Section 6.2.1 9.2.21

Magnetisation

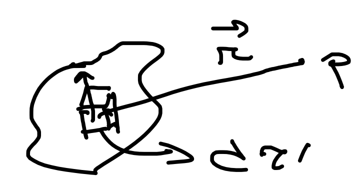
When matter becomes magnetised, it contains a large number of tiny dipoles with a net alignment $\equiv$ <u>Magnetic Polarisation</u>.

Magnetisation $\equiv$ Magnetic moment per unit volume
$(\vec{M}(\vec{r}))$

⊗ Say, we have a magnetised material. The magnetic dipole moment per unit volume is given. $(\vec{M})$. We want to calculate $\vec{A}(\vec{r})$

→ Vector potential for a single dipole $(\vec{m})$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$



The volume element $d\tau'$ carries a dipole moment $= \vec{M}\, d\tau'$. Hence, total vector potential:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

[use identity, $\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$]

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \left[\vec{m}(\vec{r}\,') \times \left(\vec{\nabla}'\frac{1}{\imath}\right)\right] d\tau'$$

[use identity, $\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times (\vec{\nabla} f)$]

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left\{ \int \frac{1}{\imath}\left[\vec{\nabla}' \times \vec{m}(\vec{r}\,')\right] d\tau' - \int \vec{\nabla}' \times \left[\frac{\vec{m}(\vec{r}\,')}{\imath}\right] d\tau' \right\}$$

[use idenity, $\int_V (\vec{\nabla} \times \vec{v})\, d\tau = -\oint_S \vec{v} \times d\vec{a}$
$\hookrightarrow$ any vector.

$\vec{v} = \vec{v} \times \vec{c}$
$\hookrightarrow$ const vector

Divergence th: $\int \vec{\nabla} \cdot (\vec{v} \times \vec{c})\, d\tau = \oint (\vec{v} \times \vec{c}) \cdot d\vec{a}$

$$\vec{\nabla} \cdot (\vec{v} \times \vec{c}) = \vec{c} \cdot (\vec{\nabla} \times \vec{v}) - \vec{v} \cdot \underbrace{(\vec{\nabla} \times \vec{c})}_{0}$$

$$= \vec{c} \cdot (\vec{\nabla} \times \vec{v})$$

$$(\vec{v} \times \vec{c}) \cdot d\vec{a} = d\vec{a} \cdot (\vec{v} \times \vec{c}) = \vec{c} \cdot (d\vec{a} \times \vec{v})$$

$$= -\vec{c} \cdot (\vec{v} \times d\vec{a})$$

$$\Rightarrow \int \vec{c} \cdot (\vec{\nabla} \times \vec{v})\, d\tau = -\oint \vec{c} \cdot (\vec{v} \times d\vec{a})$$

$$\Rightarrow \int (\vec{\nabla} \times \vec{v})\, d\tau = -\oint \vec{v} \times d\vec{a}$$]

The second term:

$$\int \vec{\nabla}' \times \left(\frac{\vec{M}(\vec{r}')}{r} \right) d\tau' = \oint \frac{1}{r} \left(\vec{M}(\vec{r}') \times d\vec{a}' \right)$$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{1}{r} \left[\vec{\nabla}' \times \vec{M}(\vec{r}') \right] d\tau$$

$$+ \frac{\mu_0}{4\pi} \oint \frac{1}{r} \left[\vec{M}(\vec{r}') \times d\vec{a}' \right]$$

1st term: Potential of a volume current: $\vec{J}_b = \vec{\nabla} \times \vec{M}$

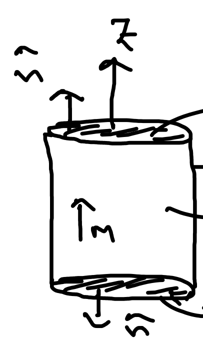
2nd term: Potential of a surface current: $\vec{K}_b = \vec{M} \times \hat{n}$

normal unit vector.

In terms of bound currents,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}_b(\vec{r}')}{r} d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{\vec{K}_b(\vec{r}')}{r} da'$$

Ex:



$\vec{K}_b = \vec{M} \times \hat{n} = 0$

$\vec{K}_b = \vec{M} \times \hat{n}$
$= \vec{M} \times \hat{r}$
$= M \hat{z} \times \hat{r}$
$= M \hat{\phi}$

$\vec{K}_b = 0$

[Infinitely long circular cylinder with uniform magnetisation

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$
$$= 0$$

Look back at Ampere's law:

Section 6.3.1

Total current, $\vec{J} = \vec{J}_b + \vec{J}_f$

↳ free current

Now, $\frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) = \vec{J}$

$= \vec{J}_f + \vec{\nabla} \times \vec{M}$

$\Rightarrow \vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_f$

$\Rightarrow \vec{\nabla} \times \vec{H} = \vec{J}_f$

→ integral form, $\oint \vec{H} \cdot d\vec{l} = I_{fenc}$

↳ total free current passing through Ampere's loop.

We can calculate $\vec{H}$ simply from our knowledge of free current.

Ex: A long copper rod of radius 'R' carrying a uniformly distributed current 'I'.

Amperian loop inside the rod, $s < R$

$H(2\pi s) = I_{fenc}$

$= \frac{I}{\pi R^2} \pi s^2$

$\vec{B}$, $\vec{M}$ & $\vec{H}$ are all circumferential

→ Inside the rod, $\vec{H} = \frac{I s}{2\pi R^2} \hat{\phi}$ $(s \le R)$

outside the rod, $\vec{H} = \frac{I}{2\pi s} \hat{\phi}$ $(s \ge R)$

⊛ outside, $\vec{M} = 0$ (empty space)

$\Rightarrow \vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$ $(s \ge R)$

• <u>Susceptibility</u> & <u>Permeability</u>:

$\boxed{\text{Section } 6.4.1}$

In most substances, magnetisation is proportional to the field.

$$\vec{M} \propto \vec{H}$$

$$\Rightarrow \vec{M} = \chi_m \vec{H}$$

↳ magnetic susceptibility

⊛ χ_m is dimensionless

⊛ characteristic of substances.

<u>Linear</u> <u>media</u> ≡ Material obeys: $\vec{M} = \chi_m \vec{H}$

$$\vec{B} = \mu_0 (\vec{H} + \vec{M})$$

$$= \mu_0 (1 + \chi_m) \vec{H}$$

$$= \mu H$$

Here, $\mu = \mu_0 (1 + \chi_m)$

↳ Permeability of material

* in vacuum, $\mu = \mu_0$ (permeability of free space)

* Volume current density in a homogeneous linear material is proportional to free current density

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \vec{\nabla} \times (\chi_m \vec{H})$$

$$= \chi_m (\vec{\nabla} \times \vec{H})$$

$$= \chi_m \vec{J}_f$$

→ When free current flows, $\vec{J}_b$ can be obtained from $\vec{J}_f$. If no free current, then only bound current that you get is surface current.