Maxima and Minima

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TIMES OF INDIA GROUP

The **second partial derivative test** is a method in multivariable calculus used to determine if a critical point of a function is a local minimum, maximum or saddle point.

Hessian matrix

- Suppose that $f(x,y)$ is a differentiable real function of two variables whose second partial derivatives exist and are continuous. Then the Hessian matrix H of f is the 2×2 matrix of partial derivatives of f:

$$H(x,y) = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix}.$$

- Define $D(x,y)$ to be the determinant

$$D(x,y) = \det(H(x,y)) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2,$$ of H.

Second partial derivative test

Suppose that (a, b) is a **critical point** of f (that is, $f_x(a, b) = f_y(a, b) = 0$).

Then the second partial derivative test asserts the following:

S.No.	Condition	Nature
1	$D(a, b) > 0, \quad f_{xx}(a, b) > 0$	local minimum
2	$D(a, b) > 0, \quad f_{xx}(a, b) < 0$	local maximum
3	$D(a, b) < 0$	Saddle point
4	$D(a, b) = 0$	No conclusion

Example

Consider the function $f(x,y) = (x+y)(xy+xy^2)$. Then

$$\frac{\partial f}{\partial x} = y(2x+y)(y+1), \quad \frac{\partial f}{\partial y} = x\left(3y^2 + 2y(x+1) + x\right).$$

Then we have the following four critical points:

$$(0,0), (0,-1), (1,-1) \text{ and } \left(\frac{3}{8}, -\frac{3}{4}\right).$$

In order to classify the critical points, we examine the value of the determinant $D(x,y)$ of the Hessian of f at each of the four critical points. We have

$$\begin{aligned} D(a,b) &= f_{xx}(a,b) f_{yy}(a,b) - (f_{xy}(a,b))^2 \\ &= 2b(b+1) \cdot 2a(a+3b+1) - (2a+2b+4ab+3b^2)^2. \end{aligned}$$

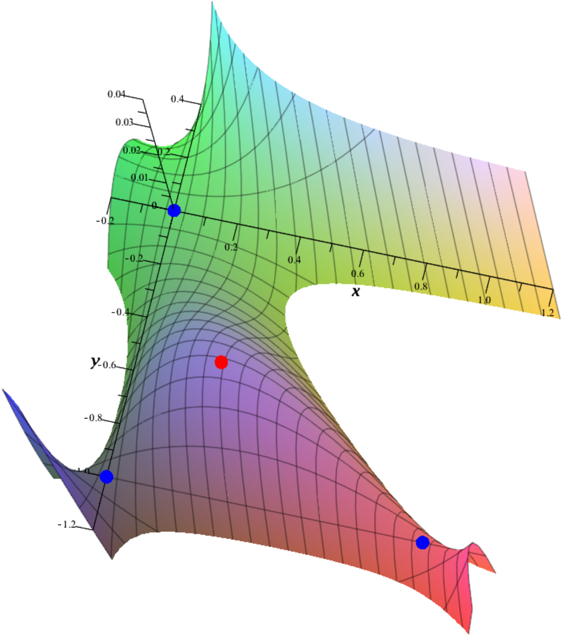
Now plugging in all the different critical values, we have

$$D(0,0)=0;\quad D(0,-1)=-1;\quad D(1,-1)=-1;\quad D\left(\frac{3}{8},-\frac{3}{4}\right)=\frac{27}{128}.$$

Thus, the second partial derivative test indicates that $f(x,y)$ has saddle points at $(0,-1)$ and $(1,-1)$ and has a local maximum at $\left(\frac{3}{8},-\frac{3}{4}\right)$ since $f_{xx}=-\frac{3}{8}<0$. At the remaining critical point $(0,0)$ the second derivative test is insufficient, and one must use higher order tests or other tools to determine the behavior of the function at this point.

(In fact, observe that f takes both positive and negative values in small neighborhoods around $(0,0)$ and so this point is a saddle point of f.)

Critical points of $f(x,y) = (x + y)(xy + xy^2)$ maxima (red) and saddle points (blue).



https://en.wikipedia.org/wiki/Second_partial_derivative_test

Example 2

Ques: Find all the critical points and their nature of $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$.

Ans:

$$f_x = y - 2x - 2 = 0, \;\; f_y = x - 2y - 2 = 0$$

Therefore, the point $(-2, -2)$ is the only critical point. Also

$$f_{xx} = -2, \;\; f_{yy} = -2, \;\; f_{xy} = 1.$$

Therefore, $D(-2, -2) = 3 > 0$ and $f_{xx}(-2, 2) = -2 < 0$. Therefore, $(-2, -2)$ is a point of local maximum.

Global/Absolute maxima and Minima on closed and bounded domains

1. Find all critical points of $f(x, y)$. These are the interior points where partial derivatives can be defined.
2. Restrict the function to the each piece of the boundary. This will be one variable function defined on closed interval I(say) and use the derivative test of one variable calculus to find the critical points that lie in the open interval and their nature.
3. Find the end points of these intervals I and evaluate $f(x, y)$ at these points.
4. The global/Absolute maximum will be the maximum of f among all these points.
5. Similarly for global minimum.

Example: Find the absolute maxima and minima of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines

$$x = 0, y = 0, y = 9 - x.$$

Solution: $f_x = 2 - 2x = 0$, $f_y = 2 - 2y = 0$ implies that $(1, 1)$ is the only critical point and $f(1, 1) = 4$. $f_{xx} = -2, f_{yy} = -2, f_{xy} = 0$. Therefore, $D(1, 1) = 4 > 0$ and $A < 0$. So this is local maximum.

Continue...

Case 1: <u>On the segment $y = 0$</u>, $f(x, y) = f(x, 0) = 2 + 2x - x^2$ defined on $I = [0, 9]$. $f(0, 0) = 2, f(9, 0) = -61$ and the interior points where $f'(x, 0) = 2 - 2x = 0$ is $x = 1$. So $x = 1$ is the only critical point and $f(1, 0) = 3$.

Case 2: <u>On the segment $x = 0$</u>, $f(0, y) = 2+2y-y^2$ and $f'(0, y) = 2-2y = 0$ implies $y = 1$ and $f(0, 1) = 3$.

Case 3: <u>On the segment $y = 9 - x$</u>, we have $f(x, 9 - x) = -61 + 18x - 2x^2$ and the critical point is $x = 9/2$. At this point $f(9/2, 9/2) = -41/2$.

Finally, $f(0, 0) = 2, f(9, 0) = f(0, 9) = -61$. so the global maximum is 4 at $(1, 1)$ and minimum is -61 at $(9, 0)$ and $(0, 9)$.

Thank You.