Functions of Several Variables

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Functions of several variables?

- functions which has several input variables and one or more output variables
- For example, the following are Real valued functions of two variables x, y:
  1. $f(x, y) = x^2 + y^2$ is a real valued function defined over $\mathbb{R}^2$.
  2. $f(x, y) = \frac{xy}{x^2+y^2}$ is a real valued function defined over $\mathbb{R}^2 \setminus \{(0, 0)\}$

Some applications for motivation

- Temperature distribution in a medium is a real valued function with more than 2 variables. The temperature function at time t and at point (x, y) has 3 variables.

  For example, the temperature distribution in a plate, (unit square) with zero temperature at the edges and initial temperature (at time $t = 0$) $T_0(x, y) = \sin \pi x \sin \pi y$, is $T(t, x, y) = e^{-\pi^2 kt} \sin \pi x \sin \pi y$.

- Sound waves and water waves problems in Physics.

  The function $u(x, t) = A \sin(kx - \omega t)$ represents the traveling wave of the initial wave front $\sin kx$.

- Optimal cost functions.

  For example a manufacturing company wants to optimize the resources, for their produce, like man power, capital expenditure, raw materials etc. The cost function depends on these variables.

Some useful definitions

Let $\mathbb{R}^2$ denote the set of all points $(x, y) : x, y \in \mathbb{R}$. The open ball of radius r with center (x_0, y_0) is denoted by

$$B_r((x_0, y_0)) = \{(x, y) : \sqrt{(x - x_0)^2 + (y - y_0)^2} < r\}.$$

1. A point (a, b) is said to be **interior point** of a subset S of $\mathbb{R}^2$ if there exists r such that $B_r((a, b)) \subset S$.
2. A subset S is called **open** if each point of S is an interior point of S.
3. A subset S is said to be **closed** if its compliment is an open subset of $\mathbb{R}^2$.

Examples

1. The open ball of radius δ: $B_\delta((0,0)) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \delta\}$ is an open set.
2. Union of open balls is also an open set.
3. The closed ball of radius r : $\overline{B_r((0,0))} = \{(x,y) : \sqrt{x^2+y^2} \leq r\}$ is closed.

Limit of a function of several variables

Definition of limit of a function (Simultaneous/Double limit)

Let Ω be an open set in $\mathbb{R}^2$, $(a, b) \in \Omega$ and let f be a real valued function defined on Ω except possibly at (a, b). Then the limit

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x, y) - L| < \epsilon.$$

Note: If limit exists, then it is unique. That is, the limit is independent of choice of path chosen $(x, y) \to (a, b)$.

Some examples of functions where limit do NOT exist

Example 1: Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist for $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$.

Answer: We'll show that the limits along the x and y axes are different, thus limit cannot exist.

Along x axis, $y = 0$. So $f(x,y) = f(x,0) = \frac{x^2-0}{x^2+0}$.

$$\implies \lim_{x\to 0} f(x,0) = 1.$$

Along y axis, $x = 0$. So $f(x,y) = f(0,y) = \frac{0-y^2}{0+y^2}$.

$$\implies \lim_{y\to 0} f(0,y) = -1.$$

Therefore, the limit does not exist.

An other approach for the previous example

Along any arbitrary line $y = mx$. So $f(x,y) = f(x,mx) = \frac{x^2 - m^2x^2}{x^2 + m^2x^2}$.

$$\implies \lim_{x \to 0} f(x,mx) = \frac{1 - m^2}{1 + m^2}.$$

For different values of m, we have different limits, so limit does NOT exist.

Example 2: Does the limit of $f(x, y) = \frac{xy^2}{x^2+y^4}$ as $(x, y) \to (0, 0)$ exist, and if yes, then what is the value?

Answer:

$$\lim_{x \to 0} f(x, 0) = 0 = \lim_{y \to 0} f(0, y).$$

Is this enough to say limit exists and is equal to 0?

Then there are infinitely many straight lines passing through origin. We can approach through those line, right!

Then,

$$\lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{x(mx)^2}{x^2 + (mx)^4} = \lim_{x \to 0} \frac{m^2 x}{1 + m^4 x^2} = 0.$$

Is this now enough to say limit exists and is equal to 0?

There exist still infinitely many curved paths to approach the point $(0, 0)$.

continue...

Now consider,

For any arbitrary m along the parabola $x = my^2$.

Observe that

$$\lim_{y \to 0} f(my^2, y) = \lim_{y \to 0} \frac{my^4}{m^2y^4 + y^4} = \frac{m}{m^2 + 1},$$

which is different for different values of m.

Therefore, limit of the function $f(x, y) = \frac{xy^2}{x^2 + y^4}$ does NOT exist at $(0, 0)$.

Using the ϵ, δ definition to prove existence of a limit

Example 3: Prove that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, where $f(x,y) = \frac{3xy^2}{x^2+y^2}$.

Why should we expect that this limit exist?

Hint: The numerator is cubic, and the denominator quadratic, so we can guess who should win in a long run.

Proof: $|f(x,y) - 0| < \epsilon \implies \left|\frac{3xy^2}{x^2+y^2}\right| = 3|x|\frac{y^2}{x^2+y^2} < \epsilon.$

From the following inequalities

$$x^2 \leq x^2 + y^2, \quad \text{and } 0 \leq \frac{y^2}{x^2+y^2} \leq 1,$$

we have

$$3|x|\frac{y^2}{x^2+y^2} \leq 3|x| = 3\sqrt{x^2} \leq 3\sqrt{x^2+y^2}.$$

continue...

Now choose $\delta = \frac{\epsilon}{3}$.

So we have now whenever $\sqrt{x^2 + y^2} < \delta = \frac{\epsilon}{3}$, the inequality $3\sqrt{x^2 + y^2} < \epsilon$ holds.

Meaning,

$$|f(x, y) - 0| = 3|x|\frac{y^2}{x^2 + y^2} \leq 3\sqrt{x^2 + y^2} < \epsilon.$$

This proves that

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0.$$

Repeated/Iterated limit(s) of a function $f(x, y)$ at (a, b)

$$\lim_{x \to a} \lim_{y \to b} f(x, y)$$

and

$$\lim_{y \to b} \lim_{x \to a} f(x, y).$$

Example: Consider the function

$$f(x, y) = \frac{x^2}{x^2 + y^2}.$$

The repeated roots are given by

$$\lim_{y \to 0} \left(\lim_{x \to 0} \frac{x^2}{x^2 + y^2} \right) = \lim_{y \to 0} 0 = 0,$$

$$\lim_{x \to 0} \left(\lim_{y \to 0} \frac{x^2}{x^2 + y^2} \right) = \lim_{x \to 0} 1 = 1.$$

Example

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy}{x^2+y^2}\right)=\lim_{y\to 0}0=0,\qquad \lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy}{x^2+y^2}\right)=\lim_{x\to 0}0=0.$$

Now, they are equal!

But what about the simultaneaous limit?

Limit as $(x,y)\to(0,0)$ <u>along the line $y=mx$</u>:

$$\lim_{\substack{(x,y)\to(0,0),\\ y=mx}}\frac{xy}{x^2+y^2}=\lim_{x\to 0}\frac{mx^2}{x^2+m^2x^2}=\frac{m}{1+m^2},$$

which is different for different values of m.

So,

$$\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}\text{ does NOT exist.}$$

Results on simultaneous and repeated limits

- Repeated limits exists does not imply that simultaneous limit will exist.
- If repeated limits are not equal, then the simultaneous limit would not exist.
- Simultaneous limit exist does not imply that repeated limits also exist, but if they exist, all will be equal.

  Consider $f(x,y) = x\sin\frac{1}{y} + y\sin\frac{1}{x}, \quad xy \neq 0$. Repeated limits do not exist. On the contrary, simultaneous limit exists.

  Let $\epsilon > 0$ be given. We have to find $\delta > 0$ such that

$$\sqrt{x^2+y^2} < \delta \implies \left| x\sin\frac{1}{y} + y\sin\frac{1}{x} - 0 \right| < \epsilon.$$

  Consider $\left| x\sin\frac{1}{y} + y\sin\frac{1}{x} \right| < |x| + |y| \leq 2\sqrt{x^2+y^2} < 2\delta = \epsilon$.
  Now, choose $\epsilon = \delta$.

Continuity of a function of several variables

Definition

Let f be a real valued function defined in a ball around (a, b). Then f is said to be continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

Examples

1. The function

$$f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x=y=0.\end{cases}$$

   Let $\epsilon>0$. Then $|f(x,y)-0|=|x|\frac{|y|}{\sqrt{x^2+y^2}}\leq |x|$.
   So if we choose $\delta=\epsilon$, then $|f(x,y)|\leq\epsilon$. Therefore, f is continuous at $(0,0)$.

2. Let

$$f(x,y)=\begin{cases}x^2+2y & (x,y)\neq(1,2)\\ 3 & (x,y)=(1,2)\end{cases}$$

   Then f is not continuous at $(1,2)$, since

$$\lim_{(x,y)\to(1,2)} x^2+2y=5\neq f(1,2).$$

THANK YOU.