Alison Li

Reflexive: $\forall x \; xRx$
Symmetric: $\forall x, y \; xRy \Rightarrow yRx$
Transitive: $\forall x, y, z \; (xRy \wedge yRz \Rightarrow xRz)$

Problem.
Let U be a set with at least 2 elements.
Define a relation on $\mathcal{P}(U)$ by:
$(A, B) \in R \Leftrightarrow A \cap B \neq \emptyset$.
Is R reflexive, symmetric, transitive?
(1) ($A \cap A = A$) but $(\emptyset, \emptyset) \notin R$. Since $\emptyset \cap \emptyset = \emptyset$, R is not reflexive.
(2) Symmetric, since if $A \cap B \neq \emptyset$ then $B \cap A \neq \emptyset$ by commutative law.
(3) Let $u, v \in U$, $u \neq v$. (since U has at least 2 elements.)
Set $A = \{u\}$, $B = \{u, v\}$, $C = \{v\}$.
Then, $A \cap B = \{u\} \neq \emptyset$, $B \cap C = \{v\} \neq \emptyset$.
But $A \cap C = \emptyset$. R is not transitive.

Equivalence Relations

def. If relation R is reflexive, symmetric, and transitive, R is called an equivalence relation. In this case, xRy is written as $x \sim y$. "x is equivalent to y."

ex. (1) "=" is an equivalence relation.
(2) On $\mathbb{Z}$, set $x \sim y \Leftrightarrow |x| = |y|$.
(3) On $\mathbb{Z} \times \mathbb{Z}$, define $(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow y_1 = y_2$.
(Imagine all the points on a plane with the same y-coordinate).
(4) Some names are "equivalent".
Mike ~ Michael ~ Michel ~ Mikhael.
Peter ~ Pierre.

def. If R is an equivalence relation on set X and $x \in X$, then equivalence class of x is the set of all $y \in X$ where $x \sim y$, denoted by $[x] = \{y \in X \mid x \sim y\}$.

ex. (1) For $x \sim y \Leftrightarrow |x| = |y|$ on $\mathbb{Z}$.
$[2] = \{-2, 2\}$
$[0] = \{0\}$
$[17] = \{-17, 17\}$
(2) [Mike] = {Michael, Mikhael, Michel, Mike}.
(3) On $\mathbb{Z} \times \mathbb{Z}$ $(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow y_1 = y_2$.
$[(1, 3)] = \{(x, y) \mid y = 3\}$.

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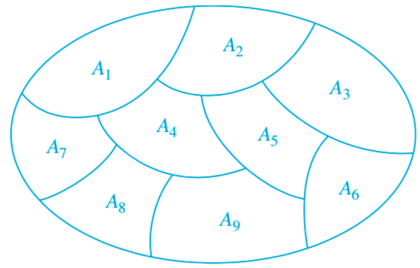
def. Let X be a set and $P \subseteq \mathcal{P}(X)$ where $\emptyset \notin P$ (P = set of subsets of X.). Then P is called a partition of X if:

(i) $\forall x \in X \ \exists$ a set $S \in P$ $x \in S$
(all $x \in X$ is in at least one set).

(ii) $\forall S_1, S_2 \in P$ $(S_1 \neq S_2 \Rightarrow S_1 \cap S_2 = \emptyset)$
(no x is in two sets).



ex. (1) $X = \{1, 2, 3, \dots, 10\}$
$P = \{\{1,10\}, \{2,3,4\}, \{7,9\}, \{8\}, \{5,6\}\}$ is a partition of X.

(2) $P = \{\{0,2,4,\dots\}\}, \{1,3,5,\dots\}\}$ is a partition of $\mathbb{N}$.

Theorem.
If R is an equivalence relation on X, then the set of all equivalence classes $P = \{[x] \mid x \in X\}$ forms a partition.

Note: Duplicate elements are removed from sets so if $[x] = [y]$ then it appears only once in P.

proof.
$\emptyset \notin P$ since $x \in [x]$ so $[x] \neq \emptyset$. The equivalence class is never empty.

(i) Let $x \in X$. Then $[x] \in P$ and $x \in [x]$ since $x \sim x$ (reflexive).

(ii) Let $[x], [y] \in P$. Assume $[x] \neq [y]$. Prove $[x] \cap [y] \neq \emptyset$ by contradiction. We will prove $[x] = [y]$.

(1) $[x] \subseteq [y]$.
Let $w \in [x]$, so $x \sim w$. We know $z \in [w]$ so $x \sim z$ and $z \in y$ so $y \sim z$. Then, $w \sim x$ (symmetric), $x \sim z$, $w \sim z$ and $y \sim z$ so $z \sim y$.
By transitivity, $w \sim z$, $z \sim y$, so $w \sim y$. By symmetry $y \sim w$ so $w \in [y]$.

(2) $[y] \subseteq [x]$.
Similar argument.

Hence, $[x] = [y]$. Contradiction. $\square$

ex. $F: \{f: \mathbb{R} \to \mathbb{R} \mid f$ is differentiable$\}$.
Define a relation R on f by: $(f(x), g(x)) \Leftrightarrow f'(x) = g'(x)$.
(same derivative)

R is an equivalence relation.

What is $[x^2] = \{f: \mathbb{R} \to \mathbb{R} \mid 2x = f(x)\}$?
$= \{x^2 + c \mid c \in \mathbb{R}\}$
$= \int 2x\,dx =$ "$x^2 + c$".

i.e. Indefinite integral is an equivalence relation.

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Mathematical Induction

Problem.
What is the sum of the first n odd integers?
Partial solution.

$$1 = 1 = 1^2$$
$$1 + 3 = 4 = 2^2$$
$$1 + 3 + 5 = 9 = 3^2$$
$$1 + 3 + 5 + 7 = 16 = 4^2$$
$$\vdots$$

Conjecture that $\sum_{k=1}^{n} (2k-1) = n^2$

Not a proof. Need mathematical induction.

Theorem. Principle of Mathematical Induction
Let $P(n)$ be a predicate $(n \in \mathbb{N})$.
(i.e. $P(n)$ evaluates to T/F when you sub n).
If
(i) $P(0)$ "Base case"
(ii) $\forall n \in \mathbb{N}\ (P(n) \Rightarrow P(n+1))$ "Inductive step"
Then,
$\forall n \in \mathbb{N}\ P(n)$ (i.e. $P(n)$ true for all n)

Idea: If (i) and (ii) true, then $P(0)$.
apply (ii) with $n = 0$, so $P(0) \Rightarrow P(1)$. Know $P(0)$, hence $P(1)$.
apply (ii) with $n = 1$, so $P(1) \Rightarrow P(2)$. Know $P(1)$, hence $P(2)$....

proof.
assume (i) and (ii). To prove $\forall n\ P(n)$.
For contradiction, assume:
$\neg\forall\ P(n)$ i.e. $\exists n \in \mathbb{N}\ \neg P(n)$.
Let $F = \{n \in \mathbb{N} \mid \neg P(n)\}$
We know $F \neq \emptyset$ by assumption.
Let m be the least element of F, by well-ordering principle.
i.e. $m = \min(F)$ so we know $\neg P(m)$ and if $n < m$, then $P(n)$ true.
Consider $m-1$.
We know $P(0)$ by (i). If $P(0)$ true, then $m \neq 0$.
so $m > 0$,
so $m \geqslant 1$,
so $m - 1 \geqslant 0$,
so $m - 1 \in \mathbb{N}$.
Then,
$m - 1 < m$ so $m - 1 \notin F$.
so $P(m-1)$ must be true.
So by (ii) of the theorem,
$P(m-1) \Rightarrow P(m)$.
Hence, $P(m) \wedge \neg P(m)$, contradiction. $\square$

Note:

(1) To use mathematical induction, prove (i), (ii), then $\forall n\ P(n)$.

(2) Base case be $n=1$ or $n=2$ or something larger.

ex. Prove $\sum_{k=1}^{n}(2k-1) = n^2$ (for all $n \geq 1$)

proof.

(i) Base case $n=1$.

$$LHS = \sum_{k=1}^{1}(2k-1) = 2(1)-1 = 1$$

$$RHS = 1^2 = 1$$

(ii) Prove $\forall n \geq 1\ P(n) \Rightarrow P(n+1)$

Let $n \geq 1$, assume

$$\sum_{k=1}^{n}(2k-1) = n^2 \Big\} \text{ Inductive Assumption (IA).}$$

Then,
$$\sum_{k=1}^{n+1}(2k-1) = \left(\sum_{k=1}^{n}(2k-1) + (2(n+1)-1)\right)$$
$$= n^2 + 2n + 1 \quad \text{by IA}$$
$$= (n+1)^2.$$

So, by induction,
$$\sum_{k=1}^{n}(2k-1) = n^2 \text{ for all } n \geq 1. \quad \square$$