Tangents and Normals to Level Curves

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Tangents and normals to level curves

Let $f(x, y)$ be differentiable and consider the level curve $f(x, y) = c$.

Let $\overrightarrow{r}(t) = g(t)\hat{i} + h(t)\hat{j}$ be its parametrization.

Example $f(x, y) = x^2 + y^2$ has $x(t) = a\cos t,\ y(t) = a\sin t$ as level curve $x^2 + y^2 = a^2$, which is a circle of radius a. Now differentiating the equation $f(x(t), y(t)) = a^2$ with respect to t, we get

$$f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = 0.$$

Now since $\overrightarrow{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j}$ is the tangent to the curve, we can infer from the above equation that ∇f is the direction of Normal.

Equation of normal at (a,b): $\boxed{x = a + f_x(a,b)t, y = b + f_y(a,b)t,\ t \in \mathbb{R}.}$

Equation of tangent: $\boxed{(x-a)f_x(a,b) + (y-b)f_y(a,b) = 0.}$

Example : Find the normal and tangent to $\frac{x^2}{4} + y^2 = 2$ at $(-2,1)$.

Solution: We find

$$\nabla f = \frac{x}{2}\hat{i} + 2y\hat{j}\Big|_{(-2,1)} = -\hat{i} + 2\hat{j}.$$

Therefore, the tangent line through $(-2,1)$ is $-(x+2) + 2(y-1) = 0.$

Tangent Plane and Normal lines

Let $\overrightarrow{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$ is a smooth level curve(space curve) of the level surface $f(x, y, z) = c$. Then differentiating $f(x(t), y(t), z(t)) = c$ with respect to t and applying chain rule, we get

$$\nabla f(a, b, c) \cdot (x'(t), y'(t), z'(t)) = 0$$

Now as in the above, we infer the following:

Normal line at (a, b, c) is $\boxed{x = a + f_x t, y = b + f_y t, z = c + f_z t.}$

Tangent plane: $\boxed{(x - a)f_x + (y - b)f_y + (z - c)f_z = 0.}$

Examples

Example 1: Find the tangent plane and normal line of $f(x, y, z) = x^2 + y^2 + z - 9 = 0$ at $(1, 2, 4)$.

$$\nabla f = 2x\hat{i} + 2y\hat{j} + \hat{k},$$

$$\implies \nabla f\Big|_{(1,2,4)} = 2\hat{i} + 4\hat{j} + \hat{k}.$$

Therefore, tangent plane is $2(x-1) + 4(y-2)(z-4) = 0$.

Then normal line is $x = 1 + 2t, y = 2 + 4t, z = 4 + t$.

Example 2: Find the tangent line to the curve of intersection of two surfaces $f(x, y, z) = x^2 + y^2 - 2 = 0, z \in R, g(x, y, z) = x + z - 4 = 0$.

Solution: The intersection of these two surfaces is an an ellipse on the plane $g = 0$. The direction of normal to $g(x, y, z) = 0$ at $(1, 1, 3)$ is $\hat{i} + \hat{k}$ and normal to $f(x, y, z) = 0$ is $2\hat{i} + 2\hat{j}$. The required tangent line is orthogonal to both these normals. So the direction of tangent is

$$v = \nabla f \times \nabla g = 2\hat{i} - 2\hat{j} - 2\hat{k}.$$

Tangent through $(1, 1, 3)$ is $x = 1 + 2t, y = 1 - 2t, z = 3 - 2t$.

Thank You.

Mean Value Theorem

Mean value theorem (MVT)

Theorem

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. Let $X_0 = (x_0, y_0)$ and $X = (x_0 + h, y_0 + k)$. Then there exists C which lies on the line joining X_0 and X such that

$$f(X) = f(X_0) + f'(C)(X - X_0),$$

i.e., there exists $c \in (0, 1)$ such that

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + hf_x(C) + kf_y(C),$$

where $C = (x_0 + ch, y_0 + ck)$.

Proof of MVT

Define $\phi : [0, 1] \to \mathbb{R}$ by

$$\phi(t) = f(x_0 + th, y_0 + tk), \quad t \in [0, 1].$$

By chain rule ϕ is differentiable and

$$\phi'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = f_x h + f_y k, \quad \text{(since } x = x_0 + th \text{ and } y = y_0 + tk.\text{)}$$

Now by MVT, there exists $c \in (0, 1)$ such that

$$\phi(1) - \phi(0) = \phi^(c).$$

The proof now follows immediately.

Extended mean value theorem (EMVT)

Theorem

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. Let $X_0 = (x_0, y_0)$ and $X = (x_0 + h, y_0 + k)$. Furthermore, suppose f_x and f_y are continuous and they have continuous partial derivatives. Then there exists C which lies on the line joining X_0 and X such that

$$f(X) = f(X_0) + f'(X_0)(X - X_0) + \frac{1}{2}(X - X_0)f''(C)(X - X_0),$$

where $f'' = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$. That is, there exists $c \in (0, 1)$ such that

$$f(x_0+h, y_0+k) = f(x_0, y_0)+(hf_x+kf_y)(X_0)+\frac{1}{2}(h^2 f_{xx}+2hkf_{xy}+k^2 f_{yy})(C),$$

where $C = (x_0 + ch, y_0 + ck)$.

Thank You.