Sequence
(Lecture 6)

Engineering Calculus

School of Engineering and Applied Sciences
Department of Mathematics
Bennett University

Subsequence

Let $\{a_n\}$ be a sequence and $\{n_1, n_2, ...\}$ be a sequence of positive integers such that $i > j$ implies $n_i > n_j$. Then the sequence $\{a_{n_i}\}_{i=1}^{\infty}$ is called a subsequence of $\{a_n\}$.

Example

$\left\{\frac{1}{k^2}\right\}_{k=1}^{\infty}$ and $\left\{\frac{1}{2^k}\right\}_{k=1}^{\infty}$ are subsequences of $\{\frac{1}{n}\}$, where $n_k = k^2$ and $n_k = 2^k$.

Theorem

If the sequence of real numbers $\{a_n\}_1^{\infty}$, is convergent to L, then any subsequence of $\{a_n\}$ is also convergent to L.

Some results

• Sequences $(1,1,1,\cdots)$ and $(0,0,0,\cdots)$ are both subsequences of $(1,0,1,0,\cdots)$. From this we see that a given sequence may have convergent subsequence though the sequence itself is not convergent.

• If $\{a_n\}$ has two subsequences converging to two different limits, then $\{a_n\}$ cannot be convergent.

• Let $\{a_n\}$ be a sequence such that $a_{2n} \to \ell$ and $a_{2n-1} \to \ell$. Then $a_n \to \ell$.

Example: The sequence $\{1, \frac{1}{2}, 1, \frac{2}{3}, 1, \frac{3}{4}, \cdots\}$ converges to 1.

• Every sequence has a monotone subsequence.

Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

Cauchy sequence

A sequence $\{a_n\}$ is called a **Cauchy sequence** if for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon \text{ for all } n, m \geq N.$$

Example

Show that the sequence $\{\frac{1}{n}\}$ is a Cauchy sequence.

Solution: Let $\epsilon > 0$ be given, we choose a natural number N such that $N > 2/\epsilon$. Then if $m, n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < \frac{\epsilon}{2}$ and similarly $\frac{1}{m} < \frac{\epsilon}{2}$. Therefore, it follows that if $m, n \geq N$, then

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since, $\epsilon > 0$ is arbitrary, we conclude that $\{\frac{1}{n}\}$ is a Cauchy sequence.

Cauchy sequence

Theorem

Every convergent sequence is a Cauchy sequence.

Proof: Let $\{a_n\}$ be a sequence such that $\{a_n\}$ converges to L (say). Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \frac{\epsilon}{2} \ \forall\ n \geq N.$$

Now, for $n, m \geq N$, we have

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\{a_n\}$ is a Cauchy sequence.

Theorem

If $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is bounded.

Theorem

If $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is convergent.

Cauchy's criterion for convergence

A sequence $\{a_n\}$ converges if and only if for every $\epsilon > 0$, there exists N such that

$$|a_n - a_m| < \epsilon, \ \ \forall \ \ m, n \geq N.$$

Theorem

For any sequence $\{a_n\}$ with $a_n > 0$

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

provided the limit on the right side exists.

Result

Let $a_n > 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.
(i) If $L < 1$, then $\lim_{n\to\infty} a_n = 0$.
(ii) If $L > 1$, then $a_n \to \infty$.

Remark

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L = 1$, we cannot make any conclusion. For example, consider the sequence $\{n\}$, $\{\frac{1}{n}\}$ and $\{\frac{2+n}{n}\}$.

Examples

(i) Let $a_n = \frac{n}{2^n}$. Then $a_n \to 0$ as $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$.

(ii) Let $a_n = ny^{n-1}$ for some $y \in (0,1)$. Since $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = y$, $a_n \to 0$.

(iii) Let $a_n = \frac{n^\alpha}{(1+p)^n}$ for some $\alpha > 0$ and $p > 0$. Then $a_n \to 0$.

(iv) $\lim\limits_{n\to\infty} n^\alpha x^n = 0$, if $|x| < 1$ and $\alpha \in \mathbb{R}$.

Hint: If $x \neq 0$, take $a_n = n^\alpha x^n$, then $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim\limits_{n\to\infty} (1 + \frac{1}{n})^\alpha |x| = |x|$.


Thank You