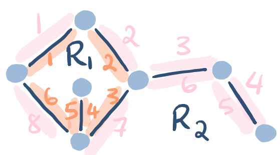
Theorem. (non-planarity condition)
Let $G=(V,E)$ be a connected planar graph with $|V| \geq 3$. Then,
(i) $|E| \leq 3|V| - 6$
(ii) If G has no cycles of length 3 (i.e. no triangles ) then $|E| \leq 2|V| - 4$.

Comments:
(1) If inequality fails, G is not planar.
(2) Let $n = |V|$. In K_n, $|E| = \binom{n}{2} = \frac{n(n-1)}{2} \approx n^2$ (quadratic range) Not planar.
To be planar, $|E| \leq 3n - 6 \approx n$ (linear range)

proof. (non-planarity condition).
For region, define perimeter $p(R)$ to be length of shortest path around boundary of R.



$p(R_1) = 6$
$p(R_2) = 8$

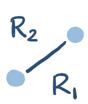
To have: $p(R) = 1$  loop

$p(R) = 2$  multi-edges

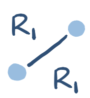
only for connected G of $|V| = 2$.

So for G with $|V| \geq 3$, $p(R) \geq 3$ for all regions.

(i) Suppose G drawn in plane, so by Euler's formula, $|V| - |E| + r = 2$ i.e. $r = 2 - |V| + |E|$.
We claim that if $R_1, R_2, \dots, R_r$ are the regions, then,
$$\sum_{i=1}^{r} p(R_i) = 2|E|.$$
True since each edge is connected exactly twice. Two possibilities:

(1) 

If regions on either side are different, edge counted twice in $p(R_1)$, once in $p(R_2)$. Twice in sum.

(2) 

If regions are the same, edge counted twice in $p(R_1)$.

Then, $2|E| = \sum_{i=1}^{r} p(R_r)$
$$\geq \sum_{i=1}^{r} \underbrace{3}_{\text{all } p(R_i) \geq 3} = 3r = 3(\overbrace{2 - |V| + |E|}^{r})$$
$2|E| \geq 6 - 3|V| + 3|E|$
$|E| \leq 3|V| - 6$

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(ii) If no  , all $p(R_i) \geq 4$ so repeat.

$$2|E| \geq 4(2-|V|+|E|)$$
$$2|E| \geq 8-4|V|+4|E|$$
$$2|E| \leq 4|V|-8$$
$$|E| \leq 2|V|-4 . \qquad \square$$

Corollary.

K_5 and $K_{3,3}$ are non-planar.

proof.

K_5: $|V| = 5$

$|E| = \binom{5}{2} = \frac{5 \cdot 4}{2} = 10$

$3|V|-6 = 15-6 = 9$

$|E| = 10 \nleq 9$ so K_5 not planar.

$K_{3,3}$: $|V| = 6$

$|E| = 9$

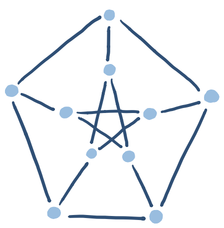
$3|V|-6 = 12$, $9 \leq 12$, not enough.

Need stronger version.

$K_{3,3}$ is bipartite, no odd cycles, so none of length of 3.

So (ii) applies. $2|V|-4 = 8$. $9 \nleq 8$. $K_{3,3}$ not planar.

ex. Try non-planarity on Peterson graph.



$|V| = 10$

$|E| = 15 = (2|E| = \Sigma \deg = 3 \cdot 10)$

$3|V|-6 = 24$. $15 \leq 24$

$2|V|-4 = 16$. $15 \leq 16$

Inconclusive.

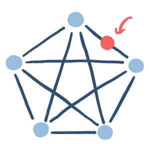
Observation: K_5 non-planar. 

def. a subgraph of G is a graph obtained by removing some vertices / edges.

Proposition.

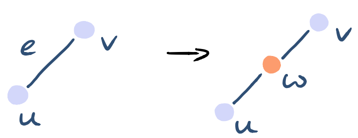
If G has a non-planar subgraph (in particular K_5 or $K_{3,3}$), then G is not planar. In particular, K_n for $n \geq 5$ and $K_{n,m}$ for $n,m \geq 3$ are all non-planar.

Observation.



Not planar (if so, just erase the extra vertex, then K_5 would be planar). But K_5 is not a subgraph.

def. a subdivision of an edge $e = (u,v)$ is

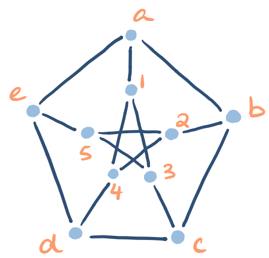


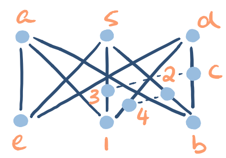
remove e, add new vertex w and two new edges (u,w) and (w,v).

Theorem. Kuratowski.
G is planar $\Leftrightarrow$ G does not contain a subgraph which is obtained from K_5 or $K_{3,3}$ by a sequence of subdivision.

proof.
($\Rightarrow$) Contrapositive. If G has such a subgraph then if you "un-subdivide" the necessary edges, you get a planar embedding of K_5 or $K_{3,3}$.
($\Leftarrow$) Difficult. $\square$

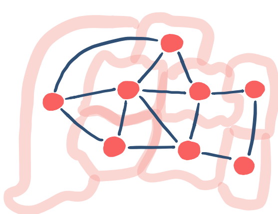
ex. Prove Peterson graph is non-planar.
Look for K_5 or $K_{3,3}$. In this case, look for $K_{3,3}$ since K_5 requires each $v \in V$ to be $\deg(v) = 4$ which does not seem likely.





This is a subgraph obtained from $K_{3,3}$ by subdivision, so the Peterson graph is non-planar.

Map Colouring



Geographic map of countries / states.
Colour countries such that adjacent countries have different colours. Make each country a vertex. Put an edge if two countries have a common border.

A colouring of G gives a colouring of the map.
min number of colours $= \chi(G)$.
But G is planar (unless countries come in multiple pieces).
What is $\chi(G)$ for G planar?

Theorem.
If G is planar, then $\chi(G) \leq 6$.

Lemma.
Every planar graph has a vertex of degree ≤ 5.
proof.
By contradiction, assume $\deg(v) \geq 6 \quad \forall v \in V$.
Then, $2|E| = \sum \deg(v) \geq 6|V|$.
So $|E| \geq 3|V|$, but for planar, $|E| \leq 3|V| - 6 < 3|V|$.
Contradiction. $\square$

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proof. (thm)

By induction on $n = |V|$.

(i) Base case.

Let $n = 1$. Then $\chi(G) = 1 \leq 6$. ✓

(ii) Inductive step.

Let $n \geq 1$. Assume $\chi(G) \leq 6$ for any planar G $|V| = n$.
Let G have $n+1$ vertices. G planar.
Let v be vertex of degree ≤ 5 (by the lemma, $\exists v$).
Remove v to get G', still planar, has n vertices, colour with 6 colours. Then v has at most 5 neighbours so at least one colour is free to use for v. So $\chi(G) \leq 6$. □

Theorem.

G planar, $\chi(G) \leq 5$.

proof.

Long, but not beyond the course.

Theorem. "Four-Colour Theorem".

G planar, $\chi(G) \leq 4$.

proof.

Very difficult; by program.