Alison Li

Theorem.

If $a_0, a_1, a_2, \ldots$ is a sequence that satisfies $a_n = ba_{n-1} + ca_{n-2}$ $(c \neq 0)$ and $x^2 - bx - c = 0$ has one root r, then $a_n = \alpha r^n + \beta n r^n$ where α, β is the solution to:

$$\begin{cases} a_0 = \alpha \\ a_1 = \alpha r + \beta r. \end{cases}$$

proof.

First, since r is the unique root.

$x^2 - bx - c = 0.$

factors as:

$x^2 - bx - c = (x-r)^2 = x^2 - 2rx + r^2.$

So $-b = 2r$, $-c = r^2$, so $b = 2r, -c = r^2$.

(1) System of equations:

$a_0 = \alpha$

$a_1 = \alpha r + \beta r$

has $\beta = \dfrac{a_1 + a_0 r}{r}$, $\alpha = a_0$ has unique solution.

(Note: $r \neq 0$ since $c \neq 0$.)

(2) Proof by induction that $a_n = 2r^n + \beta n r^n$.

(i) Base cases.

$n = 0$, $\alpha r^0 + 0 = \alpha = a_0$

$n = 1$, $\alpha r + \beta r = a_1$ as required.

(ii) Inductive step.

Let $n \geq 2$. Assume formula computes a_m for $m = n-1, m = n-2$.

Then, $a_n = b a_{n-1} + c a_{n-2}$

$= b(\alpha r^{n-1} + \beta(n-1)r^{n-1}) + c(\alpha r^{n-2}(br+c) + \beta r^{n-2}(b(n-1)r + c(n-2)))$.

$= \alpha r^{n-2} r^2 + \beta r^{n-2}(r^2 n) = \alpha r^n + \beta n r^n.$

So a_n is computed by the formula. $\square$

ex. $a_0 = 1$, $a_1 = 3$, $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$. Find non-recursive formula for a_n.

$x^2 = 4x - 4.$

$x^2 - 4x + 4 = 0.$

$(x-2)^2 = 0.$

Find α, β. $r = 2$.

$a_0 = 1 = \alpha r^0 + \beta(0) r^0 = \alpha = 1.$

$a_1 = 3 = \alpha r + \beta(1) r = 2 + 2\beta.$

$1 = 2\beta \Rightarrow \beta = 1/2.$

Formula: $a_n = 2^n + 1/2\, n\, 2^n$

$= 2^n(1 + n/2) = 2^n\left(\frac{2+n}{2}\right) = 2^n(2+n)(2^{-1})$

$= 2^{n-1}(n+2)$

Check: $a_0 = 2^{-1}(2) = 1$

$a_1 = 2^0(3) = 3$

$a_2 = 2^1(4) = 8$ ✓

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Recurrence Relations and matrices

Suppose $a_n = ba_{n-1} + ca_{n-2}$.
Let $A = \begin{bmatrix} b & c \\ 1 & 0 \end{bmatrix}_{2\times 2}$.

Then,

$$A \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix}_{1\times 1} = \begin{bmatrix} b & c \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} ba_{n-1} + ca_{n-2} \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$$

$$A \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} b & c \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} ba_n + ca_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$$

Then,

$$A^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = A^{n-1} \begin{bmatrix} a_2 \\ a_1 \end{bmatrix} = A^{n-2} \begin{bmatrix} a_3 \\ a_2 \end{bmatrix} = \cdots = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$$

Eigenvalues of A:

$$\begin{vmatrix} b-x & c \\ 1 & -x \end{vmatrix} = x^2 - bx - c = 0$$

Correction to previous Lecture:
we proved if $C_n = \#$ of binary strings, no adjecent 1's, then $C_n = C_{n-1} + C_{n-2}$. To prove $C_n = C_{n-1} + C_{n-2}$, you can show that type (II) has C_{n-2} elements. No induction.
However, if you want to prove a formula, e.g. $n = 2^{n-1}$, you can find a recurrence for D_n, then prove $D_n = 2^{n-1}$ via induction and the recurrence relation.

Counting

Basic Counting Principles
(I) To count mutually exclusive things, ADD.
(II) To count "do A then B", MULTIPLY.

ex. Lady has 10 skirts, 4 pants, 5 shirts. How many outfits?
→ choose shirt then (skirt XOR pants).
$5 \cdot (10 + 4) = 70$
→ choose skirt then shirt XOR pants then shirt.
$(10 \cdot 5) + (4 \cdot 5) = 70$

Proposition.
If A, B are finite sets, $|A \times B| = |A||B|$.
proof.
To construct all $(a,b) \in A \times B$, choose a in $|A|$ ways, then choose b in $|B|$ ways. So $|A||B|$ ways total. $\square$

Problem.

Using symbols a, b, c, d, count # of strings of length 8 such that

(i) no restrictions:

$\underline{4}\ \underline{4}\ \underline{4}\ \underline{4}\ \underline{4}\ \underline{4}\ \underline{4}\ \underline{4} = 4^8$

(ii) no adjacent symbols, e.g. aa, bb ...:

$\underline{4}\ \underline{3}\ \underline{3}\ \underline{3}\ \underline{3}\ \underline{3}\ \underline{3}\ \underline{3} = 4 \cdot 3^7$

Counting Functions.

Let A, B be sets with $|A| = m$, $|B| = n$.

How many functions $f: A \to B$ are there?

Denote $A = \{a_1, a_2, \dots, a_n\}$. Build f by:

$f(a_1)$ = choose any of n elements of B.

$f(a_2)$ = "

$\vdots$ $\vdots$

$f(a_m)$ = "

So, # ways to construct f is n^m.

Proposition.

Number of functions $f: A \to B$ is $|B|^{|A|}$. (n^m if $|A| = m$, $|B| = n$).

ex. Characteristic function.

If U = universe, $A \subseteq U$.

Then $\chi_A : U \to \{0,1\}$

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

The total # of characteristic functions $|\{0,1\}|^{|U|} = 2^{|U|}$.

Same as $|P(U)| = 2^{|U|}$

There is a bijection.

$P(U) \to \{\chi : U \to \{0,1\}\}$

$A \to \chi_A$

proof. Exercise. ☺

$\vdots$

Hence, $|P(U)| = |\{\chi : U \to \{0,1\}| = 2$.

Proposition.

Let X be a set $|X| = n$. ($< \infty$).

Then the number of bijective functions $f: X \to X$ is $n!$.

proof.

Denote $X = \{x_1, x_2, \dots, x_n\}$.

Construct $f: X \to X$ by:

$f(x_1)$ = choose any of $x_1, \dots, x_n$ $\longrightarrow$ n ways

$f(x_2)$ = " except $f(x_1)$ $\longrightarrow$ n-1 ways

$f(x_3)$ = " except $f(x_1), f(x_2)$ $\longrightarrow$ n-2 ways

$\vdots$ $\vdots$ $\vdots$

$f(x_n)$ = only one choice. $\longrightarrow$ 1 ways

Thus, $n(n-1)(n-2)\cdots 1 = n!$ bijective functions total. $\square$

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def. A bijection $f: X \to X$ is also called a permutation.

Illustration: $X = \{1, 2, 3, 4\}$ and define $f: X \to X$.

x	1	2	3	4
$f(x)$	3	1	2	4

ex. How many ways can 10 books be arranged left to right on one shelf?

$10 \cdot 9 \cdot 8 \cdot \cdots \cdot 1 = 10!$

In fact, we are counting bijections.

$f: \{1, 2, \ldots, 10\} \to \{\text{books}\}$

$f(1) = 1^{st}$ book

$f(1) = 2^{nd}$ book

$\vdots$