18.2.21

Wave equation for $\vec{E}$ and $\vec{B}$

$\boxed{\text{Section } 9.2.1}$

In region of space, if there are no charge or current,

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Coupled 1st order eq. for $\vec{E}$ & $\vec{B}$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})$$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} \qquad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$= \vec{\nabla}\times\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow \underline{\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}}$$

Similarly,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B}$$

$$= \vec{\nabla}\times\left(\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$= \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E})$$

$$= - \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

* In vacuum, the components of $\vec{E}$ & $\vec{B}$ satisfy the 3-D wave eq.

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

→ Empty space lets the electromagnetic wave travel at a velocity

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$= 3 \times 10^8 \text{ m/s} \equiv \text{Speed of light}$$

→ This tells us that light is an EM wave.

* Let us look at a sinusoidal wave of a ~~freq.~~, ω. The wave is travelling in z-direcn.

$\boxed{\text{Section } 9.2.2}$

$$\vec{\tilde{E}}(z,t) = \vec{\tilde{E}}_0 \, e^{i(kz-\omega t)} \quad \Big| \quad \omega = ck$$

$$\vec{\tilde{B}}(z,t) = \vec{\tilde{B}}_0 \, e^{i(kz-\omega t)}$$

$\tilde{E}_0$, $\tilde{B}_0$ → complex amplitudes.

→ These satisfy the wave eq.

→ Additional constraint: $\vec{\nabla} \cdot \vec{E} = 0$

$\vec{\nabla} \cdot \vec{B} = 0$

$$\Rightarrow \quad (\tilde{E}_0)_z = (\tilde{B}_0)_z = 0$$

↳ The EM waves are transverse, $\vec{E}$ & $\vec{B}$ are perpendicular to the direc^n. of propagation.

(x) from Faraday's law,

$$\vec{\nabla} \times \vec{\tilde{E}} = -\frac{\partial \vec{\tilde{B}}}{\partial t}$$

$$\Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \tilde{E}_x & \tilde{E}_y & \cancel{\tilde{E}_z} \end{vmatrix} = -\frac{\partial}{\partial t}\left(\tilde{B}_x \hat{x} + \tilde{B}_y \hat{y} + \cancel{\tilde{B}_z} \hat{z}\right)$$

$\tilde{E}_z = 0$, $\tilde{B}_z = 0$

$$\Rightarrow \quad -k(\tilde{E}_0)_y = \omega(\tilde{B}_0)_x$$

$$k(\tilde{E}_0)_x = \omega(\tilde{B}_0)_y$$

$$\Rightarrow \quad \tilde{\vec{B}}_0 = \frac{k}{\omega}(\hat{z} \times \tilde{\vec{E}}_0)$$

$\rightarrow$ $\vec{E}$ & $\vec{B}$ are mutually perpendicular.

The real amplituden, $B = \frac{k}{\omega} E_0$

$$= \frac{1}{c} E_0$$