Gauss Divergence Theorem

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Gauss divergence theorem

Theorem

Let Ω be a closed, bounded region in $\mathbb{R}^3$ whose boundary is a piecewise smooth orientable surface S. Let $\overrightarrow{F}(x, y, z)$ be a continuous function that has continuous partial derivatives in some domain containing Ω. Then

$$\iiint\limits_{\Omega} divF dV = \iint\limits_{S} \overrightarrow{F} \cdot \hat{n} dS$$

where $\hat{n}$ is the outer unit normal vector of S.

Example

Example: Evaluate $\iint\limits_{\partial\Omega} \overrightarrow{F}.\hat{n}dA$ where $\partial\Omega$ is the boundary of the domain inside the cylinder $x^2+y^2=1$ and between the planes $z=0, z=x+2$ and $\overrightarrow{F}=(x^2+ye^z)\hat{i}+(y^2+ze^2)\hat{j}+(z^2+xe^y)\hat{k}$.

Solution: With the given $\overrightarrow{F}$, it is not difficult to obtain, $\nabla\cdot\overrightarrow{F}=2x+2y+2z$.

By Divergence theorem

$$\iint\limits_{\partial\Omega} \overrightarrow{F}\cdot\hat{n}dS=\iiint\limits_{\Omega} 2(x+y+z)dV=2\iint\limits_{x^2+y^2\leq 1}\left(\int_{z=0}^{x+2}(x+y+z)dz\right)dxdy$$

Stokes's theorem

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Stokes theorem

Theorem

Let S be a piecewise smooth oriented surface with boundary and let boundary $\mathcal{C}$ be a simple closed curve. Let $\overrightarrow{F}$ be a continuous function which has continuous partial derivatives in a domain containing S. Then

$$\iint_S (\nabla \times \overrightarrow{F}) \cdot \hat{n} dS = \int_{\mathcal{C}} \overrightarrow{F} \cdot d\overrightarrow{r}$$

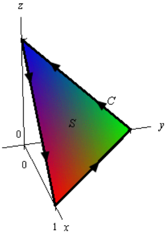
where $\hat{n}$ is a unit normal vector of S and, depending on $\hat{n}$, the integration around C is taken in the way that S lies in the left of $\mathcal{C}$. Here $\hat{n}$ is the direction of your head while moving along the boundary with surface on your left.

Example 1: Use Stokes' Theorem to evaluate $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ where $\overrightarrow{F} = z^2\hat{i} + y^2\hat{j} + x\hat{k}$ and C is the triangle with vertices $(1,0,0), (0,1,0)$ and $(0,0,1)$ with counter-clockwise rotation.

Ans: We are going to need the curl of the vector field eventually so let's get that out of the way first.

$$\text{curl}\,\overrightarrow{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = (2z-1)\hat{j}.$$

Now, all we have is the boundary curve for the surface that we'll need to use in the surface integral. However, as noted above all we need is any surface that has this as its boundary curve. So, let's use the following plane with upwards orientation for the surface.



Since the plane is oriented upwards this induces the positive direction on C as shown. The equation of this plane is,

$$x + y + z = 1 \implies z = g(x, y) = 1 - x - y.$$

First let's get the gradient. Recall that this comes from the function of the surface. So, $\nabla f = \hat{i} + \hat{j} + \hat{k}$. We get the equation of the line by plugging in $z = 0$ into the equation of the plane. So based on this the ranges that define D are, $0 \leq x \leq 1,\ 0 \leq y \leq -x + 1$. Therefore,

$$\begin{aligned}\int_c \overrightarrow{F} \cdot d\overrightarrow{r} &= \iint_D (2z - 1)\hat{j} \cdot (\hat{i} + \hat{j} + \hat{k})\ dA \\ &= \int_0^1 \int_0^{-x+1} 2(1 - x - y) - 1\ dy\ dx = -\frac{1}{6}.\end{aligned}$$

Example 2: Evaluate $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ where $\overrightarrow{F} = x^2y^3\hat{i} + \hat{j} + z\hat{k}$ and C The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16, z \geq 0$.

Ans: The intersection of cylinder and sphere is the boundary of cylinder on the plane $z = \sqrt{12}$. The unit normal to the surface is $\hat{n} = \frac{1}{4}(x\hat{i} + y\hat{j} + z\hat{k})$. The projection R of S on the xy-plane is the disc $x^2+y^2 \leq 2$, $\nabla \times \overrightarrow{F} = -3x^2y^2\hat{k}$ and $\frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} = \frac{4}{z}$. Hence by Stoke's theorem

$$\oint_C \overrightarrow{F} \cdot \overrightarrow{dr} = \iint_R (-\frac{3}{4})x^2y^2z\frac{4}{z}dA$$

$$= -3\int_{\theta=0}^{2\pi}\int_{r=0}^{2}(r^2\cos^2\theta)(r^2\sin^2\theta)r\ dr\ d\theta = -8\pi.$$

Example 3: Suppose S is a surface of a light bulb over the unit disc $x^2 + y^2 = 1$ oriented with outward pointing normal. Suppose $\overrightarrow{F} = e^{z^2-2z}x\hat{i} + (\sin(xyz) + y + 1)\hat{j} + e^{z^2}\sin(z^2)\hat{k}$. Compute $\iint_S (\nabla \times \overrightarrow{F}) \cdot \hat{n} dS$.

Ans: Enough to take any surface with boundary $x^2 + y^2 = 1$. So we take the unit disc $x^2 + y^2 \leq 1, z = 0$. Then $\overrightarrow{F}$ on this is $\overrightarrow{F} = x\hat{i} + (y+1)\hat{j}$. Then $\nabla \times \overrightarrow{F} = 0$. Hence $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = 0$.

Thank You.