# Uncertainty Propagation

**Justification of the "Addition in Quadrature" rule**

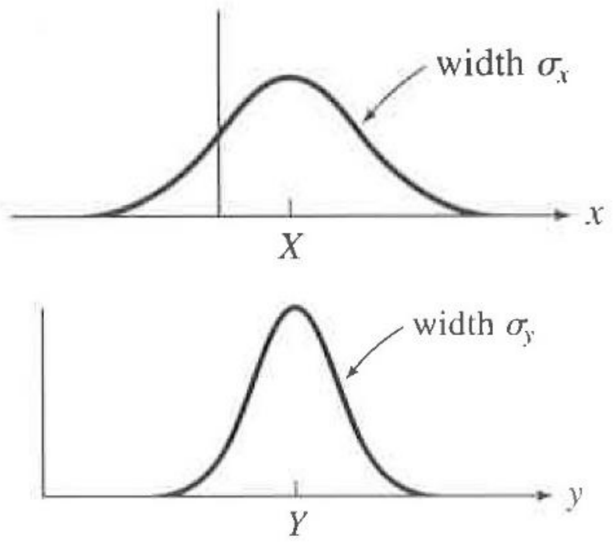
When uncertainties are random and independent, the usual rule is to add the uncertainties in quadrature (i.e. using squares of the uncertainties).

From measurements find: $x \pm \sigma_x$ and $y \pm \sigma_y$

And we want : $z = x + y$

The uncertainty in $z$ : $\Delta z = \sqrt{\sigma_x^2 + \sigma_y^2}$

- We can now see where this rule comes from.




Assume:

- We make measurements of two parameters: values are given by a range of $x$ and $y$.
- The **true value** of the parameters are $X$ and $Y$ .
- Measurements are normally distributed (Gaussian).
- The widths of the distributions are $\sigma_x$ and $\sigma_y$ respectively.

For simplicity, now assume that the true values of both $X$ and $Y$ are zero.

The probability of getting a particular value of $x$ is:

$$Prob(x) \propto e^{-\frac{x^2}{2\sigma_x^2}}$$

For $y$:

$$Prob(y) \propto e^{-\frac{y^2}{2\sigma_y^2}}$$

We are interested in the probability of finding any particular value of $z = x + y$ : $\Rightarrow$

- As $x$ and $y$ are independent, just multiply the probabilities.

The probability finding any given $x$ **and** any given $y$:

$$Prob(x, y) \propto \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)\right]$$

- As we are interested in $z = x + y$, we will re-write the exponent

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} = \frac{(x+y)^2}{\sigma_x^2 + \sigma_y^2} + \frac{\left(\sigma_y^2 x - \sigma_x^2 y\right)^2}{\sigma_x^2 \sigma_y^2 (\sigma_x^2 + \sigma_y^2)} \quad \Rightarrow \quad = \frac{(x+y)^2}{\sigma_x^2 + \sigma_y^2} + q^2$$

So: $$Prob(x, y) \propto \exp\left[-\frac{1}{2}\left(\frac{(x+y)^2}{\sigma_x^2+\sigma_y^2}+q^2\right)\right]$$

We now have an expression in terms of $x + y$ (and unfortunately $q$)

Rewrite: $$Prob(x+y, q) \propto \exp\left[-\frac{1}{2}\left(\frac{(x+y)^2}{\sigma_x^2+\sigma_y^2}\right)\right]\exp\left[\frac{-q^2}{2}\right]$$

To find the probability of of $x + y$ irrespective of $q$: $\Rightarrow$

- Integrate over all values of $q$:

$$Prob(x+y) = \int_{-\infty}^{+\infty} Prob(x+y, q)dq\,.$$

- $\exp(-q^2\ /2)$ factor integrates to $\sqrt{2\pi}$

$z$

$$Prob(x+y) \propto \exp\left[\frac{-(x+y)^2}{2(\sigma_x^2+\sigma_y^2)}\right]$$

$\Delta z$

$$\boxed{Prob(x) \propto e^{-\frac{x^2}{2\sigma_x^2}}}$$

- Thus the values of $z$= $x + y$ are normally distributed with a width of $\Delta z = \sqrt{\sigma_x^2+\sigma_y^2}$.

**To generalise to the situation where $X$ and $Y$ are nonzero:**

Value from measurements

$$x + y = (x - X) + (y - Y) + (X + Y)$$

True values

- These terms are centred about zero.
- The widths are $\sigma_x$ and $\sigma_y$.
- For the sum, the width is $\sqrt{\sigma_x^2 + \sigma_y^2}$ (as we previously worked out).

- This is just a number.
- Moves the centre of the distribution (given by $x + y$) to position $X + Y$.

Overall:

The uncertainty in measuring $z = x + y$ is given by $\Delta z = \sqrt{\sigma_x^2 + \sigma_y^2}$ (independent of $X$ and $Y$).

# The General Expression for Uncertainty Propagation

There is a single “rule” for calculating uncertainties for any parameter calculated from measurements.

- Suppose we measure two independent quantities $x$ and $y$ whose observed values are each normally distributed.
- We want to calculate some quantity $q(x, y)$ which depends on $x$ and $y$.

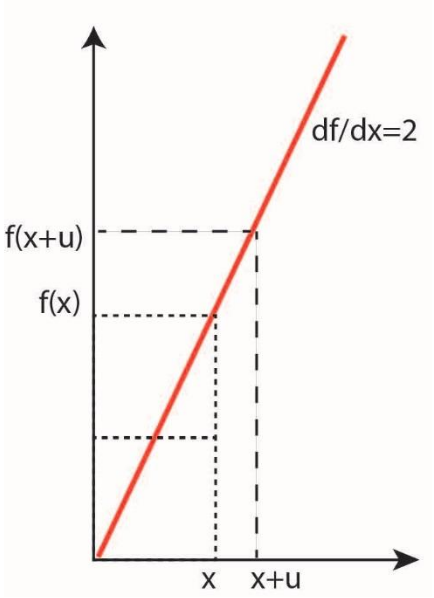
Assumption: ⇨

- The uncertainties in $x$ and $y$, $\sigma_x$ and $\sigma_y$ are small relative to the true values $X$ and $Y$.

**Recall the definition of the derivative:**

If a function has the value $f(x)$ at $x$, we can find the value of the function at a small increment $u$, by using the following approximation:

$$f(x+u) \approx f(x) + \frac{df}{dx}u$$



If the values of $x$ and $y$ are close to the true values of $X$ and $Y$, we can write:

$$q(x,y) \approx q(X,Y) + \left(\frac{\partial q}{\partial x}\right)(x - X) + \left(\frac{\partial q}{\partial y}\right)(y - Y).$$

Our function that depends on $x$ and $y$.

"$u$" in $x$ direction

"$u$" in $y$ direction

partial derivative of $q$ with respect to $x$.

$q(X,Y)$ $\Rightarrow$

- just a fixed number and so just shifts the centre of the distribution.

$\left(\frac{\partial q}{\partial x}\right)(x - X)$ $\Rightarrow$

- the distribution of values of $(x - X)$ is just $\sigma_x$ (centred on 0).

$$\left(\frac{\partial q}{\partial x}\right)(x - X) = \left(\frac{\partial q}{\partial x}\right)\sigma_x$$

- Also, we assume that the derivatives are slowly varying, so that the shape of $\left(\frac{\partial q}{\partial x}\right)\sigma_x$ is still approximately a Gaussian.
- Hence, the width of the distribution for $q(x,y)$ is given by the sum of two Gaussians. (see p4.)

**Result:**

The values of $q(x,y)$ are normally distributed about the true value, $q(X,Y)$ with a width:

$$\Delta q = \sqrt{\left(\frac{\partial q}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial q}{\partial y}\sigma_y\right)^2}$$

**General formula for uncertainty propagation for random and independent uncertainties:**

If $q = q(x, \dots, z)$ is a function of variables $x, \dots, z$ the uncertainty in $q$ is given by:

$$\Delta q = \sqrt{\left(\frac{\partial q}{\partial x}\Delta x\right)^2 + \dots + \left(\frac{\partial q}{\partial z}\Delta z\right)^2}$$

An example:

- Suppose we have an experiment to measure the acceleration due to gravity, $g$, using a simple pendulum.
- The relationship between $g$ and the period of the pendulum's swing, $T$, is given by:

$$g = \frac{4\pi^2 l}{T^2}$$

Length of the pendulum

- Values: $l = 92.9 \pm 0.1\ cm$ and $T = 1.936 \pm 0.004\ s.$
- Calculation (in SI units) gives $g = 9.785\ ms^{-2}$

The uncertainty in $g$ is given by: $\Rightarrow$

$$\Delta g = \sqrt{\left(\frac{\partial g}{\partial l}\Delta l\right)^2 + \left(\frac{\partial g}{\partial T}\Delta T\right)^2}$$

$$\boxed{g = \frac{4\pi^2 l}{T^2}}$$

$$\frac{\partial g}{\partial l} = \frac{4\pi^2}{T^2}$$

$$\frac{\partial g}{\partial l}\Delta l = 0.01053\ \ ms^{-2}$$ (converting to SI units)

The units of each of the partial derivative terms must have the same dimensions as the quantity the uncertainty is being calculated for (here $g$ in $ms^{-2}$).

Also:

$$\frac{\partial g}{\partial T} = -8\pi^2 l T^{-3}$$

$$\frac{\partial g}{\partial T}\Delta T = -0.04043\ \ ms^{-2}$$

Therefore:

$$\Delta g = \sqrt{(0.01053)^2 + (-0.04043)^2}$$

$$\Delta g = 0.0418\ \ ms^{-2}$$

Consequently:

$$g = 9.79 \pm 0.04\, ms^{-2}$$

The general expression for uncertainty propagation can be used in any situation.

- It is straightforward to derive the individual “rules” that will have been encountered previously.
- For some expressions the individual rules cannot be used and **only the general formula** will work.

E.g. $q = \frac{x+y}{x+z}$

## Individual rules

**Sums and differences:**

$$q = x + \cdots + z - (u + \cdots + w)$$

Rule: add the **absolute** uncertainties in quadrature:

$$\Delta q = \sqrt{\Delta x^2 + \cdots + \Delta z^2 + \Delta u^2 + \cdots + \Delta w^2}$$

**Products and quotients:**

$$q = \frac{x \times \cdots \times z}{u \times \cdots \times w}$$

Rule: add the **relative** uncertainties in quadrature:

$$\frac{\Delta q}{q} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \cdots + \left(\frac{\Delta z}{z}\right)^2 + \left(\frac{\Delta u}{u}\right)^2 + \cdots + \left(\frac{\Delta w}{w}\right)^2}$$

**Powers:**

$$q = x^n$$

$$\frac{\Delta q}{|q|} = |n| \frac{\Delta x}{|x|}$$

**Functions of one variable:**

$$q = q(x)$$

$$\Delta q = \left| \frac{dq}{dx} \right| \Delta x$$

Exercises 1+2

# Uncertainties when your quantity is calculated numerically

- In some situations, you don't have an analytical expression for your quantity.

Typical examples:
- The quantity is calculated by numerically integrating an expression.
- A computer script performs a calculation based on input parameters (a "black box" function).

In these situations you can evaluate the uncertainty by propagating small changes in your parameters through your function/script and observing the change in the output value that occurs.

**Proof:**
- Assume that the uncertainty in the "black box" function $q(x)$ is only due to the uncertainty in one parameter, $x$.

From the definition of the derivative:

$$q(x + \Delta x) = q(x) + \frac{dq}{dx}\Delta x$$

This is just $\Delta q$ (from error prop. rules)

So we have:

$$\Delta q(x) = q(x + \Delta x) - q(x)$$

$$\boxed{\Delta q(x) = q(x + \Delta x) - q(x)}$$

To find the uncertainty in the function $\Delta q(x)$:

- Substitute $x + \Delta x$ into the function and then subtract off the value of the function calculated using $x$ to give the uncertainty.

For functions of more than one variable: ⇨

- Calculate the final uncertainty by varying each variable in turn, and then combine the uncertainties found in quadrature:

$$\Delta q = \sqrt{\left(\Delta q(x)\right)^2 + \cdots + \left(\Delta q(z)\right)^2}$$

Uncertainty due to varying $z$

Uncertainty due to varying $x$

**Recipe:**

1. Calculate the value of your function using your experimentally found values $(q(x))$.
2. Calculate the value of your function with one of the parameters increased by its uncertainty $(q(x + \Delta x))$.
3. Subtract off the original value of your function to give you the uncertainty due to varying this parameter (this gives $\Delta q(x)$).
4. Repeat the calculations by varying each parameter in turn by its uncertainty.
5. Calculate the final uncertainty by adding all the individual absolute uncertainties in quadrature.

Exercise 3

# Rejection of Data

- Some scientists believe that data should never be rejected unless you know there is a mistake.
- History has shown that anomalous results have been a window to new physics.

The best approach is to redo the measurement(s)

Assuming you can't retake the data:

- Usually establishing a cause for anomalous results isn't possible.
- The Gaussian distribution can be used to provide insight into the plausibility of a particular measurement.

Assume you make $N$ measurements, $x_1 \; \ldots , x_n$, and one is suspiciously different, $x_{sus}$.

**Chauvenet's Criterion:**

If the expected number of measurements as deviated as the suspect one is less the 0.5, then reject the suspect measurement.

An example:

We have 6 measurements of the period of a pendulum (in s):

$p$ = 3.8, 3.5. 3.9, 3.9, 3.4, 1.8.

Should we consider rejecting this measurement?

**Step 1.**

- Calculate how many standard deviations away from the mean the measurement is.

Mean: $\bar{p} = 3.4\text{ s}$

Standard deviation: $\sigma_p = 0.8\ s$

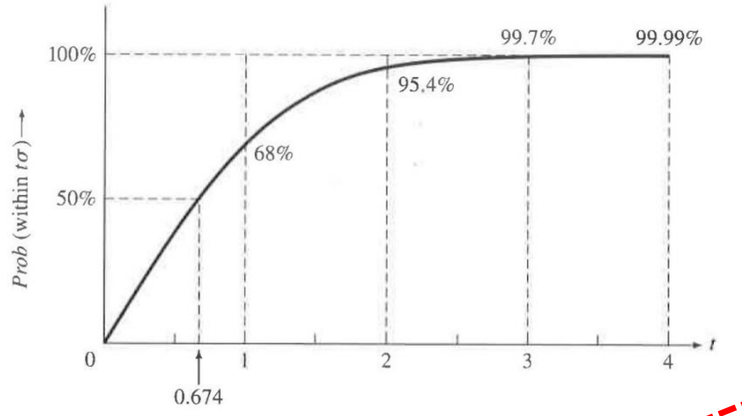
So, the suspect measurement is $3.4 - 1.8 = 1.6\ s$ away from the mean.

Thus, the number of standard deviations the measurement is away from the mean is:

$$n = \frac{1.6}{0.8} = 2.0$$

**Step 2.** • Calculate the probability of a measurement occurring this far away from the mean.

Two options: 1. Look up a table of probability vs sigma (no. of std dev) values:




| $t$ | 0 | 0.25 | 0.5 | 0.75 | 1.0 | 1.25 | 1.5 | 1.75 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Prob* (%) | 0 | 20 | 38 | 55 | 68 | 79 | 87 | 92 | 95.4 | 98.8 | 99.7 | 99.95 | 99.99 |

2 standard deviations away

2. Calculate the value numerically using the error function:

Probability of being within $t$ sigma of the mean:

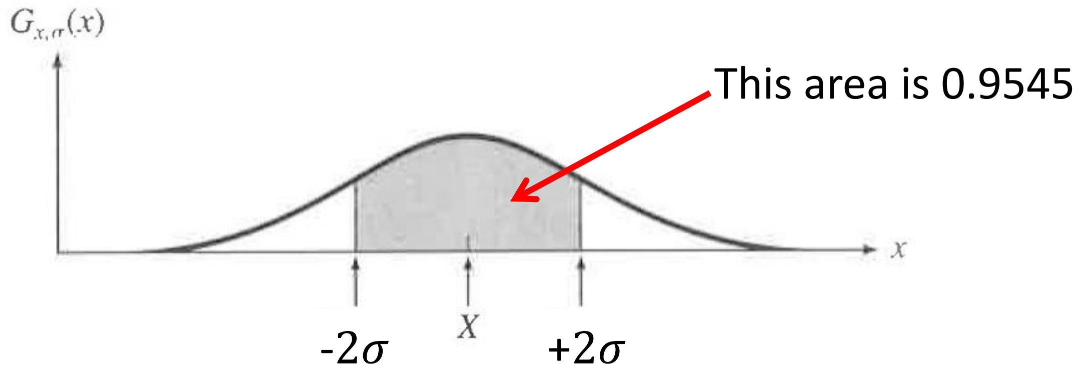
$$Prob(\text{withing t} \times \text{sigma}) = \text{erf}\left(\frac{t}{\sqrt{2}}\right)$$

$$Prob\ (within\ \ 2.0\sigma) = 0.9545$$

So we expect that the probability of a measurement being as deviated as much the suspect one is:

$$Prob\ (outside\ 2.0\sigma) = 1 - Prob\ (within\ 2.0\sigma)$$

$$= \ 0.045$$




**Step 3.** Calculate the number of measurements in the experiment that would be expected to be as deviated as the suspect result.

Here we have n=6 measurements.

Consequently the number of measurements outside 2.0$\sigma$ expected is:

$$n(outside\ 2.0\sigma) = \ 0.045 \times 6$$

$$= 0.225$$

**Step 4:** Apply Chauvenet's criterion

As the expected number of deviant measurements is less than 0.5, rejection can be considered.

**Rationale:** As you are only interested in measurements that are outside the distribution **on one side** (in our case the lower side), the number of expected deviant measurements is actually 0.5× that calculated at step 3 (the "1-tailed" distribution).

The criterion of rejection if the number of expected events is less than 0.5 is arbitrary, but does provide a guide to how likely you are to see aberrant values.

If there is more than 1 suspect point:

- Identify the most deviant data point and apply the test.
- Once you have rejected the most deviant point, recalculate the statistics to see if Chauvenet's criterion suggests exclusion for the next most deviant.

Exercise 4