Electric Potential

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$$\vec{\nabla} \times \vec{E} = 0 \quad \text{(in Electrostatics)}$$

$$\oint \vec{E} \cdot d\vec{l} = 0$$

$\Rightarrow$ Line integral of $\vec{E}$ is independent of path.

$$V(r) = -\int_{O}^{r} \vec{E} \cdot d\vec{l}$$

$\hookrightarrow$ Electric potential

$O \equiv$ Reference point (usually taken at ∞)

Potential difference betn. point 'b' & point 'a':

$$V(b) - V(a) = -\int_{O}^{b} \vec{E} \cdot d\vec{l} + \int_{O}^{a} \vec{E} \cdot d\vec{l}$$

$$= -\int_{O}^{b} \vec{E} \cdot d\vec{l} - \int_{a}^{O} \vec{E} \cdot d\vec{l}$$

$$= -\int_{a}^{b} \vec{E} \cdot d\vec{l}$$

(*) From fundamental theorem for gradients:

$$V(b) - V(a) = \int_{a}^{b} (\vec{\nabla} V) \cdot d\vec{l}$$

$$\Rightarrow \int_a^b (\vec{\nabla} V)\cdot d\vec{l} = -\int_a^b \vec{E}\cdot d\vec{l}$$

= True for any two points 'a' & 'b'.

$$\Rightarrow \underline{\vec{E} = -\vec{\nabla} V}$$

Ref. point: $\mathcal{O}$ usually at ∞

$$V(r) = -\int_{\mathcal{O}}^{r} \vec{E}\cdot d\vec{l}$$

$$V'(r) = -\int_{\mathcal{O}'}^{r} \vec{E}\cdot d\vec{l}$$

$$= -\int_{\mathcal{O}'}^{\mathcal{O}} \vec{E}\cdot d\vec{l} - \int_{\mathcal{O}}^{r} \vec{E}\cdot d\vec{l}$$

$$= K + V(r)$$

$$\left.\begin{array}{l} V'(b) - V'(a) = V(b) - V(a) \\ \vec{\nabla} V' = \vec{\nabla} V \end{array}\right\}$$

⊛ Potential obeys superposition principle:

for a collection of charges

$$V = V_1 + V_2 + V_3 + \cdots$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots$$

<u>Ex:</u> Potential inside & outside a spherical shell carrying a uniform charge density

Ref. point at ∞.

From Gauss' law:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

For points inside: $\vec{E} = 0$

<u>$r > R$</u>

$$V(r) = -\int_0^r \vec{E} \cdot d\vec{l} = -\frac{1}{4\pi\epsilon_0} \int_\infty^r \frac{q}{r'^2} \, dr'$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

<u>$r < R$</u>

$$V(r) = -\frac{1}{4\pi\epsilon_0} \int_\infty^R \frac{q}{r'^2} \, dr' - \int_R^r (0) \, dr'$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{R}$$

Poisson's eq. & Laplace eq:

$$\vec{E} = -\vec{\nabla} V$$

$$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (-\vec{\nabla} V) = -\nabla^2 V = \rho / \epsilon_0$$

$$\Rightarrow \underline{\nabla^2 V = \rho / \epsilon_0} \equiv \text{Poisson's eq.}$$

if $\rho = 0$, $\Rightarrow \nabla^2 V = 0 \equiv$ Laplace's eq.

Potential of a localised charge distribution:

Point charge 'q' at origin

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\Rightarrow \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr$$

$$V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l} = -\frac{1}{4\pi\epsilon_0} \int_{\infty}^{r} \frac{q}{r'^2} dr' = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

if '$\mathscr{r}$' is your distance from 'q' to 'r'

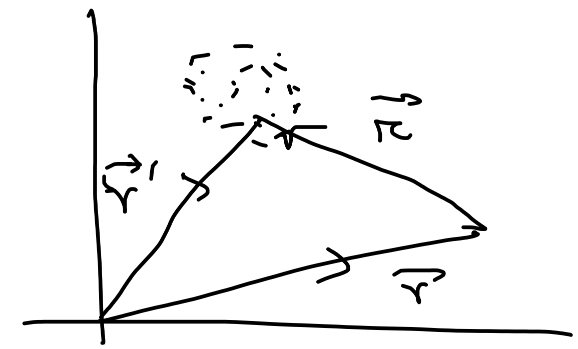
$$\Rightarrow V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

From superposition principle,

$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$$

for continuous ch. distribution,

$$V(r) = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{\mathscr{r}}$$

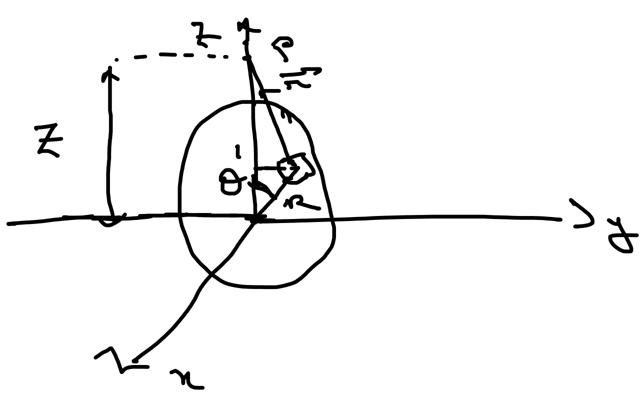


for vol. ch : $dq = \rho(\vec{r}')\,d\tau'$

for sur. ch : $dq = \sigma(\vec{r}')\,da'$

for linear ch : $dq = \lambda(\vec{r}')\,dl'$

Ex: Potential of a uniformly charged spherical shell of radius R. ($\sigma \equiv$ surface ch. distn.)



$$\mathscr{r}^2 = (R^2 + z^2 - 2Rz\cos\theta')$$

$$V(r) = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma}{\mathscr{r}}\,da'$$

$$da' = R^2\sin\theta'\,d\theta'\,d\phi$$

$$4\pi\epsilon_0 V(z) = \sigma \int_0^{\pi}\int_0^{2\pi} \frac{R^2 \sin\theta' \, d\theta' \, d\phi}{[R^2 + z^2 - 2Rz\cos\theta]^{1/2}}$$

$$= \frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right]$$

for $z > R$, $\quad V(z) = \frac{R\sigma}{2\epsilon_0 z}\left((R+z) - (z-R)\right)$

$$= \frac{R^2\sigma}{\epsilon_0 z}$$

for $z < R$, $\quad V(z) = \frac{R\sigma}{2\epsilon_0 z}\left((R+z) - (R-z)\right)$

$$= \frac{R\sigma}{\epsilon_0 z}.$$

Here $q = 4\pi R^2\sigma$

$$\Rightarrow V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \quad (r \geq R)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{R} \quad (r \leq R)$$