Product Rules (contd.) 10.11.20

Divergence: $\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

Curl: $\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f)$

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$$

You can find: $\vec{\nabla}\left(\frac{f}{g}\right)$, $\vec{\nabla}\cdot\left(\frac{\vec{A}}{g}\right)$, $\vec{\nabla}\times\left(\frac{\vec{A}}{g}\right)$

$$\vec{A} = 2x\hat{i} + xy\hat{j} + 3z\hat{k}$$
$$\vec{B} = 4z\hat{i} + 2y\hat{j} + z\hat{k}$$

(*) $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = ?$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2x & 1xy & 3z \\ 4z & 2y & 1z \end{vmatrix}$$

$$= \hat{i}(xyz - 6yz) + \hat{j}(12z^2 - 2xz) + \hat{k}(4xy - 4xyz)$$

$$= (xyz - 6yz)\hat{i} + (12z^2 - 2xz)\hat{j} + \hat{k}(4xy - 4xyz)$$

$$\vec{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = yz - 4xy$$

$$\vec{B}\cdot(\vec{\nabla}\times\vec{A}) = \vec{B}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x & xy & 3z \end{vmatrix}$$

$$= \vec{B}\cdot\left(\hat{k}\,(y - 0)\right) = yz$$

$$\vec{A}\cdot(\vec{\nabla}\times\vec{B}) = +\,4xy$$

(*) Second Derivative:

1. $\vec{\nabla}\cdot(\vec{\nabla}T)$ (Divergence of gradient)
2. $\vec{\nabla}\times(\vec{\nabla}T)$ (curl of gradient)
3. $\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$ (gradient of divergence)
4. $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v})$ (divergence of curl)
5. $\vec{\nabla}\times(\vec{\nabla}\times\vec{v})$ (curl of a curl)

$$① \Rightarrow \quad \vec{\nabla} \cdot (\vec{\nabla} T) = \nabla^2 T$$

Laplacian of a scalar.

$$= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$$

$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$② \Rightarrow \quad \vec{\nabla} \times (\vec{\nabla} T) = 0$$

$$T = xyz$$

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)$$

$$③ \Rightarrow \quad \vec{\nabla}(\vec{\nabla} \cdot \vec{v})$$ (seldom occurs)

↳ not same as Laplacian of vector

Laplacian of a vector:

$$\nabla^2 \vec{v} = (\vec{\nabla} \cdot \vec{\nabla})\vec{v}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$④ \Rightarrow \quad \vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$$

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = \underbrace{\frac{\partial}{\partial x}\left(\cancel{\frac{\partial v_z}{\partial y}}\right.}_{=} - \left.\cancel{\frac{\partial v_y}{\partial z}}\right) + \frac{\partial}{\partial y}\left(\cancel{\frac{\partial v_x}{\partial z}} - \underset{=}{\cancel{\frac{\partial v_z}{\partial x}}}\right) + \frac{\partial}{\partial z}\left(\cancel{\frac{\partial v_y}{\partial x}} - \cancel{\frac{\partial v_x}{\partial y}}\right) = 0$$

$$\left\{ \frac{\partial^2 v_z}{\partial x \partial y} = \frac{\partial^2 v_z}{\partial y \partial x} \right.$$

(5) $\Rightarrow$ $$\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{v})}_{=(3)} - \underbrace{\nabla^2\vec{v}}_{=(\nabla^2 v_x)\hat{x} + (\nabla^2 v_y)\hat{y} + (\nabla^2 v_z)\hat{z}}$$

(7) Really two kinds of second derivatives.

- ✓ The Laplacian (vector/scalar)
- ✓ Gradient of Divergence