Differentiation under Integration

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Leibniz's rule for differentiation under the integral sign

General form:

- $f(x,t)$: continuous and continuously differentiable†
  († partial derivatives exist and are themselves continuous)
- $a(x), b(x)$: continuous differentiable functions of x

Gottfried Wilhelm Leibniz (1646–1716)

Then

$$\boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = f(x,b(x)).b'(x) - f(x,a(x)).a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt.}$$

Special cases

- If $a(x), b(x)$ are constants rather than functions of x, then

$$\boxed{\frac{d}{dx}\int_a^b f(x,t)dt = \int_a^b \frac{\partial}{\partial x} f(x,t)dt.}$$

- If $a(x) = a$ and $b(x) = x$, then

$$\boxed{\frac{d}{dx}\int_a^x f(x,t)dt = f(x,x) + \int_a^x \frac{\partial}{\partial x} f(x,t)dt.}$$

What do these formulae signify?

- They interchange the integral and partial differential operators under certain conditions.

When should we use them?

- Generally, one uses differentiation under the integral sign to evaluate integrals that can be thought of as belonging to some family of integrals parameterized by a real variable.

Examples I

1. Evaluate $\int_0^1 \frac{t^3-1}{\ln t} dt$.

   Ans: Define $g(x) = \int_0^1 \frac{t^x-1}{\ln t} dt$. So, we wish to evaluate $g(3)$.

$$g'(x) = \int_0^1 \frac{\partial}{\partial x}\left(\frac{t^x - 1}{\ln t}\right) = \int_0^1 \frac{t^x \ln t}{\ln t} dt$$
$$= \frac{t^{x+1}}{x+1}\bigg|_0^1 = \frac{1}{x+1}$$

   $g(x) = \ln|x+1| + C$, for some constant C. To determine C, note that $0 = \int_0^1 \left(\frac{t^0-1}{\ln t}\right) dt = g(0) = \ln|0+1| + C = C$.

$$\implies g(x) = \ln|x+1| \implies g(3) = \ln 4 = 2\ln 2.$$

Examples II

2. Compute the definite integral $\int_0^1 (t \ln t)^{50} dt$.

   Ans:

$$\frac{d}{dx}\int_0^1 t^x dt = \int_0^1 t^x \ln t dt$$
$$\Longrightarrow \frac{d^{50}}{dx^{50}}\int_0^1 t^x dx = \int_0^1 t^x (\ln t)^{50} dt$$

   Now $\frac{d^{50}}{dx^{50}}\left(\frac{t^{x+1}}{x+1}\right)_0^1 = \frac{d^{50}}{dx^{50}}\left(\frac{1}{x+1}\right) = \frac{50!}{(x+1)^{51}}$.

$$\therefore \int_0^1 (t \ln t)^{50} dt = \frac{50!}{51^{51}}.$$

Examples III

3. The function f satisfies the following relationship.

$$f(x) = \int_1^x [f(t)]^2 dt, \quad f(2) = \frac{1}{2}.$$

Then determine the value of $f(\frac{1}{2})$.

Ans: $f'(x) = f(x,x) + \int_a^x \frac{\partial}{\partial x}\left(f(x,t)\right) dt = f^2(x)$.

$$\implies \frac{df}{f^2} = dx. \implies -\frac{1}{f(x)} = x + c.$$

From $f(2) = \frac{1}{2}$, we have $c = -4$.

$$\therefore f(x) = \frac{1}{4-x}. \quad \implies f(\frac{1}{2}) = \frac{1}{4-\frac{1}{2}} = \frac{2}{7}.$$

Examples IV

4. Find the value of

$$\lim_{p\to 0} \frac{d}{dp}\left[\int_{2p-1}^{3p+2} \left(\frac{x+6}{4x}\right)^{x} dx\right].$$

Ans: $\frac{23}{5}$. (Excercise)

Thank You.