Double Integral

BENNETT UNIVERSITY
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Double Integral, Signed Volume

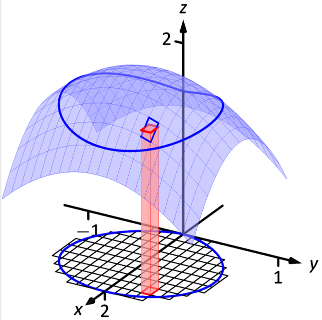
Definition

Let $z = f(x, y)$ be a continuous function defined over a closed region R in the x-y plane. The signed volume V under f over R is denoted by the double integral

$$V = \iint\limits_R f(x, y)\ dA$$

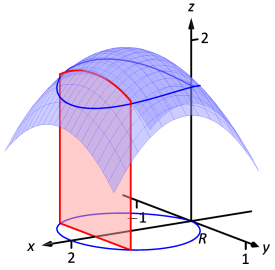
$$= \iint\limits_R f(x, y)\ dx\ dy$$

$$= \iint\limits_R f(x, y)\ dy\ dx.$$



Result for evaluating double integrals to find volume

Let $z = f(x, y)$ be a continuous function defined over a closed region R in the x-y plane. Then the signed volume V under f over R is



Finding volume under a surface by sweeping out a cross-sectional area.

$$V = \iint_R f(x, y)\, dA = \lim_{\|\Delta A\| \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i.$$

Method for finding signed volume under a surface

Fubini's Theorem

Let R be a closed, bounded region in the x-y plane and let $z = f(x, y)$ be a continuous function on R.

1. If R is bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$, then

$$\iint\limits_R f(x, y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\ dy\ dx.$$

2. If R is bounded by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$, then

$$\iint\limits_R f(x, y)\ dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\ dx\ dy.$$

Examples

- **Example 1:** Evaluate $\iint\limits_R (e^y + xy)\, dA$, where R is the rectangle with corners $(3,1)$ and $(4,2)$.

  Solution:

$$\begin{aligned}
\iint\limits_R (e^y + xy) dA &= \int_1^2 \left(\int_3^4 (e^y + xy) dx \right) dy \\
&= \int_1^2 \left(xe^y + \frac{x^2 y}{2} \right) \Big|_3^4 dy \\
&= \int_1^2 \left(e^y + \frac{7}{2} y \right) dy \\
&= \left(e^y + \frac{7}{4} y^2 \right) \Big|_1^2 = e^2 - e + \frac{21}{4}.
\end{aligned}$$

- **Example 2:** Evaluate $\iint\limits_R x^2 y \, dA$, where R is bounded by $y = \sqrt{x}$ and $y = x^2$.

Ans:

$$\begin{aligned}
\iint\limits_R x^2 y \, dA &= \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y \, dy \, dx \\
&= \int_0^1 \frac{x^2}{2}(x - x^4) \, dx \\
&= \frac{1}{2}\int_0^1 (x^3 - x^6) dx \\
&= \frac{1}{2}\left(\frac{x^4}{4} - \frac{x^7}{7}\right)\Big|_0^1 \\
&= \frac{3}{56}.
\end{aligned}$$

Exercise: Do it by solving $\int_0^1 \int_{y^2}^{\sqrt{y}} x^2 y \, dx \, dy$

- **Example 3:** Evaluate $\iint_R (x^2 - y^2)dA$, where R is the rectangle with vertices $(-1,-1), (-1,1), (1,1)$ and $(1,-1)$.

Ans:

$$\begin{aligned}
\iint_R (x^2 - y^2)dA &= \int_{-1}^{1}\int_{-1}^{1}(x^2 - y^2)dx\ dy \\
&= \int_{-1}^{1}\left(\frac{x^3}{3} - y^2 x\right)\Bigg|_{-1}^{1} dy \\
&= \int_{-1}^{1}\left(\frac{2}{3} - 2y^2\right) dy \\
&= \left[\frac{2}{3}y - \frac{2}{3}y^3\right]_{-1}^{1} = 0.
\end{aligned}$$

How could the ~~volume~~ of a region be zero?

signed volume!

Double integrals with polar coordinates

Result for evaluating double integrals with polar coordinates

Let R be a plane region bounded by the polar equations $\alpha \leq \theta \leq \beta$ and $g_1(\theta) \leq r \leq g_2(\theta)$. Then

$$\iint\limits_R f(x, y)\ dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta.$$

Examples

- **Example 1:** Find the volume of a sphere with radius a.

  Ans: The sphere of radius R, centered at the origin, has equation $x^2 + y^2 + z^2 = a^2$; solving for z, we have $z = \sqrt{a^2 - x^2 - y^2}$. This gives the upper half of a sphere. Polar bounds for this equation are $0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi$. So the volume of the sphere is given by

$$\begin{aligned} 2\iint\limits_R \sqrt{a^2 - x^2 - y^2}\ dA &= 2\int_0^{2\pi}\int_0^a \sqrt{a^2 - (r\cos\theta)^2 - (r\sin\theta)^2}\ r\ dr\ d\theta \\ &= 2\int_0^{2\pi}\int_0^a r\sqrt{a^2 - r^2}\ dr\ d\theta \\ &= \int_0^{2\pi} \frac{2}{3}a^3\ d\theta = \frac{4}{3}\pi a^3. \end{aligned}$$

- **Example 2:** Find the volume under the surface $f(x,y) = \frac{1}{x^2+y^2+1}$ over the sector of the circle with radius a centered at the origin in the first quadrant.

  Ans: In polar, the bounds on R are $0 \le r \le a, 0 \le \theta \le \frac{\pi}{2}$. Therefore, the required volume is

$$\begin{aligned}
\iint\limits_R f(x,y)\ dA &= \int_0^{\frac{\pi}{2}} \int_0^a \frac{r}{r^2+1}\ dr\ d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{2} \ln|r^2+1| \Big|_0^a\ d\theta \\
&= \left(\frac{1}{2} \ln(a^2+1)\ \theta \right) \Big|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{4} \ln(a^2+1).
\end{aligned}$$

Thank You.