Sequence
(Lecture-3)

Engineering Calculus

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Definition

A sequence of real numbers or a sequence in $\mathbb{R}$ is a function $f : \mathbb{N} \to \mathbb{R}$.

- We write a_n for $f(n),\ n \in \mathbb{N}$ and the notation for a sequence is $\{a_n\}_{n=1}^{\infty}$.

Examples

1. Constant sequence: $\{c, c, c, \cdots\}$, where $c \in \mathbb{R}$.
2. Sequence defined by listing: $\{1, 4, 8, 11, 52, \cdots\}$.
3. Sequence defined by rule: $\{a_n\}_{n=1}^{\infty}$, where $a_n = 3n^2$ for all $n \in \mathbb{N}$.
4. $\{\frac{(-1)^{n+1}}{n}\}_{n=1}^{\infty}$
5. $\{\frac{n-1}{n}\}_{n=1}^{\infty}$
6. $\{\sqrt{n}\}_{n=1}^{\infty}$

- What does **convergence** mean?
- Think of the examples: $\{2, 2, 2, \cdots\}$, $\{\frac{1}{n}\}_{n=1}^{\infty}$, $\{n^2 - 1\}_{n=1}^{\infty}$, $\{1, 2, 1, 2, \cdots\}$, $\{(-1)^n \frac{1}{n}\}_{n=1}^{\infty}$, $\{(-1)^n(1 - \frac{1}{n})\}_{n=1}^{\infty}$.

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to limit L if for every $\epsilon > 0$ (given) there exists a positive integer N such that $n \geq N \implies |a_n - L| < \epsilon$.

- Notation: $L = \lim_{n\to\infty} a_n$ or $a_n \to L$.
- If $\{a_n\}_{n=1}^{\infty}$ is a sequence and if both $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$ holds, then $L = M$.

Examples

1. Constant sequence $\{c\}_{n=1}^{\infty}, c \in \mathbb{R}$, has c as it's limit.
2. Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Solution: Let $\epsilon > 0$ be given. To show that $1/n$ approaches 0, we must show that there exists an integer $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon.$$

But $1/n < \epsilon \Leftrightarrow n > 1/\epsilon$. Thus, if we choose $N \in \mathbb{N}$ such that $N > 1/\epsilon$, then for all $n \geq N$, $1/n < \epsilon$.

Example

Show that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Solution: For any $\epsilon > 0$,

$$\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} < \epsilon \ \forall \ n \geq N,$$

where N is a positive integer such that $N > \frac{1}{\epsilon}$. Thus, $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.

Example

Show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Solution: Note that $|a_n - 1| = \frac{1}{n+1} < \frac{1}{n}$. Thus, for any $\epsilon > 0$, take $N > \frac{1}{\epsilon}$, we get

$$\left|\frac{n}{n+1} - 1\right| = \frac{1}{1+n} < \frac{1}{n} < \epsilon \ \forall \ n \geq N.$$

Hence, $\frac{n}{1+n} \to 1$ as $n \to \infty$.

Thank You