Iterated Integral

Iterated integrals

Suppose $f(x,y) = x^2y + x + y + 5$. Then $f_x(x,y) = 2xy + 1$ and $f_y(x,y) = x^2 + 1$.

$$\therefore \int (2xy+1)dx = x^2y + x + \phi(y).$$

Similarly,

$$\int (x^2+1)dy = x^2y + y + \psi(x).$$

Observations from the previous example

1. If you (indefinitely) integrate a function of two variable (say x and y) w.r.t. x, then the integrating constant is a function of y.
2. If you (indefinitely) integrate a function of two variable (say x and y) w.r.t. y, then the integrating constant is a function of x.

More examples

Example 1:

$$\int_0^x (x^2y + x + y + 5)dy = \left(\frac{x^2y^2}{2} + \frac{y^2}{2} + xy + 5y\right)_0^x = \frac{1}{2}x^4 + \frac{3}{2}x^2 + 5x.$$

Example 2:

$$\begin{aligned}
\int_1^2 \left(\int_0^x (x^2y + x + y + 5)dy\right) dx &= \int_1^2 \left(\frac{1}{2}x^4 + \frac{3}{2}x^2 + 5x\right) dx \\
&= \left(\frac{x^5}{10} + \frac{x^3}{2} + \frac{5x^2}{2}\right)_1^2 \\
&= \left(\frac{2^5}{10} + \frac{2^3}{2} + \frac{5 \times 2^2}{2}\right) - \left(\frac{1}{10} + \frac{1}{2} + \frac{5}{2}\right) \\
&= 17.2 - 3.1 \\
&= 14.1
\end{aligned}$$

Iterated integration

In the previous example, we integrated a function w.r.t. y and ended up with a function of x. We integrate the outcome again w.r.t. x. This process is known as **iterated integration** or **multiple integration**.

Iterated integration

Definition

Iterated integration is the process of repeatedly integrating the results of previous integrations.

Let a, b, c and d be numbers and let $g_1(x), g_2(x), h_1(y)$ and $h_2(y)$ be functions of x and y, respectively. Then

1. $$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) \;\; dx \; dy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y) \; dx \right) dy.$$
2. $$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \;\; dy \; dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y) \; dy \right) dx.$$

What does iterated integrals represent geometrically?

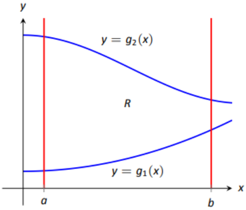
Area of a plane region

1. Let R be a plane region bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$. Then the area A of R is
$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\ dx.$$
2. Let R be a plane region bounded by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$. Then the area A of R is
$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\ dy.$$

Proof

Consider the plane region R bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$ as shown in the adjacent Figure. Then the area of R is given by

$$\int_a^b (g_2(x) - g_1(x))dx.$$



Calculating the area of a plane region R with an iterated integral.

We can view the expression $(g_2(x) - g_1(x))$ as $(g_2(x) - g_1(x)) = \int_{g_1(x)}^{g_2(x)} 1dy$, meaning we can express the area of R as an iterated integral:

$$\text{area of } R = \int_a^b (g_2(x) - g_1(x))dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} dy \right) dx.$$

Example

Find the area of the region enclosed by $y = 2x$ and $y = x^2$.

Solution:

$$\int_0^2 \int_{x^2}^{2x} 1 dy dx = \int_0^2 (2x - x^2) dx = (x^2 - \frac{x^3}{3})\Big|_0^2 = \frac{4}{3}.$$

$$\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} dx dy = \int_0^4 (\sqrt{y} - \frac{y}{2}) dy = \left(\frac{2}{3} y^{\frac{3}{2}} - \frac{y^2}{4}\right)\Big|_0^4 = \frac{4}{3}.$$

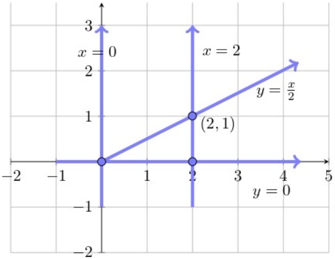
Changing order of integration

- If both integrals are bounded by constants, then by changing the order of the integral, the overall integral value does not change.
  That is, $\displaystyle\int_a^b \int_c^d dx\ dy = \int_c^d \int_a^b dy\ dx.$
- When the inner integral's bounds are not constants, it is generally very useful to sketch the bounds to determine what the region we are integrating over looks like. From the sketch, we can then rewrite the integral with the other order of integration.

Examples

- 1. Rewrite the iterated integral $\int_0^2 \int_0^{\frac{x}{2}} dy\ dx$ with the order of integration $dx\ dy$.

  Ans: Here y is bounded by 0 and $\frac{x}{2}$; x is bounded by 0 and 2. So plot the curves $y = 0, y = \frac{x}{2}, x = 0$ and $x = 2$. To change the order of integration, we need to consider the curves that bound the x-values.



  We see that the lower bound is $x = 2y$ and the upper bound is $x = 2$. The bounds on y are 0 to 1. Thus we can rewrite the integral as

$$\int_0^2 \int_0^{\frac{x}{2}} dy\ dx = \int_0^1 \int_{2y}^2 dx\ dy.$$

- 2. $\int_0^1 \int_0^{2x} f(x,y)dy\ dx = \int_0^2 \int_{\frac{y}{2}}^1 f(x,y)dx\ dy$
- 3. $\int_0^2 \int_{2y}^4 f(x,y)dx\ dy = \int_0^4 \int_0^{\frac{x}{2}} f(x,y)dy\ dx$
- 4.

$$\int_0^2 \int_{2y}^{6-y} f(x,y)dx\ dy = \int_0^4 \int_0^{\frac{x}{2}} f(x,y)dy\ dx + \int_4^6 \int_0^{6-x} f(x,y)dy\ dx$$

What have we achieved?

1. We developed one application for iterated integration: **area between curves**. However, this is not new, for we already know how to find areas bounded by curves.
2. Here our goal was to learn how to define a region in the plane using the bounds of an iterated integral. This skill is very important in our next chapters.

Anything else? Any other applications?

Application: to solve some tricky integrals

Example 1: Solve $\int_0^1 \int_{3y}^3 e^{x^2} dx\ dy$.

Solution : Changing the order of integration, it becomes

$$\int_0^3 \int_0^{\frac{x}{3}} e^{x^2} dy\ dx$$

which after simplification gives

$$\begin{aligned}
\int_0^3 \left(\int_0^{\frac{x}{3}} e^{x^2} dy \right) dx &= \int_0^3 e^{x^2} \left(\int_0^{\frac{x}{3}} dy \right) dx \\
&= \int_0^3 e^{x^2} \cdot y \Big|_{y=0}^{y=\frac{x}{3}} dx \\
&= \frac{1}{3} \int_0^3 x e^{x^2} dx = \frac{1}{6} \left(e^9 - 1 \right).
\end{aligned}$$

Example 2: Solve $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1} dy\ dx$.

Solution : Changing the order of integration, it becomes

$$\int_0^2 \int_0^{y^2} \frac{x}{y^5+1} dx\ dy$$

which after simplification gives

$$\begin{aligned}
\int_0^2 \left(\int_0^{y^2} \frac{x}{y^5+1} dx \right) dy &= \int_0^2 \frac{1}{y^5+1} \left(\int_0^{y^2} x\ dx \right) dy \\
&= \int_0^2 \frac{1}{y^5+1} \left(\frac{x^2}{2} \bigg|_0^{y^2} \right) dy \\
&= \frac{1}{10} \ln(33).
\end{aligned}$$

Thank You.