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DEUTSCHEN FORSCHUNGS- UND VERSUCHSANSTALT
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8. Jahrgang, Heft 2 | Köln | März/April 1984

Some Aspects of Three-Dimensional Separation, Part I: Streamsurface Bifurcations

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Summary: The concepts of separated flows, which may be defined clearly in the two-dimensional case, are extended and defined for three-dimensional steady flows. Local solutions in the vicinity of streamsurface bifurcation lines are obtained for the Navier-Stokes and continuity equations, which describe the features of such flows in terms of local vorticity and pressure derivatives. Simple separated flow configurations are described and classified.

**Über dreidimensionale abgelöste Strömungen,
Teil I: Stromflächenverzweigungen**

Übersicht: Die Begriffe der Strömungsablösung, die im zweidimensionalen Fall klar definierbar sind, werden für dreidimensionale, stationäre Strömungen erweitert und definiert. In der Umgebung von Stromflächenverzweigungslinien werden lokale Lösungen der Navier-Stokes- und Kontinuitätsgleichungen angegeben, welche die Eigenschaften solcher Strömungen in Abhängigkeit von lokalen Gradienten der Wirbelstärke und des Druckes beschreiben. Einfache Konfigurationen von abgelösten Strömungen werden beschrieben und klassifiziert.

1. Introduction

Let the flow situation be such that the fluid may be considered as an incompressible *Newtonian* continuum. Consider external flow of this fluid over a simply connected stationary body and let the far field flow be uniform. Let the *Reynolds* number of the flow be very much larger than one. Such flows often exhibit the phenomenon of "flow separation". This term may be defined unambiguously in a two-dimensional steady-flow situation by connecting it to the occurrence of reverse flow on the body surface, i.e. to flow whose direction is opposite to the projection of the free stream direction into the surface. (The body surface is assumed to contain no folds.)

The case of two-dimensional separated flow is a very rare exception, however, and in three-dimensional flows the specification of "separation" by means of a reverse flow is inadequate. Nevertheless, much of the literature uses concepts which have been taken over from two-dimensional flows for the description of three-dimensional "separated" flows. This is dangerous because, while authors' conceptions may be clear in their own minds, information communicated to a reader may be fraught with ambiguities inherited from the history of his experience with two-dimensional flows. The situation becomes confounded even more when a fourth dimension is introduced. Though this is necessary in many cases, as "separated" flows are often unsteady as well as being three-dimensional, we restrict our discussion to steady flows.

Our aim is to provide a vocabularly of well-defined terms for the description of three-dimensional steady "separated" flows and to use this language in order to describe and classify a number of well-known "separated" flow situations. The hope is that as simple as possible a language may be found for describing the complicated topological structure of such flows and that this will also lead to a better understanding of them.

2. Definitions and Rules

The kinematics of flows may be described for example by specifying the position vector $\mathbf{x}$ as a function of time t of a particle which at $t=0$ was at $\mathbf{x}=\mathbf{X}$. Thus,

$$\mathbf{x}=\mathbf{x}(\mathbf{X},t)$$

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describes the pathline of the particle, labelled – by its reference position at $t=0$ – as $\mathbf{X}$.

Another way of describing the flow field is to specify the velocity vector $\mathbf{u}$ of the particle which happens to be at position $\mathbf{x}$ at time t:

$$\mathbf{u}=\mathbf{u}(\mathbf{x},t).$$

The integral curves of this vector field give the streamlines of the flow, which in steady flow coincide with the pathlines. In unsteady flow, streamlines change with time, and the pathline defined by the particle $\mathbf{X}_1$ is tangent to the instantaneous streamline defined by the time t_1, in the point $\mathbf{x}(\mathbf{X},t_1)$.

Three-dimensional steady flows may conveniently be described by streamsurfaces. These have the conceptual advantage over streamlines that a surface is able to divide three-dimensional space into distinct regions, whereas a line is not. (In two dimensions, a streamline serves the same purpose. In four dimensions a three-dimensional feature would be required in which one dimension would best be chosen as the time.) In two-dimensional flow there exist special streamlines, called separatrices, which conveniently divide the flow into distinct regions. Such streamlines originate at saddle points. The counterparts to these separatrices in three-dimensional flows are special streamsurfaces.

In steady flow, a streamsurface is generated by those streamlines which pass through a fixed curve C. Thus C defines the streamsurface in the flow field. (Since we restrict the discussion to steady flows, streamsurfaces may also be thought of as pathsurfaces.)

In many flow fields there exist special streamsurfaces which bifurcate. An example of what we call positive bifurcation is the stagnation streamsurface on an infinite swept cylinder, which divides along the stagnation line. A schematic sketch (FIG. 1a) illustrates positive streamsurface bifurcation: A single streamline on the incoming streamsurface appears to bifurcate into two at the positive bifurcation line S. A negative streamsurface bifurcation is illustrated in FIG. 1b: Two streamlines appear to combine to form a single streamline on the negative bifurcation line. We use at the moment the term "appears to bifurcate". The simple interpretations given in FIG. 1a and 1b were first put forward by *Maskell* [1] and their precise validity will be discussed later.

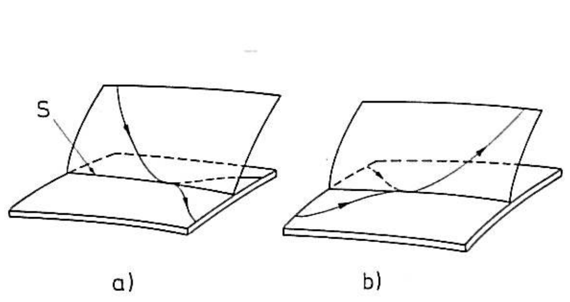


FIG. 1: a) Positive and b) negative streamsurface bifurcation at a wall

Streamsurface bifurcations need not be located only on the flow boundaries. However, a single bifurcation within the flow field leads to a difficulty which may be seen by considering a section through a bifurcation and taking the velocity components in this section (see FIG. 2a). The integral curves of these velocity components will be referred to as sectional streamlines. (Note that the sectional streamlines are in general not the same as the projections of streamlines into the section.) Consider the sectional streamlines A and B and a neighbouring sectional streamline C. If the direction of C is as shown, its relation to its neighbours A and B changes suddenly at the bifurcation point P. This would require a jump from zero to infinite shear rate across a small region around P. If there is no vortex sheet present along A and B, it is necessary that a second bifurcation occurs at P, as shown in FIG. 2b. Thus, under such conditions, free bifurcations occur only in pairs with four free sheets[1]), and a wall bifurcation has two wall sheets and one free sheet.

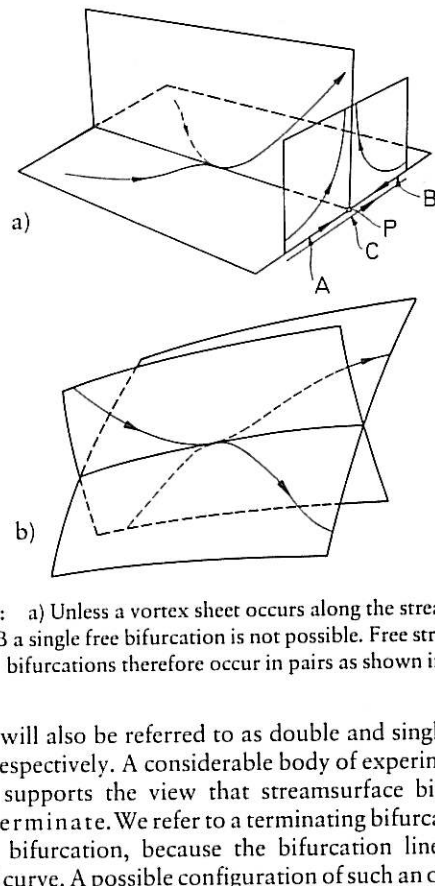


FIG. 2: a) Unless a vortex sheet occurs along the streamsurfaces A and B a single free bifurcation is not possible. Free streamsurface bifurcations therefore occur in pairs as shown in b)

These will also be referred to as double and single bifurcations respectively. A considerable body of experimental evidence supports the view that streamsurface bifurcations may terminate. We refer to a terminating bifurcation as an open bifurcation, because the bifurcation line is not a closed curve. A possible configuration of such an open bifurcation is sketched in FIG. 3. The possibilities of open and closed bifurcations are also discussed by *Wang* [2 to 5] and by *Tobak* and *Peake* [6], (see Section 3.1).

In the figures it has been tacitly assumed that it is possible for streamlines to bifurcate in a cusp-like fashion at a wall. Part of the next chapter is devoted to examining this question in some detail.

[1]) It is possible for more than one pair of bifurcations to occur together (with more than four free sheets), but the conditions necessary for such patterns are very restrictive, and their occurrence would be rare. E.g. at a six-sheet bifurcation the first three spatial derivatives of the pressure would be zero, and very exceptional far-field boundary conditions would be needed.

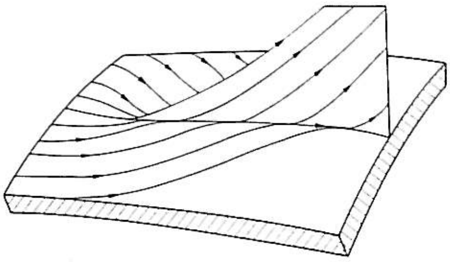


FIG. 3: Possible configuration of open negative streamsurface bifurcation at a wall

3. Theoretical Considerations

3.1. No-Slip Critical Points and Wall Bifurcation Lines

In order to examine the conditions on wall bifurcation lines it is necessary to consider the wall shear stress field. We refer to the integral curves of the wall shear stress vector exerted by the fluid on the wall as wall streamlines. At any point where the wall shear stress is zero, the wall streamline field possesses a saddle or nodal point, at which the streamline direction is indeterminate. Consider the flow in the immediate vicinity of such a critical point, located at the origin of the coordinate system shown in FIG. 4. Let the xy-plane contain the wall, and let the components of **u** and curl $\mathbf{u} \equiv \boldsymbol{\Omega}$ be as shown.

We assume that, for steady flow of an incompressible *Newtonian* fluid with constant viscosity at finite *Reynolds* number over a smooth continuous wall, the velocity and pressure are regular, so that local solutions can be written as *Taylor* series expansions in the space coordinates. The mathematical support for this assumption is discussed by *Ladyzhenskaya* [7].

The no-slip condition at the wall requires that $\mathbf{u}/z$ approaches a constant as $z \to 0$, and the vector field of this quantity has the same integral curves as the wall shear stress. They therefore coincide with the wall streamlines as defined above. Accordingly we expand $\mathbf{u}/z$ in a *Taylor* series about the critical point and retain only the lowest order term. Thus,

$$(3.1)\qquad \begin{bmatrix} u/z \\ v/z \\ w/z \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

or $\mathbf{u}/z = \mathbf{F}\cdot\mathbf{x}$, or $\dot{\mathbf{x}}/z = \mathbf{F}\cdot\mathbf{x}$.

Here $\dot{\mathbf{x}}$ denotes the substantial derivative of **x** with respect to time. Substituting (3.1) into the *Navier-Stokes* and continuity equations, the elements of F can be determined (see *Perry* and *Fairlie* [8]) as:

$$(3.2)\qquad \begin{aligned} a_1 &= \eta_x, & b_1 &= \eta_y, & c_1 &= P_x/2\nu, \\ a_2 &= -\xi_x, & b_2 &= -\xi_y, & c_2 &= P_y/2\nu, \\ a_3 &= 0, & b_3 &= 0, & c_3 &= (\xi_y - \eta_x)/2, \end{aligned}$$

where the subscripts denote partial differentiation, P is the kinematic pressure (pressure divided by density), and ν is the kinematic viscosity. Eqs. (3.1 and 3.2) were first derived by *Oswatitsch* [9] who expressed the various elements of F in terms of the pressure gradient and derivatives of the wall shear stress.

If the eigenvalues of the matrix F are real, there are three planes which are special in that they contain solution trajectories (i.e. streamlines). For simplicity let the yz-plane be a plane of symmetry, so that $P_x = \xi_x = \eta_y = 0$. The three special planes are then the xy-, yz- and xs-planes as shown in FIG. 5. For the particular case considered, the critical point manifests itself in the form of a saddle in the trajectories in both the xy- and yz-planes and as a node in the trajectories contained in the xs-plane. We refer to such a combination of two saddles and a node as a saddle-node trio.

The features of the flow in FIG. 5 have much in common with those of a negative streamsurface bifurcation. The wall streamlines converge towards it, and streamlines in a free sheet – the xs-plane – diverge away from it. Note, however, that all the streamlines on the free sheet come from the point 0.

Now set $\eta_x = 0$. Eq. (3.1) reduces to

$$(3.3)\qquad \begin{aligned} u/z &= 0, \\ v/z &= b_2 y + c_2 z, \\ w/z &= -b_2 z/2. \end{aligned}$$

This flow is illustrated in FIG. 6a. A saddle occurs in the yz-plane as before, but the saddle and node in the other planes become degenerate. This is the classical two-dimensional separation which is so special or degenerate that it would never really occur in practice, except in the immediate vicinity of the separation line of a three-dimensional flow. (For a discussion of degenerate or borderline solutions see *Kaplan* [10].) To include the three-dimensionality in the solution for such a symmetrical case, the next higher-order

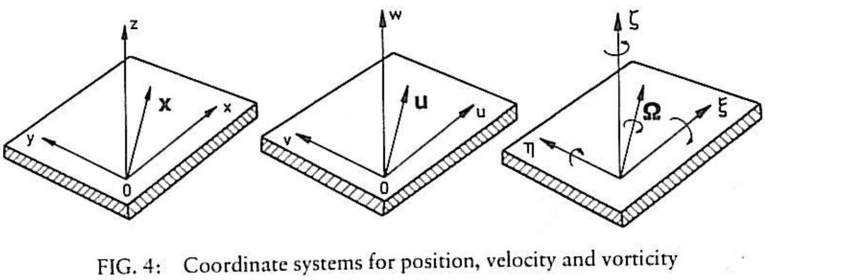


FIG. 4: Coordinate systems for position, velocity and vorticity

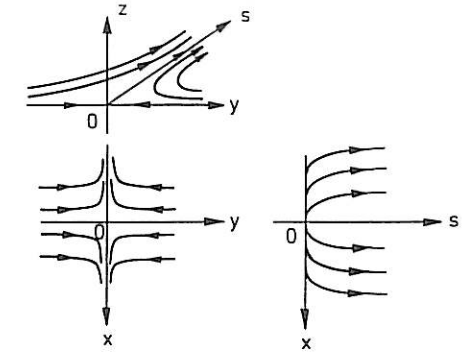


FIG. 5: The three special planes which contain streamlines in the case of a saddle-node trio

terms, which would then be third-order terms, must be included in the expression for $\mathbf{u}/z$ in (3.3). However, it may be shown that their inclusion does not affect the two-dimensionality of the flow close to the x-axis in FIG. 6a.

Consider the streamlines in the xz-plane of this flow. To locate the xs-plane, i.e. to find the angle ϑ, we seek a straight-line solution of the form

$$(3.4)\qquad \frac{w}{v}=\frac{\dot z}{\dot y}=\frac{dz}{dy}=\frac{z}{y}=\tan\vartheta .$$

From (3.3) we obtain

$$(3.5)\qquad \tan\vartheta=-\frac{3}{2}\,\frac{b_2}{c_2}=3\,\xi_y\nu/P_y .$$

This is the classical result for two-dimensional laminar separation given by *Lighthill* [11].

Now superimpose on the flow of FIG. 6a a uniform shear flow $u/z=k$, where k is a constant. The resulting equations,

$$u/z=k,$$

$$v/z=b_2y+c_2z,$$

$$(3.6)\qquad w/z=-b_2z/2,$$

$$s=z\operatorname{cosec}\vartheta$$

satisfy the *Navier-Stokes* and continuity equations as well as the no-slip condition. This describes the flow in the immediate vicinity of a quasi-two-dimensional negative bifurcation line. It is also a local solution for the genuinely three-dimensional case.

To find the solution trajectories in the xy-plane, i.e. the wall streamlines, let $z\to 0$, so that

$$(3.7)\qquad \frac{v}{u}=\frac{\dot y}{\dot x}=\frac{dy}{dx}=b_2y/k,$$

which, on integration, gives

$$(3.8)\qquad y=y_0\exp\left[(b_2/k)(x-x_0)\right],$$

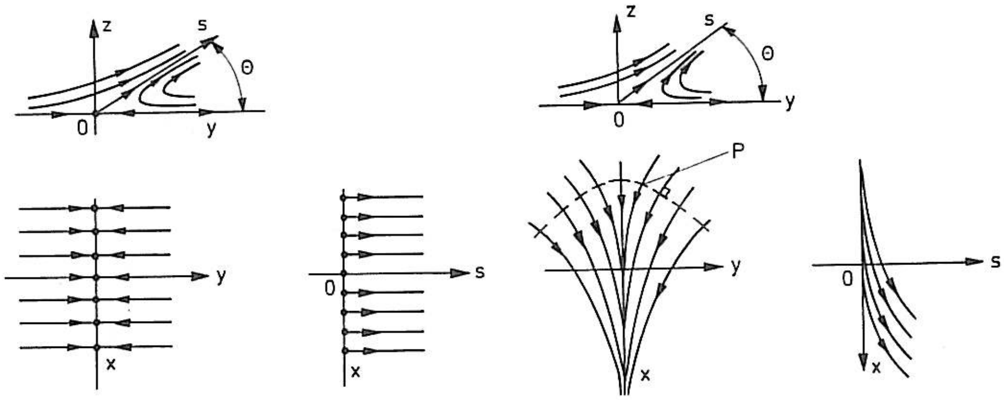
where (x_0,y_0) is the initial position of a non-diffusing particle moving along the solution trajectory. Since the xy-plane is a wall, a particle would, of course, take an infinite time to trace out the trajectory, because of the no-slip condition.

A similar analysis shows that, in the xs-plane,

$$(3.9)\qquad s=s_0\exp\left[(-b_2/2k)(x-x_0)\right],$$

where (x_0,s_0) is the initial point on the trajectory in the xs-plane. These trajectories are shown in FIG. 6b. For the figure we have chosen b_2/k to be negative, so that trajectories converge in the xy-plane and diverge in the xs-plane. It corresponds to a negative streamsurface bifurcation. Positive values of b_2/k correspond to positive bifurcation.

The flow of FIG. 6b has no trajectories in the yz-plane. However, since the x-components of all gradients are zero, the projections of the trajectories into the yz-plane coincide with the sectional streamlines as defined earlier.



a)

b)

FIG. 6:
a) The degenerate case of classical two-dimensional separation
b) Superposition of a uniform shear flow in the direction of the third dimension on the flow of a) produces a quasi-two-dimensional separated flow

The two flows shown in FIGs. 6a and 6b raise some very important points regarding the definition of separation. FIG. 6a has all the universally accepted properties of separation: A streamline leaves the surface, there is reversed flow, and the wall shear stress is zero at the separation point 0. In the flow of FIG. 6b no streamline leaves the surface (within a region of finite extent), the meaning of reversed flow becomes obscure, and the wall shear stress is finite everywhere. In both cases material is being transported away from the wall, and at precisely the same rate. If one were to consider this flow rate as a possible measure of the "degree of separation" one would have to conclude that the flow of FIG. 6b is "just as separated" as that of FIG. 6a.

A more satisfactory measure for the concept of degree of separation is the "strength of wall streamline convergence" which could be suitably measured by the maximum angle made by a wall streamline and its asymptote. It is not possible to define this strength purely on the basis of FIG. 6b, because this local model contains only one characteristic length $|k/b_2|$ and therefore depends on the scale on which it is observed. (I. e. the angle does not have a maximum value.)

This can be illustrated by looking at a streamline convergence on two scales as shown in FIG. 7a. Clearly, the apparent "rate of convergence" depends on the size of the viewing window. However, if the pattern of FIG. 6b is embedded in a flow which has its own characteristic length, L, the concept "strength of convergence" becomes meaningful. The parameter $|Lb_2/k|$ then provides a possible measure of the strength of convergence for the simple case considered here. This is illustrated in FIG. 7b. The streamlines deviate from the exponential curve in some way. This causes some feature like a point of inflexion to occur in the streamlines. The location of this feature relative to the x-axis is governed by L. The streamline slope at the point of inflexion is a measure of the strength of convergence and is governed by $|Lb_2/k|$.

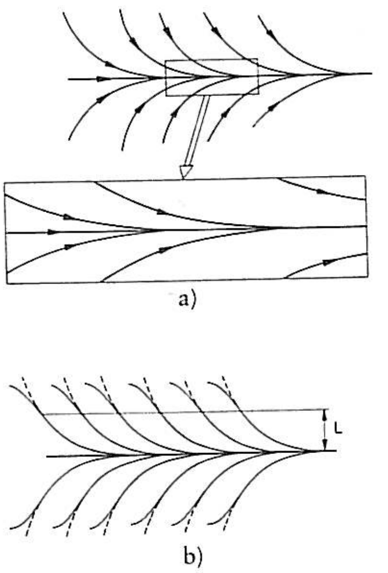


FIG. 7: a) Strength of convergence depends on the size of the viewing window.
b) In the presence of an independent length scale the strength of

The results of this section may be expected to apply locally even for curved bifurcation lines or for bifurcation lines of varying strength, provided that the radius of curvature or the characteristic length for the strength variation is large compared with $|k/b_2|$. When the bifurcation is quasi-two-dimensional as in FIG. 6b, the location of the bifurcation line is easy to define. A suitable definition of its location in other cases is best illustrated with the aid of FIG. 6b. Consider a point P on a curve drawn orthogonally to the wall streamlines. (In the quasi-two-dimensional case this is a parabola.) Consider a section perpendicular to the xy-plane and tangent to the orthogonal curve at P. The sectional streamlines in this section then have a saddle point at P. The strength of the saddle depends on the position of P along the curve and reaches a maximum at the bifurcation line. The strength of a saddle is defined as follows: If a new orthogonal coordinate system is chosen such that the x-axis is aligned with the surface trajectory at P, then the strength of the saddle is defined as $|b_2|$ in Eq. (3.6) when applied to this new coordinate system. It should be pointed out that one has no difficulty in locating a strong bifurcation line unambiguously in an experimental situation. The exponential convergence (3.8) may also be expected to apply locally if the strength of the bifurcation grows gradually from zero. I. e. it does not seem necessary for a bifurcation of this type to start at a critical point, i.e. a point of zero skin friction. It may start without the wall shear stress becoming zero anywhere.

There is considerable discussion in the literature concerning open and closed bifurcations. A closed negative bifurcation line has been referred to by *Wang* [2] as "bubble type" separation and an open negative bifurcation line as "free vortex layer" separation. Later, *Wang* [3] referred to these as closed and open separations respectively. *Tobak* and *Peake* [6] use the terms "global" and "local" separation, respectively. All of these terms imply some preconceived physical idea about the nature of the flow which may be misleading. For instance, "closed" or "bubble" type separation implies a closed streamsurface. But separation "bubbles" are only closed in the most exceptional cases of perfectly two-dimensional or axisymmetric flow. In general, streamlines "escape out" of separation "bubbles". As regards the terms "global" and "local", we again disagree with the implications contained in these words. An open negative bifurcation can be just as extensive (i. e. just as "global") in its effect on the flow pattern as a closed one. Negative and positive bifurcation lines have been referred to in the literature as separation and attachment lines respectively. We prefer the term bifurcation because it does not taint the description of the topology with other implications, and because it may also be applied to bifurcations which occur in the flow field, i. e. not on the wall. Separation and attachment are terms that only apply at a wall.

On the subject of no-slip critical points it should be pointed out that if two of the eigenvalues of the matrix F in (3.1) are complex, there exists only one plane which contains solution trajectories – namely the wall. The wall streamlines form a spiral node or focus. Any trajectory above the wall circles up into a sheared helix of decreasing radius around the single eigenvector s as shown in FIG. 8. Such critical points have been discussed extensively by *Oswatitsch* [9] and in more detail by *Lim*, *Chong* and *Perry* [12], who extended the analysis to include unsteady flow cases.

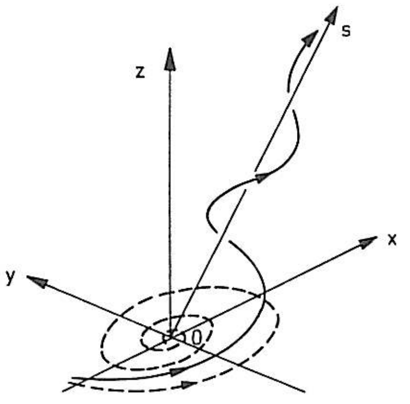


FIG. 8: Configuration near a critical point when F has two complex eigenvalues and one real eigenvalue

3.2. Bifurcation Lines as Envelopes

In the solution given by (3.8) and (3.9) streamlines approach a bifurcation line asymptotically. In contrast to this, some workers believe that bifurcation lines are approached by the trajectories along a line of cusps. The bifurcation line would then be an envelope of the trajectories, i.e. a singular solution of the *Navier-Stokes* equation. This was first proposed by *Maskell* [1,13], but it was questioned by *Legendre* [14], and *Lighthill* [11]. *Brown* [15], *Brown* and *Stewartson* [16] and *Buckmaster* [17] favour cusps and envelopes. It should be pointed out, however, that these latter workers based their arguments on the boundary layer equations, which break down near separation.

In order to examine the possibility of the occurrence of cusps, consider the simplest possible local formulation

$$(3.10)\quad \begin{aligned} u &= kz, \\ v &= za|y|^n, \qquad 0<n<1, \end{aligned}$$

for the trajectories in the plane of the wall (xy-plane). Negative values of a correspond to negative bifurcations. Integration of (3.10) gives the wall streamlines as

$$(3.11)\quad x = x_0 - \frac{1}{1-n}\,\frac{k}{a}\left(|y_0|^{1-n} - |y|^{1-n}\right),$$

where (x_0,y_0) is an initial point on the trajectory. FIG. 9 shows a sketch of (3.11). The dotted branches of the trajectories are not physically real, since we require u to be positive. We can now use continuity and the boundary condition at the wall to obtain

$$w = -(1/2)\,n z^2 a|y|^{n-1}.$$

For finite $z, w\to\infty$ as $y\to 0$, since $n-1<0$. Hence, cusps of this form are impossible. Other types of cusps might be more realistic but they have yet to be discovered. In any case they would be irregular as in the simple case given above and from the work of *Ladyzhenskaya* [7] irregular solutions are unlikely for the incompressible *Navier-Stokes* and continuity equations.

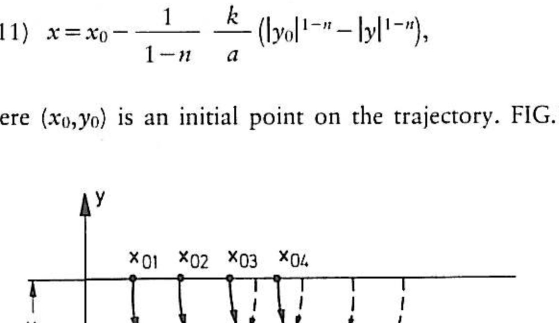


FIG. 9: Sketch of bifurcation line along which trajectories form cusps (Eq. 3.11)

3.3. Implications for Experiments

The view that streamlines converge along bifurcation lines in a cusp-like manner would permit the combination or division of streamlines as implied by the representations of Chapter 2, whereas the asymptotic convergence of Eq. (3.8) does not. We now want to show that in an experiment it is not possible to distinguish between the two cases.

Streamsurfaces may be made visible by the addition of dye of some sort to the fluid, or the wall streamline direction may be shown by smearing the wall with a thin film of a viscous suspension, or the velocity vectors may be measured by some technique and subsequently integrated to obtain streamlines. Whichever of these or other techniques are used, the resolution of the experiment is always limited, so that a smallest length λ exists, below which the spacing of two points is seen as zero. This might arise through the diffusion of the dye particles, through the size of the measuring volume in velocity measurements, through the thickness of the film of suspension, or simply through other limitations of observation.

To illustrate this, FIGs. 10a and 10b show two situations which would be experimentally indistinguishable: If the closest approach of streamline A and streamline B is less than λ, they are seen as the single streamline C. In subsequent discussions of streamsurface bifurcations the difficulty of cusp-like and asymptotic convergence of streamlines at bifurcations will therefore be ignored.

Perhaps it is appropriate to conclude this discussion on bifurcations at a wall with the descriptive picture of FIG. 11. Consider a line oriented across the flow and parallel to the wall, at a small distance Δ from it, as the generating curve of a streamsurface. In the vicinity of a bifurcation this streamsurface gradually folds in such a way as to enclose a thin layer of fluid oriented at an angle to the wall (shown in FIG. 11 as 90°). We now reduce Δ and observe that the new streamsurface is enclosed within the folds of the former one, and it "leaves the wall" later. When $\Delta = 0$, the streamsurface does not leave the wall at all. The locus of the "leading edge" of the folded surface, traced out as Δ is varied, is the free sheet of the negative streamsurface bifurcation.

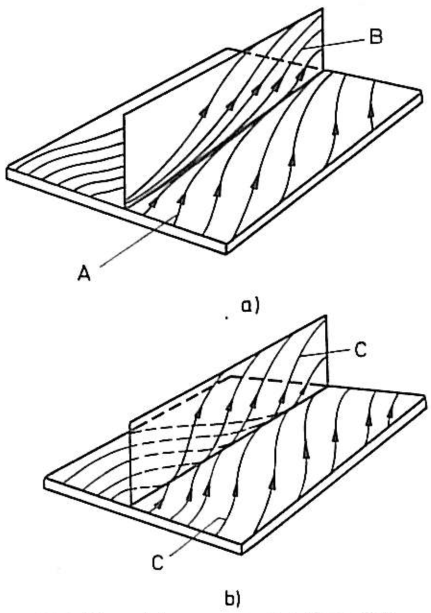


FIG. 10: a) An asymptotic bifurcation
b) A cusp-like bifurcation
Experimentally, cusp-like and asymptotic convergence are not distinguishable. If the closest approach of the streamlines labelled A and B in a) is not resolvable in an experiment, they are seen as the single streamline C in b)

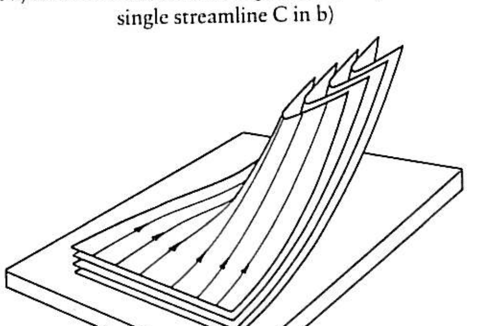


FIG. 11: Streamsurfaces generated near a negative bifurcation by straight lines at different distances from and parallel to the wall enfold each other

3.4. Free Bifurcations and Free-Slip Critical Points

The main difference between streamsurface bifurcations at a wall and free bifurcations is, that the no-slip condition is imposed on the wall sheets in the former and it applies on none of the sheets in the latter. This modifies the discussion of the relevant critical points.

Consider a critical point (indeterminate velocity direction) at a point away from a wall. Expand the velocity field **u** (this time not **u**/z) in a *Taylor* series about this point, and retain only the lowest order terms:

$$\dot{\mathbf{x}} = \mathbf{G} \cdot \mathbf{x} = \begin{bmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{3.12}$$

Substituting in the *Navier-Stokes* and continuity equations, the elements of G can again be determined. However, whereas for the wall critical point the elements were determined by a balance of viscous and pressure gradient forces, they now arise from a balance of inertia and pressure gradient forces. The gradients of viscous stresses turn out to be zero in this approximation since the vorticity is assumed to be perfectly constant over the region of the critical point. For this reason such critical points will be referred to as inviscid critical points. For a more detailed discussion of such critical points the reader is referred to *Perry* and *Fairlie* [8].

Inviscid critical points occur only at pressure extrema. They can also only be two-dimensional if the vorticity is finite. The latter is the result of an essential degeneracy, which means that only one plane contains solution trajectories (streamlines) of interest. If the eigenvalues of G are real, these streamlines form a saddle as shown in FIG. 12. The coordinate system has been chosen to give

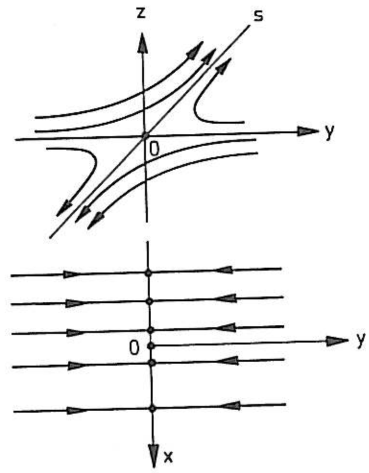


FIG. 12: Streamlines near an inviscid critical point. (Eq. 3.13)

$$\begin{aligned} u &= 0, \\ v &= e_2 y + f_2 z, \\ w &= -e_2 z, \\ \tan\vartheta &= -2e_2/f_2, \end{aligned} \tag{3.13}$$

where $e_2 = -\sqrt{-P_{yy}}$, $f_2 = -\xi$. The point 0 is a pressure maximum, and the isobars are concentric circular cylinders whose axis coincides with the x-axis. Thus $P_{yy} = P_{zz}$. For the case shown, ξ is negative. Without the no-slip condition imposed, higher order terms should be included in the solution for such degenerate cases. Such an analysis is discussed later for the case of a stretched vortex. We shall restrict the discussion here to flows which may be considered to be two-dimensional or quasi-two-dimensional near bifurcation lines.

If we now superimpose a uniform velocity $u = u_0$ onto this flow and proceed as for the critical point at a wall, the result for this quasi-two-dimensional flow is

$$\begin{aligned} y &= y_0 \exp[(e_2/u_0)(x - x_0)], \\ s &= s_0 \exp[(-e_2/u_0)(x - x_0)] \end{aligned} \tag{3.14}$$

This flow has the character of a free streamsurface bifurcation and differs from the form of (3.8 and 3.9) only in a factor of 2 in (3.9).

If the vorticity is zero at the critical point, the matrix G becomes degenerate and it is necessary to reformulate the problem to obtain the elements. For this irrotational case the solutions take the form of saddle-node trios with orthogonal eigenvectors (see *Perry, Lim* and *Chong* [18]) and occur at pressure maxima.

On the subject of inviscid critical points it is worth discussing the case when the point 0 is a pressure minimum. Two of the eigenvalues of G are then complex and the solution takes the form of a degenerate (in the sense of essential two-dimensionality) centre with concentric circular isobars as before, and with elliptical closed streamlines as shown in FIG. 13. Thus, although the problem is formulated for three dimensions, the streamlines do not spiral into the critical point. That they should do so in the three-dimensional case can be shown by considering a U-shaped vortex rod located above and parallel to a plane wall in an inviscid flow directed parallel to a plane wall along the symmetry plane of the U. First, however, consider a straight vortex rod in cross flow. With a vortex strength of suitable value this leads to the pattern shown in FIG. 14a, with closed streamlines and closed separatrix, S.

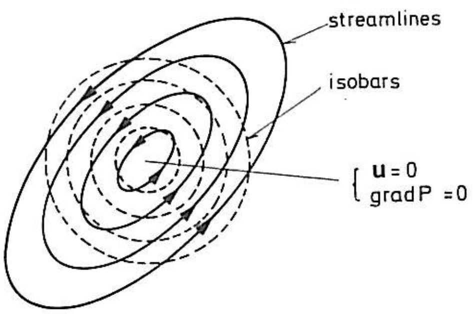


FIG. 13: Streamlines (full) and isobars (dotted) near a degenerate centre (pressure minimum)

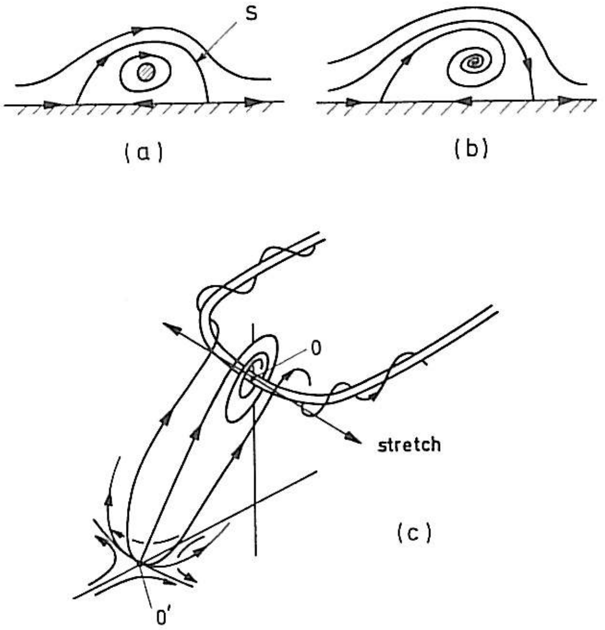


FIG. 14: a) Two-dimensional inviscid flow across a straight vortex rod near a wall.
b) Streamlines in the symmetry plane of flow over a U-shaped vortex parallel to a wall, the legs of the U pointing downstream. The opening of the separatrix is a consequence of the three-dimensionality
c) Three-dimensional sketch illustrating the flow of b) and showing how the vortex rod is being stretched. In the vicinity of the symmetry plane this stretching occurs axisymmetrically

Now consider the vortex rod in FIG. 14a to be bent into a U-shape with the legs of the U pointing downstream. It can be shown by the *Biot-Savart* law (see e.g. *Perry and Chong* [19]) and the electromagnetic analogy, Part II (*Perry* and *Hornung* [19]) that streamlines then spiral into the vortex rod. Also, in the vicinity of the symmetry plane the rod is being stretched axisymmetrically as indicated in FIG. 14c. As a consequence, the closed separatrix S in FIG. 14a becomes open as shown in FIG. 14b.

The problem of the stretching of a vortex with a core of finite extent may be formulated as an unsteady problem with vorticity diffusion. If the vortex is subjected to a constant strain rate (i.e. stretching in which the length of a material element aligned with the axis increases exponentially in time) a limiting steady solution is approached (*Townsend* [21,22]; *Batchelor* [23]; *Perry* and *Chong* [19]). This flow occurs when a vortex rod is aligned with the axis of a steady axisymmetric stagnation point flow and the solution trajectories spiral in. In this asymptotic steady solution the radial spreading of vorticity by diffusion is just balanced by the shrinking of the rod caused by the axial stretching.

A similarity solution results, which gives a *Gaussian* distribution of vorticity over the radius. By writing the corresponding velocity field as a *Taylor* series expansion one can get a clue about the effect of higher order terms. It is necessary to include third order terms, as second order terms turn out to be zero. This introduces finite viscous terms, which are of the same order as the inertia terms. As we approach the critical point, the linear terms dominate in determining the shape of the pattern, but the coefficients of the matrix G are different from the case of perfectly uniform vorticity. This permits the degenerate centre to turn into a focus. The vorticity now varies quadratically with radius.

Returning to FIG. 14, it is clear that in the real case of viscous flow, both positive and negative vorticity is constantly being carried into the vortex rod from the wall, and there may be axial diffusion of vorticity along the rod. Hence, the necessary conditions for steady flow are probably more complicated than in the case of the stretching of straight vortex rods.

4. The Important Simple Types of Separated Flows

In this Chapter we attempt to describe the topology of some important types of separated flows by means of the free sheets of streamsurface bifurcations. We choose the types in increasing order of complexity from configurations which have frequently been observed experimentally. Some of them occur as elements in more complex flows.

4.1. Slightly Asymmetric Open Negative Streamsurface Bifurcation

One of the important features of negative bifurcations is that they are the means by which the vorticity produced at the body is carried to regions far away from it. In a perfectly symmetrical open negative bifurcation which has its free sheet aligned with the flow direction (as in FIG. 3), the vorticity vector on the free sheet is directed normal to the free sheet. However, any asymmetry of the flow allows the vorticity to have a finite component in the direction of the streamline, i. e. $\mathbf{u} \cdot \operatorname{curl} \mathbf{u} \neq 0$. Consequently the free sheet behaves like a vortex sheet, inasmuch as it begins to roll up from the free edge in a direction given by the component of vorticity in the streamline direction on the sheet. The streamwise vorticity is thus gradually bunched up into a vortex, which induces a velocity away from the wall on one of its flanks and a velocity towards the wall on the other. The latter effect makes it necessary that a positive streamsurface bifurcation occurs beside the negative one and approximately parallel to it.

An attempt to illustrate this in a perspective view looking upstream is presented in FIG. 15. The free sheet S_1 of the open negative bifurcation issues from the bifurcation line PQ and rolls up into a vortex. It induces a velocity field directed generally towards the wall in the region of its starboard flank, so that a special streamsurface S_2 exists which bifurcates positively along the open positive bifurcation line P′Q′. P and P′ have been drawn as points in FIG. 15 for simplicity of the diagram. They should be thought of as regions of finite extent containing the gradual beginnings of bifurcations. Such bifurcations often form without the occurrence of a critical point at the start.

This type of three-dimensional flow separation occurs as an element in many practical situations. Usually, as in the case of flow over a slender ellipsoid at incidence, it occurs in the form of a symmetric pair of open negative bifurcations, which have a common open positive bifurcation on the symmetry plane between them. The latter feature requires an open free double bifurcation in the symmetry plane as well, and it is simpler for our present purpose not to consider the pair, but the single case of FIG. 15.

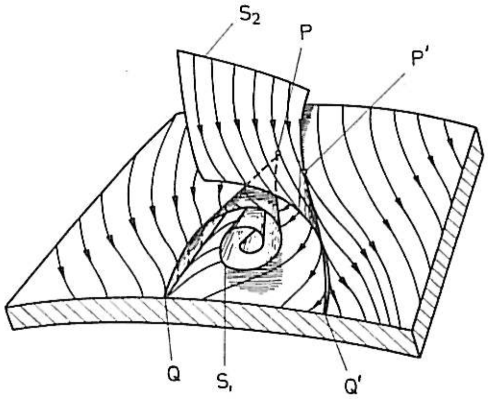
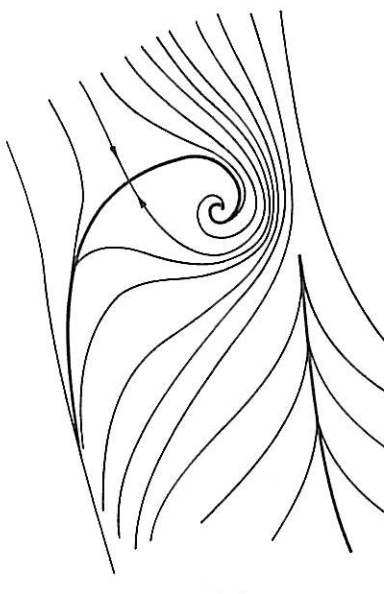


FIG. 15: An open negative streamsurface bifurcation brings about an open positive one also

4.2. Werlé-Legendre Separation

The next important type of negative streamsurface bifurcation is one that has been made visible in a convincing experiment by *Werlé* [24] and discussed at some length by *Legendre* [25] and many others since then. The distinguishing feature of this type is that the bifurcation line starts from a saddle of the wall streamline pattern and proceeds downstream on one side, and into a focus on the other. FIG. 16a presents the wall streamline pattern and FIG. 16b a perspective view looking upstream. Again the free sheet S_1, of the negative bifurcation issuing from the bifurcation line PQ,



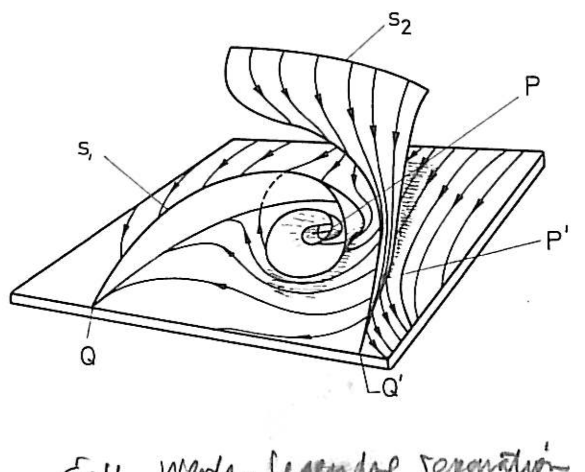


fig16 Werle - Legendre separation

(a) wall s/line field

b) Perspective view

rolls up into a vortex which brings about an open positive bifurcation along P′Q′ of the sheet S_2. Though this has much in common with the flow of FIG. 15, it is convenient to distinguish them because here the bifurcation line starts with a pair of critical points in the wall streamlines whereas in FIG. 15 no critical point occurs. Again this pattern usually occurs in the form of a symmetrical pair, and we shall return to it in Part II. Because of the characteristic shape of the negative bifurcation line and the separatrix crossing it, *Dallmann* [26] calls this the bishop's staff or pedum separation.

4.3. Simple U-Shaped Bifurcation

If a bifurcation line starts from a saddle point of the wall streamlines which lies on a symmetry plane (just as in the flow of FIG. 5), a U-shaped negative bifurcation may occur. This may sometimes be observed on the lee side of a blunt slender body at incidence. The wall streamline pattern and a sectional perspective view of the free sheets of the bifurcations are shown in the same view in FIG. 17. A negative bifurcation line issues in both cross-stream directions from a saddle A and then turns towards the downstream direction. From a node F an open positive bifurcation runs downstream along the symmetry plane. The free sheet S_1 issuing from the negative bifurcation line AB rolls up into a horseshoe-shaped vortex. This induces a flow towards the symmetry plane in a region above it, so that two special streamsurfaces, the sheets S_3 and S_2 bifurcate along the open free double bifurcation line DE. They form the feeding sheets of this bifurcation. The sheets S_5 and S_4 lie in the symmetry plane and form the two issuing sheets of the bifurcation DE. Because this bifurcation is symmetrical, the vorticity on S_4 is perpendicular to the sheet, so that S_4 does not roll up. The other issuing sheet S_5 is also the free sheet of a subsequent open positive bifurcation FG whose open end is the node F of the wall streamline field. Since the wall shear stress is zero at F, the free edge streamline may join the wall at a finite angle.

Perhaps the reference to "streamsurface separatices" made in Chapter 2 can now be understood more easily by referring to FIGs. 15, 16b and 17. In each case the free sheets of the various bifurcations separate regions of distinct topological character. The free sheets issue from or flow into bifurcation lines, which correspond topologically to the saddle points of two-dimensional flows. Indeed, the topological features of a flow may be described unambiguously by specifying the locations and types of bifurcations occurring in it.

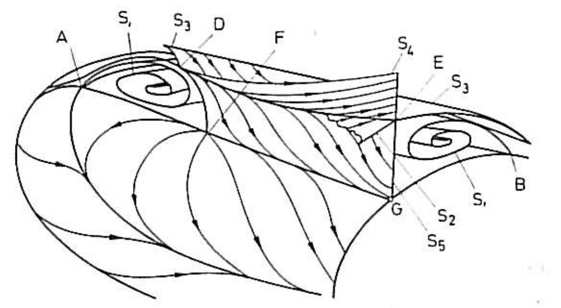


FIG. 17: Perspective view of the free streamsurface sheets and wall streamlines of simple U-shaped separation

4.4. The Limits of the Bifurcation Method of Presentation

Unfortunately the streamsurface bifurcation method of presenting the free sheets becomes hopelessly complicated even for flows whose wall streamline fields are relatively straightforward. To illustrate this, consider the flow often encountered at the junction of an axisymmetric body of zero incidence with a sting mounting (see *Fairlie* [27]). The wall streamline field of the flow is sketched in FIG. 18. A U-shaped negative bifurcation line issues from the saddle A in both cross-stream directions and runs (on one side) into the vortical node B. A second symmetrical negative bifurcation line runs from a saddle F into the same vortical node B. A negative bifurcation line runs along a symmetry plane from A through D to E, and the free sheet of this bifurcation is shown in this view also as S_2. From a node J downstream of F a positive bifurcation line runs downstream along the second symmetry plane to K. The whole situation is doubly symmetric so that the pattern repeats itself in alternating mirror images four times around the axis of symmetry.

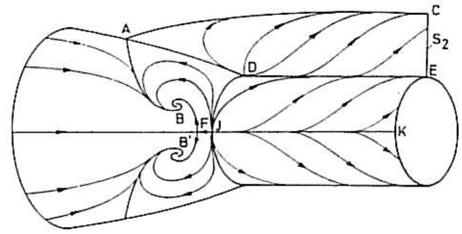


FIG. 18: Wall streamlines and one of the free sheets of the separation sometimes observed at a sting-body junction

An attempt to present a perspective view of this flow is made in FIG. 19. A total of 18 free sheets of bifurcations wrap themselves around each other so that hidden features can only be shown by sectioning the sheets. Nevertheless it is perhaps just possible to present the topological structure of this flow unambiguously in this way.

The free sheet S_1 of the negative bifurcation AB is at the same time one of the issuing sheets of a free double bifurcation AC open at the saddle point A of the wall streamline field. The negative bifurcation line AB is open at the vortical node B, and the edge streamline leaves the surface and becomes the core of a vortex V. The two feeding sheets of the double bifurcation AC, namely S_2 and S_3 come respectively from the free stream and body side along the first symmetry plane. S_2 is at the same time the free sheet of a negative bifurcation ADE.

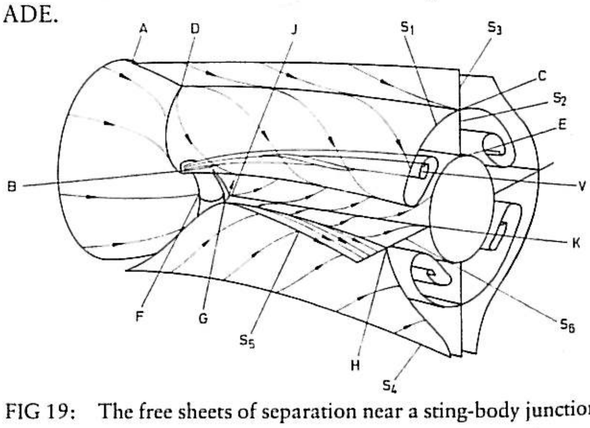


FIG 19: The free sheets of separation near a sting-body junction

The free sheet S_7 issuing from the negative bifurcation FB has a common edge streamline with S_1. This requires the rolled-up sheet S_7 to fit itself into the rolled-up sheet S_1 in a complicated way which consists essentially of the vortex in S_7 being stretched and wrapped into the core of the vortex V. A separation form similar to this was presented by *Werlé* [28] in his FIGs. 5f, 5g.

Finally, the free double bifurcation GH is in every way similar to DE in FIG. 17. Two feeding sheets, one of which is labelled S_4, bifurcate along GH to produce the issuing sheets S_5 and S_6 along the second symmetry plane. The bifurcation is open at G. The issuing sheet S_6 bifurcates again in a positive bifurcation JK at the wall. The whole pattern is doubly symmetric and repeats itself four times as in the wall streamline pattern.

To remind ourselves that we have deliberately excluded unsteady flows, it should be pointed out that the wall streamline field of FIG. 18 for example does not exclude the possibility that the flow is unsteady, though the boundary conditions either are steady or are seen as steady by the visualization techniques used to obtain such wall information. Limited temporal resolution of the observer can, of course, make unsteady flows appear steady. Indeed, the mean flow field of an unsteady flow is a useful concept. Many of the rules for streamsurface bifurcations obtained here for steady flows may be applied with equal validity to the steady mean of an unsteady flow. We make no attempt to cover this field here, however (see *Perry* and *Watmuff* [29]).

5. Conclusions

A vocabulary of well-defined terms has been constructed, by which it is possible to describe three-dimensional steady separated flows. The concept of streamsurface bifurcation is used, and mathematical solutions are obtained in the immediate vicinity of bifurcation lines, both when they occur on a wall and when they occur in the flow field. This is done by a local analysis, in which appropriate variables are considered linear in the distance from the bifurcation line. An important result is that a three-dimensional flow separation may only be defined by reference to an arbitrary measure for the strength of separation. Whether or not a flow is classed as being separated depends on whether the strength of streamline convergence near the negative streamsurface bifurcation exceeds this measure or not.

Simple but important elements of three-dimensional separated flows are then described in terms of the locations and types of bifurcations occurring in them. In particular, the free sheets of bifurcations, which are the separatrices of three-dimensional flows, are shown to be the features characterizing the topology. From this work evolved the vortex skeleton method for describing these three-dimensional flows and this is given in Part II – *Perry* and *Hornung* [20].

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