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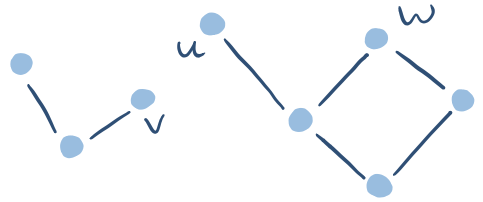
Connectedness

def. The distance from $u \in V$ to $w \in V$ is length of a shortest path from u to w. Denoted $d(u,w)$. If no path, set $d(u,w) = \infty$.

ex.



$d(u,w) = 2$
$d(u,v) = \infty$

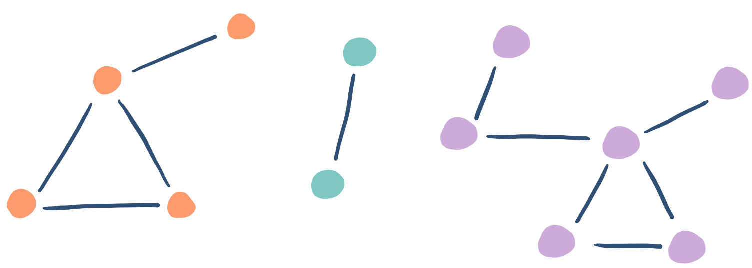
def. A graph is connected if $\forall\, u,v \in V$ $\exists$ path u to v (i.e. $d(u,v) < \infty$).

ex. Prove Q_n (hypercube) is connected.
In fact, $\forall\, s_1, s_2 \in V$ $d(s_1,s_2) \leq n$.

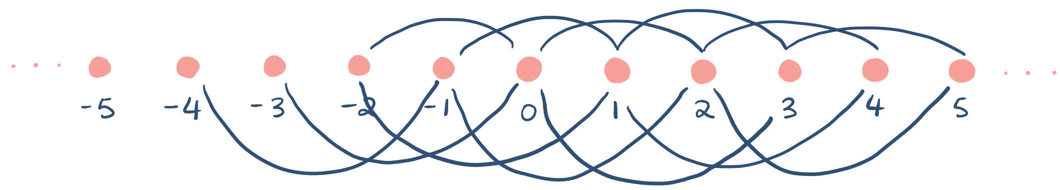
proof.
Suppose $s_1, s_2 \in V$ and they differ in m bits.
You can change any one bit using one edge, so change m bits using m edges, i.e. path of length m. $\square$

def. Let $u,v \in V$. Say "v is reachable from u" if $\exists$ path u to v. This defines an equivalence relation. The equivalence classes are called the connected components of G.

ex. One graph G, 3 connected graphs.



ex. Let a,b be positive integers. $d = \gcd(a,b)$.
Define G with $V = \mathbb{Z}$, $E = \{(x, x \pm a), (x, x \pm b) \;\; x \in \mathbb{Z}\}$.
(1) For $a = 2$, $b = 3$, draw part of $G_{2,3}$.
(2) Prove that $G_{a,b}$ has d connected components which are the equivalence classes mod d, i.e. $[0], [1], \ldots, [d-1] \in \mathbb{Z}/d\mathbb{Z}$.



(1) $\gcd(3,2) = 1$ so should be 1 connected component.
To see this, write $1 = 3 - 2$, $-1 = 3 - 2$.

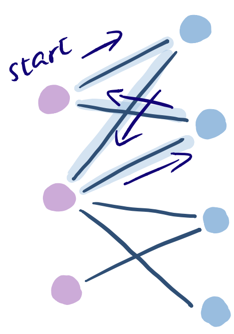
(2) To prove $[0], [1], \ldots, [d-1]$ are the connected components, there are two things to prove:

(i) If there is a path from x to y, then $x \equiv y \pmod d$. (some equivalence class mod d).
Assume path x to y exists. Each edge changes value of vertex by $\pm a$ or $\pm b$. So $y = x + ka + \ell b$ $(k, \ell \in \mathbb{Z})$.
Then, $y = x + n$, $n = ka + \ell b$. Since $d \mid a$, $d \mid b$, so $d \mid n$.
Then, $n = dm$ for some $m \in \mathbb{Z}$ so $y \equiv x + dm \pmod d$,
so $y \equiv x \pmod d$.

(ii) Prove if $x \equiv y \pmod d$ then $\exists$ path x to y.
If $x \equiv y \pmod d$ then $d \mid y - x$, so $y - x = dm$ some $m \in \mathbb{Z}$.
By Bézout, $d = sa + tb$ some $s, t \in \mathbb{Z}$.
So, $y = x + (sm)a + (tm)b$.
If $sm \geq 0$, $tm \leq 0$, follow path:
$(x, x+a, x+2a, x+3a, \ldots, x+(sm)a, x+(sm)a-b, x+(sm)a-2b, \ldots$
$\ldots, \underbrace{x+(sm)a+(tm)b}_{=y})$. (All have same equivalence class).
$\square$

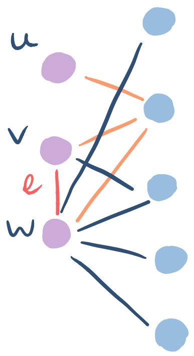
Comment: Connected components don't interact with each other, so for some theorems, you can prove the theorem for each component. i.e. prove for connected graph only.

Theorem.
G bipartite $\Leftrightarrow$ G has no odd-length cycles (all cycles are even-length).
proof.
($\Rightarrow$) Assume G is bipartite.
So $V = V_1 \cup V_2$ is a bipartition.
Each edge takes you from V_1 to V_2 or from V_2 to V_1, i.e. goes to other "side", so to return to start point (i.e. cycle), you need an even number of edges. So all cycles are even.



($\Leftarrow$) Assume G has no odd-length cycles.
Observation: G bipartite $\Leftrightarrow$ every connected component is bipartite.
Hence, we only need to prove the theorem for connected graphs. So, can assume G is connected.
Let $u \in V$. Set:
$V_0 = \{v \in V \mid d(u,v) \equiv 0 \pmod 2\}$ (even distance)
$V_1 = \{v \in V \mid d(u,v) \equiv 1 \pmod 2\}$ (odd distance).
$V_0 \cap V_1 = \emptyset$ ($d(u,v)$ is either even or odd).
$V_0 \cup V_1 = V$ since $d(u,v) < \infty$ assuming is connected.

Need to prove no edges V_0 to V_0 or V_1 to V_1.
Suppose there is an edge, say $e = (v, w)$ with $v \in V_0$, $w \in V_0$.
Consider path σ_1 from u to v. σ_1 has even length. $\}$ both in V_0
Consider path σ_2 from w to u. σ_2 also even length.

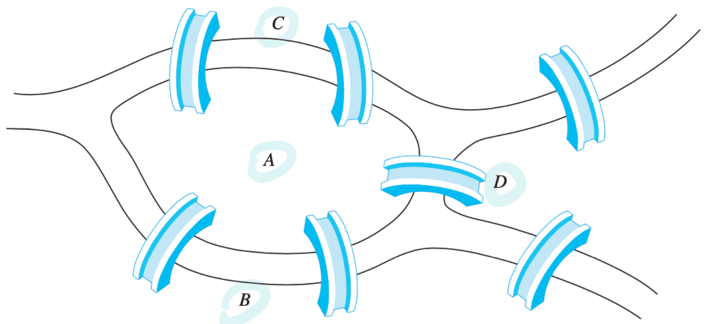


Then, we have a cycle σ_1 then v to w then σ_2 ($u \rightsquigarrow v \rightsquigarrow w \rightsquigarrow u$) of odd length. Contradiction.
Similarly, if edge from $v \in V$ to $w \in V$, then odd path σ_1 u to v, odd path σ_2 w to u, so cycle σ_1 then v to w then σ_2 of odd + 1 + odd = odd length.
also contradiction.
Hence, $V = V_0 \cup V_1$ is a bipartition. $\square$

Euler Paths and Cycles

def. a path that doesn't repeat any edge is called a trail.

7 Bridges of Königsberg



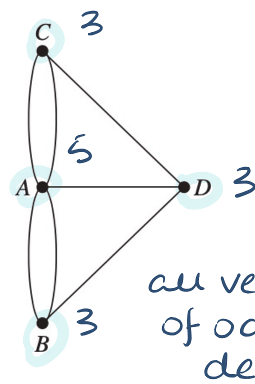
Challenge: To walk around the city and use each bridge exactly once.

Euler says this is impossible.

No Eulerian cycle or path.

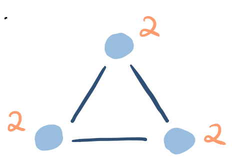
def. a Eulerian path (trail) is a path that uses every edge exactly once.

a Eulerian cycle is a cycle that uses every edge exactly once.

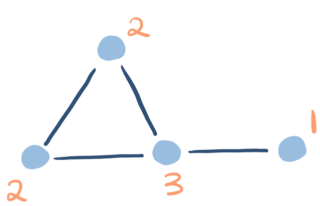


Theorem.
Let $G = (V, E)$ be a graph (multi-edges allowed). Then,
(1) G has Eulerian cycle $\Leftrightarrow$ all vertices have even degree.
(2) G has Eulerian path but not cycle $\Leftrightarrow$ G has exactly 2 odd-degree vertices (which are the start and end points).

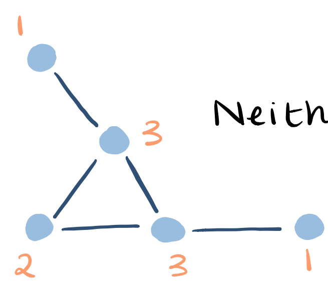
ex.



Eulerian cycle



Eulerian path, not cycle



Neither