1. Applying the Master Theorem [10 points]

For each recurrence relation below, do the following:

- First, indicate whether the master theorem can be used to find a Θ bound on that recurrence relation.
- Next, If the master theorem does apply, then use it to find an asymptotic bound. If the master theorem does not apply, explain why.

(a) $T(n) = 5T(\frac{n}{5}) + 5$

(b) $T(n) = T(\frac{n}{8}) + n^2$

(c) $T(n) = 2T(\frac{n}{4}) + \sqrt{n}$

(d) $T(n) = 5T(\frac{n}{5}) + \log n$

(e) $T(n) = 3T(\frac{2n}{3}) + n^3$

Solving Divide and Conquer Recurrences

Master Theorem: Suppose that $T(n) = a \cdot T(n/b) + O(n^k)$ for $n > b$.

- If $a < b^k$ then $T(n)$ is $O(n^k)$
  - Cost is dominated by work at top level of recursion
- If $a = b^k$ then $T(n)$ is $O(n^k \log n)$
  - Total cost is the same for all $\log_b n$ levels of recursion
- If $a > b^k$ then $T(n)$ is $O(n^{\log_b a})$
  - Note that $\log_b a > k$ in this case
  - Cost is dominated by total work at lowest level of recursion

a) $T(n) = 5T(\frac{n}{5}) + 5 = 5T(\frac{n}{5}) + 5 \cdot n^0$

i) we can apply the master theorem

ii) $a=5,\ b=5,\ k=0$ $\quad b^k = 5^0 = 1$ $\quad 5 > 1$, $a > b^k$ $\quad$ Since $a > b^k$ $T(n)$ is $\Theta(n^{\log_5(5)}) = \Theta(n)$ $\quad \log_5(5) = 1$

b) $T(n) = T(\frac{n}{8}) + n^2$

i) we can apply the master theorem

ii) $a=1,\ b=8,\ k=2$ $\quad b^k = 8^2 = 64$ $\quad 1 < 64$, $a < b^k$ $\quad$ Since $a < b^k$ $T(n)$ is $\Theta(n^2)$

c) $T(n) = 2T(\frac{n}{4}) + \sqrt{n} = 2T(\frac{n}{4}) + n^{1/2}$

i) we can apply the master theorem

ii) $a=2,\ b=4,\ k=\frac{1}{2}$ $\quad b^k = 4^{1/2} = \sqrt{4} = 2$ $\quad 2 = 2$, $a = b^k$ $\quad$ Since $a = b^k$ $T(n)$ is $\Theta(n^{1/2} \cdot \log(n))$

d) $T(n) = 5T(\frac{n}{5}) + \log(n)$

i) we cannot apply the master theorem because $\log(n)$ cannot be written in the form n^k for some $k \in \mathbb{R}$.

e) $T(n) = 3T(\frac{2}{3}n) + n^3$

i) we can apply the master theorem

ii) $a=3,\ b = \frac{1}{(\frac{2}{3})} = \frac{3}{2},\ k=3$ $\quad b^k = (\frac{3}{2})^3 = \frac{3^3}{2^3} = \frac{27}{8} = 3.375$ $\quad 3 < 3.375$, $a < b^k$

Since $a < b^k$ $T(n)$ is $\Theta(n^3)$