Taylor's Theorem

BENNETT UNIVERSITY
TIMES OF INDIA GROUP

Higher order mixed partial derivatives

Example 1: Consider $f(x,y) = x^2 + 5xy + y^2$. Then

$$f_x = \frac{\partial f}{\partial x} = 2x + 5y,$$

$$f_y = \frac{\partial f}{\partial y} = 5x + 2y,$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 5 = f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

Example 2: Consider $f(x,y) = \sin xy^2 + 9x + \frac{1}{y}$. Then

$$f_x = y^2 \cos xy^2 + 9,$$

$$f_y = 2xy \cos xy^2 - \frac{1}{y^2},$$

$$f_{xy} = 2y \cos xy^2 - 2xy^3 \sin xy^2 = f_{yx}.$$

Is it always true that $f_{xy} = f_{yx}$?

NO!

Example: Consider $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & x \neq 0, y \neq 0 \\ 0 & x = y = 0 \end{cases}$.

Then

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k\to 0} \frac{1}{k} \frac{hk(h^2-k^2)}{h^2+k^2} = h$$

Also $f_y(0,0) = 0$. Therefore,

$$\begin{aligned} f_{xy}(0,0) &= \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} \\ &= \lim_{h\to 0} \frac{h-0}{h} = 1 \end{aligned}$$

Now

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h\to 0} \frac{1}{h} \frac{hk(h^2 - k^2)}{h^2 + k^2} = -k$$

and

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = -1.$$

So we get

$$1 = f_{xy}(0,0) \neq f_{yx}(0,0) = -1.$$

A sufficient condition for $f_{xy} = f_{yx}$

Theorem

If $f, f_x, f_y, f_{xy}, f_{yx}$ are continuous in a neighbourhood of (a, b). Then $f_{xy}(a, b) = f_{yx}(a, b)$.

Note: The above statement is NOT a necessary condition.

Example: f_{xy}, f_{yx} not continuous but mixed derivatives are equal.

Consider the function $f(x, y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & x \neq 0, y \neq 0 \\ 0, & x = y = 0. \end{cases}$

Here $f_{xy}(0, 0) = f_{yx}(0, 0)$ but they are not continuous at $(0, 0)$ (Try!).

Taylor's theorem for a function of two variables

Theorem

Suppose $f(x, y)$ and its partial derivatives through order $n+1$ are continuous throughout an open rectangular region R centered at a point (a, b). Then, throughout R,

$$f(a+h, b+k) = f(a,b) + (hf_x + kf_y)\Big|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy})\Big|_{(a,b)}$$
$$+ \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})\Big|_{(a,b)}$$
$$+ \ldots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\Big|_{(a,b)} + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(a+ch, b+ck)}$$

where $(a+ch, b+ck)$ is a point on the line segment joining (a, b) and $(a+h, b+k)$.

First-degree Taylor polynomial of a function of two variables

For a function of two variables $f(x, y)$ whose first partials exist at the point (a, b), the 1st -degree Taylor polynomial of f for (x, y) near the point (a, b) is:

$$\boxed{f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).}$$

$L(x, y)$ is also called the linear approximation (or **tangent plane**) of f for (x, y) near the point (a, b).

Example

Example: Determine the 1st-degree Taylor polynomial approximations of the function $f(x, y) = \sin 2x + \cos y$ near the point $(0, 0)$.

Ans: Here

$$f_x(x, y) = 2\cos 2x, \quad f_y(x, y) = -\sin y$$

$$f(0, 0) = 1, \quad f_x(0, 0) = 2, \quad f_y(0, 0) = 0.$$

$$\implies L(x, y) = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) = 1 + 2x.$$

Second-degree Taylor polynomial of a function of two variables

For a function of two variables $f(x, y)$ whose first and second partials exist at the point (a, b), the 2nd -degree Taylor polynomial of f for (x, y) near the point (a, b) is:

$$\begin{aligned} f(x,y) \approx Q(x,y) = & L(x,y) + \frac{f_{xx}(a,b)}{2}(x-a)^2 \\ & + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2. \end{aligned}$$

Example

Example: Determine the 2nd-degree Taylor polynomial approximations of the function $f(x, y) = \sin 2x + \cos y$ near the point $(0, 0)$.

Ans: Here

$$f_{xx}(x, y) = -4\sin 2x, \quad f_{xy}(x, y) = 0, \quad f_{yy}(x, y) = -\cos y$$

$$f_{xx}(0, 0) = 0, \quad f_{xy}(0, 0) = 0, \quad f_{yy}(0, 0) = -1.$$

$$\begin{aligned} Q(x, y) &= L(x, y) + \frac{f_{xx}(0, 0)}{2}(x-0)^2 + f_{xy}(0, 0)(x-0)(y-0) \\ &\quad + \frac{f_{yy}(0, 0)}{2}(y-0)^2 \\ &= 1 + 2x - \frac{y^2}{2}. \end{aligned}$$

Thank You.