statement [47], which for us will read,

$$\underline{\nabla}_M S^M + U^N \left(\underline{\nabla}_M T^M{}_N - \mathrm{T}^A{}_{NM} T^M{}_A - R_{NM}{}^A{}_B \Sigma^{MB}{}_A - F_{NM} \cdot J^M \right)$$
$$+ [\nu_\Sigma]_{BA} \left(\underline{\nabla}_M \Sigma^{MAB} - T^{[BA]} - \Sigma_{\mathrm{H}}^{\perp AB} \right) + \nu \cdot \left(\underline{\nabla}_M J^M - \mathrm{J}_{\mathrm{H}}^{\perp} \right) \geq 0. \quad (4.1)$$

This statement is slightly different from what was considered for torsionless case in [3], but we verify its equivalence with theirs in appendix (D).

Now we come back to null fluids – fluids on null backgrounds. On null backgrounds, hydrodynamic data ψ_U needs to be compatible with ψ_V, i.e. $[\psi_V, \psi_U] = 0$ and $[\nu_\Sigma]^A{}_B V^B = 0$. This makes sense because (1) resulting constitutive relations must follow the null isometry, (2) not all components of the spin conservation in eqn. (3.6) are physical. Further, constitutive relations are allowed to depend on ψ_V as well. One can check that upon making these tweaks, off-shell second law eqn. (4.1) remains unchanged. We can now go back and study the most generic constitutive relations for null fluids, which has been thoroughly considered in [2] for a charged spinless torsionless null fluid with U(1) anomalies up to leading order in derivatives. In this work however, we are only interested in the sector of hydrodynamics that is governed and is

completely determined by the anomalies[18]. To accomplish this task in relativistic fluids, [38] (see also [48]) proposed a mechanism based on transgression forms, which allows us to 'integrate' the anomalous equations of motion eqn. (3.4) and directly figure out the anomalous contribution to constitutive relations. We will attempt to extend this construction to null fluids.

4.1 Anomalous Null Fluids

We start by defining hydrodynamic shadow gauge field and spin connection,

$$\hat{\boldsymbol{A}} = \boldsymbol{A} + \mu \boldsymbol{V}, \qquad \hat{\boldsymbol{C}}^A{}_B = \boldsymbol{C}^A{}_B + [\mu_\Sigma]^A{}_B \boldsymbol{V}, \tag{4.2}$$

where μ, $[\mu_\Sigma]^A{}_B$ are gauge and spin chemical potentials associated with ψ_U defined in eqn. (2.22). One can check that both ψ_U, ψ_V are compatible with this new gauge field and spin connection, i.e.,

$$\hat{\nu} = U^M \hat{A}_M + \Lambda = 0, \qquad [\hat{\nu}_\Sigma]^A{}_B = U^M \hat{C}^A{}_{MB} + [\Lambda_\Sigma]^A{}_B = 0,$$

$$\hat{\nu}_{(V)} = V^M \hat{A}_M = 0, \qquad [\hat{\nu}_{\Sigma(V)}]^A{}_B = V^M \hat{C}^A{}_{MB} = 0. \tag{4.3}$$

Recall that we have chosen $\Lambda_{(V)} = [\Lambda_{\Sigma(V)}]^A{}_B = 0$ to be able to define anomalies. We define the operation (ˆ) as $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}\left(\boldsymbol{A} \to \hat{\boldsymbol{A}}, \boldsymbol{C}^A{}_B \to \hat{\boldsymbol{C}}^A{}_B\right)$. One can check that the hatted field strengths also follow the null background conditions eqns. (2.14) and (2.15). We would like to import one result from transgression machinery without proof (see §11 of [49] for more details), which implies that,

$$\boldsymbol{I}_{\mathrm{CS}}^{(2n+1)} - \hat{\boldsymbol{I}}_{\mathrm{CS}}^{(2n+1)} = \boldsymbol{\mathcal{V}}_{\boldsymbol{\mathcal{P}}_{\mathrm{CS}}}^{(2n+1)} + \mathrm{d}\boldsymbol{\mathcal{V}}_{\boldsymbol{I}_{\mathrm{CS}}}^{(2n)}, \tag{4.4}$$

where,

$$\boldsymbol{\mathcal{V}}_{\boldsymbol{\mathcal{P}}_{\mathrm{CS}}}^{(2n+1)} = \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge \left(\boldsymbol{\mathcal{P}}_{\mathrm{CS}}^{(2n+2)} - \hat{\boldsymbol{\mathcal{P}}}_{\mathrm{CS}}^{(2n+2)}\right), \qquad \boldsymbol{\mathcal{V}}_{\boldsymbol{I}_{\mathrm{CS}}}^{(2n)} = \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge \left(\boldsymbol{I}_{\mathrm{CS}}^{(2n+1)} - \hat{\boldsymbol{I}}_{\mathrm{CS}}^{(2n+1)}\right). \tag{4.5}$$

One can check that these quantities are well defined. We argue that fluid in equilibrium configuration can be described by a (bulk + boundary) partition function $W^{eqb} = \mathcal{W}_{\mathcal{B}}^{eqb} + W_{\mathcal{M}}^{eqb}$ which has been discussed in preceding sections. Away from equilibrium however, the system is described by an effective action $S = S_{\mathcal{B}} + S_{\mathcal{M}}$ which boils down to $\mathcal{W}^{eqb}$ is equilibrium. We claim that appropriate $S_{\mathcal{B}}$ to generate anomalous sector of null hydrodynamics is[19],

$$S_{\mathcal{B}} = W_{\mathcal{B}} + \int_{\mathcal{B}_{(2n+2)}} \bar{\boldsymbol{V}} \wedge \hat{\boldsymbol{I}}_{\mathrm{CS}}^{(2n+1)} = -\int_{\mathcal{B}_{(2n+2)}} \bar{\boldsymbol{V}} \wedge \left(\boldsymbol{I}_{\mathrm{CS}}^{(2n+1)} - \hat{\boldsymbol{I}}_{\mathrm{CS}}^{(2n+1)}\right). \tag{4.6}$$

In equilibrium ($\psi_U = \psi_K$) and on choosing transverse gauge for ψ_K ($\Lambda_{(K)} = [\Lambda_{\Sigma(K)}]^A{}_B = 0$) the added piece vanishes, as it does not have any leg along V, and we will recover the equilibrium partition function. Using eqn. (4.4) we can decompose $S_{\mathcal{B}}$ as,

$$S_{\mathcal{B}} = \int_{\mathcal{B}_{(2n+2)}} \boldsymbol{\mathcal{V}}_{\boldsymbol{\mathcal{P}}}^{(2n+2)} + \int_{\mathcal{M}_{(2n+2)}} \boldsymbol{\mathcal{V}}_{\boldsymbol{I}}^{(2n+1)}, \tag{4.7}$$

---

[18]In relativistic hydrodynamics it is known [46] that there are certain coefficients which appear as independent constants in naive derivative expansion, but can be fixed in terms of anomaly coefficients appearing at higher derivative orders by demanding consistency of euclidean vacuum. Similar constants have also showed up for Galilean fluids in [2, 19], but their connection to anomaly is not yet clear. Here however we do not consider these contributions.

[19]It was argued by [50] that while this effective action is appropriate to give solutions to the off-shell second law of thermodynamics, minimization of this action with respect to dynamic fields does not give the correct dynamics. To get the correct dynamics we need to further modify this action in Schwinger-Keldysh formalism, which we do not touch upon here.

where we have identified,

$$\boldsymbol{\mathcal{V}}_{\boldsymbol{\mathcal{P}}}^{(2n+2)} = \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge \left(\boldsymbol{\mathcal{P}}^{(2n+3)} - \hat{\boldsymbol{\mathcal{P}}}^{(2n+3)} \right) = -\bar{\boldsymbol{V}} \wedge \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge \left(\boldsymbol{\mathcal{P}}_{\mathrm{CS}}^{(2n+2)} - \hat{\boldsymbol{\mathcal{P}}}_{\mathrm{CS}}^{(2n+2)} \right),$$

$$\boldsymbol{\mathcal{V}}_{\boldsymbol{I}}^{(2n+1)} = \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge \left(\boldsymbol{I}^{(2n+2)} - \hat{\boldsymbol{I}}^{(2n+2)} \right) = \bar{\boldsymbol{V}} \wedge \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge \left(\boldsymbol{I}_{\mathrm{CS}}^{(2n+1)} - \hat{\boldsymbol{I}}_{\mathrm{CS}}^{(2n+1)} \right). \tag{4.8}$$

The bulk term in eqn. (4.7) is manifestly symmetry invariant, and full S is symmetry invariant by definition, hence if we decompose $S_{\mathcal{M}} = S_{\text{n-a}} + S_{\mathcal{M},\text{anom}}$ with first piece being totally symmetry invariant we can infer,

$$S_{\mathcal{M},\text{anom}} = -\int_{\mathcal{B}_{(2n+2)}} \bar{\boldsymbol{V}} \wedge \hat{\boldsymbol{I}}_{\mathrm{CS}}^{(2n+1)}. \tag{4.9}$$

$S_{\mathcal{M},\text{anom}}$ will generate anomalous sector of consistent currents. On the other hand for full effective action we will be left with $S = S_{\text{anom}} + S_{\text{n-a}}$ where,

$$S_{\text{anom}} = \int_{\mathcal{B}_{(2n+2)}} \boldsymbol{\mathcal{V}}_{\boldsymbol{\mathcal{P}}}^{(2n+2)}. \tag{4.10}$$

S_{anom} will generate anomalous sector of covariant currents.

Constitutive Relations: In light of our discussion above, we should be able to generate anomalous sector of covariant currents by varying S_{anom}. We will get,

$$\delta S_{\text{anom}} = \int_{\mathcal{B}_{(2n+2)}} \left(\delta \boldsymbol{A} \wedge \cdot \star_{(2n+2)} \mathbf{J}_{\mathrm{H}} - \delta \hat{\boldsymbol{A}} \wedge \cdot \star_{(2n+2)} \hat{\mathbf{J}}_{\mathrm{H}} \right.$$

$$\left. + \delta \boldsymbol{C}^{\bar{A}}_{\bar{B}} \wedge \star_{(2n+2)} \boldsymbol{\Sigma}_{\mathrm{H}}{}^{\bar{B}}_{\bar{A}} - \delta \hat{\boldsymbol{C}}^{\bar{A}}_{\bar{B}} \wedge \star_{(2n+2)} \hat{\boldsymbol{\Sigma}}_{\mathrm{H}}{}^{\bar{B}}_{\bar{A}} \right)$$

$$+ \int_{\mathcal{M}_{(2n+1)}} \left(\delta \boldsymbol{A} \wedge \cdot \star \boldsymbol{J}_{\boldsymbol{\mathcal{P}}} + \delta \boldsymbol{C}^{A}_{B} \wedge \star \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{P}}}{}^{B}_{A} + \delta \boldsymbol{V} \wedge \star \boldsymbol{E}_{\boldsymbol{\mathcal{P}}} \right), \tag{4.11}$$

where we have defined,

$$\star \boldsymbol{E}_{\boldsymbol{\mathcal{P}}} = \frac{\partial \boldsymbol{\mathcal{V}}_{\boldsymbol{\mathcal{P}}}^{(2n+2)}}{\partial \boldsymbol{H}} = \frac{\boldsymbol{V}}{\boldsymbol{H}^{\wedge 2}} \wedge \left[\hat{\boldsymbol{\mathcal{P}}}^{(2n+3)} - \boldsymbol{\mathcal{P}}^{(2n+3)} - \boldsymbol{H} \wedge \star_{(2n+2)} \left(\mu \cdot \hat{\mathbf{J}}_{\mathrm{H}} + [\mu_\Sigma]^{A}_{B} \hat{\boldsymbol{\Sigma}}_{\mathrm{H}}{}^{B}_{A} \right) \right],$$

$$\star \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{P}}}{}^{A}_{B} = \frac{\partial \boldsymbol{\mathcal{V}}_{\boldsymbol{\mathcal{P}}}^{(2n+2)}}{\partial \boldsymbol{R}^{B}_{A}} = \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge \star_{(2n+2)} \left(\boldsymbol{\Sigma}_{\mathrm{H}}{}^{A}_{B} - \hat{\boldsymbol{\Sigma}}_{\mathrm{H}}{}^{A}_{B} \right),$$

$$\star \boldsymbol{J}_{\boldsymbol{\mathcal{P}}} = \frac{\partial \boldsymbol{\mathcal{V}}_{\boldsymbol{\mathcal{P}}}^{(2n+2)}}{\partial \boldsymbol{F}} = \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge \star_{(2n+2)} \left(\mathbf{J}_{\mathrm{H}} - \hat{\mathbf{J}}_{\mathrm{H}} \right), \tag{4.12}$$

and Hall currents have been defined in eqn. (3.13). Since S_{anom} is invariant under symmetries by construction, we can find a set of Bianchi identities these currents must follow,

$$\underline{\nabla}_M (T^M{}_N)_{\mathrm{A}} = \mathrm{T}^A{}_{NM} (T^M{}_A)_{\mathrm{A}} + R_{NM}{}^A{}_B (\Sigma^{MB}{}_A)_{\mathrm{A}} + F_{NM} (J^M)_{\mathrm{A}} - V_N \left(\mu \cdot \hat{\mathbf{J}}_{\mathrm{H}}^{\perp} + [\mu_\Sigma]_{AB} \hat{\Sigma}_{\mathrm{H}}^{\perp BA} \right),$$

$$\underline{\nabla}_M (\Sigma^{MAB})_{\mathrm{A}} = (T^{[BA]})_{\mathrm{A}} + \Sigma_{\mathrm{H}}^{\perp AB} - \hat{\Sigma}_{\mathrm{H}}^{\perp AB} + \#^{[A} V^{B]},$$

$$\underline{\nabla}_M (J^M)_{\mathrm{A}} = \mathrm{J}_{\mathrm{H}}^{\perp} - \hat{\mathrm{J}}_{\mathrm{H}}^{\perp}, \tag{4.13}$$

where we have defined anomalous 'class' of constitutive relations,

$$(T^{MA})_{\mathrm{A}} = E_{\boldsymbol{\mathcal{P}}}^{M} V^{A}, \qquad (\Sigma^{MAB})_{\mathrm{A}} = \Sigma_{\boldsymbol{\mathcal{P}}}^{MAB}, \qquad (J^M)_{\mathrm{A}} = J_{\boldsymbol{\mathcal{P}}}^{M}. \tag{4.14}$$

One can check that on plugging in $\psi_U = \psi_K$, hatted Hall currents vanish as they do not have any leg along K and V simultaneously. Consequently the Bianchi identities eqn. (4.13) reduce to equations of motion eqn. (3.6); in other words the currents $(T^{MA})_{\text{A}}$, $(\Sigma^{MAB})_{\text{A}}$, $(J^M)_{\text{A}}$ identically satisfy the equations of motion in equilibrium configuration, as required.

We would like to remind the reader that $\bar{V}$ was added as an arbitrary choice of frame and anomaly polynomial was invariant under ψ_T redefinition which shifts $\bar{V}$. The currents we have constructed should then also be invariant under ψ_T redefinition. One can check that under a ψ_T-redefinition currents in eqn. (4.12) shift by a closed form. By definition currents always have this ambiguity, hence we do not change the physics. In hydrodynamics most natural choice of ψ_T to define anomalies is ψ_U.

Adiabaticity and Entropy Current: To claim that the currents we have constructed are physical, we must find a $(S^M)_{\text{A}}$ which satisfies the off-shell second law eqn. (4.1). Anomalous sector is bound to be parity violating, implying that no scalar expression can be ensured positive definite. This turns eqn. (4.1) into a more stringent condition,

$$
\underline{\nabla}_M (S^M)_{\text{A}} + U^N \Big[\underline{\nabla}_M (T^M{}_N)_{\text{A}} - \text{T}^A{}_{NM} (T^M{}_A)_{\text{A}} - R_{NM}{}^A{}_B (\Sigma^{MB}{}_A)_{\text{A}} - F_{NM} \cdot (J^M)_{\text{A}} \Big]
$$
$$
+ [\nu_\Sigma]_{BA} \Big[\underline{\nabla}_M (\Sigma^{MAB})_{\text{A}} - (T^{[BA]})_{\text{A}} - \Sigma_{\text{H}}^{\perp AB} \Big] + \nu \cdot \Big(\underline{\nabla}_M (J^M)_{\text{A}} - \text{J}_{\text{H}}^{\perp} \Big) \geq 0, \quad (4.15)
$$

known as *adiabaticity equation* [40]. Directly putting the constitutive relations into this expression we can get,

$$
\nabla_M (S^M)_{\text{A}} = 0. \tag{4.16}
$$

Hence it suffices to choose an identically zero anomaly induced entropy current $(S^M)_{\text{A}} = 0$, to satisfy the adiabaticity equation. We would like to comment here that vanishing of anomaly induced entropy current does not rely on background being null; it is equally true for usual Einstein-Cartan backgrounds as well. See appendix (D) for more comments on the relativistic entropy current.

Equilibrium Partition Function: In the beginning of this section we argued that at equilibrium, fluid can be described by a partition function written purely in terms of background data. We would now attempt to find such an equilibrium partition function. We start by computing the variation of boundary effective action $S_{\mathcal{M},\text{anom}}$ given in eqn. (4.9),

$$
\delta S_{\mathcal{M},\text{anom}} = \int \{ \mathrm{d}x^M \} \sqrt{|G|} \Big[(T^{MA})_{\text{A}} \delta E_{AM} + \{ (\Sigma^{MAB})_{\text{A}} - \Sigma_{\text{BZ}}^{MAB} \} \delta C_{BMA} + \{ (J^M)_{\text{A}} - \text{J}_{\text{BZ}}^M \} \cdot \delta A_M \Big]
$$
$$
+ \int \{ \mathrm{d}x^M \} \sqrt{|G|} \Big[\hat{\Sigma}_{\text{BZ}}^{MAB} \delta \hat{C}_{BMA} + \hat{\text{J}}_{\text{BZ}}^M \cdot \delta \hat{A}_M \Big]. \quad (4.17)
$$

In equilibrium choosing transverse gauge for ψ_K, i.e. $\Lambda_{(K)} = [\Lambda_{\Sigma(K)}]^A{}_B = 0$, the terms in last line vanish. Hence we can define the equilibrium boundary partition function,

$$
W_{\mathcal{M},\text{anom}}^{eqb} = S_{\mathcal{M},\text{anom}} \Big|_{\psi_U = \psi_K} = - \int_{\mathcal{M}_{(2n+1)}} \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge \Big(\boldsymbol{I}^{(2n+2)} - \hat{\boldsymbol{I}}^{(2n+2)} \Big) \Big|_{\psi_U = \psi_K}. \tag{4.18}
$$

Putting it together with $W_{\mathcal{B}}$, we can get the equilibrium partition function for the full theory. In practice however, if one knows the expressions for Bardeen-Zumino currents, it suffices to have the boundary partition function to generate the covariant currents.

4.2   Null Reduction – Anomalous Galilean Fluids

Having obtained the constitutive relations for anomalous null fluids, it is now time to perform null reduction and extract out the Galilean results. To see this we can directly breakup the anomaly induced constitutive relations $(T^{MA})_{\mathrm{A}}$, $(\Sigma^{MAB})_{\mathrm{A}}$, $(J^M)_{\mathrm{A}}$ into the basis given in eqn. (2.34). A straight away computation will yield trivial identifications,

$$(\rho^\mu)_{\mathrm{A}} = 0, \qquad (p^{\mu a})_{\mathrm{A}} = 0, \qquad (\tau^{\mu a})_{\mathrm{A}} = 0, \qquad (s^\mu)_{\mathrm{A}} = 0,$$

$$(\epsilon^\mu)_{\mathrm{A}} = E_{\boldsymbol{\mathcal{P}}}^\mu, \qquad (\sigma^{\mu ab})_{\mathrm{A}} = \Sigma_{\boldsymbol{\mathcal{P}}}^{\mu ab}, \qquad (j^\mu)_{\mathrm{A}} = J_{\boldsymbol{\mathcal{P}}}^\mu. \tag{4.19}$$

We have also included an entropy current $(s^\mu)_{\mathrm{A}} = (S^\mu)_{\mathrm{A}}$ here which of-course is trivially zero. For the record we would write down the off-shell second law of thermodynamics for Galilean fluids,

$$\underline{\nabla}_\mu s^\mu + \nu_{\mathrm{M}} \underline{\nabla}_\mu \rho^\mu - \frac{1}{\vartheta}\left(\underline{\nabla}_\mu \epsilon^\mu - [\text{power}] + p^{\mu a} c_{\mu a}\right) + \frac{1}{\vartheta} u^a \left(\underline{\nabla}_\mu p^\mu{}_a - [\text{force}]_a + \rho^\mu c_{\mu a}\right)$$
$$+ [\nu_\tau]_a \left(\underline{\nabla}_\mu \tau^{\mu a} - \frac{1}{2}\left(\rho^a - p^a\right)\right) + [\nu_\sigma]_{ba}\left(\underline{\nabla}_\mu \sigma^{\mu ab} - p^{[ba]} - 2\tau^{\mu[a} c_\mu{}^{b]} - \sigma_{\mathrm{H}}^{\perp ab}\right)$$
$$+ \nu \cdot \left(\underline{\nabla}_\mu j^\mu - \mathrm{j}_{\mathrm{H}}^\perp\right) \geq 0, \tag{4.20}$$

where $u^\mu = \bar{V}_{(U)}^\mu$ (defined in eqn. (2.23)) and $u^a = e^a{}_\mu u^\mu$ is the spatial velocity of the fluid. ϑ is the temperature, $\nu_{\mathrm{M}} = \varpi - \frac{1}{2\vartheta} u^a u_a$ is the total mass chemical potential, ϖ is the mass chemical potential, $[\nu_\tau]_a = [\nu_\Sigma]_a^-$ is the boost chemical potential, $[\nu_\sigma]^a{}_b = [\nu_\Sigma]^a{}_b$ is the spatial spin chemical potential, and ν is the gauge chemical potential associated with fluid data ψ_U (find respective definitions in eqns. (2.9) and (2.21)). This expression will hugely simplify if we choose $\psi_T = \psi_U$, i.e. choose to describe fluid in its local rest frame, because then $u^a = 0$,

$$\underline{\nabla}_\mu s^\mu + \varpi \underline{\nabla}_\mu \rho^\mu - \frac{1}{\vartheta}\left(\underline{\nabla}_\mu \epsilon^\mu - [\text{power}] + p^{\mu a} c_{\mu a}\right) + [\nu_\tau]_a \left(\underline{\nabla}_\mu \tau^{\mu a} - \frac{1}{2}\left(\rho^a - p^a\right)\right)$$
$$+ [\nu_\sigma]_{ba}\left(\underline{\nabla}_\mu \sigma^{\mu ab} - p^{[ba]} - 2\tau^{\mu[a} c_\mu{}^{b]} - \sigma_{\mathrm{H}}^{\perp ab}\right) + \nu \cdot \left(\underline{\nabla}_\mu j^\mu - j_{\mathrm{H}}^\perp\right) \geq 0. \tag{4.21}$$

It should be apparent that on putting in equations of motion it simply gives the second law of thermodynamics, $\underline{\nabla}_\mu s^\mu \geq 0$. If one does not prefer to do reduction to get $(\epsilon^\mu)_{\mathrm{A}}, (j^\mu)_{\mathrm{A}}, (\sigma^{\mu ab})_{\mathrm{A}}$, these can be generated directly from the Newton-Cartan transgression form,

$$\boldsymbol{\mathcal{V}}_{\mathfrak{p}}^{(2n+1)} = -\frac{\boldsymbol{n}}{\boldsymbol{H}} \wedge \left(\mathfrak{p}^{(2n+2)} - \hat{\mathfrak{p}}^{(2n+2)}\right), \tag{4.22}$$

where $\mathfrak{p}^{(2n+2)}$ is the NC anomaly polynomial defined at the end of § 3.2, and hatted fields are,

$$\hat{\boldsymbol{A}} = \boldsymbol{A} - \mu \boldsymbol{n}, \qquad \hat{\boldsymbol{C}}^a{}_b = \boldsymbol{C}^a{}_b - [\mu_\sigma]^a{}_b \boldsymbol{n}. \tag{4.23}$$

In terms of these anomaly induced constitutive relations can be generated as,

$$(j^\mu)_{\mathrm{A}} = *_\uparrow \left[\frac{\partial \boldsymbol{\mathcal{V}}_{\mathfrak{p}}^{(2n+1)}}{\partial \boldsymbol{F}}\right]^\mu, \quad (\sigma^{\mu ab})_{\mathrm{A}} = *_\uparrow \left[\frac{\partial \boldsymbol{\mathcal{V}}_{\mathfrak{p}}^{(2n+1)}}{\partial \boldsymbol{R}_{ba}}\right]^\mu, \quad (\epsilon^\mu)_{\mathrm{A}} = *_\uparrow \left[\frac{\partial \boldsymbol{\mathcal{V}}_{\mathfrak{p}}^{(2n+1)}}{\partial \boldsymbol{H}}\right]^\mu. \tag{4.24}$$

To write the equilibrium partition function in Newton-Cartan language we can use the natural time-data in equilibrium $\psi_T = \psi_K = \psi_U$. Hence using eqn. (4.18) we can find,

$$W_{\mathrm{anom}}^{eqb} = -\int_{\mathcal{M}_{(2n)}^K} \frac{\boldsymbol{n}}{\boldsymbol{H}} \wedge \left(\mathfrak{i}^{(2n+1)} - \hat{\mathfrak{i}}^{(2n+1)}\right)\Bigg|_{\psi_U = \psi_K}, \tag{4.25}$$

where $\mathrm{d}\mathfrak{i}^{(2n+1)} = \mathfrak{p}^{(2n+2)}$; $\mathfrak{i}^{(2n+1)}$ is just $\boldsymbol{I}_{\mathrm{CS}}^{(2n+1)}$ after reduction. Please refer appendix (C) for conventions on reducing the integral.

This concludes the main abstract results of this work. We have been able to construct gauge and gravitational anomalies in Galilean theories, and find their effect on Galilean hydrodynamics. We explicitly constructed the sector of fluid constitutive relations that is totally determined in terms of anomalies. These constitutive relations obey second law of thermodynamics with a trivially zero entropy current. We also found the equilibrium partition function which generates these constitutive relations in equilibrium configuration.

5 | Examples

The entire discussion of this work till now has been very abstract. We will now try to illustrate it with few examples. In the following we will only discuss the case of abelian gauge field for simplicity. In § 5.1 we start with a thorough walkthrough example for 3 dimensional null theories (2 dimensional Galilean theories), where we perform each and every step as was done in the main work. We hope it will help the reader to understand the procedure more transparently. Later in § 5.2 we present the results for arbitrary dimensional case up to next to leading order in derivatives.

5.1 Walkthrough – 2 Spatial Dimensions

Let us go step by step for the case of 3 dimensional null backgrounds. The corresponding 5 dimensional anomaly polynomial contains squared $\boldsymbol{F}$ and $\boldsymbol{R}$,

$$\boldsymbol{\mathcal{P}}^{(5)} = \bar{\boldsymbol{V}} \wedge \left(C^{(2)} \boldsymbol{F}^{\wedge 2} + C_g^{(2)} \boldsymbol{R}_{\bar{B}}^{\bar{A}} \wedge \boldsymbol{R}_{\bar{A}}^{\bar{B}} \right), \tag{5.1}$$

from where we can read out the expression for $\boldsymbol{I}^{(4)}$,

$$\boldsymbol{I}^{(4)} = -\bar{\boldsymbol{V}} \wedge \left[C^{(2)} \boldsymbol{A} \wedge \boldsymbol{F} + C_g^{(2)} \left(\boldsymbol{C}_{\bar{B}}^{\bar{A}} \wedge \boldsymbol{R}_{\bar{A}}^{\bar{B}} - \frac{1}{3} \boldsymbol{C}_{\bar{B}}^{\bar{A}} \wedge \boldsymbol{C}_{\bar{C}}^{\bar{B}} \wedge \boldsymbol{C}_{\bar{A}}^{\bar{C}} \right) \right]. \tag{5.2}$$

From here we can define the bulk partition function $W_{\mathcal{B}} = \int_{\mathcal{B}_{(4)}} \boldsymbol{I}^{(4)}$, and compute its variation (see eqn. (3.5)),

$$\delta W_{\mathcal{B}} = \int_{\mathcal{B}_{(4)}} 2 \left(C^{(2)} \delta \boldsymbol{A} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{F} + C_g^{(2)} \delta \boldsymbol{C}_{\bar{B}}^{\bar{A}} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{R}_{\bar{A}}^{\bar{B}} \right)$$
$$- \int_{\mathcal{M}_{(3)}} \left(C^{(2)} \delta \boldsymbol{A} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{A} + C_g^{(2)} \delta \boldsymbol{C}_{B}^{A} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{C}_{A}^{B} \right). \tag{5.3}$$

Now using eqn. (3.1) or eqn. (3.13), we can find the Hall and Bardeen-Zumino currents,

$$\star_{(4)} \mathbf{J}_{\mathrm{H}} = \frac{\partial \boldsymbol{\mathcal{P}}^{(5)}}{\partial \boldsymbol{F}} = 2 C^{(2)} \bar{\boldsymbol{V}} \wedge \boldsymbol{F} \quad \Rightarrow \quad \mathrm{J}_{\mathrm{H}}^{\bar{M}} = C^{(2)} \epsilon^{\bar{N}\bar{R}\bar{S}\bar{M}} \bar{V}_{\bar{N}} F_{\bar{R}\bar{S}},$$

$$\star_{(4)} \boldsymbol{\Sigma}_{\mathrm{H}}^{\bar{A}\bar{B}} = \frac{\partial \boldsymbol{\mathcal{P}}^{(5)}}{\partial \boldsymbol{R}_{\bar{B}\bar{A}}} = 2 C_g^{(2)} \bar{\boldsymbol{V}} \wedge \boldsymbol{R}^{\bar{A}\bar{B}} \quad \Rightarrow \quad \Sigma_{\mathrm{H}}^{\bar{M}\bar{A}\bar{B}} = C_g^{(2)} \epsilon^{\bar{N}\bar{R}\bar{S}\bar{M}} \bar{V}_{\bar{N}} R_{\bar{R}\bar{S}}{}^{\bar{A}\bar{B}},$$

$$\star \mathbf{J}_{\mathrm{BZ}} = \frac{\partial \boldsymbol{I}^{(4)}}{\partial \boldsymbol{F}} = -C^{(2)} \bar{\boldsymbol{V}} \wedge \boldsymbol{A} \quad \Rightarrow \quad \mathrm{J}_{\mathrm{BZ}}^{M} = C^{(2)} \epsilon^{NRM} \bar{V}_{N} A_{R},$$

$$\star \boldsymbol{\Sigma}_{\mathrm{BZ}}^{AB} = \frac{\partial \boldsymbol{I}^{(4)}}{\partial \boldsymbol{R}_{BA}} = -C_g^{(2)} \bar{\boldsymbol{V}} \wedge \boldsymbol{C}^{AB} \quad \Rightarrow \quad \Sigma_{\mathrm{BZ}}^{MAB} = C_g^{(2)} \epsilon^{NRM} \bar{V}_{N} C^{A}{}_{R}{}^{B}. \tag{5.4}$$

The anomalous sources in eqn. (3.6) are hence given as,

$$\Sigma_{\mathrm{H}}^{\perp AB} = -C_g^{(2)}\epsilon^{MRS}\bar{V}_M R_{RS}{}^{AB}, \qquad \mathrm{J}_{\mathrm{H}}^{\perp} = -C^{(2)}\epsilon^{MNR}\bar{V}_M F_{NR}. \tag{5.5}$$

Here we have defined the volume element of the boundary manifold as $\epsilon^{\perp MNR} = \epsilon^{MNR}$. After null reduction we can trivially read out anomalous sources for NC conservation laws eqn. (3.7),

$$\sigma_{\mathrm{H}}^{\perp ab} = -C_g^{(2)}\varepsilon_{\uparrow}^{\mu\nu} R_{\mu\nu}{}^{ab}, \qquad \mathrm{j}_{\mathrm{H}}^{\perp} = -2C^{(2)}\varepsilon_{\uparrow}^{\mu\nu} F_{\mu\nu}. \tag{5.6}$$

Hydrodynamics: We want to generate fluid constitutive relations which are compatible with anomalies described above. As described in the main text, it can be done using a transgression form eqn. (4.8),

$$\boldsymbol{\mathcal{V}}_{\boldsymbol{\mathcal{P}}}^{(4)} = \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge \left(\boldsymbol{\mathcal{P}}^{(5)} - \hat{\boldsymbol{\mathcal{P}}}^{(5)}\right)$$
$$= -\boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \left[2C^{(2)}\mu\boldsymbol{F} + 2C_g^{(2)}[\mu_\Sigma]_{\bar{B}}^{\bar{A}}\boldsymbol{R}_{\bar{A}}^{\bar{B}} + \left(C^{(2)}\mu^2 + C_g^{(2)}[\mu_\Sigma]_{\bar{B}}^{\bar{A}}[\mu_\Sigma]_{\bar{A}}^{\bar{B}}\right)\boldsymbol{H}\right]. \tag{5.7}$$

From its derivatives we can find various currents defined in eqn. (4.12),

$$\star\boldsymbol{J}_{\boldsymbol{\mathcal{P}}} = -2C^{(2)}\mu\boldsymbol{V} \wedge \bar{\boldsymbol{V}} \quad \Rightarrow \quad J_{\boldsymbol{\mathcal{P}}}^M = 2C^{(2)}\mu\epsilon^{RSM}V_R\bar{V}_S,$$
$$\star\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{P}}}^{AB} = -2C_g^{(2)}[\mu_\Sigma]^{AB}\boldsymbol{V} \wedge \bar{\boldsymbol{V}} \quad \Rightarrow \quad \Sigma_{\boldsymbol{\mathcal{P}}}^{MAB} = 2C_g^{(2)}[\mu_\Sigma]^{AB}\epsilon^{RSM}V_R\bar{V}_S,$$
$$\star\boldsymbol{E}_{\boldsymbol{\mathcal{P}}} = -\left(C^{(2)}\mu^2 + C_g^{(2)}[\mu_\Sigma]_B^A[\mu_\Sigma]_A^B\right)\boldsymbol{V} \wedge \bar{\boldsymbol{V}} \quad \Rightarrow \quad E_{\boldsymbol{\mathcal{P}}}^M = \left(C^{(2)}\mu^2 + C_g^{(2)}[\mu_\Sigma]_B^A[\mu_\Sigma]_A^B\right)\epsilon^{RSM}V_R\bar{V}_S. \tag{5.8}$$

Using eqn. (4.14) we can trivially get the anomalous sector of constitutive relations from here. These constitutive relations satisfy the adiabaticity equation eqn. (4.15) with zero entropy current, and at equilibrium also satisfy the anomalous equations of motion eqn. (3.6). Upon null reduction we can get the anomalous contribution to Galilean constitutive relations from here; the only surviving quantities are,

$$(\epsilon^\mu)_{\mathrm{A}} = \left(C^{(2)}\mu^2 + C_g^{(2)}[\mu_\sigma]^a{}_b[\mu_\sigma]^b{}_a\right)\varepsilon_{\uparrow}^{\nu\mu}n_\nu,$$

$$(\sigma^{\mu ab})_{\mathrm{A}} = 2C_g^{(2)}[\mu_\sigma]^{ab}\varepsilon_{\uparrow}^{\nu\mu}n_\nu, \qquad (j^\mu)_{\mathrm{A}} = 2C^{(2)}\mu\varepsilon_{\uparrow}^{\nu\mu}n_\nu. \tag{5.9}$$

Finally we can write an equilibrium partition function W_{anom}^{eqb} which generates these currents in equilibrium configuration. Using eqn. (4.18) we can directly find,

$$W_{\mathrm{anom}}^{eqb} = -\int_{\mathcal{M}_{(3)}} \frac{\boldsymbol{V}}{\boldsymbol{H}} \wedge \left(\boldsymbol{I}^{(4)} - \hat{\boldsymbol{I}}^{(4)}\right),$$
$$= -\int_{\mathcal{M}_{(3)}} \boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \left(C^{(2)}\mu\boldsymbol{A} + C_g^{(2)}[\mu_\Sigma]_B^A\boldsymbol{C}_A^B\right),$$
$$= \int \mathrm{d}^3x\sqrt{|\mathrm{G}|}\,\epsilon^{MNR}V_M\bar{V}_N\left(C^{(2)}\mu A_R + C_g^{(2)}[\mu_\Sigma]_B^A C_{RA}^B\right). \tag{5.10}$$

Same can be written in NC language,

$$W_{\mathrm{anom}}^{eqb} = \int_{\mathcal{M}_{(2)}} \boldsymbol{n} \wedge \left(C^{(2)}\mu\boldsymbol{A} + C_g^{(2)}[\mu_\sigma]^a{}_b\boldsymbol{C}^b{}_a\right),$$
$$= \int \mathrm{d}^3x\sqrt{|\gamma|}\,\varepsilon_{\uparrow}^{\mu\nu}n_\mu\left(C^{(2)}\mu A_\nu + C_g^{(2)}[\mu_\sigma]^a{}_b C^b{}_{\nu a}\right). \tag{5.11}$$

5.2 Arbitrary Even Spatial Dimensions up to Subsubleading Order

Before proceeding with this example we should clarify the usage of 'subsubleading' or 'second non-trivial' derivative order for null/Galilean fluids derived from relativistic fluids in [51]. One can check that in partition function or constitutive relations of a $(2n+1)$-dim null fluid, first non-trivial contribution from parity odd-sector comes at $(n-1)$ derivatives, which is generally known as 'leading parity-odd derivative order'. Correspondingly n derivatives are called subleading while $(n+1)$ derivatives are called subsubleading. It is also trivial to check that anomaly polynomial always has two more derivatives that the partition function or constitutive relations. In anomalous sector one can check that first non-trivial contribution comes at leading order (gauge anomaly) while no contribution comes at subleading order. Hence 'second non-trivial correction' comes at subsubleading order.

Coming back, one can check that up to subsubleading order $\boldsymbol{\mathcal{P}}^{(2n+3)}$ and $\boldsymbol{I}^{(2n+2)}$ (for $n > 1$) are given as,

$$\boldsymbol{\mathcal{P}}^{(2n+3)} = \bar{\boldsymbol{V}} \wedge \left(C^{(2n)} \boldsymbol{F}^{\wedge(n+1)} + C_g^{(2n)} \boldsymbol{F}^{\wedge(n-1)} \wedge \boldsymbol{R}^{\bar{A}}_{\bar{B}} \wedge \boldsymbol{R}^{\bar{B}}_{\bar{A}} \right),$$

$$\boldsymbol{I}^{(2n+2)} = -\bar{\boldsymbol{V}} \wedge \boldsymbol{A} \wedge \left(C^{(2n)} \boldsymbol{F}^{\wedge n} + C_g^{(2n)} \boldsymbol{F}^{\wedge(n-2)} \wedge \boldsymbol{R}^{\bar{A}}_{\bar{B}} \wedge \boldsymbol{R}^{\bar{B}}_{\bar{A}} \right). \tag{5.12}$$

It would be worth noting that contribution from anomalies terminate at subsubleading order in 3 spatial dimensions ($d = 3, n = 2$), hence these expressions are exact for $n = 2$. From here we can get the Hall currents,

$$\Sigma_{\mathrm{H}}^{\perp AB} = -2C_g^{(2n)} \star \left[\bar{\boldsymbol{V}} \wedge \boldsymbol{F}^{\wedge(n-1)} \wedge \boldsymbol{R}^{AB} \right],$$

$$J_{\mathrm{H}}^{\perp} = -(n+1)C^{(2n)} \star \left[\bar{\boldsymbol{V}} \wedge \boldsymbol{F}^{\wedge n} \right] - (n-1)C_g^{(2n)} \star \left[\bar{\boldsymbol{V}} \wedge \boldsymbol{F}^{\wedge(n-2)} \wedge \boldsymbol{R}^A_{\ B} \wedge \boldsymbol{R}^B_{\ A} \right], \tag{5.13}$$

that provide anomalies in eqn. (3.6). The results can be trivially transformed to Newton-Cartan language,

$$\sigma_{\mathrm{H}}^{\perp ab} = -2C_g^{(2n)} *_\uparrow \left[\boldsymbol{F}^{\wedge(n-1)} \wedge \boldsymbol{R}^{ab} \right],$$

$$\mathrm{j}_{\mathrm{H}}^{\perp} = -(n+1)C^{(2n)} *_\uparrow \left[\boldsymbol{F}^{\wedge n} \right] - (n-1)C_g^{(2n)} *_\uparrow \left[\boldsymbol{F}^{\wedge(n-2)} \wedge \boldsymbol{R}^a_{\ b} \wedge \boldsymbol{R}^b_{\ a} \right], \tag{5.14}$$

which provide anomalies in eqn. (3.7).

Hydrodynamics: Using the anomaly polynomial one can find the constitutive relations for the anomalous sector of hydrodynamics,

$$
\begin{aligned}
J_{\boldsymbol{\mathcal{P}}}^{M} &= (n+1)C^{(2n)} \sum_{m=1}^{n} {}^{n}\boldsymbol{C}_{m}\mu^{m} \star \left[\boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{F}^{\wedge(n-m)} \wedge \boldsymbol{H}^{\wedge(m-1)} \right]^{M} \\
&\quad + (n-1)C_{g}^{(2n)} \left\{ \sum_{m=1}^{n-2} {}^{n-2}\boldsymbol{C}_{m}\mu^{m} \star \left[\boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{F}^{\wedge(n-2-m)} \wedge \boldsymbol{R}^{A}{}_{B} \wedge \boldsymbol{R}^{B}{}_{A} \wedge \boldsymbol{H}^{\wedge(m-1)} \right]^{M} \right. \\
&\quad \left. + \sum_{m=0}^{n-2} {}^{n-2}\boldsymbol{C}_{m}\mu^{m}[\mu_{\Sigma}]^{A}{}_{B} \star \left[\boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{F}^{\wedge(n-2-m)} \wedge (2\boldsymbol{R}^{B}{}_{A} + [\mu_{\Sigma}]^{B}{}_{A}\boldsymbol{H}) \wedge \boldsymbol{H}^{\wedge m} \right]^{M} \right\}, \\
\Sigma_{\boldsymbol{\mathcal{P}}}^{MAB} &= 2C_{g}^{(2n)} \left\{ \sum_{m=1}^{n-1} {}^{n-1}\boldsymbol{C}_{m}\mu^{m} \star \left[\boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{F}^{\wedge(n-1-m)} \wedge \boldsymbol{R}^{AB} \wedge \boldsymbol{H}^{\wedge(m-1)} \right]^{M} \right. \\
&\quad \left. + \sum_{m=0}^{n-1} {}^{n-1}\boldsymbol{C}_{m}\mu^{m}[\mu_{\Sigma}]^{AB} \star \left[\boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{F}^{\wedge(n-1-m)} \wedge \boldsymbol{H}^{\wedge m} \right]^{M} \right\}, \\
E_{\boldsymbol{\mathcal{P}}}^{M} &= \sum_{m=0}^{n-1} \mu^{m} \left({}^{n+1}\boldsymbol{C}_{m+2}C^{(2n)}\mu^{2} + {}^{n-1}\boldsymbol{C}_{m}C_{g}^{(2n)}[\mu_{\Sigma}]^{A}{}_{B}[\mu_{\Sigma}]^{B}{}_{A} \right) \star \left[\boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{F}^{\wedge(n-1-m)} \wedge \boldsymbol{H}^{\wedge m} \right]^{M} \\
&\quad + C_{g}^{(2n)} \left\{ \sum_{m=2}^{n-1} {}^{n-1}\boldsymbol{C}_{m}\mu^{m} \star \left[\boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{F}^{\wedge(n-1-m)} \wedge \boldsymbol{R}^{A}{}_{B} \wedge \boldsymbol{R}^{B}{}_{A} \wedge \boldsymbol{H}^{\wedge(m-2)} \right]^{M} \right. \\
&\quad \left. + 2\sum_{m=1}^{n-1} {}^{n-1}\boldsymbol{C}_{m}\mu^{m}[\mu_{\Sigma}]^{A}{}_{B} \star \left[\boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{F}^{\wedge(n-1-m)} \wedge \boldsymbol{R}^{B}{}_{A} \wedge \boldsymbol{H}^{\wedge(m-1)} \right]^{M} \right\}.
\end{aligned}
\tag{5.15}
$$

Anomalous sector of constitutive relations in terms of these are given by eqn. (4.14), while the entropy current is zero. Again by a trivial choice of basis these results can be transformed to Newton-Cartan basis; only non-zero constitutive relations are,

$$
\begin{aligned}
(j^{\mu})_{\mathrm{A}} &= (n+1)C^{(2n)} \sum_{m=1}^{n} {}^{n}\boldsymbol{C}_{m}\mu^{m} *_{\uparrow} \left[\boldsymbol{n} \wedge \boldsymbol{F}^{\wedge(n-m)} \wedge \boldsymbol{H}^{\wedge(m-1)} \right]^{\mu} \\
&\quad + (n-1)C_{g}^{(2n)} \left\{ \sum_{m=1}^{n-2} {}^{n-2}\boldsymbol{C}_{m}\mu^{m} *_{\uparrow} \left[\boldsymbol{n} \wedge \boldsymbol{F}^{\wedge(n-2-m)} \wedge \boldsymbol{R}^{a}{}_{b} \wedge \boldsymbol{R}^{b}{}_{a} \wedge \boldsymbol{H}^{\wedge(m-1)} \right]^{\mu} \right. \\
&\quad \left. + \sum_{m=0}^{n-2} {}^{n-2}\boldsymbol{C}_{m}\mu^{m}[\mu_{\sigma}]^{a}{}_{b} *_{\uparrow} \left[\boldsymbol{n} \wedge \boldsymbol{F}^{\wedge(n-2-m)} \wedge \left(2\boldsymbol{R}^{b}{}_{a} + [\mu_{\sigma}]^{b}{}_{a}\boldsymbol{H} \right) \wedge \boldsymbol{H}^{\wedge m} \right]^{\mu} \right\}, \\
(\sigma^{\mu ab})_{\mathrm{A}} &= 2C_{g}^{(2n)} \left\{ \sum_{m=1}^{n-1} {}^{n-1}\boldsymbol{C}_{m}\mu^{m} *_{\uparrow} \left[\boldsymbol{n} \wedge \boldsymbol{F}^{\wedge(n-1-m)} \wedge \boldsymbol{R}^{ab} \wedge \boldsymbol{H}^{\wedge(m-1)} \right]^{\mu} \right. \\
&\quad \left. + \sum_{m=0}^{n-1} {}^{n-1}\boldsymbol{C}_{m}\mu^{m}[\mu_{\sigma}]^{ab} *_{\uparrow} \left[\boldsymbol{n} \wedge \boldsymbol{F}^{\wedge(n-1-m)} \wedge \boldsymbol{H}^{\wedge m} \right]^{\mu} \right\}, \\
(\epsilon^{\mu})_{\mathrm{A}} &= \sum_{m=0}^{n-1} \mu^{m} \left({}^{n+1}\boldsymbol{C}_{m+2}C^{(2n)}\mu^{2} + {}^{n-1}\boldsymbol{C}_{m}C_{g}^{(2n)}[\mu_{\sigma}]^{a}{}_{b}[\mu_{\sigma}]^{b}{}_{a} \right) *_{\uparrow} \left[\boldsymbol{n} \wedge \boldsymbol{F}^{\wedge(n-1-m)} \wedge \boldsymbol{H}^{\wedge m} \right]^{\mu} \\
&\quad + C_{g}^{(2n)} \left\{ \sum_{m=2}^{n-1} {}^{n-1}\boldsymbol{C}_{m}\mu^{m} *_{\uparrow} \left[\boldsymbol{n} \wedge \boldsymbol{F}^{\wedge(n-1-m)} \wedge \boldsymbol{R}^{a}{}_{b} \wedge \boldsymbol{R}^{b}{}_{a} \wedge \boldsymbol{H}^{\wedge(m-2)} \right]^{\mu} \right. \\
&\quad \left. + 2\sum_{m=1}^{n-1} {}^{n-1}\boldsymbol{C}_{m}\mu^{m}[\mu_{\sigma}]^{a}{}_{b} *_{\uparrow} \left[\boldsymbol{n} \wedge \boldsymbol{F}^{\wedge(n-1-m)} \wedge \boldsymbol{R}^{b}{}_{a} \wedge \boldsymbol{H}^{\wedge(m-1)} \right]^{\mu} \right\}.
\end{aligned}
\tag{5.16}
$$

Finally we can write an equilibrium partition function W_{anom}^{eqb} which generates these currents in equilibrium configuration; for null fluids,

$$
\begin{aligned}
W_{\text{anom}}^{eqb} = -\int_{\mathcal{M}_{(2n+1)}} \boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{A} \wedge \Bigg\{ &\sum_{m=1}^{n} {}^n\boldsymbol{C}_m C^{(2n)} \mu^m \boldsymbol{F}^{\wedge(n-m)} \wedge \boldsymbol{H}^{\wedge(m-1)} \\
&+ C_g^{(2n)} \sum_{m=1}^{n-2} {}^{n-2}\boldsymbol{C}_m \mu^m \boldsymbol{F}^{\wedge(n-2-m)} \wedge \boldsymbol{H}^{\wedge(m-1)} \wedge \boldsymbol{R}^A{}_B \wedge \boldsymbol{R}^B{}_A \\
&+ C_g^{(2n)} \sum_{m=0}^{n-2} {}^{n-2}\boldsymbol{C}_m \mu^m [\mu_\Sigma]^A{}_B \boldsymbol{F}^{\wedge(n-2-m)} \wedge \boldsymbol{H}^{\wedge m} \wedge \left(2\boldsymbol{R}^B{}_A + [\mu_\Sigma]^B{}_A \boldsymbol{H} \right) \Bigg\},
\end{aligned} \tag{5.17}
$$

and for Galilean fluids,

$$
\begin{aligned}
W_{\text{anom}}^{eqb} = -\int_{\mathcal{M}_{(2n)}} \boldsymbol{n} \wedge \boldsymbol{A} \wedge \Bigg\{ &\sum_{m=1}^{n} {}^n\boldsymbol{C}_m C^{(2n)} \mu^m \boldsymbol{F}^{\wedge(n-m)} \wedge \boldsymbol{H}^{\wedge(m-1)} \\
&+ C_g^{(2n)} \sum_{m=1}^{n-2} {}^{n-2}\boldsymbol{C}_m \mu^m \boldsymbol{F}^{\wedge(n-2-m)} \wedge \boldsymbol{H}^{\wedge(m-1)} \wedge \boldsymbol{R}^a{}_b \wedge \boldsymbol{R}^b{}_a \\
&+ C_g^{(2n)} \sum_{m=0}^{n-2} {}^{n-2}\boldsymbol{C}_m \mu^m [\mu_\sigma]^a{}_b \boldsymbol{F}^{\wedge(n-2-m)} \wedge \boldsymbol{H}^{\wedge m} \wedge \left(2\boldsymbol{R}^b{}_a + [\mu_\sigma]^b{}_a \boldsymbol{H} \right) \Bigg\}.
\end{aligned} \tag{5.18}
$$

This finishes our discussion about anomalies in generic even dimensional Galilean fluid up to subsubleading order in derivative expansion. 1 spatial dimensional case was discussed separately in § 5.1 for illustrative purposes. 1 dimensional case is also qualitatively different from higher dimensions, because only in this special case we get pure gravitational anomaly term in the anomaly polynomial up to subsubleading order. 3 and higher spatial dimensional cases are qualitatively similar as we illustrated above. For physically interesting results one might want to put $n = 2$ and recover 3 spatial dimensional results, which are found to be in agreement with path integral calculation of [36].

6 | Conclusions and Further Directions

In this work we examined the effect of gauge and gravitational anomalies on Galilean theories with spin current, coupled to torsional Newton-Cartan geometries. In particular it is to be noted that we primarily studied anomalous theories on torsional null backgrounds, from where the aforementioned system is just a choice of basis (null reduction) away. It strengthens our belief that null theories are just an embedding of Galilean theories into a higher dimensional spacetime, which are closer to their relativistic cousins, are frame independent and are easier to handle compared to Newton-Cartan backgrounds. Transition from null to Galilean (Newton-Cartan) theories is essentially trivial.

We used the anomaly inflow mechanism prevalent in relativistic theories, with slight modifications, to construct these anomalies. We found that after null reduction the anomalies only contribute to spatial spin and charge conservation equations, and only in even dimensions. In other words only rotational and gauge symmetry of the Galilean theory goes anomalous. This is in contrast with the results of [37] where Galilean boost symmetry was also seen to be anomalous. As we mentioned in the introduction and in the main work, the discrepancy can be attributed to presence of extra fields in [37] which have been explicitly switched off in null

background construction. It is interesting to note that Galilean anomaly polynomial $\mathfrak{p}^{(2n+2)}$ is structurally the same as relativistic anomaly polynomial $\mathcal{P}_{\mathrm{CS}}^{(2n+2)}$, and hence the number on anomaly coefficients at the both sides match. Owing to it, the structure of Hall currents that enter the conservation laws, is also quite similar in both cases. Hence the results we have obtained promises to be genuine non-relativistic anomalies and not just the manifestation of (stronger) Galilean invariance.

Unrelated to Galilean theories, we found that in Cartan formulation of relativistic fluids there exists a more natural definition of entropy current which does not get any anomalous contributions. On the other hand the Belinfante (usual) entropy current used e.g. in [41] gets contributions from gravitational anomaly. Look at appendix (D) for more comments on this issue.

We also studied anomalous sector of null/Galilean hydrodynamics, in which we explicitly wrote down the constitutive relations which are completely determined in terms of anomalies. For this we used the transgression machinery developed to do the same task in relativistic hydrodynamics. There have been no surprises in this computation; everything went more or less through for null theories, as it did for relativistic theories. The entropy current in Galilean theories is independent of anomalies as well. From a different perspective, it illustrates that null background construction allows us to use much sophisticated and developed relativistic machinery directly into non-relativistic physics, which is encouraging.

It opens up an arena to bring in set results from relativistic theories into null theories and see if we can say something new and useful about Galilean theories from there. An immediate question that comes to our mind is regarding transcendental contributions to hydrodynamics from anomalies. In relativistic hydrodynamics [46] showed that there are certain constants in fluid constitutive relations that are left undetermined by second law of thermodynamics, but can be related to the anomaly coefficients on requiring the consistency of Euclidean vacuum. Similar constants have also been found for Galilean theories in [2, 19]. It would be nice to see if these constants can be associated with Galilean anomalies found in this work. Being little more ambitious, one can hope for a complete classification of Galilean hydrodynamic transport following its relativistic counterpart suggested recently in [40, 41]. It will also be interesting to see if Weyl anomaly analysis of [37] remains unchanged when the additional mass sources have been switched off.

For now we will leave the reader with these questions and possibilities, in hope that we would be able to unravel new and interesting non-relativistic physics using null backgrounds. If there is one thing reader should take away from this work, we would recommend the approach – if we are interested in a problem pertaining to Galilean physics which we know how to solve in relativistic case, a good way ahead would be to formulate the problem in terms of null theories, do the computation there, and perform a trivial null reduction to get Galilean results.

Acknowledgments

A | Results in Non-Covariant Basis

In this appendix we give some of the results discussed in main text in conventional non-covariant notation. We pick up a basis $x^M = \{x^\sim, t, x^i\}$ on $\mathcal{M}_{(d+2)}$ such that time data $\psi_V = \{\partial_\sim, 0, 0\}$ and $\psi_T = \{\partial_t, 0, 0\}$. $\vec{x} = \{x^i\}$ spans the spatial slice $\mathcal{M}^T_{(d)}$. This is equivalent to choosing the Newton-Cartan decomposition but with $v^i = \Lambda_{(T)} = [\Lambda_{\Sigma(T)}]^A_B = 0$. On $\mathbb{R}^{(d+1,1)}$ on the other hand we choose the same basis as before $x^A = \{x^-, x^+, x^a\}$ such that $V = \partial_-$, $\bar{V}_{(T)} = \partial_+$. In this basis various NC background fields can be decomposed as,

$$n_\mu = \begin{pmatrix} e^{-\Phi} \\ e^{-\Phi} a_i \end{pmatrix}, \qquad v^\mu = \begin{pmatrix} e^\Phi \\ 0 \end{pmatrix}, \qquad B_\mu = \begin{pmatrix} B_t \\ B_i \end{pmatrix}, \qquad e^a{}_\mu = \begin{pmatrix} 0 \\ e^a{}_i \end{pmatrix}, \qquad e_a{}^\mu = \begin{pmatrix} -a_a \\ e_a{}^i \end{pmatrix},$$

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & g_{ij} \end{pmatrix}, \qquad h^{\mu\nu} = \begin{pmatrix} a^k a_k & -a^j \\ -a^i & g^{ij} \end{pmatrix}. \tag{A.1}$$

Here spatial metric has been defined as,

$$g_{ij} = \delta_{ab} e^a{}_i e^b{}_j, \qquad g^{ij} = \delta^{ab} e_a{}^i e_b{}^j. \tag{A.2}$$

Spatial indices can be raised and lowered by g_{ij} and can be swapped using the spatial Vielbein $e^a{}_i$. However in the following we will explicitly work in $i, j \ldots$ indices. One can check that after this choice of basis we are only allowed to perform $\vec{x}$ dependent transformations, except boosts which are completely fixed. One can check that on trivially decomposing the Newton-Cartan expressions into $x^\mu = \{t, x^i\}$, our theory will be manifestly covariant against all these transformations except time translations $t \to t + \xi^t(\vec{x})$, sometimes known as Kaluza-Klein (KK) gauge transformations. The theory can be made manifestly covariant under KK transformations as well by working with corrected tensors,

$$\acute{X}^t = e^{-\Phi}\left(X^t + a_i X^i\right), \qquad \acute{X}_t = e^\Phi X_t, \qquad \acute{X}^i = X^i, \qquad \acute{X}_i = X_i - a_i X_t, \tag{A.3}$$

and similarly for higher rank tensors. These are the well known Kaluza-Klein covariant fields[20]. Under flat time approximation i.e. $\Phi = a_i = 0$ this transformation turns trivial. One can check that NC contraction can be expanded in this format as,

$$A^\mu B_\mu = \acute{A}^t \acute{B}_t + \acute{A}^i \acute{B}_i, \tag{A.4}$$

which will be helpful later. Now we can decompose various components of connections in this basis as,

$$c_{\mu t} = 0, \qquad \acute{c}_{ij} = \frac{1}{2}\acute{\partial}_t g_{ji} + \frac{1}{2}\acute{\Omega}_{ij} + \acute{T}_{(ij)t}, \qquad \acute{\alpha}_i = \acute{c}_{ti} = \acute{\Omega}_{ti},$$

$$\acute{\Gamma}^t{}_{tt} = -\acute{\partial}_t \Phi, \qquad \acute{\Gamma}^t{}_{tj} = e^{-\Phi}\acute{\partial}_t a_j, \qquad \acute{\Gamma}^t{}_{it} = -\acute{\partial}_i \Phi, \qquad \acute{\Gamma}^t{}_{ij} = e^{-\Phi}\acute{\partial}_i a_j.$$

$$\acute{\Gamma}^k{}_{tt} = \acute{\Omega}_t{}^k, \qquad \acute{\Gamma}^k{}_{it} = \acute{c}_i{}^k, \qquad \acute{\Gamma}^k{}_{tj} = \acute{c}_j{}^k - \acute{T}^k{}_{jt}.$$

$$\acute{\Gamma}^k{}_{ij} = \frac{1}{2}g^{kl}\left(\acute{\partial}_i g_{lj} + \acute{\partial}_j g_{li} - \acute{\partial}_l g_{ij}\right) + \frac{1}{2}\left(\acute{T}^k{}_{ij} - 2\acute{T}_{(ij)}{}^k\right). \tag{A.5}$$

Here we have also defined the corrected coordinate derivatives $\acute{\partial}_i = \partial_i - a_i \partial_t$, $\acute{\partial}_t = e^\Phi \partial_t$. In *equilibrium* (i.e. $\partial_t \varphi = 0 \; \forall \phi$) or when time is flat, we can recover $\acute{\partial}_i = \partial_i$, $\acute{\partial}_t = \partial_t$. We

---

[20]The original Kaluza-Klein transformation only involves KK gauge field a_i. The factors of e^Φ can be thought of as red-shift factors due to time component of time metric n_μ.

define covariant derivative $\acute{\nabla}_i$ associated with corrected derivative[21] $\acute{\partial}_i$ and connections $\acute{\Gamma}^j{}_{ik}$ and $\acute{A}_i$, which act on a general tensor $\varphi^i{}_j$ transforming in adjoint representation of the gauge group,

$$\acute{\nabla}_i\varphi^j{}_k = \acute{\partial}_i\varphi^j{}_k + \acute{\Gamma}^j{}_{il}\varphi^l{}_k - \acute{\Gamma}^l{}_{ik}\varphi^j{}_l + \left[\acute{A}_i, \varphi^j{}_k\right], \tag{A.7}$$

and similarly on higher rank objects. We also define a 'time covariant derivative' $\acute{\nabla}_t$ associated with $\acute{\partial}_t$ and connections $\acute{\Gamma}^j{}_{tk}$ and $\acute{A}_t$, acting on $\varphi^i{}_j$ naturally,

$$\acute{\nabla}_t\varphi^i{}_j = \acute{\partial}_t\varphi^i{}_j + \acute{\Gamma}^i{}_{tk}\varphi^k{}_j - \acute{\Gamma}^k{}_{tj}\varphi^i{}_k + \left[\acute{A}_t, \varphi^i{}_j\right], \tag{A.8}$$

and similarly on higher rank objects. One can check that both of these derivatives behave tensorially on the spatial slice and are KK gauge invariant. More importantly both of these preserve metric g_{ij}. There is no essential need to work with these corrected quantities, just that the statements are manifestly KK gauge invariant and look nicer.

Using similar decomposition and KK correction for various currents, we can reduce the conservation equations eqn. (3.7) into non-covariant basis,

$$
\begin{aligned}
\text{Mass Cons.:} \quad & \underline{\acute{\nabla}}_t\rho + \underline{\acute{\nabla}}_i\acute{\rho}^i = 0, \\
\text{Energy Cons.:} \quad & \underline{\acute{\nabla}}_t\epsilon + \underline{\acute{\nabla}}_i\acute{\epsilon}^i = [\text{power}] - \acute{p}^i\acute{\alpha}_i - \acute{p}^{ij}\acute{c}_{ij}, \\
\text{Momentum Cons.:} \quad & \underline{\acute{\nabla}}_t\acute{p}_i + \underline{\acute{\nabla}}_j\acute{p}^j{}_i = [\text{force}]_i - \rho\acute{\alpha}_i - \acute{\rho}^j\acute{c}_{ji}, \\
\text{Temporal Spin Cons.:} \quad & \underline{\acute{\nabla}}_t\acute{\tau}^i + \underline{\acute{\nabla}}_j\acute{\tau}^{ji} = \frac{1}{2}\left(\acute{\rho}^i - \acute{p}^i\right), \\
\text{Spatial Spin Cons.:} \quad & \underline{\acute{\nabla}}_t\acute{\sigma}^{ij} + \underline{\acute{\nabla}}_k\acute{\sigma}^{kij} = \acute{p}^{[ji]} + 2\acute{\tau}^{[i}\acute{\alpha}^{j]} + 2\acute{\tau}^{k[i}\acute{c}_k{}^{j]} + \acute{\sigma}_{\text{H}}^{\perp ij}, \\
\text{Charge Cons.:} \quad & \underline{\acute{\nabla}}_t q + \underline{\acute{\nabla}}_i\acute{j}^i = \mathrm{j}_{\text{H}}^{\perp},
\end{aligned}
\tag{A.9}
$$

where $\underline{\acute{\nabla}}_i = \acute{\nabla}_i - \acute{T}^j{}_{ji} + \acute{H}_{ti}$ and $\underline{\acute{\nabla}}_t = \acute{\nabla}_t + \acute{\Gamma}^i{}_{ti}$. It is worth noting that the corrected time component of mass current $\acute{\rho}^t$ is just the mass density ρ, and similarly for all other currents. If we are to expand the covariant derivatives in these equations, the nice looking expressions will turn notoriously bad; so we do not attempt to do that. Rather we invite the readers to qualitatively access the form of these equations and convince themselves that these are what we expect for a Galilean system. Similarly [power] and [force] densities can also be decomposed as,

$$[\text{power}] = \acute{e}_i \cdot \acute{j}^i + \dots, \qquad [\text{force}]^i = \acute{e}_i \cdot q + \acute{\beta}_{ij} \cdot \acute{j}^j + \dots, \tag{A.10}$$

where $\acute{e}_i = \acute{F}_{it}$ is the electric field, $\acute{\beta}_{ij} = \acute{F}_{ij}$ is the dual magnetic field, and ... corresponds to similar terms coming from all other field-current couples.

On the other hand non-covariant expressions for anomalous sector of hydrodynamic constitutive relations follow trivially from eqn. (4.19). Only non-zero contributions are given as[22],

$$(\acute{\epsilon}^i)_{\text{A}} = E^i_{\boldsymbol{\mathcal{P}}}, \qquad (\acute{\sigma}^{ijk})_{\text{A}} = \Sigma^{ijk}_{\boldsymbol{\mathcal{P}}}, \qquad (\acute{j}^i)_{\text{A}} = J^i_{\boldsymbol{\mathcal{P}}}. \tag{A.11}$$

The expressions for RHS can be obtained from eqn. (4.24).

---

[21] One might be lured (e.g. in [2]) to define covariant derivative with respect to original derivative ∂_i and more conventional affine connection,

$$\gamma^k{}_{ij} = \acute{\Gamma}^k{}_{ij} - \frac{1}{2}a^k\partial_t g_{ij} + g^{kl}a_{(i}\partial_t g_{j)l} = \frac{1}{2}g^{kl}\left(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}\right) + \frac{1}{2}\left(\acute{T}^k{}_{ij} - 2\acute{T}_{(ij)}{}^k\right), \tag{A.6}$$

which however will not be KK gauge invariant. The results hence will be messy and will carry extra time derivatives of metric. Therefore we will refrain from doing so. Obviously both of these covariant derivatives are same in flat time or equilibrium.

[22] We have assumed that the same ψ_T being used for reduction and to describe anomaly polynomial. Had they been different, currents would shift by a total derivative.

B | Comparison with Geracie et al. [4]

Authors of [4] have prescribed a nice covariant frame independent description of Galilean physics in terms of an 'extended representation'. The extended space is basically a one dimensional higher flat space which allows for a nice frame independent embedding of the Galilean group. On a closer inspection however, it would be clear that the extended space is nothing but the Vielbein space for null theories. To demonstrate this we pick up a basis on $\mathcal{M}_{(d+2)}$ (but not perform null reduction, which would require us to choose time data ψ_T and hence will introduce frame dependence), $x^M = \{x^{\sim}, x^{\mu}\}$ such that $V = \partial_{\sim}$. We can then express the anomalous null conservation laws as,

$$
\begin{gathered}
\left(\nabla_{\mu} - \mathrm{T}^{\nu}{}_{\nu\mu}\right) j^{\mu}_{\rho} = 0, \\
\mathrm{E}^A_{\nu}\left(\nabla_{\mu} - \mathrm{T}^{\nu}{}_{\nu\mu}\right) T^{\mu}_A - \mathrm{T}^A{}_{\nu\mu} T^{\mu}_A = R_{\nu\mu}{}^A{}_B \Sigma^{\mu B}{}_A + F_{\nu\mu} \cdot J^{\mu}, \\
\left(\nabla_{\mu} - \mathrm{T}^{\nu}{}_{\nu\mu}\right) \Sigma^{\mu AB} = -T^{[AB]} + \Sigma^{\perp AB}_{\mathrm{H}} + \#^{[A} V^{B]}, \\
\left(\nabla_{\mu} - \mathrm{T}^{\nu}{}_{\nu\mu}\right) j^{\mu}_q = \mathrm{J}^{\perp}_{\mathrm{H}}.
\end{gathered}
\tag{B.1}
$$

In this, and only in this section ∇_{μ} is associated with $\Gamma^{\rho}{}_{\mu\sigma}$, $C^A{}_{\mu B}$, A_{μ} and Vielbein has been used to transform indices. The results are presented to make them look as close as possible to (5.8) – (5.10) of [4]. The authors however did not consider anomalies, and did not report the full spin conservation. Only the boost part of spin conservation is reported in (5.13) of [4] which is identical to our corresponding conservation in eqn. (3.7).

If one looks at these equations and at the currents appearing in them, one would realize that all the unphysical degrees of freedom have been eliminated (except the spin conservation equations). Therefore EM tensor and charge current as appearing in [4] only carry physical information. On cost of some unphysical degrees of freedom (and a consistent prescription to eliminate them) we have been able to transform these set of equations into a nice covariant higher dimensional null theory.

We would like to note that authors in [4] have also used their construction to study $(2+1)$ dimensional Galilean fluids. Same results (for torsionless case) were gained from 'null fluids' in [2] and a detailed comparison can be found in their last appendix.

C | Conventions of Differential Forms

In this appendix we will recollect some results about differential forms, and will set notations and conventions used throughout this work. An m-rank differential form $\boldsymbol{\mu}^{(m)}$ on $\mathcal{M}_{(d+2)}$, can be written in a coordinate basis as,

$$
\boldsymbol{\mu}^{(m)} = \frac{1}{m!} \mu_{M_1 M_2 \ldots M_m} \mathrm{d}x^{M_1} \wedge \mathrm{d}x^{M_2} \wedge \ldots \wedge \mathrm{d}x^{M_m},
\tag{C.1}
$$

where μ is a completely antisymmetric tensor. On $\mathcal{M}_{(d+2)}$, volume element is given by a full rank form,

$$
\boldsymbol{\epsilon}^{(d+2)} = \frac{1}{(d+2)!} \epsilon_{M_1 M_2 \ldots M_{d+2}} \mathrm{d}x^{M_1} \wedge \mathrm{d}x^{M_2} \wedge \ldots \wedge \mathrm{d}x^{M_{d+2}},
\tag{C.2}
$$

where ϵ is the totally antisymmetric Levi-Civita symbol with value $\epsilon_{0,1,2,\ldots,d+1} = \sqrt{|\mathrm{G}|}$ and $\mathrm{G} = \det \mathrm{G}_{MN}$. Using it, Hodge dual is defined to be a map from m-rank differential forms to

$(d + 2 - m)$-rank differential forms,

$$\star\,[\boldsymbol{\mu}^{(m)}] = \frac{1}{(d+2-m)!}\left(\frac{1}{m!}\mu^{M_1\ldots M_m}\epsilon_{M_1\ldots M_m N_1\ldots N_{d+2-m}}\right)\mathrm{d}x^{N_1}\wedge\ldots\wedge\mathrm{d}x^{N_{d+2-m}}. \qquad \text{(C.3)}$$

One can check that $\star\star\,\boldsymbol{\mu}^{(m)} = \mathrm{sgn}(\mathrm{G})(-)^{m(d-m)}$, and,

$$\boldsymbol{\mu}^{(m)}\wedge\star[\boldsymbol{\nu}^{(m)}] = \frac{1}{m!}\mu^{M_1\ldots M_m}\nu_{M_1\ldots M_m}\boldsymbol{\epsilon}^{(d+2)}. \qquad \text{(C.4)}$$

We define the $\wedge$ product of two differential forms as,

$$\boldsymbol{\mu}^{(m)}\wedge\boldsymbol{\nu}^{(r)} = \frac{1}{(m+r)!}\left(\frac{(m+n)!}{m!r!}\mu_{[M_1\ldots M_m}\nu_{N_1\ldots N_r]}\right)\mathrm{d}x^{M_1}\wedge\ldots\wedge\mathrm{d}x^{N_1}\wedge\ldots. \qquad \text{(C.5)}$$

For multiple forms we can find,

$$\boldsymbol{\mu}^{(m)}\wedge\boldsymbol{\nu}^{(r)}\wedge\ldots\wedge\boldsymbol{\rho}^{(s)} = \frac{1}{(m+r+\ldots+s)!}\left(\frac{(m+r+\ldots+s)!}{m!r!\ldots s!}\mu_{[M_1\ldots M_m}\nu_{N_1\ldots N_r}\rho_{R_1\ldots R_s]}\right)$$
$$\mathrm{d}x^{M_1}\wedge\ldots\wedge\mathrm{d}x^{N_1}\wedge\ldots\wedge\mathrm{d}x^{R_1}\wedge\ldots, \qquad \text{(C.6)}$$

$$\star[\boldsymbol{\mu}^{(m)}\wedge\boldsymbol{\nu}^{(r)}\ldots\wedge\boldsymbol{\rho}^{(s)}] = \frac{1}{(d+2-m-r\ldots-s)!}\left(\frac{1}{m!r!\ldots s!}\mu^{M_1\cdots}\nu^{N_1\cdots}\ldots\rho^{R_1\cdots}\epsilon_{M_1\ldots N_1\ldots R_1\ldots\ldots S_1\ldots}\right)$$
$$\mathrm{d}x^{S_1}\wedge\ldots. \qquad \text{(C.7)}$$

We define the interior product with respect to a vector field X of a differential form as,

$$\iota_X\boldsymbol{\mu}^{(m)} = \frac{1}{(m-1)!}\left(X^M\mu_{[MN_1\ldots N_{m-1}]}\right)\mathrm{d}x^{N_1}\wedge\ldots\wedge\mathrm{d}x^{N_{m-1}}. \qquad \text{(C.8)}$$

One can check two useful identities

$$\iota_X\star\left[\boldsymbol{\mu}^{(m)}\right] = \star\left[\boldsymbol{\mu}^{(m)}\wedge\boldsymbol{X}\right], \qquad \star\left[\iota_X\boldsymbol{\mu}^{(m)}\right] = (-)^{m-1}\boldsymbol{X}\wedge\star\left[\boldsymbol{\mu}^{(m)}\right]. \qquad \text{(C.9)}$$

Given a one-form $\boldsymbol{Y}^{(1)}$ and a vector field X such that $\iota_X\boldsymbol{Y}^{(1)} = 1$, any differential form $\boldsymbol{\mu}^{(m)}$ can be decomposed as,

$$\boldsymbol{\mu}^{(m)} = \iota_X\left(\boldsymbol{Y}^{(1)}\wedge\boldsymbol{\mu}^{(m)}\right) + \boldsymbol{Y}^{(1)}\wedge\iota_X\boldsymbol{\mu}^{(m)}. \qquad \text{(C.10)}$$

This is in particular helpful when $\boldsymbol{\mu}^{(d+2)}$ is a full rank form,

$$\boldsymbol{\mu}^{(d+2)} = \boldsymbol{Y}^{(1)}\wedge\iota_X\boldsymbol{\mu}^{(d+2)}. \qquad \text{(C.11)}$$

The exterior product of a differential form is defined to be,

$$\mathrm{d}\boldsymbol{\mu}^{(m)} = \frac{1}{(m+1)!}\left[(m+1)\partial_{[M_1}\mu_{M_2\ldots M_{m+1}]}\right]\mathrm{d}x^{M_1}\wedge\ldots\wedge\mathrm{d}x^{M_{p+1}}. \qquad \text{(C.12)}$$

One can check a useful relation,

$$\star\,\mathrm{d}\boldsymbol{\mu}^{(d+1)} = (-)^{d+1}\underline{\nabla}_M\star\left[\boldsymbol{\mu}^{(d+1)}\right]^M, \qquad \mathrm{d}\star\left[\boldsymbol{\mu}^{(1)}\right] = \star\underline{\nabla}_M\mu^M. \qquad \text{(C.13)}$$

The lie derivative of a differential form satisfies,

$$\mathcal{L}_X\boldsymbol{\mu}^{(m)} = \iota_X\mathrm{d}\boldsymbol{\mu}^{(m)} + \mathrm{d}\left(\iota_X\boldsymbol{\mu}^{(m)}\right). \qquad \text{(C.14)}$$

Integration of a full rank form is defined as,

$$\int_{\mathcal{M}_{(d+2)}} \boldsymbol{\mu}^{(d+2)} = \mathrm{sgn}(\mathrm{G}) \int \{\mathrm{d}x^M\} \sqrt{|\mathrm{G}|} \ \star [\boldsymbol{\mu}^{(d+2)}]$$

$$= \mathrm{sgn}(\mathrm{G}) \int \{\mathrm{d}x^M\} \sqrt{|\mathrm{G}|} \ \frac{1}{(d+2)!} \epsilon^{M_1 \ldots M_{d+2}} \mu_{M_1 \ldots M_{d+2}}. \tag{C.15}$$

Here the raised Levi-Civita symbol has value $\epsilon^{0,1,2,\ldots,d+1} = \mathrm{sgn}(G)/\sqrt{|G|}$. Integration of an exact full rank form is given by integration on the boundary,

$$\int_{\mathcal{M}_{(d+2)}} \mathrm{d}\boldsymbol{\mu}^{(d+1)} = \int_{\partial\mathcal{M}_{(d+2)}} \boldsymbol{\mu}^{(d+1)}, \tag{C.16}$$

where given a unit vector N normal to boundary, volume element on the boundary is defined as $\iota_N \boldsymbol{\epsilon}^{(d+2)} = \star \boldsymbol{N}$.

Newton-Cartan Differential Forms

We decompose a vector and a one form on $\mathcal{M}_{(d+2)}$ in NC basis,

$$\mathcal{X}^M \partial_M = (\mathcal{X}^\sim - B_\mu \mathcal{X}^\mu) \partial_\sim + \mathcal{X}^\mu (\partial_\mu + B_\mu \partial_\sim),$$
$$\mathcal{Y}_M \mathrm{d}x^M = \mathcal{Y}_\sim (\mathrm{d}x^\sim - B_\mu \mathrm{d}x^\mu) + (\mathcal{Y}_\mu + B_\mu \mathcal{Y}_\sim) \mathrm{d}x^\mu. \tag{C.17}$$

One can check that these results are written in 'nicely' transforming basis from the NC perspective, which tells us that,

$$V^M \mathcal{Y}_M = \mathcal{Y}_\sim, \qquad \mathcal{Y}_\mu + B_\mu \mathcal{Y}_\sim, \qquad \bar{V}_M \mathcal{X}^M = \mathcal{X}^\sim - X^\mu B_\mu, \qquad \mathcal{X}^\mu, \tag{C.18}$$

are nicely transforming quantities. As is quite apparent, first and last do not depend on the explicit choice of ψ_T but the middle ones do. A similar analysis can be done for all tensor fields in the theory. Note that if $\mathcal{Y}_M$ satisfies $\iota_V \boldsymbol{\mathcal{Y}} = V^M \mathcal{Y}_M = 0$, the one form becomes purely NC. On the other hand if $\bar{V}_M \mathcal{X}^M = 0$, the vector field becomes purely NC. This motivates us to define a NC differential form to be a form in $\mathcal{M}_{(d+2)}$ which does not have a leg along V, i.e. $\iota_V \boldsymbol{\mu}^{(m)}$. Such a form can be expanded as,

$$\boldsymbol{\mu}^{(m)} = \frac{1}{m!} \mu_{\mu_1 \mu_2 \ldots \mu_m} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_m}. \tag{C.19}$$

On the other hand we define a NC 'differential contra-form' as a totally antisymmetric contravariant tensor in $\mathcal{M}_{(d+2)}$ which has zero contraction with $\bar{V}_M$. In basis $\partial'_\mu = \partial_\mu + B_\mu \partial_\sim$ it can be expanded as,

$$\boldsymbol{\mu}^{[m]} = \frac{1}{m!} \mu^{\mu_1 \mu_2 \ldots \mu_m} \partial'_{\mu_1} \wedge \partial'_{\mu_2} \wedge \ldots \wedge \partial'_{\mu_m}. \tag{C.20}$$

It is clear that though the basis depends on choice of ψ_T, the components of contra-form are independent of it. On a manifold with a non-degenerate metric there exists a map between these two quantities, but for us these two shall be distinct. We can also define a spatial differential form/contra-form with requirement that it should not have any leg along V and $\bar{V}$. In this case there exists a map between these two quantities realized by $p^{\mu\nu}$ and $p_{\mu\nu}$.

Correspondingly there are three volume elements,

$$\varepsilon_\uparrow^{[d+1]} = [\star \bar{\boldsymbol{V}}]^\sharp = \frac{1}{(d+1)!} \left(\bar{V}_M \epsilon^{M\mu_1 \ldots \mu_{d+1}} \right) \partial'_{\mu_1} \wedge \ldots \wedge \partial'_{\mu_{d+1}},$$

$$\varepsilon_\downarrow^{(d+1)} = \star \boldsymbol{V} = \frac{1}{(d+1)!} \left(V^M \epsilon_{M\mu_1 \ldots \mu_{d+1}} \right) \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_{d+1}},$$

$$\varepsilon^{(d)} = \star [\boldsymbol{V} \wedge \bar{\boldsymbol{V}}] = \frac{1}{d!} \left(V^M \bar{V}^N \epsilon_{MN\mu_1 \ldots \mu_d} \right) \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_d}. \tag{C.21}$$

In the main text we have primarily used the first one. Correspondingly there are three Hodge duals that provide maps from forms to contra-forms, contra-forms to forms, and a self-inverse map between spatial forms respectively,

$$*_\uparrow \left[\boldsymbol{\mu}^{(m)} \right] = \star \left[\bar{\boldsymbol{V}} \wedge \boldsymbol{\mu}^{(m)} \right]^\sharp = \frac{1}{(d+1-m)!} \left(\frac{1}{m!} \mu_{\mu_1 \dots \mu_m} \varepsilon_\uparrow^{\mu_1 \dots \mu_m \nu_1 \dots \nu_{d+1-m}} \right) \partial'_{\nu_1} \wedge \dots \wedge \partial'_{\nu_{d+1-m}},$$

$$*^\downarrow \left[\boldsymbol{\mu}^{[m]} \right] = \star \left[\boldsymbol{V} \wedge \boldsymbol{\mu}^{\flat(m)} \right] = \frac{1}{(d+1-m)!} \left(\frac{1}{m!} \mu^{\mu_1 \dots \mu_m} \varepsilon^\downarrow_{\mu_1 \dots \mu_m \nu_1 \dots \nu_{d+1-m}} \right) \mathrm{d}x^{\nu_1} \wedge \dots \wedge \mathrm{d}x^{\nu_{d+1-m}},$$

$$* \left[\boldsymbol{\mu}^{(m)} \right] = \star \left[\boldsymbol{V} \wedge \boldsymbol{u} \wedge \boldsymbol{\mu}^{(m)} \right] = \frac{1}{(d-m)!} \left(\frac{1}{m!} \mu^{\mu_1 \dots \mu_m} \varepsilon_{\mu_1 \dots \mu_m \nu_1 \dots \nu_{d-m}} \right) \mathrm{d}x^{\nu_1} \wedge \dots \wedge \mathrm{d}x^{\nu_{d-m}}.$$

$$(\text{C.22})$$

One can check that $** = -\text{sgn}(G)(-)^{m(d-m)}$ and $*^\downarrow *_\uparrow = *_\uparrow *^\downarrow = -\text{sgn}(G)(-)^{m(d+1-m)}$. Finally we need to define integration for NC full rank forms and contra-forms,

$$\int_{\mathcal{M}_{(d+1)}} \boldsymbol{\mu}^{(d+1)} = \text{sgn}(G) \int_{\mathcal{M}_{(d+2)}} \bar{\boldsymbol{V}} \wedge \boldsymbol{\mu}^{(d+1)} = \text{sgn}(\gamma) \int \{\mathrm{d}x^\mu\} \sqrt{|\gamma|} \; *_\uparrow \left[\boldsymbol{\mu}^{(d+1)} \right],$$

$$\int_{\mathcal{M}_{(d+1)}} \boldsymbol{\mu}^{[d+1]} = \text{sgn}(G) \int_{\mathcal{M}_{(d+2)}} \boldsymbol{V} \wedge \boldsymbol{\mu}^{\flat(d+1)} = \text{sgn}(\gamma) \int \{\mathrm{d}x^\mu\} \sqrt{|\gamma|} \; *^\downarrow \left[\boldsymbol{\mu}^{[d+1]} \right], \quad (\text{C.23})$$

where $\gamma_{\mu\nu} = p_{\mu\nu} + n_\mu n_\nu$ and $\gamma = \det \gamma_{\mu\nu} = -G$. Obviously a full rank spatial form would be zero. Rest of the notations and conventions follow from our relativistic discussion.

Non-covariant Differential Forms

Choosing a non-covariant basis given in appendix (A), a vector and a one-form can be decomposed as,

$$\mathcal{X}^M \partial_M = -\mathrm{e}^\Phi \left(\mathcal{X}_t + B_t \mathcal{X}_\sim \right) \partial_\sim - \mathrm{e}^\Phi \mathcal{X}_\sim \left(B_t \partial_\sim + \partial_t \right) + \mathcal{X}^i \left(\partial_i - a_i \partial_t + \left(B_i - a_i B_t \right) \partial_\sim \right),$$

$$\mathcal{Y}_M \mathrm{d}x^M = \mathcal{Y}_\sim \left(\mathrm{d}x^\sim - B_\mu \mathrm{d}x^\mu \right) + \left(\mathcal{Y}_\sim B_t + \mathcal{Y}_t \right) \left(\mathrm{d}t + a_i \mathrm{d}x^i \right) + g_{ij} \mathcal{Y}^j \mathrm{d}x^i. \quad (\text{C.24})$$

It immediately follows that a spatial differential form $(\mathcal{Y}_t = \mathcal{Y}_\sim = 0)$ is indeed a pure differential form on the spatial slice. Such a form can be expanded in coordinate basis as,

$$\boldsymbol{\mu}^{(m)} = \frac{1}{m!} \mu_{i_1 i_2 \dots i_m} \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \dots \wedge \mathrm{d}x^{i_m}. \quad (\text{C.25})$$

Since there exists a invertible metric g_{ij} on this slice, there is a map between forms and contra-forms. One can check that the volume element $\varepsilon^{(d)}$ defined before is indeed a full rank form on the spatial slice and can be written in this setting as,

$$\varepsilon^{(d)} = \star [\boldsymbol{V} \wedge \bar{\boldsymbol{V}}] = \frac{1}{d!} \left(V^M \bar{V}^N \epsilon_{MN i_1 \dots i_d} \right) \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_d}. \quad (\text{C.26})$$

The Hodge dual $*$ associated with it serves as Hodge dual operation on the spatial slice. Finally a full rank spatial form can be integrated on a spatial slice,

$$\int_{\mathcal{M}_{(d)}} \boldsymbol{\mu}^{(d)} = \text{sgn}(G) \int_{\mathcal{M}_{(d+2)}} \mathrm{e}^\Phi \boldsymbol{V} \wedge \bar{\boldsymbol{V}} \wedge \boldsymbol{\mu}^{(d)} = \text{sgn}(g) \int \{\mathrm{d}x^\mu\} \sqrt{|g|} \; * [\boldsymbol{\mu}^{(d)}]. \quad (\text{C.27})$$

Here $g = \det g_{ij} = \mathrm{e}^{2\Phi} \gamma = -\mathrm{e}^{2\Phi} G$. Other conventions and notations are same as relativistic case.

D | Comments on the Relativistic Entropy Current

In this appendix we want to make some comments on the entropy current for a relativistic fluid. To make more contact with [41], in this section we consider the relativistic manifold $\mathcal{M}_{(2n)}$ to be $2n$ dimensional, and denote indices on it by $\mu, \nu \ldots$. On the local flat space $\mathbb{R}^{(2n-1,1)}$ however we denote indices by $\alpha, \beta \ldots$. This setup is equipped with a Vielbein $e^{\alpha}{}_{\mu}$, an affine connection $\Gamma^{\lambda}{}_{\mu\nu}$, a spin connection $C^{\alpha}{}_{\mu\beta} = e_{\beta}{}^{\rho} \left(\Gamma^{\nu}{}_{\mu\rho} e^{\alpha}{}_{\nu} - \partial_{\mu} e^{a}{}_{\rho} \right)$, and a non-abelian gauge field A_{μ}. Correspondingly we have torsion tensor $\mathrm{T}^{\alpha}{}_{\mu\nu}$, Riemann curvature tensor $R_{\mu\nu}{}^{\alpha}{}_{\beta}$ and gauge field strength $F_{\mu\nu}$. $\mu, \nu \ldots$ can be raised/lowered by metric $g_{\mu\nu}$, $\alpha, \beta \ldots$ indices can be raised/lowered by flat metric $\eta_{\alpha\beta}$, while both type of indices can be interchanged by the Vielbein. The covariant derivative on the other hand is given by ∇_{μ} which is associated with all the connections. We take the fluid data to be $\psi_{\beta} = \{ \beta^{\mu}, [\Lambda_{\Sigma(\beta)}]^{\mu}{}_{\nu}, \Lambda_{(\beta)} \}$. In terms of it we define fluid temperature $T = (-\beta^{\mu}\beta_{\mu})^{-1/2}$, fluid velocity $u^{\mu} = T\beta^{\mu}$, scaled chemical potential $\nu = \Lambda_{\beta} + \beta^{\mu}A_{\mu}$, chemical potential $\mu = T\nu$, scaled spin chemical potential $[\nu_{\Sigma}]^{\alpha}{}_{\beta} = [\Lambda_{\Sigma}]^{\alpha}{}_{\beta} + \beta^{\mu}C^{\alpha}{}_{\mu\beta}$, and spin chemical potential $[\mu_{\Sigma}]^{\alpha}{}_{\beta} = T[\nu_{\Sigma}]^{\alpha}{}_{\beta}$. Finally we have canonical EM tensor $T^{\mu}{}_{\alpha}$, spin current $\Sigma^{\mu\alpha}{}_{\beta}$, charge current J^{μ}, entropy current J^{μ}_{S}, Belinfante EM tensor $T^{\mu\nu}_{(b)}$, and Belinfante (usual) entropy current $J^{\mu}_{S(b)}$.

We wrote an off-shell generalization for the second law of thermodynamics in Cartan formalism in § 4, which in aforementioned notation will become,

$$\underline{\nabla}_{\mu} J^{\mu}_{s} + \beta^{\nu} \left(\underline{\nabla}_{\mu} T^{\mu}{}_{\nu} - \mathrm{T}^{\alpha}{}_{\nu\mu} T^{\mu}{}_{\alpha} - R_{\nu\mu}{}^{\alpha}{}_{\beta} \Sigma^{\mu\beta}{}_{\alpha} - F_{\nu\mu} \cdot J^{\mu} \right)$$
$$+ [\nu_{\Sigma}]_{\beta\alpha} \left(\underline{\nabla}_{\mu} \Sigma^{\mu\alpha\beta} - T^{[\beta\alpha]} - \Sigma^{\perp\alpha\beta}_{\mathrm{H}} \right) + \nu \cdot \left(\underline{\nabla}_{\mu} J^{\mu} - \mathrm{J}^{\perp}_{\mathrm{H}} \right) \geq 0, \quad \text{(D.1)}$$

where $\underline{\nabla}_{\mu} = \nabla_{\mu} - \mathrm{T}^{\nu}{}_{\nu\mu}$. $\Sigma^{\perp\alpha\beta}_{\mathrm{H}}$, $\mathrm{J}^{\perp}_{\mathrm{H}}$ are the anomalous Hall currents, which are determined in terms of an anomaly polynomial $\mathcal{P}^{(2n+2)}_{\mathrm{CS}}$,

$$\star_{(2n+1)} \boldsymbol{\Sigma}_{\mathrm{H}}{}^{\alpha\beta} = \frac{\partial \mathcal{P}^{(2n+2)}_{\mathrm{CS}}}{\partial \boldsymbol{R}_{\beta\alpha}}, \qquad \star_{(2n+1)} \mathbf{J}_{\mathrm{H}} = \frac{\partial \mathcal{P}^{(2n+2)}_{\mathrm{CS}}}{\partial \boldsymbol{F}}. \quad \text{(D.2)}$$

On imposing equations of motion eqn. (3.4) (after appropriate change of notation) this will boil down to the second law of thermodynamics $\underline{\nabla}_{\mu} J^{\mu}_{S} \geq 0$. To compare this statement with that of [3] we making a field redefinition,

$$[\nu_{\Sigma}]_{\mu\nu} \to [\nu'_{\Sigma}]_{\mu\nu} = [\nu_{\Sigma}]_{\mu\nu} + e_{\alpha[\mu} \delta_{\beta} e^{\alpha}{}_{\nu]} = \nabla_{[\nu} \beta_{\mu]} + \mathrm{T}_{[\mu\nu]\rho} \beta^{\rho}, \quad \text{(D.3)}$$

where δ_{β} is the diffeo, spin and gauge transformation associated with ψ_{β}. This field redefinition does not spoil our equilibrium frame as the perturbation vanishes on promoting ψ_{β} to an isometry. Further setting torsion to zero, this statement boils down to the statement of [3],

$$\nabla_{\mu} J^{\mu}_{S(b)} + \beta_{\nu} \left(\nabla_{\mu} T^{\nu\mu}_{(b)} - F^{\nu\mu} \cdot J_{\nu} - \nabla_{\mu} \Sigma^{\perp\nu\mu}_{\mathrm{H}} \right) + \nu \cdot \left(\nabla_{\mu} J^{\mu} - \mathrm{J}^{\perp}_{\mathrm{H}} \right) \geq 0, \quad \text{(D.4)}$$

where we have defined the *Belinfante EM tensor*,

$$T^{\mu\nu}_{(b)} = T^{(\mu\nu)} + 2\nabla_{\rho} \Sigma^{(\mu\nu)\rho}, \quad \text{(D.5)}$$

and *Belinfante entropy current*[23],

$$J^{\mu}_{S(b)} = J^{\mu}_{S} + \beta_{\nu} \left(\nabla_{\rho} \Sigma^{\rho\mu\nu} + T^{[\mu\nu]} - \Sigma^{\perp\mu\nu}_{\mathrm{H}} \right), \quad \text{(D.6)}$$

---

[23] The motive of calling $S^{M}_{(b)}$ Belinfante entropy current is primarily to distinguish it from S^{M}, and secondly to relate it more closely to Belinfante EM tensor $T^{MN}_{(b)}$. We couldn't find any existing name in the literature for this quantity.

which is more natural quantity to use when working with Belinfante EM tensor $T^{\mu\nu}_{(b)}$. Note that the two entropy currents differ only off-shell and boil down to the same when the spin equation of motion is imposed,

$$\nabla_\rho \Sigma^{\rho\mu\nu} = T^{[\nu\mu]} + \Sigma^{\perp\mu\nu}_{\rm H}. \tag{D.7}$$

For comparison with [41] we would be interested in spinless relativistic fluids. In absence of anomalies we could define spinless fluids by $\Sigma^{\rho\mu\nu} = T^{[\mu\nu]} = 0$, but anomalies would not allow us to make this simple choice. Nevertheless we can define spinless fluids by $\Sigma^{\rho\mu\nu}$, $T^{[\mu\nu]}$ being totally determined by anomalies.

The transgression form business does not change much in Vielbein formalism. The end result is that we can define certain quantities in terms of anomaly polynomial $\mathcal{P}^{(2n+2)}_{\rm CS}$ and hatted connections $\hat{\boldsymbol{A}} = \boldsymbol{A} + \mu\boldsymbol{u}$, $\hat{\boldsymbol{C}}^\alpha_{\ \beta} = \boldsymbol{C}^\alpha_{\ \beta} + [\mu_\Sigma]^\alpha_{\ \beta}\boldsymbol{u}$,

$$\star\boldsymbol{\Sigma}^{\alpha\beta}_{\mathcal{P}} = \frac{\boldsymbol{u}}{{\rm d}\boldsymbol{u}} \wedge \left(\frac{\partial \mathcal{P}}{\partial \boldsymbol{R}_{\beta\alpha}}^{(2n+2)} - \frac{\partial \hat{\mathcal{P}}^{(2n+2)}}{\partial \hat{\boldsymbol{R}}_{\beta\alpha}} \right), \qquad \star\boldsymbol{J}_{\mathcal{P}} = \frac{\boldsymbol{u}}{{\rm d}\boldsymbol{u}} \wedge \left(\frac{\partial \mathcal{P}^{(2n+2)}}{\partial \boldsymbol{F}} - \frac{\partial \hat{\mathcal{P}}^{(2n+2)}}{\partial \hat{\boldsymbol{F}}} \right)$$

$$\star\boldsymbol{q}_{\mathcal{P}} = -\frac{\boldsymbol{u}}{{\rm d}\boldsymbol{u}} \wedge \left[\mathcal{P}^{(2n+2)}_{\rm CS} - \hat{\mathcal{P}}_{\rm CS} + {\rm d}\boldsymbol{u} \wedge \left([\mu_\Sigma]^\alpha_{\ \beta} \frac{\partial \hat{\mathcal{P}}^{(2n+2)}}{\partial \hat{\boldsymbol{R}}^\alpha_{\ \beta}} + \mu \cdot \frac{\partial \hat{\mathcal{P}}^{(2n+2)}}{\partial \hat{\boldsymbol{F}}} \right) \right]. \tag{D.8}$$

In terms of these, anomalous sector of constitutive relations is given as,

$$(T^{\mu\alpha})_{\rm A} = q^\mu_{\mathcal{P}} u^\alpha + q^\alpha_{\mathcal{P}} u^\mu, \qquad (\Sigma^{\mu\alpha\beta})_{\rm A} = \Sigma^{\mu\alpha\beta}_{\mathcal{P}}, \qquad (J^\mu)_{\rm A} = J^\mu_{\mathcal{P}}. \tag{D.9}$$

These currents follow the Bianchi identities[24],

$$\underline{\nabla}_\mu (T^\mu_{\ \nu})_{\rm A} = {\rm T}^\alpha_{\ \nu\mu}(T^\mu_{\ \alpha})_{\rm A} + R_{\nu\mu}^{\ \ \alpha}_{\ \ \beta}(\Sigma^{\mu\beta}_{\ \ \alpha})_{\rm A} + F_{\nu\mu}\cdot(J^\mu)_{\rm A} - u_\nu \left(\mu\cdot\hat{\rm J}^\perp_{\rm H} + [\mu_\Sigma]_{\alpha\beta}\hat{\Sigma}^{\perp\beta\alpha}_{\rm H} \right),$$
$$+ \frac{1}{\sqrt{-g}}\delta_\beta \left(\sqrt{-g}Tq_{\mathcal{P}\nu} \right)$$
$$\underline{\nabla}_\mu (\Sigma^{\mu\alpha\beta})_{\rm A} = \Sigma^{\perp\alpha\beta}_{\rm H} - \hat{\Sigma}^{\perp\alpha\beta}_{\rm H},$$
$$\underline{\nabla}_\mu (J^\mu)_{\rm A} = {\rm J}^\perp_{\rm H} - \hat{\rm J}^\perp_{\rm H}. \tag{D.10}$$

Plugging these constitutive relations into the off-shell adiabaticity equation we can get a relation for the entropy current,

$$\underline{\nabla}_\mu (J^\mu_S)_{\rm A} \geq 0. \tag{D.11}$$

Hence the off-shell second law can be satisfied with a trivially zero entropy current

$$J^\mu_S = 0. \tag{D.12}$$

In other words, entropy current J^μ_S does not get any contribution from anomalies. On the other hand, using Bianchi identities in eqn. (D.6), we can read out the anomalous Belinfante entropy current,

$$(J^\mu_{S(b)})_{\rm A} = \beta_\nu \hat{\Sigma}^{\perp\nu\mu}_{\rm H}, \tag{D.13}$$

which is what was found by [41]. Note that $\hat{\Sigma}^{\perp\nu\mu}_{\rm H}$ is by definition antisymmetric in its last two indices, and differ from [41] by a factor of 2. Hence we have established that entropy current in Vielbein formalism J^μ_S does not get contribution from anomalies, while the Belinfante entropy current does. Recall that a similar situation appears for EM tensor as well; while the canonical EM tensor $T^{\mu\alpha}$ that appears in Vielbein formalism is Noether current of translations,

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[24]Upon using the definition of Belinfante EM tensor from eqn. (D.5), and setting torsion to zero, these Bianchi identities reproduce the ones given in [41].

the symmetric Belinfante EM tensor $T_{(b)}^{\mu\nu}$ that appears in usual formalism couples to metric in general relativity but does not correspond to any Noether current. Hence from the point of view of symmetries, canonical EM tensor is a more natural quantity. On the same lines we guess that Vielbein entropy current will be more naturally associated with the fundamental $U(1)_T$ symmetry introduced by [41], as opposed to the Belinfante entropy current. The former being independent of anomalies seems to strengthen this natural guess. However one will have to do the explicit computation of $U(1)_T$ transformations in presence of torsion to give any weight to this claim.

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