Quiz Test 6
Beta, Gamma Functions

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1. Which among the following is a correct integral representation of Gamma function?

   (a) $\Gamma(x) = \int_0^1 t^{x-1}e^{-t}dt$

   (b) $\Gamma(p) = \int_1^\infty x^{p-1}e^{-x}dx$

   (c) $\Gamma(x) = \int_0^1 t^x e^t dt$

   (d) $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$

2. Find the incorrect option.
   (a) $\frac{3\Gamma(6)}{\Gamma(4)} = 60$ (b) $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$ (c) $\int_0^\infty x^7 e^{-x}dx = 5040$ (d) $\Gamma(0) = 1!$

3. Find the value of $\int_0^\infty x^6 e^{-4x^2}dx$.
   (a) $\frac{15\sqrt{\pi}}{2048}$ (b) $\frac{1}{156}\Gamma\left(\frac{7}{2}\right)$ (c) $\frac{105\sqrt{\pi}}{4096}$ (d) $\frac{1}{256}\Gamma\left(\frac{5}{2}\right)$

   Hint: Substitute $4x^2 = t$.

$$\implies \int_0^\infty x^6 e^{-4x^2}dx = \int_0^\infty \left(\frac{t}{4}\right)^3 e^{-t}\frac{dt}{4\sqrt{t}} = \frac{1}{256}\Gamma\left(\frac{7}{2}\right).$$

4. Which among the following is NOT a correct value of $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\ d\theta$?
   (a) $\frac{\pi}{\sqrt{2}}$ (b) $\frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right)$ (c) $\Gamma\left(\frac{3}{4},\frac{1}{4}\right)$ (d) $\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\ \cos^{-\frac{1}{2}}\theta\ d\theta$

   Hint: $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\ d\theta = \frac{1}{2}\Gamma\left(\frac{3}{4},\frac{1}{4}\right)$. Furthermore, by Euler's reflection formula

$$\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \sqrt{2}\pi.$$

5. Find the value of $\int_0^{\frac{\pi}{2}} \sin^7\theta\ d\theta$.

(a) $\beta\left(4,\dfrac{1}{2}\right)$ (b) $\dfrac{8\sqrt{\pi}}{35}$ (c) $\dfrac{16}{35}$ (d) $\dfrac{8}{35}$

Hint: Comparing with the beta function, the integral is equla to

$$\frac{1}{2}\beta\left(4,\frac{1}{2}\right)=\frac{1}{2}\,\frac{\Gamma(4)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}=\frac{8}{35}.$$

6. Find the value of $\displaystyle\int_0^{\infty} 2x^4e^{1-x}dx$.

   (a) It diverges to ∞. (b) $12e$ (c) $48e$ (d) 48

   Hint: $2e\displaystyle\int_0^{\infty} x^4e^{-x}dx = 2e\Gamma(5) = 48e$.

7. Evaluate the value of $\displaystyle\int_0^{\infty} x^3e^{-\frac{1}{2}x^2}dx$.

   (a) 2 (b) 4 (c) not defined (d) $\dfrac{1}{2}$

   Substitute $\frac{x^2}{2}=t \implies x\ dx = dt \implies \int_0^{\infty} 2te^{-t}dt = 2\Gamma(2) = 2.$

8. Choose the correct value of $\displaystyle\int_0^1\left(\ln\frac{1}{x}\right)^{a-1}dx$, where $a\neq 1,0,-1,-2,\ldots$.

   (a) $\Gamma(a)$ (b) $a!$ (c) not defined (d) $\ln a$

   Hint: Substitute $\ln\dfrac{1}{x}=t \implies dx=-e^{-t}dt$.

$$\therefore \int_0^1\left(\ln\frac{1}{x}\right)^{a-1}dx=\int_0^{\infty} t^{a-1}e^{-1}dt=\Gamma(a).$$

9. Which among the following is NOT correct?

   (a) For nonnegative integer values, Gamma function is not defined.

   (b) $\dfrac{\int_0^1 x^{p-1}(1-x)^{q-1}dx}{\int_0^1 t^{q-1}(1-t)^{p-1}dt}=1$

   (c) $\beta\left(\dfrac{2}{7},\dfrac{5}{7}\right)=\beta\left(\dfrac{3}{7},\dfrac{4}{7}\right)$

   (d) $\Gamma\left(\dfrac{3}{4}\right)\Gamma\left(\dfrac{1}{4}\right)=\sqrt{2}\pi$

   Hint: ~~(a) is correct from the definition of Gamma function.~~ (b) is correct since $\beta(m,n)=\beta(n,m)$. (d) is correct. To see it, apply Euler's reflection formula.

10. Find the value of $\int_0^{\frac{\pi}{4}} \sin^2 2x\ \cos^4 2x\ dx$.

(a) $\dfrac{\sqrt{\pi}}{64}$ (b) $\dfrac{\pi}{64}$ (c) $\dfrac{\pi}{32}$ (d) $\dfrac{\sqrt{\pi}}{32}$

Hint: Substitute $2x = t$. Then the integral becomes

$$\int_0^{\frac{\pi}{2}} \sin^2 t \cos^4 t\ \frac{dt}{2} = \frac{1}{4}\beta\left(\frac{3}{2}, \frac{5}{2}\right) = \frac{\pi}{64}.$$

11. Find the value of $\Gamma\left(-\frac{3}{2}\right)$.

(a) $\dfrac{4\pi}{3}$ (b) $-\dfrac{4\sqrt{\pi}}{3}$ (c) $\dfrac{\sqrt{\pi}}{3}$ (d) $\dfrac{4\sqrt{\pi}}{3}$

12. What is the value of $\dfrac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$?

(a) not defined (b) 2 (c) -2 (d) $2\sqrt{\pi}$

Hint: $\Gamma\left(-\dfrac{1}{2}\right) = -2\sqrt{\pi}$ and $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$.

13. Find the value of $\int_0^{\infty} \sqrt{x} e^{-\sqrt{x}} dx$.

(a) 2 (b) $\frac{1}{2}$ (c) 4 (d) not defined

Hint: Substitute $\sqrt{x} = t$. Then the integral becomes $\int_0^{\infty} te^{-t} 2t\ dt = 2\Gamma(3) = 4$.

14. Which among the following is not the correct value of $\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\ \cos^{\frac{7}{2}}\theta\ d\theta$?

(a) $\dfrac{5\sqrt{2}\pi}{64}$ (b) $\dfrac{5}{64}\Gamma\left(\dfrac{3}{4}\right)\Gamma\left(\dfrac{1}{4}\right)$ (c) $\dfrac{5}{192}\Gamma\left(\dfrac{3}{4}\right)\Gamma\left(\dfrac{1}{4}\right)$ (d) $\dfrac{1}{2}\beta\left(\dfrac{3}{4}, \dfrac{9}{4}\right)$

Hint: $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\ \cos^{\frac{7}{2}}\theta\ d\theta = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{9}{4}\right) = \frac{1}{2} \times \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{9}{4}\right)}{\Gamma(3)} = \frac{5}{64} \times \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right).$

Furthermore, by Euler's reflection formula

$$\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \sqrt{2}\pi.$$

15. Which among the following is **incorrect** value of

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \ \cos^4\theta \ d\theta \ ?$$

(a) $\frac{1}{2}\beta\left(3, \frac{5}{2}\right)$ (b) $\frac{8}{315}$ (c) $\frac{8}{105}$ (d) $\frac{1}{2}\ \frac{\Gamma(3)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{11}{2}\right)}$

Hint: $\frac{1}{2}\beta\left(3, \frac{5}{2}\right) = \frac{1}{2}\ \frac{\Gamma(3)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{11}{2}\right)}$. Also, $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}$ and $\Gamma\left(\frac{11}{2}\right) = \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}$