Automorphic forms and the Arthur-Selberg trace formula

MA842 BU Spring 2018

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1 Automorphic forms $/\mathrm{GL}_2$ (possibly GL_n)

Lecture 1 18/1/2018

These are notes for Ali Altuğ's course MA842 at BU Spring 2018, they were last updated April 6, 2018.

The course webpage is http://math.bu.edu/people/saaltug/2018_1/2018_1_sem.html.

Course overview: This course will be focused on the two papers Eisenstein Series and the Selberg Trace Formula I by D. Zagier and Eisenstein series and the Selberg Trace Formula II by H. Jacquet and D. Zagier. Although the titles of the papers sound like one is a prerequisite of the other it actually is not the case, the main difference is the language of the papers (the first is written in classical language whereas the second is written in adelically). We will spend most of our time with the second paper, which is adelic.

1.1 Goal

Jacquet and Zagier, Eisenstein series and the Selberg Trace Formula II (1980's).

Part I is a paper of Zagier from 1970 in purely classical language. Part II is in adelic language (and somewhat incomplete).

$$\begin{pmatrix}\text{Arthur-Selberg}\\ \text{trace formula}\end{pmatrix} \xleftrightarrow{\text{conjecture}} \begin{pmatrix}\text{Relative}\\ \text{trace formula}\end{pmatrix}$$

the Arthur-Selberg side is used in Langlands functoriality and the Relative is used in arithmetic applications.

1.2 Motivation

What does this paper do?

> "It rederives the Selberg trace formula for GL_2 by a regularised process."

Note 1.1.

- Selberg trace formula only for GL_2

- Arthur-Selberg more general

The Selberg trace formula generalises the more classical Poisson summation formula.

Poisson summation

Theorem 1.2 (Poisson summation)**.** *Let*

$$f : \mathbf{R} \to \mathbf{R}$$

then ***Poisson summation*** *says*

$$\sum_{n\in\mathbf{Z}} f(n) = \sum_{\xi\in\mathbf{Z}} \hat{f}(\xi)$$

where

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e(x\xi)\,\mathrm{d}x.$$

Notation: $e(x) = e^{2\pi i x}$.

To make this look more general we make the following notational choices.

$$G = \mathbf{R}, \Gamma = \mathbf{Z}$$

$$\sum_{\gamma\in\Gamma^{\#}} f(\gamma) = \sum_{\xi\in(G/\Gamma)^{\vee}} \hat{f}(\xi)$$

where

- $\Gamma^{\#}$ = conjugacy classes of Γ (= Γ in this case since Γ is abelian).
- $(G/\Gamma)^{\vee}$ =dual of G/Γ.

Selberg

$$G = \mathrm{GL}_2(\mathbf{R}),\ \Gamma = \mathrm{GL}_2(\mathbf{Z})$$

$$\sum_{\gamma\in\Gamma^{\#}} \cdots \text{“} = \text{”} \sum_{\pi\in\text{“}(G/\Gamma)^{\vee}\text{”}} \cdots$$

relating conjugacy classes on the left to automorphic forms on the right.

Arthur and Selberg prove the trace formula by a *sharp cut off*, Jacquet and Zagier derive this using a regularisation.

1.3 Motivating example

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges absolutely for $s > 1$.

Theorem 1.3 (Riemann)**.** *$\zeta(s)$ has analytic continuation up to $\Re(s) > 0$ with a simple pole at $s = 1$ residue 1. i.e.*

$$\zeta(s) = \frac{1}{s-1} + \phi(s)$$

where $\phi(s)$ is holomorphic for $\Re(s) > 0$.

Proof. Step 1: observe

$$\frac{1}{s-1} = \int_1^\infty t^{-s}\,\mathrm{d}t \text{ (for } \Re(s) > 1)$$
$$= \sum_{n=1}^\infty \int_n^{n+1} t^{-s}\,\mathrm{d}t$$

Step 2: this implies

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^\infty n^{-s} - \int_n^{n+1} t^{-s}\,\mathrm{d}t$$
$$= \frac{1}{s-1} + \sum_{n=1}^\infty \left(\int_n^{n+1} n^{-s} - \int_n^{n+1} t^{-s}\,\mathrm{d}t\right)$$

we denote each of the terms in the right hand sum as $\phi_n(s)$

$$\phi_n(s) = \int_n^{n+1} n^{-s} - t^{-s}\,\mathrm{d}t$$

Step 3:

$$|\phi_n(s)| \le \sup_{n \le t \le n+1} |n^{-s} - t^{-s}|$$
$$\sup_{n \le t \le n+1} \frac{|s|}{t^{\Re(s)+1}} \le \frac{|s|}{n^{\Re(s)+1}}$$

by applying the mean value theorem.

So $\sum_{n=1}^\infty \phi_n$ converges absolutely. Hence $\phi = \sum_{n=1}^\infty \phi_n$ is holomorphic

One can push this idea to get analytic continuation to all of $\mathbf{C}$, one strip at a time. This is an analogue of the sharp cut off method mentioned above. It's fairly elementary but somewhat unmotivated and doesn't give any deep information (like the functional equation). □

Proof. Introduce
$$\theta(t) = \sum_{n \in \mathbf{Z}} e^{-\pi n^2 t},\ t > 0$$
note that $\theta(t) = 1 + 2\sum_{n=1}^\infty e^{-\pi n^2 t}$.

Idea: Mellin transform and properties of θ to derive properties of ζ.

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{s/2}} \frac{1}{ns} = \int_0^\infty e^{-\pi n^2 t} t^{s/2} \frac{\mathrm{d}t}{t}$$

property of θ:
$$\theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right)$$

Step 1: proof of this property is the Poisson summation formula

- $$f(x) = e^{-\pi x^2} \implies \hat{f}(\xi) = f(\xi)$$

- $$g(x) = f(\sqrt{t}x) \implies \hat{g}(\xi) = \frac{1}{\sqrt{t}} \hat{f}\left(\frac{\xi}{\sqrt{t}}\right)$$

Step 2: Would like to write something like

$$\text{``} \int_0^\infty \theta(t) t^{s/2} \frac{\mathrm{d}t}{t} \text{''}$$

This integral makes no sense

- As $t \to \infty$, $\theta \sim 1$ thus

$$\begin{aligned} & \left| \int_A^\infty \theta(t) t^{s/2} \frac{\mathrm{d}t}{t} \right| < \infty \\ \iff & \left| \int_A^\infty t^{s/2} \frac{\mathrm{d}t}{t} \right| < \infty \\ \iff & \Re(s) < 0 \end{aligned}$$

- As $t \to 0$ consider $\xi = \frac{1}{t}$ so $\xi \to \infty$ and

$$\begin{aligned} \theta(t) &= \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right) = \sqrt{\xi} \theta \xi] \\ \implies \theta(t) &= \sqrt{\xi}\theta(\xi) \sim \sqrt{\xi} = \frac{1}{\sqrt{t}} \end{aligned}$$

so $\theta(t) \sim \frac{1}{\sqrt{t}}$

$$\begin{aligned} \implies & \left| \int_0^A \theta(t) t^{s/2} \frac{\mathrm{d}t}{t} \right| < \infty \\ \iff & \left| \int_0^A t^{(s-1)/2} \frac{\mathrm{d}t}{t} \right| < \infty \\ \iff & \Re(s) > 1 \end{aligned}$$

so no values of s will make sense for this improper integral.

Refined idea: Consider

$$I(s) = \int_0^1 (\theta(t) - \frac{1}{\sqrt{t}}) t^{s/2} \frac{\mathrm{d}t}{t} + \int_1^\infty (\theta(t) - 1) t^{s/2} \frac{\mathrm{d}t}{t}$$

upshot: $I(s)$ is well-defined and holomorphic for all $s \in \mathbf{C}$.

Final step: Compute the above to see

$$I(s) = \frac{2}{s} + \frac{2}{1-s} + \frac{2}{\pi^{s/2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

which implies

1. $\zeta(s)$ has analytic continuation to $s \in \mathbf{C}$, with only a simple pole at $s = 1$ with residue 1.

2. $$I(s) = I(1-s),$$

   this follows from the property of θ so if we let

   $$\Lambda(s) = \frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{s/2}} \zeta(s),$$

   then

   $$\Lambda(s) = \Lambda(1-s).$$ □

1.4 Modular forms

Functions on the upper half plane,

$$\mathbf{H} = \{z \in \mathbf{C} : \Im(z) > 0\}.$$

Historically elliptic integrals lead to elliptic functions, and modular forms and elliptic curves.

Note 1.4. When one is interested in functions on O/Λ where O is some object and Λ is some discrete group. Take f a function on O and average over Λ to get

$$\sum_{\lambda\in\Lambda} f(\lambda z).$$

If you're lucky this converges, this is good.

Elliptic functions Weierstrass, take $\Lambda = \omega_1\mathbf{Z} + \omega_2\mathbf{Z}$ a lattice an define

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\smallsetminus\{0\}} \left(\frac{1}{(z-\omega)^2} + \frac{1}{\omega^2}\right).$$

Jacobi, (Elliptic integrals) consider

$$\int_0^\phi \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-\kappa t^2)}},\ \kappa \geq 0$$

related by:

$$(\wp'_\Lambda(z))^2 = 4\wp_\Lambda(z)^3 - 60G_2(\Lambda)\wp_\Lambda(z) - 140G_3(\Lambda)$$

$$G_k(\Lambda) = \sum_{\lambda\in\Lambda\smallsetminus\{0\}} \lambda^{-2k}$$

or

$$G_k(\tau) = \sum_{(m,n)\in\mathbf{Z}^2\smallsetminus\{0\}} \frac{1}{(m\tau+n)^{2k}}$$

the weight $2k$ holomorphic Eisenstein series.

Fact 1.5. *Let*

$$u = \int_y^\infty \frac{\mathrm{d}s}{\sqrt{4s^3 - 60G_2 s - 140G_3}}$$

then

$$y = \wp_\Lambda(u).$$

1.5 Euclidean Harmonic analysis

Lecture 2 23/1/2018

We'll take a roundabout route to automorphic forms.

Today: Classical harmonic analysis on $\mathbf{R}^n$. Classical harmonic analysis on $\mathbf{H}$.

The aim (in general) is to express a certain class of functions (i.e. $\mathcal{L}^2$) in terms of building block (harmonics).

In classical analysis the harmonics are known ($e(nx)$), then the question becomes how these things fit together. In number theory the harmonics are extremely mysterious. We are looking at far more complicated geometries, quotient spaces etc. and arithmetic information comes in.

Example 1.6. $\mathbf{R}$, $f : \mathbf{R} \to \mathbf{C}$, being periodic in $\mathcal{L}^2(S^1)$ leads to a fourier expansion

$$f(x) = \sum_{n \in \mathbf{Z}} a_n e(nx).$$

1.5.1 $\mathbf{R}^2$

We have a slightly different perspective.

$$\mathbf{R}^2 = G \circlearrowright G$$

via translations (i.e. right regular representation of G will be $G \circlearrowright \mathcal{L}^2(G)$). I.e. $g \cdot x = x + g$.

Remark 1.7. This makes $\mathbf{R}^2$ a homogeneous space.
$\mathbf{R}^2$ with standard metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ is a flat space $\kappa = 0$.

To the metric we have the associated Laplacian (Laplace-Beltrami operator, $\nabla \cdot \nabla$)

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

we are interested in this as it is essentially the only operator, we will define automorphic forms to be eigenfunctions for this operator.

Note 1.8. The exponential functions

$$\phi_{u,v}(x, y) = e(ux + vy)$$

are eigenfunctions of Δ with eigenvalue $\lambda_{u,v} = -4\pi^2(u^2 + v^2)$ i.e.

$$(\Delta + \lambda_{u,v})\phi_{u,v} = 0.$$

These are a complete set of harmonics for $\mathcal{L}^2(\mathbf{R}^2)$. The proof is via fourier inversion.

$$f(x, y) = \int\int_{\mathbf{R}^2} \hat{f}(u, v)\phi_{u,v}(x, y)\,\mathrm{d}u\,\mathrm{d}v$$

where

$$\hat{f}(u, v) = \int\int_{\mathbf{R}^2} f(u, v)\bar{\phi}_{u,v}(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$

A little twist We could have established the spectral resolution (of Δ) by considering invariant integral operators.

Using the spectral theorem if we can find easier to diagonalise operators that commute we can find the eigenspaces for those to cut down the eigenspace.

Recall: an integral operator is

$$L(f)(x) = \int K(x, y)f(y)\,\mathrm{d}y$$

invariant means

$$L(gf) = gL(f)\ g \in G$$

in our case

$$g \cdot f(x) = f(g + x).$$

Observation 1.9. If L is invariant then the kernel $K(x, y)$ is given by

$$K(x, y) = K_0(x - y)$$

for some function K_0.

Proof. ($\Leftarrow$) obvious

($\Rightarrow$) Suppose L is invariant then

$$\int_{\mathbf{R}^2} K(x,y) f(y+\alpha)\,\mathrm{d}y = \int_{\mathbf{R}^2} K(x+\alpha,y) f(y)\,\mathrm{d}y \;\forall f$$

implies

$$\int\int_{\mathbf{R}^2} K(x,y-\alpha) f(y)\,\mathrm{d}y = \int_{\mathbf{R}^2} K(x+\alpha,y) f(y)\,\mathrm{d}y \;\forall f$$

so

$$\int_{\mathbf{R}^2} (K(x+\alpha,y) - K(x,y-\alpha)) f(y)\,\mathrm{d}y = 0 \;\forall f$$

which implies with some proof that

$$K(x+\alpha,y) = K(x,y-\alpha)$$

so

$$K(x,y) = K(x-y,0). \qquad \square$$

Observation 1.10. Invariant integral operators commute with each other

$$L_1L_2(f)(z) = L_2L_1(f)(z)$$

Proof.

$$L_1L_2(f)(z) = \int_{\mathbf{R}^2\times\mathbf{R}^2} f(w)K_2(u-w)K_1(z-u)\,\mathrm{d}w\,\mathrm{d}u = L_2L_1(f)(z)$$

after change of variables

$$u \mapsto z-u+w \qquad \square$$

Observation 1.11. L commutes with Δ.

Proof. Based on the following:

$$K(x,y) = K_0(x-y,0)$$

$$\frac{\partial K}{\partial x_i} = -\frac{\partial K}{\partial y_i}$$

which implies

$$\begin{aligned}\Delta_z(L(f))(z) &= \Delta_z \int_{\mathbf{R}^2} f(w)K(z,w)\,\mathrm{d}w \\ &= \int\int_{\mathbf{R}^2} \Delta_z f(w)K(z,w)\,\mathrm{d}w \\ &= \int\int_{\mathbf{R}^2} f(w)\Delta_w K(z,w)\,\mathrm{d}w\end{aligned}$$

which via integration by parts is

$$= \int\int_{\mathbf{R}^2} \Delta_w f(w)K(z,w)\,\mathrm{d}w = L(\Delta f)(z). \qquad \square$$

Observation 1.12. $\phi_{u,v}(x,y)$ is an eigenfunction of L, $(u,v) \in \mathbf{R}^2, (x,y) \in \mathbf{R}^2$.

Proof.

$$
\begin{aligned}
L\phi_{u,v}(\overbrace{\ z\ }^{=x,y}) &= \int_{\mathbf{R}^2} \phi_{u,v}(w)K(z,w)\,\mathrm{d}w \\
&= \int_{\mathbf{R}^2} \phi_{u,v}(w)K_0(z-w)\,\mathrm{d}w \\
&= \int_{\mathbf{R}}\int_{\mathbf{R}} e(uw_1+vw_2)K_0(z_1-w_1,z_2-w_2)\,\mathrm{d}w_1\,\mathrm{d}w_2 \\
&= e(uw_1+vw_2)\int_{\mathbf{R}}\int_{\mathbf{R}} K_0(w_1,w_2)e(-uw_1-vw_2)\,\mathrm{d}w_1\,\mathrm{d}w_2
\end{aligned}
$$

after the change of variable $w_i \mapsto -w_i + z_i$

$$\phi_{u,v}(z)\hat{K}_0(u,v)$$

i.e.

$$L\phi_{u,v} = \hat{K}_0(u,v)\phi_{u,v}.$$

□

Side remark: these are enough to form a generating set.

1.5.2 Poisson summation (yet again)

Let's consider integral operators on functions on $\mathbf{Z}^2\backslash\mathbf{R}^2 = \mathbf{T}^2$.
Observe: $L \rightsquigarrow K(x,y) = K_0(x-y)$.

$$
\begin{aligned}
Lf(z) &= \int_{\mathbf{R}^2} f(w)K(z,w)\,\mathrm{d}w \\
&= \int\int_{\mathbf{Z}^2\backslash\mathbf{R}^2} f(w)\underbrace{\left(\sum_{n\in\mathbf{Z}^2} K(z,w+n)\right)}_{=\sum_{n\in\mathbf{Z}} K_0(z-w+n)=\mathbf{K}(z,w)}\,\mathrm{d}w
\end{aligned}
$$

now $\mathbf{K}$ is a function on $\mathbf{T}^2\times\mathbf{T}^2$.
Trace of this operator

$$= \int_{\mathbf{T}^2}\mathbf{K}(z,z)\,\mathrm{d}z = \int_{\mathbf{T}^2}\left(\sum_{n\in\mathbf{Z}^2}K_0(n)\right)\mathrm{d}z = \sum_{n\in\mathbf{Z}^2}K_0(n)$$

Using sum of eigenvalues

$$K(z,w) = \sum_{n\in\mathbf{Z}^2}K_0(z-w+n) = \sum_{\xi\in\mathbf{Z}^2}\lambda_\xi\phi_\xi(z-w) = \sum_{\xi\in\mathbf{Z}^2}\lambda_\xi\phi_\xi(z)\bar{\phi}_\xi(w)$$

so the trace is

$$\sum_{\xi\in\mathbf{Z}^2}\lambda_\xi = \sum_{\xi\in\mathbf{Z}^2}\hat{K}_0(\xi)$$

so we get to

$$\sum_{n\in\mathbf{Z}^2}K_0(n) = \sum_{\xi\in\mathbf{Z}^2}\hat{K}_0(\xi)$$

i.e. Poisson summation.
Why care about Poisson summation?

$$\hat{K}_0(0) = \int K_0(z)\,\mathrm{d}z$$

Gauss circle problem, how many lattice points are there in a circle of radius R. We can pick a radially symmetric function that is 1 on the circle and 0 outside, or a smooth approximation of such an indicator function at least. Poisson summation packages the important information into a single term, plus some rapidly decaying ones. Then we get πR^2+ error, Gauss conjectured that the error is $R^{1/2+\epsilon}$.

Lecture 3 25/1/2018

Last time we gave a conceptual proof of Poisson summation (this strategy will generalise to the trace formula eventually).

To clean up one loose end: there is a generalisation of Poisson summation called Voronoi summation, which will actually be useful later. For Poisson summation we had

$$\sum_{n_1,n_2\in\mathbf{Z}} K(n_1,n_2) = \sum_{\xi_1,\xi_2\in\mathbf{Z}} \hat{K}(\xi_1,\xi_2)$$

suppose $K(x,y)\colon \mathbf{R}^2 \to \mathbf{C}$ is radially symmetric i.e.

$$K(x,y) = K_0(x^2+y^2),\ (x,y)\in\mathbf{R}^2$$

then the fourier transform

$$\hat{K}(u,v) = \pi\int_0^\infty K_0(r)J_0(\sqrt{\lambda r})\,\mathrm{d}r,\ \lambda = 4\pi^2(u^2+v^2)$$

where

$$J_0(z) = \frac{1}{\pi}\int_0^\pi \cos(z\cos(\alpha))\,\mathrm{d}\alpha$$

is a Bessel function of the second kind.

Exercise 1.13. Prove this.

Plug this into Poisson summation

$$\sum_{(n_1,n_2)\in\mathbf{Z}^2} K(n_1,n_2) = \sum_{N=0}^\infty r_2(N)K_0(N)$$

as K only depends on $n_1^2+n_2^2$ we group terms based on this quantity, so

$$r_2 = \#\{(n_1,n_2)\in\mathbf{Z}^2 : n_1^2+n_2^2 = N\}.$$

$$\begin{aligned}\sum_{\xi_1,\xi_2\in\mathbf{Z}} \pi\int_0^\infty K_0(r)J_0\Big(2\pi\sqrt{(\xi_1^2+\xi_2^2)r}\Big)\,\mathrm{d}r\\ = \sum_{M=0}^\infty r_2(M)\tilde{K}_0(M).\end{aligned}$$

Theorem 1.14 (Voronoi summation)**.**

$$\sum_{N=0}^\infty r_2(N)K_0(N) = \sum_{M=0}^\infty r_2(M)\tilde{K}_0(M)$$

where

$$\tilde{K}_0(z) = \pi\int_0^\infty K_0(r)J_0(2\pi\sqrt{zr})\,\mathrm{d}r.$$

Note that $J_0(0) = 1$.

How is this useful? Consider point counting in a circle problem. Let $K_0(x)$ be an approximation to the step function, 1 for $x \leq 1$ and 0 for $x > 1$. With $\int K_0 = 1$. Then

$$\sum_{N=0}^{\infty} K_0\left(\frac{N}{R^2}\right) r_2(N).$$

This is counting lattice points. The right hand side is then

$$\sum_{M=0}^{\infty} r_2(M)\tilde{K}_0(M) = \tilde{K}_0(0) + \sum_{M=1}^{\infty} r_2(M)\tilde{K}_0(M)$$

$$= \pi + \sum_{M=1}^{\infty} r_2(M)\tilde{K}_0(M).$$

Finally

$$f(z) = K_0\left(\frac{z}{R^2}\right)$$

so

$$\tilde{f}(\xi) = R^2\tilde{K}_0(\xi R^2).$$

So

$$\sum_{N=0}^{\infty} K_0\left(\frac{N}{R^2}\right) r_2(N) = R^2\left(\pi + \sum_{M=1}^{\infty} r_2(M)\tilde{K}_0(MR^2)\right)$$

where the lead term is the area of the circle. Finally if $M \neq 0$ then $f(MR^2)$ doesn't increase fast as $R \to \infty$. i.e. it is smaller than R^2. So as $R \to \infty$ we find #{lattice points in the circle} $\sim \pi R^2$.

1.6 The hyperbolic plane H

What if we consider the same problem on the hyperbolic disk? Things are extremely different.

Generalities

Definition 1.15.

$$\mathbf{H} = \{x + iy : y > 0\}.$$

$$\mathrm{d}s^2 = \frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2), \text{ Riemannian metric}$$

this gives

$$\kappa = -1$$

i.e. this is negatively curved, this is the cause of huge differences between the euclidean theory.

There is a formula for the (hyperbolic) distance between two points

$$\rho(z, w) = \log \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|}.$$

Observation 1.16. As $w \to \mathbf{R}$ we have $\rho \to \infty$. So $\mathbf{R}$ is the boundary.

Recall: the isoperimetric inequality

$$4\pi A - \kappa A^2 \leq L^2$$

where L is the length of the boundary of a region and A is the area. Note if $\kappa = 0$ then $4\pi A \leq L^2$. So A can be and would be as large as L^2.

For $\kappa = -1$ we have

$$4\pi A + A^2 \leq L^2$$

so A can at most (and most often will) be as large as $\sim L$. The upshot is that under the hyperbolic metric, the area and perimeter can be the same size.

$$|\text{Boundary}| \sim |\text{Area}|.$$

Things are a lot more subtle.

Another interesting setting is the tree of $\mathrm{PGL}_2(\mathbf{Q}_p)$ for $p = 2$ this is a 3-regular tree. How many points are there of distance less than R from a fixed point

$$1 + 3(1 + 2 + \cdots + 2^R) = 1 + 3(2^{R+1} - 1) \sim 3 \cdot 2^{R+1} = 6 \cdot 2^R.$$

But how many points of distance exactly R are there? Roughly 2^R again.

A hyperbolic disk of radius R centred at i would be a euclidean disk, but not centered at i. The area is $4\pi(\sinh(R/2))^2$ and the circumference is $2\pi\sinh(R)$ these are roughly the same size as $\sinh(x) = (e^x - e^{-x})/2$. The euclidean area is far large (roughly the square) of the hyperbolic.

1.7 H as a homogeneous space

$$\mathrm{SL}_2(\mathbf{R}) \circlearrowright \mathbf{H}$$

via linear fractional transformations, i.e.

$$g \cdot z = \frac{az + b}{cz + d} \text{ for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$$

this is the full group of holomorphic isometries of $\mathbf{H}$, to get all of them take $z \mapsto -\bar{z}$ as well.

$$\mathbf{H} = \mathrm{SL}_2(\mathbf{H})/\mathrm{SO}(2)$$

because $\mathrm{SO}(2) = \mathrm{Stab}_i(\mathrm{SL}_2(\mathbf{R}))$.

1.7.1 Several decompositions

Cartesian: $x + iy$ then the invariant measure is $\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$.

Iwasawa: $G = NAK$

$$N = \left\{ \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} : n \in \mathbf{R} \right\}$$

$$A = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} : a \in \mathbf{R}_{\geq 0} \right\}$$

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

$$x + iy \leftrightarrow \underbrace{\begin{pmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & \frac{x}{\sqrt{y}} \\ & 1 \end{pmatrix}}_{N}$$

this is very general, an analogue of Gram-Schmidt.

Observation 1.17.

$$\mathbf{H} = \underbrace{NA}_{=AN} = \underbrace{P}_{\left\{\begin{pmatrix} * & * \\ & * \end{pmatrix}\right\}}$$

warning $NA \neq AN$ elementwise

Cartan: KAK (useful when dealing with rotationally invariant functions).

Exercise 1.18. Prove these decompositions. Use the spectral theorem of symmetric matrices for the Cartan case.

Classification of Motions We classify by the number of fixed points in $\mathbf{H} \cup \hat{\mathbf{R}}$, for $\hat{\mathbf{R}}$ the extended real line.

- Identity, infinitely many fixed points.
- Parabolic, 1 fixed point in $\hat{\mathbf{R}}$ (∞) $\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}$
- Hyperbolic, 2 fixed points in $\mathbf{R}$ $(0, \infty)$, $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$.
- Elliptic, 1 fixed point in $\mathbf{H}$ but not in $\overline{\mathbf{H}}$, $(i, -i)$ $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Note 1.19 (for the future)**.** These notions are different when we consider $\gamma \in G(\mathbf{Q})$ something can be $\mathbf{Q}$-elliptic but $\mathbf{R}$-hyperbolic. This depends on the Jordan decomposition essentially, we can have such γ with no rational roots of the characteristic polynomial but which splits over $\mathbf{R}$.

So we have

- Parabolic $|\operatorname{tr}| = 2$
- ($\mathbf{R}$-)Elliptic $|\operatorname{tr}| < 2$
- ($\mathbf{R}$-)Hyperbolic $|\operatorname{tr}| > 2$

1.8 $\Delta_{\mathbf{H}}$

Lecture 4 30/1/2018

For this section $\Delta_{\mathbf{H}} = \Delta$.

Definition 1.20. We have the translation operators

$$g \in \mathrm{SL}_2(\mathbf{R})$$

$$T_g f(z) = f(g \cdot z).$$

Definition 1.21 (Invariant operators)**.** A linear operator L will be called **invariant** if it commutes with T_g for all $g \in \mathrm{SL}_2(\mathbf{R})$, i.e.

$$L(T_g f) = T_g(Lf).$$

Remark 1.22. On any Riemannian manifold Δ can be characterised by: A diffeomorphism is an isometry iff it commutes with Δ.

Δ in coordinates:
Cartesian

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = -(z - \bar{z})^2 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

Exercise 1.23. Show that Δ is an invariant differential operator.

Polar:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{\tanh(r)} \frac{\partial}{\partial r} + \frac{1}{(2\sinh(r))^2} \frac{\partial^2}{\partial \phi^2}$$

We will be interested in $\Delta \circlearrowleft C^\infty(\Gamma \backslash \mathbf{H})$.

Eigenfunctions of Δ This is a little subtle, lets take the definition of an eigenfunction to be

$$f \in C^2(\mathbf{H}) \text{ s.t. } (\Delta + \lambda) f \equiv 0.$$

Remark 1.24. Δ is an "elliptic" operator with real analytic coefficients. This implies any eigenfunction is real analytic.

Remark 1.25. $\lambda = 0$ means f is harmonic.

Some basic eigenfunctions: Lets try $f(z) = f_0(y)$ independent of x

$$\Delta f = y^2 \frac{\partial^2}{\partial y^2} f$$

if f satisfies

$$(\Delta + \lambda) f = 0$$

this implies f is a linear combination of (y^s, y^{1-s}) where $s(1-s) = \lambda$ if $\lambda \neq \frac{1}{4}$.

If $\lambda = \frac{1}{4}$ this gives $y^{1/2}$ and $\log(y) y^{1/2}$. Note the symmetry! $s \leftrightarrow 1 - s$.

Let's look at $f(z)$ depending periodically on x (with period f). Separation of variables: try

$$f(z) = e(x) F(2\pi y)$$

where the 2π is really in both factors. This gives

$$\frac{\partial^2}{\partial x^2} f = -4\pi^2 f$$

$$\frac{\partial^2}{\partial y^2} f = 4\pi^2 e(x) F''(2\pi y)$$

which gives

$$(\Delta + \lambda) f = 0 \iff y^2 4\pi^2 e(x) \left(-F(2\pi y) + F''(2\pi y) + \lambda F(2\pi y) \right) = 0$$

which implies

$$F''(2\pi y) + (\lambda - 1) F(2\pi y) = 0$$

this is a close relative of the Bessel differential equation.

$$F''(u) + \left(\frac{\lambda}{u^2} - 1 \right) F(u) = 0.$$

This has two solutions

$$\left(\frac{2y}{\pi}\right)^{\frac{1}{2}} K_{s-\frac{1}{2}}(y) \sim e^{-2\pi y} \text{ as } y \to \infty$$

$$(2y\pi)^{\frac{1}{2}} I_{s-\frac{1}{2}}(y) \sim e^{2\pi y} \text{ as } y \to \infty$$

intuition: as $y \to \infty$ we have $F'' - F = 0$ so e^u or e^{-u}.

Remark 1.26. If we insist on some "moderate growth" (at most polynomial in y) on the eigenfunction. The $I_{s-\frac{1}{2}}$ solution can not contribute. (when we come to automorphic forms we will see that the definition is essentially eigenfunctions with moderate growth).

So our periodic (in x) eigenfunction with (moderate growth) looks like

$$f_s(z) = \underbrace{C2y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi y)e(x)}_{=W_s(z)}.$$

Definition 1.27 (Whittaker functions)**.** $W_s(z)$ is called a **Whittaker function**.

These exist for arbitrary lie groups, but we may not always be able to write eigenfunctions in terms of them in general though. They are a replacement for the exponential functions.

Theorem 1.28 (Spectral decomposition)**.**

$$f(z) = \frac{1}{2\pi i} \int_{\frac{1}{2}} \int_{\mathbf{R}} W_s(rz) f_s(r) \gamma_s(r) \, \mathrm{d}s \, \mathrm{d}r, \; s = \frac{1}{2} + it$$

where

$$f_s(r) = \int_{\mathbf{H}} f(z) W_s(rz) \, \mathrm{d}z$$

$$\gamma_s(r) = \frac{1}{2\pi|r|} t \sinh(\pi t)$$

analogue of the Fourier inversion formula for **H***.*

Theorem 1.29 (2)**.** *If f is actually periodic in x and $(\Delta + s(1-s))f = 0$ with growth $O(e^2\pi y)$*

$$f(z) = f_0(y) + \sum_{n=1}^{\infty} f_n W_s(nz)$$

where $f_0(y)$ is a combination of y^s, y^{1-s}.

Note 1.30. We will be considering automorphic forms

$$\left\langle \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \right\rangle \subseteq \Gamma \subseteq \mathrm{SL}_2(\mathbf{Z}).$$

1.9 Integral operators

Recall the Cauchy integral formula for holomorphic functions

$$f(z) = \frac{1}{2\pi i}\int_{B_z} \frac{f(w)}{w-z}\,\mathrm{d}w = \int_{B_z} K(w,z)f(w)\,\mathrm{d}w$$

i.e. using an integral kernel, f is an eigenfunction for this operator.

Recall: L is an integral operator if

$$Lf(z) = \int_{\mathbf{H}} K(z,w)f(w)\,\mathrm{d}\underbrace{\mu(w)}_{\frac{\mathrm{d}u\,\mathrm{d}v}{v^2}},\ w = u+iv$$

K will often be smooth of compact support for us. L is invariant if it commutes with T_g for all g.

Observation 1.31. L is invariant iff

$$K(gz,gw) = K(z,w)\,\forall g \in \mathrm{SL}_2(\mathbf{R}).$$

Exercise 1.32. Show this.

Definition 1.33 (Point pair invariants)**.** A function $K\colon \mathbf{H}\times\mathbf{H} \to \mathbf{C}$ that satisfies $K(gz,gw) = K(z,w)$ is called a **point pair invariant**. This was first introduced by Selberg.

Invariant integral operators are convolution operators.

Remark 1.34. A point pair invariant $K(z,w)$ depends only of the distance between z,w i.e.

$$K(z,w) = K_0(\rho(z,w)) \text{ for } K_0\colon \mathbf{R}^+ \to \mathbf{C}$$

so an invariant operator is just a convolution operator.

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Theorem 1.35. *If $(\Delta+\lambda)f \equiv 0$ and L is an invariant integral operator. ($\Rightarrow$) Then there exists*

$$\Lambda(\lambda,K)$$

such that

$$L(f)(z) = \Lambda(\lambda,K)f(z).$$

($\Leftarrow$) Moreover if f is an eigenfunction of all invariant operators then f is an eigenfunction of Δ.

Proof. ($\Leftarrow$) Let L_K be

$$L_K(f)(z) = \int_{\mathbf{H}} f(w)K(z,w)\,\mathrm{d}\mu(z)$$

then

$$L_K(f)(z) = \Lambda_K f(z)$$

(if $\Lambda_K = 0$ for all K then $f \equiv 0$). So that

$$\Delta f(z) = \Delta\frac{1}{\Lambda_K}L_K f(z)$$

$$\frac{1}{\Lambda_K}\int_{\mathbf{H}} f(w)\Delta_z K(w,z)\,\mathrm{d}\mu(w)$$

Note $f \longrightarrow \int_{\mathbf{H}} f(z)\Delta_z K(w,z)\,\mathrm{d}\mu(w)$ is another invariant integral operator (exercise, show this).

We will prove an integral representation that looks like the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{B_z} \frac{f(w)}{w-z}\,\mathrm{d}w.$$

Let for $w \in \mathbf{H}$,

$$\Phi_w(f)(z) = \int_{G_w} f(gz)\,\mathrm{d}\mu(g)$$

where G_w is the stabiliser of w in $\mathrm{SL}_2(\mathbf{R})$ and $\mathrm{d}\mu$ is normalized so that G_w has volume 1.

Facts:

1. If f is an eigenfunction of Δ with eigenvalue $(\Delta+\lambda)f \equiv 0$, $\lambda = s(1-s)$ then there exists a unique function $W(z,w)$ s.t.

$$\Phi_w(f)(z) = W(z,w)f(w)$$

$$W(w,w) = 1$$

$$(\Delta_z + \lambda)W(z,w) \equiv 0$$

$$W \text{ is point pair invariant.}$$

2.
$$L(\Phi_z(f)(z) = L(f)(z)$$

   as

$$L(\Phi_z(f))(z) = \int_{\mathbf{H}} \Phi_z(f)(w)K(w,z)\,\mathrm{d}\mu(w)$$

$$\int_{\mathbf{H}}\int_{B_z} f(gw)\,\mathrm{d}\mu(g)K(w,z)\,\mathrm{d}\mu(w)$$

$$\int_{B_z}\int_{\mathbf{H}} f(w)\underbrace{K(g^{-1}w,z)}_{=K(w,gz)=K(w,z)}\,\mathrm{d}\mu(w)\,\mathrm{d}\mu(g)$$

Now returning to the proof. Let $(\Delta+\lambda)f \equiv 0$, L invariant.

$$Lf(z) = L(\Phi_z f)(z)$$

$$= \int_{\mathbf{H}} \Phi_z(f)(w)K(z,w)\,\mathrm{d}\mu(w)$$

$$= \int_{\mathbf{H}} W(w,z)f(w)K(z,w)\,\mathrm{d}\mu(w)$$

$$= \left\{\int_{\mathbf{H}} W(w,z)K(z,w)\,\mathrm{d}\mu(w)\right\} f(z)$$

Claim: $\{\cdots\}$ depends only on K and λ not z. Proof: Let $z_1, z_2 \in \mathbf{H}$ and pick $g \in \mathrm{SL}_2(\mathbf{R})$ $gz_1 = z_2$.

$$\int_{\mathbf{H}} W(w,z_2)K(z_2,w)\,\mathrm{d}\mu(w)$$

$$= \int_{\mathbf{H}} W(w, gz_1) K(gz_1, w) \,\mathrm{d}\mu(w)$$

$$= \int_{\mathbf{H}} W(g^{-1}w, z_1) K(z_1, g^{-1}w) \,\mathrm{d}\mu(w)$$

$$= \int_{\mathbf{H}} W(w, z_1) K(z_1, w) \,\mathrm{d}\mu(w). \qquad \square$$

Upshot so far: Poisson summation is a duality, but it can be seen as an equality of the trace of an operator calculated in two different ways. In the non-euclidean setting we can do something similar, but not so recognisable.

Digression: Ramanujan conjecture A weight k cusp form, eigenfunction of the Hecke operators implies

$$|\lambda_p| \leq 2p^{\frac{k-1}{2}},$$

"correct" normalisation is $|\tilde{\lambda}_p| \leq 2$.

This is about the components at p but there is also a component at infinity.

Selberg's eigenvalue conjecture: ϕ is a cuspidal automorphic (Maass) form with eigenvalue $\lambda = s(1-s)$ implies $s = \frac{1}{2} + it$, $t \in \mathbf{R}$ i.e. $|\lambda| \geq \frac{1}{4}$.

Back to H If we have $(\Delta + \lambda)f = 0$ can we say anything about λ?

Proposition 1.36.

$$\lambda \in \mathbf{R},\ \lambda \geq 0$$

Proof. Introduce the Petersson inner product

$$\langle F, G \rangle = \int_{\mathbf{H}} F(z)\overline{G(z)} \,\mathrm{d}\mu(z).$$

Now

$$\langle -\Delta F, G \rangle = \int_{\mathbf{H}} \nabla F \cdot \overline{\nabla G} \,\mathrm{d}x \,\mathrm{d}y$$

$$(\Delta F = \nabla \cdot \nabla F)$$

exercise: check this. So $\langle -\Delta F, G \rangle = \langle F, -\Delta G \rangle$ which gives $\lambda \in \mathbf{R}$.

$$\langle -\Delta F, F \rangle \geq 0 \implies \lambda \geq 0. \qquad \square$$

For the $\frac{1}{4}$ bound one needs to work a little harder.

Proposition 1.37.

$$(\Delta + \lambda) \equiv 0 \implies \lambda \geq \frac{1}{4}.$$

Proof. Let $D \subseteq \mathbf{H}$ be a (nice) domain. Consider the Dirichlet problem

$$(\Delta + \lambda)f \equiv 0 \text{ inside } D$$

$$f \equiv 0 \text{ on } \partial D.$$

Define

$$\langle F, G \rangle_D = \int_D F(z)\overline{G(z)} \,\mathrm{d}\mu(z).$$

Then

$$\langle -\Delta F, G \rangle_D = \int \nabla F \cdot \overline{\nabla G} \,\mathrm{d}x \,\mathrm{d}y$$

(exercise, show this).

$$\lambda\|F\|^2 = \langle -\Delta F, F\rangle = \int_D \left(\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2\right)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \geq \int_D \left(\frac{\partial F}{\partial y}\right)^2 \mathrm{d}x\,\mathrm{d}y \tag{1.1}$$

For every fixed x:

$$\int F^2 \frac{\mathrm{d}y}{y^2} = 2\int F\frac{\partial F}{\partial y}\frac{\mathrm{d}y}{y}$$

$$\implies \int_D F^2\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = 2\int F\frac{\partial F}{\partial y}\frac{\mathrm{d}x\,\mathrm{d}y}{y}.$$

$$\implies 2\int\left|\frac{F}{y}\frac{\partial F}{\partial y}\right|\mathrm{d}x\,\mathrm{d}y \leq 2\left(\int \frac{F^2}{y^2}\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\left(\int\left(\frac{\partial F}{\partial y}\right)^2\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}$$

$$\implies \frac{1}{2}\int_D \frac{F^2}{y^2}\mathrm{d}x\,\mathrm{d}y \leq \left(\int \frac{F^2}{y^2}\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\left(\int\left(\frac{\partial F}{\partial y}\right)^2\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \tag{1.2}$$

previous two imply

$$\frac{1}{4\lambda}\int_D\left(\frac{\partial F}{\partial y}\right)^2\mathrm{d}x\,\mathrm{d}y \leq \frac{1}{4}\int_D\frac{F^2}{y^2}\mathrm{d}x\,\mathrm{d}y \leq \int_D\left(\frac{\partial F}{\partial y}\right)^2\mathrm{d}x\,\mathrm{d}y$$

$$\frac{1}{4\lambda} \leq 1 \implies \lambda \geq \frac{1}{4}.$$

□

Remark 1.38. In Theorem 1.29 we restricted to the $1/2$ line, this is a reincarnation of $\lambda \geq \frac{1}{4}$. Only certain functions contributed, similar to the way only the unitaries $e^{2\pi i x\xi}$ contribute to a fourier expansion, not all characters of $\mathbf{R}$.

The spectrum of Δ on $\mathbf{H}$ has $\lambda = s(1-s)$, $s = \frac{1}{2} + it$. If we consider the quotient $\Gamma\backslash\mathbf{H}$ there is a possibility for $t = t_{\mathbf{R}} + it_{\mathbf{C}}$ where $0 \leq t_{\mathbf{C}} \leq \frac{1}{2}$. Selberg's conjecture is that these extra ones don't appear for cusp forms. This is very sensitive to the arithmetic, we need a congruence subgroup for this to be true.

1.10 Automorphic forms

1.10.1 Modular forms

These are functions on $\mathbf{H}$ that are very symmetric. We already saw one in the first lecture, $\theta(t)$ in the proof of Theorem 1.3. Its not quite, instead it is a *half-integral* weight modular form as we involved square roots

$$\theta(t) \leftrightarrow \theta\left(\frac{1}{t}\right).$$

Definition 1.39 (Modular functions and forms)**.** A **modular function** is some

$$f\colon \mathbf{H} \to \mathbf{C}$$

with

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)\,\forall \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

where, f is meromorphic on $\mathbf{H}$ including ∞. It is called a **modular form** if it is indeed holomorphic at infinity, this is equivalent to some growth condition.

Definition 1.40 (Cusp forms)**.** f is a **cusp form** if

$$\int_0^1 f(x+z)\,\mathrm{d}x = 0\,\forall z.$$

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Remark 1.41. f has a fourier expansion (invariant under $x \mapsto x+1$) holomorphic implies

$$f(z) = \sum_{n=0}^{\infty} a_n e(nz)\ (e(\alpha) = e^{2\pi i \alpha})$$

cusp form implies

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz)$$

as cuspidal implies $f(z) = O(e^{-2\pi y})$ as $y \to \infty$. f not cuspidal implies $f(z) = O(y^s)$ as $y \to \infty$.

1.10.2 Examples

Example 1.42. Constant functions for $k = 0$.

Example 1.43. Eisenstein series (holomorphic).

$$G_k(z) = \sum_{(m,n)\in \mathbf{Z}^2 \setminus (0,0)} \frac{1}{(mz+n)^{2k}}.$$

Is this cuspidal? Answer: No! Why?

$$G_k(z) = 2\underbrace{\zeta(2k)}_{\neq 0} + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{\alpha=1}^{\infty} \underbrace{\sigma_{2k-1}(\alpha)}_{=\sum_{d|\alpha} d^{2k-1}} e(\alpha z)$$

this is of weight $2k$!

Exercise 1.44. Prove this.

Example 1.45.

$$\Delta(z) = (60G_2(z))^3 - 27(140G_3(z))^2$$

is a cusp form of weight 12.

$$\Delta(z) = \sum_{n=1}^{\infty} a_n e(nz),\ a_n = n^{11/2}\tau(n) \implies \tau(n) = O(n^{\varepsilon})\forall \varepsilon$$

$$|\tau(p)| \leq 2$$

the original Ramanujan conjecture.

Exercise 1.46. Show these.

Example 1.47 (A non-example)**.**

$$j(x) = \frac{1728(60G_2(z))^2}{\Delta(z)}$$

not holomorphic at ∞.

Example 1.48. We had also seen θ-function but it *does not* fit in to this setting. It is rather a modular form for a covering group.

Digression: bounds on fourier coefficients of cusp forms

Theorem 1.49 (Hecke)**.** *If $f(z)$ is a cusp form of weight k*

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz)$$

then

$$a_n = O(n^{k/2})$$

called the Hecke or trivial bound.

If $\lambda_n = n^{(1-k)/2} a_n$ then this says $\lambda_n \leq \sqrt{n}$.

Proof.

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz) = e(z) \sum_{n=1}^{\infty} a_n e((n-1)z)$$

implies

$$|f(x)| \leq Ce^{-2\pi y}$$

now consider

$$\phi(z) = f(z) y^{k/2}.$$

Then

$$\phi(gz) = \phi(z) \, \forall f \in \mathrm{SL}_2(\mathbf{Z})$$

(exercise). Moreover $\phi(z) \to 0$ as $y \to \infty$ and ϕ is continuous so $\exists M$ s.t. $|\phi(z)| \leq M$.

Therefore $f(z) \leq My^{-k/2}$.

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz)$$

$$a_n e(iy) = \int_0^1 f(x+iy) e(-nz) \, \mathrm{d}x$$

$$\leq \int_0^1 \frac{M}{y^{k/2}} \, \mathrm{d}x = O\left(\frac{1}{y^{k/2}}\right)$$

for all y pick $y = 1/n$ so $O(n^{k/2})$. □

1.10.3 Maass forms

Definition 1.50 (Maass forms)**.** A function $f \colon \mathbf{H} \to \mathbf{C}$ s.t.

- $$f(gz) = f(z) \, \forall g \in \mathrm{SL}_2(Z)$$
- f is an eigenfunction of Δ.
- f is of moderate growth, $f(x+iy) = O(y^N)$ for some N.

is called a **maass form**. If

$$\int_0^1 f(x+iy) \, \mathrm{d}x = 0$$

we call it a **maass cusp form**.

Example 1.51. Constant functions are Maass forms, this is because they are L^2 because $\mathbf{H}/\mathrm{SL}_2(\mathbf{Z})$ has finite volume.

Example 1.52 (Non-holomorphic Eisenstein series.)**.**

$$E(z;s) = \sum_{(c,d)\in\mathbf{Z}^2\smallsetminus(0,0),(c,d)=1,c\geq 0} \frac{\Im(z)^{s+1/2}}{|cz+d|^{2s+1}},$$

we choose this normalisation (for now) with $s+\frac{1}{2}$ as it generalises better to GL_3 which has more elements in its Weyl group.

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Remark 1.53. In fact most things are non-holomorphic in the sense that many spaces of interest do not have a complex structure.

Properties

- $$(\Delta+\lambda)E(z;s) = 0$$ $$\lambda = \frac{1}{4} - s^2$$
- $$E(\gamma z;s) = E(z,s)\ \forall\gamma\in\mathrm{SL}_2(\mathbf{Z})$$
- $$E(z;s) = O(y^{\max\{\mathrm{Res}(s+\frac{1}{2}),\Re(-s+\frac{1}{2})\}})$$

hence $E(z;s)$ is a Maass form. We have

$$\Im(\gamma z) = \frac{\Im(z)}{|cz+d|^2}$$

so

$$E(z;s) = \sum_{(c,d)\in\mathbf{Z}^2\smallsetminus(0,0),(c,d)=1,c\geq 0} \frac{y^{s+\frac{1}{2}}}{|cz+d|^{2s+1}} = \sum_{\gamma\in\pm\Gamma_\infty\backslash\mathrm{SL}_2(\mathbf{Z})} \Im(\gamma z)^{s+\frac{1}{2}}$$

where $\Gamma_\infty = \left\{\begin{pmatrix}1 & n\\ & 1\end{pmatrix}\in\mathrm{SL}_2(\mathbf{Z})\right\}$. Exercise: check.

$$\begin{aligned}\Delta y^{s+\frac{1}{2}} &= y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})y^{s+\frac{1}{2}}\\ &= (s+\frac{1}{2})(s-\frac{1}{2})y^{s+\frac{1}{2}}\\ &= (s^2-\frac{1}{4})y^{s+\frac{1}{2}}\\ \implies (\Delta+(\frac{1}{4}-s^2))y^{s+\frac{1}{2}} &= 0\end{aligned}$$

γ is an isometry implies $\Delta\gamma = \gamma\Delta$. So $\gamma y^{s+\frac{1}{2}}$ is also an eigenfunction with eigenvalue $\frac{1}{4}-s^2$.

Theorem 1.54. *$E(z;s)$ has analytic continuation to $\mathbf{C}$ (in s), it satisfies $E(z;s) = E(z;-s)$ and*

$$E(z;s) = O(y^\sigma),\ \sigma = \max\{\Re(s), \Re(-s)\} + \frac{1}{2}.$$

Proof. (Wrong way to prove this) Fourier expansion of Eisenstein series.

$$E(z;s) = a_0(y) + \sum_{n\neq 0} a_n 2y^{\frac{1}{2}} K_s(2\pi|n|y)e(nx)$$

using Theorem 1.29.

$$\int_0^1 E(x+iy;s)e(-nx)\,\mathrm{d}x = \begin{cases} a_0(y) & n=0 \\ 2a_n y^{\frac{1}{2}} K_s(2\pi|n|y) & n\neq 0 \end{cases}.$$

Note:

$$\begin{aligned} E(z;s) &= \sum \frac{y^{s+\frac{1}{2}}}{|cz+d|^{2s+1}} \\ &= \frac{1}{2}\frac{1}{\zeta(2s+1)} \sum_{(c,d)\in\mathbf{Z}\setminus(0,0)} \frac{y^{s+\frac{1}{2}}}{|cz+d|^{2s+1}} \end{aligned}$$

We will work with

$$E_1(z;s) = \frac{\overbrace{\pi^{-(s+\frac{1}{2})}\Gamma(s+\frac{1}{2})}^{\text{for archimidean factors of } \zeta(2s+1)}}{2} \sum_{(c,d)\neq(0,0)} \frac{y^{s+\frac{1}{2}}}{|cz+d|^{1+2s}}$$

1.
$$c=0 \implies \begin{cases} \frac{\pi^{-(s+\frac{1}{2})}\Gamma(s+\frac{1}{2})}{2}\zeta(1+2s) & \text{if } n=0 \\ 0 & \text{if } n\neq 0 \end{cases}$$

2.
$$\begin{aligned} c\neq 0: &\sum_{(c,d),\, c\neq 0} \int_0^1 \frac{y^{s+\frac{1}{2}}}{|cz+d|^{2y+1}} e(-nx)\,\mathrm{d}x \\ &= 2\sum_{c=1}^{\infty}\sum_{d=-\infty}^{\infty} y^{s+\frac{1}{2}} \int_0^1 \frac{e(-nx)}{\underbrace{|cz+d|^{2s+1}}_{=cx+d+icy}}\,\mathrm{d}x \end{aligned}$$

the right hand side is invariant under $x\mapsto x+1$ we can absorb the shift into the sum over d, in a general context this is known as unfolding.

$$\begin{aligned} &= 2\sum_{c=1}^{\infty}\sum_{\alpha \pmod c}\sum_{d\equiv\alpha \pmod c} y^{s+\frac{1}{2}}\int_0^1 \frac{e(-nx)}{|cz+d|^{2s+1}}\,\mathrm{d}x \\ &= 2\sum_{c=1}^{\infty}\sum_{\alpha \pmod c}\sum_{k\in\mathbf{Z}} y^{s+\frac{1}{2}}\int_0^1 \frac{e(-nx)}{|cx+ck+\alpha+icy|^{2s+1}}\,\mathrm{d}x \\ &= 2y^{s+\frac{1}{2}}\sum_{c=1}^{\infty}\sum_{\alpha \pmod c}\int_{-\infty}^{\infty} \frac{e(-nx)}{|cx+\alpha+icy|^{2s+1}}\,\mathrm{d}x \end{aligned}$$

$$= 2y^{s+\frac{1}{2}} \sum_{c=1,\alpha \pmod c}^{\infty} \frac{e(n\alpha/c)}{c^{2s+1}} \int_0^1 \frac{e(-nx)}{|x+iy|^{2s+1}}\,\mathrm{d}x$$

note:

$$\sum_{\alpha \pmod c} e(n\alpha/c) = \begin{cases} c & \text{if } c|n \\ 0 & \text{if } c \nmid n \end{cases}$$

so we get

$$2y^{s+\frac{1}{2}} \sum_{c=1}^{\infty} \frac{1}{c^{2s}} \int_{-\infty}^{\infty} \frac{e(-nx)}{|x+iy|^{2s+1}}\,\mathrm{d}x$$

two cases

(a) $n = 0$

$$2y^{s+\frac{1}{2}} \sum_{c=1}^{\infty} \frac{1}{c^{2s}} \int_{-\infty}^{\infty} \frac{1}{|x+iy|^{2s+1}}\,\mathrm{d}x$$

$x \longrightarrow yx$

$$= 2\frac{y^{s+\frac{3}{2}}}{y^{2s+1}} \sum_{c=1}^{\infty} \frac{1}{c^{2s}} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^{\frac{2s+1}{2}}}\,\mathrm{d}x$$

$$= 2y^{-s+\frac{1}{2}}\zeta(2s) \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^{\frac{2s+1}{2}}}\,\mathrm{d}x$$

(b) $n \neq 0$

$$2y^{s+\frac{1}{2}}\sigma_{-2s}(|n|) \int_{-\infty}^{\infty} \frac{e(-nx)}{|x+iy|^{2s+1}}\,\mathrm{d}x$$

fact:

$$\pi^{-(s+\frac{1}{2})} y^{s+\frac{1}{2}} \Gamma(s+\frac{1}{2}) \int_{-\infty}^{\infty} \frac{e(-nx)}{|x+iy|^{2s+1}}\,\mathrm{d}x = \begin{cases} \pi^{-s}\Gamma(s) y^{\frac{1}{2}-s} & \text{if } n = 0 \\ 2|n|^2\sqrt{y}K_s(2\pi|n|y) & \text{if } n \neq 0 \end{cases}$$

Combining these we have shown

$$E_1(z;s) = \pi^{-(s+\frac{1}{2})}\Gamma(s+\frac{1}{2})\zeta(2s+1)y^{s+\frac{1}{2}} + \pi^{-s}\Gamma(s)\zeta(2s)y^{\frac{1}{2}-s} + \sum_{n\neq 0} \sigma_{-2s}(|n|)|n|^s\sqrt{y}K_s(2\pi|n|y)e(nx) \tag{1.3}$$

where we have $K_s = K_{-s}$ and

$$\sigma_{-2s}|n|^s = \sum_{d|n} d^{-2s}|n|^s = \sum_{d|n} \frac{d^{2s}}{|n|^s} = |n|^{-s}\sigma_{2s}(|n|) = E(z;s).$$

So we have proved the functional equation and analytic continuation.

□

We can see that we have ζ appearing here in the constant term, we can determine analytic information about it using what we know about Eisenstein series, this idea in generality is known as the Langlands-Shahidi method.

Remark 1.55. This has poles at $s = \frac{1}{2}$, $\mathrm{Res}_{s=\frac{1}{2}} E(z;s) = \frac{1}{2}$. Note that this residue is constant. We will use this in Rankin-Selberg.

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Remark 1.56. If G is a reductive group and $M \subseteq G$ a Levi subgroup, e.g. for GL_n a Levi is diagonal blocks of size $n_1 + n_2 + \cdots + n_k = n$. One can associate a cusp form to these subgroups following Eisenstein. There are automorphic L-functions corresponding to these and by doing the same procedure as last time we see these L-functions appearing in the constant terms of the Eisenstein series. So we can establish analytic properties of these automorphic L-functions via those of the Eisenstein series. This is known as the Langlands-Shahidi method, it only works in some cases but when it does it is very powerful. Shahidi pushed the idea by looking at non-constant terms. In the example above we have

$$\sigma_{-2s}(n) = \sum_{d|n} d^{-2s} \prod (1 + 1/p^{2s} = \prod_{p|n} \zeta_p^{-1}(2s)$$

so there are L-functions even in the non-constant terms.

1.10.4 Hecke Operators

The natural setting to view these is over $\mathrm{GL}_2(\mathbf{Q}_p)$. But as we haven't done this yet we will take the path that Hecke took and just write a formula. They act on the space of modular forms.

Definition 1.57 (Slash operators)**.** Let $k \in \mathbf{Z}_+$ fixed, $\gamma \in \mathrm{GL}_2^+(\mathbf{R})$ (positive determinant).

$$f|_\gamma(z) = \det(\gamma)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

There is a determinant twist so that the center acts trivially.

Definition 1.58 (T_γ)**.** Let $\gamma \in \mathrm{GL}_2^+(\mathbf{Q})$ write

$$\mathrm{SL}_2(\mathbf{Z})\gamma\,\mathrm{SL}_2(\mathbf{Z}) = \bigsqcup_{i=1}^{r} \mathrm{SL}_2(\mathbf{Z})\gamma_i$$

then

$$T_\gamma(f) = \sum_{i=1}^{r} f|_{\gamma_i}(z).$$

Exercise 1.59.

$$\gamma = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{SL}_2(\mathbf{Z})\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\mathrm{SL}_2(\mathbf{Z}) = \bigsqcup_{b \pmod p} \mathrm{SL}_2(\mathbf{Z})\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \bigsqcup \mathrm{SL}_2(\mathbf{Z})\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

More classically

$$T_p(f)(z) = p^{-k/2} \sum_{b \pmod p} f\left(\frac{z+b}{p}\right) + p^{k/2} f(pz).$$

This differs by a normalization of det won't change too much but will shift the spectrum. Or more generally we have

$$T_n(f)(z) = \sum_{ad=n,\, b \pmod d} n^{k/2} d^{-k} f\left(\frac{az+b}{d}\right).$$

Fact 1.60. *Hecke operators commpute with each other (follows from KAK decomposition).*

Fact 1.61. *Hecke operators are self-adjoint with respect to the Petersson inner product on $M_k(1)$, the modular forms of weight k and level 1.*

$$\langle f, g\rangle_{\text{Pet}} \int_{\mathrm{SL}_2(\mathbf{Z})\backslash \mathbf{H}} f(z)\overline{g(z)}y^k \,\mathrm{d}\mu(z).$$

Lemma 1.62. *Let $f(z) \neq 0$ be a cusp form of weight k which is an eigenfunction for all of the Hecke operators with eigenvalue $n^{1-k/2}\lambda(n)$ i.e.*

$$T_n(f)(z)n^{1-k/2}\lambda(n)f(z)\forall n$$

let

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz)$$

be its fourier expansion and

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Then

1. $$a_1 \neq 0$$
2. *If $a_1 = 1$ then $a_n = \lambda(n)\forall n$.*
3. *If $a_1 = 1$ then $\lambda(mn) = \lambda(m)\lambda(n)$ for all $(m, n) = 1$.*
4. *If $a_1 = 1$ then*
$$L(s, f) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

Proof. By the Fourier expansion

$$T_n f(z) = \sum_{ad=n,\, b \pmod d} \left(\frac{a}{d}\right)^{k/2} f\left(\frac{az+b}{d}\right)$$

$$= \sum_{ad=n,\, b \pmod d} \left(\frac{a}{d}\right)^{k/2} \sum_{m=1}^{\infty} a_m e\left(\frac{maz}{d}\right) e\left(\frac{mb}{d}\right)$$

$$= \sum_{ad=n,\, b \pmod d} \left(\frac{a}{d}\right)^{k/2} d \sum_{m=1}^{\infty} a_{md} e\left(maz\right).$$

Which implies

$$n^{1-k/2}\lambda(n)f(z) = \sum_{ad=n} \left(\frac{a}{d}\right)^{k/2} d \sum_{m=1}^{\infty} e(maz)$$

so

$$n^{1-k/2}\lambda(n)a_m = \sum_{ad=n,\, a|m} \left(\frac{a}{d}\right)^{k/2} d \frac{a_{md}}{a}$$

exercise: check. Take $m = 1$ so

$$n^{1-k/2}\lambda(n)a_1 = n^{-k/2+1}a_n$$

hence

$$\lambda(n)a_1 = a_n.$$

1. If $a_1 = 0$ then $f \equiv 0$.
2. If $a_1 = 1$ then $\lambda(n) = a_n$.
3. Follows from 4.
4. Note
$$(p^r)^{1-k/2}\lambda(p^r)\lambda(p) = \sum_{ad=p^r,\, a|p} \lambda\left(\frac{pd}{a}\right)\left(\frac{a}{d}\right)^{k/2} d$$
$$= \lambda(p^{r+1})(p^r)^{1-k/2} + \lambda(p^{r-1})(p^r)^{1-k/2}p^{k-1}$$
$$\left(\sum_{r=0}^{\infty} \frac{\lambda(p^r)}{p^{rs}}\right)(1 - \frac{\lambda(p)}{p^s} + \frac{p^{k-1}}{p^{2s}}) = 1. \qquad \square$$

This is very special to GL_2, in general fourier coefficients have more information than Hecke eigenvalues.

Remark 1.63. With the normalisation
$$T_n(f)(z) = \lambda(n)n^{1-k/2}f(z)$$
the Ramanujan conjecture reads $\lambda(n) = O(n^{(k-1)/2+\epsilon})$.

Remark 1.64. Having $a_1 = 1$ is known as being Hecke normalised.

1.11 Rankin-Selberg method

This is a protoype of the integral representation of automorphic L-functions.

1.11.1 Mellin transforms of automorphic forms and automorphic L-functions

Let
$$\phi(z) = \sum_{n\in\mathbf{Z}} a_n(y)e(nx)$$
then
$$a_n(y) = \int_0^1 \phi(z)\overline{e(nx)}\,\mathrm{d}x.$$

Notation:
$$\tilde{a}_n(s) = \int_0^{\infty} a_n(y)y^2\,\mathrm{d}^*y,\ \mathrm{d}^*y = \frac{\mathrm{d}y}{y}$$
converges for $\Re(s) \gg 0$ if $a_n(y) = O(y^{-N})$ for all N.

Theorem 1.65. *Let $\phi(x+iy) = O(y{-}N)$ for all $N > 0$ and $\phi(z)$ is invariant under $z \mapsto \gamma z$ for $\gamma \in \mathrm{SL}_2(\mathbf{Z})$. Then*
$$\int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}} \phi(z)E(z;s)\,\mathrm{d}z$$
$$= \pi^{-s}\Gamma(S)\zeta(2s)\tilde{a}_0(s-1).$$

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Definition 1.66 (Mellin transforms)**.** Given

$$f(y)\colon \mathbf{R}_+ \to \mathbf{C}$$

its **Mellin transform** is

$$\hat{f}(s) = \int_0^\infty f(y) y^s \frac{\mathrm{d}y}{y},\ f(y) = O(y^{-N}).$$

If

$$f(y) = g(Q(n)y)$$

$$\hat{f}(s) = \int g(Q(n)y)y^s \frac{\mathrm{d}y}{y} = \int g(y)\frac{y^s}{Q^s(n)}\frac{\mathrm{d}y}{y} = \frac{1}{Q(n)^s}\tilde{g}(y)$$

What is

$$\int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}} \phi(z)E_3(z;s)\,\mathrm{d}\mu(z)?$$

The Eisenstein series is essentially

$$\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbf{Z})} \Im(\gamma z)^s$$

we can see the integrating over $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$ a sum over $\Gamma_\infty\backslash\mathrm{SL}_2(\mathbf{Z})$ things should cancel to give us an integral over $\Gamma_\infty\backslash\mathbf{H}$, a rectangle! So this unfolding should simplify things.

Proposition 1.67. *Let* $\phi\colon \mathbf{H} \to \mathbf{C}$ *be automorphic with respect to* $\mathrm{SL}_2(\mathbf{Z})$*, with fourier expansion*

$$\phi(z) = \sum_{n=-\infty}^{\infty} a_n(y)e(nx),\ \text{where}\ a_n(y) = \int_0^1 \phi(x+iy)e(-nx)\,\mathrm{d}x.$$

If $\phi(x+iy) = O(y^{-N})$ *for all* $N > 0$*.*

$$\int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}} \phi(z)E_3(z;s)\,\mathrm{d}\mu(z) = \pi^{-s}\Gamma(s)\zeta(2s)\tilde{a}_0(s-1) \tag{1.4}$$

where $\phi(z) = \sum_{n\in\mathbf{Z}} a_n(y)e(nx)$*.*

Proof. Follow your nose!

Recall

$$E_3(z;s) = \frac{\pi^{-s}}{2}\Gamma(s)\zeta(2s)\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbf{Z})} \Im(\gamma z)^s.$$

Step 1: The integral converges: Writing E for E_3 we have

$$E(z;s) = O(y^2 + y^{1-s}).$$

Step 2: Unfold

$$\begin{aligned}
&\frac{\pi^{-s}}{2}\Gamma(s)\zeta(2s)\int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}} \phi(z)\left(\sum_{\Gamma_\infty\backslash\mathrm{SL}_2(\mathbf{Z})} \Im(\gamma z)^s\right)\mathrm{d}\mu(z)\\
&= \frac{\pi^{-s}}{2}\Gamma(s)\zeta(2s)\int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}} \left(\sum_{\Gamma_\infty\backslash\mathrm{SL}_2(\mathbf{Z})} \phi(\gamma z)\Im(\gamma z)^s\right)\mathrm{d}\mu(z)
\end{aligned}$$

$$
\begin{aligned}
&= \pi^{-s}\Gamma(s)\zeta(2s)\int_{\Gamma_\infty\backslash \mathbf{H}} \phi(z)y^s\,\mathrm{d}\mu(z)\\
&= \pi^{-s}\Gamma(s)\zeta(2s)\int_0^1\int_0^\infty \phi(z)y^s\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\\
&= \pi^{-s}\Gamma(s)\zeta(2s)\int_0^\infty a_0(y)y^s\frac{\mathrm{d}y}{y^2}\\
&= \pi^{-s}\Gamma(s)\zeta(2s)\tilde{a}_0(s-1).
\end{aligned}
$$

□

Corollary 1.68.

$$\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)\tilde{a}_0(s-1)$$

we showed

$$\Lambda(s) = \int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}} \phi(s)E(z;s)\,\mathrm{d}\mu(z).$$

(So we can find a functional equation and analytic continuation for from the corresponding properties of the Eisenstein series.)

- $\Lambda(s)$ *has analytic continuation.*
- $\Lambda(s)$ *has functional equation* $s \leftrightarrow s-1$.
- $\Lambda(s)$ *has poles only at* $s = 0, 1$.
- $$\operatorname{Res}_{s=1}\Lambda(s) = \int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}} \phi(s)\,\mathrm{d}\mu(z)$$

Note that if we use a cusp form for ϕ we get 0 from the integral above, in L^2 the cusp forms and Eisenstein series are orthogonal. Instead we will cook up something interesting from two functions.

1.11.2 Rankin-Selberg L-functions

Let

$$f(z) = \sum_{n=0}^{\infty} a_n e(nz)$$

$$g(z) = \sum_{n=0}^{\infty} b_n e(nz)$$

be holomorphic modular forms of weight k.

Assume that at least one of f or g is cuspidal. Assume additionally that f, g are normalised Hecke eigenforms so $a(1) = b(1) = 1$.

Definition 1.69.

$$\phi(z) = f(z)\overline{g}(z)y^k.$$

Note 1.70. $\phi(\gamma z) = \phi(z)$ for any $\gamma \in \mathrm{SL}_2(\mathbf{Z})$. ϕ also satisfies the decay condition.

Note 1.71. If $f = \sum a_n e(nz)$, $g = \sum b_n e(nz)$ then

$$f(z)\overline{g}(z) = \sum_{m-n\neq 0} A_{n-m} e((n-m)z) + \sum_{n\in\mathbf{Z}} a_n\overline{b}_n$$

so if we were to integrate this from 0 to 1 dx the first term would disappear and we would be left with the second.

$$\phi_0(y) = \int_0^1 f(z)\overline{g}(z)\,\mathrm{d}x\,y^k = \int_0^1 \sum_{m-n\neq 0} A_{n-m} e((n-m)z) y^k + \int_0^1 \sum_{n\in\mathbf{Z}} a_n \overline{b}_n y^k$$
$$= \sum_{n\in\mathbf{Z}} a_n \overline{b}_n y^k$$

i.e.

$$\phi_0(y) = \int_0^1 \phi(x+iy)\,\mathrm{d}x = \sum_{n\in\mathbf{Z}} a_n(y)\overline{b}_n(y) y^k e^{-4\pi i n y}.$$

Note 1.72.

$$\begin{aligned}\tilde{\phi}_0(s) &= \int_0^1 \sum_{n=0}^{\infty} a_n \overline{b}_n e^{-4\pi i y} y^{k+s} \frac{\mathrm{d}y}{y}\\ &= \sum_{n=0}^{\infty} a_n \overline{b}_n \int_0^{\infty} e^{-4\pi i y} y^{k+s} \frac{\mathrm{d}y}{y}\\ &= \frac{1}{(4\pi)^s} \sum_{n=0}^{\infty} \frac{a_n \overline{b}_n}{n^{k+s}} \int_0^{\infty} e^{-y} y^{k+s} \frac{\mathrm{d}y}{y}\\ &= \frac{\Gamma(k+s)}{(4\pi)^{k+s}} \underbrace{\sum_{n=1}^{\infty} \frac{a_n \overline{b}_n}{n^{k+s}}}_{L(s+k;f\times\overline{g})}\end{aligned}$$

this is the Rankin-Selberg L-function. So by the corollary 1.68; $L(s+k; f\times g)$ has analytic continuation and functional equation, and poles only at $s = 1+k$, $s = k$.

An application We proved f cusp form $f(z) = \sum a_n e(nz)$ implies $a_n = O(n^{k/2})$, Ramanujan $a_n = O(n^{k-1/2})$. As cusp forms often appear as error terms for counting arguments knowing it gives us many results, tells us we can just count with Eisenstein series. The averaged version of the Ramanujan conjecture is much easier

$$\sum_{n<X} a_n.$$

Lecture 10 27/2/2018

Recall Proposition 1.67 and moreover that

$$E(z;s) = \pi^{-s}\Gamma(s)\zeta(2s)\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbf{Z})} \Im(\gamma z)^s = \pi^{-s}\Gamma(s)\sum_{(m,n)\in\mathbf{Z}^2\setminus(0,0)} \frac{y^s}{|mz+n|^{2s}},$$

from this and Note 1.71 we conclude.

$$\int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}} f(z)\overline{g}(z) y^k E(z;s)\,\mathrm{d}\mu(z)$$
$$= \frac{\pi^{-s}\zeta(2s)\Gamma(s)\Gamma(s+k-1)}{(4\pi)^{2+k-1}} \sum_{n=1}^{\infty} \frac{a_n\overline{b}_n}{n^{s+k-1}} \tag{1.5}$$

(1.5) has analytic continuation as a function of s to all $s \in \mathbf{C}$. It has at most simple poles with residue at $s = 1$:

$$\frac{1}{2} \langle f, g \rangle_{\mathrm{Pet}}.$$

If we let

$$L(s, f \times \overline{g}) = \zeta(2(s-k+1)) \sum_{n=1}^{\infty} \frac{a_n \overline{b}_n}{n^s}$$

$$\Lambda(s, f \times \overline{g}) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+1)L(s, f \times g)$$

so

$$\Lambda(s, f \times \overline{g}) = \Lambda(2k-1-s, f \times g)$$

this follows from Theorem 1.54 $E(z; s) = E(s; 1-s)$. .

$\Lambda(s, f \times \overline{g})$ has analytic continuation to $s \in \mathbf{C}$ with poles at most at $s = k$, $s = k-1$.

$$\mathrm{Res}_{s=k} \Lambda(s, f \times g) = \frac{1}{2\pi^{k-1}} \langle f, g \rangle_{\mathrm{Pet}}.$$

This is analogous to when we took the Mellin transform of the theta function. We have obtained some highly nontrivial information above:

Remark 1.73. Given an arbitrary series of the form

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \ |a_n| = O(n^{\alpha})$$

will converge for $\Re(s) > \alpha + 1$. As these coefficients often come from point counts, they will in general be polynomial.

Recall the Hecke bound

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz)$$

a cuspidal (Hecke eigen)form of weight k has $a_n = O(n^{k/2})$. Ramanujan (Deligne) gives us $a_n = O(n^{(k-1)/2})$. Hecke implies that $a_n\overline{b}_n = O(n^k)$ implies $\sum_{n=1}^{\infty} \frac{a_n \overline{b}_n}{n^s}$ converges for $\Re(s) > k+1$. Deligne implies that $a_n\overline{b}_n = O(n^{k-1})$ implies $\sum_{n=1}^{\infty} \frac{a_n \overline{b}_n}{n^s}$ converges for $\Re(s) > k$. But the above already gives us this convergence, highly nontrivial k!

Remark 1.74. If we take f, g to be normalized Hecke eigenforms. Let

$$1 - \frac{a_p}{p^s} + \frac{1}{p^{2s-k+1}} = (1 - \frac{\alpha_1}{p^s})(1 - \frac{\alpha_2}{p^s})$$

$$1 - \frac{b_p}{p^s} + \frac{1}{p^{2s-k+1}} = (1 - \frac{\beta_1}{p^s})(1 - \frac{\beta_2}{p^s})$$

then

$$L(s, f \times \overline{g}) = \prod_p \prod_{1 \le i,j \le 2} (1 - \frac{\alpha_i \overline{\beta}_j}{p^s})^{-1}.$$

Exercise 1.75. Prove this.

Applications If one proves the prime number theorem using non-vanishing of the ζ function in a certain region you use a weird identity using sines and cosines being positive. This really comes from a Rankin-Selberg product.

Remark 1.76. Rankin-Selberg proves positivity.

$$z \in \mathbf{Z},\ |z|^2 \geq 0.$$

Ramanujan on average:

$$\sum_{n<X} a_n \sim X^{(k+1)/2}$$

this is equivalent to

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges for $\Re(s) > (k+1)/2$. Let

$$\phi(z) = \sum_{n=1}^{\infty} a_n e(nz)$$

be a cusp form of weight k.

Consider

$$D(s) = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^s},$$

Hecke implies this converges for $\Re(s) > k+1$.

Note 1.77.

$$D(s) = L(s, f \times \overline{f})$$

converges for $\Re(s) > k$.

Now observe that for any $\lambda > 0$

$$|a_n| \leq \max\left\{n^{\lambda}, \frac{|a_n|^2}{n^{\lambda}}\right\}.$$

So

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^s} \leq \max \sum_{n=1}^{\infty} \max\left\{\frac{1}{n^{s-\lambda}}, \frac{|a_n|}{n^{s+\lambda}}\right\}$$

choose $\lambda = (k-1)/2$ so

$$< \max \sum_{n=1}^{\infty} \max\left\{\frac{1}{n^{s-(k-1)/2}}, \frac{|a_n|}{n^{s+(k-1)/2}}\right\}$$

which converges for $s > (k+1)/2$.

Question 1.78. Fix $\mathrm{d}\mu(z) = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$ on $\mathbf{H}$. What is $\mathrm{Vol}(\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H})$? ($\pi/3$?) What about other Γ?

Naive observation:

$$\int \frac{1}{2}\,\mathrm{d}\mu(z) = \frac{\mathrm{Vol}}{2}$$

and

$$\frac{1}{2} = \operatorname{Res}_{s=1} E(z;s)$$

so

$$\frac{\text{Vol}}{2} = \int_{\text{SL}_2\backslash\mathbf{H}} \operatorname{Res}_{s=1} E(z;s)\,\mathrm{d}\mu(z)\text{“} = \text{”} \operatorname{Res}_{s=1} \int_{\text{SL}_2(\mathbf{Z})\backslash\mathbf{H}} E(z;s)\,\mathrm{d}\mu(z)$$

but the right hand side does not converge (exercise, check).

Problem: naive idea doesn't work $\int_{\text{SL}_2(\mathbf{Z})\backslash\mathbf{H}} E(z;s)\,\mathrm{d}\mu(z)$ converges only for $0 < 1 < 1$. but then we can't unfold because the series defining $E(z;s)$ does not converge $0 < s < 1$. We must target the source of this divergence, the constant term of the Eisenstein series.

$$\int_1^\infty *y^s + *y^{1-s}\frac{\mathrm{d}y}{y^2}$$

we will truncate the Eisenstein series.

1.11.3 Applications

Lecture 11 1/3/2018

First aim of the day: Calculate

$$\text{Vol}(\text{SL}_2(\mathbf{Z})\backslash\mathbf{H},\ \mathrm{d}\mu(z) = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2})$$

Idea: Use the pole of $E(z;s)$ at $s = 1$ and unfolding.

$$E(z;s) = \frac{\pi^{-s}\Gamma(s)\zeta(2s)}{2} \sum_{\gamma\in\Gamma_\infty\backslash\text{SL}_2(\mathbf{Z})} \Im(\gamma z)^s$$

$$\operatorname{Res}_{s=1} E(z;s) = \frac{1}{2}.$$

Idea:

$$\operatorname{Res}_{s=1} \int_{\text{SL}_2(\mathbf{Z})\backslash\mathbf{H}} E(z;s) = \mathrm{d}\mu(z) = \frac{\text{Vol}}{2}.$$

Problem: Constant term of $E(z;s) \sim y^s + y^{1-s}$.

$$\int_1^\infty y^s + y^{1-s}\frac{\mathrm{d}y}{y^2}$$

converges only if $0 < s < 1$. This approach needs modification, we will look at two such.

1. Sharp cut-off,
2. Smooth cut-off.

1 Sharp cut-off For sharp cut off we will fix some $T > 0$ and only consider $y < T$. Setting

$$y_T(z) = \begin{cases} y & y < T \\ 0 & y \geq T \end{cases}$$

using this

$$E_T(z;s) = \frac{\pi^{-s}\Gamma(s)\zeta(2s)}{2} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbf{Z})} y_T(\gamma z)^s.$$

Observations:

Lemma 1.79. *$K \subseteq \mathbf{H}$ compact, there exists T_K such that for all $T \geq T_K$*

$$E_T(z;s) = E(z;s) \, \forall z \in K.$$

Proof.

$$\begin{aligned} \Im(\gamma z) = \frac{y}{|cz+d|^2} &= \frac{y}{((cx+d)^2 + (cy)^2)} \\ &\leq \max\left\{\frac{y}{d^s}, \frac{1}{c^2 y}\right\} \\ \Im(\mathrm{SL}_2(\mathbf{Z})K) &\leq T_0 \end{aligned}$$

for some T_0. $\square$

Lemma 1.80.

$$\int_{\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}} E_T(z;s) \, \mathrm{d}\mu(z) = \pi^{-s}\Gamma(s)\zeta(2s)\frac{T^{s-1}}{s-1}$$

Proof. (Unfold), recall

$$\Gamma_\infty = \left\{\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})\right\}$$

so

$$\begin{aligned} \int_{\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}} E_T(z;s) \, \mathrm{d}\mu(z) &= \pi^{-s}\Gamma(s)\zeta(2s) \int_{\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbf{Z})} y_T(\gamma z)^s \, \mathrm{d}\mu(z) \\ &= \pi^{-s}\Gamma(s)\zeta(2s) \int_{\Gamma_\infty \backslash \mathbf{H}} y_T \, \mathrm{d}\mu(z) \\ &= \pi^{-s}\Gamma(s)\zeta(2s) \int_0^1 \int_0^T y^{s-1} \, \mathrm{d}\mu(z) \\ &= \pi^{-s}\Gamma(s)\zeta(2s)\frac{T^{s-1}}{s-1} \end{aligned}$$

$\square$

There is a huge generalization of this lemma by Langlands that allows him to calculate a lot of volumes.

Lemma 1.81. *Let $T > 1$ and $x + iy$ in the standard fundamental domain $\mathcal{F}$ for $\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}$ then*

$$E_T(z;s) = \begin{cases} E(z;s) & y < T \\ E(z;s) - pi^{-s}\Gamma(s)\zeta(2s)y^s & y \geq T \end{cases}.$$

Proof. Recall
$$\Im(\gamma z) = \frac{y}{((cx+d)^2 + (cy)^2)}$$

Case 1: $c \neq 0$ implies $\Im(\gamma z) \leq \frac{1}{c^2 y} \leq T$ for all $y \in \mathcal{F}$.

Case 2: $c = 0$ implies $\Im(\gamma z) = \frac{y}{d^2}$

$$\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

implies $ad = 1$ so $a = d = 1$ or $a = d = -1$.

$$E_T(z;s) = \frac{\pi^{-s}\Gamma(s)\zeta(2s)}{2} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbf{Z})} y_T(\gamma z)^s$$

$$= \frac{\pi^{-s}\Gamma(s)\zeta(2s)}{2} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbf{Z})} y(\gamma z)^s - \frac{\pi^{-s}\Gamma(s)\zeta(2s)}{2} \sum_{d=\pm 1} y_T(\frac{y}{d^s})^s$$

$$= \begin{cases} E(z;s) - \pi^{-s}\Gamma(s)\zeta(2s)y^s & y < T \\ E(z;s) & y \geq T \end{cases}$$ □

Remark 1.82. $E(z;s) - E_T(z;s)$ for fixed z is holomorphic as a function of s at $s = 1$.

Theorem 1.83.
$$\mathrm{Vol}(\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}) = \frac{\pi}{3}.$$

Proof. $E(z;s) - E_T(z;s)$ is holomorphic at $s = 1$. So

$$\int_{\mathrm{SL}_2(\mathbf{Z})\backslash \mathbf{H}} \mathrm{Res}_{s=1}(E(z;s) - E_T(z;s)) \, \mathrm{d}\mu(z) = 0$$

$$\int_{\mathrm{SL}_2(\mathbf{Z})\backslash \mathbf{H}} \frac{1}{2} - \mathrm{Res}_{s=1}(E_T(z;s)) \, \mathrm{d}\mu(z) = 0$$

$$\int_{\mathrm{SL}_2(\mathbf{Z})\backslash \mathbf{H}} \mathrm{Res}_{s=1}(E_T(z;s)) \, \mathrm{d}\mu(z) = \frac{1}{2} \mathrm{Vol}(\mathrm{SL}_2(\mathbf{Z})\backslash \mathbf{H})$$

$$\mathrm{Res}_{s=1} \int_{\mathrm{SL}_2(\mathbf{Z})\backslash \mathbf{H}} E_T(z;s) \, \mathrm{d}\mu(z) = \frac{1}{2} \mathrm{Vol}(\mathrm{SL}_2(\mathbf{Z})\backslash \mathbf{H})$$

$$\mathrm{Res}_{s=1} \pi^{-s}\Gamma(s)\zeta(2s)\frac{T^{s-1}}{s-1} = \frac{1}{2} \mathrm{Vol}(\mathrm{SL}_2(\mathbf{Z})\backslash \mathbf{H})$$

$$= \frac{1}{\pi}\frac{\pi^2}{6}$$

hence
$$\mathrm{Vol}(\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}) = \frac{\pi}{3}.$$ □

Volumes of such domains are known as Tamagawa number and many such were computed via these methods by Langlands in the 60s.

2 Smooth cut-off Let $f \in C_c^\infty(\mathbf{R}_{\geq 0})$ e.g. some nice bump. Consider

$$\theta_f(z) = \frac{1}{2} \sum_{\Gamma_\infty \setminus \mathrm{SL}_2(\mathbf{Z})} f(\Im(\gamma z))$$

idea

$$f(x) = \frac{1}{2\pi i} \int_{(\sigma)} \tilde{f}(s) x^{-s} \, \mathrm{d}s$$

for $a < \sigma < b$ such that $\tilde{f} = \int_0^\infty y^s f(y) \frac{\mathrm{d}y}{y}$ converges absolutely for $a < s < b$. Observation: $\sigma \in \mathbf{R}$ for this to hold. So

$$\theta_f(z) = \sum \left(\frac{1}{2\pi i} \int_{(\sigma)} \tilde{f}(s) \Im(\gamma z)^{-s} \, \mathrm{d}s \right)$$

for $\Re(s) < -s$:

$$= \frac{1}{4\pi i} \int_{(\sigma)} \tilde{f}(s) \left(\sum_{\gamma \in \Gamma_\infty \setminus \mathrm{SL}_2(\mathbf{Z})} \Im(\gamma z)^{-s} \right) \mathrm{d}s$$

$$= \frac{1}{2\pi i} \int_{(\sigma)} \tilde{f}(s) E_1(z; -s) \, \mathrm{d}s$$

where

$$E_1(z; s) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \setminus \mathrm{SL}_2(\mathbf{Z})} \Im(\gamma z)^s.$$

Now integrate

$$\int_{\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}} \theta_f(z) \, \mathrm{d}\mu(z) = \int_{\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}} \frac{1}{2\pi i} \int_{(\sigma)} \tilde{f}(s) E(z; -1) \, \mathrm{d}s \, \mathrm{d}\mu(z)$$

$$= \tilde{f}(-1) \left(\operatorname{Res}_{s=1} E_1(z; s) \right) \mathrm{Vol}(\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H})$$

$$\underbrace{+ \frac{1}{2\pi i} \int_{(-\frac{1}{2})} \tilde{f}(s) \int_{\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}} E(z; -s) \, \mathrm{d}\mu(z) \, \mathrm{d}s}_{= \langle E(z; s), 1 \rangle_{\mathrm{Pet}}}$$

(shifting contours to $\sigma = \frac{-1}{2} + it$). The rightmost term is 0 as we have a decomposition

$$L^2(\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}) = 1 \oplus \int_{(\frac{1}{2})} E(z; s) \, \mathrm{d}s \oplus \text{cusp form}.$$

We'll do this by hand here. i.e.

$$\int_{\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}} \theta_f(z) \, \mathrm{d}\mu(z) = \tilde{f}(-1) \underbrace{\left(\operatorname{Res}_{s=1} E_1(z; s) \right)}_{=3/\pi} \mathrm{Vol}(\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H})$$

$$+ \int_{\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}} E(z; -s) \, \mathrm{d}\mu(z) \, \mathrm{d}s.$$

Lemma 1.84.

$$\int_{\mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H}} \theta_f(z) \, \mathrm{d}\mu(z) = \tilde{f}(-1)$$

Proof. Exercise. □

The lemma implies $\forall f \in C_c^\infty(\mathbf{R}_{>0})$ we have the following

$$\tilde{f}(-1)\left(1 - \frac{3}{\pi}\operatorname{Vol}\right) = \frac{1}{2\pi i}\int_{(\frac{1}{2})} \tilde{f}(-s)\int E(z;-s)\,\mathrm{d}\mu(z)\,\mathrm{d}s \tag{1.6}$$

Claim 1.85. *(1.6) implies that*

$$\int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}} E(z;-s)\,\mathrm{d}\mu(z) \equiv 0,$$

whenever it converges.

Proof. By a change of variables we can rewrite (1.6) as

$$c\tilde{f}(-1) = \int_{(0)} \tilde{f}(-\frac{1}{2}+s)I(S)\,\mathrm{d}s$$

for some constant c.

1. Take $F(y) = y^{\frac{1}{2}}f(y)$ which implies

$$\tilde{F}(s) = \tilde{f}(s+\frac{1}{2}).$$

(1.6) $\iff$

$$\frac{1}{2\pi i}\int_{(0)} \tilde{F}(s)I(s)\,\mathrm{d}s = c\tilde{F}(-\frac{3}{2}).$$

2. Trick: Let $G(y) = (y\frac{\partial}{\partial y}F + \frac{3}{2}F)$ so

$$\tilde{G}(s) = s\tilde{F}(s) + \frac{3}{2}\tilde{F}(s)$$

so

$$\tilde{G}(-\frac{3}{2}) = 0$$

implies

$$\frac{1}{2\pi i}\int \tilde{G}(-s)I(z;s)\,\mathrm{d}s = 0$$

$$I(z;s) \equiv 0$$

whenever it converges.

□

2 The Eichler-Selberg trace formula

2.1 The OG approach

Lecture 12 15/3/2018

What does it do? It calculates the *trace* of the mth Hecke operator $T(m)$ on S_k the space of holomorphic modular forms of weight k level 1.

Input $m \in \mathbf{Z}_{\geq 0}$, k weight.

Remark 2.1. It is more general, there is an (Eichler-)Selberg trace formula for general level N.

Even more generally there is a Selberg trace formula for Maass forms of arbitrary level.

Even more general Arthur-Selberg trace formula for automorphic representations on any group.

Recall

$$T(m)f(z) = m^{k-1} \sum_{ad=m,\, b \pmod d} d^{-k} f\left(\frac{az+b}{d}\right)$$

S_k cusp forms of weight $k \geq 2$ even level 1.

$$\mathrm{Tr}_{S_k} T(m) = -\frac{1}{2} \sum_{t \in \mathbf{Z},\, t^2 - 4m < 0} P_k(t,m) H(4m - t^2)$$

$$-\frac{1}{2} \sum_{dd'=m} \min(d,d')^{k-1} + \begin{cases} \frac{k-1}{2} m^{k/2-1} & m \in \mathbf{Z}^2 \\ 0 \end{cases}.$$

Where

$$P_k(t,m) = \frac{\rho^k - \bar{\rho}^k}{\rho - \bar{\rho}}$$

for $\{\rho, \bar{\rho}\}$ solutions to $X^2 - tX + m = 0$. $H(n)$ is the "Hurwitz class number"

$$= \#\{Q \text{ pos. def. integral quad. form} : \operatorname{disc} Q = -N\}/\mathrm{SL}_2(\mathbf{Z})$$

each one counted with multiplicity 1 unless it is equivalent to $x^2 + y^2 \to \frac{1}{2}$, $x^2 + xy + y^2 \to \frac{1}{3}$ We will prove this and examine some consequences.

How would we calculate terms in this? It can be hard, the Hurwitz class number requires $\sqrt{D}$ time for $N = f^2 D$, D a fundamental discriminant. To get the trace we can integrate $\int K(x,x)$ for the kernel which gives us $T(m)$.

Towards the trace formula We will follow Zagier: Modular forms whose coefficients are Dirichlet series.

Normalization: Let $\{f_1, \dots, f_r\}$ be a basis for S_k. We will assume that they are all Hecke eigenfunctions and they are normalized by $a_1^i = 1$ (Hecke normalized).

$$f_i(z) = \sum_{i=1}^{\infty} a_n^i e(nx)$$

Note 2.2. f_is are pairwise orthogonal with respect to $\langle -, - \rangle_{\mathrm{Pet}}$

$$\left\langle f_i, f_j \right\rangle_{\mathrm{Pet}} = \int_{\Gamma \backslash \mathbf{H}} f_i \bar{f}_j y^k \frac{\mathrm{d}x\, \mathrm{d}y}{y^2}$$

why: say $a_m^i \neq 0$ for some m then

$$\left\langle f_i, f_j \right\rangle_{\mathrm{Pet}} = \frac{1}{a_m^i} \left\langle T(m) f_i, f_j \right\rangle_{\mathrm{Pet}}$$

$$= \frac{1}{a_m^i} \left\langle f_i, T(m) f_j \right\rangle_{\mathrm{Pet}}$$

$$= \frac{a_m^j}{a_m^i} \left\langle f_i, f_j \right\rangle_{\mathrm{Pet}}$$

Kernel function

Theorem 2.3. *$T(m)$ is an integral operator with the kernel*

$$h_m(z,z') = \sum_{ad-bc=m} \frac{1}{(czz'+dz'+az+b)^k} = \frac{1}{(cz+d)^k}\frac{1}{z' + \left(\frac{az+b}{cz+d}\right)^k}.$$

Note 2.4. Let's set $j_k(\gamma,z) = (cz+d)^{-k}$,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

$$F_k(z,z') = (z+z')^{-k}$$

so

$$h_m(z,z') = \sum_{\gamma\in\mathrm{Mat}_2(\mathbf{Z}),\,\det\gamma=m} j_k(\gamma,z)F_k(\gamma z,z').$$

Even more is true $h_m(z,z') = h_1(T_m(z),z')$.

Proof.

$$f * h_m(z') = \frac{c_k}{m^{k-1}}T_m(f)(z')$$

$c_k = (-1)^{k/2}\pi/2^{k-3}(k-1)$. set $m=1$ (this is enough because of our earlier observation).

$$f * h_1(z') = \int_{\Gamma\backslash\mathbf{H}} f(z)\overline{h_1(z,-\bar{z}')}y^k\,\mathrm{d}\mu(z)$$

$$\int_{\Gamma\backslash\mathbf{H}} f(z)\left\{\sum_{\gamma} j_k(\gamma,z)(-z'+\gamma z)^{-k}\right\} y^k\,\mathrm{d}\mu(y)$$

f is a modular form of weight k implies

$$f(\gamma z) = (cz+d)^k f(z)$$

$$\Im(\gamma z) = \frac{y}{|cz+d|^2}$$

so

$$\overline{(cz+d)^{-k}}f(z)y^k = f(\gamma z)\Im(\gamma z)^k. \tag{2.1}$$

this implies

$$\int_{\Gamma\backslash\mathbf{H}} f(\gamma z)\Im('gz)^i(-z'+\overline{\gamma z})^{-k}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$
$$= 2\int_0^\infty\int_{-\infty}^\infty f(z)y^k(-z'+\bar{z})^{-k}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

Cauchy integral formula implies

$$\int_{-\infty}^\infty \frac{f(x+iy)}{(x-iy-z')^k}\,\mathrm{d}x$$

$$2\pi i/(k-1)!f^{(k-1)}(2\pi iy)$$

$$4\pi i/(k-1)!\int_0^\infty f^{(k-1)}(2iy+z')y^{k-2}\,\mathrm{d}y$$

inner part is

$$1/(2i)^{k-2}\frac{\mathrm{d}^{k-2}}{\mathrm{d}t^{k-2}}f'(2ity+z')|_{t=1}$$

$$4\pi i/(k-1)!1/(2i)^{k-2}\frac{\mathrm{d}^{k-2}}{\mathrm{d}t^{k-2}}\int_0^\infty f'(2ity+z')\,\mathrm{d}y|_{t=1}$$

$$= -4\pi i/(k-1)!(2i)^{k-1}\frac{\mathrm{d}^{k-2}}{\mathrm{d}t^{k-2}}\frac{1}{t}f(z')|_{t=1}$$

$$= (i/2)^{k-1}4\pi i/(k-1)f(z') = c_k f(z').$$

□

Corollary 2.5.

$$c_k^{-1}m^{k-1}h_m(z,z') = \sum_{i=1}^{r}\frac{a_m^i}{\langle f_i, f_i\rangle_{\mathrm{Pet}}}f_i(z)f_i(z')$$

spectral decomposition.

$$\mathrm{Tr}(T_m) = \frac{c_k^{-1}}{m^{k-1}}\int_{\Gamma\backslash\mathbf{H}} h_m(z,\bar{z})\Im(z)^k\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

Proof. Expand $h_m(z,z')$ with respect to $\{f_1,\ldots,f_r\}$. $h_m(z,z')$ is a cusp form in both variables, we will show this later!

$$h_m(z,z') = \sum_{1\le i,j\le r}\alpha_{i,j}f_i(z)f_j(z'),\ \alpha_{i,j}\in\mathbf{C}$$

By the theorem

$$\frac{c_k^{-1}}{m^{k-1}}f * h_m(z') = T_m(f)(z')$$

let's apply this to f_{i_0} for any i_0 .. Next time. □

Lecture 13 20/3/2018

Summary:

Aim: to establish the Eichler-Selberg trace formula for the trace of $T_m \circlearrowleft S_k$, for T_m the mth Hecke operator

$$(T_m f)(z) = m^k\sum_{j=1}^{r} j_k(\gamma,z)f(\gamma_j z)$$

where γ_j are coset representatives for

$$\Gamma\backslash\{x\in\mathrm{Mat}_2(\mathbf{Z}) : \det(x) = m\}/\Gamma = \bigsqcup\Gamma\gamma_i$$

and $j_k(\gamma z) = (cz+d)^{-k}$,

$$\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbf{Z}).$$

$$h_m(z,z') = \sum_{ad-bc=m}(czz'+dz'+az+b)^{-k}$$

Theorem 2.6 (Petersson). *Let* $c_k = \frac{(-1)^{k/2}\pi}{2^{k-3}(k-1)}$.

1. *Then* $c_k^{-1} m^{k-1} h_m(z, -\bar{z}')$ *is the kernel for* $T_m \circlearrowleft S_k$. *I.e.*

$$\begin{aligned} c_k^{-1} m^{k-1} \int_{\Gamma\backslash\mathbf{H}} f(z)\overline{h_m(z,-\bar{z}')} y^k \, \mathrm{d}\mu(z) \\ = \left\langle f, c_k^{-1} m^{k-1} h_m(\cdot, -\bar{z}')\right\rangle_{\mathrm{Pet}} \\ = T_m(f)(z') \end{aligned}$$

for all f *cuspidal (so that the integral converges).*

2. $$m^{k-1} c_k^{-1} h_m(z,z') = \sum_{j=1}^{\dim S_k} \frac{a_j(m)}{\langle f_j, f_j\rangle_{\mathrm{Pet}}} f_j(z) f_j(z').$$

When $\{f_1, \ldots, f_{\dim(S_k)}\}$ *is a (Hecke normalized) orthogonal basis for* S_k

$$f_j(z) = \sum_{n=1}^{\infty} a_j(n) e(nz).$$

Proof. For the proof set $g_k(z,z') = (z+z')^{-k}$. This implies that

$$h_m(z,z') = \sum_{\gamma \in \mathrm{Mat}_z(\mathbf{Z}), \det(\gamma)=m} j_k(\gamma; z) g_k(\gamma z, z')$$

$$= \sum_{\gamma \in \mathrm{Mat}_z(\mathbf{Z}), \det(\gamma)=m} j_k(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\mathrm{T}} \gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; z) g_k(z, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\mathrm{T}} \gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z')$$

$$= \sum_{\gamma_1 \in \mathrm{Mat}_2(\mathbf{Z}), \det(\gamma_1)=m} j_k(\gamma z; z') g_k(z; \gamma_1 z')$$

which implies that the integral is well defined.

1. Reduce part 1. to the case of T_1.

$$T_m(h_1(z,z')) = \sum_{\gamma\in\Gamma} T_m(j_k(\gamma; z) g_k(z; z'))$$

$$\sum_{\gamma\in\Gamma} m^{k-1} \sum_{j=1}^{r} j_k(\gamma; \gamma_j z) g_k(\gamma\gamma_j z, z') j_k(\gamma_j, z)$$

$$\sum_{\gamma\in\Gamma} m^{k-1} \sum_{j=1}^{r} j_k(\gamma\gamma_j; z) g_k(\gamma\gamma_j z, z')$$

$$m^{k-1} \sum_{\gamma_1 \in \mathrm{Mat}_2(\mathbf{Z}), \det(\gamma_1)=m} j_k(\gamma; z) g_k(\gamma z, z') = m^{k-1} h_m(z,z').$$

For h_1 our calculation last time established

$$c_k^{-1} \int_{\Gamma\backslash\mathbf{H}} f(z)\overline{h_1(z,-\bar{z}')} y^k \, \mathrm{d}\mu(z) = f(z').$$

2. Fix z'. Implies

$$c_k^{-1} m^{k-1} h_m(z,z') = \sum_{j=1}^{\dim(S_k)} \frac{\alpha_j(z')}{\langle f_j, f_j \rangle_{\mathrm{Pet}}} \tag{2.2}$$

by the first part

$$c_k^{-1} m^{k-1} \left\langle f_k, h(\cdot, z') \right\rangle_{\mathrm{Pet}} = T_m(f_k)(z') = a_k(m) f_k(z') \tag{2.3}$$

as f_k is an eigenform. On the other hand by (2.2)

$$\begin{aligned} c_k^{-1} m^{k-1} \left\langle f_k, h_m(\cdot, -\bar{z}') \right\rangle &= c_k^{-1} m^{k-1} \sum_{j=1}^{\dim S_k} \left\langle f_k, \frac{\alpha_j(-\bar{z}') f_j}{\langle f_j, f_j \rangle_{\mathrm{Pet}}} \right\rangle_{\mathrm{Pet}} \\ &= c_k^{-1} m^{k-1} \sum_{j=1}^{\dim S_k} \frac{\overline{\alpha_j(-\bar{z}') f_j}}{\langle f_j, f_j \rangle_{\mathrm{Pet}}} \langle f_k, f_j \rangle_{\mathrm{Pet}} \\ &= c_k^{-1} m^{k-1} \sum_{j=1}^{\dim S_k} \overline{\alpha_j(-\bar{z}')} \frac{\langle f_k, f_k \rangle_{\mathrm{Pet}}}{\langle f_k, f_k \rangle_{\mathrm{Pet}}} \end{aligned}$$

combined with (2.3) this implies

$$a_k(m) \overline{f_k(z')} = c_k^{-1} m^{k-1} \alpha_k(-\bar{z}')$$

Note:

$$\begin{aligned} \overline{f_k(z')} &= \overline{\sum_{n=1}^{\infty} a_k(n) e(nz)} \\ &= \sum_{n=1}^{\infty} a_k(n) \overline{e(nz)} \\ &= \sum_{n=1}^{\infty} a_k(n) e(-n\overline{z}') \end{aligned}$$

implies

$$a_k(m) f_k(z') = c_k^{-1} m^{k-1} \alpha_k(z')$$

so

$$c_k^{-1} m^{k-1} h_m(z,z') = \sum_{j=1}^{\dim S_k} \frac{a_j(m)}{\langle f_j, f_j \rangle_{\mathrm{Pet}}} f_j(z) f_j(z'). \qquad \square$$

Corollary 2.7.

$$c_k^{-1} m^{k-1} \int_{\Gamma \backslash \mathbf{H}} h_m(z, -\bar{z}') y^k \, \mathrm{d}\mu(z) = \mathrm{Tr}(T(m)).$$

Proof.

$$\begin{aligned} &\int_{\Gamma \backslash \mathbf{H}} \sum_{j=1}^{\dim S_k} \frac{a_j(m)}{\langle f_j, f_j \rangle_{\mathrm{Pet}}} f_j(z) \overline{f_j(z)} y^k \, \mathrm{d}\mu(z) \\ &= \sum_{j=1}^{\dim(S_k)} a_j(m) \frac{\langle f_j, f_j \rangle_{\mathrm{Pet}}}{\langle f_j, f_j \rangle_{\mathrm{Pet}}} = \mathrm{Tr}(T(m)). \qquad \square \end{aligned}$$

Subtle point (in general): This integral converges, and gives a manageable expression.

2.2 Zagier's approach

$$I(s) = \int_{\Gamma\backslash\mathbf{H}} h_m(z, -\bar{z}')E(z;s)y^k \,\mathrm{d}\mu(z)$$

for $\Re(s) > 1$. (This "goes around the convergence issues"). Implies

$$\mathrm{Res}_{s=1} I(s) = \mathrm{Tr}(T_m)\frac{c_k}{m^{k-1}}E(z;s) = \frac{1}{2}\zeta(2s)\pi^s\Gamma(s) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s.$$

Theorem 2.8.

1. *$I(s)$ is absolutely convergent for $\Re(s) > 1$.*
2. $$I(s) = \sum_{\Delta\neq 0} \zeta(s,\Delta)\{archimidean\} + \zeta(s,0)\{archimidean\} + \frac{(-1)^{k/2}\Gamma(s+k-1)\zeta(s)\zeta(2s)}{(2\pi)^{s-1}\Gamma(k)}$$

   $\zeta(s,\Delta)$ is a cousin of $\zeta_\Delta(s) = \sum_{I_k} \frac{1}{N(I_k)^s}$ for $K = \mathbf{Q}(\sqrt{-\Delta})$.

Lecture 14 22/3/2018

Calculation of $I(s)$ First step:

Recall that there exists a one to one correspondence between (for fixed m, t)

$$\{A \in \mathrm{Mat}_{2\times 2}(\mathbf{Z}) : \mathrm{Tr}(A) = t,\ \det(A) = m\}$$

$$\updownarrow$$

$$\{\phi(u,v) = au^2 + buv + cv^2 : \Delta_\phi = |\phi| = b^2 - 4ac = t^2 - 4m\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \phi(u,v) = cu^2 + (d-a)uv - b^2v^2$$

$$\phi(u,v) = au^2 + buv + cv^2 \mapsto \begin{pmatrix} \frac{1}{2}(t-b) & -c \\ a & \frac{1}{2}(t+b) \end{pmatrix}.$$

Using this

$$y^k h_m(z, -\bar{z}) = \sum_{a,b,c,d\in\mathbf{Z},\, ad-bc=m} \frac{y^k}{(c|z|^2 + d\bar{z} - az - b)^k}$$

$$= \sum_{t\in\mathbf{Z}} \sum_{ad-bc=m,\, a+d=t} \frac{y^k}{(c|z|^2 + d\bar{z} - az - b)^k}$$

$$= \sum_{t\in\mathbf{Z}} \sum_{\phi,\, |\phi|=t^2-4m} R_\phi(z,t)$$

where

$$R_\phi(z,t) = \frac{y^k}{(a|z|^2 + bx + c - ity)^k}.$$

Exercise: substitute $a = \frac{1}{2}(t-b), b = -c, c = a, d = \frac{1}{2}(t+b)$.

Proposition 2.9. *For $s \in \mathbf{C}$ s.t. $2-k < \Re(s) < k-1$*

$$\sum_{t\in\mathbf{Z}} \int_{\Gamma\backslash\mathbf{H}} |E(z;s)| \left| \sum_{|\phi|=t^2-4m} R_\phi(z,t) \right| \mathrm{d}\mu(z) < \infty$$

Remark 2.10. It will turn out that the t-sum is finite.

Recall that $\mathrm{SL}_2(\mathbf{Z})$ acts on binary quadratic forms Φ by

$$\gamma\phi(u,v) = \phi(au+cv, bu+dv) = \phi(\gamma^{\mathrm{T}} \begin{pmatrix} u \\ v \end{pmatrix}).$$

Theorem 2.11 (Geometric side of the Eichler-Selberg trace formula)**.** *For t, m fixed*

$$\int_{\Gamma\backslash\mathbf{H}} \left(\sum_{|\phi|=t^2-4m} R_\phi(z,t) \right) E(z,s)\,\mathrm{d}\mu(z) = \zeta(s,\Delta)\left\{ I(\Delta,t,s) + I(\Delta,-t,s) \right\}$$

$$+ \begin{cases} (-1)^{k/2} \frac{\Gamma(s+k-1)\zeta(s)\zeta(2s)}{(2\pi)^{s-1}\Gamma(k)} & \Delta = 0 \\ 0 & \Delta \neq 0 \end{cases}$$

for $\Delta = t^2 - 4m$.

Where

$$\zeta(s,\Delta) = \sum_{\phi \pmod{\Gamma},\, |\phi|=\Delta} \; \sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0\}/\mathrm{Aut}(\phi),\, \phi(m,n)>0} \frac{1}{\phi(m,n)^s}$$

$$\mathrm{Aut}(\phi) = \{\gamma \in \mathrm{SL}_2(\mathbf{Z}) : \gamma\phi = \phi\}$$

$$\begin{aligned} I(\Delta,t,s) &= \int_0^\infty \int_{-\infty}^\infty \frac{y^{k+s-2}}{(x^2+y^2+ity-\frac{1}{4}\Delta)^k}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k)} \int_0^\infty \frac{y^{k+s-2}}{(y^2+ity-\frac{1}{4}\Delta)^{k-\frac{1}{2}}}\,\mathrm{d}y. \end{aligned}$$

Remark 2.12. If $\Delta = \mathrm{disc}(K/\mathbf{Q})$ then

$$\zeta(s,\Delta) = \zeta_K(s).$$

Then we have

$$\sum_t \mathrm{Res}_{s=1}\, \zeta_K(s)$$

where $K = \mathbf{Q}(\sqrt{t^2-4m})$.

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Proposition 2.13. *Let $\zeta(s,\Delta)$ be as above and $\Re(s) > 1$. Then*

1.

$$\zeta(s,\Delta) = \zeta(2s) \sum_{a=1}^\infty \frac{n(a)}{a^s}$$

wn(ae

$$a(n) = \#\{b \pmod{2a} : b^2 \equiv \Delta \pmod{4a}\}.$$

2. *$\zeta(s,\Delta)$ has meromorphic continuation and functional equation.*

$$\gamma(s,\Delta)\zeta(s,\Delta) = \gamma(1-s,\Delta)\zeta(1-s,\Delta)$$

where

$$\gamma(s,\Delta) = \begin{cases} (2\pi)^{-s}|\Delta|^{s/2}\Gamma(s) & \Delta < 0 \\ \pi^{-s}\Delta^{s/2}\Gamma(s/2) & \Delta > 0 \end{cases}.$$

3.

$$\zeta(s,\Delta) = \begin{cases} 0 & \Delta \equiv 2,3 \pmod 4 \\ \zeta(s)\zeta(2s-1) & \Delta = 0 \\ \zeta(s)L_D(s)\sum_{d|f}\mu(d)\frac{D}{d}d^{-s}\sigma_{1-2s}(\frac{f}{d}) \end{cases}$$

for D the fundamental discriminant, so that $\Delta = Df^2$ and $\sigma_\nu(n) = \sum_{d|n} d^\nu$, μ is the Möbius function, and $L_D(s) = L(s, \left(\frac{D}{\cdot}\right))$.

4.

$$\operatorname{Res}_{s=1}\zeta(s,\Delta) = \frac{\pi}{\sqrt{|\Delta|}}H(|\Delta|).$$

Proof.

1. A slick proof involves noting that

$$n(a) = \#\{\phi\in\Phi,\ (m,n)\in\mathbf{Z}^2 : \phi(m,n) = 0, |\phi| = \Delta,\ \gcd(m,n) = 1\}$$

comes from the theory of quadratic forms.

Instead note that $\Gamma \circlearrowleft \Phi$ via $\gamma\cdot\phi = (u,v) = \phi(\gamma^{\mathrm{T}}\begin{pmatrix}u\\v\end{pmatrix})$. We also have $\Gamma \circlearrowleft \mathbf{Z}^2$ via $\gamma\cdot(m,n) = \gamma\begin{pmatrix}m\\n\end{pmatrix}$.

Note that

$$(\gamma^{\mathrm{T}})^{-1} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\gamma\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

(check!).

Define

$$\begin{aligned} \langle -,-\rangle &: \Phi\times X \\ \big\langle\phi,(m,n)\big\rangle &= \phi(n,-m) \\ &= \phi\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}m\\n\end{pmatrix}\right). \end{aligned}$$

Observe $\langle -,-\rangle$ is Γ-invariant! i.e.

$$\big\langle\phi,(m,n)\big\rangle = \big\langle\gamma\phi,\gamma(m,n)\big\rangle$$

for all $\gamma\in\Gamma$. Then

$$\begin{aligned} \zeta(s,\Delta) &= \sum_{\phi\pmod\Gamma,\,|\phi|=\Delta}\ \sum_{(m,n)\in X/\mathrm{Aut}(\phi),\,\phi(m,n)>0}\frac{1}{\phi(m,n)^s},\ \Re(s) > 1 \\ &= \sum_{\phi\in\Phi_\Delta/\Gamma}\sum_{x\in X/\mathrm{Aut}(\phi)}\frac{1}{\big\langle\phi,x\big\rangle^s} \end{aligned}$$

$$= \sum_{(\phi,x)\in(\Phi_\Delta\times X)/\Gamma} \frac{1}{\langle \phi, x\rangle^s}$$

$$= \sum_{x\in X/\Gamma} \sum_{\phi\in\Phi_\Delta/\mathrm{Aut}(x)} \frac{1}{\langle \phi, x\rangle^s} \tag{2.4}$$

we are using Fubini to swap the order of summation as the pairing is invariant. Finally note that

$$x = \pm(m,n) \sim_\Gamma \pm(0, \gcd(m,n)),\ r = \gcd(m,n) > 0$$

so (2.4)

$$= \sum_{r=1}^{\infty} \sum_{\phi\in\Phi_\Delta/\mathrm{Aut}((r,0))} \frac{1}{\phi(r,0)^s}$$

what is $\mathrm{Aut}((r,0))$? It is

$$\Gamma_\infty^{\mathrm{T}} = \left\{\begin{pmatrix}1 & 0\\ * & 1\end{pmatrix}\right\} \text{ as } \begin{pmatrix}1 & 0\\ * & 1\end{pmatrix}\begin{pmatrix}0\\ r\end{pmatrix} = \begin{pmatrix}0\\ r\end{pmatrix}$$

continuing we have

$$\zeta(2s)\sum_{a=1}^{\infty}\frac{1}{a^s}\left\{\sum_{b \pmod{2a},\, b^2\equiv\Delta \pmod{4a}} 1\right\}$$

Exercise, show this is the sum from before. Hint, consider, explicitly, the action of $\gamma \in \Gamma_\infty^{\mathrm{T}}$ on $\phi(u,v)$.

□

Back to the proof of the trace formula.

Lemma 2.14. *Let $k \in \mathbf{Z}_{>2}$, $k \equiv 0 \pmod 2$, $\Delta \in \mathbf{Z}$, $\Delta \equiv 0, 1 \pmod 4$, $t \in \mathbf{R}$ s.t. $t^2 > \Delta$. For each $\phi \in \Phi_\Delta$ let*

$$R_\phi(t,z) = \frac{y^k}{(a|z|^2 + bx + c - ity)^k}$$

Then for $s \in \mathbf{C}$, $s \neq 1$, $1-k < \Re(s) < k$.

$$\int_{\Gamma\backslash\mathbf{H}} \left(\sum_{|\phi|=\Delta} R_\phi(t,z)\right) E(z,s)\, d\mu(z)$$

$$= \zeta(s,\Delta)\{I_k(\Delta,t,s) + I_k(\Delta,-t,s)\}$$
$$+ \begin{cases} (-1)^{k/2}\frac{\Gamma(s+k-1)\zeta(s)\zeta(2s)}{(2\pi)^{s-1}\Gamma(k)} & \Delta = 0\\ 0 & \Delta \neq 0\end{cases}$$

Proof.

1. Note $R_{\gamma\phi}(t,z) = R_\phi(t,\gamma^{\mathrm{T}}z)$ which means that the integral is well defined (check this).

2.
$$\int_{\Gamma\backslash\mathbf{H}}\left(\sum_{|\phi|=\Delta} R_\phi(t,z)\right)E(z;s)\,\mathrm{d}\mu(z)$$
$$= \int_{\Gamma\backslash\mathbf{H}}\sum_{|\phi|=\Delta}\sum_{(m,n)\in\mathbf{Z}^2\smallsetminus\{(0,0)\}} R_\phi(t,z)\frac{y^s}{|mz+n|^{2s}}\,\mathrm{d}\mu(z)$$
unfold whenever you can!

3. Recall $\Gamma \circlearrowright \Phi_\Delta \times X$ fixing $\langle\phi,(m,n)\rangle = \phi(m,n)$. We can break the sum into $\phi(m,n) > 0$, $\phi(m,n) < 0$.

4. Fact: The action of Γ on the above is free, (exercise: prove this fact, hint: first take $(m,n) \to (0,r)$, and analyse the action of the stabiliser on $(0,r)$).

5.
$$\int_{\Gamma\backslash\mathbf{H}}\sum_{|\phi|=\Delta}\sum_{(m,n)\in X,\,\phi(m,n)>0} R_\phi(t,z)\frac{y^s}{|mz+n|^{s}}\,\mathrm{d}\mu(z)$$
$$\sum_{(\Phi_\Delta\times X)/\Gamma}\int_{\mathbf{H}} R_\phi(t,z)\frac{y^s}{|mz+n|^{2s}}\,\mathrm{d}\mu(z).$$

6. Check
$$z \mapsto \frac{nx - \frac{1}{2}bn + cm}{-mz + an - \frac{1}{2}bm}$$
finishes the proof.

7. To check other cases use
$$R_{-\phi}(z,t) = R_\phi(z,-t)$$
and
$$\langle\phi,(m,n)\rangle = 0$$
needs to be considered separately.

□

We really need to check some convergence here, but lets instead use the trace formula for something useful, to determine its value.

Application to spaces of modular forms

Theorem 2.15.
$$\left\lfloor\frac{k}{12}\right\rfloor - \mathbf{1}[k \equiv 2 \pmod{12}] = -\frac{1}{2}\sum_{t=-2}^{t=2} P_k(t,1)H(4-t^2) - \frac{1}{2}$$
where
$$H(n) = \textit{Hurwitz class number}$$
$$P_k(t,m) = \frac{\rho^{k-1} - \bar\rho^{k-1}}{\rho - \bar\rho},\ \{\rho,\bar\rho\}\ \textit{solutions to}\ u^2 - tu + m = 0$$
then
$$\dim(S_k(1)) = \left\lfloor\frac{k}{12}\right\rfloor - \mathbf{1}[k \equiv 2 \pmod{12}].$$
This is a weird looking formula but it generalises very well, works even without a complex structure, Riemann-Roch, etc.

Proof. Recall that

$$\mathrm{Tr}(T_{S_k}(m))$$

is given by the Eichler-Selberg trace formula. In particular

$$\begin{aligned}\mathrm{Tr}(T_{S_k}(m)) &= \dim(S_k)\\ &= -\frac{1}{2}\sum_{t^2-4\leq 0} P_k(t,1)H(4-t^2) - \frac{1}{2}\end{aligned}$$

we have

$$\begin{aligned}H(0) &= \frac{-1}{12}\\ H(3) &= \frac{1}{3}\\ H(4) &= \frac{1}{2}\end{aligned}$$

to find the $P_k(t,1)$

$$\rho_t, \bar{\rho}_t = \frac{t \pm \sqrt{t^2-4}}{2} = e^{i\theta}, e^{-i\theta}$$

so $\tan\theta = \frac{\sqrt{4-t^2}}{t}$ note then

$$\frac{\rho_t^{k-1} - \bar{\rho}_t^{k-1}}{\rho_t - \bar{\rho}_t} = \frac{\rho_{-t}^{k-1} - \bar{\rho}_{-t}^{k-1}}{\rho_{-t} - \bar{\rho}_{-t}} = \frac{\sin((k-1)\theta)}{\sin(\theta)}$$

so $P_k(t,1)$ is as in the table

t	θ	$P_k(t,1)$
0	$\pi/2$	$k \mod 4 - 1$
1	$\pi/3$	$(k \mod 6 - 4)/2)$
2	0	$(k-1)$

Table 2.16: Values of $P_k(t,1)$.

Now if $k \equiv 0 \pmod{12}$

$$-\left(\frac{-1}{4} - \frac{1}{3} - \frac{k-1}{12}\right) - \frac{1}{2} = \frac{k}{12} = \left\lfloor\frac{k}{12}\right\rfloor = \cdots.$$ □

Another application: Equidistribution of Hecke eigenvalues.

Lecture 16 29/3/2018

Following Serre '97, very readable article.
Equidistribution:

Definition 2.17. Let (Ω, μ) be a measure space

$$\Omega \text{ compact}$$

$$\mu \text{ positive Radon measure}$$

$$\int_\Omega \mathrm{d}\mu = 1$$

Example 2.18.

$$[-1,1],\ \frac{\mathrm{d}x}{2} = \mathrm{d}\mu$$

Note 2.19. μ gives a linear functional.

$$f \mapsto \int_\Omega f(x)\,\mathrm{d}\mu(x),$$

notated

$$\langle f, \mu \rangle .$$

Definition 2.20 (δ measures)**.** Let L be a sequence of indices, going to ∞ (this will be the indexing set for the weight k). For $\lambda \in L$ let I_λ be a non-empty finite set of cardinality

$$\# I_\lambda = d_\lambda$$

(this will be the set of eigenvalues for each k). Let

$$\mathbf{X}_\lambda = (x_{i,\lambda}),\ i \in I_\lambda .$$

We define

$$\delta_{\mathbf{X}_\lambda} = \frac{1}{d_\lambda} \sum_{i \in I_\lambda} \delta_{x_{i,\lambda}}$$

note

$$\langle f, \delta_{\mathbf{X}_\lambda} \rangle = \frac{1}{d_\lambda} \sum_{i \in I_\lambda} f(x_{i,\lambda}).$$

The family $\mathbf{X}_\lambda$ for $\lambda \in L$ is called μ-equidistributed (or equidistributed with respect to the measure μ) if

$$\lim_{\lambda \to \infty} \delta_{\mathbf{X}_\lambda} = \mu$$

the limit is in the weak-$*$ sense i.e.

$$\lim_{\lambda \to \infty} \langle f, \delta_{\mathbf{X}_\lambda} \rangle = \langle f, \mu \rangle ,\ \forall f \in C(\Omega, \mathbf{R}).$$

So far these are all general statements. Let us put ourselves in the following situation:

For each $\lambda \in L$ we have a linear operator

$$H_\lambda,\ \mathrm{rank}(H_\lambda) = d_\lambda < \infty$$

whose eigenvalues are in Ω. And $I_\lambda = \{1, 2, \dots, d_\lambda\}$, $\mathbf{X}_\lambda = \{x_{1,\lambda}, \dots, x_{d_\lambda,\lambda}\}$ the eigenvalues of H_λ suitably normalised counted with multiplicity.

Proposition 2.21. *TFAE*

1. $$\{x_\lambda\}_{\lambda \in L} \textit{ is } \mu \textit{ equidistributed on } \Omega$$

2. *For all polynomials P with $\mathbf{R}$ coefficients*

$$\frac{\mathrm{Tr}(P(H_\lambda))}{d_\lambda} \to_{\lambda \to \infty} \langle P, \mu \rangle \tag{2.5}$$

3. *For all $m \in \mathbf{Z}_{\geq 0}$ there exists a polynomial P of degree m s.t. (2.5) is satisfied.*

Proof. Exercise. □

Now we will choose a special set of polynomials to work with.

Symmetric polynomials:

Let $\Omega = [-2,2] \leftrightarrow 2\cos[-\pi,\pi] = \mathrm{Tr}(\mathrm{SU}(2,\mathbf{C}))$, this is really Satake parameters. For $x \in [-2,2]$ we have a corresponding $2\cos(\phi)$. Let

$$X_n(x) = e^{in\phi} + e^{i(n-2)\phi} + \cdots + e^{-in\phi} = \mathrm{Tr}(\mathrm{Sym}^n(U)),\ x = \mathrm{Tr}(u).$$

Clebesh-Gordon: (Decomposition of tensor powers of irreducible representations of $\mathrm{SU}(2,\mathbf{C})$.

$$X_n X_m = \sum_{0\le r\le \min\{n,m\}} X_{n+m-2r} = x_{n+m} + x_{n+m-2} + \cdots + x_{|n-m|}.$$

Exercise 2.22. Prove this.

Several measures:

1. Sato-Tate:
$$\begin{aligned}\mu_\infty &= \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}\,\mathrm{d}x\\ &= \frac{2}{\pi}\sin^2(\phi)\,\mathrm{d}\phi\end{aligned}$$
properties
$$\int_\Omega \mathrm{d}\mu_\infty = 1$$
$$\left\langle X_n, \mu_\infty\right\rangle = \begin{cases}1 & n=0,\\ 0 & \text{otw}\end{cases}$$
$$\left\langle X_n X_m, \mu_\infty\right\rangle = \delta_{n,m}.$$
Exercise: prove this, with orthogonality of characters.

2. p-adic Plancherel measure: Let $q \in \mathbf{R}_{>1}$ and define
$$f_q(x) = \sum_{n=0}^{\infty} q^{-n} X_{2n}(x) = \frac{q+1}{(q^{1/2}+q^{-1/2})^2 - x^2}$$
$$\mu_q = f_q \mu_\infty$$
note:
$$\lim_{q\to 1^+} \mu_q = \frac{\mathrm{d}\phi}{\pi}$$
$$\lim_{q\to\infty} \mu_q = \mu_\infty$$
μ_q is positive and has total mass 1. note:
$$\begin{aligned}\left\langle X_n, \mu_q\right\rangle &= \left\langle X_n, \sum_{m=0}^{\infty} q^{-m} X_{2m}\mu_\infty\right\rangle\\ &= \sum_{m=0}^{\infty} q^{-m}\left\langle X_n X_{2m}, \mu_\infty\right\rangle\\ &= \begin{cases} q^{-n/2} & n\equiv 0 \pmod 2\\ 0 & \text{otw}\end{cases}.\end{aligned}$$

Theorem 2.23 (Serre '97)**.** *$k \equiv 0 \pmod 2$ let $s(k) = \dim(S_k(1))$*

$$T'_k(n) = \frac{T_k(n)}{n^{k-1/2}}$$

so that the eigenvalues are in $[-2, 2]$ by Deligne. Letting $\mathbf{X}(k, p)$ be the eigenvalues of $T'_k(p)$. Then $\mathbf{X}(k, p)$ are equidistributed with respect to μ_p.

$$\mu_p = \frac{p+1}{(p^{1/2} + p^{-1/2})^2 - x^2} \frac{1}{\pi} \sqrt{1 - \frac{x^2}{4}}\, dx.$$

Proof. Claim:

$$\lim_{k\to\infty, k\equiv 0 \pmod 2} \frac{\operatorname{tr}(T'_k(n))}{s(k)}$$

$$= \lim_{k\to\infty, k\equiv 0 \pmod 2} \frac{\operatorname{tr}(T'_k(n))}{(k-1)/12} = \begin{cases} n^{-1/2} & \text{if } n \text{ is a square} \\ 0 & \text{otw} \end{cases}$$

Assume the claim for now: Note that

$$T'_k(p^m) = X_m(T'_k(p))$$

as

$$\sum_{m=0}^{\infty} T'_k(p^m) t^m = \frac{1}{1 - T'_k(p)t + t^2}$$

$$\sum_{m=0}^{\infty} X_m(T'_k(p)) t^m$$

$$= \lim_{k\to\infty, k\equiv 0 \pmod 2} \frac{\operatorname{tr}(T'_k(n))}{s(k)} = \begin{cases} p^{-m/2} & m \equiv 0 \pmod 2 \\ 0 & \text{otw} \end{cases}$$

$$= \langle X_m, \mu_p \rangle$$

by the claim and previous calculation.

Proof of the claim: Recall that the Eichler-Selberg trace formula says that

$$\operatorname{tr}(T_k(n)) = \frac{k-1}{12} n^{k/2-1} \delta_{n=\square} \tag{2.6}$$

$$-\frac{1}{2} \sum_{t^2 < 4n} P_k(t, n) H(4n - t^2) \tag{2.7}$$

$$-\frac{1}{2} \sum_{dd'=n,\, d,d'>0} \min(d, d')^{k-1} \tag{2.8}$$

Sublemma: (2.7) $= O(n^{k/2})$
Proof: (2.7) $= O_n(\sum_{t^2<4n} P_k(t, n))$

$$P_k(t, n) = \frac{\rho^{k-1} - \bar\rho^{k-1}}{\rho - \bar\rho}, \ \rho = \frac{t + i\sqrt{4n - t^2}}{2}$$

$$|\rho^{k-1} - \bar\rho^{k-1}| \le 2|\rho^{k-1}| = O((4n - t^2)^{(k-1)/2})$$

$$\rho - \bar\rho = \sqrt{n - t^2}$$

Sublemma: (2.8) $= O(n^{(k-1)/2})$ Proof: exercise.
The equations above combined with the fact $n^{k-1/2} \to n^{-1/2}\delta_{n=\square}$. ◻

3 Tate's Thesis (GL_1 theory)

What's the aim?

Lecture 17 3/4/2018

Redo Hecke's work, in an adelic setting, more canonically. I.e. obtain analytic continuation and functional equation of Hecke L-functions. This was also seemingly a bit more general than the Hecke theory. The local theory will give us terms like

$$(1 - \frac{\chi(p)}{p^r})^{-1}$$

and the global theory will give

$$\prod_p (1 - \frac{\chi(p)}{p^r})^{-1}.$$

3.1 Local theory

For all of today, K is a number field and v a place of K so that

$$K_v = \begin{cases} \mathbf{R}, \mathbf{C} & \text{if } v \text{ is archimidean} \\ [K_v : \mathbf{Q}_v] < \infty & \text{if } v \text{ is non-archimidean} \end{cases}.$$

If $\alpha \in K_v$ then

$$|\alpha|_v = \begin{cases} |\alpha|_{\mathbf{C}}^2 & \text{if } K_v = \mathbf{C} \\ |\alpha|_{\mathbf{R}} & \text{if } K_v = \mathbf{R} \\ N(\varpi_v)^{-v_v(\alpha)} & \text{if } v \text{ is non-archimidean} \end{cases}$$

where $|N(\varpi_v) = |\mathcal{O}_v/\varpi\mathcal{O}_v|$.

3.1.1 Additive theory (Theory of Fourier transform)

Let K_v^+ denote the additive group of K_v.

Characters of K_v^+ (continuous)

Lemma 3.1. *Let χ_v be a non-trivial additive character of K_v^+, i.e.*

$$\chi_v \colon K_v^+ \to \mathbf{C}^\times$$

then all the characters of K_v^+ are of the form

$$\chi_v(\eta\cdot) = \chi_{v,\eta}$$

for $\eta \in K_v^+$ i.e.

$$\chi_{v,\eta}(\xi) = \chi_v(\eta\xi)$$

which implies

$$K_v^+ \simeq \widehat{K}_v^+$$

both topologically and algebraically.

Proof. Exercise. □

This is not really canonical, but as far as number theorists are concerned there is a right choice of additive character to fix.

A particular non-trivial additive character If $K_v = \mathbf{R}, \mathbf{C}$ let

$$\lambda_0 \colon \mathbf{R} \to \mathbf{R}$$

$$\xi \mapsto -\xi \pmod 1$$

then

$$\chi_v(\xi) = e(\lambda(\xi)) = e^{2\pi i \lambda(\xi)}$$

where

$$\lambda(\xi) = \lambda_0(\mathrm{tr}_{K_v/\mathbf{R}}(\xi)).$$

For K_v non-archimidean let

$$\lambda_0 \colon \mathbf{Q}_v \to \mathbf{R}$$

$$\xi \mapsto \xi \pmod 1$$

i.e. if

$$\xi = a_{-N}\varpi_v^{-N} + \cdots + a_{-1}\varpi_v^{-1} \underbrace{+\cdots}_{\in O_v}$$

$$\lambda_0(\xi) = a_{-N}\varpi_v^{-N} + \cdots + a_{-1}\varpi_v^{-1}$$

$$\chi_v(\xi) = e(\lambda(\xi))$$

where

$$\lambda(\xi) = \lambda_0(\mathrm{tr}_{K_v/\mathbf{Q}_v}(\xi)).$$

Lemma 3.2. *Let v be non-archimidean*

$$\chi_{v,\eta}|_{O_v} \equiv 1 \iff \eta \in D_v^{-1}$$

where D_v is the different ideal of K_v

$$D_v^{-1} = \{x \in K_v : \mathrm{tr}(xy) \in \mathbf{Z}_v, \ \forall y \in O_v\}.$$

Haar measures

- $\mathbf{C}$, μ_v is 2 times the Lebesgue measure.
- $\mathbf{R}$, μ_v is the Lebesgue measure.
- K_v, $\mu_v(O_v) = N(D_v)^{-1/2}$.

Lemma 3.3. $\alpha \neq 0$ *implies*

$$\mu_v(\alpha M) = |\alpha|_v \mu(M)$$

for all measurable sets M.

Proof. Exercise ∎

I.e.

$$\int f(x)\,\mathrm{d}\mu(x) = |\alpha|_v \int f(\alpha x)\,\mathrm{d}\mu(x).$$

Definition 3.4 (Fourier Transform)**.**

$$\hat{f}(\eta) = \int f(\xi) e(-\lambda_v(\eta\xi))\,\mathrm{d}\mu_v(\xi).$$

Theorem 3.5 (Fourier inversion)**.** *Let $f \in L^1(K_v^+)$ such that $\hat{f} \in L^1(K_v^+)$, then*

$$f(\xi) = \int \hat{f}(\eta) e(-\lambda_v(\eta\xi))\,\mathrm{d}\mu(\eta) = \hat{\hat{f}}(-\xi).$$

3.1.2 Multiplicative theory

Definition 3.6 (Unit group)**.**

$$U_v = \ker(x \mapsto |x|_v)$$

it is compact, open if v is non-archimidean, e.g.

$$U_v = \begin{cases} S^1 & \text{if } K_v = \mathbf{C} \\ S^1 & \text{if } K_v = \mathbf{R} \\ O_v^\times & \text{if } v \text{ non-archimidean} \end{cases}.$$

Definition 3.7. A **quasicharacter** is

$$K_v^\times \to \mathbf{C}^\times$$

A (unitary) character is

$$K_v^\times \to S^1.$$

Such a map is unramified if it is trivial on U_v. E.g.

$$\xi \mapsto |\xi|_v^s$$

is unramified $s \in \mathbf{C}$.

Lemma 3.8. *All unramified characters are of this form*

$$\widehat{K}_{\mathrm{nr}}^\times = \mathbf{C}/(2\pi i/\log(N(\varpi_v)))$$

if v is non-archimidean.

Choose a uniformiser ϖ_v s.t.

$$\begin{array}{ccc} \alpha & \tilde{\alpha} & \varpi_v^{v_v(\alpha)} \\ \mathbf{C}^\times & S^1 & \mathbf{R}_+ \\ \mathbf{R}^\times & \{\pm 1\} & \mathbf{R}_+ \\ K_v^\times & O_v^\times & \mathbf{Z} \end{array}.$$

Theorem 3.9. *All quasicharacters of $K_v^\times$ are of the form*

$$\alpha \mapsto c(\alpha) = \tilde{c}(\tilde{\alpha})|\alpha|^s$$

where

$$\tilde{c} \colon U_v \to \mathbf{C}^\times$$

Proof. Exercise. ◻

Example 3.10. Dirichlet characters mod p.

E.g. over $\mathbf{C}^\times$ then

$$\tilde{c}(\alpha) = \left(\frac{\alpha}{|\alpha|}\right)^m, \ m \in \mathbf{Z}$$

over $\mathbf{R}^\times$ then

$$\tilde{c}(\alpha) = \left(\frac{\alpha}{|\alpha|}\right)^m, \ m \in \{0, 1\}$$

over $O_v^\times$ then

$$\tilde{c}(\alpha)|_{1+\varpi^k O_v} \equiv 1$$

because c is a continuous map from a p-adic field to the complex numbers. Let

$$\tilde{k} = \min\{k \in \mathbf{N} : \tilde{c}(\alpha)|_{1+\varpi_v^k O_v} \equiv 1\}$$

and $\varpi_v^{\tilde{k}} = f_v$ is the conductor of $\tilde{c}$

Note 3.11. Sometimes this $\tilde{k}$ is called the ramification degree of c.

Note 3.12. If $c = \tilde{c}|\cdot|^s$ then $\Re(s) = \sigma$ is uniquely determined by c. It is called the exponent of c. In modern lingo this is measuring how non-tempered c is.

Multiplicative Haar measures Let

$$\mathrm{d}_v^\times \alpha = \begin{cases} \frac{\mathrm{d}_v\alpha}{|\alpha|_v} & \text{if } v \text{ archimidean} \\ \left(\frac{1}{1-1/N(\varpi_v)}\right)\frac{\mathrm{d}_v\alpha}{|\alpha|_v} & \text{if } v \text{ non-archimidean} \end{cases}$$

where $\mathrm{d}_v\alpha$ is the additive Haar measure. these extra factors are really Tamagawa numbers. They make the product in the next lemma converge.

Lemma 3.13. *For v non-archimidean*

$$\int_{O_v^\times} \mathrm{d}_v^\times \alpha = N(D_v)^{-1/2}.$$

Proof.

$$\begin{aligned} \int_{O_v^\times} \mathrm{d}_v^\times \alpha &= \int_{O_v^\times} \frac{\mathrm{d}_v\alpha}{|\alpha|_v}(1 - N(\varpi_v)^{-1})^{-1} \\ &= \int_{O_v^\times} \mathrm{d}_v\alpha(1 - N(\varpi_v)^{-1})^{-1} \\ &= \sum_{\beta \in (O_v/\varpi_v O_v)^\times} \int_{\beta + \varpi_v O_v} \mathrm{d}_v\alpha(1 - N(\varpi_v)^{-1})^{-1} \\ &= |\varpi_v|_v \sum_{\beta \in (O_v/\varpi_v O_v)^\times} \int_{O_v} \mathrm{d}_v\alpha(1 - N(\varpi_v)^{-1})^{-1} \\ &= N(D_v)^{-1/2}|\varpi_v|(N(\varpi_v) - 1)(1 - N(\varpi_v)^{-1})^{-1} = N(D_v)^{-1/2}. \end{aligned}$$

□

Lecture 18 5/4/2018We are trying to set up a general machinery that will take a quasicharacter and associate a zeta function. In fact we want have

$$c\colon N_v^\times \to \mathbf{C}^\times \rightsquigarrow \zeta(f, c)$$

a family of $zeta$-functions. We will then look at the gcd over all possible f, this will be the L-factor.

3.1.3 Local ζ-functions

Let

$$\begin{aligned} f\colon K_v^+ &\to \mathbf{C} \\ \xi &\mapsto f(\xi) \end{aligned}$$

restrict to

$$\begin{aligned} f\colon K_v^\times &\to \mathbf{C} \\ \alpha &\mapsto f(\alpha) \end{aligned}$$

and such that

1. $f(\xi), \hat{f}(\xi) \in L^1(K^+)$ are continuous.

2. $f(\alpha)|\alpha|^{\sigma}$ and $\hat{f}(\alpha)|\alpha|^{\sigma} \in L^1(K^{\times})$ for all $\alpha > 0$.

Call the class of such f S.

Definition 3.14. Let $f \in S$ and c s.t. exponent of c is $\sigma > 0$, i.e. $c = c_0 \cdot |cdot|^s$ with c_0 unitary and $\Re(s) = \sigma > 0$ then we may define

$$\zeta(f, c) = \int_{K_v^{\times}} f(\alpha)c(\alpha)\, \mathrm{d}^{\times}\alpha.$$

In fact we have seen examples of this, for $K_v = \mathbf{R}$, the Γ function, in Tate's language this is a local zeta function.

$$\zeta(f, |\cdot|^s) = \int_{\mathbf{R}^{\times}} e^{-|x|}|x|^s \frac{\mathrm{d}x}{|x|} = 2\gamma(s)$$

Remark 3.15. $c = c_0|\cdot|^s$ for fixed f and c_0

$$\implies \zeta(f, c) = \int_{K_v^{\times}} f(\alpha)c_0(\alpha)|\alpha|_v^s\, \mathrm{d}^{\times}\alpha.$$

Aim:

1. Prove that $\zeta(f, s)$ extends to a meromorphic function of s.
2. Calculate this for "nice" choices of f.

3.1.4 Analytic properties of $\zeta(f, c)$

Lemma 3.16. *$\zeta(f, c)$ is a regular function for quasicharacters c with $\sigma > 0$.*

Proof.

$$\zeta(f, c) = \int_{K_v^{\times}} f(\alpha)c(\alpha)\, \mathrm{d}^{\times}\alpha$$

check, absolutely convergent around 0, has derivatives around 0, other points are fine. □

Lemma 3.17 (fundamental lemma)**.** *Let c be a quasicharacter that satisfies $0 < \sigma < 1$ and define*

$$\hat{c}(\alpha) = c_0^{-1}(\alpha)|\cdot|^{1-s}$$

then for $f, g \in S$ we have

$$\zeta(f, c)\zeta(\hat{g}, \hat{c}) = \zeta(\hat{f}, \hat{c})\zeta(g, c).$$

Proof. All of the integrals are absolutely convergent.

Fubini implies the LHS is

$$\int\int_{(K_v^{\times})^2} f(\alpha)\hat{g}(\beta)c(\alpha)c_0^{-1}(\beta)|\beta|^{1-s}\, \mathrm{d}^{\times}\mu(\alpha, \beta)$$

$$= \vdots$$

$$= \zeta(\hat{f}, \hat{c})\zeta(g, c)$$

change variable $\beta \mapsto \alpha\beta$. □

Theorem 3.18 (Local functional equation)**.** *A ζ-function will satisfy*

$$\zeta(f, c) = \rho(c)\zeta(\hat{f}, \hat{c})$$

where ρ is meromorphic, and is defined initially for $0 < \sigma < 1$ then extended by analytic continuation.

Proof. Take $f = g$ so

$$\rho(C) = \frac{\zeta(f, c)}{\zeta(\hat{f}, \hat{c})}$$

we will show $\rho(c)$ is independent of f and that we can choose f s.t. $\zeta(\hat{f}, \hat{c})$ is non-zero. TBC. ◻

Proposition 3.19. *Several properties:*

$$\rho(\hat{c}) = \frac{c(-1)}{\rho(c)}$$

$$\rho(\bar{c}) = c(-1)\overline{\rho(c)}$$

$$|\rho(c)| = 1 \text{ if } \sigma = \frac{1}{2}.$$

Proof.

$$\zeta(f, c) = \rho(c)\zeta(\hat{f}, \hat{c}) = \rho(c)\rho(\hat{c})\zeta(\hat{\hat{f}}, \hat{\hat{c}})$$

as $\hat{\hat{c}}$ and $\hat{\hat{f}}(x) = f(-x)$ we get $\rho(c)\rho(\hat{c})c(-1)\zeta(f, c)$.

$$\overline{\zeta(f, c)} = \zeta(\bar{f}, \bar{c}) = \rho(\bar{c})\zeta(\hat{\bar{f}}, \hat{\bar{c}})$$
$$= \rho(\bar{c})c(-1)\overline{\zeta(\hat{f}, \hat{c})}.$$

Remark:

$$\zeta(\hat{}f, \hat{b}arc) = \int \hat{}f(\alpha\hat{)}c(\alpha)\,\mathrm{d}^{\times}\alpha$$

$$= \int \hat{\bar{}}(\alpha)\bar{c}^{-1}(\alpha)|\alpha|\,\mathrm{d}^{\times}\alpha$$

$$= \int \hat{\bar{}}f(-\alpha)\bar{c}^{-1}(\alpha)|\alpha|\,\mathrm{d}^{\times}\alpha$$

$$= c(-1)\int \hat{\bar{}}f(\alpha)\bar{c}^{-1}(\alpha)|\alpha|\,\mathrm{d}^{\times}\alpha$$

$$= c(-1)\overline{\zeta(\hat{f}, \hat{c})}$$

If $\sigma = \frac{1}{2}$ then $c(\alpha)\bar{c}(\alpha) = |\alpha| = c(\alpha)\hat{c}(\alpha) = \bar{c}(\alpha) = \hat{c}(\alpha)$. Together the previous parts give $\rho(c)\overline{\rho(c)} = 1$. ◻

3.1.5 Explicit ζ functions

First let $K_v = \mathbf{R}$ then we will use the following notation: Additively ξ with $\Lambda(\xi) = -\xi$, $\mathrm{d}\xi$ the Lebesgue measure. Multiplicatively α with $|\alpha|_v = |\alpha|_{\mathbf{R}}$ and $\mathrm{d}^\times\alpha = \mathrm{d}\alpha/|\alpha|_{\mathbf{R}}$. We will use characters $|\cdot|^s$ or $\mathrm{sgn}|\cdot|^s$, $f = e^{-\pi\xi^2}$ and $f_{\mathrm{sgn}}(\xi) = \xi e^{-\pi\xi^2}$. These have fourier transforms

$$\hat{f}(\xi) = f(\xi)$$

and

$$\hat{f}_{\mathrm{sgn}}(\xi) = i f_{\mathrm{sgn}}(\xi).$$

So that

$$\begin{aligned}\zeta(f, |\cdot|^s) = \int_{\mathbf{R}^\times} f(\alpha)|\alpha|^s \,\mathrm{d}^\times\alpha &= \int_{\mathbf{R}^\times} e^{-\pi\alpha^2}|\alpha|^s \,\mathrm{d}^\times\alpha \\ &= 2\int_0^\infty e^{-\pi\alpha^2}.\end{aligned}$$