# Volume Between Surfaces and Triple Integration

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# Volume Between Surfaces

Let $f$ and $g$ be continuous functions on a closed, bounded region $R$, where

$$f(x, y) \geq g(x, y) \quad \forall (x, y) \in R.$$

Then the volume $V$ between $f$ and $g$ over $R$ is

$$V = \iint\limits_R \left(f(x, y) - g(x, y)\right)\, dA.$$

# Example

**Example:** Find the volume of the space region bounded by the planes

$$2x + 3y - z = 8 \text{ and } x + 3y + z = 10, \text{ where } x, y > 0.$$

**Ans:** We need to determine the region $R$ over which we will integrate. To do so, we need to determine where the planes intersect. They have common $z$-values, when $2x + 3y - 8 = 10 - x - 3y \implies x + 2y = 6$. That is the planes intersect along the line $x + 2y = 6$. Therefore the region $R$ is bounded by $x = 0, y = 0$ and $x = 6 - 2y$.

$$\begin{aligned} \therefore V &= \iint_R \left((10 - x - 3y) - (2x + 3y - 8)\right) dA \\ &= \int_0^3 \int_0^{6-2y} (18 - 3x - 6y)\ dx\ dy = 54 \text{ unit}^3. \end{aligned}$$

# Observation

- In the previous example, we compute the volume by evaluating the integral
$$\int_0^3 \int_0^{6-2y} ((10 - x - 3y) - (2x + 3y - 8)) \; dx \; dy.$$

- Now observe that $(10 - x - 3y) - (2x + 3y - 8) = \displaystyle\int_{2x+3y-8}^{10-x-3y} dz.$
- Thus we can write

$$\begin{aligned}\int_0^3 \int_0^{6-2y} &((10 - x - 3y) - (2x + 3y - 8)) \; dx \; dy \\ &= \int_0^3 \left( \int_0^{6-2y} \left( \int_{2x+3y-8}^{10-x-3y} dz \right) dx \right) dy.\end{aligned}$$

Hurray! We get triple integral!!

# Triple integrals

- We know how to integrate over a two-dimensional region; we need to move on to integrating over a three-dimensional region.
- We used a double integral to integrate over a two-dimensional region and so it should not be too surprising that we'll use a triple integral to integrate over a three dimensional region.

# Definition

Let $D$ be a closed, bounded region in space. Let $a$ and $b$ be real numbers, let $g_1(x)$ and $g_2(x)$ be continuous functions of $x$, and let $f_1(x,y)$ and $f_2(x,y)$ be continuous functions of $x$ and $y$.

1. The volume $V$ of $D$ is denoted by a triple integral, $V = \iiint\limits_D dV$.
2. The iterated integral $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\ dy\ dx$ is evaluated as

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\ dy\ dx = \int_a^b \left( \int_{g_1(x)}^{g_2(x)} \left( \int_{f_1(x,y)}^{f_2(x,y)} dz \right) dy \right) dx.$$

Evaluating the above iterated integral is triple integration.

# Result

Let $D$ be a closed, bounded region in space and let $\Delta D$ be any subdivision of $D$ into $n$ cuboidal solids, where the $i$-th subregion $D_i$ has dimensions $\Delta x_i \times \Delta y_i \times \Delta z_i$ and volume $\Delta V_i$.

1. The volume $V$ of $D$ is

$$V = \iiint\limits_D dV = \lim_{\|\Delta D\| \to 0} \sum_{i=1}^{n} \Delta V_i = \lim_{\|\Delta D\| \to 0} \sum_{i=1}^{n} \Delta x_i \Delta y_i \Delta z_i.$$

2. If $D$ is defined as the region bounded by the planes $x = a$ and $x = b$, the cylinders $y = g_1(x)$ and $y = g_2(x)$, and the surfaces $z = f_1(x, y)$ and $z = f_2(x, y)$, where $a < b$, $g_1(x) \leq g_2(x)$ and $f_1(x, y) \leq f_2(x, y)$ on $D$, then

continue...

$$\iiint_D dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\ dy\ dx.$$

- $V$ can be determined using iterated integration with other orders of integration (there are $6$ total), as long as $D$ is defined by the region enclosed by a pair of planes, a pair of cylinders, and a pair of surfaces.

# Cautions

1. The outer limits have to be constant. They cannot depend on any of the variables.
2. The middle limits can depend on the variable from the outer integral only. They cannot depend on the variable from the inner integral.
3. The inner limits can depend on the variable from the outer integral and the variable from the middle integral.

For example, the following integral does NOT make any sense.

$$\iiint\limits_D dV = \cancel{\int_x^y \int_1^z \int_0^1 f(x,y,z)dx\ dy\ dz}.$$

## Examples

**Example 1:** Evaluate $\iiint\limits_D xyz\ dV$, where $D = [0,1] \times [1,2] \times [2,3]$.

**Ans:** Notice that the order does not matter. So

$$
\begin{aligned}
\iiint\limits_D xyz\ dV &= \int_2^3 \int_1^2 \int_0^1 xyz\ dx\ dy\ dz \\
&= \int_2^3 \int_1^2 \frac{x^2}{2} yz \Big|_0^1 dy\ dz \\
&= \frac{1}{2} \int_2^3 \int_1^2 yz\ dy\ dz \\
&= \frac{1}{2} \int_2^3 \frac{1}{2} y^2 z \Big|_1^2 dz \\
&= \frac{1}{4} \int_2^3 3z\ dz = \frac{3}{8} z^2 \Big|_2^3 = \frac{15}{8}
\end{aligned}
$$

**Example 2:** A cube has sides of length $1$ cm. Let one corner be at the origin and the adjacent corners be on the positive $x, y$ and $z$ axes. If the cube's density is directly proportional to the distance from the $xy$-plane, find its mass.

**Ans:** The density of the cube is $f(x, y, z) = kz$, for some constant $k$ whose unit is gm/cm$^4$. If $D$ is the cube, then the mass is the triple integral given by

$$\begin{aligned}
\iiint\limits_D dV &= \int_0^1 \int_0^1 \int_0^1 kz\, dx\, dy\, dz \\
&= \int_0^1 \int_0^1 kz\, dy\, dz \\
&= \int_0^1 kz\, dz \\
&= k\frac{z^2}{2}\bigg|_0^1 = \frac{k}{2} \text{ gms}
\end{aligned}$$

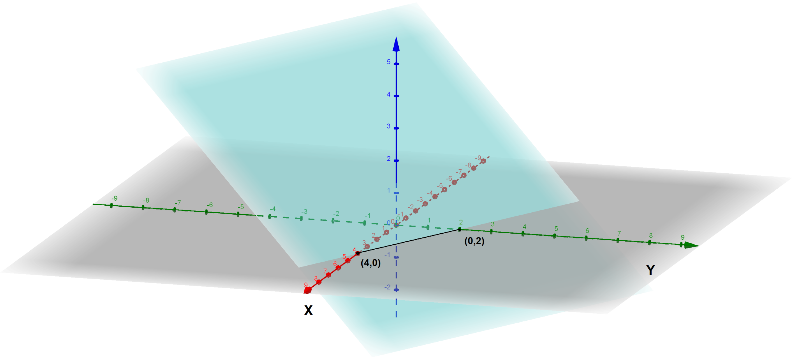
**Example 3:** Find the volume of the space region in the $1$-st octant bounded by the plane $x + 2y + 3z = 4$.

**Ans:** There are a total of 6 different approaches, but the result is same irrespective of any approach. We'll do it in the approach when the order of integration is $dz \cdot dy \cdot dx$

The region $D$ is bounded below by the plane $z = 0$ (because we are restricted to the first octant) and above by $z = \frac{1}{3}(4 - x - 2y)$

$$\implies 0 \leq z \leq \frac{1}{3}(4 - x - 2y).$$

To find the bounds on $y$ and $x$, we collapse the region onto the $x$-$y$ plane. (You can consider it as the shadow or the top view region. Therefore, this method is called shadow method.)




Here it will form a triangular region, bounded by the lines $x = 0, y = 0$ and $x + 2y = 4$.Therefore we have

$$0 \leq y \leq 2 - \frac{x}{2}, \quad 0 \leq x \leq 4.$$

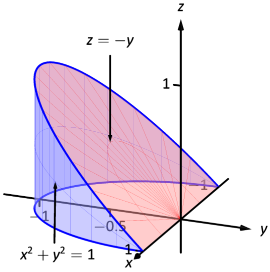
Thus the volume $V$ of the region $D$ is given by

$$\begin{aligned}\iiint\limits_D dV &= \int_0^4 \left( \int_0^{2-\frac{x}{2}} \left( \int_0^{\frac{1}{3}(4-x-2y)} dz \right) dy \right) dx \\ &= \frac{1}{3} \int_0^4 \int_0^{2-\frac{x}{2}} (4 - x - 2y)\, dy\ dx \\ &= \frac{1}{3} \int_0^4 \left(4y - xy - y^2\right) \Big|_0^{2-\frac{x}{2}} dx \\ &= \frac{1}{3} \left( 4x - \frac{x^3}{12} - x^2 \right) \Big|_0^4 = \frac{16}{9}.\end{aligned}$$

**Example 4:** Find the volume of the space region $D$ bounded by the surfaces $x^2 + y^2 = 1, z = 0$ and $z = -y$.

**Ans:** Consider the triple integral in the order $dz\ dy\ dx$

The region $D$ is bounded below by the plane $z = 0$ and above by the plane $z = -y$. The cylinder $x^2 + y^2 = 1$ does not offer any bounds in the $z$-direction, as that surface is parallel to the $z$-axis. Thus $0 \leq z \leq -y$. Collapsing the region into the $x$-$y$ plane, we get part of the region bounded by the circle with equation $x^2 + y^2 = 1$.




$$\therefore -\sqrt{1-x^2} \leq y \leq 0 \text{ and } -1 \leq x \leq 1.$$

So the required volume is given by

$$\begin{aligned}
\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} dz\ dy\ dx &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}(-y)dy\ dx \\
&= \int_{-1}^{1}\left(-\frac{y^2}{2}\right)\Big|_{-\sqrt{1-x^2}}^{0} dx \\
&= \frac{1}{2}\int_{-1}^{1}(1-x^2)dx \\
&= \frac{1}{2}\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} \\
&= \frac{2}{3}\ \text{unit}^3.
\end{aligned}$$

# Cylindrical coordinates

- Cylindrical coordinates can be thought of as a combination of the polar and rectangular coordinate systems.
- One can identify a point $(x_0, y_0, z_0)$, given in rectangular coordinates, with the point $(r_0, \theta_0, z_0)$, given in cylindrical coordinates, where the $z$-value in both systems is the same, and the point $(x_0, y_0)$ in the $x$-$y$ plane is identified with the polar point $P(r_0, \theta_0)$.
- Conversion technique:
  from rectangular to cylindrical: $r = \sqrt{x^2 + y^2}, \tan\theta = \frac{y}{x}$ and $z = z$;
  from cylindrical to rectangular: $x = r\cos\theta, y = r\sin\theta$ and $z = z$.

**Example:** Convert the rectangular point $(3, \sqrt{3}, 2)$ to cylindrical coordinates, and convert the cylindrical point $(2, -\frac{\pi}{4}, 1)$ to rectangular.

**Ans:** $r = \sqrt{9+3} = 2\sqrt{3}, \quad \tan\theta = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} \implies \theta = \frac{\pi}{6}$. Therefore, the point $(3, \sqrt{3}, 2)$ in cylindrical coordinates is $(2\sqrt{3}, \frac{\pi}{6}, 2)$.

In the second case, we have $x = r\cos\theta = 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}, y = r\sin\theta = 2 \times \left(-\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$. Therefore, the cylindrical coordinate point $(2, -\frac{\pi}{4}, 1)$ in rectangular coordinate is $(\sqrt{2}, -\sqrt{2}, 1)$.

# Spherical coordinate

- Spherical coordinates can be thought of as a "double application" of the polar coordinate system.
- In spherical coordinates, a point $P$ is identified with $(\rho, \theta, \phi)$, where $\rho$ is the distance from the origin to $P$, $\theta$ is the same angle as would be used to describe $P$ in the cylindrical coordinate system, and $\phi$ is the angle between the positive $z$-axis and the ray from the origin to $P$.
- $\rho \geq 0, 0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

# Relationship between rectangular and spherical coordinates

- From rectangular to spherical:

$$\rho = \sqrt{x^2 + y^2 + z^2}, \ \ \tan\theta = \frac{y}{x} \text{ and } \cos\phi = z/\sqrt{x^2 + y^2 + z^2}.$$

- From spherical to rectangular:

$$x = \rho\sin\phi\cos\theta, \ \ y = \rho\sin\phi\sin\theta \text{ and } z = \rho\cos\phi.$$

**Example:** Convert the rectangular point $(3, \sqrt{3}, 2)$ to spherical coordinates, and convert the spherical point $(1, \frac{\pi}{2}, \frac{\pi}{4})$ to rectangular coordinates.

**Ans:** $\rho = \sqrt{9+3+4} = 4$, $\tan\theta = \dfrac{1}{\sqrt{3}} \implies \theta = \dfrac{\pi}{6}$ and $\cos\phi = \frac{2}{4} = \frac{1}{2} \implies \phi = \frac{\pi}{3}$. Therefore, $(3, \sqrt{3}, 2)$ in spherical coordinates is $(4, \frac{\pi}{6}, \frac{\pi}{3})$.

In the second case, $x = \rho \sin\phi \cos\theta = 0$, $y = \rho \sin\phi \sin\theta = \frac{1}{2}$ and $z = \rho\cos\phi = \frac{\sqrt{3}}{2}$. Therefore, the spherical point $(1, \frac{\pi}{2}, \frac{\pi}{4})$ in rectangular coordinates is $(0, \frac{1}{2}, \frac{\sqrt{3}}{2})$.

# Triple integration in cylindrical coordinates

Let $w = h(r, \theta, z)$ be a continuous function on a closed, bounded region $D$ in space, bounded in cylindrical coordinates by $\alpha \leq \theta \leq \beta, g_1(\theta) \leq r \leq g_2(\theta)$ and $f_1(r, \theta) \leq z \leq f_2(r, \theta)$. Then

$$\iiint_D h(r, \theta, z) dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(r,\theta)}^{f_2(r,\theta)} h(r, \theta, z)\ r\ dz\ dr\ d\theta.$$

# Triple integration in spherical coordinates

Let $w = h(\rho, \theta, \phi)$ be a continuous function on a closed, bounded region $D$ in space, bounded in spherical coordinates by $\alpha_1 \leq \phi \leq \alpha_2, \beta_1 \leq \theta \leq \beta_2$ and $f_1(\theta, \phi) \leq \rho \leq f_2(\theta, \phi)$. Then

$$\iiint_D h(\rho, \theta, \phi) dV = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \int_{f_1(\theta,\phi)}^{f_2(\theta,\phi)} h(\rho, \theta, \phi)\ \rho^2 \sin(\phi)\ d\rho\ d\theta\ d\phi.$$

# Examples

**Example 1:** Let $D$ be the region in space bounded by the sphere, centered at the origin, of radius $r$. Use a triple integral in spherical coordinates to find the volume $V$ of $D$.

**Ans:** Equation of the sphere is $\rho = r$. Then the bounds on $\theta$ and $\phi$ are $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

$$\begin{aligned} \therefore V &= \iiint_D dV == \int_0^{\pi} \int_0^{2\pi} \int_0^{r} (\rho^2 \sin(\phi)) \; d\rho \; d\theta \; d\phi \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

# Thank You.