Solutions of Tutorial Sheet 7

Improper Integral

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1. (a) $\int_0^\infty e^{-x}\cos x dx = \lim_{b\to\infty}\int_0^b e^{-x}\cos x dx$

   Now take $I = \int_0^b e^{-x}\cos x dx = \frac{1}{2}(1-\cos be^{-b}+\sin be^{-b})$. So $\lim_{b\to\infty} I = \dfrac{1}{2}$.

   $\Rightarrow \int_0^\infty e^{-x}\cos x dx$ is convergent.

   (b) $\int_1^\infty \frac{dx}{x^2(1+e^x)} \le \int_1^\infty \frac{dx}{x^2}$ which is convergent .Hence by comparison test given improper integral is convergent.

   (c) $\int_1^\infty \frac{(x+1)dx}{x^{\frac{3}{2}}} = \int_1^\infty \frac{1}{\sqrt{x}} + \int_1^\infty x^{-3/2}dx$. The first integral on the right side diverges. Hence given integral diverges.

2. (a) Take $\ln x = t$ then $x = e^t$ and the integral becomes $\int_0^{\ln 2}\frac{e^{t/2}}{t}$. It is easy to see that integrand is $\ge \frac{1}{t}$ and the integral $\int_0^{\ln 2}\frac{1}{t}$ diverges.

   (b) $f(x) = \frac{\sin(\frac{1}{x})}{\sqrt{x}}$ and Take $g(x) = \frac{1}{\sqrt{x}}$. Then using comparison test, since $\int_0^1 \frac{dx}{\sqrt{x}}$ is convergent, we have $\int_0^1 \frac{\sin(\frac{1}{x})}{\sqrt{x}}dx$ is convergent.

   (c) Take $f(x) = \frac{\tan(x)}{x^{3/2}}$ and $g(x) = \tan x$. Then $\lim_{x\to\frac{\pi}{2}}\dfrac{f(x)}{g(x)} \in (0,\infty)$. Also as $\int_1^{\frac{\pi}{2}}\tan(x)dx$ is divergent so $\int_1^{\frac{\pi}{2}}\frac{\tan(x)}{x^{3/2}}dx$ is divergent.

3. (a) $\int_0^\infty x^{\frac{-1}{2}}e^{x^2}dx = \int_0^1 \frac{e^{x^2}dx}{\sqrt{x}} + \int_1^\infty \frac{e^{x^2}dx}{\sqrt{x}}dx$. Now $\int_0^1 \frac{e^{x^2}dx}{\sqrt{x}}$ is convergent, Since if we take $g(x) = \frac{1}{\sqrt{x}}$ then $\lim_{x\to 0}\dfrac{f(x)}{g(x)} = 1$ and $\int_0^1\frac{dx}{\sqrt{x}}$ is convergent But $\int_1^\infty \frac{e^{x^2}dx}{\sqrt{x}}$ is divergent. Since if we take $g(x) = \frac{1}{\sqrt{x}}$ then $\lim_{x\to\infty}\dfrac{f(x)}{g(x)} = \infty$ and $\int_1^\infty \frac{dx}{\sqrt{x}}$ is divergent. Hence the given integral is divergent.

   (b) Note that

   $$\int_{-\infty}^{\infty}\frac{dx}{|x|^p(1+x^2)} = \int_{-\infty}^{-1}\frac{dx}{|x|^p(1+x^2)} + \int_{-1}^{1}\frac{dx}{|x|^p(1+x^2)} + \int_1^\infty \frac{dx}{x^p(1+x^2)}.$$

   Now take $g(x) = \frac{1}{1+x^2}$. Then by limit comparison test we have 1st and 3rd integral on RHS convergent as $\int_{-\infty}^{-1}\frac{dx}{(1+x^2)}$ and $\int_1^\infty \frac{dx}{(1+x^2)}$ are convergent. Now for $\int_{-1}^{1} \frac{dx}{|x|^p(1+x^2)}$, take $g(x) = \frac{1}{|x|^p}$. Then $\lim\limits_{x \to 0} \dfrac{1}{|x|^p(1+x^2)} \times |x|^p = 1$ And $\int_{-1}^{1} \frac{dx}{x^p}$ is convergent for $0 < p < 1$ Thus given integral is convergent for $p < 1$.

(c) Let $g(x) = \frac{1}{1+x^2}$ Then $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = 1$. Since $\int_0^\infty g(x)dx$ converges. So, by limit test $\int_0^\infty g(x)dx$ converges.

4. Clearly, for $p \leq 0$, $\int_0^1 \frac{\sin x}{x^p} dx$ exists as Riemann integrals. So, let $p > 0$. Then for $x > 0$, we have

$$\left|\frac{\sin x}{x^p}\right| = \left|\frac{\sin x}{x}\right| \frac{1}{x^{p-1}} \leq \frac{1}{x^{p-1}}.$$

From comparison test, it follows that $\int_0^1 \frac{\sin x}{x^p} dx$ converges for $p < 2$.

Now we will show that $\int_0^1 \frac{\sin x}{x^p} dx$ diverges for $p \geq 2$. Since $\frac{\sin x}{x}$ is decreasing in $(0, 1]$, then for all $x \in (0, 1]$, we have

$$\frac{\sin x}{x^p} = \frac{\sin x}{x} \frac{1}{x^{p-1}} \geq \frac{\sin 1}{x^{p-1}}.$$

Also, $\int_0^1 \frac{1}{x^p} dx$ diverges for $p - 1 \geq 1$. Therefore, by Comparision test, it follows that $\int_0^1 \frac{\sin x}{x^p}$ diverges for $p \geq 2$.

5. Leibniz formula is to be used.

(a) Let $f(t) = \int_0^\infty e^{-tx} \frac{\sin x}{x} dx$. Then $\frac{df}{dt} = -\int_0^\infty e^{-tx} \sin x dx = \frac{-1}{1+t^2}$.
(since

$$\begin{aligned}
I &= -\int_0^\infty e^{-tx} \sin x dx \\
&= -\left[-\sin x e^{-tx} \frac{1}{t}\Big|_0^\infty + \int_0^\infty e^{-tx} \frac{1}{t} \cos x dx\right] \\
&= -\frac{1}{t} \int_0^\infty e^{-tx} \cos x dx \\
&= -\frac{1}{t}\left[\cos x e^{-tx} \frac{1}{-t}\Big|_0^\infty + \int_0^\infty \frac{1}{t} e^{-tx}(-\sin x) dx\right] \\
&= -\frac{1}{t^2} - \frac{I}{t^2}.
\end{aligned}$$

$\therefore I = -\frac{1}{1+t^2} = f'(t).$)

$$\implies f(t) = -\arctan t + c.$$

By the second fundamental theorem, $f(a) - f(0) = \int_0^a f'(t)dt = \int_0^a \frac{-1}{1+t^2}dt$, taking $a \to \infty$, $\lim_{a\to\infty} f(a) - f(0) = -\pi/2$. Also,

$$0 \leq |f(a)| = \left| \int_0^\infty e^{-ax} \frac{sinx}{x} dx \right| \leq C_1 \int_0^\infty e^{-ax} dx \quad \text{as} \quad a \to \infty.$$

Therefore, $\lim\limits_{a\to\infty} f(a) = 0$. Using this we get $c = \frac{\pi}{2}$ and hence $f(t) = \frac{\pi}{2} - \arctan t$.

(b) Let $f(t) = \int_0^1 \frac{x^t - 1}{\ln x} dx$, then $\frac{df}{dt} = \int_0^1 x^t dx = \frac{1}{t+1}$.

$$\implies f(t) = \ln(t+1) + c.$$

Now $f(0) = 0 \Rightarrow c = 0$.

$$\Rightarrow \int_0^1 \frac{x^t - 1}{\ln x} dx = \ln(t+1).$$

6. (a) $I = \int_0^\infty e^{-x^2} dx$. Put $x^2 = t \Rightarrow 2xdx = dt$.

$$\implies I = \int_0^\infty \frac{1}{2} e^{-t} t^{-\frac{1}{2}} dt$$

$$\therefore I = \frac{1}{2} \Gamma\left(\frac{1}{2}\right).$$

(b) $I = \int_0^{\frac{\pi}{2}} \sqrt{\tan x} dx = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}} x \cos^{-\frac{1}{2}} x dx = \frac{1}{2}\beta(\frac{3}{4}, \frac{1}{4})$

(c) Let $I = \int_0^\infty x^{2/3} e^{-\sqrt{x}} dx$. Substitute $\sqrt{x} = t$.

$$\implies I = 2\int_0^\infty t^{\frac{7}{3}} e^{-t} dt.$$

Comparing it with the gamma function, $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$, we have $p = \frac{10}{3}$.

$$I = \Gamma\left(\frac{10}{3}\right).$$

7. Use $\Gamma(n+1) = n\Gamma(n)$, recurssively.

   (a) $\frac{3}{4}\sqrt{\pi}$

   (b) $\frac{105}{16}\sqrt{\pi}$

   (c) Use Euler's reflection formula:

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)}.$$

   Choose $p = -\frac{1}{2}$, so answer$=-2\sqrt{\pi}$.