Solutions of Tutorial Sheet 8
Limit and Continuity of a Function of Several Variables

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1. (a) Consider the path $y = mx$. Then $f_1(x, mx) = \frac{1+m^2}{1-m^2}$. So limit depends on m. Hence limit does not exist at $(0, 0)$.

   (b) Observe that

$$|f_2(x,y) - 0| = \left| xy \left(\frac{x^2 - y^2}{x^2 + y^2} \right) \right| \leq |xy| = |x| \cdot |y|$$
$$\leq \sqrt{x^2 + y^2} \cdot \sqrt{x^2 + y^2} < \delta^2 = \epsilon.$$

   Therefore, by choosing $\delta = \sqrt{\epsilon}$, then we have

$$\sqrt{x^2 + y^2} < \delta \implies |f_2(x,y) - 0| < \epsilon.$$

   So limit of the function exists at $(0, 0)$ and the value of the limit is zero.

   (c) We know that $|\sin x| \leq 1$ for every x , Then we have

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| \leq |x| + |y| \leq 2\sqrt{(x^2 + y^2)} < \epsilon.$$

   Choose $\delta = \frac{\epsilon}{2}$. Then for every $\epsilon > 0$, there exist $\delta > 0$, such that

$$\sqrt{x^2 + y^2} < \delta \Rightarrow |f(x,y) - 0| < \epsilon.$$

   (d) Put $y = mx$, we have

$$\lim_{(x,y) \longrightarrow (0,0)} \frac{sin(xy)}{x^2 + y^2} = \lim_{x \longrightarrow 0} \frac{\sin(mx^2)}{x^2(1 + m^2)} = \frac{m}{1 + m^2},$$

   which depends on m. Hence limit does not exist at $(0, 0)$.

2. (a) $\lim_{x \to 0} \lim_{y \to 0} f(x,y) = \dfrac{3}{5}$ and $\lim_{y \to 0} \lim_{x \to 0} f(x,y) = -\dfrac{1}{2}$.

   (b) $\lim_{x \to 0} \lim_{y \to 0} g(x,y) = -\dfrac{2}{3}$ and $\lim_{y \to 0} \lim_{x \to 0} g(x,y) = \dfrac{2}{3}$.

3. Observe that $\lim_{x \to 0} \left(\lim_{y \to 0} y \sin \dfrac{1}{x} + \dfrac{xy}{x^2 + y^2} \right) = \lim_{x \to 0} 0 = 0$. So this repeated limit exist.

On the other hand, along $y = mx$, we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{x\to 0} f(x, mx)\\ &= \lim_{x\to 0} m\left(\frac{\sin\frac{1}{x}}{\frac{1}{x}}\right) + \left(\frac{mx^2}{x^2+m^2x^2}\right)\\ &= \frac{m}{1+m^2},\end{aligned}$$

which is diffeent for different values of m. Hence, double limit of the function does not exist at origin.

4. (a) Take $x = my^3$ and then $f(x,y) = \frac{m}{1+m^2}$ which show that the function is not continuous at $(0,0)$.

   (b) Let $\epsilon > 0$ be given. Now

$$\left|\frac{\sin^2(x-y)}{|x|+|y|}\right| \leq \frac{|x-y|^2}{|x|+|y|} \leq \frac{(|x|+|y|)^2}{|x|+|y|} = (|x|+|y|) \leq 2(x^2+y^2)^{\frac{1}{2}} < \epsilon.$$

   If we take $\delta = \frac{\epsilon}{2}$. Then for every $\epsilon > 0$, there exist $\delta > 0$, such that

$$\sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - f(0,0)| < \epsilon.$$

   (c) Take $y = x$, we have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x-y)^2} = \lim_{x\to 0}\frac{x^4}{x^4} = 1 \neq 0 = f(0,0)$. Hence f is not continuous at $(0,0)$.