Tutorial Sheet 11
Double And Triple Integration.

---

1. Evaluate the double integrals:

   (a) $\iint_R x^2 \, dA$, where R is the region bounded by $y = x^2, y = x + 2$

   (b) $\iint_R (x^2 + y^2) \, dA$, where $R : 0 \le y \le \sqrt{1 - x^2}, 0 \le x \le 1$.

   (c) $\int_0^1 \int_2^{4-2x} dy \, dx$

   (d) $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4 - y} dy \, dx$

2. Find the volume of the region under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x, x = 0$, and $x + y = 2$ in the xy plane.

3. Use the given transformations to transform the integrals and evaluate them:

   (a) $u = x + 2y, v = x - y$ and $I = \int_0^{2/3} \int_y^{2-2y} (x + 2y)e^{(y-x)} \, dA$

   (b) $u = x, v = xy, w = 3z$ and $I = \iiint_D (x^2y + 3xyz) \, dV$ where $D = \{(x, y, z) \in \mathbb{R}^3 : 1 \le x \le 2,\ 0 \le xy \le 2,\ 0 \le z \le 1\}$.

4. Evaluate the following volume integrals:

   (a) $\iiint_D (z^2x^2 + z^2y^2) \, dV$, where $D = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 \le 1, -1 \le z \le 1\}$

   (b) $\iiint_D xyz \, dV$ where $D = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 \le 1,\ 0 \le z \le x^2 + y^2\}$

5. Evaluate the following Line integrals using Green's theorem on plane:

   (a) $\oint_C (y^2dx + x^2dy)$, C: The triangle bounded by $x = 0, x + y = 1, y = 0$.

   (b) $\oint_C (3ydx + 2xdy)$, C: The boundary of $0 \le x \le \pi,\ 0 \le y \le \sin x$.

6. Using Green's theorem find the areas of regions enclosed by

   (a) The circle $\vec{r}(t) = (a \cos t)\hat{i} + a \sin t)\hat{j},\ 0 \le t \le 2\pi$

   (b) The Astroid $\vec{r}(t) = (\cos^3 t)\hat{i} + (\sin^3 t)\hat{j}, 0 \le t \le 2\pi$

Solutions of Tutorial Sheet 11

Double And Triple Integration.

1. Evaluate the double integrals

(a) $\iint_R x^2 dA$ where R is the region bounded by $y = x^2, y = x + 2$

The region is bounded by $y = x^2, y = x+2$. Simple to notice that $y = x+2$ is the upper curve and $y = x^2$ is the lower curve between the points of intersection $x = -1$ and $x = 2$. So it is enough to evaluate the iterated integral

$$\int_{x=-1}^{2} \left(\int_{x^2}^{x+2} x^2 dy \right) dx = \int_{-1}^{2} x^2(x+2-x^2)dx = \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^5}{5} \right]_{-1}^{2} = \frac{163}{60}$$

(b) $\iint_R (x^2 + y^2)dA$ where $R : 0 \leq y \leq \sqrt{1-x^2}, 0 \leq x \leq 1$

The region is the part of the unit disc $x^2+y^2 \leq 1$ that lies in the first quadrant. So the upper curve and lower curve can be easily identified as $y = \sqrt{1-x^2}$ and $y = 0$ respectively. So it is enough to evaluate

$$\int_{x=0}^{1} \left(\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)dy \right) dx = \int_{x=0}^{1} \left(x^2 y - \frac{y^3}{3} \right) \Big|_{0}^{\sqrt{1-x^2}} dx = \pi/8.$$

Second Method, It is very easy to do using polar co-ordinate: $x = r\cos\theta, y = r\sin\theta, 0 \leq r \leq 1$ and $0 \leq \theta \leq \frac{\pi}{2}$. Then

$$I = \int_{0}^{\frac{\pi}{2}} \int_{r=0}^{1} r^2 r dr d\theta = \frac{\pi}{8}.$$

(c) $\int_0^1 \int_2^{4-2x} dydx$

The domain is bounded by $x = 0, y = 2$ and $y = 4-2x$. So it is easy to note that in the domain y ranges from 2 to 4 while x ranges from 0 to $4-2x$. So

$$\int_0^1 \int_2^{4-2x} dydx = \int_{y=2}^{4} \left(\int_{x=0}^{(4-y)/2} dx \right) dy = \int_{y=2}^{4} \frac{4-y}{2} dy = \frac{4y-y^2}{4}\Big|_2^4 = 1.$$

Remark: Without changing the order of integration, we can easily solve this problem.

(d) $\displaystyle\int_0^2\int_0^{4-x^2}\frac{xe^{2y}}{4-y}dydx$

The domain is bounded by $x=0$, $x=2$, $y=0$ and $y=4-x^2$. Hence

$$\int_0^2\int_0^{4-x^2}\frac{xe^{2y}}{4-y}dydx=\int_0^4\left(\int_0^{\sqrt{4-y}}\frac{xe^{2y}}{4-y}dx\right)dy=\int_0^4\frac{e^{2y}}{4-y}\left(\int_0^{\sqrt{4-y}}xdx\right)dy$$
$$=\int_0^4\frac{e^{2y}}{2}dy=\frac{e^{2y}}{4}\Big|_0^4=\frac{1}{4}-\frac{e^8}{4}.$$

2. We use the formula $V=\iint_R f(x,y)dA$ where $f(x,y)\geq 0$ is a continuous real valued function defined over the domain R of the plane.

   Here $f(x,y)=x^2+y^2$ and the domain is bounded by $x=0, y=2-x$ and $y=x$. So in this domain drawing a line parallel to y axis, it is easy to see that the domain may be described as $x\leq y\leq 2-x,\ 0\leq x\leq 1$. In other words,the upper curve is $y=2-x$ and lower curve is $y=x$ between $x=0$ and $x=1$. Hence

$$V=\iint_R(x^2+y^2)dA=\int_0^1\left(\int_x^{2-x}(x^2+y^2)dy\right)dx=\frac{4}{3}$$

3. Use the given transformations to transform the integrals and evaluate them:

   (a) $u=x+2y, v=x-y$. The Jacobian of the transformation is

$$J(x,y)=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}1 & 2\\ 1 & -1\end{vmatrix}=-3.$$

   Therefore, $J(u,v)=-\frac{1}{3}$ and $|J|=1/3$. The given domain is the triangle bounded by $y=x, y=0$ and $x+2y=2$. The image of this triangle under the transformation is again a triangle bounded by $v=0, v=u$ and $u=2$. Hence

$$\int_0^{2/3}\int_y^{2-2y}(x+2y)e^{(y-x)}dA=\int_{u=0}^2\left(\int_0^u ue^{-v}\frac{1}{3}dv\right)du=\frac{1}{3}(3e^{-2}+1)$$

   (b) The inverse of the given transformation is $x=u, y=\frac{v}{u}$ and $z=\frac{w}{3}$. So the required Jacobian is

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}$$

and the given integral is

$$\iiint_D (x^2y + 3xyz)dV = \frac{1}{3}\int_{w=0}^{3}\int_{v=0}^{2}\int_{u=1}^{2}\left(v + \frac{vw}{u}\right)\, du\ dv\ dw = 2 + 3\log 2$$

4. (a) Using the cylindrical coordinates $x = r\cos\theta, y = r\sin\theta$ and $z = z$, the given domain may be represented as

$$D = \{(r,\theta,z):\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ -1 \le z \le 1\}.$$

Hence the integral becomes

$$\iiint_D z^2(x^2+y^2)dV = \int_{z=-1}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(z^2r^2)rdrd\theta dz$$
$$\int_{z=-1}^{1}\int_{0}^{2\pi} z^2\frac{1}{4}d\theta dz = \frac{\pi}{3}$$

(b) Again using the cylindrical coordinates

$$D = \{(r,\theta,z):\ 0 \le z \le r^2,\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}.$$

Hence

$$\iiint_D xyz\ dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{r^2} zr^3\cos\theta\sin\theta dzdrd\theta = 0$$

5. (a) By Green's theorem $\oint_C Mdx + Ndy = \iint_D (\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})dA$. The region D is the domain bounded by the curve C. Here $M = y^2$ and $N = x^2$. Hence

$$\iint_D 2(x-y)dA = 2\int_0^1\int_0^{1-x}(x-y)dydx = 0.$$

(b) In this case $M = 3y$ and $N = 2x$. Therefore,

$$\iint_D (2-3)dA = -\int_0^{\pi}\int_0^{\sin x} dxdy = -\int_0^{\pi}\cos x = -2.$$

6. Use the formula: Area of $R = \frac{1}{2}\oint_C xdy - ydx$, where R is the region bounded by the curve C

(a) $R = \frac{1}{2}\int_0^{2\pi} a^2\cos^2 t + a^2\sin^2 t = a^2\pi.$

(b) $R = \frac{1}{2}\int_0^{2\pi} 3\cos^4 t\sin^2 t + 3\cos^2 t\sin^4 t = \frac{3\pi}{8}$

Good Luck!!!