Tutorial Set - II

① $(\hat{i}, \hat{j}, \hat{k}) \longrightarrow (\hat{r}, \hat{\theta}, \hat{\phi})$

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \quad \vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

⊛ unit vector $\hat{r} = \dfrac{\partial\vec{r}}{\partial r} \Big/ \left|\dfrac{\partial\vec{r}}{\partial r}\right|$

$$\frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\left|\frac{\partial\vec{r}}{\partial r}\right| = \left[\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\right]^{1/2}$$

$$= \left[\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta\right]^{1/2}$$

$$= 1$$

Hence,

$$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

⊛ unit vector $\hat{\theta} = \dfrac{\partial\vec{r}}{\partial\theta} \Big/ \left|\dfrac{\partial\vec{r}}{\partial\theta}\right|$

$$\frac{\partial \vec{r}}{\partial \theta} = r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}$$

$$\left|\frac{\partial \vec{r}}{\partial \theta}\right| = \left[r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta\right]^{1/2}$$

$$= \left[r^2\cos^2\theta\left(\cos^2\phi + \sin^2\phi\right) + r^2\sin^2\theta\right]^{1/2}$$

$$= r$$

Hence

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

(*) unit vector $\hat{\phi} = \frac{\partial \vec{r}}{\partial \phi} \Big/ \left|\frac{\partial \vec{r}}{\partial \phi}\right|$

$$\frac{\partial \vec{r}}{\partial \phi} = -r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}$$

$$\left|\frac{\partial \vec{r}}{\partial \phi}\right| = \left[r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi\right]^{1/2}$$

$$= r\sin\theta$$

Hence

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$(\hat{r}, \hat{\theta}, \hat{\phi}) \to (\hat{i}, \hat{j}, \hat{k})$

We have,

$$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k} \quad — (1)$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k} \quad — (2)$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} \quad — (3)$$

(1) $\times \sin\theta$ + (2) $\times \cos\theta$

$$\Rightarrow \hat{r}\sin\theta + \hat{\theta}\cos\theta = \cos\phi\,\hat{i} + \sin\phi\,\hat{j} \quad — (4)$$

(3) $\times \cos\phi$ + (4) $\times \sin\phi$

$$\Rightarrow \hat{\phi}\cos\phi + \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi$$

$$= \cos^2\phi\,\hat{j} + \sin^2\phi\,\hat{j}$$

$$= \hat{j}$$

$$\Rightarrow \hat{j} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$$

Replace $\hat{j}$ in (3)

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi(\hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi)$$

$$\Rightarrow \sin\phi\,\hat{i} = \hat{r}\sin\theta\sin\phi\cos\phi + \hat{\theta}\cos\theta\sin\phi\cos\phi + \hat{\phi}(\cos^2\phi - 1)$$

$$= \hat{r}\sin\theta\sin\phi\cos\phi + \hat{\theta}\cos\theta\sin\phi\cos\phi - \hat{\phi}\sin^2\phi$$

$$\Rightarrow \hat{i} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$

$$\frac{①}{\sin\theta} - \frac{②}{\cos\theta}$$

$$\Rightarrow \frac{\hat{r}}{\sin\theta} - \frac{\hat{\theta}}{\cos\theta} = \frac{\cos\theta}{\sin\theta}\hat{k} + \frac{\sin\theta}{\cos\theta}\hat{k}$$

$$= \frac{1}{\sin\theta\cos\theta}\hat{k}$$

$$\Rightarrow \hat{k} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

Hence,

$$\hat{i} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$

$$\hat{j} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$$

$$\hat{k} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

(2) $(\hat{i}, \hat{j}, \hat{k}) \longrightarrow (\hat{s}, \hat{\phi}, \hat{z})$

$$\left.\begin{aligned} x &= s\cos\phi \\ y &= s\sin\phi \\ z &= z \end{aligned}\right\} \quad \vec{r} = s\cos\phi\,\hat{i} + s\sin\phi\,\hat{j} + z\hat{k}$$

⊛ unit vector $\hat{s} = \dfrac{\partial\vec{r}}{\partial s} \Big/ \left|\dfrac{\partial\vec{r}}{\partial s}\right|$

$$\frac{\partial\vec{r}}{\partial s} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\left|\frac{\partial\vec{r}}{\partial s}\right| = 1$$

$$\Rightarrow \quad \hat{s} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

⊛ unit vector $\hat{\phi} = \dfrac{\partial\vec{r}}{\partial\phi} \Big/ \left|\dfrac{\partial\vec{r}}{\partial\phi}\right|$

$$\frac{\partial\vec{r}}{\partial\phi} = -s\sin\phi\,\hat{i} + s\cos\phi\,\hat{j}$$

$$\left|\frac{\partial\vec{r}}{\partial\phi}\right| = s$$

$$\Rightarrow \quad \hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

⊛ unit vector $\hat{z} = \dfrac{\partial\vec{r}}{\partial z} \Big/ \left|\dfrac{\partial\vec{r}}{\partial z}\right| = \hat{k}$

We have

$$\hat{s} = \cos\phi \, \hat{i} + \sin\phi \, \hat{j} \quad \text{— (1)}$$

$$\hat{\phi} = -\sin\phi \, \hat{i} + \cos\phi \, \hat{j} \quad \text{— (2)}$$

$$\hat{z} = \hat{k} \quad \text{— (3)}$$

Now, $(\hat{s}, \hat{\phi}, \hat{z}) \longrightarrow (\hat{i}, \hat{j}, \hat{k})$

$(1) \times \sin\phi + (2) \times \cos\phi$

$$\Rightarrow \hat{s} \sin\phi + \hat{\phi} \cos\phi = (\sin^2\phi + \cos^2\phi) \, \hat{j}$$

$$\Rightarrow \hat{j} = \hat{s} \sin\phi + \hat{\phi} \cos\phi$$

$(1) \times \cos\phi - (2) \times \sin\phi$

$$\Rightarrow \hat{s} \cos\phi - \hat{\phi} \sin\phi = (\cos^2\phi + \sin^2\phi) \, \hat{i}$$

$$\Rightarrow \hat{i} = \hat{s} \cos\phi - \hat{\phi} \sin\phi$$

Hence,

$$\hat{i} = \hat{s} \cos\phi - \hat{\phi} \sin\phi$$

$$\hat{j} = \hat{s} \sin\phi + \hat{\phi} \cos\phi$$

$$\hat{k} = \hat{z}$$

③ $x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$r = [x^2 + y^2 + z^2]^{1/2}$

$\frac{y}{x} = \tan\phi \Rightarrow \phi = \tan^{-1}\left(\frac{y}{x}\right)$

$\theta = \cos^{-1}\left(\frac{z}{r}\right)$

a) $x = 10,\ y = 0,\ z = 0$

Hence, $r = [100]^{1/2} = 10$

$\theta = \cos^{-1}(0) = \frac{\pi}{2}$

$\phi = \tan^{-1}(0) = 0$

$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

$= \hat{i}$

Similarly for others.

(ii) a) $r = 5$
$\theta = \pi/2$
$\phi = \pi/4$

$x = r\sin\theta\cos\phi = 5/\sqrt{2}$
$y = r\sin\theta\sin\phi = 5/\sqrt{2}$
$z = 0$

Similarly for others.

(5) $f(x) = x^2 + y^2 + z^2 - 9$

$g(x) = x^2 + y^2 - z - 3$

$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

$\vec{\nabla} f(x) = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$

$= 4\hat{i} - 2\hat{j} + 4\hat{k}$ at (P)

$\vec{\nabla} g(x) = 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$

$= 4\hat{i} - 2\hat{j} - \hat{k}$ at (P)

⊕ Angle(θ) between the surfaces $f(x, y, z) = 0$ & $g(x, y, z) = 0$

$(\vec{\nabla} f(x)) \cdot (\vec{\nabla} g(x)) = |\vec{\nabla} f(x)|\,|\vec{\nabla} g(x)| \cos\theta$

$\Rightarrow \cos\theta = \dfrac{16 + 4 - 4}{[16 + 4 + 16]^{1/2}\,[16 + 4 + 1]^{1/2}}$

$= \dfrac{16}{6 \cdot \sqrt{21}} = \dfrac{8}{3\sqrt{21}}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right) \simeq 54.40$

(6) Maximum directional derivative is given by $|\vec{\nabla} f(x,y,z)|$

$$f(x,y,z) = x^2 y z^3$$

$$\Rightarrow \vec{\nabla} f(x,y,z) = 2xyz^3 \hat{i} + x^2 z^3 \hat{j} + 3x^2 y z^2 \hat{k}$$

$$|\vec{\nabla} f(x,y,z)| = \left[4x^2y^2z^6 + x^4 z^6 + 9x^4y^2z^4\right]^{1/2}$$

at (i) $$= \left[16 + 16 + 144\right]^{1/2}$$

$$= 13.27$$

(7) $$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(F_x) + \frac{\partial}{\partial y}(F_y) + \frac{\partial}{\partial z}(F_z)$$

$$\vec{F} = \hat{i} F_x + \hat{j} F_y + \hat{k} F_z$$

For (d), it is simpler to use divergence in spherical polar coordinates

$$\vec{F}_4 = \frac{\hat{r}}{r^2} \quad \text{(only has } \hat{r} \text{ component)}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{F}_4 = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r^2}\right) = \frac{1}{r^2} \frac{\partial}{\partial r}(1) = 0$$

(8) $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

a> $\vec{F}_1 = \hat{i}\,\alpha y \Rightarrow F_x = \alpha y,\ F_y = 0,\ F_z = 0$

$\Rightarrow \vec{\nabla} \times \vec{F}_1 = \hat{k}(0 - \alpha) = -\alpha \hat{k}$

Similarly for others.

(9) $f(x,y,z) = \alpha x y^2$

a> $\vec{\nabla} f(x,y,z) = \alpha y^2 \hat{i} + 2\alpha x y \hat{j}$

b> $\vec{\nabla} \times (\vec{\nabla} f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \alpha y^2 & 2\alpha x y & 0 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2\alpha xy)\right) + \hat{j}\left(\frac{\partial}{\partial z}(\alpha y^2) - \frac{\partial}{\partial x}(0)\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(2\alpha xy) - \frac{\partial}{\partial y}(\alpha y^2)\right)$$

$$= \hat{k}\,(2\alpha y - 2\alpha y)$$

$$= 0$$

10) $\vec{G} = x^2\,\hat{i} + 3xz^2\,\hat{j} - 2xz\,\hat{k}$

a) $$\vec{\nabla} \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 3xz^2 & -2xz \end{vmatrix}$$

$$= \hat{i}\,(0 - 6xz) + \hat{j}\,(0 + 2z) + \hat{k}\,(3z^2 - 0)$$

$$= -6xz\,\hat{i} + 2z\,\hat{j} + 3z^2\,\hat{k}$$

b) $\vec{A} = -6xz\,\hat{i} + 2z\,\hat{j} + 3z^2\,\hat{k}$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = -6z + 0 + 6z$$

$$= 0$$