Automorphic forms and the Arthur-Selberg trace formula

MA842 BU Spring 2018

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1 Automorphic forms $/\mathrm{GL}_2$ (possibly GL_n)

Lecture 1 18/1/2018

These are notes for Ali Altuğ's course MA842 at BU Spring 2018, they were last updated January 23, 2018.

The course webpage is http://math.bu.edu/people/saaltug/2018_1/2018_1_sem.html.

Course overview: This course will be focused on the two papers Eisenstein Series and the Selberg Trace Formula I by D. Zagier and Eisenstein series and the Selberg Trace Formula II by H. Jacquet and D. Zagier. Although the titles of the papers sound like one is a prerequisite of the other it actually is not the case, the main difference is the language of the papers (the first is written in classical language whereas the second is written in adelically). We will spend most of our time with the second paper, which is adelic.

1.1 Goal

Jacquet and Zagier, Eisenstein series and the Selberg Trace Formula II (1980's).

Part I is a paper of Zagier from 1970 in purely classical language. Part II is in adelic language (and somewhat incomplete).

$$\begin{pmatrix}\text{Arthur-Selberg}\\ \text{trace formula}\end{pmatrix} \xleftrightarrow{\text{conjecture}} \begin{pmatrix}\text{Relative}\\ \text{trace formula}\end{pmatrix}$$

the Arthur-Selberg side is used in Langlands functoriality and the Relative is used in arithmetic applications.

1.2 Motivation

What does this paper do?

> "It rederives the Selberg trace formula for GL_2 by a regularised process."

Note 1.1.

- Selberg trace formula only for GL_2
- Arthur-Selberg more general

The Selberg trace formula generalises the more classical Poisson summation formula.

Poisson summation

Theorem 1.2 (Poisson summation)**.** *Let*

$$f:\mathbf{R}\to\mathbf{R}$$

then ***Poisson summation*** *says*

$$\sum_{n\in\mathbf{Z}} f(n)=\sum_{\xi\in\mathbf{Z}}\hat{f}(\xi)$$

where

$$\hat{f}(\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e(x\xi)\,\mathrm{d}x.$$

Notation: $e(x)=e^{2\pi ix}$.

To make this look more general we make the following notational choices.

$$G=\mathbf{R},\Gamma=\mathbf{Z}$$

$$\sum_{\gamma\in\Gamma^{\#}} f(\gamma)=\sum_{\xi\in(G/\Gamma)^{\vee}}\hat{f}(\xi)$$

where

- $\Gamma^{\#}$ = conjugacy classes of Γ (= Γ in this case since Γ is abelian).
- $(G/\Gamma)^{\vee}$ =dual of G/Γ.

Selberg

$$G=\mathrm{GL}_2(\mathbf{R}),\ \Gamma=\mathrm{GL}_2(\mathbf{Z})$$

$$\sum_{\gamma\in\Gamma^{\#}}\cdots\text{“}=\text{”}\sum_{\pi\in\text{“}(G/\Gamma)^{\vee}\text{”}}\cdots$$

relating conjugacy classes on the left to automorphic forms on the right.

Arthur and Selberg prove the trace formula by a *sharp cut off*, Jacquet and Zagier derive this using a regularisation.

1.3 Motivating example

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

converges absolutely for $s>1$.

Theorem 1.3 (Riemann)**.** *$\zeta(s)$ has analytic continuation up to $\Re(s)>0$ with a simple pole at $s=1$ residue 1. i.e.*

$$\zeta(s)=\frac{1}{s-1}+\phi(s)$$

where $\phi(s)$ is holomorphic for $\Re(s)>0$.

Proof. Step 1: observe

$$\frac{1}{s-1}=\int_1^{\infty} t^{-s}\,\mathrm{d}t \text{ (for } \Re(s)>1)$$

$$= \sum_{n=1}^{\infty} \int_n^{n+1} t^{-s}\, \mathrm{d}t$$

Step 2: this implies

$$\begin{aligned} \zeta(s) &= \frac{1}{s-1} + \sum_{n=1}^{\infty} n^{-s} - \int_n^{n+1} t^{-s}\, \mathrm{d}t \\ &= \frac{1}{s-1} + \sum_{n=1}^{\infty} \left(\int_n^{n+1} n^{-s} - \int_n^{n+1} t^{-s}\, \mathrm{d}t \right) \end{aligned}$$

we denote each of the terms in the right hand sum as $\phi_n(s)$

$$\phi_n(s) = \int_n^{n+1} n^{-s} - t^{-s}\, \mathrm{d}t$$

Step 3:

$$\begin{aligned} |\phi_n(s)| \leq \sup_{n \leq t \leq n+1} |n^{-s} - t^{-s}| \\ \sup_{n \leq t \leq n+1} \frac{|s|}{t^{\Re(s)+1}} \leq \frac{|s|}{n^{\Re(s)+1}} \end{aligned}$$

by applying the mean value theorem.

So $\sum_{n=1}^{\infty} \phi_n$ converges absolutely. Hence $\phi = \sum_{n=1}^{\infty} \phi_n$ is holomorphic

One can push this idea to get analytic continuation to all of $\mathbf{C}$, one strip at a time. This is an analogue of the sharp cut off method mentioned above. It's fairly elementary but somewhat unmotivated and doesn't give any deep information (like the functional equation). □

Proof. Introduce

$$\theta(t) = \sum_{n \in \mathbf{Z}} e^{-\pi n^2 t},\ t > 0$$

note that $\theta(t) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}$.

Idea: Mellin transform and properties of θ to derive properties of ζ.

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{s/2}} \frac{1}{ns} = \int_0^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{\mathrm{d}t}{t}$$

property of θ:

$$\theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right)$$

Step 1: proof of this property is the Poisson summation formula

- $$f(x) = e^{-\pi x^2} \implies \hat{f}(\xi) = f(\xi)$$

- $$g(x) = f(\sqrt{t}x) \implies \hat{g}(\xi) = \frac{1}{\sqrt{t}} \hat{f}\left(\frac{\xi}{\sqrt{t}}\right)$$

Step 2: Would like to write something like

$$\text{“} \int_0^{\infty} \theta(t) t^{s/2} \frac{\mathrm{d}t}{t} \text{”}$$

This integral makes no sense

- As $t \to \infty$, $\theta \sim 1$ thus

$$\left|\int_A^\infty \theta(t)t^{s/2}\frac{\mathrm{d}t}{t}\right| < \infty$$
$$\iff \left|\int_A^\infty t^{s/2}\frac{\mathrm{d}t}{t}\right| < \infty$$
$$\iff \Re(s) < 0$$

- As $t \to 0$ consider $\xi = \frac{1}{t}$ so $\xi \to \infty$ and

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right) = \sqrt{\xi}\theta\xi]$$
$$\implies \theta(t) = \sqrt{\xi}\theta(\xi) \sim \sqrt{\xi} = \frac{1}{\sqrt{t}}$$

so $\theta(t) \sim \frac{1}{\sqrt{t}}$

$$\implies \left|\int_0^A \theta(t)t^{s/2}\frac{\mathrm{d}t}{t}\right| < \infty$$
$$\iff \left|\int_0^A t^{(s-1)/2}\frac{\mathrm{d}t}{t}\right| < \infty$$
$$\iff \Re(s) > 1$$

so no values of s will make sense for this improper integral.

Refined idea: Consider

$$I(s) = \int_0^1 (\theta(t) - \frac{1}{\sqrt{t}})t^{s/2}\frac{\mathrm{d}t}{t} + \int_1^\infty (\theta(t) - 1)t^{s/2}\frac{\mathrm{d}t}{t}$$

upshot: $I(s)$ is well-defined and holomorphic for all $s \in \mathbf{C}$.

Final step: Compute the above to see

$$I(s) = \frac{2}{s} + \frac{2}{1-s} + \frac{2}{\pi^{s/2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

which implies

1. $\zeta(s)$ has analytic continuation to $s \in \mathbf{C}$, with only a simple pole at $s = 1$ with residue 1.

2. $$I(s) = I(1-s),$$

   this follows from the property of θ so if we let

   $$\Lambda(s) = \frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{s/2}}\zeta(s),$$

   then

   $$\Lambda(s) = \Lambda(1-s). \qquad \square$$

1.4 Modular forms

Functions on the upper half plane,

$$\mathbf{H} = \{z \in \mathbf{C} : \Im(z) > 0\}.$$

Historically elliptic integrals lead to elliptic functions, and modular forms and elliptic curves.

Note 1.4. When one is intereseted in functions on O/Λ where O is some object and Λ is some discrete group. Take f a function on O and average over Λ to get

$$\sum_{\lambda\in\Lambda} f(\lambda z).$$

If you're lucky this converges, this is good.

Elliptic functions Weierstrass, take $\Lambda = \omega_1\mathbf{Z} + \omega_2\mathbf{Z}$ a lattice an define

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\smallsetminus\{0\}} \left(\frac{1}{(z-\omega)^2} + \frac{1}{\omega^2}\right).$$

Jacobi, (Elliptic integrals) consider

$$\int_0^\phi \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-\kappa t^2)}}, \ \kappa \geq 0$$

related by:

$$(\wp'_\Lambda(z))^2 = 4\wp_\Lambda(z)^3 - 60G_2(\Lambda)\wp_\Lambda(z) - 140G_3(\Lambda)$$
$$G_k(\Lambda) = \sum_{\lambda\in\Lambda\smallsetminus\{0\}} \lambda^{-2k}$$

or

$$G_k(\tau) = \sum_{(m,n)\in\mathbf{Z}^2\smallsetminus\{0\}} \frac{1}{(m\tau+n)^{2k}}$$

the weight $2k$ holomorphic Eisenstein series.

Fact 1.5. *Let*

$$u = \int_y^\infty \frac{\mathrm{d}s}{\sqrt{4s^3 - 60G_2 s - 140G_3}}$$

then

$$y = \wp_\Lambda(u).$$

1.5 Euclidean Harmonic analysis

Lecture 2 23/1/2018

We'll take a roundabout route to automorphic forms.

Today: Classical harmonic analysis on $\mathbf{R}^n$. Classical harmonic analysis on $\mathbf{H}$.

The aim (in general) is to express a certain class of functions (i.e. $\mathcal{L}^2$) in terms of building block (harmonics).

In classical analysis the harmonics are known ($e(nx)$), then the question becomes how these things fit together. In number theory the harmonics are extremely mysterious. We are looking at far more complicated geometries, quotient spaces etc. and arithmetic information comes in.

Example 1.6. $\mathbf{R}$, $f\colon \mathbf{R} \to \mathbf{C}$, being periodic in $\mathcal{L}^2(S^1)$ leads to a fourier expansion

$$f(x) = \sum_{n \in \mathbf{Z}} a_n e(nx).$$

1.5.1 $\mathbf{R}^2$

We have a slightly different perspective.

$$\mathbf{R}^2 = G \circlearrowright G$$

via translations (i.e. right regular representation of G will be $G \circlearrowright \mathcal{L}^2(G)$). I.e. $g \cdot x = x + g$.

Remark 1.7. This makes $\mathbf{R}^2$ a homogeneous space.

$\mathbf{R}^2$ with standard metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ is a flat space $\kappa = 0$.

To the metric we have the associated Laplacian (Laplace-Beltrami operator, $\nabla \cdot \nabla$)

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

we are interested in this as it is essentially the only operator, we will define automorphic forms to be eigenfunctions for this operator.

Note 1.8. The exponential functions

$$\phi_{u,v}(x, y) = e(ux + vy)$$

are eigenfunctions of Δ with eigenvalue $\lambda_{u,v} = -4\pi^2(u^2 + v^2)$ i.e.

$$(\Delta + \lambda_{u,v})\phi_{u,v} = 0.$$

These are a complete set of harmonics for $\mathcal{L}^2(\mathbf{R}^2)$. The proof is via fourier inversion.

$$f(x, y) = \int\int_{\mathbf{R}^2} \hat{f}(u, v)\phi_{u,v}(x, y)\,\mathrm{d}u\,\mathrm{d}v$$

where

$$\hat{f}(u, v) = \int\int_{\mathbf{R}^2} f(u, v)\bar{\phi}_{u,v}(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$

A little twist We could have established the spectral resolution (of Δ) by considering invariant integral operators.

Using the spectral theorem if we can find easier to diagonalise operators that commute we can find the eigenspaces for those to cut down the eigenspace.

Recall: an integral operator is

$$L(f)(x) = \int K(x, y) f(y)\,\mathrm{d}y$$

invariant means

$$L(gf) = gL(f) \ g \in G$$

in our case

$$g \cdot f(x) = f(g + x).$$

Observation 1.9. If L is invariant then the kernel $K(x, y)$ is given by

$$K(x, y) = K_0(x - y)$$

for some function K_0.

Observation 1.10. Invariant integral operators commute with each other

$$L_1 L_2(f)(z) = L_2 L_1(f)(z)$$

Observation 1.11. L commute with Δ.

Observation 1.12. $\phi_{u,v}(x,y)$ is an eigenfunction of L, $(u,v) \in \mathbf{R}^2, (x,y) \in \mathbf{R}^2$.

Side remark: these are enough to form a generating set.

1.5.2 Poisson summation (yet again)

Let's consider integral operators on functions on $\mathbf{Z}^2 \backslash \mathbf{R}^2 = \mathbf{T}^2$.

Observe: $L \rightsquigarrow K(x,) = K_0(x-y)$.

$$Lf(z) = \int_{\mathbf{R}^2} f(w) K(z,w)\,\mathrm{d}w$$

$$= \int \int_{\mathbf{Z}^2 \backslash \mathbf{R}^2} f(w) \underbrace{\left(\sum_{n \in \mathbf{Z}^2} K(z, w+n) \right)}_{=\sum_{n \in ZZ} K_0(z-w+n) = \mathbf{K}(z,w)} \mathrm{d}w$$

now $\mathbf{K}$ is a function on $\mathbf{T}^2 \times \mathbf{T}^2$.

Trace of this operator

$$= \int_{\mathbf{T}^2} \mathbf{K}(z,z)\,\mathrm{d}z = \int_{\mathbf{T}^2} \left(\sum_{n \in \mathbf{Z}^2} K_0(n) \right) \mathrm{d}z = \sum_{n \in \mathbf{Z}^2} K_0(n)$$

Using sum of eigenvlaues

$$K(z,w) = \sum_{n \in \mathbf{Z}^2} K_0(z - +n) = \sum_{\xi \in \mathbf{Z}^2} \lambda_\xi \phi_\xi(z-w) = \sum_{\xi \in \mathbf{Z}^2} \lambda_\xi \phi_\xi(z) \bar{\phi}_\xi(w)$$

so the trace is

$$\sum_{\xi \in \mathbf{Z}^2} \lambda_\xi = \sum_{\xi \in \mathbf{Z}^2} \hat{K}_0(\xi)$$

so we get to

$$\sum_{n \in \mathbf{Z}^2} K_0(n) = \sum_{\xi \in \mathbf{Z}^2} \hat{K}_0(\xi)$$

i.e. Poisson summation.

Why care about Poisson summation?

$$\hat{K}_0(0) = \int K_0(z)\,\mathrm{d}z$$

Gauss circle problem, how many lattice points are there in a circle of radius R. We can pick a radially symmetric function that is 1 on the circle and 0 outside, or a smooth approximation outside. Poisson summation packages the important information into a single term, plus some rapidly decaying ones. Then we get $\pi R^2 +$ error, Gauss conjectured that the error is $R^{1/2+\epsilon}$.