Error propagation

Justification of the "Addition in Quadrature" rule.

When uncertainties are random and independent, the usual rule is to add the uncertainties in quadrature (that is using squares of the uncertainties).

That is if I have $x \pm \sigma_x$ and $y \pm \sigma_y$, the uncertainty in $z = x + y$ is given by $\Delta z = \sqrt{\sigma_x^2 + \sigma_y^2}$.

We can now see where this rule comes from. Suppose we make measurements of two parameters, x and y, which are normally distributed about their true values X and Y with widths σ_x and σ_y respectively. For simplicity, assume that the true values of both X and Y are zero.
Consequently, the probability of getting a particular value of x is:

$$Prob(x) \propto e^{-\frac{x^2}{2\sigma_x^2}} ,$$

and that for y is:

$$Prob(y) \propto e^{-\frac{y^2}{2\sigma_y^2}} .$$

We are interested in the probability of finding any particular value of $x + y$, and to see what distribution this gives us. As x and y are independently measured the probability of finding any given x and any given y is just the product of the probabilities above:

$$Prob(x, y) \propto \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)\right].$$

As the quantity we are interested in is $x + y$, we can rewrite the exponent using an identity involving $x + y$:

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} = \frac{(x+y)^2}{\sigma_x^2 + \sigma_y^2} + \frac{\left(\sigma_y^2 x - \sigma_x^2 y\right)^2}{\sigma_x^2 \sigma_y^2 (\sigma_x^2 + \sigma_y^2)}$$

$$= \frac{(x+y)^2}{\sigma_x^2 + \sigma_y^2} + z^2$$

So,

$$Prob(x, y) \propto \exp\left[-\frac{1}{2}\left(\frac{(x+y)^2}{\sigma_x^2 + \sigma_y^2} + z^2\right)\right],$$

As we now have an expression in terms of $x + y$ (and unfortunately z), we can write:

$$Prob(x + y, z) \propto \exp\left[-\frac{1}{2}\left(\frac{(x+y)^2}{\sigma_x^2 + \sigma_y^2}\right)\right] \exp\left[\frac{-z^2}{2}\right].$$

To find the probability of obtaining a given value of $x + y$ irrespective of the value of z, we integrate over all values of z:

$$Prob(x + y) = \int_{-\infty}^{+\infty} Prob(x + y, z)dz.$$

Happily the $\exp(-z^2\ /2)$ factor integrates to $\sqrt{2\pi}$, and so we find the probability of finding a value of $x + y$ is given by:

$$Prob(x + y) \propto \exp\left[\frac{-(x + y)^2}{2(\sigma_x^2 + \sigma_y^2)}\right],$$

which shows the values of z= $x + y$ are normally distributed with a width of $\Delta z = \sqrt{\sigma_x^2 + \sigma_y^2}$ as expected.

To generalise to the situation where X and Y are nonzero, we can write:

$$x + y = (x - X) + (y - Y) + (X + Y)\,.$$

The first two terms are centred on zero, as X and Y are the centre positions of their respective distributions with widths σ_x and σ_y. Consequently, the sum of the first two terms is normally distributed with a width $\sqrt{\sigma_x^2 + \sigma_y^2}$ as we have just proven.

The third term is a just a number, so it's effect is to move the centre of the distribution given by $x + y$ to the position $X + Y$.

So we have that for any value of X and Y the distribution of values $(x + y)$ are normally distributed with a width of $\sqrt{\sigma_x^2 + \sigma_y^2}$. In other words the uncertainty in measuring $z = x + y$ is given by $\Delta z = \sqrt{\sigma_x^2 + \sigma_y^2}$.

A general expression for error propagation

Suppose we measure two independent quantities x and y whose observed values are each normally distributed, and we want to calculate some quantity $q(x, y)$ which depends on x and y.

Assumption: we will assume that the uncertainties in x and y, σ_x and σ_y respectively are small relative to the true values X and Y.

If we have the value of some function at position x, say $f(x)$, we can find the value of the function at a small increment u, by using the following approximation (essentially the definition of the derivative):

$$f(x + u) \approx f(x) + \frac{df}{dx}u\,.$$

So, assuming that the values of x and y are close to the true values of X and Y, we can write:

$$q(x, y) \approx q(X, Y) + \left(\frac{\partial q}{\partial x}\right)(x - X) + \left(\frac{\partial q}{\partial y}\right)(y - Y).$$

Here $\frac{\partial q}{\partial x}$ represents the partial derivative of q with respect to x. (Essentially treat the other variables as constants when calculating this derivative).
The first term is just a fixed number and so just shifts the centre of the distribution as we've seen before. For the second term the distribution of values of $(x - X)$ is just σ_x, centred on zero as we've seen before and therefore the width of the distribution of the second term is :

$$\left(\frac{\partial q}{\partial x}\right)\sigma_x .$$

Similarly, the values of the third term are centred on zero and with width:

$$\left(\frac{\partial q}{\partial y}\right)\sigma_y .$$

Combining the results, we have that the values of $q(x, y)$ are normally distributed about the true value, $q(X, Y)$ with a width:

$$\Delta q = \sqrt{\left(\frac{\partial q}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial q}{\partial y}\sigma_y\right)^2}.$$

Based on this result, we can write the general formula for uncertainty propagation for random and independent uncertainties.
If $q = q(x, \dots, z)$ is a function of variables $x, \dots, z$ the uncertainty in q, Δq is given by:

$$\Delta q = \sqrt{\left(\frac{\partial q}{\partial x}\Delta x\right)^2 + \cdots + \left(\frac{\partial q}{\partial z}\Delta z\right)^2}$$

An example:
Suppose we have an experiment to measure the acceleration due to gravity, g, using a simple pendulum.
The relationship between g and the period of the pendulum's swing, T, is given by:

$$g = \frac{4\pi^2 l}{T^2},$$

where l is the length of the pendulum.
We have the following values: $l = 92.9 \pm 0.1\ cm$ and $T = 1.936 \pm 0.004\ s$. Calculation (in SI units) gives $g = 9.785\ ms^{-2}$.

The uncertainty in g is given by:

$$\Delta g = \sqrt{\left(\frac{\partial g}{\partial l}\Delta l\right)^2 + \left(\frac{\partial g}{\partial T}\Delta T\right)^2}.$$

So:

$$\frac{\partial g}{\partial l} = \frac{4\pi^2}{T^2}$$

$\frac{\partial g}{\partial l}\Delta l = 0.01053\ ms^{-2}$ (converting to SI units and substituting in values for all quantities).

Note that the units of each of the partial derivative terms in the summation must have the same dimensions as the quantity the uncertainty is being calculated for (here g).

Also:

$$\frac{\partial g}{\partial T} = -8\pi^2 l T^{-3}\,,$$

$$\frac{\partial g}{\partial T}\Delta T = -0.04043\ ms^{-2}\,.$$

Therefore:

$$\Delta g = \sqrt{(0.01053)^2 + (-0.04043)^2}\,,$$

$$\Delta g = 0.0418\ ms^{-2}$$

Consequently,

$$g = 9.79 \pm 0.04\, ms^{-2}.$$

The general expression for uncertainty propagation can be used in any situation. It is also straightforward to derive the individual "rules" for uncertainty propagation that will have been encountered previously. Note that for some expressions the individual "rules" cannot be used and only the general formula will work.

Sums and differences

$$q = x + \cdots + z - (u + \cdots + w),$$

Rule: add the absolute uncertainties in quadrature:

$$\Delta q = \sqrt{\Delta x^2 + \cdots + \Delta z^2 + \Delta u^2 + \cdots + \Delta w^2}.$$

Products and quotients

$$q = \frac{x \times \ldots \times z}{u \times \ldots \times w}$$

Rule: add the relative uncertainties in quadrature:

$$\frac{\Delta q}{q} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \cdots + \left(\frac{\Delta z}{z}\right)^2 + \left(\frac{\Delta u}{u}\right)^2 + \cdots + \left(\frac{\Delta w}{w}\right)^2}.$$

Powers

$$q = x^n,$$

$$\frac{\Delta q}{|q|} = |n|\frac{\Delta x}{|x|}.$$

Functions of one variable

$$q = q(x),$$

$$\Delta q = \left|\frac{dq}{dx}\right| \Delta x$$

Uncertainties when your quantity is calculated numerically

In some situations, you don't have an analytical expression for your quantity. A typical example would be if your quantity is calculated by numerically integrating some expression, or you have a computer script which performs a calculation based on parameters (a "black box" function).
In these situations you can evaluate the uncertainty by propagating small changes in your parameters through your function/script and observing the change in the output value that occurs.

For simplicity, we'll assume that the uncertainty in the "black box" function is only due to the uncertainty in one parameter, x.

Recognising that $q(x) + \Delta q(x) = q(x + \Delta x)$, we immediately have:

$$\Delta q(x) = q(x + \Delta x) - q(x).$$

Consequently, we can find the uncertainty in the function $\Delta q(x)$ by substituting $x + \Delta x$ into the function and finding the value ($q(x + \Delta x)$) and then subtracting off the value of the function calculated using x ($q(x)$).
For functions of more than one variable with uncertainties (such as a numerical model) the final uncertainty is found by combining the individual uncertainties found for each variable in quadrature, i.e;

$$\Delta q = \sqrt{(\Delta q(x))^2 + \cdots + (\Delta q(z))^2}$$

This can be seen from the general expression for uncertainty propagation and the definition of the derivative which shows us that:

$$\Delta q(x) = \frac{dq}{dx}\Delta x$$

In these situations, the way to proceed is the following:

1. Calculate the value of your function using your experimentally found values ($q(x)$).
2. Calculate the value of your function with one of the parameters increased by its uncertainty ($q(x + \Delta x)$).

3. Subtract off the original value of your function to give you the uncertainty due to varying this parameter (this gives $\Delta q(x)$).
4. Repeat the calculations by varying each parameter in turn by its uncertainty.
5. Calculate the final uncertainty by adding all the individual absolute uncertainties in quadrature.