The First Forbidden Shape Factor and the $f_n t$ Products for Beta-Decay*

JACK P. DAVIDSON, JR.
Washington University, St. Louis, Missouri

(Received November 6, 1950)

The shape factor used in making Kurie plots for first forbidden beta-decay transitions with maximum spin change is expressed in a form suitable for computation. Graphs of the function for various Z and W are given.

Approximations for the $f_n t$ products of beta-decay are derived enabling one to compute these products for certain types of transitions from the values of the allowed f_0 function. Finally, these results are used to compile a table of 1st, 2nd, and 3rd forbidden transitions exhibiting the unique spectral shapes associated with maximum spin change for the given degree of forbiddenness.

I. THE FIRST FORBIDDEN BETA-DECAY SHAPE FACTOR

IN making a Kurie plot for a beta-active isotope, one plots $(N/C_{nX}F_0)^{\frac{1}{2}}$ against the energy W. N is the number of counts per unit energy interval, F_0 is the Fermi function, and C_{nX} is a correction term such that the probability distribution function for the energy is $C_{nX}F_0$; n refers to the forbiddenness of the transition and X to the relativistic form of the interaction.[1] The present discussion is confined to the special case of maximum spin change for given n and X referring to tensor or axial vector coupling.

Many critical experiments have shown that C_{nX} must indeed be appropriate to the degree of forbiddenness under consideration in order that the Kurie plot may have the desired constant-slope form. For $n=1([\Delta I]_{\max}=\pm 2$, yes), the correction term C_{1T} has the asymptotic form

$$C_{1T}=[(W_0-W)^2+(W^2-1)]/12, \tag{1}$$

which is valid only[2] for $\alpha Z\ll 1$. In many applications this condition does not hold, and in some calculations made for Osoba,[3] a better fit was obtained by using the accurate form

$$C_{1T}=[(1+S_0)/24][(W_0-W)^2+A(W^2-1)]. \tag{2}$$

The exact expression for A is[2]

$$A=\frac{S_1+2}{2S_0+2}\left(\frac{12\Gamma(2S_0+1)}{\Gamma(2S_1+1)}\right)^2 \times(2p\rho)^{2(S_1-S_0-1)}\left|\frac{\Gamma(S_1+iy)}{\Gamma(S_0+iy)}\right|^2, \tag{3}$$

in which $y=\alpha ZW/p$, $p=(W^2-1)^{\frac{1}{2}}$, $S_0=(1-\alpha^2Z^2)^{\frac{1}{2}}$, $S_1=(4-\alpha^2Z^2)^{\frac{1}{2}}$, and ρ is the nuclear radius.[4] By making use of the natural logarithm of the complex gamma-function and the logarithmic derivative,[5] one obtains from Eq. (3) the following expression which is suitable for computational purposes:

$$A=\frac{S_1+2}{2S_0+2}\left[\frac{12\Gamma(2S_0+1)}{\Gamma(2S_1+1)}\right]^2(2p\rho)^{\alpha^2Z^2/2}|1-\tfrac{1}{4}\alpha^2Z^2+iy|^2 \times\left[1-\tfrac{1}{2}\alpha^2Z^2C+\tfrac{1}{2}\alpha^2Z^2y^2\sum_{n=1}^{\infty}\frac{1}{n(n^2+y^2)}\right], \tag{4}$$

where C is the gauss number. This expression is identical with that used by Osoba[3] through terms in α^2Z^2. Figure 1 shows A plotted against W for $Z=0$, 30, 60, and 90, while in Fig. 2 A is plotted for the same values of A but for small W. Certain points should be noted. Firstly, one has

$$\lim_{Z\to 0}A(W,Z)=1 \text{ for all } W. \tag{5}$$

Secondly, one can replace the gamma-functions in Eq. (3) through Stirling's formula and can compute the limit of the A function as W approaches unity. In this manner it can be shown that as W approaches unity (y increases without limit), A behaves asymptotically as y^2. Finally, as W becomes unboundedly large, A again increases but only as $W^{\alpha^2Z^2}$. These characteristics are seen in the graphs of the A function. Similar curves are given by Laslett and his co-workers.[6] It is important to note that for any Z not zero, $A\gg 1$ will hold for sufficiently small energies.

Finally, as a check, the A function was computed at $Z=90$ keeping all terms through α^4Z^4. If A_2 be the approximation containing terms through α^2Z^2, and A_4 the approximation containing terms through α^4Z^4, it was found that

$$\begin{aligned} A_4/A_2&=0.94 \quad 1.2\lesssim W\lesssim 2.0,\\ A_4/A_2&=0.99 \quad W\gtrsim 6.0.\end{aligned} \tag{6}$$

The accuracy of the first approximation, A_2, is sufficient for almost all applications. For $Z<90$, it is, of course, even more accurate than is indicated by Eq. (6).

* Part of a thesis submitted in partial fulfillment of the requirements for the degree of Master of Arts in Physics in the Graduate School of Washington University.

[1] This notation is used by E. Greuling, Phys. Rev. **61**, 568 (1942).

[2] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941).

[3] J. S. Osoba, Phys. Rev. **76**, 345 (1949).

[4] The units used in this article are the usual dimensionless ones; i.e., energy is in units of mc², lengths in units of ℏ/mc, etc.

[5] Jahnke and Emde, *Tables of Functions* (B. G. Teubner, Leipzig, 1938).

[6] Laslett, Jensen, and Paskin, Phys. Rev. **79**, 412 (1950).

II. THE $f_n t$ PRODUCTS

(A) The Averaging Approximation

If one defines the function f_0 for a beta-transition as

$$f_0=\int_1^{W_0} C_0F_0(W,Z)pW(W_0-W)^2dW, \quad (7)$$

then the product of f_0 by the half-life of the transition is relatively constant for allowed transitions (both favored and unfavored). However, one does not expect the same constancy for forbidden transitions because the term C_0 in Eq. (7) must be replaced by C_{nX}, which is appropriate to the transition under consideration. The generalization of the f_0-function to any transition is immediate:

$$f_{nX}=\int_1^{W_0} C_{nX}F_0(W,Z)pW(W_0-W)^2dW. \quad (8)$$

However, the complexity of the correction terms C_{nX} makes most calculations out of the question (the general expression for the C_{nX} is given by Greuling[1]).

A very great simplification is obtained if one considers those interaction types which permit the greatest ΔI for a given degree of forbiddenness n. One then notes that for all n the tensor and axial vector interactions allow $(\Delta I)_{\max}=\pm(n+1)$, which is greater by one unit than for any of the other three interactions. For these two cases all of the nuclear matrix elements but one vanish, and this forms a symmetrical tensor with zero spur. The correction term is of the form

$$C_{nT \text{ or } A}=|Q_{n+1}(\boldsymbol{\sigma},\mathbf{r})/(n+1)!|^2\sum_{\nu=0}^{n}(B_{n\nu}K^{2(n-\nu)}L_\nu), \quad (9)$$

with

$$B_{n\nu}=\frac{2^{n-2\nu}(2\nu+1)!}{(2n-2\nu+1)!(\nu!)^2}, \quad K=W_0-W$$
$$L_\nu=(F_\nu/F_0)\left(\frac{2^\nu(\nu!)}{(2\nu+1)!}p^\nu\right)^2\frac{\nu+1+S_\nu}{2\nu+2}, \quad (10)$$

with $Q_{n+1}(\boldsymbol{\sigma},\mathbf{r})$ a tensor of rank $(n+1)$, symmetric in all indices and having zero spur. For $Z\rightarrow 0$,$S_\nu\rightarrow\nu+1$, and $(F_\nu/F_0)\rightarrow 1$, then

$$\lim_{Z\rightarrow 0}L_\nu=\left(\frac{2^\nu(\nu!)}{(2\nu+1)!}p^\nu\right)^2. \quad (11)$$

Thus, for $n=1$, 2, 3 and $Z=0$, one has

$$12C_{1T,A}=(W_0-W)^2+(W^2-1),$$
$$5\cdot 6^3C_{2T,A}=(W_0-W)^4+(10/3)(W_0-W)^2(W^2-1)+(W^2-1)^2,$$
and (12)
$$70\cdot 72^2C_{3T,A}=(W_0-W)^6+7(W_0-W)^4(W^2-1)+7(W_0-W)^2(W^2-1)^2+(W^2-1)^3.$$

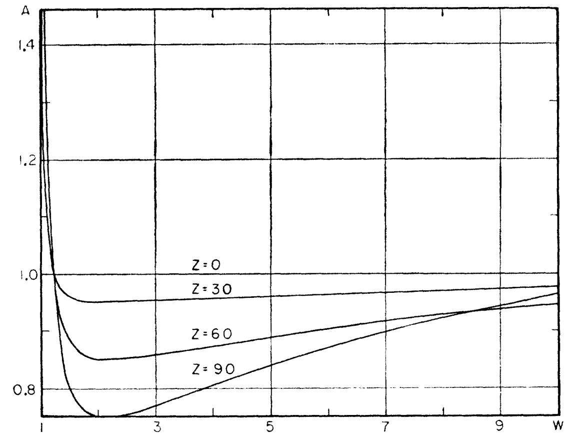

FIG. 1. The function $A(W,Z)$ plotted against W for $Z=0$, 30, 60, and 90. $C_{1T}=[(1+S_0)/24][(W_0-W)^2+A(W,Z)(W^2-1)]$ is the beta-decay shape factor for unique first forbidden transitions.

The function $\mathfrak{F}(W)=(W-1)(W_0-W)$ looks very much like the energy distribution function for beta-decay and has been used[7] to compute an approximate formula for f_1. Replacing F_0 by $\mathfrak{F}$ in the general expressions

$$f_n=\int_1^{W_0} C_nF_0(W,Z)pW(W_0-W)^2dW$$
$$=\bar{C}_n\int_1^{W_0} F_0(W,Z)pW(W_0-W)^2dW=\bar{C}_nf_0, \quad (13)$$

one obtains simple approximate formulas for $\bar{C}_n$. The

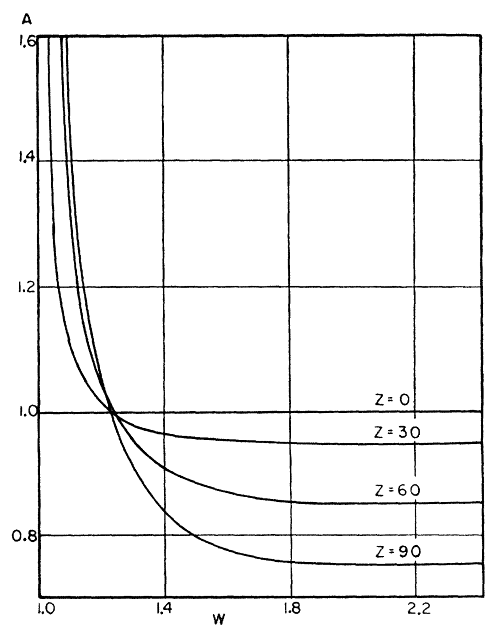

FIG. 2. The function $A(W,Z)$ for low values of W.

[7] F. B. Shull and E. Feenberg, Phys. Rev. 75, 1768 (1949).

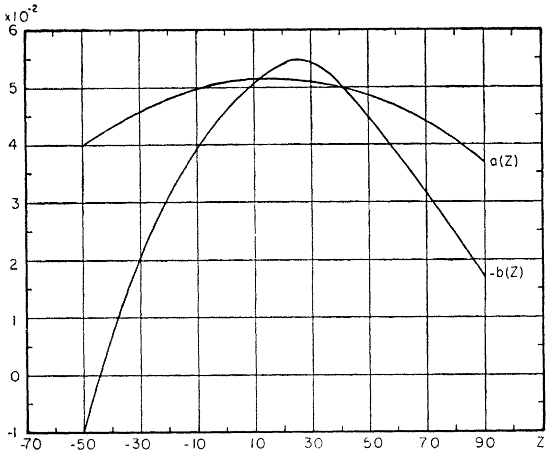

FIG. 3. The coefficients $a(Z)$ and $-b(Z)$ plotted against Z, where $f_1/f_0=a(Z)(W_0^2-1)+b(Z)(W_0-1)$.

$\bar{C}_n$ are

$$12\bar{C}_{1T}(W_0)=(6/10)(W_0^2-1)-(1/5)(W_0-1),$$

$$5\cdot 6^3\bar{C}_{2T}(W_0)=(3/7)(W_0^2-1)^2 -(26/105)(W_0^2-1)(W_0-1) -(2/105)(W_0-1)^2, \quad (14)$$

and

$$70\cdot 72^2\bar{C}_{3T}(W_0) =(1/3)(W_0^2-1)^3-(9/35)(W_0^2-1)^2(W_0-1) -(2/35)(W_0^2-1)(W_0-1)^2+(8/105)(W_0-1)^3.$$

(B) The Ratio f_1/f_0

If something were known concerning the ratio f_1/f_0, then, through the use of Trigg's curves[8] of f_0, one could obtain f_1 and hence the f_1t products. Consider the ratio

$$\frac{f_1}{f_0}=\frac{\int_1^{W_0} C_1F_0(W,Z)pW(W_0-W)^2dW}{\int_1^{W_0} C_0F_0(W,Z)pW(W_0-W)^2dW}=\frac{\bar{C}_1}{\bar{C}_0}. \quad (15)$$

Since this ratio was expected to be fairly independent of Z, two new functions, $a(Z)$ and $b(Z)$, were defined by

$$f_1/f_0=R(W_0,Z)=a(Z)(W_0^2-1)+b(Z)(W_0-1), \quad (16)$$

and were computed from exact values of f_1 and f_0 at $Z=0$, ± 25, ± 50, and 90 and for $W_0=1.5$, 5, and 10. The functions $a(Z)$ and $b(Z)$ are shown in Fig. 3 plotted against Z. It should be noted that the average value of the coefficient $a(Z)$ is almost equal to the coef-

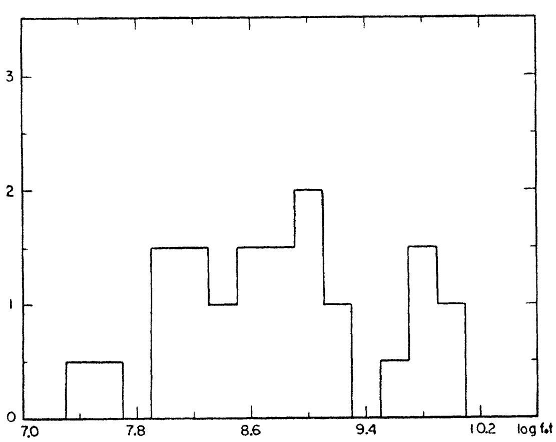

FIG. 4. Histogram of the number of unique first forbidden transitions plotted against $\log(f_0t)$.

[8] E. Feenberg and G. L. Trigg, Revs. Modern Phys. 22, 399 (1950).

TABLE I. A table of ft products for forbidden transitions with maximum spin change for the degree of forbiddenness (classification derived from spectral shapes, known spin changes, and comparative half-lives).

Nucleus	W_0	$\log(f_0t)$	$\log[(W_0^2-1)f_0t]$	$\log(f_nt)$[a]	Ref.
Forbidden transitions with $n=1$, $\Delta I=\pm 2$.					
Cl^{38}	10.43	7.44	9.48	8.15	f, s
A^{39}	2.11	8.66[b]	9.10[b]	7.56[b]	t
K^{42}	8.02	8.02	9.82	8.49	f
Kr^{85}	2.36	9.09	9.75	8.30	u, x
Rb^{86}	4.57	8.51	9.81	8.43	f
Sr^{89}	3.93	8.57	9.73	8.34	h, i
Sr^{90}	2.04	9.86[c]	10.48[c]	8.89[c]	g, j
Y^{90}	5.40	8.03	9.48	8.11	g, i, j
Y^{91}	4.02	9.19[c]	10.37[c]	8.98[c]	i, k, l
Sn^{123}	3.78	8.88	10.00	8.60	m, v
Sn^{125}	5.57	8.86	10.34	8.96	m, v
I^{124}[d]	5.31	8.11	9.54	8.11	n
Cs^{137}	2.02	9.83[c]	10.32[c]	8.85[c]	f, k
Tl^{204}	2.51	9.60	10.32	8.85	w
Forbidden transitions with $n=2$, $\Delta I=\pm 3$.					
Be^{10}	2.10	14.50[c]	15.03[c]	12.08[c]	o
Na^{22}[e]	4.7	13.6	14.96	12.85	o, p
Forbidden transitions with $n=3$, $\Delta I=\pm 4$.					
K^{40}	3.64	18.46	19.54	15.60	f, q, r, y

[a] $f_n=f_1=f_0[a(Z)(W_0^2-1)+b(Z)(W_0-1)]$ for $n=1$,
$=\bar{C}_{nT}(W_0)f_0$ for $n=2$, 3.

[b] These products are lower limits as the reported half-life is given as a minimum value.
[c] These products contain the term $(2I_f+1)/(2I_i+1)$, I the nuclear spin. This factor must be included because $I_f>I_i$.
[d] Corrected for K-capture.
[e] A branching ratio of one in 11,000 used.
[f] G. T. Seaborg and I. Perlman, Revs. Modern Phys. **20**, 585 (1948).
[g] Braden, Slack, and Shull, Phys. Rev. **75**, 1964 (1949).
[h] Slack, Braden, and Shull, Phys. Rev. **75**, 1965 (1949).
[i] L. M. Langer and H. C. Price, Jr., Phys. Rev. **76**, 641 (1949).
[j] E. U. Jensen and L. J. Laslett, Phys. Rev. **75**, 1949 (1949).
[k] Reference 3.
[l] L. M. Langer and H. C. Price, Jr., Phys. Rev. **75**, 1109 (1949).
[m] J. C. Lee and M. L. Pool, Phys. Rev. **76**, 606 (1949).
[n] Reference 10.
[o] L. J. Laslett, Phys. Rev. **76**, 858 (1949).
[p] Reference 11.
[q] L. H. Ahrens and R. D. Evans, Phys. Rev. **74**, 279 (1948).
[r] D. E. Alburger, Phys. Rev. **79**, 236(A) (1950).
[s] L. M. Langer, Phys. Rev. **77**, 50 (1950).
[t] Zeldes, Ketelle, and Brasi, Phys. Rev. **79**, 901 (1950).
[u] Reference 9.
[v] Ketelle, Nelson, and Boyd, Phys. Rev. **79**, 242(A) (1950).
[w] D. Saxon and J. Richards, Phys. Rev. **76**, 982 (1949).
[x] I. Bergström (private communication from Dr. Kai Siegbahn).
[y] Bell, Weaver, and Cassidy, Phys. Rev. **77**, 399 (1950).

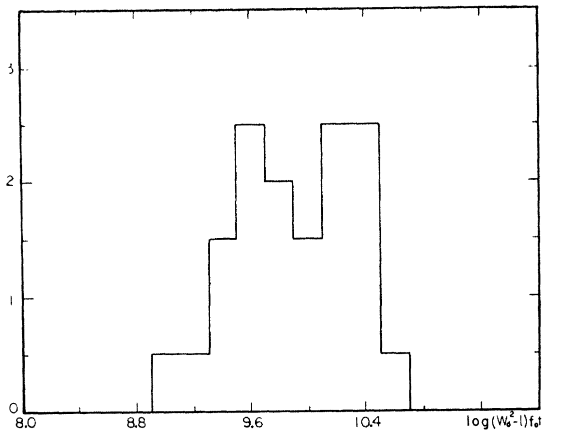

FIG. 5. Histogram of the number of unique first forbidden transitions plotted against $\log[(W_0^2-1)f_0t]$.

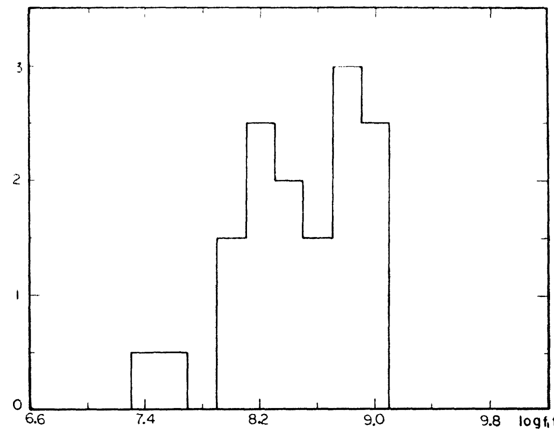

FIG. 6. Histogram of the number of unique first forbidden transitions plotted against $\log(f_1t)$.

ficient of the term in (W_0^2-1) in $\bar{C}_{1T}$, indicating the suitability of the approximate averaging process based on the distribution function $\mathfrak{F}$.

(C) Discussion of the Results

In Eq. (9) the nuclear matrix elements for $n=1$ contain the nucleon space coordinates to the first power. Thus, one might expect the product $A^{\frac{2}{3}}f_1t$ to be more constant than the f_1t product alone. To check this, the spread of values for the products $A^m f_1t$ (A the atomic number) was investigated. It was found that $m=0$ resulted in a smaller spread for this product (when applied to the experimental information for the isotopes listed in Table I) than any other value. The total region investigated was $-1 \leq m \leq 1$.

In Table I the low ft products for A^{39} must be considered lower limits because the half-life reported is only a minimum value.[9] Using Trigg's curves for[8] f_k+f_+, the branching ratios for the two low energy transitions found by Mitchell and his co-workers in I^{124} can be corrected for K-capture.[10] This correction reduces the branching ratio for the high energy, first forbidden transition from 51 to 14 percent, and raises all of the ft products for this isotope. If K-capture does indeed give rise to the gamma-ray line at 1.72 Mev, as suggested in Reference 10, this would have the effect of further reducing the branching ratio for the first forbidden transition and increasing the value of f_1t.

Figures 4–6 are histograms in which the number of transitions with the shape factor $C_{1T \text{ or } A}$ is plotted against $\log(f_0t)$, $\log[(W_0^2-1)f_0t]$, and $\log(f_1t)$. From the broad distribution in the f_0t plot it seems evident that the f_0t product is not as suitable a criterion for the unique first forbidden type transition as are the other products.

Nothing at all can be said for the third forbidden transition of K^{40} as there is no other known transition with which to compare it. The situation is different for the second forbidden transitions. The transition of Na^{22} to the ground state of Ne^{22} is in fair agreement with the similar transition of Be^{10} to B^{10}. The branching ratio given by Morganstern and Wolf[11] has been corrected for masking by the low energy positron spectrum, and the ratio of one to an upper limit of 11,000 is used. While agreement with the Be^{10} transition is not excellent, it would seem that the uncertainty in the approximate averaging process is probably far less than that arising naturally from the experimental method.

The author wishes to express his sincere appreciation to Professor E. Feenberg for suggesting this problem and for many helpful suggestions and discussions.

[9] Brosi, Zeldes, and Ketelle, Phys. Rev. **79**, 902 (1950).

[10] Mitchell, Mei, Maierschein, and Peacock, Phys. Rev. **76**, 1450 (1949).

[11] K. H. Morganstern and K. W. P. Wolf, Phys. Rev. **76**, 1261 (1949).