Line Integral

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Vector function form (Parametric equations) of different curves

- $y = f(x) \quad \Longrightarrow \quad \overrightarrow{r}(t) = t\ \hat{i} + f(t)\ \hat{j}$
- $x = g(y) \quad \Longrightarrow \quad \overrightarrow{r}(t) = g(t)\ \hat{i} + t\ \hat{j}$
- Line segment from the point (x_0, y_0, z_0) to the point (x_1, y_1, z_1):

$$\overrightarrow{r}(t) = (1-t)(x_0\ \hat{i} + y_0\ \hat{j} + z_0\ \hat{k}) + t(x_1\ \hat{i} + y_1\ \hat{j} + z_1\ \hat{k}), \quad 0 \le t \le 1$$

- Circle: $x^2 + y^2 = r^2 \quad \Longrightarrow$

$$\overrightarrow{r}(t) = \begin{cases} r\cos t\ \hat{i} + r\sin t\ \hat{j}, \quad 0 \le t \le 2\pi & \text{counter clockwise} \\ r\cos t\ \hat{i} - r\sin t\ \hat{j}, \quad 0 \le t \le 2\pi & \text{clockwise} \end{cases}$$

- Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies$

$$\overrightarrow{r}(t) = \begin{cases} a\cos t\ \hat{i} + b\sin t\ \hat{j}, \quad 0 \le t \le 2\pi & \text{counter clockwise} \\ a\cos t\ \hat{i} - b\sin t\ \hat{j}, \quad 0 \le t \le 2\pi & \text{clockwise} \end{cases}$$

Definition of line integral

Let C be a smooth curve parameterized by s, the arc-length parameter, and let f be a continuous function of s. A line integral is an integral of the form

$$\int_C f(s)\ ds = \lim_{|\Delta s| \to 0} \sum_{i=1}^{n} f(c_i)\Delta s_i,$$

where $s_1 < s_2 < \cdots < s_n$ is any partition of the s-interval over which C is defined, c_i is any value in the i-th subinterval, Δs_i is the width of the i-th subinterval, and $|\Delta s|$ is the length of the longest subinterval in the partition.

- When C is a closed curve, i.e., a curve that ends at the same point at which it starts, we use

$$\oint_C f(s)\ ds \ \text{ instead of } \int_C f(s)\ ds.$$

Evaluating a Line Integral

Result

Let C be a curve parameterized by $\overrightarrow{r}(t) = g(t)\hat{i} + h(t)\hat{j},\ a \leq t \leq b$, where g and h are continuously differentiable, and let $z = f(x, y)$, where f is continuous over C. Then

$$\begin{aligned}\int_C f(s)\, ds &= \int_a^b f(g(t), h(t))\ |\overrightarrow{r}'(t)|\ dt \\ &= \int_a^b f(g(t), h(t))\ \sqrt{(g'(t))^2 + (h'(t))^2}\ dt.\end{aligned}$$

Evaluating a Line Integral in 3D

Let C be a curve parameterized by $\overrightarrow{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k},\ a \leq t \leq b$, where g, h and k are continuously differentiable, and let $w = f(x, y, z)$, where f is continuous over C. Then

$$\int_C f(s)\ ds = \int_a^b f(g(t), h(t), k(t))\ |\overrightarrow{r}'(t)|\ dt.$$

Examples

Example 1: Evaluate $\int_C xy^2 \; ds$, where C is the right half of the circle $x^2 + y^2 = 4$.

Ans: C in vector function form is given by

$$\overrightarrow{r}(t) = 2\cos t \; \hat{i} + 2\sin t \; \hat{j}, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

$$\begin{aligned}
\int_C xy^2 \; ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos t \cdot 4\sin^2 t \; \sqrt{(-2\sin t)^2 + (2\cos t)^2} \; dt \\
&= 16 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot \sin^2 t \; dt \\
&= 16 \int_{-1}^{1} u^2 \; du \text{ (substituting } \sin t = u) \\
&= 16 \frac{u^3}{3}\Big|_{-1}^{1} = \frac{32}{3}.
\end{aligned}$$

Example 2: Compute $\int_C ye^x \, ds$ where C is the line segment from $(1, 2)$ to $(4, 7)$.

Ans: The vector function form of the line segment C is given by

$$\overrightarrow{r}(t) = (1-t)(1\hat{i} + 2\hat{j}) + t(4\hat{i} + 7\hat{j}) = (1+3t)\hat{i} + (2+5t)\hat{j}, \quad 0 \le t \le 1.$$

Then

$$\int_C ye^x \, ds = \int_0^1 (2+5t)e^{1+3t}\sqrt{3^2+5^2} \, dt = \frac{16}{9}\sqrt{34}e^4 - \frac{1}{9}\sqrt{34}e.$$

Example 3: Evaluate $\int_C xyz\ ds$, where C is the helix given by

$$\overrightarrow{r}(t) = \cos(t)\ \hat{i} + \sin(t)\ \hat{j} + 3t\ \hat{k}; \quad 0 \le t \le 4\pi.$$

Ans:

$$\begin{aligned}\int_C xyz\ ds &= \int_0^{4\pi} 3t \cos t\ \ \sin t \sqrt{\sin^2 t + \cos^2 t + 9}\ dt \\ &= \frac{3\sqrt{10}}{2} \int_0^{4\pi} t \sin 2t\ dt \\ &= -3\sqrt{10}\ \pi.\end{aligned}$$

Green's Theorem

Theorem

Let R be a closed, bounded region of the plane whose boundary C is composed of finitely many smooth curves, let $\overrightarrow{r}(t)$ be a counterclockwise parameterization of C, and let $\overrightarrow{F} = M\ \hat{i} + N\ \hat{j}$ where N_x and M_y are continuous over R. Then

$$\oint_C \overrightarrow{F} \cdot \overrightarrow{dr} = \iint_R curl\,\overrightarrow{F}\ dA.$$

In other words,

$$\oint_c M\ dx + N\ dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$

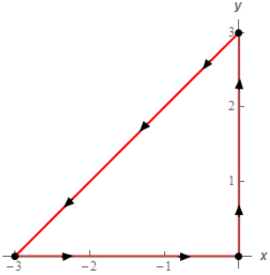
Examples

Example 1: Evaluate $\oint_C x\ dx - x^2y^2\ dy$, where C is the positively oriented triangle with vertices $(0,0),(0,1)$ and $(1,1)$.

Ans:

$$\begin{aligned}
\oint_C x\ dx - x^2y^2\ dy &= \int_0^1 \int_x^1 \left(-2xy^2 - 0\right) dy\ dx \\
&= \int_0^1 -\frac{2xy^3}{3}\Bigg|_x^1 dx \\
&= \int_0^1 \left(-\frac{2x}{3} + \frac{2x^4}{3}\right) dx \\
&= \left(-\frac{x^2}{3} + \frac{2}{15}x^5\right)\Bigg|_0^1 \\
&= -\frac{1}{5}.
\end{aligned}$$

Example 2: Use Green's Theorem to find the value of $\oint_C (xy^2 + x^2)dx + (4x - 1)dy$ where C is given in the adjacent figure.



Ans: $M = xy^2 + x^2$ and $N = 4x - 1$.

Now using Green's theorem, we have

$$\begin{aligned}\oint_C (xy^2 + x^2)dx + (4x - 1)dy &= \iint_R (4 - 2xy)\, dA \\ &= \int_{-3}^{0} \int_{0}^{x+3} (4 - 2xy)dy\ dx \\ &= \frac{99}{4}.\end{aligned}$$

Thank You.