Gradient, Divergence, Curl

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Scalar and vector fields

- A scalar field is something that has a particular value at every point in space.
- An example of a scalar field is temperature. Everywhere on Earth has a particular temperature value but if you move up or down, left or right, or forward or backward then the value of the temperature will change.
- A vector field is the same as a scalar field but except for only having a value at every point in space, it has a value and direction at every point in space.
- Example: velocity of flowing liquid, earth's gravitational field. The gravitational field not only has a given strength depending on how far from Earth you are but it also always points towards the center of the planet.

∇: 'Del' operator

$$:= \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Gradient	**Divergence**	**Curl**
turns a scalar field into a vector field	turns a vector field into a scalar field	turns a vector field into another vector field
∇f	$\nabla \cdot (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$	$\nabla \times (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$
$\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$	$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$	cross product

Example

1. **Gradient:** Let $f(x,y,z) = x^2 + 2xy + z^2 y$. Then

$$\nabla f = (2x+2y)\hat{i} + (2x+z^2)\hat{j} + (2zy)\hat{k}.$$

2. **Divergence:** Let $F(x,y,z) = f_1(x,y,z)\hat{i} + f_2(x,y,z)\hat{j} + f_3(x,y,z)\hat{k} = (x^2y)\hat{i} + (3x - z^3)\hat{j} + (4y^2)\hat{k}$. Then

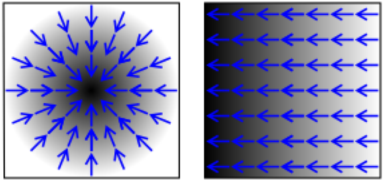
$$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 2xy.$$

3. **Curl:**

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 3x - z^3 & 4y^2 \end{vmatrix} = (8y + 3z^2)\hat{i} + (3 - x^2)\hat{k}.$$

Significance of gradient of a scalar field

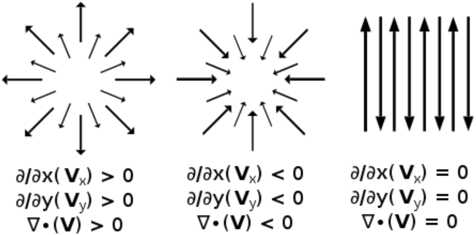
- Technically, the gradient of the scalar function/field is a vector representing both magnitude and direction of the maximum space rate (derivatives w.r.t spatial coordinates) of the increase of that function/field.
- In 3D form, Gradients are surface normal to particular points.
- In 2D format, Gradients tangents representing the direction of steepest descent or ascent.



The gradient, represented by the blue arrows, denote the direction of greatest change of a scalar function. The values of the function are represented in greyscale and increase in value from white (low) to dark (high).

https://en.wikipedia.org/wiki/Gradient#/media/File:Gradient2.svg

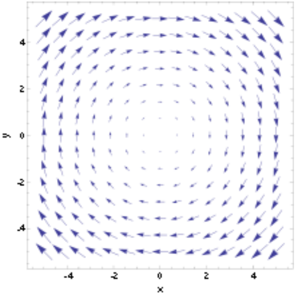
Significance of divergence of a vector field



- The divergence of a vector field at a given point is the net outward flux per unit volume as the volume shrinks (tends to) zero at that point.
- In simple words, it gives us an idea about the "outgoingness" of the field at that point.
- In other words, it tells us whether the field is converging or diverging at that point.

Significance of curl of a vector field

- The curl at a point in the field is represented by a vector whose length and direction denote the magnitude and axis of the maximum circulation.
- The curl of a field is formally defined as the circulation density at each point of the field.
- A vector field whose curl is zero is called **irrotational**.



Depiction of a two-dimensional vector field with a uniform curl.

Directional derivates and gradient of a differentiable function

Theorem

If $f(x, y)$ is differentiable, then the directional derivative in the direction $\hat{p}$ at (a, b) is

$$D_{\hat{p}} f(a, b) = \nabla f(a, b) \cdot \hat{p}.$$

- So using the directional derivatives, we can find the direction of maximum rate of change.
- $D_{\hat{p}} f = \nabla f \cdot \hat{p} = |\nabla f| \cos\theta$. So the function f increases most rapidly when $\cos\theta = 1$ or when $\hat{p}$ is the direction of ∇f. The derivative in the direction $\frac{\nabla f}{|\nabla f|}$ is equal to $|\nabla f|$.

- f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\hat{p}} f = -|\nabla f|$.
- The direction of no change is when $\theta = \frac{\pi}{2}$. i.,e., $\hat{p} \perp \nabla f$.

Example: Find the direction in which $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$ increases and decreases most rapidly at the point $(1, 1)$.

Ans: Now $f_x(1, 1) = 1$, $f_y(1, 1) = 1$.

Then direction of f in which it increases is direction of $\nabla f = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$.

And direction of f in which it decreases is direction of $\nabla f = -\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$.

Function for which all directional derivatives exist but not differentiable

Is it possible? Yes!

Example: Consider the function $f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2}, & y \neq 0 \\ 0, & y = 0. \end{cases}$

Ans:

$$D_{\hat{p}} f(a,b) = \lim_{s \to 0} \frac{f(sp_1, sp_2) - f(0,0)}{s} = \frac{p_2}{|p_2|}.$$

$$f_x(0,0) = \lim_{h \to 0} \frac{0-0}{h} = 0, \quad f_y(0,0) = \lim_{k \to 0} \frac{\frac{k}{|k|}\sqrt{k^2}}{k} = 1.$$

In this case, both $D_{\hat{p}} f(a,b)$ and $\nabla f(a,b) \cdot \hat{p}$ exists but not equal.

Thank You.