Tutorial Set-2 (EPHY105L)

1. Express the unit vectors $(\hat{r}, \hat{\theta}, \hat{\phi})$ in spherical polar coordinates in terms of the unit vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$ in the Cartesian coordinate system. Invert these equations to express the unit vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$ in the Cartesian coordinate system in terms of the unit vectors $(\hat{r}, \hat{\theta}, \hat{\phi})$ in the spherical polar coordinate system.
2. Express the unit vectors $(\hat{r}, \hat{\phi}, \hat{z})$ in cylindrical coordinates in terms of the unit vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$ in the Cartesian coordinate system. Invert these equations to express the unit vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$ in the Cartesian coordinate system in terms of the unit vectors $(\hat{r}, \hat{\phi}, \hat{z})$ in the cylindrical coordinate system.
3. Express the following points given in Cartesian coordinates $(\hat{\imath}, \hat{\jmath}, \hat{k})$ in the spherical polar coordinate system $(\hat{r}, \hat{\theta}, \hat{\phi})$ (all values in meters):
   a) $x = 10; y = 0, z = 0$
   b) $x = 0; y = 0, z = 5$
   c) $x = 5; y = 2, z = 0$
   d) $x = 0; y = 3; z = 3$
   Express the unit vector $\hat{r}$ in terms of the Cartesian unit vectors at the above points. Notice that the direction of unit vector in spherical polar coordinates depends on the coordinates of the point.
4. Express the following points given in spherical polar coordinates $(\hat{r}, \hat{\theta}, \hat{\phi})$ in Cartesian coordinate system $(\hat{\imath}, \hat{\jmath}, \hat{k})$ (all values in meters):
   a) $r = 5, \theta = \pi/2, \phi = \pi/4$
   b) $r = 3, \theta = \pi/4, \phi = 0$
   c) $r = 8, \theta = \pi/2, \phi = \pi$
5. Find the gradients $(\nabla\phi)$ of the following scalar functions at a point P with Cartesian coordinates (2, -1, 2):
   a) $f(x, y, z) = x^2 + y^2 + z^2 - 9$
   b) $g(x, y, z) = x^2 + y^2 - z - 3$
   Using the gradients obtain the angle between the surfaces given by $f(x, y, z) = 0$ and $g(x, y, z) = 0$ at the point P. [Ans: $cos^{-1}(8/3\sqrt{21}) \approx 54.4^o$]
6. Obtain the maximum directional derivative of the scalar function $f(x, y, z) = x^2yz^3$ at a point with coordinates (2, 1, -1). [Ans: 13.27]
7. Calculate the divergence $(\nabla . \vec{F})$ of the following vector functions:
   a) $\vec{F}_1 = \hat{\imath}x - \hat{\jmath}y$
   b) $\vec{F}_2 = \hat{k}z$
   c) $\vec{F}_3 = \alpha\vec{r} = \alpha(\hat{\imath}x + \hat{\jmath}y + \hat{k}z)$
   d) $\vec{F}_4 = \beta\frac{\hat{r}}{r^2} = \beta\frac{\vec{r}}{r^3} = \beta\frac{(\hat{\imath}x+\hat{\jmath}y+\hat{k}z)}{(x^2+y^2+z^2)^{3/2}}$ for $r \neq 0$.
8. Calculate the curl $(\nabla \times \vec{F})$ of the following vector functions:
   a) $\vec{F}_1 = \hat{\imath}\alpha y$
   b) $\vec{F}_2 = \hat{\imath}\alpha x + \hat{\jmath}\beta y^2$
   c) $\vec{F}_3 = \hat{\imath}x^2 + 3xz^2\hat{\jmath} - 2xz\hat{k}$
9. Consider a scalar function given by $f(x, y, z) = \alpha xy^2$.
   a) Calculate the gradient of the function f.

b) Obtain the curl of the gradient of the function and show that it is zero. *[Note: Curl of the gradient of a function is always zero. Thus if we find a vector function whose curl is zero, then the vector function can always be represented by the gradient of a scalar function.]*

10. Consider a vector function given by $\vec{G} = \hat{\imath}x^2 + 3xz^2\hat{\jmath} - 2xz\hat{k}$.
   a) Calculate the curl of the vector function $\vec{G}$.
   b) If $\nabla \times \vec{G} = \vec{A}$ then show that $\nabla . \vec{A} = 0$.
   [Note: The divergence of the curl of a vector function is always zero. Thus if we find a vector function whose divergence is zero, then we can always represent the vector function as the curl of another vector function.]