Quiz Test 7

Limit and Continuity of Functions of Several Variables

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1. Find the value of c for which the function $f(x,y) = \begin{cases} \frac{\tan x}{x} \cdot \cos y & \text{if } x \neq 0, \\ c & \text{otherwise.} \end{cases}$ is continuous at $(0,0)$.

   (a) 0 (b) 1 (c) ∞ (d) none of these.

   Hint: Product of two continuous functions is continuous.

2. Find the value of c for which the function $f(x,y) = \begin{cases} \frac{\sin x}{x(y-9)} & \text{if } x \neq 0, \\ c & \text{otherwise.} \end{cases}$ is continuous at $(0,0)$.

   (a) 0 (b) $-\frac{1}{9}$ (c) ∞ (d) none of these.

   Hint: $\lim_{(x,y)\to(0,0)} \dfrac{\sin x}{x(y-9)} = -\dfrac{1}{9}$.

3. If $\lim_{(x,y)\to(0,0)} f(x,y) = l$ and $l \in \mathbb{R}$, then which among the following is the correct?

   (a) There exists $m \in \mathbb{R}$ such that $\lim_{x\to 0} f(x, mx) \neq l$ along the path $y = mx^2$.

   (b) $\lim_{x\to 0} f(x,0) \neq l$.

   (c) $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ may not exist.

   (d) $\lim_{x\to 0}\lim_{y\to 0} f(x,y) \neq \lim_{y\to 0}\lim_{x\to 0} f(x,y)$.

   Hint: We have an example for this case in our lecture slide. Consider $f(x,y) = x\sin\dfrac{1}{y} + y\sin\dfrac{1}{x},\ xy \neq 0$.

4. Let $A, B \in \mathbb{R}^2$ and $A = \{(x,y) : \sqrt{x^2+y^2} \leq 5\}$ and $B = \{(x,y) : \sqrt{x^2+y^2} < \sqrt{5}\}$. Then choose the **incorrect** option.

   (a) $A \cap B$ is closed in $\mathbb{R}^2$.

   (b) $A \cap B$ is open in $\mathbb{R}^2$.

   (c) $A \cup B$ is closed in $\mathbb{R}^2$.

   (d) A and B are respectively closed and open in $\mathbb{R}^2$.

   Hint: $A \cap B = B$ and $A \cup B = A$.

5. The set $\{(x,y) : \sqrt{(x-1)^2+(y-1)^2} \leq 5\} \cap \{(x,y) : \sqrt{x^2+y^2} < 1\}$

   (a) is an open set in $\mathbb{R}^2$.

   (b) is closed set in $\mathbb{R}^2$.

   (c) neither open nor closed in $\mathbb{R}^2$.

   (d) semi-open and semi-closed in $\mathbb{R}^2$.

   Hint: The intersection is $\{(x,y) : \sqrt{x^2+y^2} < 1\}$.

6. Which among the following is **NOT** a correct option?

   (a) The set $\{(x,y) : \sqrt{(x-1)^2+(y-1)^2} < 2.1\}$ is an open set in $\mathbb{R}^2$.

   (b) A subset $S \subset \mathbb{R}^2$ is called open if each point of S is an interior point of S.

   (c) Union of open sets is also an open set.

   (d) $\{(x,y) : \sqrt{x^2+y^2} \leq \frac{1}{2}\}$ is open in $\mathbb{R}^2$.

7. Let $\lim_{x\to a}\lim_{y\to b} f(x,y) = l_1$ and $\lim_{y\to b}\lim_{x\to a} f(x,y) = l_2$; $\quad l_1, l_2 \in \mathbb{R}$. Then which among the following statements is correct?

   (a) If $l_1 = l_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

   (b) If $l_1 \neq l_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ may exist.

   (c) l_1 is always equal to l_2.

   (d) If $\lim_{(x,y)\to(a,b)} f(x,y) = l \quad (l \in \mathbb{R})$, then $l = l_1 = l_2$.

   Hint: Properties of repeated and simultaneous limits.

8. Consider the function $f(x,y) = \sin(x^2 \cos y)$. Then

   (a) f is continuous everywhere.

   (b) f is not defined at origin.

   (c) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

   (d) The repeated limits do not exist at origin.

   Hint: Let $f_1(x,y) = x^2, f_2(x,y) = \cos y$ and $f_3(x,y) = \sin x$. Then f_1, f_2 and f_3 are continuous everywhere in the whole domain $\mathbb{R}^2$. So $f = f_3 \circ (f_1 \cdot f_2)$ is also continuous everywhere.

9. Consider the function $f(x,y) = \begin{cases} \frac{\sin x \cos y}{x} & \text{if } x \neq 0, \\ \cos y & \text{if } x = 0. \end{cases}$ Then

   (a) $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

   (b) $\lim_{x\to 0} \lim_{y\to 0} f(x,y) = 0$.

   (c) $f(x,y)$ is continuous everywhere.

   (d) $\lim_{y\to 0} \lim_{x\to 0} f(x,y) = 0$.

   Hint: $\lim_{(x,y)\to(0,0)} f(x,y) = 1 = \lim_{x\to 0} \lim_{y\to 0} f(x,y) = \lim_{y\to 0} \lim_{x\to 0} f(x,y)$.

10. Choose the **incorrect** option.

    (a) $\lim_{(x,y)\to(1,\pi)} \frac{y}{x} + \cos(xy) = \pi - 1$.

    (b) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

    (c) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$.

    (d) $\lim_{(x,y)\to(\pi,\frac{1}{2})} \frac{\sin(xy)}{xy} = 1$.

    Hint: $\lim_{(x,y)\to(\pi,\frac{1}{2})} \frac{\sin(xy)}{xy} = \frac{2}{\pi}$.

11. Consider the function $f(x,y) = \begin{cases} \frac{xy \sin\sqrt{x^2+y^2}}{x^2+y^2}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } x = y. \end{cases}$ Then

    (a) $f(x,y)$ is continuous at $(0,0)$.

    (b) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

    (c) If instead $f(0,0)$ is chosen to be equal to 1, then $f(x,y)$ becomes continuous at $(0,0)$.

    (d) Both the repeated limits exist and are equal to 1.

    Hint: $\lim_{(x,y)\to(0,0)} \frac{xy \sin\sqrt{x^2+y^2}}{x^2+y^2} = 0$.

12. Suppose $f(x,y) = \frac{xy - 2x - y + 2}{(x-1)(y-2)} e^{xy}$. Then which among the following is the correct statement?

(a) $\lim_{(x,y)\to(1,2)} f(x,y)$ does not exist.

(b) $\lim_{(x,y)\to(1,2)} f(x,y)$ exists and is equal to e^2.

(c) $\lim_{(x,y)\to(1,2)} f(x,y)$ exists and is equal to 1.

(d) The function is continuous at $(1,2)$.

13. If the simultaneous limit of a function f exists and has the same value along any three different paths, then

(a) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

(b) $\lim_{(x,y)\to(a,b)} f(x,y)$ may or may not exist.

(c) $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

(d) none of these.

14. Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

(a) -1 (b) 0 (c) 1 (d) The limit does not exist.