Alison Li

Theorem.

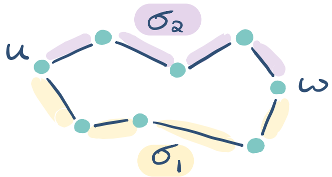
Let $G=(V,E)$. Then,

G is a tree $\Leftrightarrow \forall u,w \in V \; \exists$ unique path from u to w.

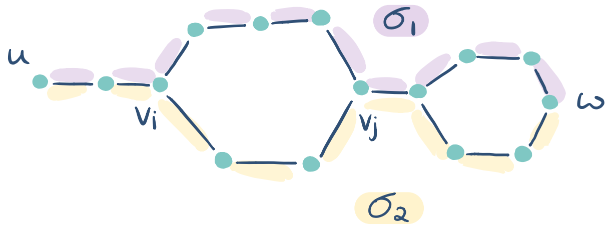
proof.

($\Rightarrow$) Assume G is a tree.

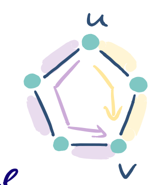
Let $u,w \in V$. Since G is connected, $\exists$ a path u to w.
Assume for a contradiction, there are two distinct paths σ_1, σ_2 u to w.



or



Want to show G has a simple cycle (the contradiction).
Let v_i be the first vertex after which σ_1 and σ_2 differ ($\exists v_i$ since $\sigma_1 \neq \sigma_2$).
Let v_j be the next vertex that σ_1 and σ_2 have in common ($\exists v_j$ since they both end at w).
Then, the path from v_i and v_j along σ_1, then v_j to v_i along σ_2 in reverse is a simple cycle, which is a contradiction.
G is a tree.



($\Leftarrow$) Assume $\forall u,w \in V \; \exists$ unique path. Prove G is a tree.

(i) G is connected since $\forall u,w \in V \; \exists$ path u to w.

(ii) If G has a simple cycle, any two vertices on a cycle has at least two paths. Contradicts unique paths. $\square$

Theorem.

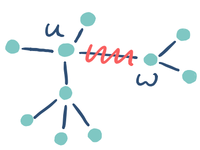
Let $G=(V,E)$. The following are equivalent:

(i) G is a tree.

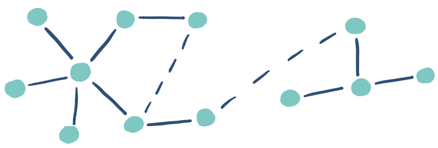
(ii) $|E| = |V| - 1$ and G is connected.

(iii) $\forall u,w \in V \; \exists$ unique path

(iv) G is connected but removing any edge will disconnect G.



(v) G has no simple cycles but adding any edge (no new vertices) produces a simple cycle.



Note: This means (i) $\Leftrightarrow$ (ii), (i) $\Leftrightarrow$ (iii), (iii) $\Leftrightarrow$ (iv), etc.
Any of these statements can define a tree.

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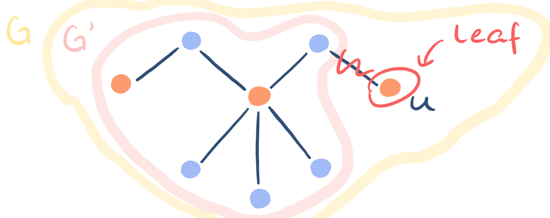
Theorem.
All trees are bipartite.

Proof.
We will actually prove $\chi(G) \leq 2$.
By induction on n = number of vertices.

(i) Base case.
Let $n=1$. G is just a single vertex. Then,
$\chi(G) = 1 \leq 2$. ✓

(ii) Inductive step.
Let $n \geq 1$. Assume any tree with n vertices has $\chi \leq 2$.
Let $G=(V,E)$ be a tree with $|V| = n+1$.

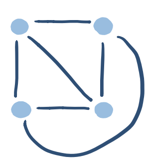


Then, let u be a leaf of G. Remove u and its single edge to obtain graph G' which has n vertices and is still a tree (still connected, no simple cycles). Colour G' with 2 colours. Colour u with the colour not used by its only neighbour. □

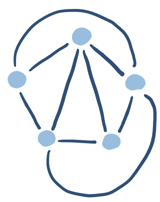
Planar Graphs

def. A graph G is planar if you can draw G in the plane (i.e. on paper) without any edges crossing.

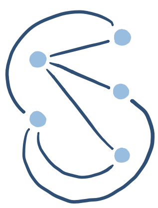
ex. K_4 is planar by this drawing.



ex. K_5 is not planar.



ex. $K_{2,3}$ is planar.



ex. $K_{3,3}$ is not planar.
Try drawing.

Applications.
Circuit boards. Wires cannot cross.
Circuit can be printed ⇔ it is planar.

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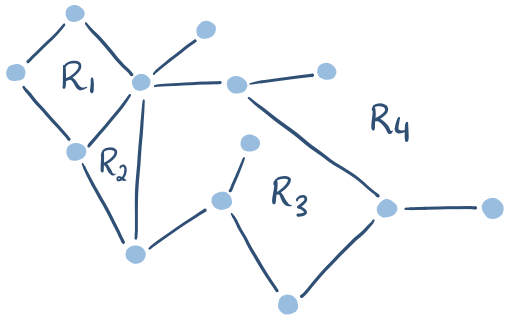
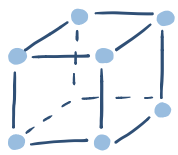
def. A drawing of G in the plane (without edges crossing) is called a <u>planar embedding</u> of G. Such an embedding defines <u>regions</u>.

ex.

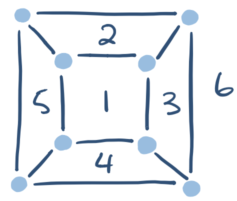


There is one "outside" region.

ex. The cube is planar.



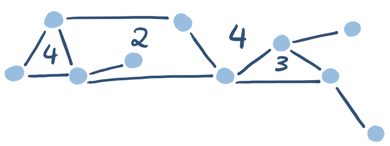
planar $\longrightarrow$ embedding



def. If G is drawn with r regions, the <u>Euler characteristic</u> is $|V| - |E| + r$.

ex. For cube, $8 - 12 + 6 = 2$.

ex.



$10 - 12 + 4 = 2$.

Theorem. Euler's formula.

Let $G = (V, E)$ be a connected graph. If G is drawn in a plane without edges crossing with r regions, then $|V| - |E| + r = 2$. (Or $r = 2 - |V| + |E|$, i.e. r is dependent only on $|V|$ and $|E|$ but not on how you draw G on the plane.)

proof.

By induction on m = number of edges.

(i) Base case.

Let $m = 0$.

Connected G with no edges is a single vertex.

So $|V| - |E| + r = 1 - 0 + 1 = 2$. ✓

(ii) Inductive step.

Let $m \geq 0$. Assume any connected graph G drawn in plane with m edges satisfies $|V| - |E| + r = 2$.

Let $G = (V, E)$ be connected, drawn in plane with r_G regions and $|E| = m + 1$. Remove an edge e to obtain graph $G' = (V', E')$ $(V' = V)$. Then G' is drawn in the plane, no edges crossing.

Two cases to consider:

⋮

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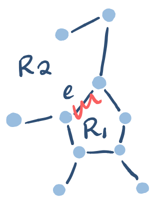
(1) G is a tree. Then $|E| = |V| - 1$, then $r_G = 1$ so $|V| - |E| + r_G$
$= |V| - (|V| - 1) + 1 = 2$. ✓

(2) G is not a tree. So G has a simple cycle (G is connected).
Remove an edge e from the cycle to produce a connected graph $G' = (V', E')$ $(V' = V)$, still drawn on plane.
On either side of e in G, there are two different regions.
Hence in G', these regions are merged into one so $r_{G'} = r_G - 1$.
But G' has m edges so IA applies to G' i.e. $|V'| - |E'| + r_{G'} = 2$.



Then, for G,

$$
\begin{aligned}
|V| - |E| + r_G &= |V'| - (|E'| + 1) + r_{G'} + 1 \\
&= |V'| - |E'| - 1 + r_{G'} + 1 \\
&= 2. \ ✓ \qquad \square
\end{aligned}
$$