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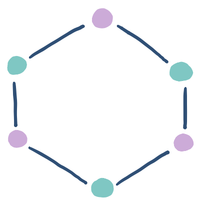
def. The chromatic number of G, denoted as $\chi(G)$, is the least number of colours that G can be coloured with.

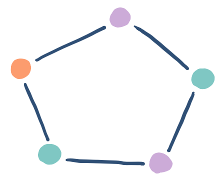
ex. Line graph L_n

n vertices.
$\chi(L_n) = 2 \quad (n \geq 2)$.

ex. Circle graph C_n



$\chi(C_6) = 2$

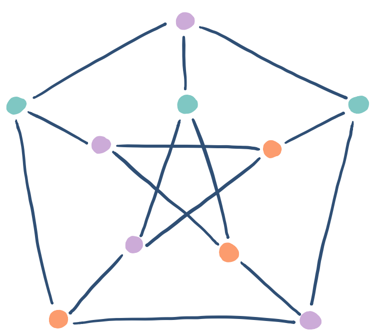


$\chi(C_5) = 3$

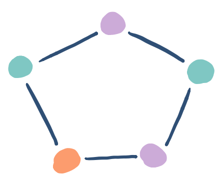
$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$$

ex. Complete graph K_n
If v is "red", no other vertex is red.
All vertices $v \in V$ must have different colours.

ex. Peterson graph, defined as G. Show $\chi(G) = 3$.



Two parts to prove:
(i) Show $\chi(G) \leq 3$.
See picture; valid proof.
(ii) Show $\chi(G) \geq 3$.
Have C_5 around outside.

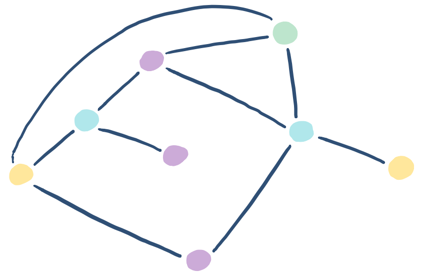


If $\chi(G) \geq 2$, then the 5 vertices on outside give colouring of C_5 with 2 colours, which is impossible since $\chi(C_5) = 3$. $\square$

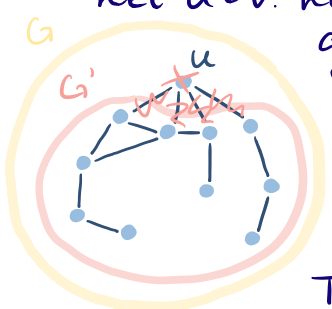
Theorem.
Let G be a graph with at least one edge. Then,
$\chi(G) = 2 \Leftrightarrow G$ is bipartite.

proof.
($\Leftarrow$) Assume G is bipartite with $V = V_1 \cup V_2$ bipartition.
Colour all $v \in V_1$ "red", all $u \in V_2$ "green".
All edges have one endpoint in V_1 (red) and the other endpoint in V_2 (green).
($\Rightarrow$) Assume $\chi(G) = 2$, say colours are "red", "green".
Set V_1 = red vertices, V_2 = green vertices. All edges have endpoints with different colours, hence one endpoint in V_1 and one endpoint in V_2. $\square$

Theorem.
Let $G=(V,E)$ and $d_G = \max\{\deg(v) \mid v \in V\}$.
Then, $\chi(G) \leq d_G + 1$. (i.e. can colour with at most d_G+1 colours).
proof.
Idea: Induction on number of vertices. Re-state the theorem as:
$\forall n \geq 1$, $\forall G=(V,E)$ with $|V| = n$, $\chi(G) \leq d_G + 1$.



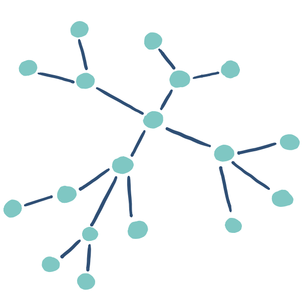
(i) Base case.
Let $n=1$. Then, $d_G = 0$ and $\chi(G) = 0+1 = 1$.
(ii) Inductive step.
Let $n \geq 1$. Assume for all $G'=(V',E')$ with $|V'|=n$, $\chi(G') \leq d_{G'}+1$.
Let $G=(V,E)$ with $|V| = n+1$.
Let $u \in V$. Let $G'=(V',E')$ be the graph obtained by removing u, and all edges incident on u.



Since $|V'| = n$, the IA applies to G'. So $\chi(G') \leq d_{G'}+1$.
Since we removed edges, for each vertex $v \in V$, its degree in G' is $\leq$ its degree in G.
Hence, $d_{G'} \leq d_G$. Hence $\chi(G') \leq d_G + 1$.
So can colour G' with d_G+1 colours. Do so.
Then, G is coloured correctly, except for u.
u has at most d_G neighbours, which have at most d_G different colours. So there is at least one colour that does not appear amongst neighbours of u (at least one colour still available).
So G is colourable with d_G+1 colours, so $\chi(G) \leq d_G+1$. $\square$

Comment: $\chi(G) = d_G + 1$ only in two cases:
(1) C_n, n odd, $d_{C_n} = 2$, $\chi(C_n) = 2+1 = 3$.
(2) K_n, $d_{K_n} = n-1$, $\chi(K_n) = (n-1)+1 = n$.

Trees



def. G is called a tree if G has no simple cycles and G is connected.

def. A vertex of degree 1 is called a leaf.

Proposition.

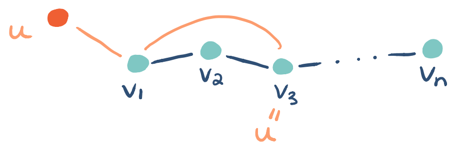
A tree with at least one edge has at least 2 leaves.

proof.

Maximality argument. Let $S = (v_1, v_2, \ldots, v_n)$ be a maximal length simple path (no vertex repeated).

Claim: v_1, v_n are leaves. Note $v_1 \neq v_n$, else S is a simple cycle.

For a contradiction, assume they are not both leaves. i.e. at least one is not a leaf. WLOG, say v_1 is not a leaf.



Since v_1 is not a leaf, $\exists$ edge $e = (u, v_1)$ (with $u \neq v_1$, $u \neq v_2$).

Two cases:

(1) u is not one of $v_1, v_2, \ldots, v_n$. Then S is not maximal length since $(u, v_1, v_2, \ldots, v_n)$ is a longer simple path. Contradiction.

(2) u is one of the vertices $u = v_i$ for some $i \in \{3, \ldots, n\}$.
Then $(v_1, v_2, \ldots, v_i, \underset{\substack{\shortparallel \\ v_n}}{u})$ is a simple cycle. Contradiction.

By contradiction, v_1, v_n are both leaves. $\square$

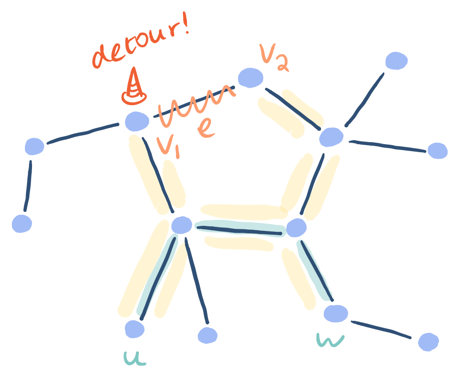
Characterization of Trees

Lemma.

Let G be a connected graph and σ is a simple cycle in G.
If G' is obtained from G by removing one edge of σ, G' is connected.

proof.

Idea: You can go two ways around a cycle.
Let u, w be vertices in G'. Then $\exists$ path p in G from u to w.



Let $e = (v_1, v_2)$ be the edge that was removed. Two cases:

(1) p does not use e. So p is a path in G, so there exists a path u to w.

(2) p uses e. WLOG from v_1 to v_2, then follow p from u to v_1, then around cycle in the second direction to v_2, then follow p again to w, so path u to w. So G' is connected. $\square$

Theorem.

Let $G = (V, E)$. Then, G is a tree $\Leftrightarrow |E| = |V| - 1$ and G is connected.

proof.

($\Rightarrow$) Assume G is a tree. By induction on number of vertices.

(i) Base case.

Let $n = 1$. Then $|V| = 1$, $|E| = |V| - 1 = 1 - 1 = 0$. ✓

Alison Li

(ii) Inductive step.

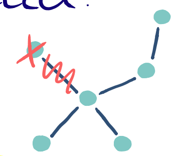
Let $n \geq 1$. Assume any tree $G=(V,E)$ with $|V|=n$ has $|E|=|V|-1$ and is connected.

Let $T=(V,E)$ with $|V|=n+1$. By definition, T is connected.

Let u be a leaf of T (exists by previous proposition).

Form $T'=(V',E')$ by removing u and its one edge.

So $|V'|=n$, and T' is still connected, still no simple cycles.

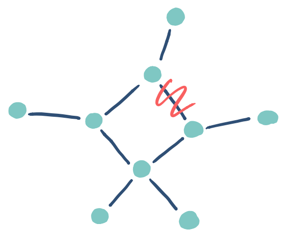


So IA applies. $|E'|=|V'|-1$. Then in T, $|E|=|E'|+1=\overbrace{|V'|-1}^{|E'|}+1$

$$=|V'|+1-1$$
$$=|V|-1. \quad \square$$

($\Leftarrow$) Assume $|E|=|V|-1$ and G is connected.

For contradiction, assume G is not a tree. So G has a simple cycle. Remove an edge from a simple cycle of G. G is still connected.



If G now has no simple cycles, it is now a tree. If not, it again has a simple cycle, so remove one edge from the cycle. G is still connected. Repeat this, say k times, until G has no simple cycles. So we now have a connected graph $G'=(V',E')$ with no simple cycles, i.e. a tree.

Here, $V'=V$. But now $|E'|=|E|-k$.

But G' is a tree so by ($\Rightarrow$), $|E'|=|V'|-1$.

$$=\underbrace{|V|-1=|E|}_{\text{assumed at start}}.$$

$|E|-k=|E|$. Contradiction, we removed at least one edge.

Therefore, G was a tree to begin with. $\square$