Ordered Subsets (Permutations)

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Injective Functions.
Let A be a set, $|A| = m$; B be a set, $|B| = n$.
How many $f: A \to B$ are injective?
Denote $A = \{a_1, a_2, a_3, \cdots, a_m\}$.
Define as

$f(a_1)$: choose any $b \in B \to n$ choices.
$f(a_2)$: choose any $b \in B$, except $f(a_1) \to n-1$ choices
$f(a_3)$: choose any $b \in B$, except $f(a_1), f(a_2) \to n-2$ choices.
$\vdots$
$f(a_m)$: choose any $b \in B$, except $f(a_1), \cdots, f(a_{m-1}) \to n-(m-1)$ choices.

Theorem.
If $|A| = m$, $|B| = n$, then the number of injective $f: A \to B$ is
$n(n-1)(n-2)\cdots(n-(m-1)) = \frac{n!}{(n-m)!}$.
Notation: $P(n,m)$
Note: This also counts:
(1) Number of size m ordered subsets of set sized n.
(2) Number of ways to arrange m objects, selected from set of size n.

ex. You have 10 books. How many ways can you place 6 of them on a shelf (order matters)?
$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = \frac{10!}{(10-6)!} = \frac{10!}{4!}$

Surjective Functions
Let A be a set, $|A| = m$; B be a set, $|B| = n$.
How many $f: A \to B$ are surjective?
(1) If $|A| < |B|$, none.
(2) If $|A| = |B|$, $n!$ (since A and B must be bijective)
(3) If $|A| > |B|$, complicated (see Inclusion-Exclusion)

Unordered Subsets (Combinations)

ex. How many teams of 10 can be made from 100 players?
Concept of double-counting: Count the same thing in 2 different ways, and set them equal.
Let L = number of teams.
We will count M = number of ordered teams, in 2 ways:
(1) number of ordered subsets:
$M = 100 \cdot 99 \cdot 98 \cdot \cdots \cdot 91 = \frac{100!}{(100-10)!}$

(2) choose an unordered team in L ways, then order it:
Can do this in $M = L \cdot 10!$ ways.
number of reorderings of chosen elements
Hence, $M = M$
$\frac{100!}{(100-10)!} = L \cdot 10!$, so $L = \frac{100!}{(100-10!)10!}$

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Theorem.

Let $n \geq k \geq 0$ be integers. Then, the number of subsets of size k (unordered) of a set of size n is:

$$\left\{ \frac{n!}{(n-k)!\,k!} = \binom{n}{k} = C(n,k) = \text{"n choose k"} \right.$$

binomial coefficient.

proof.

Repeat last example with n instead of 100, k instead of 10.

Problem.

From a deck of 52 cards, how many hands of 5 contain at least 2 hearts?

Idea: Write down an algorithm to choose all such hands without containing the same hand twice.

INCORRECT SOLUTION:

Choose 2 hearts, then choose remaining.

This is incorrect; some hands are counted more than once.

$$\binom{13}{2}\binom{52-2}{3}$$

2 ♡, 3 ♡ | 4 ♡, 3 ◇, 5 ◇
2 ♡, 4 ♡ | 3 ♡, 3 ◇, 5 ◇
3 ♡, 4 ♡ | 2 ♡, 3 ◇, 5 ◇

all the same hand, counted separately here though

CORRECT SOLUTION #1:

Different cases.

(exactly two ♡ cards) XOR (exactly three) XOR (exactly four) XOR (exactly five)

cards don't overlap

$$\binom{13}{2}\binom{52-13}{3} \qquad \binom{13}{3}\binom{52-13}{2} \qquad \text{etc...}$$

remove all the heart cards

Total number is...

$$\binom{13}{2}\binom{39}{3} + \binom{13}{3}\binom{39}{2} + \binom{13}{4}\binom{39}{1} + \binom{13}{5}\binom{39}{0}$$

CORRECT SOLUTION #2:

All hands − (hands with one or no hearts)

$$\binom{52}{5} - \left[\binom{13}{0}\binom{39}{5} + \binom{13}{1}\binom{39}{4}\right]$$

Problem.

How many 5-card hands have "2 pairs" (i.e. hands of the form: AABBC where $A \neq B$, $B \neq C$, $A \neq C$ and A, B, C are values).

Construct such hands by the procedure:

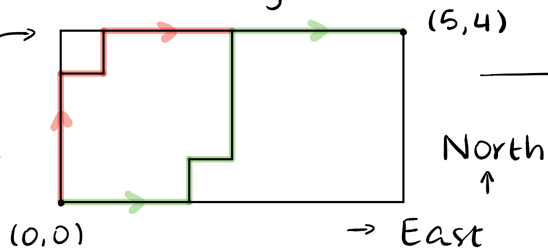
choose which are two types A, B , choose 2 of type A , choose 2 of type B , last card (not A or B)

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot \binom{52-8}{1}$$

Problem.

In a rectangular grid, how many possible paths are there from (0,0) to (5,4), without traveling west or south (moving efficiently)?

encode each path as a sequence of movements E or N.



$\longrightarrow$ sequences
EENENNNEE
NNNENEEEE
i.e. strings of length 9 that have 5 E's and 4 N's.

$\exists$ bijection $\left\{\begin{array}{c}\text{paths (0,0) to (5,4)} \\ \text{not going W or S}\end{array}\right\} \to \left\{\begin{array}{c}\text{strings of length 9 with} \\ \text{5 E's and 4 N's}\end{array}\right\}$

How many strings?

Strings are determined by the locations of the 5 E's. There are 9 slots, choose 5, so $\binom{9}{5}$ = number of possible paths.

We could also choose the N's instead. Note $\binom{9}{5} = \binom{9}{4}$ proven below.

Combinatorial Proofs

def. a combinatorial proof means proving an identity by interpreting each side as counting same thing but in different ways (at least for an equality).

Theorem.

$$\binom{n}{k} = \binom{n}{n-k}$$

proof. (algebraic)

Use formula:

$$\frac{n!}{(n-k)!\,(n-(n-k))!} = \binom{n}{k}$$

proof. (combinatorial)

LHS = number of ways to select k things from set size n.

RHS = number of ways to reject k from set size n

Problem.

Prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

proof. (combinatorial)

Start with right side (simpler).

RS = number of subsets size n of set size 2n.

Let $X = \{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\} = A \cup B$.

RS = number of size n subsets of X ($|X| = 2n$, $|A| = |B| = n$)

$$LS = \binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \cdots + \binom{n}{n-1}\binom{n}{n-1} + \binom{n}{n}\binom{n}{n}$$

$$= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0}$$

To choose a subset of size n from X, choose k from A and n-k from B.
You can do this in

$$\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}\binom{n}{k} = \binom{n}{k}^2 \text{ ways.}$$

Sum $\binom{n}{k}^2$ from $k=0$ to n gives the left side. $\square$

Theorem. Pascal's Identity.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

proof. (combinatorial).

LS = number of size k subsets of set size n.

Let $X = \{1, 2, \cdots, n\}$

LS = number of size k subsets of X.

RS : Consider $Y = \{2, 3, \cdots, n\}$.

$\binom{n-1}{k}$ = number of subsets of Y, size k

= number of subsets of X, size k, which do not contain 1.

Partition size k subsets of X into:

(I) not containing 1: $\binom{n-1}{k}$ of these.

(II) containing 1: choose the 1, then remaining k-1 from Y.

$\quad\downarrow$ (one way) $\cdot$ $\downarrow$ ($\binom{n-1}{k-1}$ ways)

RS counts the same thing as (I)+(II). So,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

$\square$