Sequence
(Lecture-5)

Engineering Calculus

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Bounded sequence

A sequence $\{a_n\}$ is said to be **bounded above**, if there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$.
Similarly, we say that a sequence $\{a_n\}$ is **bounded below**, if there exists $M_1 \in \mathbb{R}$ such that $a_n \geq M_1$ for all $n \in \mathbb{N}$.
Thus a sequence $\{a_n\}$ is said to be **bounded** if it is both bounded above and below.

Theorem

Every convergent sequence is bounded.

Proof: Let $\{a_n\}$ be a convergent sequence and $\lim\limits_{n\to\infty} a_n = L$. Then there exists $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. It implies that $|a_n - L| < 1$ for all $n \geq N$.
Further,

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|, \forall\, n \geq N.$$

Let $M = \max\{|a_1|, |a_2|, ..., |a_{N-1}|, 1 + |L|\}$. Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. Hence $\{a_n\}$ is bounded.

Remark

The condition given in previous theorem is necessary but not sufficient.
For example, the sequence $\{(-1)^n\}$ is a bounded sequence but not convergent sequence.

Question: Boundedness + (??) $\Longrightarrow$ Convergence.

Monotone Sequences

A sequence $\{a_n\}$ of real numbers is called an **increasing** sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and $\{a_n\}$ is called a **decreasing** sequence if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.
A sequence that is increasing or decreasing is called a **monotone sequence**.

Examples:

1. The sequences $\{1 - 1/n\}$, $\{n^3\}$ are increasing sequences.
2. The sequences $\{1/n\}$, $\{1/n^2\}$ are decreasing sequences.
3. The sequences $\{(-1)^n\}$, $\left\{\cos\left(\frac{n\pi}{3}\right)\right\}$, $\{(-1)^n n\}$, $\left\{\frac{(-1)^n}{n}\right\}$ are not monotonic sequences.

Monotone convergence theorem

(i) An increasing sequence which is bounded above is convergent.
That is, suppose $\{a_n\}$ is a bounded above and increasing sequence. Then the least upper bound of the set $\{a_n : n \in \mathbb{N}\}$ is the limit of $\{a_n\}$.

(ii) A decreasing sequence which is bounded below is convergent.
That is, suppose $\{a_n\}$ is a bounded below and decreasing sequence. Then the greatest lower bound of the set $\{a_n : n \in \mathbb{N}\}$ is the limit of $\{a_n\}$.

Example

If $0 < b < 1$, then the sequence $\{b^n\}_1^\infty$ converges to 0.

Solution: We write $b^{n+1} = b^n b < b^n$. Hence $\{b^n\}$ is decreasing. Since $b^n > 0$ for all $n \in \mathbb{N}$, the sequence $\{b^n\}$ is bounded below. Hence, by the above theorem, $\{b^n\}$ converges. Let $L = \lim\limits_{n\to\infty} b^n$. Further, $\lim\limits_{n\to\infty} b^{n+1} = \lim\limits_{n\to\infty} b \cdot b^n = b \cdot \lim\limits_{n\to\infty} b^n = b \cdot L$. Thus the sequence $\{b^{n+1}\}$ converges to $b \cdot L$. On the other hand, $\{b^{n+1}\}$ is a subsequence of $\{b^n\}$. Hence $L = b \cdot L$ which implies $L = 0$ as $b \neq 1$.

Example

Show that the sequence $\{a_n\}$, where $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$, for all $n \in \mathbb{N}$ is convergent.

Solution: Now $a_{n+1} - a_n = \dfrac{1}{2n+1} + \dfrac{1}{2n+2} - \dfrac{1}{n+1} = \dfrac{1}{2(n+1)(2n+1)} > 0$ for all n.
Therefore the sequence $\{a_n\}$ is monotonically increasing. Again

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = 1.$$

i.e. $0 < a_n < 1$. Therefore, the sequence a_n is bounded. Hence the sequence being bounded and monotonically increasing, is convergent.

Result

(i) An increasing sequence which is not bounded above diverges to ∞.

(ii) A decreasing sequence which is not bounded below diverges to $-\infty$.

Example: If $b > 1$, then the sequence $\{b^n\}_1^\infty$ diverges to ∞.

Thank You