Surface Integral

BENNETT UNIVERSITY
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What about surface area of a curved surface?

How can we find out the curved surface area?

- We'll make an approximation, then using limits, we'll refine the approximation to the exact value.
- First, we subdivide R into n rectangular subregions, where the rectangle has dimensions Δx_i and Δy_i, along with its corresponding region on the surface.
- When Δx_i and Δy_i are small, the function is approximated well by the tangent plane at any point (x_i, y_i) in this subregion. Then we can approximate the surface area S_i of this region of the surface with the area T_i of the corresponding portion of the tangent plane.

- This portion of the tangent plane is a parallelogram, defined by sides $\overrightarrow{u}$ and $\overrightarrow{v}$. But from our intermediate course, we know that the area of this parallelogram is $|\overrightarrow{u} \times \overrightarrow{v}|$. So we need to determine $\overrightarrow{u}$ and $\overrightarrow{v}$.
- $\overrightarrow{u}$ is tangent to the surface in the direction of x, therefore,

$$\overrightarrow{u} = \Delta x_i \left(1\hat{i} + 0\hat{j} + f_x(x_i, y_i)\hat{k}\right).$$

Similarly,

$$\overrightarrow{v} = \Delta y_i \left(0\hat{i} + 1\hat{j} + f_y(x_i, y_i)\hat{k}\right).$$

- Thus,

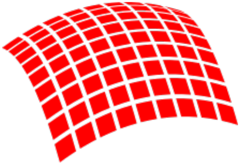
$$\begin{aligned}\text{surface area of } S_i &\approx \text{ area of } T_i \\ &= |\overrightarrow{u} \times \overrightarrow{v}| \\ &= \sqrt{1 + [f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2}\ \Delta x_i \Delta y_i.\end{aligned}$$

- Therefore, summing up all n of the approximations to the surface area gives

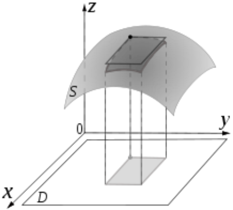
$$\text{surface area over } R \approx \sum_{i=1}^{n} \sqrt{1 + [f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2}\ \Delta A_i,$$

where $\Delta A_i = \Delta x_i \cdot \Delta y_i$ is the area of the i-th subregion.

Surface integral



The definition of surface integral relies on splitting the surface into small surface elements.



An illustration of a single surface element.

Result for finding surface area of a curved surface

Let $z = f(x, y)$ where f_x and f_y are continuous over a closed, bounded region R. Then the surface area S over R is

$$S = \iint_R dS = \iint_R \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA.$$

Examples

- **Example 1:** Let $f(x,y) = 2x + 3y - 4$ and let R be the region in the plane bounded by $x = 0, y = 0$ and $y = 2 - \frac{x}{2}$. Find the surface area of f over R.

  Ans: Here $f_x(x,y) = 2$ and $f_y(x,y) = 3$.

$$\therefore S = \iint\limits_R ds = \int_0^4 \int_0^{2-\frac{x}{2}} \sqrt{1+4+9}\; dy\; dx$$

$$= \sqrt{14} \int_0^4 y \Big|_0^{2-\frac{x}{2}} dx = 4\sqrt{14}.$$

- **Example 2:** Find the area of the surface $f(x,y) = x^2 + 5y - 9$ over the region R bounded by $-x \leq y \leq x, 0 \leq x \leq 1$.

  Ans: Here $f_x(x,y) = 2x, f_y(x,y) = 5$. Thus the required surface area is given by

$$\begin{aligned}
\iint\limits_R \sqrt{1 + 4x^2 + 25}\ dA &= \iint\limits_R \sqrt{26 + 4x^2}\ dA \\
&= \int_0^1 \int_{-x}^{x} \sqrt{26 + 4x^2}\ dy\ dx \\
&= \int_0^1 \sqrt{26 + 4x^2}\ y\Big|_{-x}^{x} dx \\
&= \int_0^1 2x\sqrt{26 + 4x^2}\ dx \\
&= \frac{1}{6}\left(30^{\frac{3}{2}} - 26^{\frac{3}{2}}\right).
\end{aligned}$$

- **Example 3:** Find the surface area of the sphere with radius a centered at the origin.

  Ans: Equation of the top hemisphere $f(x, y) = \sqrt{a^2 - x^2 - y^2}$. So $f_x(x, y) = \dfrac{-x}{a^2 - x^2 - y^2}$ and $f_y(x, y) = \dfrac{-y}{a^2 - x^2 - y^2}$. Hence we have

$$S = 2 \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\; dA$$
$$= 2 \iint_R \sqrt{1 + \frac{x^2 + y^2}{a^2 - x^2 - y^2}}\; dA.$$

Now in polar coordinates ($x = r\cos\theta, y = r\sin\theta, dA = r\ dr\ d\theta$) and with bounds $0 \le \theta \le 2\pi$ and $0 \le r \le a$, we get

continue...

$$
\begin{aligned}
S &= 2\int_0^{2\pi}\int_0^a r\sqrt{1+\frac{r^2\cos^2\theta + r^2\sin^2\theta}{a^2 - r^2\cos^2\theta - r^2\sin^2\theta}}\ dr\ d\theta \\
&= 2\int_0^{2\pi}\int_0^a r\sqrt{1+\frac{r^2}{a^2-r^2}}\ dr\ d\theta \\
&= 2\int_0^{2\pi}\int_0^a r\sqrt{\frac{a^2}{a^2-r^2}}\ dr\ d\theta \\
&= 2\int_0^{2\pi} a^2\ d\theta = 4\pi a^2.
\end{aligned}
$$

Exercise problem: The general formula for a right cone with height h and base radius a is $f(x,y) = h - \frac{h}{a}\sqrt{x^2+y^2}$. Find the surface area of this cone.

Thank You.