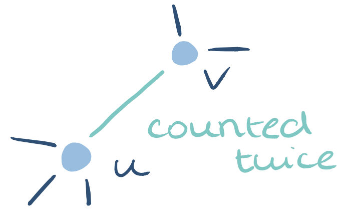
Alison Li

Theorem.

Let $G=(V,E)$ be a finite simple graph. Then,

$$\sum_{v\in V} \deg(v) = 2|E|.$$

(sum of degrees is $2 \cdot$ #edges)



proof.

Right side: Counts each edge 2 times.

Left side: Each edge $e=(u,v)$ contributes $+1$ to $\deg(u)$, $+1$ to $\deg(v)$, i.e. $+2$ to left side so left side also is counting each edge twice. $\square$

Corollary. ("handshaking lemma")

In a finite simple graph, the number of odd-degree vertices is even.

proof.

Partition V into $V = V_o \cup V_e$,

$V_o = \{v \in V \mid \deg(v) \text{ odd}\}$

$V_e = \{v \in V \mid \deg(v) \text{ even}\}$

Clearly $V_o \cap V_e = \emptyset$ and $V_o \cup V_e = V$ is a partition.

Take $2|E| \equiv \sum_{v\in V} \deg(v) \bmod 2$.

$$0 \equiv \sum_{v\in V_o} \deg(v) + \sum_{v\in V_e} \deg(v) \pmod 2$$

$$0 \equiv \sum_{v\in V_o} 1 + \sum_{v\in V_e} 0 \pmod 2$$

$$0 \equiv |V_o| + 0|V_e| \equiv |V_o| \pmod 2$$

i.e. Number of odd degree vertices is even. $\square$

Interpretation of "handshake lemma":

Party; some people shake hands.

Person = vertex.

(u,v) is an edge if u and v shook hands. Corollary says number of people who shook an odd number of hands is even.

Problem.

If $G=(V,E)$ is k-regular, how many edges does G have?

Let $n=|V|$.

Then, by the theorem,

$$2|E| = \sum_{v\in V} \deg(v) = \sum_{v\in V} k = |V|k = nk.$$

So $|E| = \frac{nk}{2}$.

ex. Q_n hypercube. How many edges?

We know Q_n is n-regular. $|V| = 2^n$ (all binary strings of length n).

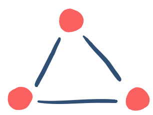
So $|E| = \frac{|V|n}{2} = \frac{2^n \cdot n}{2}$.

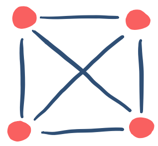
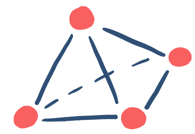
Alison Li

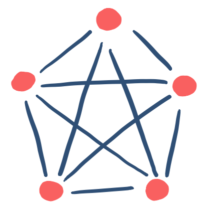
def. The complete graph K_n is the graph with n vertices where all pairs of vertices have an edge connecting them.

ex. K_1: 

K_2: 

K_3: 

K_4:  or 

K_5: 

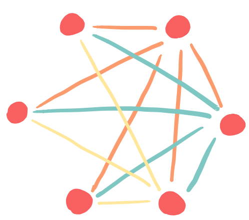
Proposition.
K_n has $\binom{n}{2}$ edges.

proof #1.
Number of edges = #pairs of vertices = $\binom{n}{2}$. □

proof #2.
Each vertex in K_n has $n-1$ neighbours. (all except itself).
So K_n is $n-1$ regular.

$$|E| = \frac{n(n-1)}{2} = \binom{n}{2}.$$ □

proof #3.

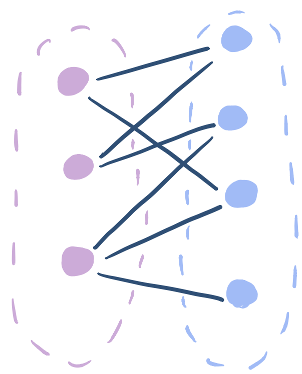


Draw edges, for each vertex.

$$(n-1) + (n-2) + (n-3) + \cdots + 1 = \sum_{i=1}^{n-1} i = \frac{(n-1)}{2}.$$ □

Bipartite Graphs

def. $G = (V, E)$ is bipartite if there is a partition $V = V_1 \cup V_2$ such that all edges have one endpoint in V_1 and the other in V_2.
In this case, V_1, V_2 is called a bipartition.

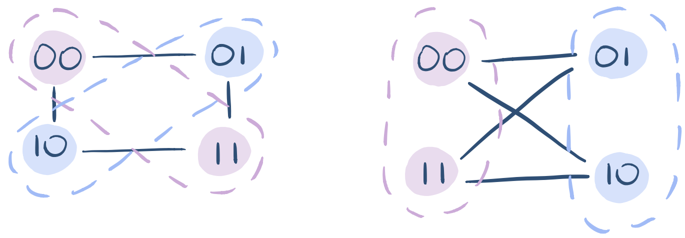


ex. Prove Q_n is bipartite.
Start with a few examples Q_1, Q_2, Q_3, then guess the right partition.

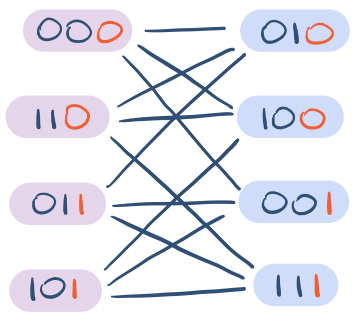
Q_1: 

Alison Li

Q_2:



Q_3:



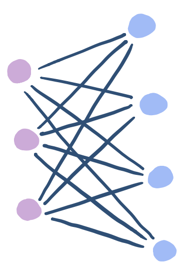
Flip the second copy left-right.

proof #1.
Induction on n. Take 2 copies of Q_n, "flip one", then form Q_{n+1}. Show Q_{n+1} is also bipartite. (fill in the details.) ☺

proof #2.
Partition V into
$V_0 = \{ s \in V \mid s \text{ has even number of 1's} \}$
$V_1 = \{ s \in V \mid s \text{ has odd number of 1's} \}$
Clearly $V_0 \cap V_1 = \emptyset$ and $V_0 \cup V_1 = V$ so it is a partition.
Let $e = (s_1, s_2)$ be an edge in Q_n. Then, s_1, s_2 same except for one bit. Hence, if one has even number of 1's, other has odd number of 1's.
So one endpoint in V_0, other in V_1. $\square$

def. The complete bipartite graph $K_{m,n}$ is a graph with $V = V_1 \cup V_2$, $|V_1| = m$, $|V_2| = n$ and $\forall u \in V_1, v \in V_2 \ \exists$ edge $(u,v) \in E$ (no other edges).

ex. $K_{3,4}$ (same as $K_{4,3}$)



Problem.
How many edges does $K_{m,n}$ have?
For each vertex on the "m" side, have n edges.
So $m \cdot n$ total by intuition.
OR
$V = V_1 \cup V_2$ bipartition.
$|V_1| = m$, $|V_2| = n$.
Each $u \in V$ has degree n, each $v \in V_2$ has degree m.

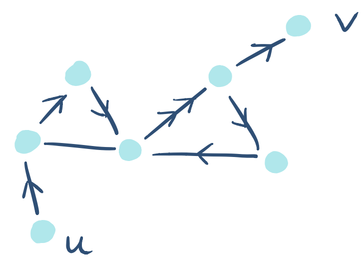
Alison Li

Then, $2|E| = \sum_{v \in V} \deg(v) = \sum_{u \in V_1} n + \sum_{v \in V_2} m = |V_1|n + |V_2|m = mn + nm$.

So $|E| = mn$.

Paths and Cycles

$G = (V, E)$. Let $u, v \in V$.
a path from u to v is a sequence of vertices
$$\sigma = (v_1, v_2, \dots, v_n)$$
where $v_1 = u$, $v_n = v$, and $\forall i = 1, 2, \dots, n-1$, $(v_i, v_{i+1}) \in E$.



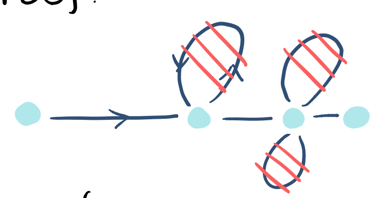
a cycle is a path with $v_1 = v_n$ (returns to start).

a path is simple if no vertex is repeated (no vertex "visited" more than once).

a cycle is simple if no vertex repeated except for v_1 and v_n at the start/end and length is at least 3. (●—● not a simple cycle).

length of a path $\sigma = (v_1, v_2, \dots, v_n)$ is the number of edges (i.e. $n-1$ here).

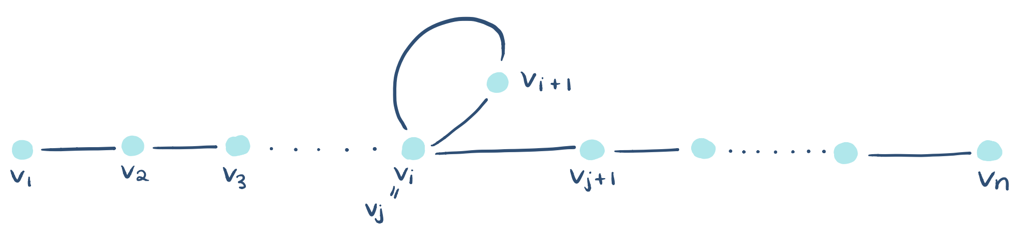
Proposition.
If there is a path from u to v, there is a simple path from u to v.
proof.



The idea is to "remove loops".
(not a proof but for visual understanding).

proof.

Minimality Argument:
Let $\sigma = (v_1, v_2, \dots, v_n)$ be a path u to v of the shortest length. ($\neg\exists$ shorter path u to v). Claim σ is a simple path.
For contradiction, assume σ is not simple.
Hence, there is a repeated vertex, i.e. $v_i = v_j$ some $i < j$.



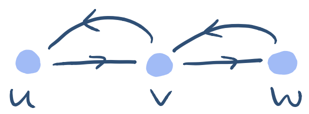
Then, $\sigma' = (v_1, v_2, \dots, v_j, v_{j+1}, \dots, v_n)$ is a shorter path u to v.
Contradiction of a minimal length of σ. So σ must be a simple path. $\square$

Alison Li

Proposition.

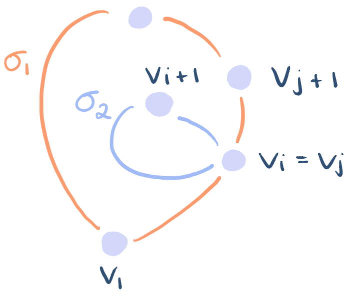
If G has an odd-length cycle, then it has an odd-length simple cycle.

Note: Not true for even instead of odd.



$G = (u, v, w, v, u)$.
Cycle of length 4, not simple.
No (even-length) simple cycle exists.

proof.



Minimality argument.
Let $\sigma = (v_1, v_2, \dots, v_n)$ be an odd-length cycle of minimal length (i.e. In G, there are no shorter odd-length cycles).
Claim σ is a simple cycle.
For contradiction, assume σ is not a simple cycle (i.e. minimal cycles are even).
So $v_i = v_j$, some $i < j$.

Then, $\sigma_1 = (v_1, v_2, \dots, v_i, v_{j+1}, v_{j+2}, \dots, v_n)$
$\sigma_2 = (v_i, v_{i+1}, \dots, v_j)$
are both cycles of shorter length.
By minimality, both have even length.
But length of σ is sum of lengths σ_1, σ_2, hence σ is even.
Contradiction. (for length of σ to be odd, σ_1 or σ_2 must be odd).
So, σ is a simple cycle. $\square$