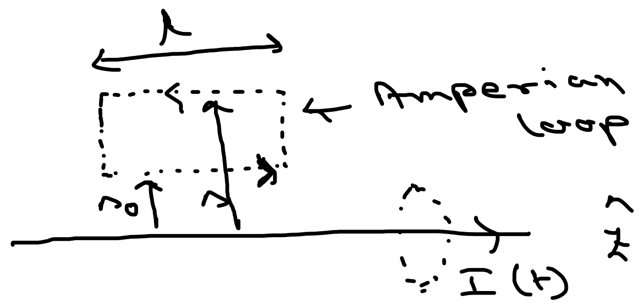
16.02.21

Ex:



An infinitely long straight wire carries a slowly varying current, $I(t)$. Determine the induced $\vec{E}$.

→ Magnetic field due to a current carrying wire at a distance 's' $= \dfrac{\mu_0 I}{2\pi s}$

$\vec{E}$ will be parallel to the axis.

For the Amperian loop,

$$\oint \vec{E} \cdot d\vec{l} = E(s_0)\, l - E(s)\, l$$

$$= -\frac{d}{dt}\int \vec{B} \cdot d\vec{a}$$

$$= -\frac{\mu_0}{2\pi}\, l\, \frac{dI}{dt}\int_{s_0}^{s} \frac{1}{s'}\, ds'$$

$$= -\frac{\mu_0 l}{2\pi}\frac{dI}{dt}\left(\ln(s) - \ln(s_0)\right)$$

$$\Rightarrow \vec{E}(s) = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln(s) + K\right]\hat{z}$$

K ↓ a const. (indep. of 's')

Maxwell's eq:

$\boxed{\text{Section } 7\text{-}3.1}$

We have seen

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

(*) Divergence of a curl of any vector is zero

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \cdot \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{B})$$

$$= 0 \quad \rightarrow \text{consistent}$$

But,

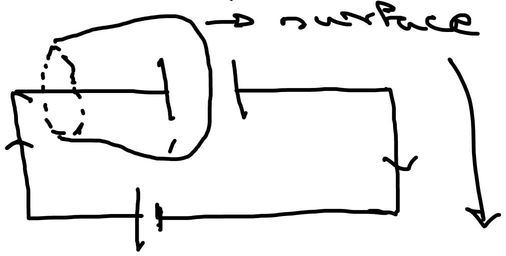
$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 (\vec{\nabla} \cdot \vec{J})$$

$$\neq 0$$

[$\vec{\nabla} \cdot \vec{J} = 0$ for steady currents, but in general, not]

→ Ampere's law works in magnetostatics. Does not work for non-steady currents.

Ex: Suppose, we're charging up a capacitor.



ⓐ Ampere's law is independent of shape & size of the surface

$I_{enc} = 0$

→ The current intercepted by an arbitrary surface now depends on the chosen surface

ⓑ The problem arises when there is charge piling up somewhere.

$\boxed{\text{Section 7.3.2}}$

To fix Ampere's law:

Look back at $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Applying continuity eq:

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$= -\frac{\partial}{\partial t}\left(\epsilon_0 \vec{\nabla} \cdot \vec{E}\right)$$

$$= -\vec{\nabla} \cdot \left(\epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

Hence, if we combine $\epsilon_0 \frac{\partial \vec{E}}{\partial t}$ with $\vec{J}$, we can get rid of the non-zero contribution

Now, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

↪ continuity eq. is also revived.

⊛ in magnetostatics, $\frac{\partial \vec{E}}{\partial t} = 0$

$\Rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ → consistent.

⊛ Just as a changing magnetic field gives rise to an electric field, a changing electric field also gives rise to a magnetic field.

- Maxwell termed this extra term "Displacement current"

$$\vec{J}_d \equiv \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

⊛ A look back to the capacitor problem.

The electric field in between the plates: $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A \epsilon_0}$

$$\Rightarrow \frac{\partial E}{\partial t} = \frac{1}{A \epsilon_0} \frac{dQ}{dt} = \frac{I}{A \epsilon_0}$$

Now, $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc.} + \mu_0 \epsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$

⊕ for two different types of surfaces:

$\rightarrow E = 0$ & $I_{enc.} = I$ (conduction current)

$\rightarrow I_{enc.} = 0$ & $\int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$

$= \frac{I}{\epsilon_0}$ (Displacement current)

$\Rightarrow$ Same answer for either surface.

Maxwell's eq:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Together with Lorentz force law,

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$\Rightarrow$ Sumarises the entire theoretical content of classical electrodynamics.