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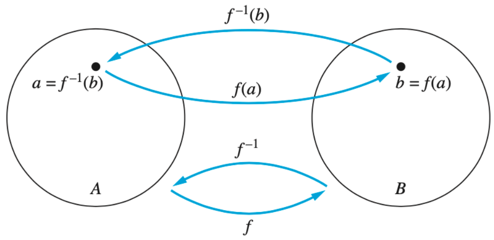
Note: $f: A \to B$ bijective means correspondence between A and B.

def. If f is bijective, we define the inverse as $f^{-1}: B \to A$, which is defined by:
For each $b \in B$, $f^{-1}(b) = a$ where a is the unique value in A such that $f(a) = b$.



Note:
(1) Since f is surjective, there exists at least one $a \in A$ for each $b \in B$.
(2) Since f is injective, can't have two a_1, a_2 ($a_1 \neq a_2$) with $f(a_1) = f(a_2) = b$.

ex. For $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n+2$.
f is injective since if $f(n) = f(m)$ then $m+2 = n+2$, so $m = n$.
f is surjective since if $m \in \mathbb{Z}$, $n = m-2 \in \mathbb{Z}$ and $f(n) = (m-2)+2 = m$.
f is bijective since the inverse of f is $f^{-1}(n) = n-2$.

Proposition.
Let A, B be finite sets with $|A| = |B|$ and $f: A \to B$.
Then f is injective $\Leftrightarrow$ f is surjective.
This is only for sets of the same size.
proof. Exercise; use range(f).
($\Rightarrow$) By contraposition. Assume f not surjective, range$(f) \neq B$.
So $|\text{range}(f)| < |B| = |A|$. Then f not injective.
($\Leftarrow$) By contraposition. Assume f not injective.
... (similar proof). $\square$

Proposition.
Let A, B be finite sets and $f: A \to B$. Then f is bijective $\Leftrightarrow$ $|A| = |B|$.

Cardinality of Sets

def. Let A, B be sets (infinite or finite).
(1) If there is a bijection of $f: A \to B$, we say A and B have the same cardinality. $|A| = |B|$.
(2) If there exists a function $f: A \to B$ which is injective, we say $|A| \leq |B|$.
(3) If $|A| \leq |B|$ but $|A| \neq |B|$, we say $|A| < |B|$.

ex. (1) $f: \mathbb{N} \to \{1, 2, 3, \ldots\}$ defined by $f(n) = n+1$.
f is injective since if $f(n) = f(m)$ then $n+1 = m+1$, so $n = m$.
f is surjective since if $m \in \{1, 2, \ldots\}$ then $n = m-1 \in \{0, 1, 2, \ldots\}$ and $f(m-1) = m$.
f is bijective so $|\{0, 1, 2, \ldots\}| = |\{1, 2, 3, \ldots\}|$.

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Picture:
$$\begin{array}{ccccccccc} 0 & , & 1 & , & 2 & , & 3 & , & 4 \\ | & & | & & | & & | & & | \\ 1 & , & 2 & , & 3 & , & 4 & , & 5 \end{array}$$

As cardinality is concerned, they're the same.

(2) $f : \mathbb{N} \to \{0, 2, 4, 6, \dots\}$ defined by $f(n) = 2n$ is bijective.
So, $|\mathbb{N}| = |\{\text{even natural numbers}\}|$.

(3) $f : \mathbb{N} \to \mathbb{Z}$ $\quad f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -\frac{(n+1)}{2} & \text{if } n \text{ odd} \end{cases}$.

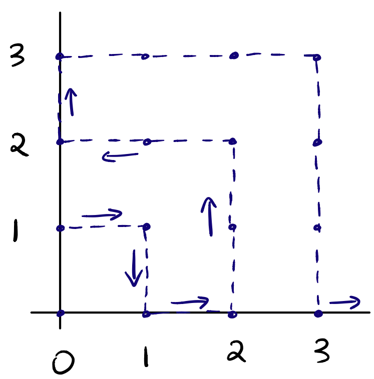
f is bijective, so $|\mathbb{N}| = |\mathbb{Z}|$.
Picture:

$$\begin{array}{ccccccccc} \cdots & f(5) & f(3) & f(1) & f(0) & f(2) & f(4) & \cdots \\ & -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array}$$

def. A set A is countable if $|A| = |\mathbb{N}|$ or A is finite.

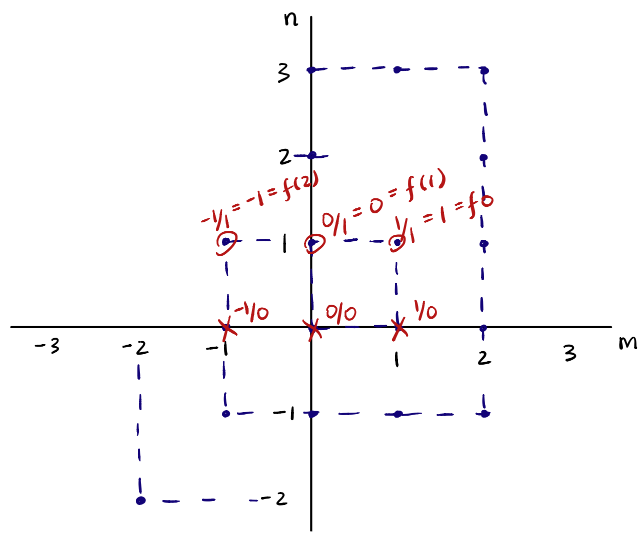
Note: If $f : \mathbb{N} \to A$ is bijective, it means you can enumerate the elements of A as $f(0), f(1), f(2), \dots$ without repetition.

Theorem.
$\mathbb{N} \times \mathbb{N}$ is countable, i.e. $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.
Picture: $\mathbb{N} \times \mathbb{N} = \{(a,b) \mid a, b \in \mathbb{N}\}$.



$f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$

You'll get every point eventually, listing them off.



Theorem.
$\mathbb{Q}$ is countable, i.e. $|\mathbb{Q}| = |\mathbb{N}|$.
$f : \mathbb{N} \to \mathbb{Q}$
$f(0) = 1/1 = 1$
$f(1) = 0/1 = 0$
$f(2) = -1/1 = -1$
$f(3) = 2/-1 = -2$
$f(4) = 2/1 = 2$
$\vdots$

Rational numbers are countable.
Picture: Every $q \in \mathbb{Q}$ is $q = \frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0$. Think of $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ as representing $q = \frac{m}{n}$.

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Uncountable Sets

def. a function $f: \mathbb{N} \to \mathbb{R}$ is called an infinite real sequence.

ex. The sequence $1, 1/2, 1/3, 1/4, \ldots = a_n = \frac{1}{n+1}$ is a function $f(n) = \frac{1}{n+1}$. i.e. $f(n)$ refers to the n^{th} term.

def. an infinite binary sequence is a function $f: \mathbb{N} \to \{0,1\}$.

ex. $0, 0, 0, \ldots$ is the function $f(n) = 0$.

Theorem.
Let A be the set of all infinite binary sequences, i.e. $A = \{s: \mathbb{N} \to \{0,1\}\}$ (the set of sequences), then A is uncountable. This means you cannot list off the elements of A as $a_0, a_1, a_2, \ldots$.