Solutions to Tutorial Sheet 9
Derivative of a Function of Several Variables

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1. (a) Here $f(x,y) = (x^2 + xy)^3$.

   Hence $\left.\dfrac{\partial f}{\partial x}\right|_{(1,0)} = \lim\limits_{h \to 0} \dfrac{f(1+h,0) - f(1,0)}{h} = \lim\limits_{h \to 0} \dfrac{(1+h)^6 - 1}{h} = 6.$

   And $\left.\dfrac{\partial f}{\partial y}\right|_{(1,0)} = \lim\limits_{k \to 0} \dfrac{f(1,k) - f(1,0)}{k} = \lim\limits_{h \to 0} \dfrac{(1+k)^3 - 1}{k} = 3.$

   (b) $f_x = \dfrac{1}{y} - \dfrac{y}{x^2}$ and $f_y = \dfrac{1}{x} - \dfrac{x}{y^2}$. Therefore, $f_x\Big|_{(\sqrt{2},\sqrt{2})} = f_y\Big|_{(\sqrt{2},\sqrt{2})} = 0.$

2. If $f_x = 0$ then f is independent of x. If $f_y = 0$ then f is independent of y. If $f_x = 0$ and $f_y = 0$ then f is independent of both x and y.

   If f is independent of both x and y then it must be constant.

3. (a) Both the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are 0 at point $(0,0)$. For differentiability $\triangle f = f(h,k) - f(0,0) = f(h,k)$, $df = hf_x(0,0) + kf_y(0,0) = 0$. Consider $\lim\limits_{\rho \to 0} \dfrac{\triangle f - df}{\rho} = \lim\limits_{(h,k)\to(0,0)} \dfrac{f(h,k)}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \dfrac{h \sin \frac{1}{h} + k \sin \frac{1}{k}}{\sqrt{h^2+k^2}}$ fails to exist along $k = h$. Hence not differentiable.

   (b) Both the partial derivatives are 0 at point $(0,0)$. Consider $\lim\limits_{\rho \to 0} \dfrac{\triangle g - dg}{\rho} = \lim\limits_{(x,y)\to(0,0)} \dfrac{g(x,y)}{\sqrt{x^2+y^2}} = \dfrac{xy}{x^2+y^2} = \dfrac{1}{2}$ by taking limit along the line $y = x$. Hence g is not differentiable at $(0,0)$.

4. Given $\epsilon > 0$ we have to find a $\delta > 0$ such that for

$$0 < \sqrt{x^2+y^2} < \delta \implies |\ f(x,y) - 0\ | < \epsilon.$$

   Consider

$$|\ f(x,y) - 0\ | \leq ||\ x\ | - |\ y\ || + |\ x\ | + |\ y\ || \leq 2(|\ x\ | + |\ y\ |) \leq 4\sqrt{x^2+y^2}.$$

   So take $\delta = \frac{\epsilon}{8}$ we have $|\ f(x,y) - 0\ | < \epsilon$. Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$ and the function is continuous at (0,0).
   Also both the the partial derivatives are 0 at point(0,0).
   For differentiability, consider $\lim\limits_{\rho \to 0} \dfrac{\triangle f - df}{\rho} = \lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y)}{\sqrt{x^2+y^2}} = -1$ along y=x.

Hence f is not differentiable at (0,0).

So, we can't apply the formula $D_u f = f_x(0,0)u_1 + f_y(0,0)u_2$. Direction derivative of the function exists only in the direction of $(1,0)$ and $(0,1)$ and this can be checked from the definition as

$$D_{\widehat{p}} f(0,0) = \quad \lim_{t\to 0} \frac{f(t\rho_1, t\rho_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{|t|}{t} \left(||\rho_1| - |\rho_2|| - |\rho_1| - |\rho_2|\right)$$

exists only if $\rho = (1,0)$ or $(0,1)$.

5. $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$. Here $z_x = 5x^4e^{9y}$ and $z_y = 9x^5e^{9y}$. $\therefore dz = 5x^4e^{9y}dx + 9x^5e^{9y}dy$.

6. Here $z_x(1,2) = (3x^2y + y)\Big|_{(1,2)} = 8$ and $z_y(1,2) = (x^3 + x)\Big|_{(1,2)} = 2$. So $dz = 8dx + 2dy$.

7. By chain rule
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$
Here $z_x = 3x^2y + y$ and $z_y = x^3 + x$. Similarly, $x_t = \dfrac{\partial x}{\partial t} = -\sin t$ and $y_t = \dfrac{\partial y}{\partial t} = 2\cos 2t$. Now at $t = \frac{\pi}{4}$, we have $x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $y = 1$. So the final answer is $\frac{-5}{2\sqrt{2}}$.

8. $f_x(1,2) = 20$, $f_y(1,2) = -20$. Directional derivative is greatest when pointing in the direction of the gradient $(20,-20)$. Hence, the direction is $\frac{1}{\sqrt{2}}\,\widehat{i} - \frac{1}{\sqrt{2}}\,\widehat{j}$

9. $f_x(1,3) = -1/5$, $f_y(1,2) = 2/5$. Directional derivative is greatest when pointing in the direction of the gradient $(-1/5, 2/5)$. Hence, the direction is $\frac{-1}{\sqrt{5}}\,\widehat{i} + \frac{2}{\sqrt{5}}\,\widehat{j}$. $f_{\widehat{u}}(1,3) = f_x(1,3)u_1 + f_y(1,3)u_2 = \frac{1}{\sqrt{5}}$

10. Differentiating partially w.r.t. x (and treating z as a function of x; and y as a constant), we get
$$\cos(xyz)\left(yz + xy\frac{\partial z}{\partial x}\right) = 1 + 3\frac{\partial z}{\partial x}.$$
Simplifying, we get
$$\frac{\partial z}{\partial x} = \frac{1 - yz\cos(xyz)}{xy\cos(xyz) - 3}.$$

11. We can approximate $f(4.1, 0.2)$ using $f(4,0) = 0$. The total differential gives us a way of adjusting this initial approximation to hopefully get a more accurate answer.

We let $\Delta z = f(4.1, 0.2) - f(4, 0)$. The total differential dz is approximately equal to Δz, so

$$f(4.1, 0.2) - f(4, 0) \approx dz \implies f(4.1, 0.2) \approx dz + f(4, 0).$$

To find dz, we need f_x and f_y. $f_x(x, y) = \frac{\sin y}{2\sqrt{x}} \implies f_x(4, 0) = 0$, and
$f_y(x, y) = \sqrt{x} \cos y \implies f_y(4, 0) = 2$.
Approximating 4.1 with 4 gives $dx = 0.1$; approximating 0.2 with 0 gives $dy = 0.2$. Thus

$$dz(4, 0) = f_x(4, 0)(0.1) + f_y(4, 0)(0.2) = 0(0.1) + 2(0.2) = 0.4.$$

$$\therefore, f(4.1, 0.2) \approx 0.4 + 0 = .4.$$

12. The total differential approximates how much f changes from the point $(2, -3)$ to the point $(2.1, -3.03)$. With $dx = 0.1$ and $dy = -0.03$, we have

$$dz = f_x(2, -3)dx + f_y(2, -3)dy = 1.3(0.1) + (-0.6)(-0.03) = 0.148.$$

The change in z is approximately 0.148, so we approximate $f(2.1, -3.03) \approx 6.148$.

13. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dt} = (-7)(-1) + 2(3) = 13.$

14. Let $f(x, y) = \sin(xy) + y^2 + x - 5$. Then $f_x(x, y) = y \cos(xy) + 1$ and $f_y(x, y) = x \cos(xy) + 2y$. Then
$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{y \cos(xy) + 1}{x \cos(xy) + 2y}.$$

15. $\nabla f = \dfrac{x}{8}\hat{i} + \dfrac{y}{4}\hat{j} + \dfrac{z}{2}\hat{k}$. So the equation of the tangent is given by $\frac{1}{4}(x - 2) + \frac{1}{2}(y - 2) + \frac{1}{2}(z - 1) = 0 \implies x + 2y + 2z = 12$.

16. $\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \implies \nabla f\Big|_{(1,0,3)} = 2\hat{i} + 6\hat{k}$.
Therefore, the equation of the tangent plane is given by

$$2(x - 1) + 0(y - 0) + 6(z - 3) = 0 \text{ i.e. } x + 3z = 10.$$

The normal line is given by

$$\begin{aligned}\overrightarrow{r}(t) &= (\hat{i} + 0\hat{j} + 3\hat{k}) + t(2\hat{i} + 0\hat{j} + 6\hat{k}) \\ &= (1 + 2t)\hat{i} + (3 + 6t)\hat{k}.\end{aligned}$$