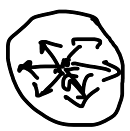
Divergence & Curl of $\vec{E}$ (contd.)

03.12.20

Flux of $\vec{E}$ through a surface 'S'

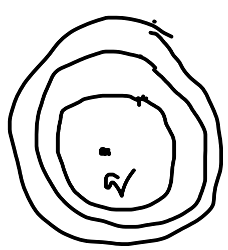
$$\Phi_E = \int_S \vec{E} \cdot d\vec{a}$$



For a closed surface, the flux is a measure of total charge inside.

Ex: Point charge located at centre of a sphere of radius r.

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{4\pi\epsilon_0} \int \left(\frac{q}{r^2}\right) \hat{r} \cdot (r^2 \sin\theta \, d\theta \, d\phi) \hat{r}$$



$$= \frac{q}{\epsilon_0} \leadsto \text{independent of 'r'}$$

$\Rightarrow$ flux through any surface enclosing the charge is q/ϵ_0.

(*) For a collection of charges,

$$\vec{E} = \sum_{i=1}^{n} \vec{E}_i$$

$$\oint \vec{E} \cdot d\vec{a} = \sum_{i=1}^{n} \frac{q_i}{\epsilon_0}$$

• For any closed surface,

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc.}}{\epsilon_0} \Rightarrow \text{Gauss' law.}$$

We can write

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{E})\, d\tau = \frac{Q_{enc.}}{\epsilon_0}$$

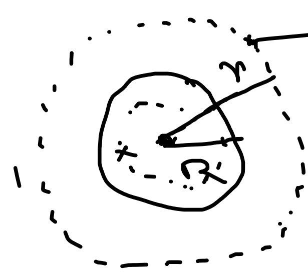
if we have a vol. ch. density 'ρ' is charge per unit vol.

$$Q_{enc.} = \int_V \rho \, d\tau$$

$$\Rightarrow \int_V (\vec{\nabla} \cdot \vec{E})\, d\tau = \int_V \frac{\rho}{\epsilon_0}\, d\tau \rightarrow \text{True for any volume}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$ # Gauss' law in diff. form.

Ex: Electric field outside a uniformly charged sphere of radius 'R' and total charge 'q'



$\rightarrow$ Gaussian surface

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc.}}{\epsilon_0} = \frac{q}{\epsilon_0}$$

$$\int_S \vec{E}\cdot d\vec{a} = \int_S |\vec{E}|\, da$$

↳ magnitude is const throughout the surface

$$\Rightarrow \int_S \vec{E}\cdot d\vec{a} = |\vec{E}| \int_S da$$

$$= |\vec{E}|\, 4\pi r^2$$

$$\Rightarrow |\vec{E}|\, 4\pi r^2 = \frac{q}{\epsilon_0}$$

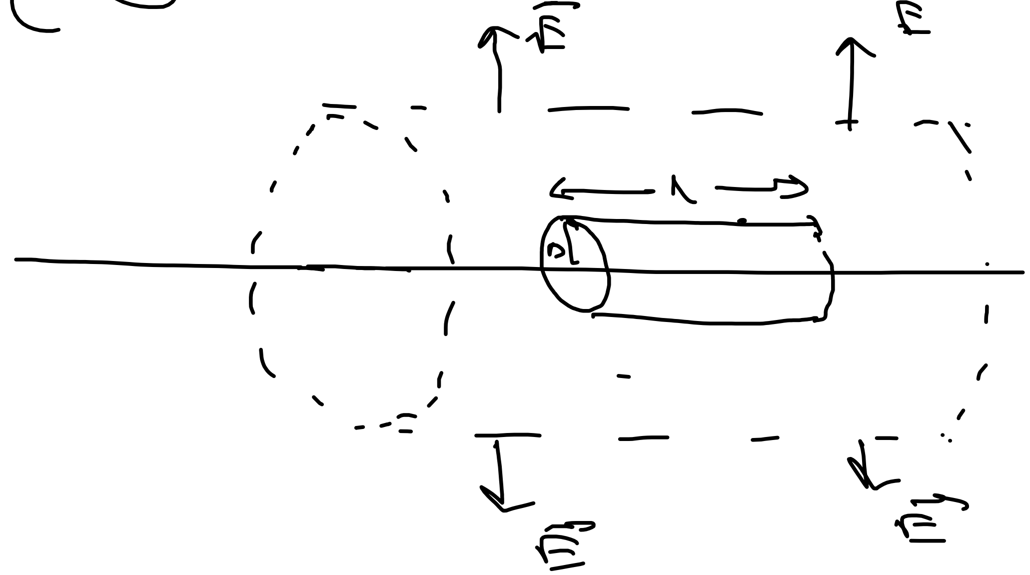
$$\Rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{r}$$

(*) Useful only when

- (*) charge distribution is uniform
- (*) Gaussian surface is symmetric.

Ex.

$\rho = ks'$



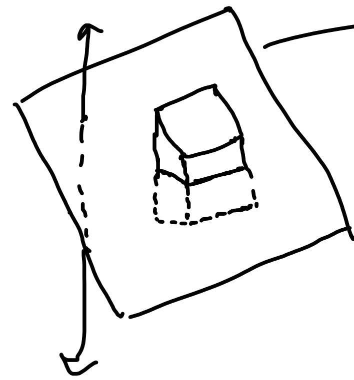
$$q = \int \rho\, d\tau$$

$$= \int ks'(s'\, ds'\, d\phi\, dz)$$

$$= \frac{2}{3}\pi k l s^3$$

Ex: Infinite plane carrying uniform charge density

Gaussian pillbox



→ uniform surface ch. density $= \sigma$

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc.}}{\epsilon_0}$$

$Q_{enc.} = \sigma A$
↳ Area of pillbox

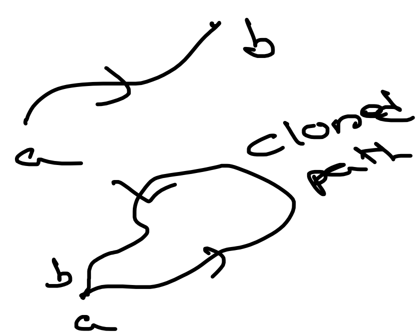
$$\int \vec{E}\cdot d\vec{a} = 2A|\vec{E}|$$

$$\Rightarrow 2A|\vec{E}| = \frac{\sigma A}{\epsilon_0} \Rightarrow \vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}$$

Curl of $\vec{E}$:

(point charge q at origin)

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$



Line integral : $\int_a^b \vec{E}\cdot d\vec{l}$

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\int_a^b \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}\,dr = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

For a closed path

$$\oint_a^b \vec{E} \cdot d\vec{l} = 0$$

Apply Stokes' Theorem:

$$\oint \vec{E} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = 0$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{E} = 0$$

$\Rightarrow$ Always true for electrostatic fields.