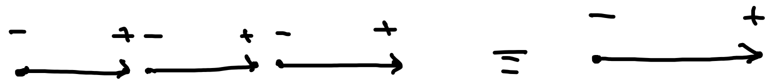
Bound Charges

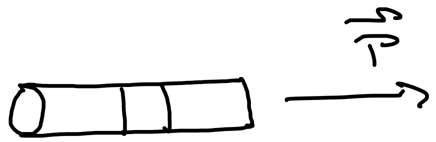
$\boxed{\text{Section } 4.2.2}$

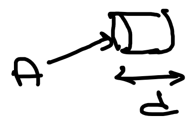
14.1.21



To calculate charge densities:

(x) Take uniformly polarised system





$=$ volume $= Ad$

→ Dipole moment of the chunk $= PAd$

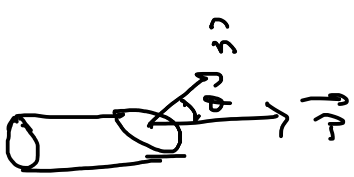
If charges in the ends $= q$ ⇒ Dipole moment $= qd$

$$\Rightarrow qd = PAd$$

$$\Rightarrow q = PA$$

If we sliced off perpendicularly,

$$\sigma_b = \frac{q}{A} = P$$



→ For an oblique cut,

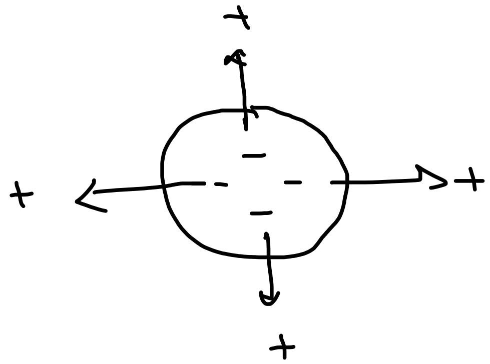
$$A = A_{end} \cos\theta$$

$$\Rightarrow \sigma_b = \frac{q}{A_{end}} = P\cos\theta = \vec{P}\cdot\hat{n}$$

(x) Take non-uniformly polarised system

→ we get an accumulation of bound

charges within the material as well as surface.



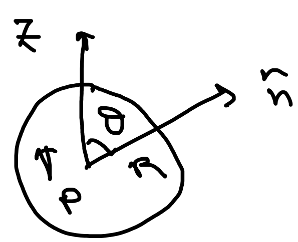
$\rightarrow$ A diverging polarisation vector $\vec{P}$

The net charge $\int \rho_b \, d\tau$ should be equal to and opposite to the total surface charge.

$$\int_V \rho_b \, d\tau = - \oint_S \vec{P} \cdot d\vec{a} = - \int_V (\vec{\nabla} \cdot \vec{P}) \, d\tau$$

$$\Rightarrow \quad \rho_b = - \vec{\nabla} \cdot \vec{P}$$

Ex:



uniformly polarised sphere

$\rightarrow$ z-axis coincides with $\vec{P}$

Volume charge density $= 0$

However, $\sigma_b = \vec{P} \cdot \hat{n} = P \cos\theta$

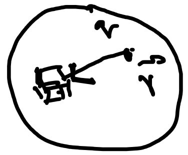
$$V(r, \theta) = \frac{P r \cos\theta}{3 \epsilon_0}, \quad r \leq R$$

$r < R$

$$\vec{E} = - \vec{\nabla} V = - \frac{P}{3 \epsilon_0} \hat{z} \qquad [\, r\cos\theta = z$$

$$= - \frac{\vec{P}}{3 \epsilon_0}$$

Ex:



The average field due to q at $\vec{r}$

$$\vec{E}_{avg.} = \frac{1}{\frac{4}{3}\pi R^3} \int \vec{E}\, d\tau$$

r = source point
$d\tau$ = field point
$\hat{\mathscr{r}}$ = from $\vec{r}$ to $d\tau$

$$= \frac{1}{\frac{4}{3}\pi R^3} \frac{1}{4\pi\epsilon_0} \int q \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau$$

Now, we calculate the field at r due to a uniform charge density 'ρ' over the sphere.

$$\vec{E}_\rho = \frac{1}{4\pi\epsilon_0} \int \rho \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau$$

$d\tau$ = source point
r = field point
$\hat{\mathscr{r}}$ = from $d\tau$ to $\vec{r}$

$\Rightarrow$ To get $\underline{\vec{E}_{avg.} = \vec{E}_\rho}$

we need to define

$$\rho = -\frac{q}{\frac{4}{3}\pi R^3}$$

(x) Solid charged sphere

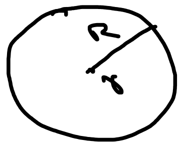
(field inside the sphere due to all charges within)

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\rho \cdot \frac{4}{3}\pi R^3}{r^2} \hat{r}$$

$$= \frac{\rho \vec{r}}{3\epsilon_0} = -\frac{q\vec{r}}{4\pi\epsilon_0 R^3} = \frac{-\vec{p}}{4\pi\epsilon_0 R^3}$$

Field inside a dielectric

Section 4.2.3



We want to calculate macroscopic field at some point $\vec{r}$ within a dielectric.

We imagine a sphere centering $\vec{r}$ with radius 'R' (~ 1000 times the size of a molecule)

$r >> d$

⊗ The macroscopic field at $\vec{r}$ is made up of two parts

- → Avg. field over the sphere due to all charges outside
- → Avg. field due to all charges inside

We can write,

$$\vec{E} = \vec{E}_{out} + \vec{E}_{in}$$

For 'outside' we can use pure dipole approximation

$$V_{out} = \frac{1}{4\pi\epsilon_0} \int_{out} \frac{\vec{p}(\vec{r}\,') \cdot \hat{\mathcal{R}}}{\mathcal{R}^2} \, d\tau$$

For dipoles inside the sphere:

$$\vec{E}_{in} = -\frac{1}{4\pi\epsilon_0} \frac{\vec{p}}{R^3}$$

$$\vec{p} = \left(\frac{4}{3}\pi R^3\right)\vec{P}$$

$$\Rightarrow \vec{E}_{in} = -\frac{\vec{P}}{3\epsilon_0}$$

We can simply write

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}(\vec{r}\,')\cdot\hat{\imath}}{\imath^2}\, d\tau$$

↳ volume integral runs over the entire volume of the dielectric.

(*) The argument holds due to the fact that the avg. field over any sphere (due to charge inside) is same as the field at the center of a uniformly polarised sphere with same total dipole moment.