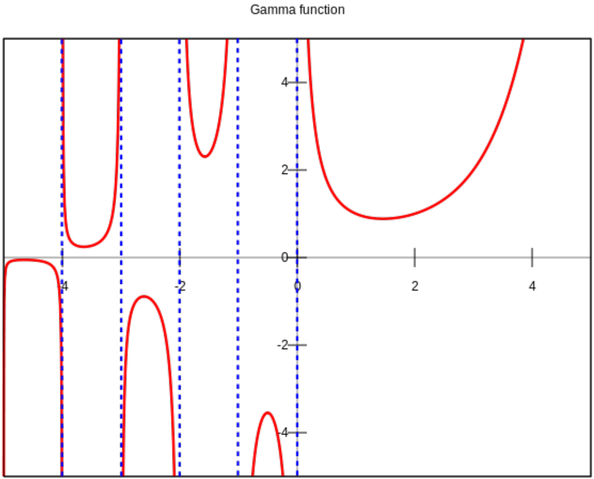
Gamma and Beta Functions

Gamma function

- introduced by Swiss mathematician *Leonhard Euler* (1729)
- generalization of factorial function to non-integer values (more specifically, to all complex numbers except the non-positive integers)
- A translated version of factorial function is the following recurrence relation:
  1. $f(1) = 1$,
  2. $f(x+1) = xf(x)$.

Leonhard Euler (1707–1783)



- For $p > 0$, the gamma function, $\boxed{\Gamma(p) := \int_0^\infty x^{p-1} e^{-x} dx.}$

Is gamma function convergent?

$$\Gamma(p) = \int_0^1 x^{p-1} e^{-x} dx + \int_1^\infty x^{p-1} e^{-x} dx$$
$$= I_1 + I_2$$

To see the convergence of I_1 (improper integral of second kind), we take $f(x) = x^{p-1}e^{-x}$ and $g(x) = x^{p-1}$, then $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$ and $\int_0^1 x^{p-1} dx$ converges.

To see the convergence of I_2 (improper integral of first kind), take $f(x) = x^{p-1}e^{-x}$ and $g(x) = \frac{1}{x^2}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} x^{2+p-1}e^{-x} = 0$ and $\int_1^\infty \frac{1}{x^2} dx$ converges. Hence by (2) of limit comparison theorem, the integral converges.

Some properties of Gamma function

- $\Gamma(1) = \int_0^\infty e^{-x} dx = 1.$
- $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$

  Integration by parts formula implies,

  $$\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x} dx = -(x^\alpha e^{-x})|_0^\infty + \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx = \alpha\Gamma(\alpha).$$

  Therefore, $\Gamma(n+1) = n! \qquad \forall n \in \mathbb{N}.$
- Euler's reflection formula for $p \notin \mathbb{Z}$

  $$\boxed{\Gamma(p) \cdot \Gamma(1-p) = \tfrac{\pi}{\sin(\pi p)}.}$$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Hint: choose $p = \frac{1}{2}$ in the above formula.

Beta functions

- For $p, q > 0$, beta function, $\boxed{\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.}$
- If $p > 1$ and $q > 1$, then the integral is definite integral. When $p < 1$ and/or $q < 1$, this integral is improper of second kind at 0 and/or 1.
- To prove the convergence, we divide as before

$$\int_0^1 x^{p-1}(1-x)^{q-1}dx = \int_0^{1/2} x^{p-1}(1-x)^{q-1}dx + \int_{1/2}^1 x^{p-1}(1-x)^{q-1}d$$

$$= I_1 + I_2.$$

To see the convergence of I_1, take $f(x) = x^{p-1}(1-x)^{q-1}$ and $g(x) = x^{p-1}$. Then $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0}(1-x)^{q-1} = 1$ and $\int_0^{1/2} x^{p-1}dx$ converges. Similarly, for convergence of I_2, we take $f(x) = x^{p-1}(1-x)^{q-1}$ and $g(x) = (1-x)^{q-1}$.

Some properties of beta function

- $\beta(p,q) = \beta(q,p)$.

  Hint: Substitute $t = 1 - x$.

- $$\beta(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta d\theta$$

  .

  Hint: Substitute $x = \sin^2\theta$ in $\beta(p,q)$.

- Relationship between gamma and beta functions

  $$\boxed{\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.}$$

Thank You.