Power Series and Taylor series

Engineering Calculus

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Definition

Definition: A series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$, where $a_n, c \in \mathbb{R}$ is called **power series** with center c.

Some remarks

- Power series is a function of x provided it converges for x. If a power series converges, then the domain of convergence is either a bounded interval or the whole of $\mathbb{R}$.
- A power series always converges for $x = c$.
- The translation $x' = x - c$ reduces a power series around c to a power series around 0.

- Consider the series for $c = 0$, i.e., the power series around 0 of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots + a_n x^n + \cdots . \tag{1}$$

Even though the functions appearing in (1) are defined over all of $\mathbb{R}$, it is not to be expected that the series (1) will converge for all x in $\mathbb{R}$. For example, the series

$$\sum_{n=0}^{\infty} n! x^n, \quad \sum_{n=0}^{\infty} x^n, \quad \sum_{n=0}^{\infty} x^n/n!,$$

converge for x in the sets

$$\{0\}, \quad \{x \in \mathbb{R} : |x| < 1\}, \quad \mathbb{R}, \quad \text{respectively.}$$

Theorem

If $\sum a_n x^n$ converges at $x = r$, then $\sum a_n x^n$ converges for $|x| < |r|$.

Proof: We can find $C > 0$ such that $|a_n r^n| \leq C$ for all n. Then

$$|a_n x^n| \leq |a_n r^n| |\frac{x}{r}|^n \leq C |\frac{x}{r}|^n.$$

Conclusion follows from comparison theorem.

Remark

If $\sum a_n x^n$ diverges at $x = r_1$, then $\sum a_n x^n$ diverges for $|x| > |r_1|$.

Proof: Proof of this follows by using contradiction and above theorem.

Theorem and Remark for $\sum a_n (x-c)^n$

The above theorem and remark can be stated for general power series with center c.

Theorem : If $\sum a_n (x-c)^n$ converges at $x = r$, then it converges for $|x - c| < |r - c|$.

Remarks : If $\sum a_n (x-c)^n$ diverges at $x = r_1$, then it diverges for $|x - c| > |r_1 - c|$.

Theorem

For a power series $\sum_{n=0}^{\infty} a_n x^n$ exactly one of the following three cases is true:

- **Case 1:** The series converges only for $x = c$.
- **Case 2:** There exists a positive real number R such that the series converges absolutely for all real x satisfying $|x - c| < R$ and diverges for all x satisfying $|x - c| > R$.
- **Case 3:** The series converges for all $x \in \mathbb{R}$.

• Define $R = 0$ and $R = \infty$ for Case 1 and Case 3 of above Theorem, respectively.

• The R is called as **radius of convergence** of the power series.

Theorem

Consider the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$. The radius of convergence is given as follows:

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

For a given power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ with radius of convergence R, we have:

- Series converges absolutely for all real x satisfying $|x-c| < R$, that is $c-R < x < c+R$.
- Series diverges for all x satisfying $|x-c| > R$, that is $x < c-R$ or $x > c+R$.
- No conclusion about the series about points $x = c-R$ and $x = c+R$.

Examples

Find the radius of convergence of $(i)\ \sum \frac{x^n}{n}$, $(ii)\ \sum \frac{x^n}{n!}$, $(iii)\ \sum 2^{-n}x^n$.

(i) $\frac{1}{R} = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$,. So $R = 1$.

(ii) $\frac{1}{R} = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 0$. So $R = \infty$, and series converges everywhere.

(iii) $\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = 2^{-1} = \frac{1}{2}$. Therefore, $R = 2$.

Taylor's series

Suppose f is infinitely differentiable at c then we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

This series is called Taylor series of $f(x)$ about the point c.

Remark

- If $c = 0$, the formula obtained in Taylor's theorem is known as *Maclaurin's formula* and the corresponding series that one obtains is known as *Maclaurin's series.*
- Taylor series representation of a function about some c is unique.

Examples

(i) Find Taylor series of $f(x) = e^x$ about $c = 0$.
We have $f^{(n)}(x) = e^x$. So $f^{(n)}(0) = e^0 = 1$.

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(ii) Find Taylor series of $f(x) = e^x$ about $c = -1.5$.
We have $f^{(n)}(x) = e^x$. So $f^{(n)}(-1.5) = e^{-1.5}$.

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1.5)}{n!}(x+1.5)^n = \sum_{n=0}^{\infty} \frac{e^{-1.5}}{n!}(x+1.5)^n.$$


Thank You