Quiz Test 5
Improper Integrals

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1. Find the value of the improper integral:

$$\int_e^\infty \frac{dx}{x(\ln x)^2}.$$

   (a) 0 (b) -1 (c) 1 (d) $\frac{1}{e}$

   Hint: Substitute $t = \ln x$. So,

$$\int_e^\infty \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \int_1^{\ln b} \frac{dt}{t^2} = \lim_{b\to\infty} \left(-\frac{1}{\ln b} + 1\right) = 1.$$

2. Consider the following two integrals.

$$I_1 = \int_{-1}^1 \frac{dx}{x^2} \qquad I_2 = \int_0^1 \frac{\sin x}{x^{5/2}} dx$$

   Which among the following statements is true?

   (a) Both I_1 and I_2 are convergent.

   (b) Both I_1 and I_2 are divergent.

   (c) I_1 is convergent and I_2 is divergent.

   (d) I_1 is divergent and I_2 is convergent.

   Hint: We know that $\int_0^1 \frac{dx}{x^p}$ diverges if $p \geq 1$ and converges if $p < 1$.
   Integral I_1: The integrand is unbounded at $x = 0$. So, we have to consider I_1 as sum of two improper integrals.

$$\int_{-1}^1 \frac{dx}{x^2} = \int_{-1}^0 \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2}$$

   The second (similarly, first) integral on the right hand side is divergent since $p = 2 \geq 1$.
   Integral I_2: We know, $\int_0^1 \frac{\sin x}{x^p} dx$ converges if and only if $0 < p < 2$. Hence I_2 is divergent.

3. The improper integral $\displaystyle\int_0^\infty \frac{dx}{x^2+4}$.

(a) diverges to ∞.

(b) converges to $\frac{\pi}{2}$.

(c) converges to $\frac{\pi}{4}$.

(d) diverges to $-\infty$.

Hint: $\displaystyle\int_0^\infty \frac{dx}{x^2+4} = \lim_{b\to\infty}\left(\frac{1}{2}\ tan^{-1}\frac{x}{2}\right)\Big|_0^b = \frac{\pi}{4}.$

4. What can be said about the following integrals?

$$I_1 = \int_0^1 \frac{e^x}{x^2}dx, \qquad I_2 = \int_1^\infty \frac{\cos^2 x}{x^2}dx.$$

(a) Both I_1 and I_2 are convergent.

(b) Both I_1 and I_2 are divergent.

(c) I_1 is convergent and I_2 is divergent.

(d) I_1 is divergent and I_2 is convergent.

Hint: I_1 diverges by comparison test since the integrand is greater than $\frac{1}{x^2}$.
I_2 converges again by comparison test since the integrand is less than $\frac{1}{x^2}$.

5. Evaluate the integral $\int_0^\infty xe^{-x^2}dx$.
(a) 1 (b) $\frac{1}{2}$ (c) e (d) diverges to ∞
Hint: Substitute $t = x^2$.

$$\implies \int_0^\infty xe^{-x^2}dx = \frac{1}{2}\int_0^\infty e^{-t}dt = \frac{1}{2}.$$

6. Compute $\displaystyle\int_0^2 \frac{dx}{x-1}$.
(a) 0 (b) diverges (c) 2 (d) 1
Hint: The function is unbounded in the domain at $x = 0$. So we have to partition it as

$$\int_0^2 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^2 \frac{dx}{x-1}.$$

Now, $\int_0^1 \frac{dx}{x-1} = \displaystyle\lim_{\epsilon\to 0}\int_0^{1-\epsilon}\frac{1}{x-1}dx = \lim_{\epsilon\to 0}\ln|x-1|\Big|_0^{1-\epsilon} = -\infty.$

7. The improper integral $\displaystyle\int_0^\infty \frac{dx}{x^2+9}$

(a) diverges to ∞.

(b) converges to $\frac{\pi}{2}$.

(c) converges to $\frac{\pi}{6}$.

(d) diverges to $-\infty$.

Hint: $\displaystyle\int_0^\infty \frac{dx}{x^2+9} = \lim_{b\to\infty} \left(\frac{1}{3}\, tan^{-1}\frac{x}{2}\right)\Big|_0^b = \frac{\pi}{6}.$

8. Consider the following two integrals.

$$I_1 = \int_{-1}^{1} \frac{dx}{x^2} \qquad I_2 = \int_{-1}^{1} \frac{dx}{1+x}$$

Which among the following statements is true?

(a) Both I_1 and I_2 are convergent.

(b) Both I_1 and I_2 are divergent.

(c) I_1 is convergent and I_2 is divergent.

(d) I_1 is divergent and I_2 is convergent.

Hint: We know that $\int_0^1 \frac{dx}{x^p}$ diverges if $p \geq 1$ and converges if $p < 1$.
Integral I_1: The integrand is unbounded at $x = 0$. So, we have to consider I_1 as sum of two improper integrals.

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}$$

The second (similarly, first) integral on the right hand side is divergent since $p = 2 \geq 1$.
Integral I_2: Substitute $t = x + 1$. So

$$I_2 = \int_0^2 \frac{dt}{t} = \int_0^1 \frac{dt}{t} + \int_1^2 \frac{dt}{t}.$$

Since $\int_0^1 \frac{dt}{t}$ is divergent, I_2 is divergent.

9. Compute the integral $\displaystyle\int_{-3}^{3} \frac{dx}{(x+2)^3}$.

(a) $\frac{12}{25}$ (b) $-\frac{12}{25}$ (c) divergent (d) $\frac{24}{25}$

Hint: The integrand here is unbounded at $x = -2$. Hence, it is an improper integral of second kind.

$$\therefore \int_{-3}^{3} \frac{dx}{(x+2)^3} = \int_{-3}^{-2} \frac{dx}{(x+2)^3} + \int_{-2}^{3} \frac{dx}{(x+2)^3}$$

Consider the second integral. Substituting $t = x + 2$, we have

$$\int_{-2}^{3} \frac{dx}{(x+2)^3} = \int_{0}^{5} \frac{dt}{t^3} = \lim_{\epsilon \to 0} \int_{\epsilon}^{5} \frac{dt}{t^3} = \infty.$$

10. Find $\displaystyle\int_{-2}^{2} \frac{dx}{x+1}$.

 (a) diverges (b) $\frac{1}{2}\ln 3$ (c) $\ln 3$ (d) 0

 Hint: The integrand here is unbounded at $x = -1$. Hence, it is an improper integral of second kind.

$$\therefore \int_{-2}^{2} \frac{dx}{x+1} = \int_{-2}^{-1} \frac{dx}{x+1} + \int_{-1}^{2} \frac{dx}{x+1}$$

Now, consider the second integral.

$$\int_{-1}^{2} \frac{dx}{x+1} = \lim_{\epsilon \to 0} \int_{-1+\epsilon}^{2} \frac{dx}{x+1} = -\infty.$$

11. The improper integral $\displaystyle\int_{e}^{\infty} \frac{dx}{x \ln x}$:

 (a) diverges (b) converges to $\frac{\pi}{2}$ (c) converges to e (d) converges to $\frac{e}{2}$

 Hint: Substitute $\ln x = t$, then $\displaystyle\int_{e}^{\infty} \frac{dx}{x \ln x} = \int_{1}^{\infty} \frac{dt}{t}$.

12. The integral $\int_{\pi}^{\infty} \frac{\sin x}{x} dx$:

 (a) is not an improper integral.

 (b) converges absolutely.

 (c) converges conditionally.

 (d) diverges.

 Hint: See it from the lecture class note/slides. This is an example where the function is convergent but not absolutely. In the lecture note, you can see that $\int_{\pi}^{\infty} \frac{\sin x}{x} dx$ is convergent but $\int_{\pi}^{\infty} \frac{|\sin x|}{x} dx$ is divergent. Also, note that $\int_{\pi}^{\infty} \frac{|\sin x|}{x} dx \geq \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n+1}$. On the right it is a divergent series.

13. Evaluate the improper integral $\displaystyle\int_2^\infty \frac{dx}{x(x-1)}$.

(a) $\ln 2$ (b) $\ln 3$ (c) $\ln \frac{1}{2}$ (d) diverges

Hint: $\displaystyle\int \frac{dx}{x(x-1)} = \int \frac{dx}{x-1} - \frac{dx}{x} = \ln\left|\frac{x-1}{x}\right| + C.$

14. Consider the following two integrals.

$$I_1 = \int_{-1}^{1} \frac{dx}{x^2} \qquad I_2 = \int_0^1 \frac{\cos x}{\sqrt{x}} dx$$

Which among the following statements is true?

(a) Both I_1 and I_2 are convergent.

(b) Both I_1 and I_2 are divergent.

(c) I_1 is convergent and I_2 is divergent.

(d) I_1 is divergent and I_2 is convergent.

Hint: We know that $\int_0^1 \frac{dx}{x^p}$ diverges if $p \geq 1$ and converges if $p < 1$.
Integral I_1: The integrand is unbounded at $x = 0$. So, we have to consider I_1 as sum of two improper integrals.

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2}$$

The second (similarly, first) integral on the right hand side is divergent since $p = 2 \geq 1$.
Integral I_2: We know that $\displaystyle\int_0^1 \frac{\cos x}{x^p} dx$ converges for $p < 1$.