$T(r, \theta, \phi)$

$$dT = \frac{\partial T}{\partial r} dr + \frac{\partial T}{\partial \theta} d\theta + \frac{\partial T}{\partial \phi} d\phi$$

$$= \vec{\nabla} T . d\vec{r}$$

$$\Rightarrow \frac{\partial T}{\partial r} dr + \frac{\partial T}{\partial \theta} d\theta + \frac{\partial T}{\partial \phi} d\phi = \vec{\nabla} T . d\vec{r}$$

in spherical coordinates,

$$\frac{\partial T}{\partial r} dr + \frac{\partial T}{\partial \theta} d\theta + \frac{\partial T}{\partial \phi} d\phi = (\vec{\nabla} T)_r dr + (\vec{\nabla} T)_\theta r d\theta + (\vec{\nabla} T)_\phi r \sin\theta d\phi$$

Hence,

$$\frac{\partial T}{\partial r} = (\vec{\nabla} T)_r \Rightarrow (\vec{\nabla} T)_r = \frac{\partial T}{\partial r}$$

$$\frac{\partial T}{\partial \theta} = (\vec{\nabla} T)_\theta r \Rightarrow (\vec{\nabla} T)_\theta = \frac{1}{r} \frac{\partial T}{\partial \theta}$$

$$\frac{\partial T}{\partial \phi} = (\vec{\nabla} T)_\phi r \sin\theta \Rightarrow (\vec{\nabla} T)_\phi = \frac{1}{r \sin\theta} \frac{\partial T}{\partial \phi}$$

Therefore,

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin\theta} \frac{\partial}{\partial \phi} \quad \text{(Gradient)}$$

Now,

$$\vec{\nabla} . \vec{v} = \left(\hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin\theta} \frac{\partial}{\partial \phi} \right) . \left(v_r \hat{r} + v_\theta \hat{\theta} + v_\phi \hat{\phi} \right)$$

$$= \hat{r} . \left(\frac{\partial v_r}{\partial r} \hat{r} + \frac{\partial v_\theta}{\partial r} \hat{\theta} + \frac{\partial v_\phi}{\partial r} \hat{\phi} + v_r \frac{\partial \hat{r}}{\partial r} + v_\theta \frac{\partial \hat{\theta}}{\partial r} + v_\phi \frac{\partial \hat{\phi}}{\partial r} \right) + \hat{\theta}(\ ) + \hat{\phi}(\ )$$

$\hat{r}$

$$\hat{r} . \left[\frac{\partial v_r}{\partial r} \hat{r} + \frac{\partial v_\theta}{\partial r} \hat{\theta} + \frac{\partial v_\phi}{\partial r} \hat{\phi} + 0 + 0 + 0 \right]$$

$$= \frac{\partial v_r}{\partial r}$$

$\hat{\theta}$

$$\frac{\hat{\theta}}{r} . \left[\frac{\partial v_r}{\partial \theta} \hat{r} + \frac{\partial v_\theta}{\partial \theta} \hat{\theta} + \frac{\partial v_\phi}{\partial \theta} \hat{\phi} + v_r \frac{\partial \hat{r}}{\partial \theta} + v_\theta \frac{\partial \hat{\theta}}{\partial \theta} + v_\phi \frac{\partial \hat{\phi}}{\partial \theta} \right]$$

$$= \frac{\hat{\theta}}{r} . \left[\frac{\partial v_r}{\partial \theta} \hat{r} + \frac{\partial v_\theta}{\partial \theta} \hat{\theta} + \frac{\partial v_\phi}{\partial \theta} \hat{\phi} + v_r \hat{\theta} + v_\theta (-\hat{r}) + 0 \right]$$

$$= \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{1}{r} v_r$$

(Φ) $= \frac{\hat{\phi}}{r\sin\theta}\cdot\left[\frac{\partial v_r}{\partial\phi}\hat{r} + \frac{\partial v_\theta}{\partial\phi}\hat{\theta} + \frac{\partial v_\phi}{\partial\phi}\hat{\phi} + v_r\frac{\partial\hat{r}}{\partial\phi}\right.$
$\left. + v_\theta\frac{\partial\hat{\theta}}{\partial\phi} + v_\phi\frac{\partial\hat{\phi}}{\partial\phi}\right]$

$= \frac{\hat{\phi}}{r\sin\theta}\cdot\left[\frac{\partial v_r}{\partial\phi}\hat{r} + \frac{\partial v_\theta}{\partial\phi}\hat{\theta} + \frac{\partial v_\phi}{\partial\phi}\hat{\phi} + v_r\sin\theta\hat{\phi}\right.$
$\left. + v_\theta\cos\theta\,\hat{\phi} + v_\phi(-\hat{r}\sin\theta - \hat{\theta}\cos\theta)\right]$

$= \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r}{r} + \frac{v_\theta\cos\theta}{r\sin\theta}$

Hence,

$\vec{\nabla}\cdot\vec{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r}{r}$
$+ \frac{v_\theta\cos\theta}{r\sin\theta}$

$= \left(\frac{\partial v_r}{\partial r} + \frac{2v_r}{r}\right) + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\theta\cos\theta}{r\sin\theta}\right)$
$+ \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)$
$+ \frac{1}{r\sin\phi}\frac{\partial v_\phi}{\partial\phi}$

Similarly, for the expression of curl, start with

$\vec{\nabla}\times\vec{v} = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\times\left(v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}\right)$