Alison Li

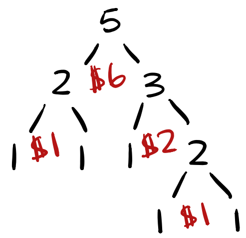
a coin game:

There is a pile of n \$1 coins. You do the following:

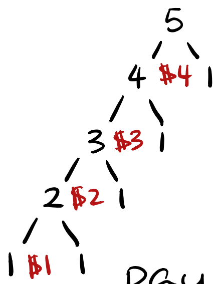
(1) Choose any pile. Divide it into two piles. If the sizes of the two piles are k and l, you earn \$$(kl)$.

(2) Repeat (1) until piles have one coin each.

ex.



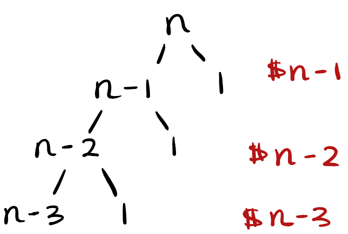
pay = \$10



pay = \$10

Prove that the pay only depends on n.

Note: Define pay with the scheme:



$$\text{pay} = (n-1)+(n-2)+\cdots+1 = \sum_{k=1}^{n-1} k = \frac{(n-1)n}{2}$$

proof. Exercise. ☺

proof.

We prove $\text{pay} = \frac{(n-1)n}{2}$ by strong induction.

(i) Base case. $n=1$.

Then $\text{pay} = \frac{1(0)}{2} = 0$. ✓

(ii) Inductive step.

Let $n \geq 1$. Assume that for any m, $1 \leq m \leq n$, that by dividing m into piles of 1, pay is $\frac{m(m-1)}{2}$.

Given a pile of $n+1$ coins. Divide $n+1$ into piles of size k and $(n+1-k)$. Pay is $\underbrace{k(n+1-k)}_{\text{from 1st division}}$ + (pay from size k pile) + (pay from $n+1-k$ size pile).

Both k and $n+1-k$ are less than $n+1$. So IA applies:

$$\text{Pay} = k(n+1-k) + \frac{k(k-1)}{2} + \frac{(n+1-k)(n-k)}{2}$$

$$= \frac{2kn + 2k - 2k^2 + k^2 - k + n^2 - nk + n - k - kn + k^2}{2}$$

$$= \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

= Formula for $n+1$. □

Alison Li

Recurrence Relation

def. Let $f: \mathbb{N} \to \mathbb{R}$ be a function ("sequence"). A recurrence relation for f is an equation giving value of $f(n)$ in terms of $f(m)$ for some (several) $m < n$.

ex. Define $f: \mathbb{N} \to \mathbb{Q}$ by $f(0) = 1$, $f(n) = 1 + \frac{1}{f(n-1)}$, $n \geq 1$.
(1) Calculate $f(n)$, n up to 5.
(2) Prove $\forall n \; 1 \leq f(n) \leq 2$.

(1) $f(0) = 1$

$f(1) = 1 + \frac{1}{f(0)} = 2$

$f(2) = 1 + \frac{1}{2} = \frac{3}{2}$

$f(3) = 1 + \frac{2}{3} = \frac{5}{3}$

$f(4) = 1 + \frac{3}{5} = \frac{8}{5}$

$f(5) = 1 + \frac{5}{8} = \frac{13}{8}$

(2) (i) Base case. $n = 0$.
$f(0) = 1$.

(ii) Inductive step.
Let $n \geq 0$. Assume $1 \leq f(n) \leq 2$. Prove $1 \leq f(n+1) \leq 2$, i.e.

$1 \leq 1 + \frac{1}{f(n)} \leq 2$.

First, $f(n) \geq 1$ (known).

$1 \geq \frac{1}{f(n)}$

add 1:

$2 \geq 1 + \frac{1}{f(n)} = f(n+1)$.

Next: $f(n) \leq 2$ so $\frac{1}{f(n)} \geq \frac{1}{2}$.

add 1:

$1 + \frac{1}{f(n)} \geq 1 + \frac{1}{2}$.

$f(n+1) \geq \frac{3}{2} \geq 1$, so $f(n+1) \geq 1$. $\square$

Fun fact: $\lim_{n \to \infty} f(n) = \frac{1 + \sqrt{5}}{2}$ "Golden ratio".

Problem.
Define sequence $a_0, a_1, a_2, \dots, a_n$ by $a_0 = 5$ and $a_n = 3a_{n-1} + 2$.
Find and prove a non-recursive formula for a_n.

Finding a formula:

$$
\begin{aligned}
a_n = 3\underline{a_{n-1}} + 2 &= 3(\overbrace{3a_{n-2} + 2}^{a_{n-1}}) + 2 \\
&= (3^2 a_{n-2} + 3 \cdot 2) + 2 \\
&= 3^2(\overbrace{3a_{n-3} + 2}^{a_{n-2}}) + 3 \cdot 2 + 2 \\
&= (3^3 a_{n-3} + 3^2 \cdot 2) + 3 \cdot 2 + 2 \\
&\ \ \vdots \\
&= 3^n \underbrace{a_{n-n}}_{a_0} + 3^{n-1} \cdot 2 + 3^{n-2} \cdot 2 + \dots + 3 \cdot 2 + 2 \\
&= 3^n \cdot \underline{5} + 2(\underbrace{1 + 3 + 3^2 + \dots + 3^{n-1}}_{\text{geometric series}}) \\
&= 3^n \cdot 5 + 2 \sum_{k=0}^{n-1} 3^k
\end{aligned}
$$

Claim: $1 + 3 + \dots + 3^{n-1} = \dfrac{3^n - 1}{3 - 1} = \dfrac{1}{2}(3^n - 1)$

So conjecture:

$$
\begin{aligned}
a_n &= 5 \cdot 3^n + 2\left(\tfrac{1}{2}(3^n - 1)\right) \\
&= 5 \cdot 3^n + 3^n - 1 = 3^n + 3^n + 3^n + 3^n + 3^n + 3^n - 1. \\
&= 6 \cdot 3^n - 1 = 2 \cdot 3 \cdot 3^n - 1 \\
a_n &= 2 \cdot 3^{n+1} - 1
\end{aligned}
$$

(i) Base case. $n = 0$

$a_0 = 5$ (by definition)

$2 \cdot 3 - 1 = 6 - 1 = 5$

(ii) Inductive step.

Let $n \geq 0$. Assume $a_n = 2 \cdot 3^{n+1} - 1$.

$$
\begin{aligned}
a_{n+1} &= 3 \cdot \underline{a_n} + 2 \\
&= 3 \cdot \underline{(2 \cdot 3^{n+1} - 1)} + 2 \quad \text{by IA.} \\
&= 2 \cdot 3^{n+2} - 3 + 2 \\
&= 2 \cdot 3^{n+2} - 1 \quad \text{as required.}
\end{aligned}
$$

def. a binary string of length n is a sequence of n 0's or 1's.

Problem

Let S_n be the set of all binary strings of length n that do not contain the pattern '11'. (i.e. no two adjacent 1's).
Let $C_n = |S_n|$. Find a recurrence relation for C_n.

$S_1 = \{0, 1\}$ $\quad C_1 = 2$

$S_2 = \{00, 01, 10\}$ $\quad C_2 = 3$

$S_3 = \{000, 001, 010, 100, 101\}$ $\quad C_3 = 5$

$S_4 = \{0000, 0001, 0010, 0100, 1000, 0101, 1010, 1000, 1001\}$ $\quad C_4 = 8$

Conjecture: $C_n = C_{n-1} + C_{n-2}$

proof.

By strong induction.

(i) Base cases:

$c_1 = 2$

$c_2 = 3$

as above. ✓

(ii) Inductive step.

Let $n \geq 3$. Assume $c_n = c_{n-1} + c_{n-2}$ true for $n-2$ and $n-1$.

Prove $c_n = c_{n-1} + c_{n-2}$.

Notice: $S_4 = \{0000, 0001, 0010, 0100,$ ends in 0

$1000, 0101, 1010, 1001\}$ ends in 01

The plan is to break off these two cases.

Partition S_n into two types:

(I) Strings ending in 0. Remove the '0' to obtain the element of S_{n-1}. Then we have a bijection.

$\{S \in S_n \mid S \text{ ends in } 0\} \to S_{n-1}$

$(f : \{(I)\} \to S_{n-1})$ "remove last 0".

Verify this is actually a bijection.

Injective: If S_1 and S_2 are type (I) and $S_1 \neq S_2$, they differ on some bit other than the last one. So $f(S_1) \neq f(S_2)$.

Surjective: If $S \in S_{n-1}$, let S' be S with '0' appended to the end. Then $S' \in S_n$ of type (I) and $f('S) = S$.

So bijection $\{(I)\} \to S_{n-1}$.

$|(I)| = c_{n-1}$

(II) Strings ending in 1.

These actually end in '01'. Define $f : \{(II)\} \to S_{n-2}$ by "remove '01' from end".

Verify this is actually a bijection as an exercise. ☺

So (II) has $S_{n-2} = c_{n-2}$ strings.

Then,

$c_n = |S_n| = |(I)| + |(II)| = c_{n-1} + c_{n-2}$. □