Solutions – Tutorial Sheet 1

1 (a) If $a_1 = 1$ and $a_{n+1} = \frac{2+a_n}{1+a_n}$ $\forall$ $n \in \mathbb{N}$, then compute a_2, a_3, a_4 and a_5.

Solution: Given $a_1 = 1$ and $a_{n+1} = \frac{2+a_n}{1+a_n}$ $\forall$ $n \in \mathbb{N}$

Put $n=1$, we get $a_2 = \frac{2+a_1}{1+a_1} = \frac{2+1}{1+1} = \frac{3}{2}$.

$$a_3 = \frac{2+a_2}{1+a_2} = \frac{2+\frac{3}{2}}{1+\frac{3}{2}} = \frac{\frac{4+3}{2}}{\frac{2+3}{2}} = \frac{7}{5}.$$

$$a_4 = \frac{2+a_3}{1+a_3} = \frac{2+\frac{7}{5}}{1+\frac{7}{5}} = \frac{\frac{17}{5}}{\frac{12}{5}} = \frac{17}{12}.$$

$$a_5 = \frac{2+a_4}{1+a_4} = \frac{2+\frac{17}{12}}{1+\frac{17}{12}} = \frac{24+17}{12+17} = \frac{41}{29}$$

Thus, we have $\boxed{a_2 = \frac{3}{2},\ a_3 = \frac{7}{5},\ a_4 = \frac{17}{12},\ a_5 = \frac{41}{29}}$ Ans

1(b) If $a_1 = 5$ and $a_{n+1} = 2 + \frac{1}{a_n}$ $\forall$ $n \in \mathbb{N}$, then compute a_3.

Solution: Given $a_1 = 5$ and $a_{n+1} = 2 + \frac{1}{a_n}$ $\forall$ $n \in \mathbb{N}$.

Put $n=1$, we get $a_2 = 2 + \frac{1}{a_1} = 2 + \frac{1}{5} = \frac{11}{5}$.

Put $n=2$, we get $a_3 = 2 + \frac{1}{a_2} = 2 + \frac{1}{\frac{11}{5}} = 2 + \frac{5}{11}$

$$\Rightarrow \boxed{a_3 = \frac{27}{11}}$$ Ans

Scanned by CamScanner

Problem-2: Compute supremum and infimum (if they exist) of the following sets. Which of these belong to the set? Also, find the maximum and minimum (if they exist) for these sets and check whether these sets are bounded or not.

(a) $A = \left\{-1, \frac{-1}{2}, \frac{-1}{3}, \frac{-1}{4}, \ldots\right\}$.

$\sup(A) = 0$, $\inf. = -1$

Here, infimum belongs to the set and minimum $= -1$, maximum does not exist.

The given set is bounded.

(b) $A = \{x \in \mathbb{R} : x^2 < 5\} = \{x \in \mathbb{R} : -\sqrt{5} < x < \sqrt{5}\}$

Here $\sup. = \sqrt{5}$, $\inf. = -\sqrt{5}$

$\Rightarrow$ The given set is bounded.

Maximum, Minimum doesnt exist.

(c) $A = \left\{\frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$.

$= \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \frac{-1}{5}, \ldots\right\}$

$\sup(A) = \frac{1}{2}$, $\inf(A) = -1$.

Since $\sup(A)$ and $\inf(A)$ belong to the set

$\Rightarrow \max(A) = \frac{1}{2}$, $\min(A) = -1$.

The given set is bounded.

(d) $A = \{1+(-1)^n : n \in \mathbb{N}\}$

$= \{1-1, 1+1, 1-1, \text{---}\}$

$= \{0, 2\}$.

$\max.(A) = 2$, $\min(A) = 0$.

$\sup.(A) = 2$, $\inf(A) = 0$.

The given set is bounded.

(e) $A = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$.

$\Rightarrow A = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \text{---}\right\}$

$\min.(A) = \frac{1}{2}$, $\max.(A) =$ doesnot exist.

Since $n+1 > n$

$\Rightarrow \frac{n}{n+1} < 1$.

$\inf.(A) = \frac{1}{2}$, $\sup.(A) = 1$.

The given set is bounded.

(f) $A = \left\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$.

$= \left\{1 + \frac{(-1)^1}{1}, 2 + \frac{(-1)^2}{2}, 3 + \frac{(-1)^3}{3}, \text{---}\right\}$

$= \left\{0, \frac{5}{2}, \frac{8}{3}, \text{---}\right\}$

$\max.(A) =$ not exist, $\min.(A) = 0$.

$\sup.(A) =$ not exist, $\inf(A) = 0$. and $0 \in A$.

The given set is not bounded.

(g) $A = \left\{ \sin\left(\frac{n\pi}{3}\right) : n \in \mathbb{N} \right\}$.

$= \left\{ 0, \pm \frac{\sqrt{3}}{2} \right\}$

A is bounded, $\max. = \sup. = \frac{\sqrt{3}}{2}$ and

$\min. = \inf = -\frac{\sqrt{3}}{2}$.

(h) $A = \left\{ \frac{1}{n+m} : n, m \in \mathbb{N} \right\}$.

$= \left\{ \frac{1}{n+1} : n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{n+2} : n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{n+3} : n \in \mathbb{N} \right\}$

$= \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \text{---} \right\} \cup \left\{ \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \text{---} \right\} \cup \left\{ \frac{1}{4}, \frac{1}{5}, - \right\}$

$= \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \text{---} \right\}$

$\max.(A) = \frac{1}{2}$, $\min.(A) =$ not exist.

$\sup.(A) = \frac{1}{2}$, $\inf(A) = 0$.

The given set is bounded.

(i) $A = \left\{ \frac{1}{n} : n \text{ is prime} \right\}$.

$= \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \text{---} \right\}$.

$\max.(A) = \frac{1}{2} = \sup.(A)$,

$\min.(A) =$ not exist and $\inf.(A) = 0$.

The given set is bounded.

(j) $A = \{x \in \mathbb{R}^+ : x^2 < 3\}$

$\Rightarrow A = \{x \in \mathbb{R}^+ : 0 < x < \sqrt{3}\}$

maximum and minimum don't exist.

$\sup(A) = \sqrt{3}$, $\inf(A) = 0$.

The given set is bounded.

(k) $A = \{x \in \mathbb{R} : |x-2| < 1\}$:

$= \{x \in \mathbb{R} : -1 < x-2 < 1\}$

$= \{x \in \mathbb{R} : 1 < x < 3\}$

$= (1, 3)$.

$\max(A)$ and $\min(A)$ don't exist.

$\sup(A) = 3$, $\inf(A) = 1$.

The given set is bounded.

Ques-3 What can you say about a nonempty subset A of real numbers for which $\sup A = \inf A$.

Solution: Let $A \subseteq \mathbb{R}$ is non-empty with $\sup. A = \inf. A$.

Let $a \in A$, then

$$\sup.(A) = \inf.(A) \leq a \leq \sup.(A),$$

which shows that $a = \sup.(A)$.

Thus $A = \{\sup.(A)\} = \{\inf.(A)\}$.

In other words, A consists of a single element $\sup.(A)$ or $\inf.(A)$.

4 Give examples of sets which are :

(i) bounded (ii) Not bounded

(iii) bounded below but not bounded above.

(iv) bounded above but not bounded below.

Solution: (i) $\{1, 3, 5, 7, 9\}$

Sup. $= 9$, Inf. $= 1$.

$\Rightarrow$ Bounded set.

(ii) The set of natural numbers, the set of real numbers

(iii) $[1, \infty) \rightarrow$ bounded below but not bounded above.

Here sup. doesn't exist and inf. $= 1$.

(iv) $(-\infty, 1] \rightarrow$ bounded above but not bounded below.

Here sup. $= 1$, but inf. doesn't exist.