Riemann Integral

Engineering Calculus

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Consider following terms:

- Let $f : [a, b] \to \mathbb{R}$ be a bounded real valued function on the closed, bounded interval $[a, b]$. Also let m, M be the infimum and supremum of $f(x)$ on $[a, b]$, respectively.
- A **partition** P of $[a, b]$ is an ordered set $P := \{a = x_0, x_1, x_2, ..., x_n = b\}$ such that $x_0 < x_1 < \cdots < x_n$.
- Let m_k and M_k be the infimum and supremum of $f(x)$ on the subinterval $[x_{k-1}, x_k]$, respectively.

Definition

Riemann Lower sum: The Riemann Lower sum, denoted with $L(P,f)$ of $f(x)$ with respect to the partition P is given by

$$L(P,f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

Riemann Upper sum: The Riemann Upper sum, denoted with $U(P,f)$ of $f(x)$ with respect to the partition P is given by

$$U(P,f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

Refinement of a Partition: A partition Q is called a refinement of the partition P if $P \subseteq Q$.

Lemma

If Q is a refinement of P, then

$$L(P,f) \leq L(Q,f) \quad \text{and} \quad U(P,f) \geq U(Q,f).$$

Proof: Let $P = \{x_0, x_1, x_2, ..., x_{k-1}, x_k, ..., x_n\}$ and $Q = \{x_0, x_1, x_2, ..., x_{k-1}, z, x_k, ..., x_n\}$. Then

$$\begin{aligned} L(P,f) &= m_0(x_1 - x_0) + ... + m_k(x_k - x_{k-1}) + ... + m_{n-1}(x_n - x_{n-1}) \\ &\leq m_0(x_1 - x_0) + ... + m_k^{'}(x_k - z) + m_k^{''}(z - x_{k-1}) + ... + m_{n-1}(x_n - x_{n-1}) \\ &= L(Q,f) \end{aligned}$$

where $m_k^{'} = \inf\limits_{[z,x_k]} f(x)$ and $m_k^{''} = \inf\limits_{[x_{k-1},z]} f(x)$.

Lemma

If P_1 and P_2 be any two partitions, then $L(P_1,f) \leq U(P_2,f)$.

Proof: Let $Q = P_1 \cup P_2$. Then Q is a refinement of both P_1 and P_2. So by above Lemma, we have $L(P_1,f) \leq L(Q,f) \leq U(Q,f) \leq U(P_2,f)$.

Definition

Let $\mathcal{P}$ be the collection of all possible partitions of $[a,b]$. Then upper integral of f is defined as

$$\overline{\int_a^b} f = \inf\{U(P,f) : P \in \mathcal{P}\}$$

and lower integral of f is defined as

$$\underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}\}.$$

- For a bounded function $f : [a,b] \to \mathbb{R}$, we have $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.
- **Riemann integrability:** $f : [a,b] \to \mathbb{R}$ is said to be Riemann integrable if $\underline{\int_a^b} f = \overline{\int_a^b} f$ and the value of the limit is denoted with $\int_a^b f(x)dx$. We say $f \in \mathcal{R}[a,b]$.

Example 1

Consider $f(x) = x$ on $[0, 1]$ and the sequence of partitions $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, \frac{n}{n}\}$. Then

$$L(P_n, f) = 0 \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} + \cdots + \frac{n-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}[1 + 2 + ... + (n-1)] = \frac{n(n-1)}{2n^2}$$

Thus $\lim\limits_{n\to\infty} L(P_n, f) = \frac{1}{2}$. Hence from the definition $\underline{\int_0^1} f(x)dx \geq \frac{1}{2}$. Similarly

$$U(P_n, f) = \frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} + \cdots + \frac{n}{n} \cdot \frac{1}{n} = \frac{1}{n^2}[1 + 2 + \cdots + n] = \frac{n(n+1)}{2n^2}$$

Hence $\lim\limits_{n\to\infty} U(P_n, f) = \frac{1}{2}$. Again from the definition $\overline{\int_0^1} f(x)dx \leq \frac{1}{2}$.

So $\frac{1}{2} \leq \underline{\int_0^1} f(x)dx \leq \overline{\int_0^1} f(x)dx \leq \frac{1}{2}$.

Thus $\int_0^1 f(x)dx = \underline{\int_0^1} f(x)dx = \overline{\int_0^1} f(x)dx = \frac{1}{2}$.

Example 2

Consider $f(x) = x^2$ on $[0, 1]$ and the sequence of partitions $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, \frac{n}{n}\}$. Then

$$\begin{aligned} U(P_n, f) &= \frac{1}{n^2} \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^2 \cdot \frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^2 \cdot \frac{1}{n} \\ &= \frac{1}{n^3}[1 + 2^2 + \cdots + n^2] \\ &= \frac{n(n+1)(2n+1)}{6n^3} \end{aligned}$$

Thus $\lim_{n\to\infty} U(P_n, f) = \frac{1}{3}$. Similarly

$$\begin{aligned} L(P_n, f) &= 0 \cdot \frac{1}{n} + \left(\frac{1}{n}\right)^2 \cdot \frac{1}{n} + \cdots + \left(\frac{n-1}{n}\right)^2 \cdot \frac{1}{n} \\ &= \frac{1}{n^3}[1 + 2^2 + \cdots + (n-1)^2] \\ &= \frac{n(n-1)(2n-1)}{6n^3} \end{aligned}$$

Therefore, $\lim_{n\to\infty} L(P_n, f) = \frac{1}{3}$. Hence from the definition $\int_{\underline{0}}^{1} f \geq \frac{1}{3}$ and $\int_0^{\overline{1}} f \leq \frac{1}{3}$.

So $\frac{1}{3} \leq \int_{\underline{0}}^{1} f(x)dx \leq \int_0^{\overline{1}} f(x)dx \leq \frac{1}{3}$. Thus $\int_0^1 f(x)dx = \int_{\underline{0}}^{1} f(x)dx = \int_0^{\overline{1}} f(x)dx = \frac{1}{3}$.

Example 3

On $[0, 1]$, define $f(x) = \begin{cases} 1, & x \in Q, \\ 0, & x \notin Q. \end{cases}$

Let P be a partition of $[0, 1]$. In any sub interval $[x_{k-1}, x_k]$, there exists a rational number and irrational number. Then the supremum in any subinterval is 1 and infimum is 0. Therefore, $L(P,f) = 0$ and $U(P,f) = 1$. Hence $\underline{\int_0^1} f \neq \overline{\int_0^1} f$.

Result

Suppose f is a continuous function on $[a, b]$. Then $f \in \mathcal{R}[a, b]$.

Theorem

Suppose $f : [a, b] \to \mathbb{R}$ be a bounded function which has finitely many discontinuities. Then $f \in \mathcal{R}[a, b]$.

Properties of definite integral

(a) For a constant $c \in \mathbb{R}$, $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

(b) Let $f_1,\ f_2 \in \mathcal{R}[a, b]$. Then $\int_a^b (f_1 + f_2)(x)dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx$.

(c) If $f(x) \leq g(x)$ on $[a, b]$. Then $\displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

(d) If $f \in \mathcal{R}[a, b]$ then $|f| \in \mathcal{R}[a, b]$ and $\displaystyle\left|\int_a^b f(x)dx\right| \leq \int_a^b |f|(x)dx$.

(e) Let f be bounded on $[a, b]$ and let $c \in (a, b)$. Then f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and $[c, b]$. In this cases

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Mean value theorem

Let $f(x)$ be a continuous function on $[a,b]$. Then there exists $\xi \in [a,b]$ such that

$$\int_a^b f(x)dx = f(\xi)(b-a).$$

First Fundamental theorem

Let $f(x)$ be a continuous function on $[a,b]$ and let $\phi(x) = \int_a^x f(s)ds$. Then ϕ is differentiable and $\phi^{'}(x) = f(x)$.

- A function $F(x)$ is called anti-derivative of $f(x)$, if $F'(x) = f(x)$.

Second fundamental theorem

Suppose $F(x)$ is an anti- derivative of continuous function $f(x)$. Then $\int_a^b f(x)dx = F(b) - F(a)$.

Change of variable theorem

Let $u(t),\ u'(t)$ be continuous on $[a, b]$ and f is a continuous function on the interval $u([a, b])$. Then

$$\int_a^b f(u(x))\ u'(x)dx = \int_{u(a)}^{u(b)} f(y)dy.$$

Problem

Evaluate $\int_0^1 x\sqrt{1+x^2}dx$.

Solution: Taking $u = 1 + x^2$, we get $u' = 2x$ and $u(0) = 1, u(1) = 2$. Then

$$\int_0^1 x\sqrt{1+x^2}dx = \frac{1}{2}\int_1^2 \sqrt{u}du = \frac{1}{3}\left[u^{\frac{2}{3}}\right]_{u=1}^2 = \frac{1}{3}(2^{\frac{2}{3}} - 1).$$

Riemann Integration