Department of Mathematics, Bennett University
Engineering Calculus (EMAT101L)
Solutions for Tutorial Sheet 3

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1. (a) Consider the sequence of partial sums $\{S_n\}$. Then

$$S_n = \sum_{k=1}^{n} \frac{4}{k^2+3k+2} = \sum_{k=1}^{n} \frac{4}{(k+1)(k+2)} = \sum_{k=1}^{n} 4\left(\frac{1}{k+1} - \frac{1}{k+2}\right)$$
$$= 4\left(\frac{1}{2} - \frac{1}{n+2}\right) \to 2.$$

Thus the series $\sum\limits_{n=1}^{\infty} \frac{4}{n^2+3n+2}$ converges and it converges to 2.

(b) Consider the sequence of partial sums $\{S_n\}$. Then

$$S_n = \sum_{k=1}^{n} \left(\sin^2 \frac{1}{n} - \sin^2 \frac{1}{n+2}\right) = \left(\sin^2 1 + \sin^2 \frac{1}{2} - \sin^2 \frac{1}{n+1} - \sin^2 \frac{1}{n+2}\right)$$
$$\to \sin^2 1 + \sin^2 \frac{1}{2}.$$

Thus the series $\sum\limits_{n=1}^{\infty} \left(\sin^2 \frac{1}{n} - \sin^2 \frac{1}{n+2}\right)$ converges and it converges to $\sin^2 1 + \sin^2 \frac{1}{2}$.

2. (a) $\lim\limits_{n\to\infty} 5^{\frac{1}{n}} = 1 \neq 0$. As $\lim\limits_{n\to\infty} a_n \neq 0$, so series diverges.

(b) $\lim\limits_{n\to\infty} a_n = e^x \neq 0$. Series diverges.

(c) Let $\{S_n\}$ be the sequence of partial sum of the series $\sum\limits_{n=1}^{\infty} a_n$. Then $S_n = \log(n+1) \to \infty$ as $n \to \infty$ and hence diverges.

(d) Let

$$S_n = a + (a+b) + (a+2b) + \cdots + a + (n-1)b = \frac{n}{2}(a + (n-1)b), n \in \mathbb{N}.$$

Here $\lim\limits_{n\to\infty} S_n = \infty$, thus the series $\sum\limits_{n=1}^{\infty} (a + (n-1)b)$ is divergent.

(e) If $0 \leq a_n \leq 1$ $(n \geq 1)$ and $0 \leq x < 1$, then $|a_n x^n| \leq |x|^n$ for all n. Now use comparison test.

3. (a) Take $a_n = \frac{\log n}{n^{3/2}}$ and $b_n = \frac{1}{n^\alpha}$ where $1 < \alpha < \frac{3}{2}$. By limit comparison test series converges. (one can also use Cauchy condensation test i.e find the behaviour of the series $\sum 2^n a_{2^n}$.)

(b) Take $b_n = \dfrac{1}{n}$. By limit comparison test series diverges.

(c) Take $b_n = \dfrac{1}{n^2}$. By limit comparison test series converges.

(d) Converges, $|a_n| \leq \frac{\pi}{2^n}$, use comparison test.

(e) Converges, $|a_n| \leq \frac{1}{n^{3/2}}$.

4. (a) Take $a_n = \dfrac{n^{\sqrt{2}}}{2^n}$. Then $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{1}{2} < 1$. Hence series converges.

   (b) Take $a_n = \dfrac{n!}{10^n}$. Then $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \infty > 1$. Hence series diverges.

   (c) Take $a_n = \dfrac{n!}{(2n+1)!}$. Then $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 0 < 1$. Hence series converges.

5. (a) Conditionally convergent.

   (b) Absolutely convergent, as $|a_n| \leq \frac{1}{n^2}$.

   (c) Conditionally convergent.

6. (a) Let $a_n = (n + 1 + 2^n)x^n$. Then $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 2|x|$ and series converges for $|x| < \frac{1}{2}$.

   (b) $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{|x|}{e}$, series converges if $|x| < e$.

   (c) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n |x-1| = \frac{|x-1|}{e}$, series converges if $\frac{|x-1|}{e} < 1$.