Section 5.4.1

4.2.21

Vector Potential:

In Electrostatics,

$$\vec{\nabla} \times \vec{E} = 0$$

Hence, we could introduce scalar potential V such that

$$\vec{E} = -\vec{\nabla} V$$

In Magnetostatics,

$$\vec{\nabla} \cdot \vec{B} = 0$$

Hence, we can introduce vector potential, $\vec{A}$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

* Ampere's law with introduction of $\vec{A}$

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\underbrace{\vec{\nabla} \cdot \vec{A}}_{0}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

We can eliminate $\vec{\nabla} \cdot \vec{A}$

$\Rightarrow$ $\underline{\vec{\nabla} \cdot \vec{A} = 0}$ $\rightsquigarrow$ this can always be achieved

* Say, we have a vector potential, $\vec{A}_0$ for which $\vec{\nabla} \cdot \vec{A}_0 \neq 0$

Redefine the vector potential,

$$\vec{A} = \vec{A}_0 + \vec{\nabla} \underbrace{\lambda}_{\text{scalar}}$$

$\Rightarrow$ keeps $\vec{B}$ unchanged since $\vec{\nabla} \times (\vec{\nabla}\lambda) = 0$

Hence, $\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{A}_0 + \nabla^2\lambda = 0$

$$\Rightarrow \nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}_0$$ ⤳ Analogous to

Poisson's eq $\left(\nabla^2 V = -\frac{\rho}{\epsilon_0}\right)$

$$\rightarrow V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{r}\, d\tau'$$

$\rho \rightarrow 0$ at $\pm\infty$

identify

$$\vec{\nabla}\cdot\vec{A}_0 = \frac{\rho}{\epsilon_0}$$

$\vec{\nabla}\cdot\vec{A}_0 \rightarrow 0$ at $\pm\infty$

Hence, $$\lambda = \frac{1}{4\pi}\int \frac{\vec{\nabla}\cdot\vec{A}_0}{r}\, d\tau'$$

$\Rightarrow$ It is always possible to make the vector potential divergenceless.

⊗ Now the Ampere's law becomes

$$\nabla^2\vec{A} = -\mu_0\vec{J}$$

$\rightarrow$ Assuming that $\vec{J} \rightarrow 0$ at ∞

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{r}\, d\tau'$$

⊗ For line current, $\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}'}{r}$

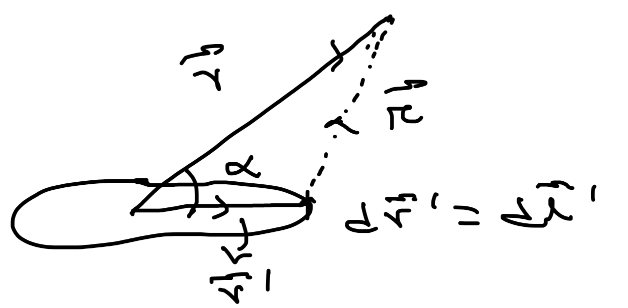
⊗ For surface current, $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{r}\, da'$

→ Usually the direcⁿ of $\vec{A}$ matches the direcⁿ of current

→ It is possible to add arbitrary const. vector to $\vec{A}$. It does not affect the definition of $\vec{B}$ & its properties.

$\boxed{\text{Section } 5.4.3}$

Multipole Expansion of $\vec{A}$:



$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

Idea is to write $\vec{A}$ in form of a power series in $\frac{1}{r}$. If r is sufficiently large, the series will be dominated by the lowest order term.

For this loop,

$$\vec{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l}'}{|\vec{r} - \vec{r}'|}$$

$$|\vec{r} - \vec{r}'| = \left[(\vec{r} - \vec{r}') \cdot (\vec{r} - \vec{r}') \right]^{1/2}$$

$$= \left[r^2 + r'^2 - 2\vec{r} \cdot \vec{r}' \right]^{1/2}$$

$$\vec{r} \cdot \vec{r}' = rr' \cos\alpha$$

$$|\vec{r} - \vec{r}'| = \left[r^2 + r'^2 - 2rr' \cos\alpha \right]^{1/2}$$

$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}'|} = \left[r^2 + r'^2 - 2rr'\cos\alpha\right]^{-1/2}$$

$$\simeq \frac{1}{r}\left[1 - \frac{2r'}{r}\cos\alpha + \frac{r'^2}{r^2}\right]^{-1/2}$$

$$\simeq \frac{1}{r}\left[1 + \frac{r'}{r}\cos\alpha + \cdots\right]$$

$$= \frac{1}{r} + \frac{r'}{r^2}\cos\alpha + \cdots$$

$$\Rightarrow \vec{A} = \underbrace{\frac{\mu_0 I}{4\pi r}\oint d\vec{l}'}_{\substack{\text{monopole} \\ \text{contribution} \\ = 0}} + \underbrace{\frac{\mu_0 I}{4\pi r^2}\oint r'\cos\alpha \, d\vec{l}'}_{\substack{\text{Dipole} \\ \text{contribution}}} + \cdots$$

$\oint d\vec{l}' = 0$ since the total vector displacement around a closed loop is zero.

⊗ In absence of monopole contribution, leading order contribution comes from dipole.

$$\vec{A}_{dip.}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\alpha \, d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot\vec{r}')\, d\vec{l}'$$

using identity,

$$\oint (\vec{c}\cdot\vec{r})\, d\vec{l} = \vec{a} \times \vec{c}$$

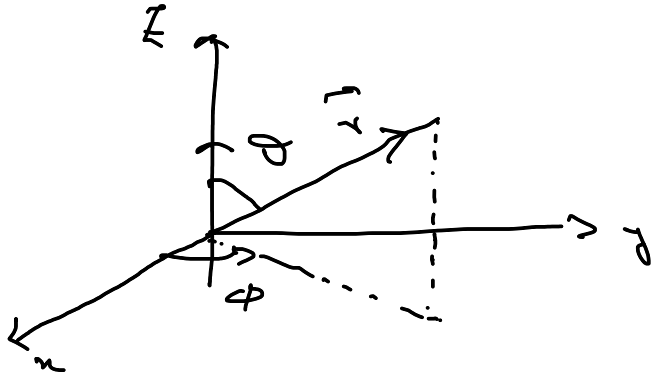
↳ area vector of the closed loop

$$\oint (\hat{r}\cdot\vec{r}')\, d\vec{l}' = \vec{a}\times\hat{r} \qquad \Big|\ \vec{c} \equiv \hat{r}$$

$$\Rightarrow \vec{A}_{dip.}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}(\vec{a}\times\hat{r})$$

$$= \frac{\mu_0}{4\pi r^2}(\vec{m}\times\hat{r})$$

Here, $\vec{m} = I\vec{a} \equiv$ <u>Magnetic dipole moment</u>.



$$\vec{A}_{dip.}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

$$\Rightarrow \vec{B}_{dip.}(\vec{r}) = \vec{\nabla}\times\vec{A}$$

$$= \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

(*) Identity: $\oint (\vec{c}\cdot\vec{r})\, d\vec{l} = \vec{a}\times\vec{c}$

we start with the identity

$$\int_S \vec{\nabla} T \times d\vec{a} = -\oint T\, d\vec{l}$$

↳ a scalar

Let $\vec{v} = \vec{c}\,' T$

↳ const. vector

Stoke's th:

$$\int (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint \vec{v} \cdot d\vec{l}$$

$$\vec{\nabla} \times \vec{v} = T \underbrace{(\vec{\nabla} \times \vec{c}\,')}_{0} - \vec{c}\,' \times (\vec{\nabla} T)$$

$$= -\vec{c}\,' \times (\vec{\nabla} T)$$

Hence, $-\int (\vec{c}\,' \times \vec{\nabla} T) \cdot d\vec{a} = \oint T \vec{c}\,' \cdot d\vec{l}$

$$\Rightarrow -\int \vec{c}\,' \cdot (\vec{\nabla} T \times d\vec{a}) = \oint \vec{c}\,' \cdot T \, d\vec{l}$$

$$\Rightarrow \int \vec{\nabla} T \times d\vec{a} = -\oint T \, d\vec{l}$$

Let, $T = \vec{c} \cdot \vec{r}$

$$\vec{\nabla} T = \vec{\nabla} (\vec{c} \cdot \vec{r}) = \vec{c} \times \underbrace{(\vec{\nabla} \times \vec{r})}_{0} + (\vec{c} \cdot \vec{\nabla}) \vec{r}$$

$$= (\vec{c} \cdot \vec{\nabla}) \vec{r}$$

$$= \vec{c}$$

$$\oint T \, d\vec{l} = \oint (\vec{c} \cdot \vec{r}) \, d\vec{l} = -\int (\vec{\nabla} T) \times d\vec{a}$$

$$= -\int \vec{c} \times d\vec{a} = -\vec{c} \times \int d\vec{a}$$

$$= -\vec{c} \times \vec{a} = \vec{a} \times \vec{c}$$