Real Number System

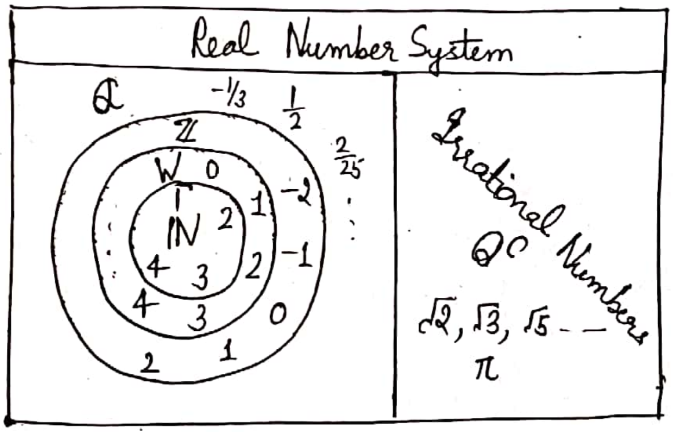
- The set of natural numbers $\mathbb{N} = \{1, 2, 3, ---\}$.
- The set of whole numbers $W = \{0, 1, 2, 3, ---\}$.
- The set of ~~rat~~ integers $\mathbb{Z} = \{---2, -1, 0, 1, 2, ---\}$.
- The set of rational numbers $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$.
- The set of irrational numbers $\mathbb{Q}^c$ $(\sqrt{2}, \sqrt{3}, \frac{1+\sqrt{5}}{2}, \pi, e, \text{etc}-)$

Now the question is

Can we have a number system without these gaps?

Yes, the complete number system is the real line $\mathbb{R}$.

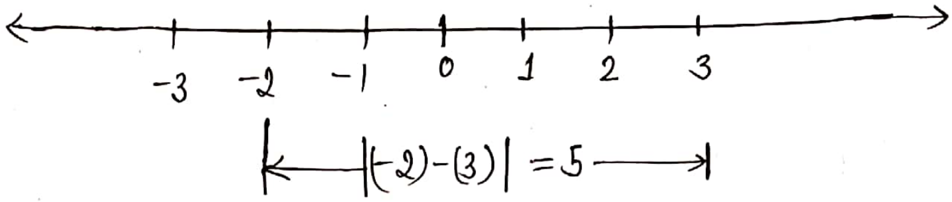
$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ = The set of rational and irrational numbers



The real line: A convenient and familiar interpretation of the real number system is the real line $\mathbb{R}$. In this interpretation, the absolute value $|a|$ of an element a in $\mathbb{R}$ is regarded as the distance from a to the origin 0.

i.e $|a|$ = distance from a to the origin 0.

In general, the distance between elements a and b in $\mathbb{R}$ is $|a-b|$.



The distance between $a = -2$ and $b = 3$.

Later, we will need precise language to discuss the notion of one real number "close to" another.

If 'a' is a given real number, then we say that a real number x is "close to" a should mean that the distance $|x-a|$ between x and a is "small".

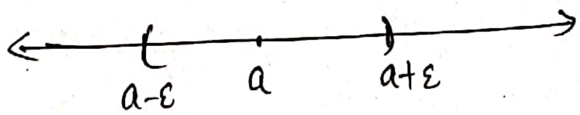
i.e. $|x-a| < \varepsilon$ for $\varepsilon > 0$.

$\varepsilon \rightarrow$ a very small quantity near to zero.

ε-neighbourhood of a point:

Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then the ε-neighbourhood of a is defined as

$$(a-\varepsilon, a+\varepsilon) = \{x : a-\varepsilon < x < a+\varepsilon\}.$$



An ε- neighbourhood of a.

Or, in other words, the ε- neighbourhood of a is the set

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x-a| < \varepsilon\}$$

$$V_\varepsilon(a) = \{x \in \mathbb{R} \mid x \in (a-\varepsilon, a+\varepsilon)\}.$$

$$\Rightarrow V_\varepsilon(a) = (a-\varepsilon, a+\varepsilon).$$

Intervals:

Open Interval: If $a, b \in \mathbb{R}$ satisfy $a<b$, then the open interval determined by a and b is the set

$$(a,b) = \{x \in \mathbb{R} : a < x < b\}.$$

The points a and b are called the endpoints of the interval but the endpoints are not included in an open interval.

Closed Interval: If both endpoints are adjoined to the above open interval, then we obtain the closed interval determined by a and b; namely, the set

$$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

Half Open Interval (Half Closed Interval):

$$[a,b) = \{x \in \mathbb{R} : a \leq x < b\}.$$

$$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

Note: (i) length of each of the four intervals (a,b) $[a,b]$, $[a,b)$, $(a,b]$ is $b-a$.

(ii) If $a=b$, then $(a,a) = \phi$ and $[a,a] = \{a\}$.

Infinite Open Intervals:

$$(a, \infty) = \{x \in \mathbb{R} : x > a\} \quad \text{and} \quad (-\infty, b) = \{x \in \mathbb{R} : x < b\}.$$

Infinite Closed Intervals:

$$[a, \infty) = \{x \in \mathbb{R} : x \geq a\} \quad \text{and} \quad (-\infty, b) = \{x \in \mathbb{R} : x \leq b\}.$$

Note: Thus $\mathbb{R} = (-\infty, \infty)$.

Maximum and Minimum of a Set:

Let S be a non-empty set of $\mathbb{R}$. Then we give the following definitions:

Maximum of a set: If S contains a largest element s^0, then we call s^0 the maximum of S.

Minimum of a set: If S contains a smallest element s_0, then we call s_0 the minimum of S.

Example: In the closed interval $S = [1, 2]$,

$$\max . S = \max [1, 2] = 2$$

$$\min . S = \min . [1, 2] = 1.$$

But if $S = (1, 2)$,

then $\max . S$ and $\min . S$ donot exist.

Bounded above and Bounded below:

Bounded above: A non-empty set $S \subseteq \mathbb{R}$ is said to be bounded above if there is an element $k^0 \in \mathbb{R}$

such that $x \leq K \quad \forall \; x \in S$.

The number K is called an upper bound of S.

If no such K exists, the set is said to be not bounded above.

Bounded below: The set S is said to be bounded below if $\exists$ a real number k such that

$$k \leq x \quad \forall \; x \in S.$$

The number k is called the lower bound of S.

If no such k exists, the set is said to be not bounded below.

Bounded Set: A set is said to be bounded if it is bounded above as well as bounded below.

i.e $\exists \; k, K \in \mathbb{R}$ such that

$$k \leq x \leq K \quad \forall \; x \in S.$$

Examples: (i) The set $S = \{x \in \mathbb{R} : x < 2\}$ is bounded above; the number 2 and any number larger than 2 is an upper bound of S. This set has no lower bounds. So, the set S is not bounded below.

Thus it is unbounded.

(ii) Let $S = (1, 2)$.

Upper bounds of $S = [2, \infty)$

Lower bounds of $S = (-\infty, 1]$

Thus the set S is bounded (as it is bounded below as well as bounded above)

(iii) The set $\mathbb{N}$ of natural numbers is bounded below but not bounded above.

Here 1 is a lower bound.

(iv) The sets $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are not bounded.

(v) Every finite set of numbers is bounded.

Supremum and infimum of a Set :

Let S be a nonempty subset of $\mathbb{R}$.

Supremum of S (Least Upper bound) : If S is bounded above, then a number K is said to be supremum (or a least upper bound) of S if it satisfies the conditions :

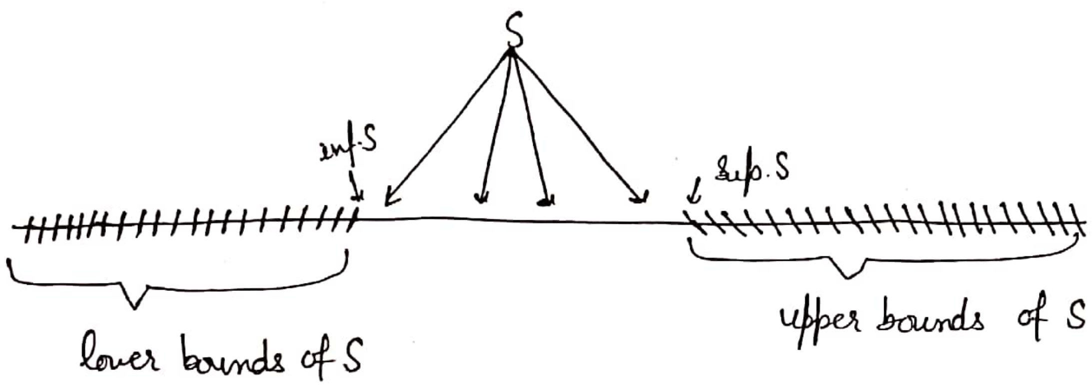
(1) K is an upper bound of S, and

(2) if K' is any upper bound of S, then $K \leq K'$.

Infimum of S (Greatest Lower bound) : If S is bounded below, then a number k is said to be infimum (or greatest lower bound) of S if it satisfies the conditions :

(1) k is a lower bound of S, and

(2) If k' is any lower bound of S, then $k' \leq k$.



inf.S and sup.S.

If the supremum or the infimum of a set S exists, we will denote them by

sup.S and inf.S.

Note: (i) Unlike maximum and minimum, sup.S and inf.S may not belong to the set.

(ii) There can be only one supremum (or infimum) of a given subset S of $\mathbb{R}$.

Examples: (i) $S = [1, 3, 5, 7, 9]$

$\sup.S = 9, \quad \inf S = 1$

[As set of upper bounds of $S = [9, \infty)$
and set of lower bounds of $S = (-\infty, 1]$
$\Rightarrow$ least upper bound $(\sup.(S)) = 9$
and greatest lower bound $(\inf.(S)) = 1$]

$\Rightarrow$ sup.S and inf S belong to the set S.

(ii) $S = \{x : x > 0, x \in \mathbb{R}\}$

Here $\inf.(S) = 0$ but $\sup.(S)$ doesn't exist.

inf.(S) doesnot belong to the set S.

(iii) The infinite set $S = \{x : 0 \leq x \leq 1, x \in \mathbb{Q}\}$ is bounded with supremum 1 and infimum 0.

sup.(S) and inf.(S) both belong to the set S.

(iv) Consider the set

$$S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \text{---} \right\}$$

which is bounded.

Here sup.(S) $= 1 \notin S$ but inf.(S) $= 0 \notin S$.

(v) Each of the following intervals is bounded:

$$[a, b], \; (a, b], \; [a, b), \; (a, b).$$

Completeness Property:

Least Upper bound property: Every non-empty subset S of $\mathbb{R}$ which is bounded above has a least upper bound.

ie sup.S exists and is a real number.

Greatest lower bound property: Every non-empty subset S of $\mathbb{R}$ which is bounded below has a greatest lower bound ie inf.S exists and is a real number.

Archimedean Property:

For each $x \in \mathbb{R}$, $\exists$ a natural number n (depending on x) such that $n > x$.