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Theorem.

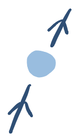
Let $G = (V, E)$ be a connected graph. Then,

(i) G has Eulerian cycle $\Leftrightarrow$ all vertices have even degree.

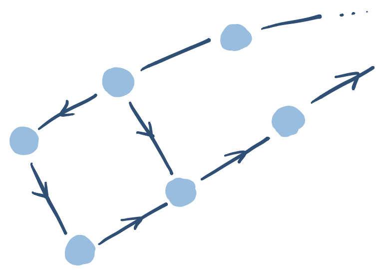
(ii) G has Eulerian path but not cycle $\Leftrightarrow$ exactly 2 vertices have odd degree.

proof of (i).

($\Rightarrow$) Assume Eulerian cycle.



Every time the cycle visits a vertex, 2 of its edges (its degree) are used. So vertex must have even degree, except at start vertex. Use one edge at start, 2 at each time cycle visits it, one coming back at end so still even.



($\Leftarrow$) Assume all even degree.

Maximality argument: Let $S = (v_1, v_2, \dots, v_n)$ be a path not repeating edges which is of maximum length.

We will prove S is a Eulerian cycle, in three parts.

(1) Prove S is a cycle.

Assume not a cycle, i.e. $v_1 \neq v_n$.

Leaving v_1 uses one edge, use 2 for each subsequent visit.

Degree even so at least one unused edge at v_1.

Then path $(v_0, v_1, v_2, \dots, v_n)$ does not repeat edges, longer than S, contradicts maximality.

Hence $v_1 = v_n$, S is a cycle.

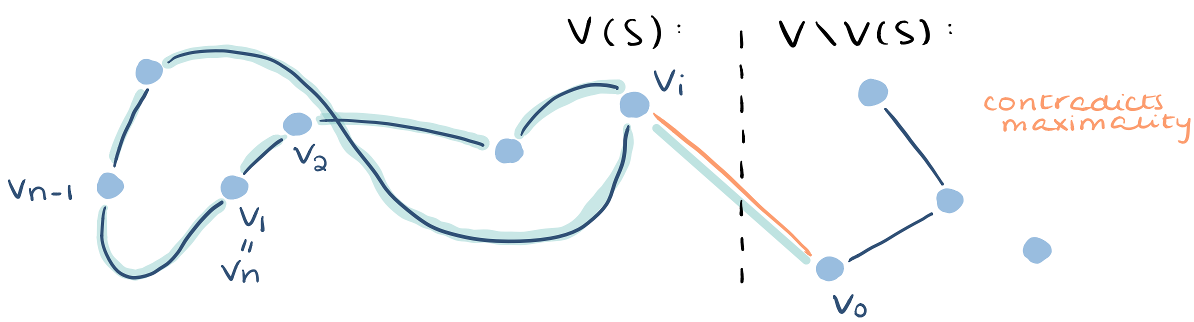
(2) Notation: $V(S)$ = vertices visited by S.

$E(S)$ = edges used by S.

To prove: $E(S) = E$.

(3) Prove $V(S) = V$.

Assume not, so $V \setminus V(S) \neq \emptyset$.



G is connected so $\exists$ edge from some $v_0 \in V \setminus V(S)$ to some $v_i \in V(S)$. Then path $(v_0, v_i, v_{i+1}, \dots, \underset{v_1}{\overset{\shortparallel}{v_n}}, v_2, v_3, \dots, v_{i-1}, v_i)$ does not repeat any edges and is longer than S by 1.

Contradiction to maximum length of S, so $V(S) = V$.

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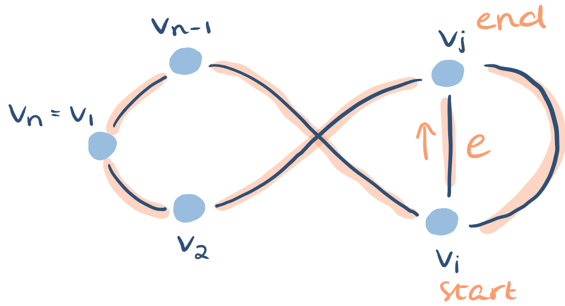
(4) Prove $E(S) = E$

Assume not. So there is an edge not on cycle S.
By part (3), the endpoints of edge e are on the cycle.
Let $e = (v_i, v_j)$ some $i < j$. Then path



$(\underbrace{v_i, v_j}_{e}, v_{j+1}, \dots, \underbrace{v_n}_{= v_1}, v_2, v_3, \dots, v_j)$

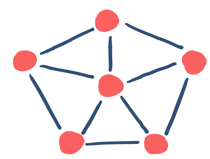
does not repeat edges and is 1 longer than S.
Contradiction to maximality.
So $E(S) = E$, so S is a Eulerian cycle. So (i) is proved. $\square$

proof of (ii).

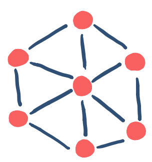
($\Leftarrow$) Assume two (exactly) odd-degree vertices, say u, w.
Add an edge (u, w); now all vertices have even degree.
So by (i), $\exists$ Eulerian cycle. This cycle must use the edge, so then there is a path starting at w (after new edge), ending at v, not using new edge. This is a Eulerian path. $\square$

($\Rightarrow$) Exercise. ☺

def. The wheel graph W_n is obtained from C_n by adding a new vertex adjacent to all old vertices.



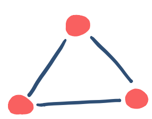
ex. For what n does W_n have a Eulerian cyle/path?

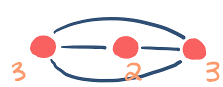


In C_n, every vertex degree 2.
In W_n, all vertices on cycle have degree 3. New one has degree n.

If $n \geq 3$, have 3 vertices degree 3 on cycle, so no Euler path or cycle.

If $n = 1$: C_1  W_1  has Eulerian path, no cycle.

If $n = 2$: C_2  W_2  has Eulerian cycle.

has Eulerian path but not cycle.
2 odd-degree vertices.

Hamiltonian Paths and Cycles

def. A cycle (or path) which visits every vertex exactly once is called a Hamiltonian cycle (or path).

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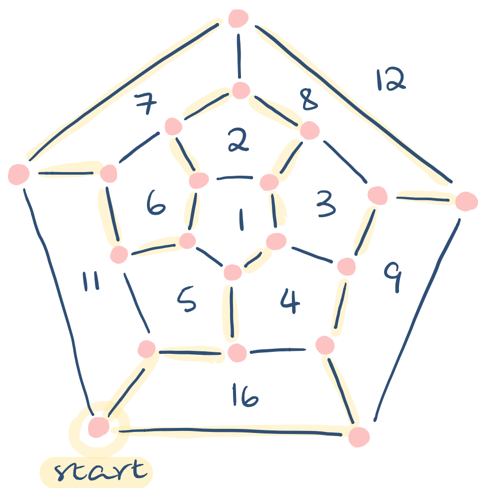
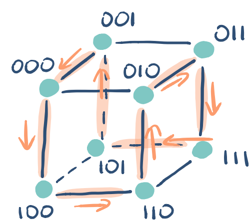
ex. a Hamiltonian cycle on the dodecahedron.



Problem.
Prove that Q_n hypercube has a Hamiltonian cycle for $n \geq 2$.
Idea: We know there is a Hamiltonian cycle on Q_2. Do this on "front", except last edge. Go to "back". Follow Hamiltonian cycle on back, in reverse. Go back to front.



proof.

By induction.

(i) Base case.
Let $n=2$. Clearly has Hamiltonian cycle: $(00, 10, 11, 01, 00)$.

(ii) Inductive step.
Let $n \geq 2$. Assume Q_n has Hamiltonian cycle. Q_{n+1} is formed from two copies of Q_n:

$Q_n^{(0)} = (V_n^{(0)}, E_n^{(0)})$ (has 0 as last bit)

$Q_n^{(1)} = (V_n^{(1)}, E_n^{(1)})$ (has 1 as last bit)

Follow a Hamiltonian cycle in $Q_n^{(0)}$, say $\sigma = (v_1^{(0)}, \dots, v_m^{(0)})$ where $v_1^{(0)} = v_m^{(0)}$. Then, the cycle:

$$(v_1^{(0)}, \dots, \underbrace{v_{m-1}^{(0)}, v_{m-1}^{(1)}}_{\text{only last bit is different}}, v_{m-2}^{(1)}, \dots, \underbrace{v_1^{(1)}, v_1^{(0)}}_{\text{only last bit is different}})$$

is a Hamiltonian cycle in Q_{n+1}. □

Illustration:

Q_2:	00	Q_3:	000
	10		100
	11		110
	01		010
			011
			111
			101
			001

Note: Hamiltonian cycle on Q_n is a listing of 2^n binary strings of length n in such a way that adjacent strings differ by exactly 1 bit. This is also known as Gray Code (or binary code).

Compare to "usual listing" of 000, 001, 010, 011, 100, 101, 110, 111.

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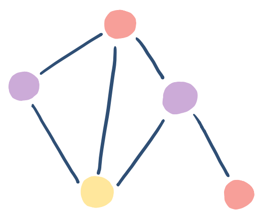
Application.
Electrical system of switches, each can be on/off. Want to test system in all 2^n configurations. If test system in Gray Code order, least # of total flips of switches to do.

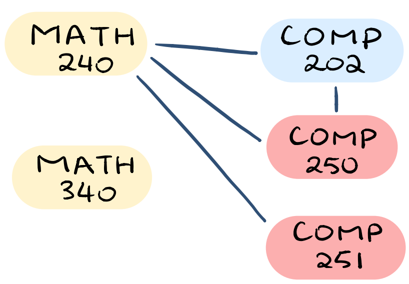
Bad news about Hamiltonian cycles/paths:
No known theorem of form:
G has Hamiltonian cycle/path $\Leftrightarrow$ "nice condition"

Graph Colouring

def. a colouring of $G = (V, E)$ is an assignment of "colours" (labels) to the vertices such that adjacent vertices do not have the same colour.



Application.
Exam scheduling.
- courses = vertices.
- Edge between courses if exams should not be at same time.
- colours = exam time slots.
  = { Dec 5 9:00, Dec 5 14:00, Dec 5 18:30, Dec 6 9:00 }



a colouring of the graph gives an exam schedule without any conflict.

Want as few colours as possible.