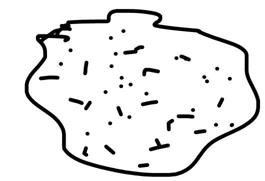
15.12.20

Multipole Expansion:

A Systematic expansion for the potential of any localised charge distribution in powers of '$\frac{1}{r}$'

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

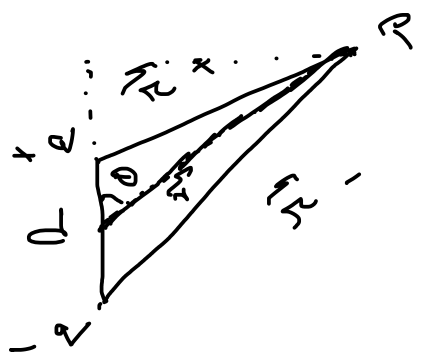


Electric dipole

→ Two equal and opposite charges ($\pm q$) separated by 'd'

$r_- \equiv$ distance from $-q$

$r_+ \equiv$ distance from $+q$



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

$$r_\pm = r^2 + \left(\frac{d}{2}\right)^2 \mp r d\cos\theta$$

$$= r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

→ negligible for $r \gg d$

$$\Rightarrow \frac{1}{r_\pm} \simeq \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2}$$

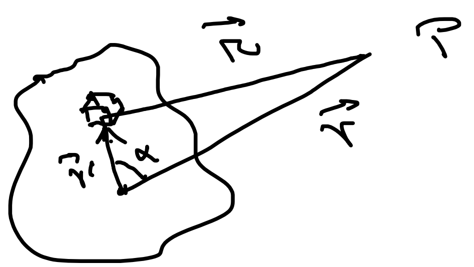
$$\simeq \frac{1}{r}\left(1 \pm \frac{1}{2}\frac{d}{r}\cos\theta\right)$$

Hence, $\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \simeq \frac{d}{r^2}\cos\theta$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{qd\cos\theta}{r^2}$$

* Potential due to a dipole $\sim \frac{1}{r^2}$

In general,



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\,\rho(\vec{r}')\,d\tau'$$

$$\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\alpha$$

$$= r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

α = Angle betn $\vec{r}$ & $\vec{r}'$

$r'\cos\alpha = \hat{r}\cdot\vec{r}'$

$$\Rightarrow \mathscr{r} = r(1+\epsilon)^{1/2}$$

where, $\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$

We are interested in $\epsilon \ll 1$.

For that limit,

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2}$$

$$= \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$$

$$\Rightarrow \frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \cdots\right]$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) - \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right) + \cdots\right]$$

The coefficients of powers of $\left(\frac{r'}{r}\right)$ are given by Legendre polynomial.

$$\Rightarrow \frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

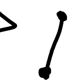
$\vec{r}$ is a constant of integration.

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int P_n(\cos\alpha)\,(r')^n\,\rho(\vec{r}')\,d\tau'$$

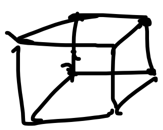
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\Big[\frac{1}{r}\int \rho(\vec{r}\,')\,d\tau' \quad \text{(monopole contb}^{n}\text{)}$$

sq

$$+ \frac{1}{r^2}\int r'\cos\alpha\,\rho(\vec{r}\,')\,d\tau' \quad \text{(Dipole contb}^{n}\text{)}$$

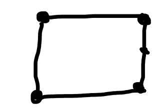


Octopole



$V \sim \frac{1}{r^4}$

$$+ \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\vec{r}\,')\,d\tau' \quad \text{(Quadrupole contb}^{n}\text{)}$$



$$+ \frac{1}{r^4}\int \cdots + \cdots\Big]$$

Monopole | dipole terms:

$$V_{mon.}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \quad \rightarrow \text{leading contrb}^{n}.$$

$\rightarrow$ but if total charge is zero $\Rightarrow$ leading contb^{n} comes from dipole.

$$V_{dip.}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\alpha\,\rho(\vec{r}\,')\,d\tau'$$

$\hat{r}\cdot\vec{r}\,' = r'\cos\alpha$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\,\hat{r}\cdot\underbrace{\int \vec{r}\,'\rho(\vec{r}\,')\,d\tau'}$$

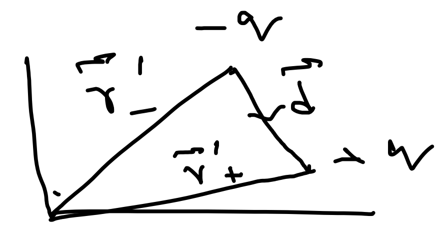
$\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')\,d\tau' \equiv$ Dipole moment of the distribution

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{\vec{p}\cdot\hat{r}}{r^2}$$

for a collection of point charges,

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r}_i'$$

for a physical dipole with equal & opposite charges $\pm q$



$$\vec{p} = q\vec{r}_+' - q\vec{r}_-'$$

$$= q(\vec{r}_+' - \vec{r}_-')$$

$$= q\vec{d}$$

* for better approximation, for a fixed $\vec{r}$, we decrease 'd' $(r >> d)$

* for a perfect dipole, $d \to 0 \Rightarrow$ 'q' simultaneously increases to keep $\vec{p}$ intact.

Physical dipole $\to$ pure dipole

$$\Rightarrow d \to 0 \;\&\; q \to \infty$$