Automorphic forms and the Arthur-Selberg trace formula

MA842 BU Spring 2018

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1 Automorphic forms $/\mathrm{GL}_2$ (possibly GL_n)

Lecture 1 18/1/2018

These are notes for Ali Altuğ's course MA842 at BU Spring 2018, they were last updated January 21, 2018.

The course webpage is http://math.bu.edu/people/saaltug/2018_1/2018_1_sem.html.

Course overview: This course will be focused on the two papers Eisenstein Series and the Selberg Trace Formula I by D. Zagier and Eisenstein series and the Selberg Trace Formula II by H. Jacquet and D. Zagier. Although the titles of the papers sound like one is a prerequisite of the other it actually is not the case, the main difference is the language of the papers (the first is written in classical language whereas the second is written in adelically). We will spend most of our time with the second paper, which is adelic.

1.1 Goal

Jacquet and Zagier, Eisenstein series and the Selberg Trace Formula II (1980's).

Part I is a paper of Zagier from 1970 in purely classical language. Part II is in adelic language (and somewhat incomplete).

$$\begin{pmatrix}\text{Arthur-Selberg}\\ \text{trace formula}\end{pmatrix} \xleftrightarrow{\text{conjecture}} \begin{pmatrix}\text{Relative}\\ \text{trace formula}\end{pmatrix}$$

the Arthur-Selberg side is used in Langlands functoriality and the Relative is used in arithmetic applications.

1.2 Motivation

What does this paper do?

> "It rederives the Selberg trace formula for GL_2 by a regularised process."

Note 1.1.

- Selberg trace formula only for GL_2
- Arthur-Selberg more general

The Selberg trace formula generalises the more classical Poisson summation formula.

Theorem 1.2 (Poisson summation)**.** *Let*

$$f : \mathbf{R} \to \mathbf{R}$$

then ***Poisson summation*** *says*

$$\sum_{n\in\mathbf{Z}} f(n) = \sum_{\xi\in\mathbf{Z}} \hat{f}(\xi)$$

where

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e(x\xi)\,\mathrm{d}x$$

To make this look more general we make the following notational choices.

$$G = \mathbf{R}, \Gamma = \mathbf{Z}$$

$$\sum_{\gamma\in\Gamma^{\#}} f(\gamma) = \sum_{\xi\in(G/\Gamma)^{\vee}} \hat{f}(\xi)$$

where

- $\Gamma^{\#}$ = conjugacy classes of Γ (= Γ in this case since Γ is abelian).
- $(G/\Gamma)^{\vee}$ =dual of G/Γ.

Selberg

$$G = \mathrm{GL}_2(\mathbf{R}),\ \Gamma = \mathrm{GL}_2(\mathbf{Z})$$

$$\sum_{\gamma\in\Gamma^{\#}} \cdots \text{“} = \text{”} \sum_{\pi\in\text{“}(G/\Gamma)^{\vee}\text{”}} \cdots$$

relating conjugacy classes on the left to automorphic forms on the right.

Arthur and Selberg prove the trace formula by a *sharp cut off*, Jacquet and Zagier derive this using a regularisation.

1.3 Motivating example

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges absolutely for $s > 1$.

Theorem 1.3 (Riemann)**.** *$\zeta(s)$ has analytic continuation up to $\Re(s) > 0$ with a simple pole at $s = 1$ residue 1. i.e.*

$$\zeta(s) = \frac{1}{s-1} + \phi(s)$$

where $\phi(s)$ is holomorphic for $\Re(s) > 0$.

Proof. Step 1: observe

$$\begin{aligned}\frac{1}{s-1} &= \int_1^{\infty} t^{-s}\,\mathrm{d}t \text{ (for } \Re(s) > 1) \\ &= \sum_{n=1}^{\infty}\int_n^{n+1} t^{-s}\,\mathrm{d}t\end{aligned}$$

Step 2: this implies

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} n^{-s} - \int_n^{n+1} t^{-s}\,\mathrm{d}t$$
$$= \frac{1}{s-1} + \sum_{n=1}^{\infty} \left(\int_n^{n+1} n^{-s} - \int_n^{n+1} t^{-s}\,\mathrm{d}t \right)$$

Step 3:

$$|\phi_n(s)| \le \sup_{n \le t \le n+1} |n^{-s} - t^{-s}|$$
$$\sup_{n \le t \le n+1} \frac{|s|}{t^{\Re(s)+1}} \le \frac{|s|}{n^{\Re(s)+1}}$$

by applying the mean value theorem.

So $\sum_{n=1}^{\infty} \phi_n$ converges absolutely. Hence $\phi = \sum_{n=1}^{\infty} \phi_n$ is holomorphic

One can push this idea to get analytic continuation to all of $\mathbf{C}$, one strip at a time. This is an analogue of the sharp cut off method mentioned above. It's fairly elementary but somewhat unmotivated and doesn't give any deep information (like the functional equation). □

Proof. Introduce

$$\theta(t) = \sum_{n \in \mathbf{Z}} e^{-\pi n^2 t},\ t > 0$$

note that $\theta(t) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}$.

Idea: Mellin transform and properties of θ to derive properties of ζ.

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{s/2}} \frac{1}{ns} = \int_0^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{\mathrm{d}t}{t}$$

property of θ:

$$\theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right)$$

Step 1: proof of this property is the Poisson summation formula

- $$f(x) = e^{-\pi x^2} \implies \hat{f}(\xi) = f(\xi)$$
- $$g(x) = f(\sqrt{t}x) \implies \hat{g}(\xi) = \frac{1}{\sqrt{t}} \hat{f}\left(\frac{\xi}{\sqrt{t}}\right)$$

Step 2: Would like to write something like

$$\text{“}\int_0^{\infty} \theta(t) t^{s/2} \frac{\mathrm{d}t}{t}\text{”}$$

This integral makes no sense

- As $t \longrightarrow \infty$, $\theta \sim 1$ thus

$$\left| \int_A^{\infty} \theta(t) t^{s/2} \frac{\mathrm{d}t}{t} \right| < \infty$$
$$\iff \left| \int_A^{\infty} t^{s/2} \frac{\mathrm{d}t}{t} \right| < \infty$$
$$\iff \Re(s) < 0$$

- As $t \to 0$ consider $\xi = \frac{1}{t}$ so $\xi \to \infty$ and

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right) = \sqrt{\xi}\theta\xi]$$

$$\implies \theta(t) = \sqrt{\xi}\theta(\xi) \sim \sqrt{\xi} = \frac{1}{\sqrt{t}}$$

so $\theta(t) \sim \frac{1}{\sqrt{t}}$

$$\begin{aligned} &\implies \left|\int_0^A \theta(t)t^{s/2}\frac{\mathrm{d}t}{t}\right| < \infty \\ &\iff \left|\int_0^A t^{(s-1)/2}\frac{\mathrm{d}t}{t}\right| < \infty \\ &\iff \Re(s) > 1 \end{aligned}$$

so no values of s will make sense for this improper integral.

Refined idea: Consider

$$I(s) = \int_0^1 (\theta(t) - \frac{1}{\sqrt{t}})t^{s/2}\frac{\mathrm{d}t}{t} + \int_1^\infty (\theta(t) - 1)t^{s/2}\frac{\mathrm{d}t}{t}$$

upshot: $I(s)$ is well-defined and holomorphic for all $s \in \mathbf{C}$.

Final step: Compute the above to see

$$I(s) = \frac{2}{s} + \frac{2}{1-s} + \frac{2}{\pi^{s/2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

which implies

1. $\zeta(s)$ has analytic continuation to $s \in \mathbf{C}$, with only a simple pole at $s = 1$ with residue 1.

2. $$I(s) = I(1-s),$$

   this follows from the property of θ so if we let

   $$\Lambda(s) = \frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{s/2}}\zeta(s),$$

   then

   $$\Lambda(s) = \Lambda(1-s).$$

□

1.4 Modular forms

Functions on the upper half plane,

$$\mathbf{H} = \{z \in \mathbf{C} : \Im(z) > 0\}.$$

Historically elliptic integrals lead to elliptic functions, and modular forms and elliptic curves.

Note 1.4. When one is intereseted in functions on O/Λ where O is some object and Λ is some discrete group. Take f a function on O and average over Λ to get

$$\sum_{\lambda \in \Lambda} f(\lambda z).$$

If you're lucky this converges, this is good.

Elliptic functions Weierstrass, take $\Lambda = \omega_1 \mathbf{Z} + \omega_2 \mathbf{Z}$ a lattice an define

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} + \frac{1}{\omega^2} \right).$$

Jacobi, (Elliptic integrals) consider

$$\int_0^\phi \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-\kappa t^2)}}, \ \kappa \geq 0$$

related by:

$$(\wp'_\Lambda(z))^2 = 4\wp_\Lambda(z)^3 - 60G_2(\Lambda)\wp_\Lambda(z) - 140G_3(\Lambda)$$

$$G_k(\Lambda) = \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-2k}$$

or

$$G_k(\tau) = \sum_{(m,n) \in \mathbf{Z}^2 \setminus \{0\}} \frac{1}{(m\tau + n)^{2k}}$$

the weight $2k$ holomorphic Eisenstein series.

Fact 1.5. *Let*

$$u = \int_y^\infty \frac{\mathrm{d}s}{\sqrt{4s^3 - 60G_2 s - 140G_3}}$$

then

$$y = \wp_\Lambda(u).$$

Lecture 2 23/1/2018