# Continuity

Engineering Calculus

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### Definition

A real valued function $f(x)$ is said to be continuous at $x = c$ if

(i) $f(c)$ is defined,

(ii) $\lim_{x \to c} f(x)$ exists,

(iii) $\lim_{x \to c} f(x) = f(c)$.

### Example

Show that $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at 0.

**Solution:** Let $\epsilon > 0$. Then $|f(x) - f(0)| \leq |x^2|$. So it is enough to choose $\delta = \sqrt{\epsilon}$.

## Theorem (Sequential criteria of continuity)

A function $f$ is continuous at $c$ if and only if for every sequence $x_n \to c$, we must have $f(x_n) \to f(c)$ as $n \to \infty$.

## Example

Show that $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at 0.

**Solution:** We note that $|f(x)| \leq |x^2|$. Therefore, $f(x_n) \to f(0)$ whenever $x_n \to 0$. This proves that $f$ is continuous at $x = 0$.

## Example

Show that $f(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous at 0.

**Solution:** Choose $\frac{1}{x_n} = \frac{\pi}{2} + 2n\pi$. Then $\lim\limits_{n\to\infty} x_n = 0$ and $f(x_n) = \frac{1}{x_n} \to \infty$.

### Theorem

Suppose $f$ and $g$ are continuous at $c$. Then

1. $f \pm g$ is also continuous at $c$.
2. $fg$ is continuous at $c$.
3. $\dfrac{f}{g}$ is continuous at $c$ if $g(c) \neq 0$.
4. $|f|$ is also continuous at $c$ and $\lim_{x \to c} |f(x)| = |f(c)|$.

- A function which is not continuous is called discontinuous function.

## Theorem

Let $f(x)$ be a continuous function on $\mathbb{R}$ and let $f(a)f(b) < 0$ for some $a, b$. Then there exits $c \in (a, b)$ such that $f(c) = 0$.

## Example

Show that $f(x) = x^2 - 2$ has at least one root in $(1, 2)$.

## Intermediate Value Theorem

Let $f(x)$ be a continuous function on $[a,b]$ and let $f(a) < y < f(b)$. Then there exists $c \in (a,b)$ such that $f(c) = y$.

## Remark

From the IVT, we can conclude that A continuous function assumes all values between its maximum and minimum.

## Fixed point theorem

Let $f(x)$ be a continuous function from $[0,1]$ into $[0,1]$. Then show that there is a point $c \in [0,1]$ such that $f(c) = c$.


Thank You