Derivative of a Function of several variables

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Partial derivatives

Definition

The partial derivative of f with respect to x at (a,b) is defined as

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{1}{h} \left(f(a+h,b) - f(a,b) \right).$$

similarly, the partial derivative with respect to y at (a,b) is defined as

$$\frac{\partial f}{\partial y}(a,b) = \lim_{k \to 0} \frac{1}{k} \left(f(a,b+k) - f(a,b) \right).$$

Examples I

1. (not continuous at $(0,0)$ but partial derivatives exist)

   Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

   Thus partial derivatives exist but f is not continuous at $(0,0)$.

Examples II

2. (continuous function but partial derivatives need not exist)

   Let $f(x, y) = |x| + |y|$. Then it is a continuous function at $(0, 0)$ as for every $\epsilon > 0$ there exists $\delta = \frac{\epsilon}{2} > 0$ such that

$$\sqrt{x^2 + y^2} < \delta \implies |f(x, y) - f(0, 0)| = |x| + |y| < 2\sqrt{x^2 + y^2} < \epsilon.$$

   Partial derivatives:

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{|k|}{k}.$$

   Both the limit do not exist. Hence partial derivatives do not exist.

Sufficient condition for continuity

Theorem

Suppose one of the partial derivatives exist at (a, b) and the other partial derivative is bounded in a neighborhood of (a, b). Then $f(x, y)$ is continuous at (a, b).

Directional derivatives

Definition

Let $\hat{p} = p_1\hat{i} + p_2\hat{j}$ be any **unit vector**. Then the directional derivative of $f(x, y)$ at (a, b) in the direction of $\hat{p}$ is

$$D_{\hat{p}}f(a, b) = \lim_{s\to 0} \frac{f(a + sp_1, b + sp_2) - f(a, b)}{s}.$$

$$\boxed{D_{\hat{p}}f(x, y) = f_x(x, y)p_1 + f_y(x, y)p_2.}$$

Example

1. Find the directional derivatives of $f(x,y) = x^2 + xy$ at $P(1,2)$ in the direction of unit vector $u = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$.

$$D_{\hat{p}}f(1,2) = \lim_{s\to 0} \frac{f(1+\frac{s}{\sqrt{2}}, 2+\frac{s}{\sqrt{2}}) - f(1,2)}{s}$$
$$= \lim_{s\to 0} \frac{1}{s}\left(s^2 + s(\sqrt{2} + \frac{3}{\sqrt{2}})\right) = \sqrt{2} + \frac{3}{\sqrt{2}}$$

Other approach: $f_x(x,y) = 2x + y, f_y(x,y) = x$.

$$D_{\hat{p}}f(x,y) = f_x(x,y)p_1 + f_y(x,y)p_2 = (2x+y)\left(\frac{1}{\sqrt{2}}\right) + (x)\left(\frac{1}{\sqrt{2}}\right)$$
$$= \left(\frac{3}{\sqrt{2}}\right)x + y\left(\frac{1}{\sqrt{2}}\right)$$

$$\implies D_{\hat{p}}f(1,2) = \sqrt{2} + \frac{3}{\sqrt{2}}$$

Further conditions and examples

Caution 1: The existence of partial derivatives does not guarantee the existence of directional derivatives in all directions.

Example: Consider

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x = y = 0 \end{cases}.$$

Let $\overrightarrow{p} = (p_1, p_2)$ such that $p_1^2 + p_2^2 = 1$. Then the directional derivative along p is

$$D_{\widehat{p}} f(0,0) = \lim_{h \to 0} \frac{f(hp_1, hp_2) - f(0,0)}{h} = \lim_{h \to 0} \frac{p_1 p_2}{h(p_1^2 + p_2^2)}$$

exist if and only if $p_1 = 0$ or $p_2 = 0$.

Caution 2: The existence of all directional derivatives does not guarantee the continuity of the function.

Example: Consider $f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & x = y = 0. \end{cases}$

Let $\overrightarrow{p} = (p_1, p_2)$ such that $p_1^2 + p_2^2 = 1$. Then the directional derivative along p is

$$\begin{aligned} D_{\widehat{p}} f(0,0) &= \lim_{s \to 0} \frac{f(sp_1, sp_2) - f(0,0)}{s} \\ &= \lim_{s \to 0} \frac{s^3 p_1^2 p_2}{s(s^4 p_1^4 + s^2 p_2^2)} \\ &= \frac{p_1^2 p_2}{p_2^2} \text{ if } p_2 \neq 0. \end{aligned}$$

In case of $p_2 = 0$, we can compute the partial derivative w.r.t y to be 0. Therefore all the directional derivatives exist. But this function is not continuous ($y = mx^2$ and $x \to 0$).

Differentiability of a function of several variables

Definition

Let D be an open subset of $\mathbb{R}^2$. Then a function $f(x, y) : D \to \mathbb{R}$ is differentiable at a point (a, b) of D if there exists $\epsilon_1 = \epsilon_1(h, k), \epsilon_2 = \epsilon_2(h, k)$ such that

$$f(a + h, b + k) - f(a, b) = hf_x(a, b) + kf_y(a, b) + h\epsilon_1 + k\epsilon_2,$$

where $\epsilon_1, \epsilon_2 \to 0$ as $(h, k) \to (0, 0)$.

Examples

(Q1): Show that the following function $f(x,y)$ is not differentiable at $(0,0)$,

$$f(x,y) = \begin{cases} x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right), & xy \neq 0 \\ 0, & xy = 0. \end{cases}$$

Ans: Since $|\sin x| \leq 1$ for every $x \in \mathbb{R}$, we have

$$|f(x,y) - f(0,0)| \leq |x| + |y| \leq 2\sqrt{x^2+y^2}$$

This implies that f is continuous at $(0,0)$ by choosing $\delta = \epsilon/2$. Also

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0.$$

continue...

If f is differentiable, then there exists ϵ_1, ϵ_2 such that

$$f(h,k) - f(0,0) = \epsilon_1 h + \epsilon_2 k,$$

where $\epsilon_1, \epsilon_2 \to 0$ as $h, k \to 0$. Now taking $h = k$, we get

$$f(h,h) = (\epsilon_1 + \epsilon_2)h \implies 2\frac{h \sin\left(\frac{1}{h}\right)}{h} = (\epsilon_1 + \epsilon_2).$$

So, $\lim_{h \to 0} \sin\left(\frac{1}{h}\right) = 0$, a contradiction, as limit does not exist.

(Q2): Show that the function $f(x, y) = \sqrt{|xy|}$ is not differentiable at the origin.

Ans: Easy to check the continuity (take $\delta = \epsilon$) as

$$|f(x, y) - f(0, 0)| = |\sqrt{|xy|} - 0| \leq \frac{1}{2}(|x| + |y|) \leq \sqrt{x^2 + y^2}.$$

Now,

$$f_x(0, 0) = \lim_{h \to 0} \frac{0 - 0}{h} = 0,$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{0 - 0}{k} = 0.$$

So if f is differentiable at $(0, 0)$, then there exist, ϵ_1, ϵ_2 such that

$$f(h, k) = \epsilon_1 h + \epsilon_2 k.$$

Taking $h = k$, we get

$$|h| = (\epsilon_1 + \epsilon_2)h.$$

This implies that $(\epsilon_1 + \epsilon_2) \not\to 0.$

Equivalent Condition For Differentiability

Notations:

- $\Delta f = f(a+h, b+k) - f(a, b)$, the total variation of f
- $df = hf_x(a, b) + kf_y(a, b)$, the total differential of f.
- $\rho = \sqrt{h^2 + k^2}$

Theorem

f is differentiable at (a, b) $\iff \lim_{\rho \to 0} \dfrac{\Delta f - df}{\rho} = 0.$

Examples I

1. Consider the function $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & x = y = 0. \end{cases}$ Prove that f is differentiable at $(0,0)$.

   Ans: Partial derivatives exist at $(0,0)$ and $f_x(0,0) = 0, f_y(0,0) = 0$.

   By taking $h = \rho\cos\theta, k = \rho\sin\theta$, we get

$$\frac{\Delta f - df}{\rho} = \frac{h^2k^2}{\rho^3} = \frac{\rho^4\cos^2\theta\sin^2\theta}{\rho^3} = \rho\cos^2\theta\sin^2\theta.$$

   Therefore, $\left|\frac{\Delta f - df}{\rho}\right| \leq \rho \to 0$ as $\rho \to 0$. Therefore f is differentiable at $(0,0)$.

Examples II

2. Consider $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2}, & (x,y) \neq 0 \\ 0, & x = y = 0. \end{cases}$ Prove that f is not differentiable at $(0,0)$.

   Ans: Partial derivatives exist at $(0,0)$ and $f_x(0,0) = f_y(0,0) = 0$. By taking $h = \rho\cos\theta, k = \rho\sin\theta$, we get

$$\frac{\Delta f - df}{\rho} = \frac{h^2k}{\rho^3} = \frac{\rho^3\cos^2\theta\sin\theta}{\rho^3} = \cos^2\theta\sin\theta.$$

   The limit does not exist. Therefore, f is NOT differentiable at $(0,0)$.

A sufficient condition for differentiability

Theorem

Suppose $f_x(x, y)$ and $f_y(x, y)$ exist in an open neighborhood containing (a, b) and both functions are continuous at (a, b). Then f is differentiable at (a, b).

Further conditions and examples

Caution 1: There are functions which are differentiable but the partial derivatives need not be continuous.

Example: Consider the function

$$f(x,y) = \begin{cases} x^3 \sin \frac{1}{x^2} + y^3 \sin \frac{1}{y^2} & xy \neq 0 \\ 0 & xy = 0. \end{cases}$$

Ans: Now

$$f_x(x,y) = \begin{cases} 3x^2 \sin \frac{1}{x^2} - 2\cos \frac{1}{x^2} & xy \neq 0 \\ 0 & xy = 0 \end{cases}$$

Also $f_x(0,0) = \lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = 0$. So partial derivatives are not continuous at $(0,0)$.

Continue...

$$f(h,k) = (h)^3 \sin \frac{1}{(h)^2} + (k)^3 \sin \frac{1}{(k)^2}$$
$$= 0 + 0 + \epsilon_1 h + \epsilon_2 k$$

where $\epsilon_1 = h^2 \sin \frac{1}{h^2}$ and $\epsilon_2 = k^2 \sin \frac{1}{k^2}$. It is easy to check that $\epsilon_1, \epsilon_2 \to 0$. So f is differentiable at $(0,0)$.

Caution 2: There are functions for which directional derivatives exist in any direction, but the function is not differentiable.

Example: Consider the function

$$f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

(Exercise problem)

Chain rule: Partial derivatives of composite functions

Let $z = F(u, v)$ and $u = \phi(x, y), v = \psi(x, y)$.

Then $z = F(\phi(x, y), \psi(x, y))$ as a function of x, y.

Suppose F, ϕ, ψ have continuous partial derivatives, then we can find the partial derivatives of z w.r.t x, y as follows:

$$\boxed{\frac{\partial z}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}} \text{ and } \boxed{\frac{\partial z}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}.}$$

Example

Example: Let $z = \ln(u^2 + v), u = e^{x+y^2}, v = x^2 + y$.

Then $z_u = \frac{2u}{u^2+v}$, $z_v = \frac{1}{u^2+v}$,

$u_x = e^{x+y^2}$, $v_x = 2x$, $u_y = 2ye^{x+y^2}$ and $v_y = 1$. Then

$$z_x = \frac{2u}{u^2 + v}e^{x+y^2} + \frac{2x}{u^2 + v},$$

$$z_y = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{2uy}{u^2 + v}e^{x+y^2} + \frac{1}{u^2 + v}.$$

Derivative of implicitly defined function

Theorem

Let $y = y(x)$ be defined as $F(x, y) = 0$, where F, F_x, F_y are continuous at (x_0, y_0) and $F_y(x_0, y_0) \neq 0$. Then $\frac{dy}{dx} = -\frac{F_x}{F_y}$ at (x_0, y_0).

Proof: Increase x by Δx, then y receives Δy increment and $F(x + \Delta x, y + \Delta y) = 0$. Also

$$0 = \Delta F = F_x \Delta x + F_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x \to 0$. This is same as

$$\frac{\Delta y}{\Delta x} = -\frac{F_x + \epsilon_1}{F_y + \epsilon_2}$$

Now taking limit $\Delta x \to 0$, we get $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

Example

Example: Let $F(x, y) = 0$, where $F(x, y) = e^y - e^x + xy$. Then

$$F_x = -e^x + y, F_y = e^y + x.$$

$$\implies \frac{dy}{dx} = \frac{e^x - y}{e^y + x}.$$

Example showing difference between partial derivative and total derivative

- When you take a partial derivative, you operate assuming that you hold one variable fixed while the other changes. On the other hand, while computing a total derivative, you allow changes in one variable to affect the other.
- **Example:** Let $f(x,y) = \sin x + \sin y$. Then

$$\frac{\partial f}{\partial x} = \cos x,$$

while

$$\frac{df}{dx} = \cos x \frac{dx}{dx} + \cos y \frac{dy}{dx} = \cos x + \cos y \frac{dy}{dx}.$$

THANK YOU.