Improper Integrals

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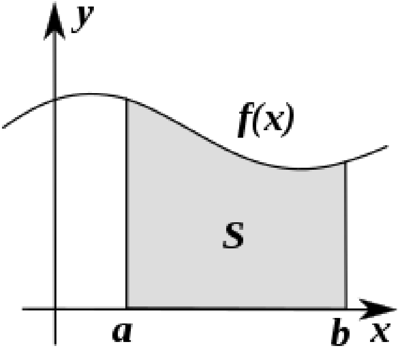
TIMES OF INDIA GROUP

Riemann Integral



The integral as the area of a region under a curve.

https://upload.wikimedia.org/wikipedia/commons/2/28/Riemann_integral_regular.gif

A sequence of Riemann sums over a regular partition of an interval. The number on top is the total area of the rectangles, which converges to the integral of the function.

$$\int_a^b f(x)dx = \lim_{\|\Delta x\|\to 0} \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

Some other integrals

1. Cauchy integral
2. Riemann-Stieltjes integral
3. Lebesgue integral
4. Lebesgue-Stieltjes integral
5. Henstock-Kurzweil (HK) integral
6. Wiener integral
7. Feynman integral

What is our today's goal?

1. The function $f(x)$ defined on unbounded interval $[a, \infty)$ or $(-\infty, b]$ and $f \in \mathcal{R}[a, b]$ for all $b > a$.
2. The function is unbounded at some points on the interval $[a, b]$.

Improper integral of first kind

Suppose f is a bounded function defined on $[a, \infty)$ or $(-\infty, b]$ and $f \in \mathcal{R}[a, b]$ for all $b > a$. The improper integral of f on $[a, \infty)$ is defined as

$$\int_a^\infty f(x)dx := \lim_{b \to \infty} \int_a^b f(x)dx.$$

and the improper integral of f on $(-\infty, b]$ is defined as

$$\int_{-\infty}^b f(x)dx := \lim_{a \to -\infty} \int_a^b f(x)dx.$$

If the limit exists and is finite, we say that the improper integral converges. If the limit goes to infinity or does not exist, then we say that the improper integral diverges.

Examples of improper integral of first kind

1. $\int_1^\infty \frac{1}{x^2}dx = \lim\limits_{b\to\infty} \displaystyle\int_1^b \frac{1}{x^2}dx = \lim\limits_{b\to\infty} 1 - \frac{1}{b} = 1.$
2. $\int_0^\infty \frac{dx}{1+x^2} = \lim\limits_{b\to\infty} \displaystyle\int_0^b \frac{dx}{1+x^2} = \lim\limits_{b\to\infty} \arctan x \mid_0^b = \frac{\pi}{2}.$
3. $\int_1^\infty \frac{1}{x^p}dx = \lim\limits_{b\to\infty} \displaystyle\int_1^b \frac{1}{x^p}dx = \lim\limits_{b\to\infty} \frac{x^{-p+1}}{1-p} \mid_1^b = \frac{b^{-p+1}}{1-p} - \frac{1}{1-p} = \frac{1}{p-1}$ if $p > 1$.

   Thus $\int_1^\infty \frac{1}{x^p}dx$ converges if $p > 1$ and diverges if $p \leq 1$.

Comparison test

Theorem

Suppose $0 \leq f(x) \leq g(x)$ *for all* $x \geq a$*, then*

1. $\int_a^\infty f(x)dx$ *converges if* $\int_a^\infty g(x)dx$ *converges.*
2. $\int_a^\infty g(x)dx$ *diverges if* $\int_a^\infty f(x)dx$ *diverges.*

Proof.

Define $F(x) = \int_a^x f(t)dt$ and $G(x) = \int_a^x g(t)dt$. Then by properties of Riemann integral, $0 \leq F(x) \leq G(x)$ and we are given that $\lim_{x\to\infty} G(x)$ exists. So $G(x)$ is bounded. F is monotonically increasing and bounded above. Therefore, $\lim_{x\to\infty} F(x)$ exists. $\square$

Examples

1. $\int_1^\infty \frac{dx}{x^2(1+e^x)}$. Note that $\frac{1}{x^2(1+e^x)} < \frac{1}{x^2}$ and $\int_1^\infty \frac{dx}{x^2}$ converges.
2. $\int_1^\infty \frac{x^3}{x+1}dx$. Note that $\frac{x^3}{x+1} \geq \frac{x^2}{2}$ on $[1,\infty)$ and $\int_1^\infty x^2 dx$ diverges.
3. $\int_1^\infty \frac{1}{1+\sqrt{x}}$. Note that $\frac{1}{1+\sqrt{x}} \geq \frac{1}{2\sqrt{x}}$ on $[1,\infty)$ and $\int_1^\infty \frac{1}{\sqrt{x}}dx$ diverges. Therefore $\int_1^\infty \frac{1}{1+\sqrt{x}}$ diverges.
4. $\int_1^\infty \frac{\sqrt{x}}{1+x^5}$. Note that $\frac{\sqrt{x}}{1+x^5} \leq \frac{1}{x^{3/2}}$ and $\int_1^\infty \frac{dx}{x^{3/2}}$ converges.

Limit comparison test

Theorem

Let $f(x), g(x)$ *are defined and positive for all* $x \geq a$ *and* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$.

1. *If* $L \in (0, \infty)$, *then the improper integrals* $\int_a^\infty f(x)dx$ *and* $\int_a^\infty g(x)dx$ *are either both convergent or both divergent. i.e.,* $\int_a^\infty f(x)dx$ *converges* $\iff$ $\int_a^\infty g(x)dx$ *converges.*
2. *If* $L = 0$, *then* $\int_a^\infty f(x)dx$ *converges if* $\int_a^\infty g(x)dx$ *converges. i.e,* $\int_a^\infty g(x)dx$ *converges* $\implies$ $\int_a^\infty f(x)dx$ *converges.*
3. *If* $L = \infty$, *then* $\int_a^\infty f(x)dx$ *diverges if* $\int_a^\infty g(x)dx$ *diverges. i.e.,* $\int_a^\infty g(x)dx$ *diverges* $\implies$ $\int_a^\infty f(x)dx$ *diverges.*

Examples

1. $\int_1^\infty \frac{dx}{\sqrt{x+1}}$. Take $f(x) = \frac{1}{\sqrt{x+1}}$ and $g(x) = \frac{1}{\sqrt{x}}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $\int_1^\infty g(x)dx$ diverges. So by above theorem, $\int_1^\infty f(x)dx$ diverges.

2. $\int_1^\infty \frac{dx}{1+x^2}$. Take $f(x) = \frac{1}{1+x^2}$ and $g(x) = \frac{1}{x^2}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $\int_1^\infty g(x)dx$ converges. So by above theorem, $\int_1^\infty f(x)dx$ converges.

3. $\int_0^\infty \frac{x}{\cosh x}dx$. Let $f(x) = \frac{x}{\cosh x} = \frac{2xe^x}{e^{2x}+1} \sim xe^{-x}$. So choose $g(x) = xe^{-x}$. Then $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 2$ and $\int_0^\infty g(x)dx$ converges.

Improper integrals of second kind

Let $f(x)$ be defined on $[a, c)$ and $f \in \mathcal{R}[a, c-\epsilon]$ for all $\epsilon > 0$. Further, suppose $f(x)$ becomes **unbouded** only at the endpoint $x = c$. Then we define

$$\int_a^c f(x)dx = \lim_{\epsilon \to 0} \int_a^{c-\epsilon} f(x)dx.$$

Then $\int_a^b f(x)dx$ is said to converge if the limit exists and is finite. Otherwise, we say improper integral $\int_a^b f(x)dx$ diverges.

Suppose $a_1, a_2, ...a_n$ are finitely many points in $[a, b]$ where of $f(x)$ is unbounded. Then

$$\int_a^b f(x)dx = \int_a^{a_1} f(x)dx + \int_{a_1}^{a_2} f(x)dx + \int_{a_2}^{a_3} f(x)dx + ... + \int_{a_n}^b f(x)dx$$

If all the improper integrals on the right hand side converge, then we say the improper integral of f over $[a, b]$ converges. Otherwise, we say it diverges.

Example

1. $$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0} 2(1-\sqrt{\epsilon}) = 2.$$
2. $\int_0^1 \frac{1}{x^p} dx = \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{1}{x^p} dx = \lim_{\epsilon\to 0} \frac{x^{-p+1}}{1-p}\Big|_\epsilon^1 = \frac{1}{1-p} - \frac{\epsilon^{-p+1}}{1-p} = \frac{1}{1-p}$ if $p < 1$.

   Thus $\int_0^1 \frac{1}{x^p} dx$ converges if $p < 1$ and diverges if $p \geq 1$.

Theorem (Comparison Theorem)

Suppose $0 \le g(x) \le f(x)$ *for all* $x \in [a, c)$ *and are discontinuous at* c.

1. *If* $\int_a^c f(x)dx$ *converges then* $\int_a^c g(x)dx$ *converges.*
2. *If* $\int_a^c g(x)dx$ *diverges then* $\int_a^c f(x)dx$ *diverges.*

Theorem (Limit comparison theorem)

Suppose $0 < f(x), g(x)$ *be continuous in* $[a, c)$ *and* $\lim_{x \to c} \frac{f(x)}{g(x)} = L$. *Then*

1. *If* $L \in (0, \infty)$. *Then* $\int_a^c f(x)dx$ *and* $\int_a^c g(x)dx$ *both converge or diverge together.*
2. *If* $L = 0$ *and* $\int_a^c g(x)dx$ *converges then* $\int_a^c f(x)dx$ *converges.*
3. *If* $L = \infty$ *and* $\int_a^c g(x)dx$ *diverges then* $\int_a^c f(x)dx$ *diverges.*

Absolutely convergent improper integral

Definition

Let $f \in \mathcal{R}[a,b]$ for all $b > a$. Then we say $\int_a^\infty f(x)dx$ converges absolutely if $\int_a^\infty |f(x)|dx$ converges.

Theorem

If the integral $\int_a^\infty |f(x)|dx$ converges, then the integral $\int_a^\infty f(x)dx$ converges.

Absolute convergence $\implies$ Convergence

Converse is NOT True, in general.

$$\int_\pi^\infty \frac{\sin x}{x} dx$$

$$\begin{aligned}
\int_{\pi}^{\infty} \frac{|\sin x|}{x} dx &= \sum_{n=1}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx \\
&\geq \sum_{n=1}^{\infty} \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin x| dx \\
&= \sum_{n=1}^{\infty} \frac{1}{(n+1)\pi} \int_{0}^{\pi} \sin x \ dx = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n+1}.
\end{aligned}$$

On the other hand, by integration by parts,

$$\begin{aligned}
\lim_{b\to\infty} \int_{\pi}^{b} \frac{\sin x}{x} dx &= \lim_{b\to\infty} \int_{\pi}^{b} \frac{\sin x}{x} dx \\
&= \lim_{b\to\infty} \left(\frac{-\cos b}{b} + \frac{1}{\pi} - \int_{\pi}^{b} \frac{\cos x}{x^2} dx \right) \\
&= \frac{1}{\pi} - \int_{\pi}^{\infty} \frac{\cos x}{x^2} dx,
\end{aligned}$$

It is not difficult to show that the limits on the right exist by comparison test.

Some further results

1. $\int_1^\infty \frac{\sin x}{x^p} dx$ and $\int_1^\infty \frac{\cos x}{x^p} dx$ converges for all $p > 0$.
2. $\int_0^1 \frac{\sin x}{x^p} dx$ converges for all $p < 2$ and $\int_0^1 \frac{\cos x}{x^p} dx$ converges for all $p < 1$.

(Exercise, prove yourselves.)

Thank You.