# Limits of Real Valued Functions

Engineering Calculus

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## Definition

Let $f(x)$ be defined on $(a, b)$ except possibly at $c \in (a, b)$. We say that $\lim_{x \to c} f(x) = L$ if, for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

## Example 1

$$\lim_{x \to 1} \left( \frac{3x}{2} - 1 \right) = \frac{1}{2}.$$

**Solution:** Let $\epsilon > 0$. Then we have to find $\delta > 0$ such that

$$|x - 1| < \delta \implies |f(x) - L| = \left| \left( \frac{3x}{2} - 1 \right) - \frac{1}{2} \right| = \frac{3}{2} |x - 1| < \epsilon.$$

Now, we have

$$|f(x) - L| = \frac{3}{2} |x - 1| < \epsilon \text{ whenever } |x - 1| < \delta = \frac{2}{3} \epsilon.$$

## Example 2

Prove that $\lim_{x \to 2} f(x) = 4$, where $f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2. \end{cases}$

**Solution:** Let $\epsilon > 0$ be given. Then we have to find a $\delta > 0$ such that

$$0 < |x - 2| < \delta \implies |f(x) - L| < \epsilon.$$

Now, $|x^2 - 4| = |x+2||x-2| = |x-2||x+2+2-2| < |x-2|(|x-2|+4) < \delta(\delta+4) < 5\delta$. Choose $\delta = \frac{\epsilon}{5}$ and we are done.

**Theorem**

If limit exists, then it is unique.

**Theorem (Sequential criteria of limits)**

$\lim_{x \to c} f(x) = L$ if and only if for any sequence $\{x_n\}$ with $x_n \to c$, we have $f(x_n) \to L$ as $n \to \infty$.

### Example

Show that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

**Solution:** Consider the sequences $\{x_n\} = \left\{\frac{1}{n\pi}\right\}$, $\{y_n\} = \left\{\frac{1}{2n\pi + \frac{\pi}{2}}\right\}$. Then it is easy to see that $x_n, y_n \to 0$ and $\sin\left(\frac{1}{x_n}\right) \to 0$, $\sin\left(\frac{1}{y_n}\right) \to 1$.

### Example

Let $f(x) = \frac{1}{x}$. Then $\lim_{x \to 0} f(x)$ does not exist.

**Solution:** Consider the sequence $\{x_n\}$ with $x_n = \frac{1}{n}$. Then $x_n \to 0$ but $f(x_n)$ diverges to infinity. Therefore, $\lim_{x \to 0} f(x)$ does not exist.

# Limits of real valued functions

## Theorem

Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

(a) $\lim_{x\to c}(f(x) \pm g(x)) = L \pm M$.

(b) $f(x) \leq g(x)$ for all $x$ in an open interval containing $c$. Then $L \leq M$.

(c) $(i)$ $\lim_{x\to c}(fg)(x) = LM$ and $(ii)$ when $M \neq 0$, $\lim_{x\to c} \frac{f}{g}(x) = \frac{L}{M}$.

(d) (Sandwich) Suppose that $h(x)$ satisfies $f(x) \leq h(x) \leq g(x)$ in an interval containing $c$, and $L = M$. Then $\lim_{x\to c} h(x) = L$.

(e) If $\lim_{x\to c} f(x) = L$ then $\lim_{x\to c} |f(x)| = |L|$. But converse is not true.

## Example

(a) Consider $f : [0, 1] \to [-1, 1]$ as $f(x) = -1$ if $0 \leq x < 1/2$ and $f(x) = 1$ if $1/2 \leq x < 1$. Then $\lim_{x\to \frac{1}{2}} f(x)$ does not exist.

(b) $\lim_{x\to 0} x^m = 0$ $(m > 0)$.

(c) $\lim_{x\to 0} x \sin x = 0$.

## Definition

$f(x)$ has limit $L$ as $x$ approaches $+\infty$, if for any given $\epsilon > 0$, there exists $M > 0$ such that

$$x > M \implies |f(x) - L| < \epsilon.$$

Similarly, $f(x)$ has limit $L$ as $x$ approaches $-\infty$, if for any given $\epsilon > 0$, there exists $M > 0$ such that

$$x < -M \implies |f(x) - L| < \epsilon.$$

## Example

(a) $\lim_{x \to \infty} \frac{1}{x} = 0.$
**Solution:** For every $\epsilon > 0$, there exist $M = \frac{1}{\epsilon}$ such that $x > M \Rightarrow \frac{1}{x} < \epsilon$.

(b) $\lim_{x \to -\infty} \frac{1}{x} = 0.$
**Solution:** For every $\epsilon > 0$, there exist $M = \frac{1}{\epsilon}$ such that $x < -M \Rightarrow \left|\frac{1}{x}\right| < \epsilon$.

(c) $\lim_{x \to \infty} \sin x$ does not exist.
**Solution:** Choose $x_n = n\pi$ and $y_n = \frac{\pi}{2} + 2n\pi$. Then $x_n, y_n \to \infty$ and $\sin x_n = 0$, $\sin y_n = 1$. Hence the limit does not exist.


Thank You