Alison Li

Sets Inside Sets

def. If A is a finite set, the cardinality of A, $|A|$ is the number of elements in A.

ex. Let $A = \{1, 2, \{1,2\}, 3, \{1\}\}$.
Then $|A| = 5$.
Is $\{1,2\} \subseteq A$? ✓
Is $\{1,2\} \in A$? ✓
Is $\{1,3\} \in A$? ✗
Is $\{1,3\} \subseteq A$? ✓
Is $\{1,2,3\} \in A$? ✗
Is $\emptyset \subseteq A$? ✓
Is $\emptyset \in A$? ✗

ex. Sets of empty sets:
$|\emptyset| = 0$
$|\{\emptyset\}| = 1$
$|\{\emptyset, \{\emptyset\}\}| = 2$
$|\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}| = 3$

Russell's Paradox.
Let $R = \{\text{sets } S \mid S \notin S\}$, e.g. $\mathbb{N} \subseteq \mathbb{N}$ but $\mathbb{N} \notin \mathbb{N}$ so $\mathbb{N} \in R$.
Is $R \in R$?
(i) If $R \in R$, R satisfies condition so $R \notin R$. Contradiction.
(ii) If $R \notin R$, R satisfies condition so $R \in R$. Contradiction.
Be careful of how you create sets.

Power Sets

def. Let $A \subseteq U$. The power set of A, $\mathcal{P}(A)$ is the set of all subsets of A. $\mathcal{P}(A) = \{B \subseteq U \mid B \subseteq A\}$.

ex. $A = \{1, 2, 3\}$.
$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

Proposition.
$|A| = n \Rightarrow |\mathcal{P}(A)| = 2^n$.

proof.
You can build any subset $X \subseteq A$ as follows:
For each $a \in A$, choose to include a in X or exclude it.
Have 2 choices for each of n elements, so total $2 \cdot 2 \cdot \cdots 2 = 2^n$ possible subsets. $\square$

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Examples of powersets:

(1) $P(\mathbb{N})$ = set of all subsets of natural numbers.

e.g. $\{1, 217, 81343\} \in P(\mathbb{N})$.

$\{2, 3, 5, 7, 11, \dots\} \in P(\mathbb{N})$.

(2) $P(\mathbb{R}^2)$ = all sets of points in the plane.

e.g. $\{(x,y) \in \mathbb{R}^2 \mid y = x^2\} \notin P(\mathbb{R}^2)$.

(3) $P(P(\mathbb{R}^2))$.

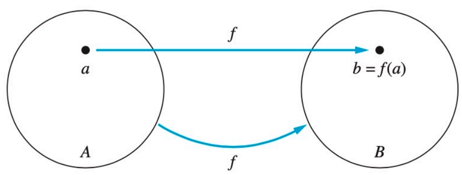
Functions

def. A, B sets. a function $f: A \to B$ is a subset $f \subseteq A \times B$ where for each $a \in A$, there is exactly one $b \in B$ such that $(a,b) \in f$. We write $f(a) = b$, i.e. $(a, f(a)) \in f$.

ex. $f(x) = x^2$, $f: \mathbb{R} \to \mathbb{R}$ is the set $f = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2\}$
$= \{(x, x^2) \mid x \in \mathbb{R}\}$.

def. $f: A \to B$ is a function.

(1) A is domain.
B is codomain.



(2) The range (or image) of f is $\text{range}(f) = \{b \in B \mid \exists a \in A \ f(a) = b\}$.

ex. $f: \mathbb{N} \to \mathbb{N}$, $f(n) = 2n$.
$\text{range}(f) = \{m \in B \mid \exists n \in \mathbb{N} \ m = f(n) = 2n\}$ = all even integers.

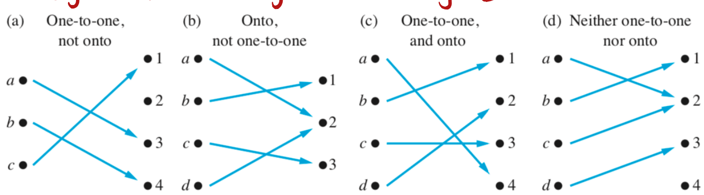
(3) f is surjective if $\text{range}(f) = B$.
$\forall b \in B \ \exists a \in A \ f(a) = b$.

ex. $f: \mathbb{N} \to \mathbb{N}$, $f(n) = 2n$ is not surjective.
But if $E = \{m \in \mathbb{N} \mid \exists n \ m = 2n\}$ and $f: \mathbb{N} \to E$ then f is surjective.

(4) f is injective if: $\forall x, y \in A \quad x \neq y \Rightarrow f(x) \neq f(y)$.
or $\forall x, y \in A \quad f(x) = f(y) \Rightarrow x = y$

(5) f is bijective if both injective and surjective.

bijective surjective injective



ex. $f: \mathbb{N} \to \mathbb{N}$, $f(n)$ = the n^{th} prime.
$(f(0) = 2, f(1) = 3, f(2) = 5, \dots)$
Not surjective since $4 \in \mathbb{N}$ but $4 \notin \text{range}(f)$. Is injective.

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ex. Let $f: \mathbb{N} \to \mathbb{Z}$, $f(n) = \begin{cases} n/2 & \text{if } n \text{ even.} \\ -\left(\frac{n+1}{2}\right) & \text{if } n \text{ odd.} \end{cases}$

Prove f bijective.

proof.

(i) For f injective.

Want to prove $\forall n, m \in \mathbb{N}$ $f(n) = f(m) \Rightarrow n = m$.

Let $m, n \in \mathbb{N}$. Assume $f(n) = f(m)$.

We claim that either m, n both even or both odd.

By contradiction, assume one is even, one is odd.

Case 1: n even, m odd.

Then, $f(n) = f(m)$

$$\frac{n}{2} = -\left(\frac{m+1}{2}\right)$$

$n = -m-1$ but $n, m \geq 0$ $(m, n \in \mathbb{N})$

$(-m-1 = n) < 0$.

Contradiction.

Case 2: n odd, m even.

Similar to case 1. Contradiction.

Now left with two cases again.

Case 1: n, m both even.

Then, $f(n) = f(m)$, so $\frac{n}{2} = \frac{m}{2}$, so $m = n$ as required.

Case 2: n, m both odd.

Then, $f(n) = f(m)$, so $-\left(\frac{n+1}{2}\right) = -\left(\frac{m+1}{2}\right)$,

$-n-1 = -m-1$

$n = m$ as required.

So f injective.

(ii) For f surjective.

Want to prove $\forall z \in \mathbb{Z}$ $\exists n \in \mathbb{N}$ $f(n) = z$.

Let $z \in \mathbb{Z}$. Two cases:

Case 1: If z is positive, $z \geq 0$, set $n = 2z$.

Since $z \in \mathbb{Z}$, $z \geq 0$, $2z \in \mathbb{N}$.

Then, $f(n) = \frac{n}{2} = \frac{2z}{2} = z$ as required.

Case 2: If z is negative, $z < 0$, then ...

(need $z = f(n) = -\left(\frac{n+1}{2}\right)$ so $2z = -(n-1) \Rightarrow -2z = n+1 \Rightarrow n = -2z-1$)

... set $n = -2z-1$.

Need $n \in \mathbb{Z}$. Since $z < 0$, $-z > 0$, so $-2z > 0$, so $-2z \geq 1$, so $-2z-1 \geq 0$, so $n \in \mathbb{N}$ and n is odd.

So $f(n) = -\left(\frac{n+1}{2}\right) = -\left(\frac{-2z-1+1}{2}\right) = \frac{-(-2z)}{2} = z$ as required.

So f surjective.