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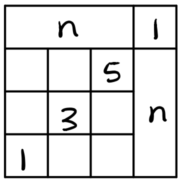
Notes on mathematical induction:

(1) To use, must know conjecture in advance.

(2) Proof by induction doesn't always give idea of why is true intuitively.

ex. In $\sum_{k=1}^{n}(2k-1)=n^2$, can see with a picture:



$2n+1$ added each time.

(3) In the inductive step, you may prove instead $P(n-1) \Rightarrow P(n)$.

(4) In the proof of induction, we had a set $F = \{n \in \mathbb{N} \mid \neg P(n)\}$, then said "let m be the least element of F."
Well-Ordering Principle: Every non-empty set of natural numbers has a least element. Need this axiom to prove induction. Conversely, can prove this principle by assuming induction is true.

Problem.
Prove $\forall x \in \mathbb{R}\ x \geq -1\ \ \forall n \in \mathbb{N}\ n \geq 1\ \ (1+x)^n \geq 1+nx$.
Note: This is a statement of the form: $\forall n \in \mathbb{N}\ (\forall x \in (-1,\infty),\ P(n,x))$.
proof.

(i) Base case. $n=1$.
LHS: $(1+x)^1$
RHS: $1 \cdot 1 + x$
True for $n=1$ since both sides equal $(1+x)$.

(ii) Inductive step.
Let $n \geq 1$.
To prove $(1+x)^{n+1} \geq 1+(n+1)x$, given $(1+x)^n \geq 1+nx$.
Multiply both sides by $(1+x)$.
Assume since $x \geq -1$, $1+x \geq 0$, sign of equality doesn't change.

$$(1+x)^{n+1} \geq (1+nx)(1+x) = 1+nx+x+nx^2$$
$$= 1+(n+1)x+\underline{nx^2}$$
$$\geq 1+(n+1)x$$

since $nx^2 \geq 0$, safe to remove as equality is preserved.

$(1+x)^{n+1} \geq 1+(n+1)x$ as required.
By induction, true for all n. $\square$

Problem.
Prove that for $n \geq 1$, any $2^n \times 2^n$ checkerboard with a single square removed can be tiled with non-overlapping tiles of shape...

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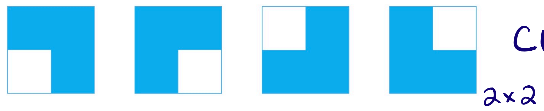
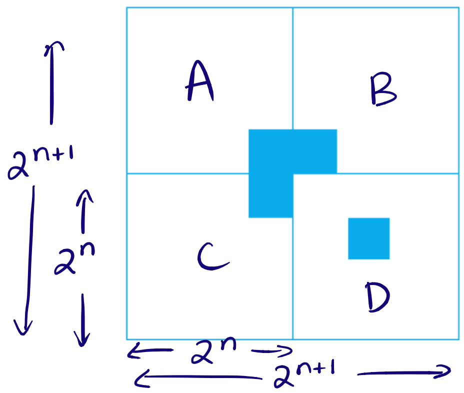
mini example of our proof by induction:
proof.
(i) Base case. $n=1$



Clearly you can just use one tile.

(ii) Inductive step.
Let $n \geq 1$. Assume that any $2^n \times 2^n$ board with one tile missing can be covered. Consider $2^{n+1} \times 2^{n+1}$ board, one square missing.



Four sub-boards A, B, C, D each $2^n \times 2^n$. One is missing a square.

From A, B, C, D, temporarily remove the "middle corner" square.

By IA, A, B, C, D can be covered with the tiles. Fill in the three "middle squares" with a single tile.
Now, whole board is covered except missing square.
By induction, proved for all $n \geq 1$. $\square$

Problem.
Let X be a finite set, $|X| = n$. Prove $|P(X)| = 2^n$.
Idea: Counting via bijection. Can count $|X|$ indirectly by creating a bijection $f: X \to Y$, then counting $|Y| = |X|$.
proof.
We want to prove $\forall n \in \mathbb{N}\ \forall$ sets X $|X| = n \Rightarrow |P(X)| = 2^n$.
(i) Base case. $n=0$.
Only one set $\emptyset$ has size 0. $P(\emptyset) = \{\emptyset\}$. $|P(\emptyset)| = 2^0$. ✓
(ii) Inductive step.
Let $n \geq 0$. Assume $\forall X$ $|X| = n \Rightarrow |P(X)| = 2^n$.
To prove $\forall X$ $|X| = n+1 \Rightarrow |P(X)| = 2^{n+1}$.
Let A be a set, $|A| = n+1$. Denote $A = \{a_1, a_2, \dots, a_{n+1}\}$.
Define $X = \{a_2, a_3, \dots, a_{n+1}\}$. Note: $|X| = n$.
Partition $P(A)$ into sets of type (I) and (II):
(I) $Z \subseteq A$ such that $a_1 \notin Z$.
(II) $Z \subseteq A$ such that $a_1 \in Z$.
Now count:
(I) These are precisely the subsets of X. Since $|X| = n$ by IA, there are 2^n of this type.
(II) To count these, define $f: \{Z \subseteq A \mid a_1 \in Z\} \to P(X)$ by $f(Z) = Z \setminus \{a_1\}$.
Prove f is bijective.

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f is injective:
Let Z_1, Z_2 in domain (i.e. $a_1 \in Z_1, a_1 \in Z_2$).
Assume $Z_1 \neq Z_2$. Then, $\exists a_i$ where $a_i \in Z_1$, $a_i \notin Z_2$ (or vice-versa).
We know $a_i \neq a_1$. So,
$f(Z_1) = Z_1 \setminus \{a_1\}$ still contains a_i
$f(Z_2) = Z_2 \setminus \{a_1\}$ still does not contain a_i.
Hence, $f(Z_1) \neq f(Z_2)$. So f is injective.

f is surjective.
Let $Z \in P(X)$ (i.e. $Z \subseteq \{a_2, a_3, \dots, a_{n+1}\}$).
Then, $Y = Z \cup \{a_1\}$ is type (II). So Y in domain f and
$f(Y) = (Z \cup \{a_1\}) \setminus \{a_1\} = Z$. So is surjective

Hence, $f: (II) \to P(X)$ is bijective.
So $|(II)| = |P(X)| = 2^n$ by IA.
Finally, $|P(A)| = |(I)| + |(II)|$

$$
\begin{aligned}
&= 2^n + 2^n \\
&= 2 \cdot 2^n \\
&= 2^{n+1}. \quad \square
\end{aligned}
$$

Theorem. Strong Induction.
Let P(n) be a predicate, $n \in \mathbb{N}$. if:
(i) P(0) is true
(ii) $\forall n \in \mathbb{N}\ (P(0) \wedge P(1) \wedge \dots \wedge P(n)) \Rightarrow P(n+1)$

Problem.
Let $n \in \mathbb{N}$. Prove n can be represented as a binary number, i.e. in the form:

$$n = b_0 2^0 + b_1 2^1 + \dots + b_k 2^k$$

for some $k \in \mathbb{N}$ where each $b_i \in \{0,1\}$.
Note: About binary addition,

$$
\begin{array}{r}
1100 \\
+\ 0001 \\
\hline
1101
\end{array}
\text{ (even)}
\qquad
\begin{array}{r}
0111 \\
+\ \ \ \ 1 \\
\hline
1000
\end{array}
\text{ (odd)}.
$$

proof.
(i) Base case. $n = 0$.
Then $n = 0 \cdot 2^0 = 0$. ✓
(ii) Inductive step.
Let $n \geq 0$. Assume $\forall m = 0, 1, 2, \dots, n$, m can be written in the correct form. Prove n+1 has correct form.
Two cases:
Case 1: n+1 odd, so n even.
By IA, $n = b_0 2^0 + b_1 2^1 + \dots + b_k 2^k$. Since n even and $2^1, 2^2, \dots, 2^k$ are all even, $b_0 = 0$.
So $n+1 = 1(2^0) + b_1 2^1 + \dots + b_k 2^k$.
Case 2: n+1 even, so $n+1 = 2m$.
So $m = \frac{n+1}{2} < n+1$, so IA applies to m.

Hence, $m = b_0 2^0 + b_1 2^1 + \cdots + b_k 2^k$

$n+1 = 2m = b_0 2^1 + b_1 2^2 + \cdots + b_k 2^{k+1}$.

By induction, true for all $n \in \mathbb{N}$. $\square$