Mean Value Theorem

BENNETT
UNIVERSITY
TIMES OF INDIA GROUP

Mean value theorem (MVT)

Theorem

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. Let $X_0 = (x_0, y_0)$ and $X = (x_0 + h, y_0 + k)$. Then there exists C which lies on the line joining X_0 and X such that

$$f(X) = f(X_0) + f'(C)(X - X_0),$$

i.e., there exists $c \in (0, 1)$ such that

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + hf_x(C) + kf_y(C),$$

where $C = (x_0 + ch, y_0 + ck)$.

Proof of MVT

Define $\phi : [0,1] \to \mathbb{R}$ by

$$\phi(t) = f(x_0 + th, y_0 + tk), \quad t \in [0,1].$$

By chain rule ϕ is differentiable and

$$\phi'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = f_x h + f_y k, \quad \text{(since } x = x_0 + th \text{ and } y = y_0 + tk.\text{)}$$

Now by MVT, there exists $c \in (0,1)$ such that

$$\phi(1) - \phi(0) = \phi'(c).$$

The proof now follows immediately.

Extended mean value theorem (EMVT)

Theorem

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. Let $X_0 = (x_0, y_0)$ and $X = (x_0 + h, y_0 + k)$. Furthermore, suppose f_x and f_y are continuous and they have continuous partial derivatives. Then there exists C which lies on the line joining X_0 and X such that

$$f(X) = f(X_0) + f'(X_0)(X - X_0) + \frac{1}{2}(X - X_0)f''(C)(X - X_0),$$

where $f'' = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$. That is, there exists $c \in (0, 1)$ such that

$$f(x_0+h, y_0+k) = f(x_0, y_0)+(hf_x+kf_y)(X_0)+\frac{1}{2}(h^2 f_{xx}+2hkf_{xy}+k^2 f_{yy})(C),$$

where $C = (x_0 + ch, y_0 + ck)$.

Thank You.