Fuzzy Sets and Systems 2 (1979) 267–278

THE ROLE OF FUZZY SETS IN CIVIL ENGINEERING

D.I. BLOCKLEY

Department of Civil Engineering, University of Bristol, Bristol, Great Britain

Received February 1978
Revised October 1978

Civil Engineering is an art, the practise of which requires the use of scientific knowledge as a basic tool. The engineer has to use considerable judgement in setting up and interpreting his scientific calculations and in making decisions based upon incomplete information. Because a very high safety level is required in civil engineering structures the uncertainty associated with the application of scientific calculations is very crudely and conservatively estimated using traditional methods. Current reliability theory enables a discussion of random parameter uncertainty but in civil engineering, system uncertainty and the possibility of human error is extremely important and must be included in any reliability calculations.

The potential role of fuzzy sets in analysing system and human uncertainty is discussed in the paper and two numerical examples are given of the calculation of system and random uncertainty using probability measures of fuzzy sets.

Keywords: Structures, Design, Fuzzy, Fatigue, Column.

1. Introduction

Engineering is often described as an applied science. In the sense that an engineer uses his scientific knowledge to help solve his problems this is true, but it is an oversimplification to think of engineering merely as applied science. Science is, of course concerned with knowledge of the universe and the search for new knowledge is thus the scientist's primary purpose. This may be contrasted with the primary purpose of the engineer which is to create something, an artefact, an engineering product, which fulfills a human need. The classic definition of art is 'the right making of what needs making' [4] and so in this sense engineering is an art and not a science.

Harris [4] has discussed this matter with relevance to civil engineering. 'Any art needs knowledge for its practise. The basic knowledge needed by the engineer is knowledge of his materials—how they are made, how shaped, how assembled, how they stand up to stress, to weather, to use, how finally they fail. Knowledge may be obtained pragmatically through experience or systematically by the operation of the scientific method. Increasingly, the power of the latter is such that it displaces the former, clarifying and numbering what was previously vague or a matter for judgement. This does not, of course, make civil engineering an applied science, whatever that may be, any more than painting is applied chemistry, even though a knowledge of the chemical interaction of pigments . highly desirable. Art remains devoted to its purpose, which is making things; all the knowledge, and it may well be vast, needed for attaining that end remains strictly subservient to it'.

An impartial observer of any engineering research journal could be forgiven if he (or she) concluded, on the evidence of reading only that journal, that engineering is merely an applied science. Indeed, that is the unthinking view held by some engineering researchers. On the other hand many design engineers immersed in the everyday turmoil of commercial life do not have the time or inclination to read and pursue erudite research papers which are often written in a manner which is extremely difficult to follow if one is not an expert in the particular field. There tends therefore to be a communications gap between researcher and designer.

The fact is that, of course, in order to design an engineering product, whether a bridge or a washing machine, the designer has to make a set of decisions regarding what is to be built and he has to provide adequate instructions to enable the builder to build it. In making these decisions he calls upon information of various qualities, both professional and commercial and he must inevitably use considerable judgement in assessing this information. In civil engineering particularly the available theory never quite fits the actual problem and there is rarely the chance to test a prototype as in other engineering industries. This is because civil engineering projects tend to be 'one off' jobs whereas in the manufacturing industries production line techniques may be used to manufacture large quantities of the same product. Thus the uncertainty in applying theoretical solutions to practical civil engineering problems is large. The designer has obviously to take this into account because the standard of safety required by the general public concerning bridges, buildings and other structures is extremely high. Whilst the public is prepared to risk a relatively high probability of death in driving a motor car it generally expects an extremely low probability of risk that the bridge over which the car is passing will collapse.

These considerations explain to some extent the reason for the particular way in which civil engineering has developed historically. Before scientific knowledge was available concerning the strength and behaviour of materials civil engineering was a craft. Design decisions regarding for example, the great cathedrals, were based almost entirely on 'rules of thumb' derived from an accumulation of experiences over the centuries. As scientific knowledge developed, and began to be used by engineers, the craft gradually became engineering, but still the uncertainty regarding the application of the new knowledge was dealt with by rules of thumb. Today the engineer has a vast armoury of scientific theory to help him make his design decisions, but there is still a large uncertainty concerning the application of the theories to actual problems and rules of thumb are still used extensively. In some problems the uncertainty is less than in others. A good example of a situation where scientific knowledge is still inadequate is in the design of foundations to large structures. Peck, a well-known researcher in soil mechanics is quoted [6] 'The everyday procedures now used to calculate bearing capacity, settlement, or factor of safety of a slope, are nothing more than the use of the framework of soil mechanics to organise experience. If the techniques of soil testing and the theories had not led to results in accord with experience and field observations, they would not have been adopted for widespread practical use. Indeed the procedures are valid and justified only to the extent that they have been verified by experience. In this sense, the ordinary procedures of soil mechanics are merely devices for interpolating among specific experiences of many engineers in order to solve our own problems, or which we recognise to fall within the limits of previous experience'.

2. Uncertainty

It is apparent from the previous discussion that there is a large amount of uncertainty surrounding the execution of any engineering project. The nature of this uncertainty has been discussed under three headings [7] human based uncertainty, system uncertainty and random uncertainty. The prediction of these three types of uncertainty is difficult and present methods, embodied in reliability theory, tend to concentrate on random uncertainty. There is, however, a fundamental difference between the nature of random uncertainty and that of human and system uncertainty. Randomness involves uncertainty about a proposition, event or system precisely defined and described. In fact the uncertainty concerning the output of such a system is entirely derived from the uncertain nature of the parameters describing that system. On the other hand human based uncertainty and system uncertainty both derive from a 'vagueness' or lack of precision of a lack of understanding about a proposition, event, or system. In other words the greater our lack of understanding of the basic description of the phenomena under consideration the greater the uncertainty in any prediction concerning it even when the parameters are precisely defined. To analyse this type of uncertainty a mathematics which is directed at 'vagueness' as distinct from randomness is required and this is the potential role of fuzzy sets [1, 7].

A previously presented discussion of human, system and random uncertainty with respect to structural failures [2] described eight further sub-categories of uncertainty in civil engineering structural projects. These were

(1) Structures, the behaviour of which are reasonably well understood by the designers. Structural failure might occur because of the occurance of a random extremely high value of load or extremely low value of strength.

(2) Structures, the behaviour of which are poorly understood i.e. the system uncertainty is large. The designer, however, is aware of this and generally can make allowance for the uncertainty by making conservative design decisions.

(3) Structures which perform satisfactorily until damaged by an external random occurrence such as fire, vehicle impact, earthquake etc.

(4) Structures which fail because the designer did not allow for some basic mode of behaviour inadequately understood by existing technology. This mode of behaviour has probably never before been critical with this type of structure; a basic structural parameter may have been changed so much from previous applications that the new behaviour becomes critical. Alternatively the structure may be entirely of a new type or involve new materials or techniques.

(5) Structures which fail because of a mistake by the designer concerning something which is well understood by existing technology.

(6) Structures which fail through an error during construction. These errors will be a result of poor site control, poor inspection procedures and poor communications between the people involved in design and construction.

(7) Structural projects which are subject to intense pressures resulting from industrial, political and financial circumstances. These pressures create a situation in which the liklihood of errors by the people involved is greatly increased.

(8) Structures which fail through being misused or where alterations are done improperly.

The actual likelihood of a structure failing is a function of one or more of these factors. However, current reliability theory can only satisfactorily deal with the first category where uncertainty is due mainly to random parameters. The author has discussed a preliminary analysis of all of these categories with respect to past structural failures using fuzzy sets [2]. However, a more detailed analysis using fuzzy approximate reasoning may well be possible. In this paper examples of calculations appropriate to the second category are presented which involve the use of fuzzy sets to estimate system uncertainty.

3. System uncertainty analysis

As illustrations of the possible application of fuzzy sets to the calculation of system and random uncertainty in engineering problems, two examples will be considered. The first is the problem of determining the fatigue life of a steel structure subject to varying loads and the second is the determination of the safe carrying capacity of a steel column.

3.1. *A fatigue calculation*

To clearly illustrate the difference between random parameter uncertainty and system uncertainty consider the following two laboratory experiments. The first experiment is that of measuring the elastic deflection of a simply supported beam at room temperature under a known central point load and the second experiment is to measure the number of cycles to failure of a steel specimen under a known elastic sine wave cyclic fatigue loading.

The result of the first experiment may be predicted accurately using simple beam theory, but the result of the second can be predicted only very approximately. The system uncertainty in the first experiment is small and in the second is large. If these two experiments are repeated but this time the applied loads are chosen randomly from known probability density functions, then the uncertainty in any prediction about the outcome of the experiments will be increased and will be a combination of system and random uncertainty. If the experiments are then transferred to some real structure, then the problem becomes even more difficult because elements of a real structure never correspond to idealised laboratory specimens and the random variability of the parameters may be difficult to specify because of the difficulty of getting enough data.

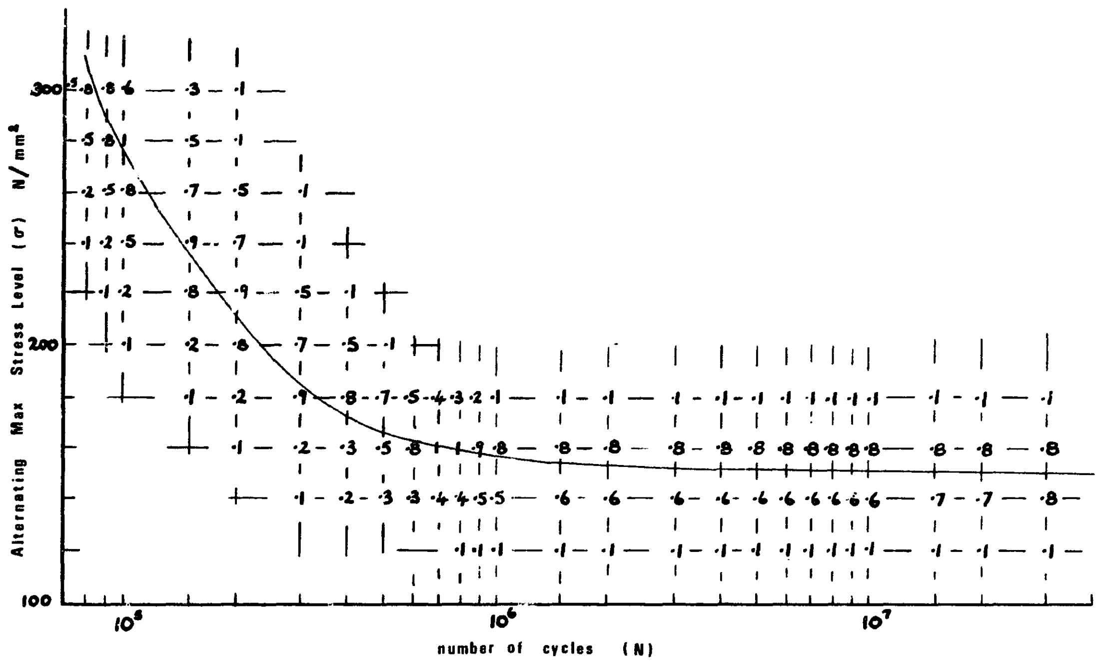

Fig. 1. S–N relation for a steel $\boldsymbol{R}_1$.

As a numerical example consider again the laboratory fatigue test. A steel specimen is to be subject to alternating stresses of maximum value $\pm 220\ \mathrm{N/mm^2}$ for 0.5×10^5 cycles. The maximum alternating stress value is then to be changed and this new stress cycle is to be applied until failure. The probability that this second stress value is $\pm 200\ \mathrm{N/mm^2}$ is 0.8 and that it is $\pm 260\ \mathrm{N/mm^2}$ is 0.2. The problem is to calculate the probability density function for the number of cycles at the second stress level which will cause failure. In order to carry out this calculation the S–N curve (solid line in Fig. 1) is provided. This shows the number of cycles to failure obtained from a series of tests at the same maximum stress level. For a deterministic calculation Miner's rule is used which is a linear cumulative damage rule. If N_i is the number of cycles to failure taken from the S–N curve at a stress σ_i, then in our laboratory experiment

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} = 1$$

where n_i is the number of cycles at stress σ_i, n_1 is known and thus n_2 can be calculated.

In this example $n_1 = 0.5 \times 10^5$, $\sigma_1 = 220\ \mathrm{N/mm^2}$ and using the solid line in Fig. 1 $N_1 = 1.75 \times 10^5$. If $\sigma_2 = 200\ \mathrm{N/mm^2}$, then $N_2 = 2.3 \times 10^5$.

Therefore

$$n_2 = \left(1 - \frac{n_1}{N_1}\right) N_2 = \left(1 - \frac{0.5}{1.75}\right) 2.3 \times 10^5 = 1.64 \times 10^5 \text{ cycles},$$

if $\sigma_2 = 260\ \mathrm{N/mm^2}$, $N_2 = 1.15 \times 10^5$ and $n_2 = 0.82 \times 10^5$ cycles.

Now in this deterministic calculation the S–N curve in spite of a known large scatter in test results was treated as a one-to-one function and the Miner's rule

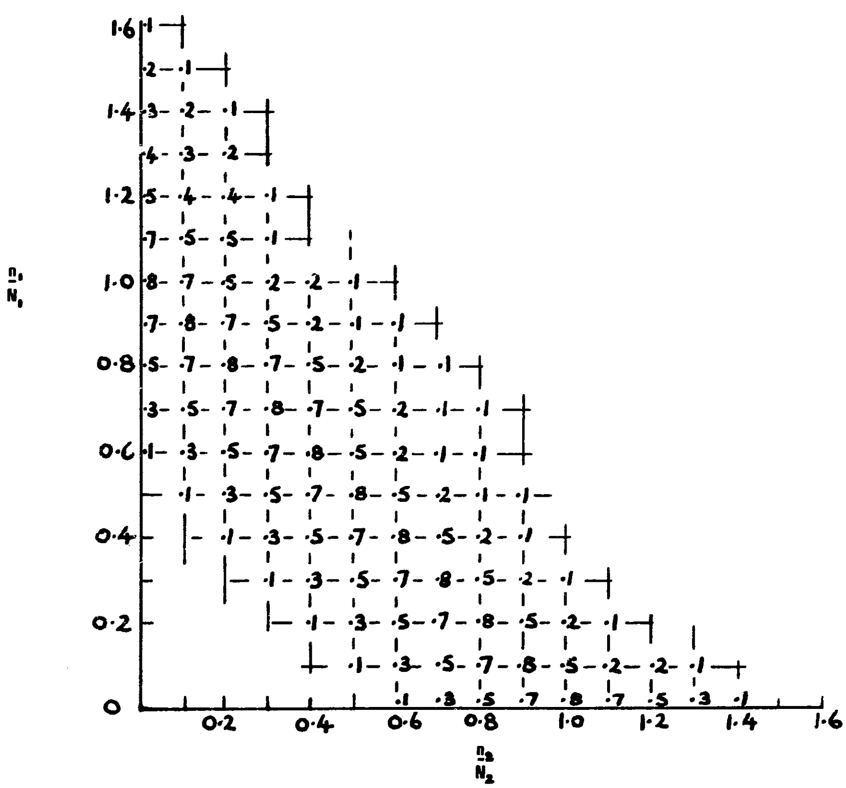

Fig. 2. Miner's rule relation $\boldsymbol{R}_2$.

was assumed accurate. To perform the fuzzy calculation consider the S–N curve as fuzzy relation $\boldsymbol{R}_1$ (Fig. 1) and the Miner's rule as a fuzzy relation $\boldsymbol{R}_2$ (Fig. 2). These relations are subjective estimates of the system uncertainty. Thus if $\sigma_1 = 220\ \mathrm{N/mm^2}$, $\boldsymbol{N}_1 = \sigma_1 \circ \boldsymbol{R}_1$ and using MAX–MIN composition $\boldsymbol{N}_1 = 0.9/0.1$, $1/0.2$, $1.5/0.8$, $2/0.9$, $3/0.5$, $4/0.1$ (each element $\times 10^5$). The first number in each pair is the element value and the second is the membership value $\chi_{\mathrm{N}_1}(n)$. (It is obvious here that the problem under discussion is continuous whereas the calculation is using discrete values for selected points in the fuzzy sets. Due to the inherent subjectivity and approximation in establishing the fuzzy relations this will not be a serious limitation in these examples and the discrete values will be assumed to be central values operating for a region of the continuous variable either side of the element value. The subjective derivation of the fuzzy relations will be discussed later.) The arithmetic of fuzzy variables has been discussed by Nahmias [5]. As $n_1 = 0.5 \times 10^5$, then

$$\left(\frac{\boldsymbol{n}_1}{\boldsymbol{N}_1}\right) = 0.55/0.1, 0.5/0.2, 0.33/0.8, 0.25/0.9, 0.167/0.5, 0.125/0.1.$$

Because discrete values are being used and this set must be composed with $\boldsymbol{R}_2$ the element values must correspond. This has to be done subjectively. Thus $(\boldsymbol{n}_1/\boldsymbol{N}_1)$ is changed to 0.6/0.1, 0.5/0.2, 0.4/0.5, 0.3/0.9, 0.2/0.9, 0.1/0.1. Now

$$\left(\frac{\boldsymbol{n}_2}{\boldsymbol{N}_2}\right) = \left(\frac{\boldsymbol{n}_1}{\boldsymbol{N}_1}\right) \circ \boldsymbol{R}_2 \quad \text{(Fig. 2).}$$

Therefore

$$\left(\frac{\boldsymbol{n}_2}{\boldsymbol{N}_2}\right) = 0/0.1, 0.1/0.1, 0.2/0.2, 0.3/0.3, 0.4/0.5, 0.5/0.5, 0.6/0.7,$$
$$0.7/0.8, 0.8/0.8, 0.9/0.5, 1/0.2, 1.1/0.1, 1.2/0.1, 1.3/0.1$$

Now

$$\boldsymbol{N}_2 = \sigma_2 \circ \boldsymbol{R}_1$$

and if $\sigma_2 = 200$ N/mm^2,

$$\boldsymbol{N}_2 = 1/0.1, 1.5/0.2, 2/0.8, 3/0.7, 4/0.5, 5/0.1 \text{ (each element } \times 10^5\text{)};$$

if $\sigma_2 = 260$ N/mm^2,

$$\boldsymbol{N}_2 = 0.8/0.2, 0.9/0.5, 1/0.8, 1.5/0.7, 2/0.5, 3/0.1 \text{ (each element} \times 10^5\text{)}$$

and $\boldsymbol{n}_2 = (\boldsymbol{n}_2/\boldsymbol{N}_2) \cdot \boldsymbol{N}_2$.

Thus if $\sigma_2 = 200$ N/mm^2,

$$\boldsymbol{n}_2 = 0/0.1, 0.15/0.1, 0.35/0.2, 0.6/0.3, 0.9/0.5, 1.2/0.7, 1.5/0.8,$$
$$2.4/0.7, 3.15/0.5, 4/0.2, 6/0.1 \text{ (each element } \times 10^5\text{)};$$

if $\sigma_2 = 260$ N/mm^2

$$\boldsymbol{n}_2 = 0/0.1, 0.08/0.1, 0.16/0.2, 0.27/0.3, 0.4/0.5, 0.6/0.7, 0.75/0.8,$$
$$1.2/0.7, 1.8/0.5, 2/0.2, 3.5/0.1 \text{ (each element } \times 10^5\text{)}.$$

Again bringing these two sets $\boldsymbol{n}_2$ onto common element values

Element value $\times 10^5$	0	0.08	0.1	0.15	0.3	0.4	0.6	0.7	0.8	0.9	1	1.5	2	3	4	5	6
$\chi_{n_2}(n)$ for $\sigma_2 = 260$	0.1	0.1	0.1	0.2	0.3	0.5	0.7	0.8	0.8	0.8	0.7	0.5	0.2	0.1	0.1		
$\chi_{n_2}(n)$ for $\sigma_2 = 200$	0.1	0.1	0.1	0.1	0.2	0.2	0.3	0.3	0.5	0.5	0.7	0.8	0.7	0.5	0.2	0.2	0.1

In order to calculate probability mass functions for $\boldsymbol{n}_2$ use will be made of the following development [8]. For two sets X and Y related by $\boldsymbol{R}$ on $X \times Y$ and when the p.m.f. $\boldsymbol{p}(x)$ is known, then

$$P\{Y = y\} = \sum_{x \in X} \frac{\chi_{\boldsymbol{R}}(y/x) \cdot P\{X = x\}}{P\{Z\}}$$

where $P\{Z\}$ is a normalising factor, needed because the relation may be a many to many mapping.

In the example the memberships obtained for $\boldsymbol{n}_2$ for various stresses σ_2 is the relation $\boldsymbol{R}$ on the cartesian product $\sigma \times n$.

The summation of the membership values in $\boldsymbol{R}$ for $\sigma_2 = 200$ N/mm^2 is 5.6 and for $\sigma_2 = 260$ N/mm^2 is 6.0.

Thus the normalising factor $P\{Z\} = 0.8 \times 5.6 + 0.2 \times 6 = 5.68$ and for example

$$P\{n_2 = 0.7\} = \frac{0.8 \times 0.3 + 0.2 \times 0.8}{5.68} = 0.0704.$$

In a similar way probabilities for n_2 being equal to all other values can be calculated and the cumulative distribution function so calculated is shown in Fig. 3.

This probability function is a measure of system uncertainty as well as uncertainty due to random parameters. It is a probabilistic measure of a fuzzy system and as such is much more representative of the real uncertainty with which a designer has to cope than is a probabilistic calculation based on ordinary set theory.

3.2. *A steel column calculation*

The carrying capacity of a structural member subject to end compressive loads is a function of various factors. The most important of these are the end restraints acting upon the column as a result of the column being part of a total structure, the initial shape of the column, the residual stresses present in the column and the longitudinal slenderness of the column. Test results obtained over the years from various laboratories around the world show a large amount of scatter, in the failure load. Often details such as residual stress distribution and initial slope are not recorded and this of course aggravates the problem of using the data for design purposes. Tests are also usually performed on columns with idealised pin ended conditions, a situation it is rarely economical to provide in a real structure. In order to use this data in practise, curves are fitted to lower bounds on the data for given types of structural cross section. An example showing M against λ is given in Fig. 4 wnere $M = \sigma_c/\sigma_y$ a non-dimensional parameter, the ratio of

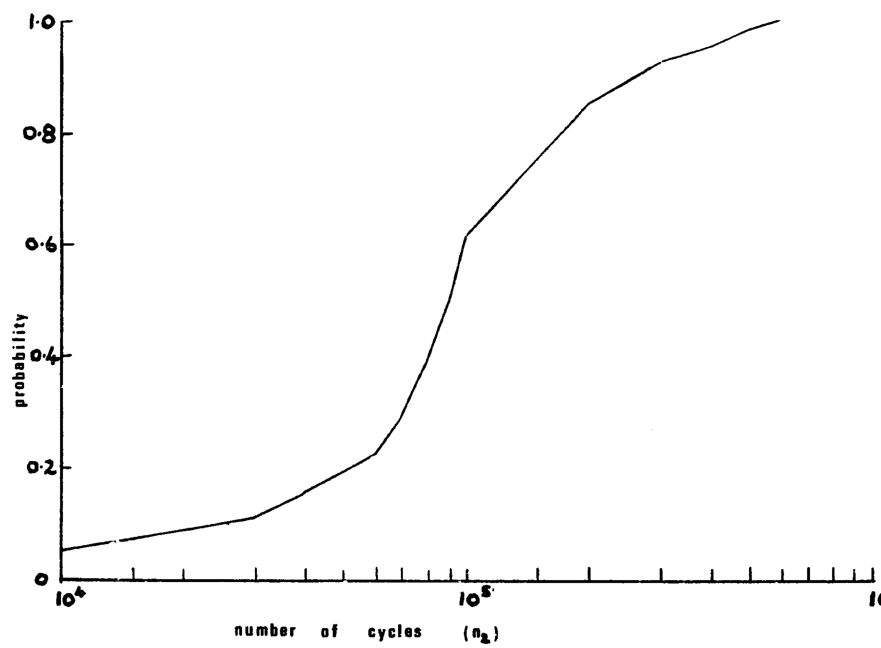

Fig. 3. Cumulative probability distribution for n_2.

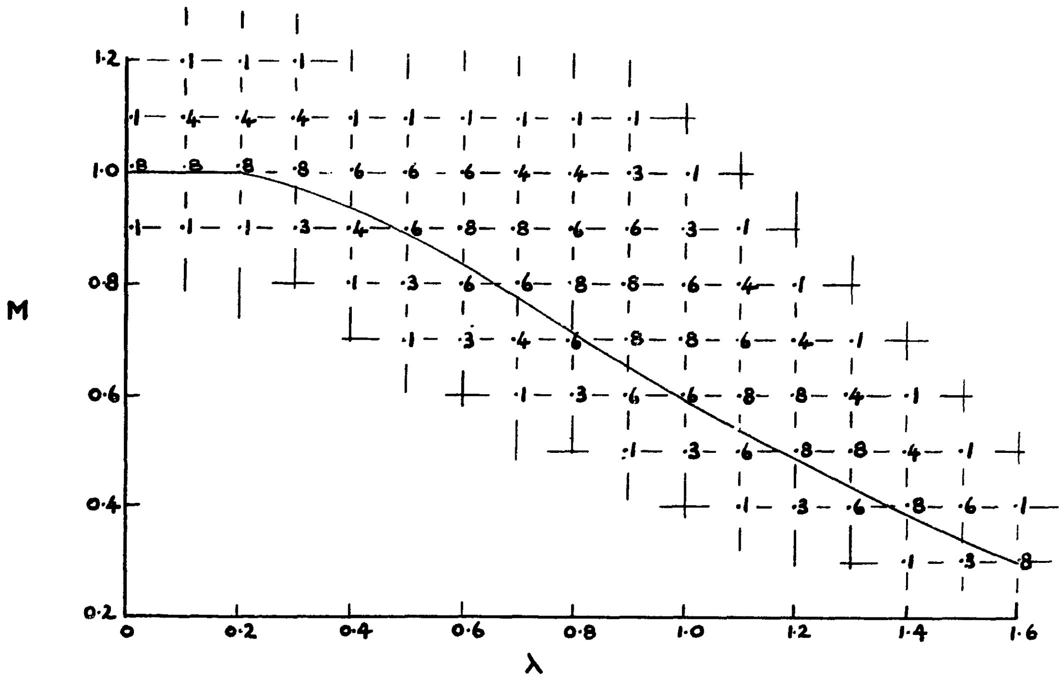

Fig. 4. A stress–slenderness relation for a steel column.

permissible compressive stress σ_c to yield stress σ_y and

$$\lambda = \frac{L}{r_y \pi \sqrt{\dfrac{E}{\sigma_y}}}$$

a measure of longitudinal slenderness and L is the effective length of the column, r_y is the minimum radius of gyration of the cross-section and E is Young's modulus. The effective length of the column is a proportion of the actual length of the column. For idealised end conditions this may be determined exactly but for application in a real structure the choice of effective length calls for a judgement by the designer as to the likely values of end restraint on the column.

Thus a deterministic calculation would proceed as follows. A column of 3 m length, a cross sectional area of 3800 mm^2 and a radius of gyration of 30.5 mm with Young's modulus of 200 kN/mm^2 and yield stress of 245 N/mm^2 is given. If the effective length factor is judged to be 0.8, then the effective length is $0.8 \times 3\ \text{m} = 2.4\ \text{m}$. If the load to be carried is 550 kN is the column satisfactory? Now

$$\lambda = \frac{2.4 \times 10^3}{30.5 \cdot \pi \sqrt{\dfrac{200 \times 10^3}{245}}} = 0.88$$

and from the solid line in Fig. 4, $M = 0.66$ and as $\sigma_y = 245\ \text{N/mm}^2$ then $\sigma_c = 0.66 \times 245 = 162\ \text{N/mm}^2$.

If the applied load to the column is 550 kN then the applied stress

$$f_c = \frac{550 \times 10^3}{3800} = 145\ \text{N/mm}^2$$

and the ratio

$$m = \frac{f_c}{\sigma_c} = \frac{145}{162} = 0.895$$

which shows the column is satisfactory as it is <1.0.

This calculation, however, gives no indication of the uncertainty involved in the answer. The same calculation will now be presented where the relation between M and λ is treated as a fuzzy relation and the effective length factor becomes a fuzzy variable. For example let $\boldsymbol{L} = 0.7/0.4,\ 0.8/0.9,\ 0.9/0.6$ (each element $\times 3m$, the physical length). Now if r_y, E, σ_y are deterministic as before

$$\boldsymbol{\lambda} = 0.77/0.4, 0.88/0.9, 0.99/0.6$$

which for the compatibility reasons described earlier will be re-written as

$$\boldsymbol{\lambda} = 0.7/0.3, 0.8/0.7, 0.9/0.9, 1/0.6.$$

Now $\boldsymbol{M} = \boldsymbol{\lambda} \circ \boldsymbol{R}$ where $\boldsymbol{R}$ is on $M \times \lambda$ (Fig. 4). Therefore

$$\boldsymbol{M} = 0.5/0.3, 0.6/0.6, 0.7/0.8, 0.8/0.8, 0.9/0.6, 1/0.4, 1.1/0.1$$

and

$$\boldsymbol{\sigma}_c = \boldsymbol{M} \cdot \sigma_y = 122/0.3, 147/0.6, 171/0.8, 196/0.8, 220/0.6, 245/0.4, 269/0.1.$$

Now assume the applied load (W) is a random variable taking on the following values

$$P\{W = 750\text{ kN}\} = 0.1, \qquad P\{W = 650\text{ kN}\} = 0.2,$$
$$P\{W = 550\text{ kN}\} = 0.6, \qquad P\{W = 450\text{ kN}\} = 0.1$$

Then the ratio

$$\left(\frac{\boldsymbol{f}_c}{\boldsymbol{\sigma}_c}\right) = \left(\frac{W}{A \cdot \boldsymbol{\sigma}_c}\right) = \boldsymbol{m}$$

can be calculated and the results are

Ratio m	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7
χ_m for $W = 750$				0.1	0.4	0.6	0.8	0.8	0.8	0.6	0.6	0.3	0.3	0.1
χ_m for $W = 650$			0.1	0.4	0.6	0.8	0.8	0.6	0.6	0.3	0.3	0.1		
χ_m for $W = 550$		0.1	0.4	0.8	0.8	0.8	0.6	0.3	0.3	0.1				
χ_m for $W = 450$	0.1	0.4	0.8	0.8	0.6	0.3	0.3	0.1						

The normalising factor is $0.1 \times 5.4 + 0.2 \times 4.6 + 0.6 \times 4.2 + 0.1 \times 3.4 = 4.32$ and the cumulative distribution function for m is shown in Fig. 5. A measure of the safety of the column in this example is

$$P\{m \leqslant 1\} = 0.73.$$

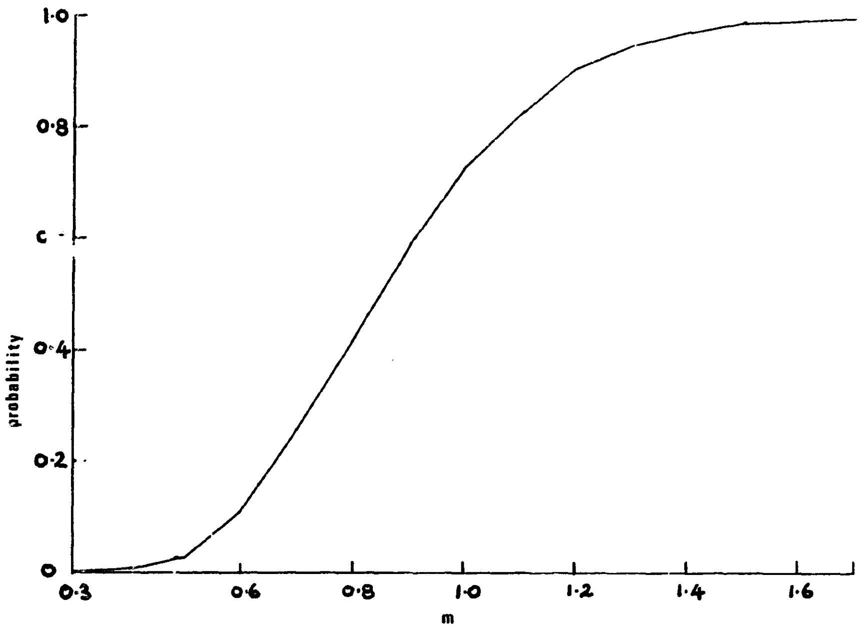

Fig. 5. Cumulative probability distribution for *m*.

4. Subjective judgement

It will be argued by those who regard engineering as an applied science that the use of such methods as those proposed represents a loss of objectivity in the calculations and the use of subjectively derived fuzzy relations is not valid. However, as pointed out earlier, in order to use the objective calculations in real engineering situations, subjective assessments concerning valid assumptions at the beginning of a calculation and the meaning of the results obtained at the end have to be made with respect to the decisions which have to be taken. The sort of procedures described herein merely bring subjective judgement into the formal calculations. Decisions have to be based on sparse experimental data and a lack of understanding of some basic phenomena. Continued research is obviously required into those phenomena, providing better data and more understanding, which at the very least will reduce the variability in the subjective estimates.

One attraction of the methods proposed is that subjective assessments obtained from various sources can be combined. An experienced research worker, for example, can use his knowledge in a situation with which he is familiar, his research testing program, and this can then be carried through and combined with other data and judgements made about the structure under service conditions. Further consideration has been given to this problem for conditions of high temperature fatigue and creep of steel [3].

5. Conclusions

Civil Engineering is an art the practise of which requires scientific knowledge. However, because of the public need for high levels of safety and the lack of a prototype testing facility the calculation of the uncertainty surrounding the application of a scientific calculation is extremely difficult. Much judgement based upon experience of what has gone before is used and this must be utilised more efficiently in the future. Fuzzy sets can be used to estimate system uncertainty and probability measures of system and random uncertainty can be calculated.

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