Solution - Tutorial sheet 6

① (a) $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{1}{n^n}\right|^{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0$

So $R = \infty$. Given series is absolutely convergent for all $x \in \mathbb{R}$.

(b) similar to (a) part, $R = \infty$.

(c) $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |4^n|^{1/n} = \lim_{n\to\infty} 4 = 4$

For $x = \frac{1}{4}$

$\Sigma 1 \to$ dgs.

For $x = -\frac{1}{4}$

$\Sigma (-1)^n \to$ dgs

$\Rightarrow R = \frac{1}{4}$. Series converges absolutely for $|x| < \frac{1}{4}$ and diverges for $|x| \geq \frac{1}{4}$.

(d) $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{1}{4^n}\right|^{1/n} = \lim_{n\to\infty} \frac{1}{4} = \frac{1}{4}$

$\Rightarrow \boxed{R = 4}$. Series converges absolutely for all x; $|x| < 4$.

And diverges for $|x| > 4$.

For $x = 4$ and $x = -4$ we need to check separately.

$\underline{x=4}$: $\sum \frac{1}{4^n} \cdot 4^n = \sum 1 \rightarrow$ divergent series.

$\underline{x=-4}$: $\sum \frac{1}{4^n}(-4)^n = \sum (-1)^n \rightarrow$ divergent series.

So series converges for $|x| < 4$ and diverges for $|x| \geq 4$.

(e) $$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^n+1}{3^{n+1}+1}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{3^n}}{3+\frac{1}{3^n}} = \frac{1+0}{3+0} = \frac{1}{3}$$

$$\Rightarrow \boxed{R=3}$$

(f) $$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$$\Rightarrow \boxed{R=\infty}$$

(g) $$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n^p}{(n+1)^p} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^p = \left(\frac{1}{1+0}\right)^p = 1$$

$$\Rightarrow \boxed{R=1}$$

(h) $$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty}\frac{(n+1)\ n^n}{(n+1)(n+1)^n}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

$$\boxed{R=e}$$

2 (a) Taylor series of $\sin x$ about $c = \frac{\pi}{4}$.

We need to find $\sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{4})}{n!}(x-\frac{\pi}{4})^n$ where $f(x) = \sin x$.

$f(x) = \sin x = \sin x$

$f'(x) = \cos x = \sin(\frac{\pi}{2} + x)$

$f''(x) = -\sin x = \sin(\frac{2\pi}{2} + x)$

$f'''(x) = -\cos x = \sin(\frac{3\pi}{2} + x)$

$f^{(4)}(x) = \sin x = \sin(\frac{4\pi}{2} + x)$

$\vdots$

In general $f^{(n)}(x) = \sin(\frac{n\pi}{2} + x)$

So $f^{(n)}\left(\frac{\pi}{4}\right) = \sin\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$.

$$\sin x = \sum_{n=0}^{\infty} \frac{\sin\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)}{n!}\left(x-\frac{\pi}{4}\right)^n .$$

(b) Taylor series of $f(x) = x^3 - 7x + 11$ about $c = -1$.

$$\begin{aligned} f(x) &= x^3 - 7x + 11 \\ &= (x+1-1)^3 - 7(x+1-1) + 11 \\ &= \left[(x+1)^3 - 1 - 3(x+1)^2 + 3(x+1)\right] - 7(x+1) + 7 + 11 \\ &= (x+1)^3 - 3(x+1)^2 - 4(x+1) + 17 \end{aligned}$$

$\rightarrow$ By uniqueness of Taylor series this is Taylor series of f with $c=-1$.

(c) $f(x)=\frac{1}{x}$ with $c=1$.

$$f(x)=\frac{1}{x-1+1}=\left[1+(x-1)\right]^{-1}$$

$$=1-(x-1)+(x-1)^2-(x-1)^3+\ldots$$

$\left[\text{Using } (1+a)^{-1}=1-a+a^2-a^3+\ldots\right]$

(d) $f(x)=\frac{x}{x^4+9}$ with $c=0$.

$$f(x)=\frac{x}{9}\cdot\left(1+\frac{x^4}{9}\right)^{-1}$$

$$=\frac{x}{9}\left(1-\frac{x^4}{9}+\frac{x^8}{9^2}-\frac{x^{12}}{9^3}+\ldots\right).$$

$\left[\text{Using } (1+a)^{-1}=1-a+a^2-a^3+\ldots\right]$

$$=\frac{x}{9}-\frac{x^5}{9^2}+\frac{x^9}{9^3}-\frac{x^{13}}{9^4}+\ldots$$