Solutions - Tutorial Sheet 2

Problem-1: Find the limit of the following sequences.

(a) $a_n = \frac{1}{1+n^2}$ (b) $a_n = (-1)^n\left(\frac{2}{n+2}\right)$ (c) $a_n = \frac{n+1}{2n+3}$.

Solution:

(a) Here, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1+n^2} = 0$.

(b) Here $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \cdot \left(\frac{2}{n+2}\right) = 0$

(c) Here, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+1}{2n+3}\right) = \lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{n\left(2+\frac{3}{n}\right)}$

$\Rightarrow \lim_{n\to\infty} a_n = \frac{1}{2}$.

Problem-2: Examine whether the following sequences are convergent. Also, determine their limits if they are convergent.

(a) $a_n = \frac{1}{n} \sin^2 n \quad \forall\ n \in \mathbb{N}$.

Solution: Since $0 \leq \frac{1}{n} \sin^2 n \leq \frac{1}{n} \quad \forall\ n \in \mathbb{N}$.

Since $\frac{1}{n} \to 0$ as $n \to \infty$.

Therefore by Sandwich Theorem, $\left\{\frac{1}{n} \sin^2 n\right\}$ is convergent with limit 0.

i.e $\frac{1}{n} \sin^2 n \to 0$ as $n \to \infty$.

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} \sin^2 n = 0.$$

[Sandwich Theorem: Let $\{a_n\}$, $\{b_n\}$ & $\{c_n\}$ are three sequences such that $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then

$$\lim_{n\to\infty} b_n = L$$
]

(b) $a_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2} \quad \forall\, n \in \mathbb{N}$.

Solution: Since $n+i \geq n+1 \quad \forall\; 1 \leq i \leq n$

$$\Rightarrow \frac{1}{n+i} \leq \frac{1}{n+1}$$

$$\Rightarrow \frac{1}{(n+i)^2} \leq \frac{1}{(n+1)^2}$$

$$\Rightarrow \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2} \leq \frac{1}{(n+1)^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+1)^2}$$

$$= \frac{n}{(n+1)^2}$$

$$\Rightarrow \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2} \leq \frac{n}{(n+1)^2}.$$

and $n+n \geq n+i$ for $1 \leq i \leq n$

$$\Rightarrow \frac{1}{n+n} \leq \frac{1}{n+i} \text{ for } 1 \leq i \leq n$$

$$\Rightarrow \frac{1}{(n+n)^2} \leq \frac{1}{(n+i)^2} \text{ for } 1 \leq i \leq n$$

$$\Rightarrow \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2} \geq \frac{1}{(n+n)^2} + \frac{1}{(n+n)^2} + \cdots + \frac{1}{(n+n)^2}$$

$$= \frac{n}{(n+n)^2}$$

$$\Rightarrow \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2} \geq \frac{n}{(n+n)^2}$$

$$\Rightarrow \frac{n}{(n+n)^2} \leq \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2} \leq \frac{n}{(n+1)^2} \quad \forall n$$

Since $\lim_{n\to\infty} \frac{n}{(n+n)^2} = \lim_{n\to\infty} \frac{n}{n^2(1+\frac{1}{n})^2} = 0$

and $\lim_{n\to\infty} \frac{n}{(n+1)^2} = \lim_{n\to\infty} \frac{n}{n^2(1+\frac{1}{n})^2} = 0$

$\Rightarrow$ By Sandwich Principle, $\lim_{n\to\infty} \left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2} \right) = 0.$

(c) $a_n = \frac{n}{n^3+1} + \frac{2n}{n^3+2} + \cdots + \frac{n^2}{n^3+n} \quad \forall\ n \in \mathbb{N}$.

<u>Solution:</u> We have

$$\frac{n}{n^3+n} + \frac{2n}{n^3+n} + \cdots + \frac{n^2}{n^3+n} \leq \underbrace{\frac{n}{n^3+1} + \frac{2n}{n^3+2} + \cdots + \frac{n^2}{n^3+n}}_{a_n}$$

$$\leq \frac{n}{n^3+1} + \frac{2n}{n^3+1} + \cdots + \frac{n^2}{n^3+1}.$$

$$\Rightarrow \frac{(1+2+\cdots+n)n}{n^3+n} \leq a_n \leq \frac{(1+2+\cdots+n)n}{n^3+1} \quad \forall\ n \in \mathbb{N}.$$

Also,

$$\frac{(1+2+\cdots+n)n}{n^3+n} = \frac{\frac{n(n+1)}{2}\cdot n}{n^3+n}$$

$$= \frac{\frac{n^2}{2}(n+1)}{n^3\left(1+\frac{1}{n^2}\right)} = \frac{\frac{n^3}{2}\left(1+\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n^2}\right)}$$

$$= \frac{\left(1+\frac{1}{n}\right)}{2\left(1+\frac{1}{n^2}\right)}$$

$$\to \frac{1}{2} \text{ as } n \to \infty$$

$$\Rightarrow \frac{(1+2+\cdots+n)n}{n^3+n} \to \frac{1}{2} \text{ as } n \to \infty.$$

and $$\frac{(1+2+\cdots+n)n}{n^3+1} = \frac{\frac{n(n+1)}{2}\cdot n}{n^3\left(1+\frac{1}{n^3}\right)} = \frac{n^3\left(1+\frac{1}{n}\right)}{2n^3\left(1+\frac{1}{n^3}\right)}$$

$$= \frac{1+\frac{1}{n}}{2\left(1+\frac{1}{n^3}\right)} \to \frac{1}{2} \text{ as } n \to \infty.$$

Thus, by Sandwich Principle, we have

$$a_n = \frac{n}{n^3+1} + \frac{2n}{n^3+2} + \text{——} + \frac{n^2}{n^3+n}$$

$$\to \frac{1}{2} \quad \text{as } n \to \infty.$$

$\Rightarrow$ $\{a_n\}$ is convergent with limit $\frac{1}{2}$.

3(d) $a_n = \sqrt{4n^2+n} - 2n \quad \forall\ n \in \mathbb{N}$.

Solution! $a_n = \left(\sqrt{4n^2+n} - 2n\right) = \left(\sqrt{4n^2+n} - 2n\right) \times \frac{\left(\sqrt{4n^2+n} + 2n\right)}{\left(\sqrt{4n^2+n} + 2n\right)}$

$$= \frac{\left(\sqrt{4n^2+n}\right)^2 - (2n)^2}{\left(\sqrt{4n^2+n} + 2n\right)}$$

$$= \frac{4n^2+n-4n^2}{\sqrt{4n^2+n} + 2n}$$

$$= \frac{n}{\sqrt{4n^2+n} + 2n}$$

$$= \frac{\cancel{n}}{n\left(\sqrt{4+\frac{1}{n}} + 2\right)}$$

$$\Rightarrow \sqrt{4n^2+n} - 2n = \frac{1}{\left(\sqrt{4+\frac{1}{n}} + 2\right)}$$

Since $\frac{1}{n} \to 0$ as $n \to \infty$.

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\left(\sqrt{4+\frac{1}{n}} + 2\right)} = \frac{1}{\sqrt{4+0}+2} = \frac{1}{4}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \frac{1}{4}$$

$\Rightarrow \{a_n\}$ is convergent with limit $\frac{1}{4}$.

(e) $a_n = \sqrt{n^2+n} - \sqrt{n^2+1} \quad \forall\ n \in \mathbb{N}$.

<u>Solution</u>: $a_n = \sqrt{n^2+n} - \sqrt{n^2+1} = \dfrac{\left(\sqrt{n^2+n} - \sqrt{n^2+1}\right) \times \left(\sqrt{n^2+n} + \sqrt{n^2+1}\right)}{\left(\sqrt{n^2+n} + \sqrt{n^2+1}\right)}$

$$= \frac{\left(\sqrt{n^2+n}\right)^2 - \left(\sqrt{n^2+1}\right)^2}{\sqrt{n^2+n} + \sqrt{n^2+1}}$$

$$= \frac{\cancel{n^2}+n-\cancel{n^2}-1}{\sqrt{n^2+n} + \sqrt{n^2+1}}$$

$$= \frac{n\left(1-\frac{1}{n}\right)}{n\sqrt{1+\frac{1}{n}} + n\sqrt{1+\frac{1}{n^2}}}$$

$$= \frac{n\left(1-\frac{1}{n}\right)}{n\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{1}{n^2}}\right)}$$

$$= \frac{1-\frac{1}{n}}{\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{1}{n^2}}}$$

Since $\frac{1}{n} \to 0$ as $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{\left(1-\frac{1}{n}\right)}{\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{1}{n^2}}}\right) = \frac{1}{2}.$$

$\Rightarrow \{a_n\}$ is convergent with limit $\frac{1}{2}$.

3 Examine the convergence of the following sequences using Monotone Convergence Theorem.

(a) $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \text{———} + \frac{1}{n+n} \quad \forall\ n \in \mathbb{N}$.

Solution: For all $n \in \mathbb{N}$, we have

$$a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \geq \frac{2}{2n+2} - \frac{1}{n+1} = 0$$

$\Rightarrow a_{n+1} \geq a_n \quad \forall\ n \in \mathbb{N}$

$\Rightarrow \{a_n\}$ is increasing.

Also, $a_n \leq \frac{1}{n} + \frac{1}{n} + \text{———} + \frac{1}{n} = 1 \quad \forall\ n \in \mathbb{N}$.

$\Rightarrow \{a_n\}$ is bounded.

Thus $\{a_n\}$ is monotonically increasing and bounded sequence.

Using Monotone Convergence Theorem, $\{a_n\}$ is convergent.

4(b) $a_1 = 1$ and $a_{n+1} = 1+\sqrt{a_n}$ $\forall$ $n \in \mathbb{N}$.

Solution: We have. $a_1 = 1$ and $a_{n+1} = 1+\sqrt{a_n}$ $\forall$ $n\in\mathbb{N}$.

$\Rightarrow$ $a_2 = 1+\sqrt{a_1} = 2 > a_1$

$\Rightarrow$ $a_2 > a_1$.

Also, if $a_{k+1} > a_k$ for some $k \in \mathbb{N}$, then

$$a_{k+2} = 1+\sqrt{a_{k+1}} > 1+\sqrt{a_k} = a_{k+1}.$$

Hence, by the principle of mathematical induction,

$$a_{k+1} > a_k \quad \forall \quad k\in\mathbb{N}.$$

Thus $\{a_n\}$ is increasing.

Again, $a_1 < 3$ and if $a_k < 3$ for some $k\in\mathbb{N}$, then

$$a_{k+1} = 1+\sqrt{a_k} < 1+\sqrt{3} < 3.$$

Hence, by the principle of mathematical induction,

$$a_n < 3 \quad \forall \quad n\in\mathbb{N}.$$

So, $\{a_n\}$ is bounded above.

$\Rightarrow$ $\{a_n\}$ is monotonically increasing and bounded above.

$\Rightarrow$ $\{a_n\}$ is convergent. (By Monotone Convergence Theorem).

If $l = \lim_{n\to\infty} a_n$, then $a_{n+1} \to l$ and

Since $a_{n+1} = 1+\sqrt{a_n}$ $\forall$ $n\in\mathbb{N}$,

$\Rightarrow$ $l = 1+\sqrt{l}$

$\Rightarrow$ $l-1 = \sqrt{l}$ $\Rightarrow$ $(l-1)^2 = l$

$\Rightarrow$ $l^2+1-2l = l$

$\Rightarrow$ $l^2-3l+1 = 0$

$\Rightarrow$ $l = \frac{3+\sqrt{5}}{2}$ or $\frac{3-\sqrt{5}}{2}$.

Since $a_n \geq 1 \ \forall \ n \in \mathbb{N}$.

$\Rightarrow \quad l \geq 1$ and so $l = \frac{3+\sqrt{5}}{2}$.

Thus $\lim_{n \to \infty} a_n = \frac{3+\sqrt{5}}{2}$.

$\Rightarrow \{a_n\}$ is convergent with limit $\frac{3+\sqrt{5}}{2}$.

Problem-4 Discuss the convergence of the following sequences. Also, find their limits if they are convergent.

(i) $a_n = \dfrac{n^k}{\alpha^n}$, where $|\alpha|>1$ and $k>0$.

Solution: Consider

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^k}{\alpha^{n+1}}\cdot\frac{\alpha^n}{n^k}\right|$$

$$= \lim_{n\to\infty}\left(\frac{n^k\left(1+\frac{1}{n}\right)^k}{|\alpha|^{n+1}}\cdot\frac{|\alpha|^n}{n^k}\right)$$

$$= \lim_{n\to\infty}\frac{1}{|\alpha|}\left(1+\frac{1}{n}\right)^k$$

$$= \frac{1}{|\alpha|}$$

Since $|\alpha|>1 \Rightarrow \dfrac{1}{|\alpha|}<1$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{|\alpha|}<1$$

$\Rightarrow$ The sequence $\{a_n\}$ is convergent and $a_n\to 0$ as $n\to\infty$

[$\because$ If $a_n \neq 0$ and $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. Then
(i) if $L<1$, then a_n converges to zero.
(ii) if $L>1$, then a_n diverges.]

(ii) $a_n = \dfrac{m(m-1)(m-2) \text{ ------ } (m-n+1)}{n!} x^n$, where $|x| < 1$ and

m is a fixed positive integer.

<u>Solution:</u> Here

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| =$$

$$= \lim_{n\to\infty} \left[\frac{m(m-1)(m-2) - -(m-n+1)(m-n)|x|^{n+1}}{(n+1)!} \times \frac{n!}{m(m-1)(m-2) - (m-n+1)|x|^n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{|(m-n)||x|^n \cdot |x|}{(n+1)\cdot n!} \cdot \frac{n!}{|x|^n}\right]$$

$$= \lim_{n\to\infty} \frac{|(m-n)| \cdot |x|}{n+1}$$

$$= \lim_{n\to\infty} \frac{n\left|\left(\frac{m}{n}-1\right)\right| \; |x|}{n\left(1+\frac{1}{n}\right)}$$

$$= \quad |x| \quad < 1$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x| < 1$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0 .$$

Problem-5: State whether the following statements are true/false. Give proper justifications.

(a) A sequence can have exactly two limits.

Solution: False, as limit of a convergent sequence is unique.

(b) A sequence must have atleast one limit.

Solution: False, as a sequence may be divergent.

(c) A bounded sequence must have a limit.

Solution: False, for example $(-1)^n$ is a bounded sequence but not convergent (doesn't have a limit).

(d) An unbounded sequence will never have a limit.

Solution: True, as a convergent sequence is always bounded.

(e) A monotone sequence must have a limit.

Solution: False, for example, $\{x_n\}=\{n\}$ is a monotonically increasing sequence but doesn't have a limit.

(f) A bounded monotone sequence must have a limit.

Solution: True by Monotone Convergence Theorem.