Automorphic forms and the Arthur-Selberg trace formula

MA842 BU Spring 2018

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1 Automorphic forms $/\mathrm{GL}_2$ (possibly GL_n)

> Lecture 1 18/1/2018

These are notes for Ali Altuğ's course MA842 at BU Spring 2018, they were last updated January 30, 2018.

The course webpage is http://math.bu.edu/people/saaltug/2018_1/2018_1_sem.html.

Course overview: This course will be focused on the two papers Eisenstein Series and the Selberg Trace Formula I by D. Zagier and Eisenstein series and the Selberg Trace Formula II by H. Jacquet and D. Zagier. Although the titles of the papers sound like one is a prerequisite of the other it actually is not the case, the main difference is the language of the papers (the first is written in classical language whereas the second is written in adelically). We will spend most of our time with the second paper, which is adelic.

1.1 Goal

Jacquet and Zagier, Eisenstein series and the Selberg Trace Formula II (1980's).

Part I is a paper of Zagier from 1970 in purely classical language. Part II is in adelic language (and somewhat incomplete).

$$\begin{pmatrix} \text{Arthur-Selberg} \\ \text{trace formula} \end{pmatrix} \xleftrightarrow{\text{conjecture}} \begin{pmatrix} \text{Relative} \\ \text{trace formula} \end{pmatrix}$$

the Arthur-Selberg side is used in Langlands functoriality and the Relative is used in arithmetic applications.

1.2 Motivation

What does this paper do?

> "It rederives the Selberg trace formula for GL_2 by a regularised process."

Note 1.1.

- Selberg trace formula only for GL_2

- Arthur-Selberg more general

The Selberg trace formula generalises the more classical Poisson summation formula.

Poisson summation

Theorem 1.2 (Poisson summation). *Let*

$$f : \mathbf{R} \to \mathbf{R}$$

*then **Poisson summation** says*

$$\sum_{n \in \mathbf{Z}} f(n) = \sum_{\xi \in \mathbf{Z}} \hat{f}(\xi)$$

where

$$\hat{f}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e(x\xi) \, dx.$$

Notation: $e(x) = e^{2\pi i x}$.

To make this look more general we make the following notational choices.

$$G = \mathbf{R}, \Gamma = \mathbf{Z}$$

$$\sum_{\gamma \in \Gamma^{\#}} f(\gamma) = \sum_{\xi \in (G/\Gamma)^{\vee}} \hat{f}(\xi)$$

where

- $\Gamma^{\#}$ = conjugacy classes of Γ (= Γ in this case since Γ is abelian).

- $(G/\Gamma)^{\vee}$ = dual of G/Γ.

Selberg

$$G = \mathrm{GL}_2(\mathbf{R}), \ \Gamma = \mathrm{GL}_2(\mathbf{Z})$$

$$\sum_{\gamma \in \Gamma^{\#}} \cdots \text{``} = \text{''} \sum_{\pi \in \text{``}(G/\Gamma)^{\vee}\text{''}} \cdots$$

relating conjugacy classes on the left to automorphic forms on the right.

Arthur and Selberg prove the trace formula by a *sharp cut off*, Jacquet and Zagier derive this using a regularisation.

1.3 Motivating example

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges absolutely for $s > 1$.

Theorem 1.3 (Riemann). *$\zeta(s)$ has analytic continuation up to $\mathfrak{R}(s) > 0$ with a simple pole at $s = 1$ residue 1. i.e.*

$$\zeta(s) = \frac{1}{s-1} + \phi(s)$$

where $\phi(s)$ is holomorphic for $\mathfrak{R}(s) > 0$.

Proof. Step 1: observe

$$\frac{1}{s-1} = \int_{1}^{\infty} t^{-s} \, dt \text{ (for } \mathfrak{R}(s) > 1)$$

$$= \sum_{n=1}^{\infty} \int_{n}^{n+1} t^{-s} \, dt$$

Step 2: this implies

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} n^{-s} - \int_{n}^{n+1} t^{-s} \, dt$$

$$= \frac{1}{s-1} + \sum_{n=1}^{\infty} \left(\int_{n}^{n+1} n^{-s} - \int_{n}^{n+1} t^{-s} \, dt \right)$$

we denote each of the terms in the right hand sum as $\phi_n(s)$

$$\phi_n(s) = \int_{n}^{n+1} n^{-s} - t^{-s} \, dt$$

Step 3:

$$|\phi_n(s)| \leq \sup_{n \leq t \leq n+1} |n^{-s} - t^{-s}|$$

$$\sup_{n \leq t \leq n+1} \frac{|s|}{t^{\Re(s)+1}} \leq \frac{|s|}{n^{\Re(s)+1}}$$

by applying the mean value theorem.

So $\sum_{n=1}^{\infty} \phi_n$ converges absolutely. Hence $\phi = \sum_{n=1}^{\infty} \phi_n$ is holomorphic

One can push this idea to get analytic continuation to all of $\mathbf{C}$, one strip at a time. This is an analogue of the sharp cut off method mentioned above. It's fairly elementary but somewhat unmotivated and doesn't give any deep information (like the functional equation). □

Proof. Introduce

$$\theta(t) = \sum_{n \in \mathbf{Z}} e^{-\pi n^2 t}, \; t > 0$$

note that $\theta(t) = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 t}$.

Idea: Mellin transform and properties of θ to derive properties of ζ.

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{s/2}} \frac{1}{n s} = \int_{0}^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{dt}{t}$$

property of θ:

$$\theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right)$$

Step 1: proof of this property is the Poisson summation formula

-
$$f(x) = e^{-\pi x^2} \implies \hat{f}(\xi) = f(\xi)$$

-
$$g(x) = f(\sqrt{t}x) \implies \hat{g}(\xi) = \frac{1}{\sqrt{t}} \hat{f}\left(\frac{\xi}{\sqrt{t}}\right)$$

Step 2: Would like to write something like

$$\text{``} \int_{0}^{\infty} \theta(t) t^{s/2} \frac{dt}{t} \text{''}$$

This integral makes no sense

3

- As $t \to \infty$, $\theta \sim 1$ thus

$$\left| \int_A^\infty \theta(t) t^{s/2} \frac{dt}{t} \right| < \infty$$

$$\Longleftrightarrow \left| \int_A^\infty t^{s/2} \frac{dt}{t} \right| < \infty$$

$$\Longleftrightarrow \Re(s) < 0$$

- As $t \to 0$ consider $\xi = \frac{1}{t}$ so $\xi \to \infty$ and

$$\theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right) = \sqrt{\xi}\, \theta \xi]$$

$$\Longrightarrow \theta(t) = \sqrt{\xi}\, \theta(\xi) \sim \sqrt{\xi} = \frac{1}{\sqrt{t}}$$

so $\theta(t) \sim \frac{1}{\sqrt{t}}$

$$\Longrightarrow \left| \int_0^A \theta(t) t^{s/2} \frac{dt}{t} \right| < \infty$$

$$\Longleftrightarrow \left| \int_0^A t^{(s-1)/2} \frac{dt}{t} \right| < \infty$$

$$\Longleftrightarrow \Re(s) > 1$$

so no values of s will make sense for this improper integral.

Refined idea: Consider

$$I(s) = \int_0^1 (\theta(t) - \frac{1}{\sqrt{t}}) t^{s/2} \frac{dt}{t} + \int_1^\infty (\theta(t) - 1) t^{s/2} \frac{dt}{t}$$

upshot: $I(s)$ is well-defined and holomorphic for all $s \in \mathbf{C}$.

Final step: Compute the above to see

$$I(s) = \frac{2}{s} + \frac{2}{1-s} + \frac{2}{\pi^{s/2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

which implies

1. $\zeta(s)$ has analytic continuation to $s \in \mathbf{C}$, with only a simple pole at $s = 1$ with residue 1.

2.
$$I(s) = I(1 - s),$$

this follows from the property of θ so if we let

$$\Lambda(s) = \frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{s/2}} \zeta(s),$$

then

$$\Lambda(s) = \Lambda(1 - s). \qquad \square$$

1.4 Modular forms

Functions on the upper half plane,

$$\mathbf{H} = \{z \in \mathbf{C} : \mathfrak{I}(z) > 0\}.$$

Historically elliptic integrals lead to elliptic functions, and modular forms and elliptic curves.

Note 1.4. When one is intereseted in functions on O/Λ where O is some object and Λ is some discrete group. Take f a function on O and average over Λ to get

$$\sum_{\lambda \in \Lambda} f(\lambda z).$$

If you're lucky this converges, this is good.

Elliptic functions   Weierstrass, take $\Lambda = \omega_1 \mathbf{Z} + \omega_2 \mathbf{Z}$ a lattice an define

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} + \frac{1}{\omega^2} \right).$$

Jacobi, (Elliptic integrals) consider

$$\int_0^\phi \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-\kappa t^2)}}, \quad \kappa \geq 0$$

related by:

$$(\wp'_\Lambda(z))^2 = 4\wp_\Lambda(z)^3 - 60G_2(\Lambda)\wp_\Lambda(z) - 140G_3(\Lambda)$$

$$G_k(\Lambda) = \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-2k}$$

or

$$G_k(\tau) = \sum_{(m,n) \in \mathbf{Z}^2 \setminus \{0\}} \frac{1}{(m\tau + n)^{2k}}$$

the weight $2k$ holomorphic Eisenstein series.

Fact 1.5. *Let*

$$u = \int_y^\infty \frac{\mathrm{d}s}{\sqrt{4s^3 - 60G_2 s - 140G_3}}$$

then

$$y = \wp_\Lambda(u).$$

1.5 Euclidean Harmonic analysis

> Lecture 2 23/1/2018

We'll take a roundabout route to automorphic forms.

Today: Classical harmonic analysis on $\mathbf{R}^n$. Classical harmonic analysis on $\mathbf{H}$.

The aim (in general) is to express a certain class of functions (i.e. $\mathcal{L}^2$) in terms of building block (harmonics).

In classical analysis the harmonics are known $(e(nx))$, then the question becomes how these things fit together. In number theory the harmonics are extremely mysterious. We are looking at far more complicated geometries, quotient spaces etc. and arithmetic information comes in.

Example 1.6. $\mathbf{R}$, $f \colon \mathbf{R} \to \mathbf{C}$, being periodic in $\mathcal{L}^2(S^1)$ leads to a fourier expansion

$$f(x) = \sum_{n \in \mathbf{Z}} a_n e(nx).$$

1.5.1 $\mathbf{R}^2$

We have a slightly different perspective.

$$\mathbf{R}^2 = G \cup G$$

via translations (i.e. right regular representation of G will be $G \cup \mathcal{L}^2(G)$). I.e. $g \cdot x = x + g$.

Remark 1.7. This makes $\mathbf{R}^2$ a homogeneous space.
$\mathbf{R}^2$ with standard metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ is a flat space $\kappa = 0$.

To the metric we have the associated Laplacian (Laplace-Beltrami operator, $\nabla \cdot \nabla$)

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

we are interested in this as it is essentially the only operator, we will define automorphic forms to be eigenfunctions for this operator.

Note 1.8. The exponential functions

$$\phi_{u,v}(x, y) = e(ux + vy)$$

are eigenfunctions of Δ with eigenvalue $\lambda_{u,v} = -4\pi^2(u^2 + v^2)$ i.e.

$$(\Delta + \lambda_{u,v})\phi_{u,v} = 0.$$

These are a complete set of harmonics for $\mathcal{L}^2(\mathbf{R}^2)$. The proof is via fourier inversion.

$$f(x, y) = \int\int_{\mathbf{R}^2} \hat{f}(u, v)\phi_{u,v}(x, y) \, \mathrm{d}u \, \mathrm{d}v$$

where

$$\hat{f}(u, v) = \int\int_{\mathbf{R}^2} f(u, v)\bar{\phi}_{u,v}(x, y) \, \mathrm{d}y \, \mathrm{d}x.$$

A little twist We could have established the spectral resolution (of Δ) by considering invariant integral operators.

Using the spectral theorem if we can find easier to diagonalise operators that commute we can find the eigenspaces for those to cut down the eigenspace.

Recall: an integral operator is

$$L(f)(x) = \int K(x, y)f(y) \, \mathrm{d}y$$

invariant means

$$L(gf) = gL(f) \ g \in G$$

in our case

$$g \cdot f(x) = f(g + x).$$

Observation 1.9. If L is invariant then the kernel $K(x, y)$ is given by

$$K(x, y) = K_0(x - y)$$

for some function K_0.

Proof. ($\Longleftarrow$) obvious

($\Longrightarrow$) Suppose L is invariant then

$$\int_{\mathbf{R}^2} K(x,y)f(y+\alpha)\,\mathrm{d}y = \int_{\mathbf{R}^2} K(x+\alpha,y)f(y)\,\mathrm{d}y \;\forall f$$

implies

$$\int\int_{\mathbf{R}^2} K(x,y-\alpha)f(y)\,\mathrm{d}y = \int_{\mathbf{R}^2} K(x+\alpha,y)f(y)\,\mathrm{d}y \;\forall f$$

so

$$\int_{\mathbf{R}^2}(K(x+\alpha,y)-K(x,y-\alpha))f(y)\,\mathrm{d}y = 0 \;\forall f$$

which implies with some proof that

$$K(x+\alpha,y) = K(x,y-\alpha)$$

so

$$K(x,y) = K(x-y,0). \qquad \square$$

Observation 1.10. Invariant integral operators commute with each other

$$L_1 L_2(f)(z) = L_2 L_1(f)(z)$$

Proof.

$$L_1 L_2(f)(z) = \int_{\mathbf{R}^2\times\mathbf{R}^2} f(w)K_2(u-w)K_1(z-u)\,\mathrm{d}w\,\mathrm{d}u = L_2 L_1(f)(z)$$

after change of variables

$$u \mapsto z-u+w \qquad \square$$

Observation 1.11. L commutes with Δ.

Proof. Based on the following:

$$K(x,y) = K_0(x-y,0)$$

$$\frac{\partial K}{\partial x_i} = -\frac{\partial K}{\partial y_i}$$

which implies

$$\Delta_z(L(f))(z) = \Delta_z \int_{\mathbf{R}^2} f(w)K(z,w)\,\mathrm{d}w$$

$$= \int\int_{\mathbf{R}^2} \Delta_z f(w)K(z,w)\,\mathrm{d}w$$

$$= \int\int_{\mathbf{R}^2} f(w)\Delta_w K(z,w)\,\mathrm{d}w$$

which via integration by parts is

$$= \int\int_{\mathbf{R}^2} \Delta_w f(w)K(z,w)\,\mathrm{d}w = L(\Delta f)(z). \qquad \square$$

Observation 1.12. $\phi_{u,v}(x,y)$ is an eigenfunction of L, $(u,v)\in\mathbf{R}^2$, $(x,y)\in\mathbf{R}^2$.

Proof.

$$L\phi_{u,v}(\overbrace{z}^{=x,y}) = \int_{\mathbf{R}^2} \phi_{u,v}(w) K(z,w)\, dw$$

$$= \int_{\mathbf{R}^2} \phi_{u,v}(w) K_0(z-w)\, dw$$

$$= \int_{\mathbf{R}} \int_{\mathbf{R}} e(uw_1 + vw_2) K_0(z_1 - w_1, z_2 - w_2)\, dw_1\, dw_2$$

$$= e(uw_1 + vw_2) \int_{\mathbf{R}} \int_{\mathbf{R}} K_0(w_1, w_2) e(-uw_1 - vw_2)\, dw_1\, dw_2$$

after the change of variable $w_i \mapsto -w_i + z_i$

$$\phi_{u,v}(z) \hat{K}_0(u,v)$$

i.e.

$$L\phi_{u,v} = \hat{K}_0(u,v)\phi_{u,v}. \qquad \square$$

Side remark: these are enough to form a generating set.

1.5.2 Poisson summation (yet again)

Let's consider integral operators on functions on $\mathbf{Z}^2\backslash\mathbf{R}^2 = \mathbf{T}^2$.

Observe: $L \rightsquigarrow K(x,y) = K_0(x-y)$.

$$Lf(z) = \int_{\mathbf{R}^2} f(w) K(z,w)\, dw$$

$$= \int \int_{\mathbf{Z}^2\backslash\mathbf{R}^2} f(w) \underbrace{\left(\sum_{n\in\mathbf{Z}^2} K(z, w+n) \right)}_{=\sum_{n\in\mathbf{Z}} K_0(z-w+n)=\mathbf{K}(z,w)}\, dw$$

now $\mathbf{K}$ is a function on $\mathbf{T}^2 \times \mathbf{T}^2$.

Trace of this operator

$$= \int_{\mathbf{T}^2} \mathbf{K}(z,z)\, dz = \int_{\mathbf{T}^2} \left(\sum_{n\in\mathbf{Z}^2} K_0(n) \right) dz = \sum_{n\in\mathbf{Z}^2} K_0(n)$$

Using sum of eigenvalues

$$K(z,w) = \sum_{n\in\mathbf{Z}^2} K_0(z-w+n) = \sum_{\xi\in\mathbf{Z}^2} \lambda_\xi \phi_\xi(z-w) = \sum_{\xi\in\mathbf{Z}^2} \lambda_\xi \phi_\xi(z)\bar{\phi}_\xi(w)$$

so the trace is

$$\sum_{\xi\in\mathbf{Z}^2} \lambda_\xi = \sum_{\xi\in\mathbf{Z}^2} \hat{K}_0(\xi)$$

so we get to

$$\sum_{n\in\mathbf{Z}^2} K_0(n) = \sum_{\xi\in\mathbf{Z}^2} \hat{K}_0(\xi)$$

i.e. Poisson summation.

Why care about Poisson summation?

$$\hat{K}_0(0) = \int K_0(z)\, dz$$

8

Gauss circle problem, how many lattice points are there in a circle of radius R. We can pick a radially symmetric function that is 1 on the circle and 0 outside, or a smooth approximation of such an indicator function at least. Poisson summation packages the important information into a single term, plus some rapidly decaying ones. Then we get πR^2+ error, Gauss conjectured that the error is $R^{1/2+\epsilon}$.

Lecture 3 25/1/2018

Last time we gave a conceptual proof of Poisson summation (this strategy will generalise to the trace formula eventually).

To clean up one loose end: there is a generalisation of Poisson summation called Voronoi summation, which will actually be useful later. For Poisson summation we had

$$\sum_{n_1,n_2 \in \mathbf{Z}} K(n_1, n_2) = \sum_{\xi_1,\xi_2 \in \mathbf{Z}} \hat{K}(\xi_1, \xi_2)$$

suppose $K(x,y) \colon \mathbf{R}^2 \to \mathbf{C}$ is radially symmetric i.e.

$$K(x,y) = K_0(x^2 + y^2), \ (x,y) \in \mathbf{R}^2$$

then the fourier transform

$$\hat{K}(u,v) = \pi \int_0^\infty K_0(r) J_0(\sqrt{\lambda} r) \, \mathrm{d}r, \ \lambda = 4\pi^2(u^2 + v^2)$$

where

$$J_0(z) = \frac{1}{\pi} \int_0^\pi \cos(z \cos(\alpha)) \, \mathrm{d}\alpha$$

is a Bessel function of the second kind.

Exercise 1.13. Prove this.

Plug this into Poisson summation

$$\sum_{(n_1,n_2) \in \mathbf{Z}^2} K(n_1, n_2) = \sum_{N=0}^\infty r_2(N) K_0(N)$$

as K only depends on $n_1^2 + n_2^2$ we group terms based on this quantity, so

$$r_2 = \#\{(n_1, n_2) \in \mathbf{Z}^2 : n_1^2 + n_2^2 = N\}.$$

$$\sum_{\xi_1,\xi_2 \in \mathbf{Z}} \pi \int_0^\infty K_0(r) J_0(2\pi \sqrt{(\xi_1^2 + \xi_2^2)r}) \, \mathrm{d}r$$

$$= \sum_{M=0}^\infty r_2(M) \tilde{K}_0(M).$$

Theorem 1.14 (Voronoi summation)**.**

$$\sum_{N=0}^\infty r_2(N) K_0(N) = \sum_{M=0}^\infty r_2(M) \tilde{K}_0(M)$$

where

$$\tilde{K}_0(z) = \pi \int_0^\infty K_0(r) J_0(2\pi \sqrt{zr}) \, \mathrm{d}r.$$

Note that $J_0(0) = 1$.

How is this useful? Consider point counting in a circle problem. Let $K_0(x)$ be an approximation to the step function, 1 for $x \leq 1$ and 0 for $x > 1$. With $\int K_0 = 1$. Then

$$\sum_{N=0}^{\infty} K_0\left(\frac{N}{R^2}\right) r_2(N).$$

This is counting lattice points. The right hand side is then

$$\sum_{M=0}^{\infty} r_2(M)\tilde{K}_0(M) = \tilde{K}_0(0) + \sum_{M=1}^{\infty} r_2(M)\tilde{K}_0(M)$$

$$= \pi + \sum_{M=1}^{\infty} r_2(M)\tilde{K}_0(M).$$

Finally

$$f(z) = K_0\left(\frac{z}{R^2}\right)$$

so

$$\tilde{f}(\xi) = R^2 \tilde{K}_0(\xi R^2).$$

So

$$\sum_{N=0}^{\infty} K_0\left(\frac{N}{R^2}\right) r_2(N) = R^2 \left(\pi + \sum_{M=1}^{\infty} r_2(M)\tilde{K}_0(MR^2)\right)$$

where the lead term is the area of the circle. Finally if $M \neq 0$ then $f(MR^2)$ doesn't increase fast as $R \to \infty$. i.e. it is smaller than R^2. So as $R \to \infty$ we find #{lattice points in the circle} $\sim \pi R^2$.

1.6 The hyperbolic plane H

What if we consider the same problem on the hyperbolic disk? Things are extremely different.

Generalities

Definition 1.15.
$$\mathbf{H} = \{x + iy : y > 0\}.$$

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2), \text{ Riemannian metric}$$

this gives

$$\kappa = -1$$

i.e. this is negatively curved, this is the cause of huge differences between the euclidean theory.

There is a formula for the (hyperbolic) distance between two points

$$\rho(z, w) = \log \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|}.$$

Observation 1.16. As $w \to \mathbf{R}$ we have $\rho \to \infty$. So $\mathbf{R}$ is the boundary.

Recall: the isoperimetric inequality

$$4\pi A - \kappa A^2 \le L^2$$

where L is the length of the boundary of a region and A is the area. Note if $\kappa = 0$ then $4\pi A \le L^2$. So A can be and would be as large as L^2.

For $\kappa = -1$ we have

$$4\pi A + A^2 \le L^2$$

so A can at most (and most often will) be as large as $\sim L$. The upshot is that under the hyperbolic metric, the area and perimeter can be the same size.

$$|\text{Boundary}| \sim |\text{Area}|.$$

Things are a lot more subtle.

Another interesting setting is the tree of $\text{PGL}_2(\mathbf{Q}_p)$ for $p = 2$ this is a 3-regular tree. How many points are there of distance less than R from a fixed point

$$1 + 3(1 + 2 + \cdots + 2^R) = 1 + 3(2^{R+1} - 1) \sim 3 \cdot 2^{R+1} = 6 \cdot 2^R.$$

But how many points of distance exactly R are there? Roughly 2^R again.

A hyperbolic disk of radius R centred at i would be a euclidean disk, but not centered at i. The area is $4\pi(\sinh(R/2))^2$ and the circumference is $2\pi\sinh(R)$ these are roughly the same size as $\sinh(x) = (e^x - e^{-x})/2$. The euclidean area is far large (roughly the square) of the hyperbolic.

1.7   H as a homogeneous space

$$\text{SL}_2(\mathbf{R}) \circlearrowright \mathbf{H}$$

via linear fractional transformations, i.e.

$$g \cdot z = \frac{az + b}{cz + d} \text{ for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbf{R})$$

this is the full group of holomorphic isometries of $\mathbf{H}$, to get all of them take $z \mapsto -\bar{z}$ as well.

$$\mathbf{H} = \text{SL}_2(\mathbf{H})/\text{SO}(2)$$

because $\text{SO}(2) = \text{Stab}_i(\text{SL}_2(\mathbf{R}))$.

1.7.1   Several decompositions

Cartesian: $x + iy$ then the invariant measure is $\frac{dx\,dy}{y^2}$.

Iwasawa: $G = NAK$

$$N = \left\{ \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} : n \in \mathbf{R} \right\}$$

$$A = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} : a \in \mathbf{R}_{\ge 0} \right\}$$

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

$$x + iy \leftrightarrow \underbrace{\begin{pmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & \frac{x}{\sqrt{y}} \\ & 1 \end{pmatrix}}_{N}$$

this is very general, an analogue of Gram-Schmidt.

Observation 1.17.

$$\mathbf{H} = \underbrace{NA}_{=AN} = \underbrace{P}_{\{\begin{pmatrix} * & * \\ & * \end{pmatrix}\}}$$

warning $NA \ne AN$ elementwise

Cartan: KAK (useful when dealing with rotationally invariant functions).

Exercise 1.18. Prove these decompositions. Use the spectral theorem of symmetric matrices for the Cartan case.

Classification of Motions   We classify by the number of fixed points in $\mathbf{H} \cup \hat{\mathbf{R}}$, for $\hat{\mathbf{R}}$ the extended real line.

- Identity, infinitely many fixed points.

- Parabolic, 1 fixed point in $\hat{\mathbf{R}}$ (∞) $\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}$

- Hyperbolic, 2 fixed points in $\mathbf{R}$ $(0, \infty)$, $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$.

- Elliptic, 1 fixed point in $\mathbf{H}$ but not in $\overline{\mathbf{H}}$, $(i, -i)$ $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Note 1.19 (for the future). These notions are different when we consider $\gamma \in G(\mathbf{Q})$ something can be $\mathbf{Q}$-elliptic but $\mathbf{R}$-hyperbolic. This depends on the Jordan decomposition essentially, we can have such γ with no rational roots of the characteristic polynomial but which splits over $\mathbf{R}$.

So we have

- Parabolic $|\operatorname{tr}| = 2$

- ($\mathbf{R}$-)Elliptic $|\operatorname{tr}| < 2$

- ($\mathbf{R}$-)Hyperbolic $|\operatorname{tr}| > 2$

1.8 $\Delta_{\mathbf{H}}$

Lecture 4 30/1/2018

For this section $\Delta_{\mathbf{H}} = \Delta$.

Definition 1.20.

$$g \in SL_2(\mathbf{R})$$

$$T_g f(z) = f(g \cdot z)$$

Definition 1.21. A linear operator L will be called **invariant** if it commutes with T_g for all $g \in SL_2(\mathbf{R})$, i.e.

$$L(T_g f) = T_g(Lf).$$

Remark 1.22. On any Riemannian manifold Δ can be characterised by: A diffeomorphism is an isometry iff it commutes with Δ.

Δ in coordinates:

Cartesian

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = -(z - \bar{z})^2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Exercise 1.23. Show that Δ is an invariant differential operator.

Polar:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{\tanh(r)}\frac{\partial}{\partial r} + \frac{1}{(2\sinh(r))^2}\frac{\partial^2}{\partial\phi^2}$$

We will be interested in $\Delta \circlearrowright C^\infty(\Gamma\backslash\mathbf{H})$.

Eigenfunctions of Δ   This is a little subtle, lets take the definition of an eigenfunction to be

$$f \in C^2(\mathbf{H}) \text{ s.t. } (\Delta + \lambda)f \equiv 0$$

Remark 1.24. Δ is an "elliptic" operator with real analytic coefficients. This implies any eigenfunction is real analytic.

Remark 1.25. $\lambda = 0$ means f is harmonic.

Some basic eigenfunctions: Lets try $f(z) = f_0(y)$ independent of x

$$\Delta f = y^2\frac{\partial^2}{\partial y^2}f$$

if f satisfies

$$(\Delta + \lambda)f = 0$$

this implies f is a linear combination of (y^s, y^{1-s}) where $s(1 - s) = \lambda$ if $\lambda \neq \frac{1}{4}$.

If $\lambda = \frac{1}{4}$ this gives $y^{1/2}$ and $\log(y)y^{1/2}$. Note the symmetry! $s \leftrightarrow 1 - s$.

Let's look at $f(z)$ depending periodically on x (with period f). Separation of variables: try

$$f(z) = e(x)F(2\pi y)$$

where the 2π is really in both factors. This gives

$$\frac{\partial^2}{\partial x^2}f = -4\pi^2 f$$

$$\frac{\partial^2}{\partial y^2}f = 4\pi^2 e(x)F''(2\pi y)$$

which gives

$$(\Delta + \lambda)f = 0 \iff y^2 4\pi^2 e(x)\left(-F(2\pi y) + F''(2\pi y) + \lambda F(2\pi y)\right) = 0$$

which implies

$$F''(2\pi y) + (\lambda - 1)F(2\pi y) = 0$$

this is a close relative of the Bessel differential equation.

$$F''(u) + \left(\frac{\lambda}{u^2} - 1\right)F(u) = 0.$$

This has two solutions

$$\left(\frac{2y}{\pi}\right)^{\frac{1}{2}} K_{s-\frac{1}{2}}(y) \sim e^{-2\pi y} \text{ as } y \to \infty$$

$$(2y\pi)^{\frac{1}{2}} I_{s-\frac{1}{2}}(y) \sim e^{2\pi y} \text{ as } y \to \infty$$

intuition: as $y \to \infty$ we have $F'' - F = 0$ so e^u or e^{-u}.

Remark 1.26. If we insist on some "moderate growth" (at most polynomial in y) on the eigenfunction. The $I_{s-\frac{1}{2}}$ solution can not contribute. (when we come to automorphic forms we will see that the definition is essentially eigenfunctions with moderate growth).

So our periodic (in x) eigenfunction with (moderate growth) looks like

$$f_s(z) = \underbrace{C2y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi y)e(x)}_{W_s(z)}$$

Definition 1.27 (Whittaker functions)**.** $W_s(z)$ is called a Whittaker function.

These exist for arbitrary lie groups, but we may not always be able to write eigenfunctions in terms of them in general though. They are a replacement for the exponential functions.

Theorem 1.28 (Spectral decomposition)**.**

$$f(z) = \frac{1}{2\pi i} \int_{\frac{1}{2}} \int_{\mathbf{R}} W_s(rz) f_s(r) \gamma_s(r) \, ds \, dr, \quad s = \frac{1}{2} + it$$

where

$$f_s(r) = \int_{\mathbf{H}} f(z) W_s(rz) \, dz$$

$$\gamma_s(r) = \frac{1}{2\pi|r|} t \sinh(\pi t)$$

analogue of the Fourier inversion formula for $\mathbf{H}$.

Theorem 1.29 (2)**.** *If f is actually periodic in x and $(\Delta + s(1-s))f = 0$ with growth $O(e^2\pi y)$*

$$f(z) = f_0(y) + \sum_{n=1}^{\infty} f_n W_s(nz)$$

where $f_0(y)$ is a combination of y^s, y^{1-s}.

Note 1.30. We will be considering automorphic forms

$$\left\langle \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \right\rangle \subseteq \Gamma \subseteq \mathrm{SL}_2(\mathbf{Z}).$$

1.9 Integral operators

Recall the Cauchy integral formula for holomorphic functions

$$f(z) = \frac{1}{2\pi i} \int_{B_z} \frac{f(w)}{w-z} \, dw = \int_{B_z} K(w, z) f(w) \, dw$$

i.e. using an integral kernel, f is an eigenfunction for this operator.

Recall: L is an integral operator if

$$Lf(z) = \int_{\mathbf{H}} K(z, w) f(w) \, \mathrm{d} \underbrace{\mu(w)}_{\frac{\mathrm{d}u \, \mathrm{d}v}{v^2}}, \ w = u + iv$$

K will often be smooth of compact support for us. L is invariant if it commutes with T_g for all g.

Observation 1.31. L is invariant iff

$$K(gz, gw) = K(z, w) \, \forall g \in \mathrm{SL}_2(\mathbf{R}).$$

Exercise 1.32. Show this.

Definition 1.33 (Point pair invariants)**.** A function $K \colon \mathbf{H} \times \mathbf{H} \to \mathbf{C}$ that satisfies $K(gz, gw) = K(z, w)$ is called a **point pair invariant**. This was first introduced by Selberg.

Remark 1.34. A point pair invariant $K(z, w)$ depends only of the distance between z, w i.e.

$$K(z, w) = K_0(\rho(z, w)) \text{ for } K_0 \colon \mathbf{R}^+ \to \mathbf{C}$$

so an invariant operator is just a convolution operator.