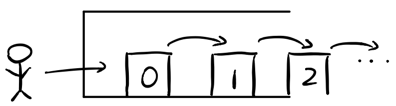
Alison Li

Hilbert's Grand Hotel

a hotel with infinitely many rooms that are countable.



The rooms are full; if someone new wants to check in, manager says: "If in room n, move to room $n+1$."
a countably infinite hockey team arrives; each player has a number $m = 0, 1, 2, \dots$. Manager: "If in room n, move to $2n+1$ and player m go to $2m$."

Theorem.
Let $A = \{ s : \mathbb{N} \to \{0,1\} \}$ (infinite binary sequence). A is uncountable.

proof. Cantor's Diagonal Argument.
Proof by contradiction. Assume A is countable. Clearly, A is not finite. So, $\exists$ bijection $\mathbb{N} \to A$, i.e. you can list the elements of A as $s_0, s_1, s_2, \dots$ (each element of A appears exactly once in list.) Arrange the sequences into an infinite grid.

s_0	0	1	1	1	0	0	1	1	0	...
s_1	1	1	0	0	0	0	0	1	1	...
s_2	0	1	1	1	1	1	0	0	0	...
s_3	1	1	0	0	0	1	1	1	0	...
$\vdots$										
$q = s_m$						?				

Define a sequence $q \in A$, $q : \mathbb{N} \to \{0,1\}$ by "negating the diagonal," i.e. $q(n) = \neg s_n(n)$. e.g. q: 1 0 0 1 ... .
Since $q \in A$, q is in the list so $q = s_m$ for some m.
What is $q(m)$?

$q(m) = \neg s_m(m)$

$q(m) = \neg q(m)$

Contradiction. A is uncountable. $\square$

Corollary.
(i) The interval $[0,1]$ in $\mathbb{R}$ is uncountable.
(ii) $\mathbb{R}$ is uncountable.

proof.
(i) Binary decimal numbers. Each $x \in [0,1]$ can be represented as a binary number.

infinite binary sequence

$$0.\underbrace{1101101}_{} = \frac{1}{2} + \frac{1}{4} + \frac{0}{8} + \frac{1}{16} + \frac{1}{32} + \frac{0}{64} + \frac{1}{128}$$

1/2 1/4 1/8 1/16 1/32 1/64 1/128

Each $x \in \mathbb{R}$ is represented by an infinite binary sequence, i.e. an element of A. So bijection from $A \to [0,1]$ (not exactly though).
Hence, $|A| = |[0,1]|$ uncountable. □

(ii) $f: [0,1] \to \mathbb{R}$ defined by $f(x) = x$ is injective. Hence, $|[0,1]| \leq |\mathbb{R}|$.
Hence, $\mathbb{R}$ also uncountable. □

Comment: In fact, $|[0,1]| = |\mathbb{R}|$. Not obvious.

Fact: $|X| < |P(X)|$. Prove by showing $f: X \to P(X)$ injective.

Composition of Functions

def. If $f: A \to B$, $g: B \to C$, we define the composition of $g \circ f: A \to C$ by
$(g \circ f)(a) = g(f(a))$. $(A \to B \to C)$.

Proposition.
$f: A \to B$, $g: B \to C$
(i) If f, g injective, then $g \circ f$ injective.
(ii) If f, g surjective, then $g \circ f$ surjective.
(iii) " bijective.

proof of (i).
Assume f, g both injective. Suppose $(g \circ f)(a_1) = (g \circ f)(a_2)$.
That is, $g(\underbrace{f(a_1)}_{b_1}) = g(\underbrace{f(a_2)}_{b_2})$

We have $g(b_1) = g(b_2)$.
Since g is injective, $b_1 = b_2$, i.e. $f(a_1) = f(a_2)$.
But then f is injective, so $a_1 = a_2$ which shows $g \circ f$ injective. □

Fact (helpful for assignment):
Let A, B be finite sets. Then $|A \cup B| = |A| + |B| - |A \cap B|$.

Relations

def. Let A be a set. A relation R on A is a subset of $A \times A$.
$R \subseteq A \times A$.

ex. (1) $R = \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid m = 2n\}$
e.g. $(1, 2) \in R$
$(5, 10) \in R$
$(2, 3) \notin R$
$(2, 1) \notin R$
Notation: If $(x, y) \in R$, write xRy.

(2) "Less than"

$L = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x < y\}$

So $(1,2) \in L$, $(3,2) \notin L$.

$1 L 2$, $\neg(3 L 2)$.

$1 < 2$, $3 \nless 2$.

(3) Equals $= \{(x,y) \in A\times A \mid x = y\}$.

(4) a function $f: A \to A$ is a subset. $f = \{(a,b) \in A\times A \mid b = f(a)\}$.
So $f \subseteq A\times A$. Hence, f is a relation.

(5) Divides $= \{(a,b) \in \mathbb{Z} \mid a \mid b\}$.

(6) R need not mean anything.

(7) A = set of all humans.
"x is an ancestor of y"

Properties of Relations

def. Let R be a relation on A.

(i) R is reflexive if $\forall a \in A$ aRa.

ex.

(1) "=" is reflexive.

(2) "<" on $\mathbb{N}$ is not reflexive. So $\neg\forall a$ aRa.

proof. Prove $\exists a$ $a \nless a$. Set $a = 1$. $1 \nless 1$. $\square$

(3) "$\leq$" is reflexive.

(4) "x is an ancestor of y" is not reflexive.

(5) "divides" on $\mathbb{Z}\times\mathbb{Z}$ is almost reflexive but not since $0 \nmid 0$.
"divides" on $\mathbb{Z}\setminus\{0\}$ is reflexive since $a \in \mathbb{Z}/\{0\}$ $a \neq 0$ on $a = 1a$.

(ii) R is symmetric if $\forall a,b \in A$ $aRb \Rightarrow bRa$.

ex.

(1) "=" is symmetric.

(2) "<" is not symmetric.

(3) "$\leq$" is not symmetric.

(iii) R is transitive if $\forall a,b,c \in A$ $aRb \wedge bRc \Rightarrow aRc$.

ex.

(1) "=" is transitive

(2) "<" is transitive. $(a<b \wedge b<c \Rightarrow a<c)$.

(3) "divides" is transitive. $(a|b \wedge b|c \Rightarrow a|c)$.

(4) "x is ancestor of y" is transitive.