Department of Mathematics, Bennett University
Engineering Calculus (EMAT101L)
Solutions for Tutorial Sheet 4

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1. (a) Consider

$$\begin{aligned}\left|x^2 \cos \frac{1}{x} - 0\right| &\le |x|^2 \\ &< \epsilon \text{ iff } |x| < \sqrt{\epsilon}\end{aligned}$$

Choose $\delta = \sqrt{\epsilon}$, then for $|x - 0| < \delta$, $\left|x^2 \cos \dfrac{1}{x} - 0\right| < \epsilon$.

$$\Rightarrow \lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$$

(b) Consider

$$\begin{aligned}\left|x^2 - a^2\right| &= |x-a||x+a| = |x-a||x-a+2a| \le |x-a|(|x-a| + |2a|) \\ &< \delta(\delta + 2a) \text{ whenever } |x-a| < \delta\end{aligned}$$

Choose $\delta > 0$ such that $\delta(2a + \delta) = \epsilon$, then for $|x - a| < \delta$, $|x^2 - a^2| < \epsilon$.

$$\Rightarrow \lim_{x \to a} x^2 = a^2.$$

(c) Consider

$$\begin{aligned}\left|x^2 + 5x + 4 - 28\right| &= |x^2 + 5x - 24| = |(x+8)(x-3)| = |x-3||x-3+11| \\ &\le |x-3|(|x-3| + 11) < \delta(\delta + 11) \text{ whenever } |x-3| < \delta.\end{aligned}$$

Choose $\delta > 0$ such that $\delta(\delta + 11) = \epsilon$, then for $|x - 3| < \delta$, $|x^2 + 5x + 4 - 28| < \epsilon$.

$$\Rightarrow \lim_{x \to 3}(x^2 + 5x + 4) = 28.$$

2. (a) Choose $\{x_n = \frac{1}{n\pi}\}$, then $x_n \to 0$, but $\cos\left(\frac{1}{x_n}\right) = (-1)^n$ does not converge.

   (b) Choose $\left\{x_n = \frac{1}{n}\right\}$, then $x_n \to 0$, but $f(x_n) = n \to \infty$.

   (c) Choose $\left\{x_n = \frac{1}{(n\pi)^k} + a\right\}$ and $\left\{y_n = \frac{1}{(2n\pi + \frac{\pi}{2})^k} + a\right\}$, then $x_n, y_n \to a$ but $f(x_n) \to 0$ and $f(y_n) \to 1$.

3. Let $f(x) = x^{179} + \dfrac{163}{1 + x^2 + \sin^2 x} - 119 \ \forall \ x \in \mathbb{R}$.
Then, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(-2) < 0$ and $f(0) > 0$. So, by using intermediate value theorem, there exists $c \in (-2, 0)$ such that $f(c) = 0$.

$$i.e., \ c^{179} + \frac{163}{1 + c^2 + \sin^2 c} = 119.$$

4. (a) $f(x) \equiv x^5 - 3x^2 + 1, \ x \in [0, 1], \ f(0) = 1, \ f(1) = -1$ and f is continuous. Now apply IVT.

   (b) $f(0) = -1, f(\frac{\pi}{2}) = 2$, f is continuous. Now apply IVT.

5. (a) Let $f(x) = x$ if x is rational, and $f(x) = 0$ if x is irrational. Then f is continuous only at $x = 0$.

   (b) Constant function, polynomial function, cosine function are continuous everywhere.