Alison Li

Pascal's Triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

many patterns ...

	sum
1	1
1 1	2
1 2 1	4
1 3 3 1	8
1 4 6 4 1	16
1 5 10 10 5 1	32

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$ Recall Pascal's Idenity.

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

i.e. coefficients in expansion are the binomial coefficients $\binom{n}{k}$.

proof.

Idea: $(x+y)(x+y) = x^2 + xy + yx + y^2$.

$$(x+y)(x+y)(x+y) = x^3 + xxy + xyx + yxx + xyy + yxy + yyx + y^3$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^n = (x+y)(x+y)\cdots(x+y).$$

To obtain $x^{n-k}y^k$ in expansion, it is necessary to choose $n-k$ x's from the n binomial factors, so the other k terms in the product are y's.
Number of ways is number of ways to choose k.
Factors (the y's) chosen from set of n is $\binom{n}{k}$.

So coefficient of $x^{n-k}y^k$ is $\binom{n}{k}$. $\square$

Note: $(x+y)^2 = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

Let $\ell = n-k$
$k = n - \ell$.

$$= \sum_{\ell=0}^{n} \binom{n}{n-\ell} x^{\ell} y^{n-\ell}$$
$$= \sum_{\ell=0}^{n} \binom{n}{\ell} x^{\ell} y^{n-\ell}.$$

ex. Find coefficient of x^5 in $(2x-3)^{15}$.

$$(2x-3)^{15} = \sum_{k=0}^{15} \binom{15}{k} (2x)^k (-3)^{15-k}$$

Need $k=5$.

$$\binom{15}{5}(2x)^5(-3)^{10} = \underbrace{\binom{15}{5} 2^5 (-3)^{10}}_{\text{coefficient of } x} x^5$$

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Idea: Then, for all $x, y \in \mathbb{R}$, you can substitute any values.

ex. Set $x = 1, y = 1$.

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k$$

Corollary.

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

proof. (combinatorial)

LS: $2^n = |P(\{1, 2, \cdots, n\})|$ = number of subsets.

RS: $\underbrace{\binom{n}{0}}_{\emptyset} + \underbrace{\binom{n}{1}}_{\text{sets of size 1}} + \underbrace{\binom{n}{2}}_{\text{size 2}} + \cdots + \underbrace{\binom{n}{n}}_{\text{size } n}$.

ex. Set $x = 1, y = -1$.

$$0 = (1-1)^n = \sum_{k=0}^{n} (-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n}.$$

Corollary.

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Combinatorial Interpretation:

Let $X = \{1, 2, \cdots, n\}$

$$\underbrace{\binom{n}{1}}_{\text{subsets size 1}} + \underbrace{\binom{n}{3}}_{\text{size 3}} + \underbrace{\binom{n}{5}}_{\text{size 5}} + \cdots (\cdots) = \underbrace{\binom{n}{0}}_{\text{size 0}} + \underbrace{\binom{n}{2}}_{\text{size 2}} + \underbrace{\binom{n}{4}}_{\text{size 4}} + \cdots (\cdots).$$

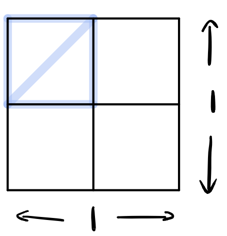
number of ODD sized subsets = number of EVEN sized subsets

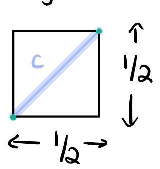
Pigeonhole Principle

If more than n objects are placed into n boxes, then one box has at least two things in it.

ex. 5 darts hit a square 1×1 dart board.
Prove that 2 of the darts are within $1/\sqrt{2}$ of each other.

Objects = 5 darts
boxes = 4 boxes (or less)





By the Pigeonhole Principle, 2 darts are in the same region.

$$c^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

So $c = \sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}$.

So all points within $\frac{1}{\sqrt{2}}$ of each other.

Theorem. Pigeonhole Principle.
Let A, B finite sets with $|A| > |B|$.
Then, any $f: A \to B$ is not injective.
That is, if $f: A \to B$ then $\exists a_1, a_2 \in A$, $a_1 \neq a_2$ with $f(a_1) = f(a_2)$.

ex. Let $A \subseteq \{1, 2, \cdots, 100\}$. $|A| = 10$.
Prove there exists subsets $X \subseteq A$, $Y \subseteq A$, $(X \neq Y)$ such that sum of numbers in X equals to sum of numbers in Y.
Illustration:
$A = \{3, 7, 8, 14, 23, 37, 41, 55, 56, 99\}$
So, $7 + 56 = 8 + 55$

proof.
Function f should complete the sum ("same sum"). Domain is subsets of A ("objects").
So, $f: P(A) \to \mathbb{N}$.
$f(x) =$ sum of numbers in X.
Replace $\mathbb{N}$ with something smaller than $P(A)$.
max sum $= 100 + 99 + 98 + \cdots + 91 \leq 1000$.
Since $\emptyset \subseteq P(A)$, let $f(\emptyset) = 0$.
So use $\{0, 1, 2, \cdots, 1000\}$ for codomain.
So $f: \underbrace{P(A)}_{2^{10} = 1024 \text{ values}} \to \underbrace{\{0, 1, \cdots, 1000\}}_{1001 \text{ values}}$.

By Pigeonhole Principle, f not injective. So there exists $X \subseteq A$, $Y \subseteq A$, $X \neq Y$ such that $f(X) = f(Y)$.
sum in X = sum in Y.

Note: Proof does not help to find X, Y. ("non-constructive proof").

Application of Pigeonhole: "Lossless data compression"
Let $B = \{0, 1\}$.
$B^n = B \times B \times B \times \cdots \times B$. binary strings of length n.
$B^0 = \{\text{empty string}\}$
$\mathbb{B} := B^0 \cup B^1 \cup B^2 \cup B^3 \cup \cdots = \bigcup_{n=0}^{\infty} B^n$. all binary strings, any length (finite)
"defined by"

a datafile is a binary string, i.e. element of $\mathbb{B}$. So $\mathbb{B}$ = all possible data files.
an algorithm/program is a function $\alpha: \mathbb{B} \to \mathbb{B}$.
def. A lossless data compression algorithm is a function $\alpha: \mathbb{B} \to \mathbb{B}$ such that
(1) α is injective ("lossless"). (If not, $\alpha(D_1) = \alpha(D_2)$, $D_1 \neq D_2$, so you could not recover D_1 from $\alpha(D_1)$.)
(2) Let $\ell(x)$ = length of x = number of bits, $x \in \mathbb{B}$.
$\forall x \in \mathbb{B}$ $\ell(\alpha(x)) \leq \ell(x)$.
(no datafile should get longer).
(3) $\exists x \in \mathbb{B}$ $\ell(\alpha(x)) < \ell(x)$.
(data must be compressed).

Theorem.
α does not exist.

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proof.

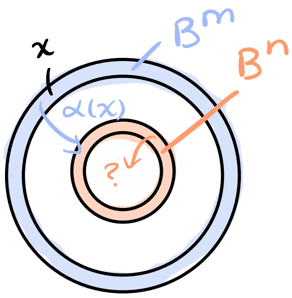
By contradiction.

Assume α exists. By (3), $\exists x \; \ell(\alpha(x)) < \ell(x)$.

Assume x is the shortest input $x \in \mathbb{B}$ that satisfies (3). so $\ell(\alpha(x)) < \ell(x)$.

Denote $m = \ell(x)$, i.e. $x \in B^m$. (all shorter files not shorter when compressed).

Let $n = \ell(\alpha(x)) < m$. So, all $y \in B^n$, satisfy $\ell(\alpha(y)) = \ell(y)$. (no change)



Consider α but with domain restricted to $B^n \cup \underbrace{\{x\}}_{\text{some compressed file}}$.

We know $\alpha(x) \in B^n$.

$\forall y \in B^n$, $\alpha(y) \in B^n$.

So we have a function that is injective by (1).

$$\alpha : \underbrace{B^n \cup \{x\}} \to B^n.$$

But $|B^n \cup \{x\}| = |B^n| + 1 > |B^n|$.

So α is not injective. Contradiction to Pigeonhole Principle.

So α does not exist. $\square$