Alison Li

Further note on functions (helpful for assignment).

(1) Let X be a set. Then $id_X : X \to X$ identity function defined by $id_X(x) = x$ for all $x \in X$.

(2) $f \circ id = f$ since $f(id(x)) = f(x)$. Also $id \circ f = f$.

(3) $f : A \to B$ then $f^{-1} : B \to A \Leftrightarrow f$ bijective $\Leftrightarrow \exists\, g : B \to A$.

$g \circ f = id_A$
$f \circ g = id_B$

(4) Function composition is associative.

$(f \circ g) \circ h = f \circ (g \circ h)$

def. The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = n-1 + n-2$.

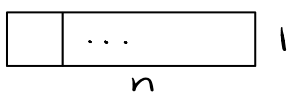
Note: Compare to last class string problem.

$F_0 = 0$	$F_3 = 2 = C_1$	Same sequence, different start position.
$F_1 = 1$	$F_4 = 3 = C_2$	
$F_2 = 1$	$\vdots$	

Other things counted by Fibonacci numbers:

(1) Have tiles:

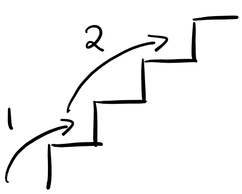
How many ways, W_n, can you tile the $1 \times n$ strip?



Look at last tile:

Then $W_n = W_{n-1} + W_{n-2}$.

(2) Staircase with n steps:



How many ways can you ascend 1 or 2 steps at a time?

(3) Subsets of $\{1, 2, \dots, n\}$ not counting consecutive values.

Prove the subsets are counted by Fibonacci.

Hint: These subsets either contain 1 or they do not.

Linear Recurrence Relation

def. A recurrence of the form:
$$a_n = b a_{n-1} + c a_{n-2}$$
where $b, c \in \mathbb{R}$ are constants is called linear homogeneous degree 2 with constant coefficients.

Idea: $a_n = r^n$ will satisfy $a_n = b a_{n-1} + c a_{n-2}$ if:

(i) $r^n = br^{n-1} + cr^{n-2}$ (multiply both sides by r^{-n+2})

(ii) $r^2 = br + c$ (Obtain a quadratic)

Consequently, the sequence a_n with $a_n = r^n$ is a solution $\Leftrightarrow$ r is a solution of $r^2 - br - c = 0$.

def. The characteristic equation of $a_n = b a_{n-1} + c a_{n-2}$ is $x^2 - bx - c = 0$.

Tip to remember: $a_n \to x^2$
$a_{n-1} \to x$
$a_{n-2} \to 1$

Theorem.

Let $a_0, a_1, a_2, \cdots$ be a sequence that satisfies $a_n = b a_{n-1} + c a_{n-2}$. If $x^2 - bx - c = 0$ has two distinct roots r_1, r_2 then $a_n = \alpha r_1^n + \beta r_2^n$ where α, β is the unique solution to the system:
$$\begin{cases} \alpha + \beta = a_0 \\ \alpha r_1 + \beta r_2 = a_1 \end{cases}$$

proof.

Why unique solution? System is linear:
$$\begin{cases} \alpha + \beta = a_0 \\ \alpha r_1 + \beta r_2 = a_1 \end{cases}$$
α, β are variables.
a_0, a_1, r_1, r_2 are constants.

$$\begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix} \quad \det \begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix} = r_1 - r_2 \neq 0. \text{ Unique solution.}$$

Prove by strong induction that $\underbrace{\alpha r_1^n + \beta r_2^n}_{\text{"formula"}}$

(i) Base cases:

$n = 0$. $\alpha r_1^0 + \beta r_2^0 = \alpha + \beta = a_0$.

$n = 1$. $\alpha r_1 + \beta r_2 = a_1$.

(ii) Inductive step.

Let $n \geq 2$. Assume formula computes a_m correctly for $m = n-1$ and $m = n-2$. Show a_m is computed by the formula:
$$\begin{aligned} a_n &= b a_{n-1} + c a_{n-2} \\ &= b(\alpha r_1^{n-1} + \beta r_2^{n-1}) + c(\alpha r_1^{n-2} + \beta r_2^{n-2}) \\ &= \alpha r_1^{n-2}(br_1 + c) + \beta r_2^{n-2}(br_2 + c) \end{aligned}$$
(Know that r_1, r_2 satisfy $x^2 - bx - c = 0$, i.e. $r_1^2 - br_1 - c = 0$, so $r_1^2 = br_1 + c$. Same for r_2.)
$$\begin{aligned} &= \alpha r_1^{n-2} r_1^2 + \beta r_2^{n-2} r_2^2 \\ &= \alpha r_1^n + \beta r_2^n \end{aligned}$$
$\square$

Alison Li

Formula for Fibonacci F_n. We have $F_n = F_{n-1} + F_{n-2}$ so $b = c = 1$.

Characteristic equation of F_n:

$$x^2 = x + 1$$
$$x^2 - x - 1 = 0$$
$$x = \frac{1 \pm \sqrt{1+4}}{2}.$$

So,

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}.$$

Find α, β:

$F_0 = 0 = \alpha r_1^0 + \beta r_2^0 = \alpha + \beta$ so $\beta = -\alpha$.

$F_1 = 1 = \alpha r_1 + \beta r_2$

$1 = \alpha\left(\frac{1+\sqrt{5}}{2}\right) - \alpha\left(\frac{1-\sqrt{5}}{2}\right)$ by substitution of 1

$2 = \alpha(1 + \sqrt{5} - 1 + \sqrt{5})$

$2 = \alpha(2\sqrt{5})$

$\alpha = \frac{1}{\sqrt{5}}$

so $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$

Observations:

(1) $F_n \in \mathbb{N}$. Not obvious from formula.

(2) $\frac{1+\sqrt{5}}{2} \approx 1.618\cdots$, $\frac{1-\sqrt{5}}{2} \approx -0.618\cdots$

since $\left|\frac{1-\sqrt{5}}{2}\right| < 1$, $\lim_{n\to\infty}\left(\frac{1-\sqrt{5}}{2}\right)^n = 0$.

So for large values of n:

$$F_n \approx \frac{1}{\sqrt{5}}(1.618\cdots)^n.$$

i.e. F_n is almost a geometric sequence with a common ratio of $\frac{1+\sqrt{5}}{2}$.

(3) $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$