Appunti di

Equazioni alle Derivate Parziali
Parte Analitica

Dalle lezioni del Prof. Sandro Salsa & del Prof. Gianmaria Verzini
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Il codice sorgente LaTeX è disponibile su
`https://github.com/teobucci/edp-analitica`

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Chapter 1

Stochastic Processes

A stochastic process (SP) is an infinite sequence of random variables all defined on the same probabilistic space:

$$\dots, v(1,s), v(2,s), v(3,s), \dots, v(t,s), \dots$$

with s: random experiment realization and $t = 0, \pm 1, \pm 2, \dots$: time index.

Observation. SP extends the notion of random vector (SP is a random vector with infinite entries).

Observation. For a fixed value of the random experiment $s = \overline{s}$, the stochastic process becomes the numeric sequence:

$$\dots, v(1,\overline{s}), v(2,\overline{s}), v(3,\overline{s}), \dots, v(t,\overline{s}), \dots$$

which is called **realization** of the stochastic process. For different values of s, one gets different realizations of the stochastic process.

We will think of available observations $u(1), u(2), \dots, u(N)$ and $y(1), y(2), \dots, y(N)$ as finite length realizations of stochastic process (stochastic process = uncertain model for a time series or I/O system).

Mean value $m(t)$: it is the expected value of random variable $v(t,s)$ at time t:

$$m(t) = \mathbb{E}[v(t,s)] = \int v(t,s)\mathbb{P}(ds)$$

$m(t)$ returns the value around which the process take value at time t.

Covariance function $\gamma(t_1, t_2)$: it is the expected value of the product of unbiased random variables $(v(t,s) - m(t))$ at two time instants (t_1, t_2) :

$$\begin{aligned}\gamma(t_1,t_2) &= \mathbb{E}[(v(t_1,s) - m(t_1))(v(t_2,s) - m(t_2))] \\ &= \int (v(t_1,s) - m(t_1))(v(t_2,s) - m(t_2))\mathbb{P}(ds)\end{aligned}$$

$\gamma(t_1, t_2)$ quantifies the relation existing between the *gaps* between the process realizations and the mean value at two different time instants.

Particular case: $t_1 = t_2 = t$.

$$\gamma(t,t) = \mathbb{E}[(v(t,s) - m(t))^2] = \int (v(t,s) - m(t))^2\mathbb{P}(ds)$$

is called the process **variance function** (it quantifies the process dispersion around its mean value at each time instant).

1.1 Stationary Stochastic Processes (SSP)

A stochastic process is called stationary (wide sense) if:

- $m(t) = m \quad \forall t$
- $\gamma(t_1, t_2)$ depends on $\tau = t_1 - t_2$ only, i.e.: $\gamma(t_1,t_2) = \gamma(t_3,t_4)$ if $t_1 - t_2 = t_3 - t_4 = \tau \quad \forall t_1, t_2, t_3, t_4$

Idea: the probabilistic properties of a SSP are time-translation invariant.

Stationary stochastic processes admit a *simplified* representation of the covariance function:

$$\gamma(\tau) = \gamma(t, t-\tau) = \mathbb{E}[(v(t)-m)(v(t-\tau)-m)] \qquad (t_1 = t, t_2 = t-\tau, t_1 - t_2 = \tau)$$

and

$$\gamma(0) = \mathbb{E}[(v(t)-m)^2] = \lambda^2 \qquad \text{is the variance of the process}$$

Why stationary stochastic processes?

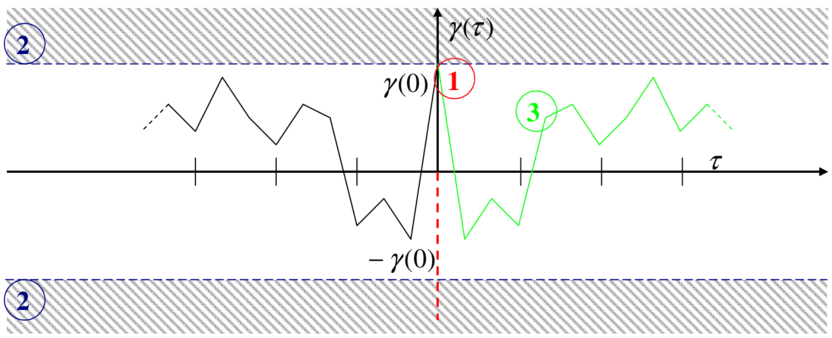
- *Stationary* means *time-invariant* data generating system (situation often encountered in practice).
- S.S.P. are easier to study.
- Non-stationary processes can be recast in the framework of S.S.P. by first eliminating the non-stationary part from data (data pre-processing).

Covariance function properties for an S.S.P.

$$\gamma(\tau) = \mathbb{E}[(v(t)-m)(v(t-\tau)-m)]$$

- $\gamma(0) = \mathbb{E}[(v(t)-m)^2] \geq 0$ (non negative at initial time)
- $|\gamma(\tau)| \leq \gamma(0)$ (bounded)
- $\gamma(\tau) = \gamma(-\tau)$ (symmetric) indeed

$$\begin{aligned}\gamma(-\tau) &= \mathbb{E}[(v(t)-m)(v(t-(-\tau))-m)] \\ &= \mathbb{E}[(v(t)-m)(v(t+\tau))-m)] \\ &= \mathbb{E}[(v(t+\tau)-m)(v(t))-m)] \\ &= \gamma(\tau) \quad (t+\tau-t=\tau)\end{aligned}$$



Observation

1. Given a S.S.P. $x(t)$, we will write m_x e $\gamma_x(\tau)$ for its mean and covariance function
2. Two S.S.P. $y_1(t)$ and $y_2(t)$ are wide-sense equivalent if $m_{y_1} = m_{y_2}$ e $\gamma_{y_1}(\tau) = \gamma_{y_2}(\tau), \forall\tau$
3. The *covariance function*

$$\mathbb{E}[(v(t)-m)\cdot(v(t-\tau)-m)]$$

is very *different* from the 2^{nd} order moment function $\mathbb{E}[v(t)\cdot v(t-\tau)]$.

Example 1

$v(t,s) = \alpha(s), \forall t$ where $\alpha(s) \sim G(1,3)$ (Gaussian random variable with mean = 1 and variance = 3)

Is the process stationary? Yes:

- $m_v(t) = \mathbb{E}[v(t,s)] = \mathbb{E}[\alpha(s)] = 1 = m_v$ doesn't depend on t.
- recalling that $v(t,s) = \alpha(s), \forall t$ and that $m_v(t) = 1 = m_v$

$$\begin{aligned}\gamma_v(t, t-\tau) &= \mathbb{E}[(v(t,s)-m_v(t))(v(t-\tau,s)-m_v(t-\tau))] \\ &= \mathbb{E}[(\alpha(s)-1)(\alpha(s)-1)] = 3 = \gamma_v(\tau)\end{aligned}$$

doesn't depend on t.

Example 2

$v(t,s) = t \cdot \alpha(s) - t$, where $\alpha(s) \sim G(1,3)$.

- $m_v(t) = \mathbb{E}[v(t,s)] = \mathbb{E}[t \cdot \alpha(s) - t] = t \cdot \mathbb{E}[\alpha(s)] - t = t - t = 0$ doesn't depend on t.
-
$$\begin{aligned}\gamma_v(t,t-\tau) &= \mathbb{E}[(v(t,s) - m_v(t))(v(t-\tau,s) - m_v(t-\tau))] \\ &= \mathbb{E}[(t \cdot \alpha(s) - t)((t-\tau) \cdot \alpha(s) - (t-\tau))] \\ &= \mathbb{E}[t \cdot (t-\tau)(\alpha(s)-1)^2] \\ &= t \cdot (t-\tau) \cdot \mathbb{E}[(\alpha(s)-1)^2] = t \cdot (t-\tau) \cdot 3\end{aligned}$$

  does depend on t.

The process is not stationary.

Observation.

If $\gamma(t,\tau) > 0$ then there is a tendency of preserving the sign of the deviation from t to τ. Otherwise there is a tendency of changing the sign.

1.2 White Noise

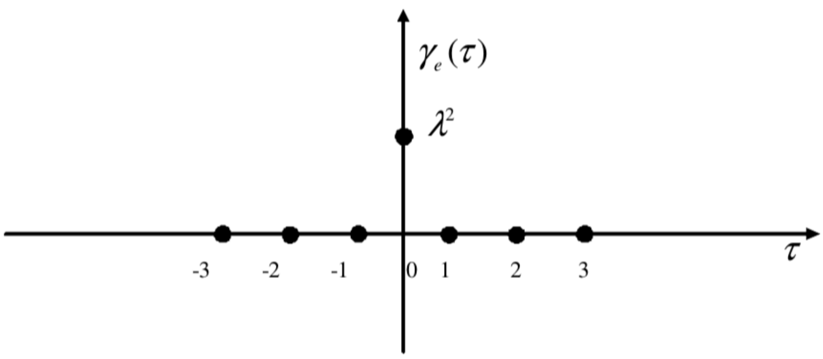
An S.S.P. $e(t)$ is called White Noise (WN) with mean μ and variance λ^2, we shall write

$$e(t) \sim WN(\mu, \lambda^2),$$

if the following conditions hold:

- $\mathbb{E}[e(t)] = \mu \quad \forall t$
- $\gamma_e(0) = \mathbb{E}[(e(t) - \mu)^2] = \lambda^2 \quad \forall t$
- $\gamma_e(\tau) = \mathbb{E}[(e(t) - \mu) \cdot (e(t-\tau) - \mu)] = 0 \quad \forall t, \forall \tau \neq 0$

The last property is the fundamental one. It says that there is complete incorrelation between random variables at different time instants. The realizations of $e(t)$ are erratic and unpredictable (**whiteness property**).



Observation. The probability distribution of each single random variables $e(t,s)$ does not matter and is not made explicit in general (wide-sense description of S.S.P.). It could be Gaussian, uniform, etc. (WGN = White Gaussian Noise, WUN = White Uniform Noise, etc.).

Observation. Is a constant realization admissible? Yes, it is, but such a realization is *highly unlikely.*

White Noise is a sort of *building block* to construct a number of different stationary stochastic processes.

Remark. To ease the notation, in the following we will consider *zero mean* white noise. The extension to the general case presents no conceptual difficulties.

1.2.1 MA(n) processes

Read *Moving Average of order n.*

Let $e(t) \sim WN(0, \lambda^2)$, and MA process is obtained as

$$\boxed{y(t) = c_0 e(t) + c_1 e(t-1) + c_2 e(t-2) + \ldots + c_n e(t-n).}$$

In other words, the output $y(t)$ of a MA process is given by a linear combination of the last $n+1$ past values of the input noise $e(t)$. While t is let vary, the linear combination is made on a sliding window (moving average).

Mean.

$$m_y(t) = \mathbb{E}[y(t)] = \mathbb{E}[c_0 e(t) + c_1 e(t-1) + c_2 e(t-2) + \cdots + c_n e(t-n)] = 0 + 0 + \cdots + 0 = m_y = 0 \quad \forall t$$

hence $m_y(t)$ doesn't depend on t.

Variance. (i.e. covariance when $\tau = 0$)

$$\begin{aligned}
\gamma(0) &= \mathbb{E}[(y(t) - m_y)(y(t) - m_y)] = \mathbb{E}[(y(t))^2] \\
&= \mathbb{E}[(c_0 e(t) + c_1 e(t-1) + \ldots + c_n e(t-n))^2] \\
&= \mathbb{E}[c_0^2 e(t)^2 + \ldots + c_n^2 e(t-n)^2 \\
&\qquad + 2c_0 c_1 e(t) e(t-1) + \ldots + 2c_{n-1} c_n e(t-n-1) e(t-n)] \\
&= c_0^2 \mathbb{E}[e(t)^2] + c_1^2 \mathbb{E}[e(t-1)^2] + \ldots + c_n^2 \mathbb{E}[e(t-n)^2] \\
&\qquad + 2c_0 c_1 \mathbb{E}[e(t) e(t-1)] + \ldots + 2c_{n-1} c_n \mathbb{E}[e(t-n-1) e(t-n)]
\end{aligned}$$

Since $e(t) \sim WN(0, \lambda^2)$, we have that:

$$\mathbb{E}[e(t)^2] = \mathbb{E}[e(t-1)^2] = \ldots = \mathbb{E}[e(t-n)^2] = \lambda^2$$

and that

$$\mathbb{E}[e(t)e(t-1)] = \ldots = \mathbb{E}[e(t-n-1)e(t-n)] = 0$$

thus

$$\boxed{\gamma(t,t) = \gamma(0) = (c_1^2 + c_1^2 + \cdots + c_n^2) \cdot \lambda^2}$$

hence $\gamma(0)$ doesn't depend on t.

Covariance.

To calculate the generic covariance, let us proceed with $\tau = 1$.

$$\begin{aligned}
\gamma(t, t-1) &= \mathbb{E}[(y(t) - m_y)(y(t-1) - m_y)] \\
&= \mathbb{E}[y(t) y(t-1)] \\
&= \mathbb{E}[(c_0 e(t) + c_1 e(t-1) + \ldots + c_n e(t-n)) \cdot (y(t-1))] \\
&= \mathbb{E}[(c_0 e(t) + c_1 e(t-1) + \ldots + c_n e(t-n)) \cdot (c_0 e(t-1) + \ldots + c_{n-1} e(t-n) + c_n e(t-n-1))]
\end{aligned}$$

Only those terms where the white noise is multiplied by itself at the same time instant are non null.

$$\gamma(t, t-1) = \gamma(1) = (c_0 c_1 + c_1 c_2 + \cdots + c_{n-1} c_n) \cdot \lambda^2$$

hence $\gamma(1)$ doesn't depend on t.

Similarly

$$\begin{aligned}
\gamma(t, t-2) &= \gamma(2) = (c_0 c_2 + c_1 c_3 + \cdots + c_{n-2} c_n) \cdot \lambda^2 \\
&\vdots \\
\gamma(t, t-n) &= \gamma(n) = (c_0 c_n) \cdot \lambda^2 \\
\gamma(t, t-n-1) &= \gamma(n+1) = 0
\end{aligned}$$

since all products are uncorrelated. In conclusion

$$\gamma(\tau) = \begin{cases}
(c_1^2 + c_1^2 + \cdots + c_n^2) \cdot \lambda^2 & \text{if } \tau = 0 \\
(c_0 c_1 + c_1 c_2 + \cdots + c_{n-1} c_n) \cdot \lambda^2 & \text{if } \tau = \pm 1 \\
(c_0 c_2 + c_1 c_3 + \cdots + c_{n-2} c_n) \cdot \lambda^2 & \text{if } \tau = \pm 2 \\
\vdots & \\
(c_0 c_n) \cdot \lambda^2 & \text{if } \tau = \pm n \\
0 & \text{if } \tau > \pm 1
\end{cases}$$

1.2.2 MA(∞) processes

$$y(t) = c_0 e(t) + c_1 e(t-1) + \cdots + c_i e(t-i) + \cdots = \sum_{i=0}^{\infty} c_i e(t-i) \quad e(t) \sim WN\left(0, \lambda^2\right)$$

Assumption: $\sum_{i=0}^{\infty} c_i^2 < \infty$ (it guarantees that $y(t)$ is well defined).

Mean.

$$m_y(t) = \mathrm{E}[y(t)] = \mathrm{E}\left[\sum_{i=0}^{\infty} c_i e(t-i)\right] = \sum_{i=0}^{\infty} c_i \mathrm{E}[e(t-i)] = \sum_{i=0}^{\infty} c_i \cdot 0 = 0$$

doesn't depend on t.

Variance.

$$\begin{aligned}
\gamma_y(t,t) &= \mathrm{E}[(y(t)-m_y)^2] \\
&= \mathrm{E}\left[\sum_{i=0}^{\infty} c_i e(t-i) \cdot \sum_{j=0}^{\infty} c_j e(t-j)\right] \\
&= \mathrm{E}\left[\sum_{i,j=0}^{\infty} c_i c_j \cdot e(t-i)e(t-j)\right] \\
&= \sum_{i,j=0}^{\infty} c_i c_j \cdot \mathrm{E}[e(t-i)e(t-j)] \\
&= \{\text{non null only when } i=j\} \\
&= \sum_{i=0}^{\infty} c_i^2 \cdot \lambda^2
\end{aligned}$$

doesn't depend on t.

Covariance.

$$\begin{aligned}
\gamma_y(t,t-\tau) &= \mathrm{E}[(y(t)-m_y)(y(t-\tau)-m_y)] \\
&= \mathrm{E}[y(t)y(t-\tau)] \\
&= \mathrm{E}\left[\sum_{i=0}^{\infty} c_i e(t-i) \cdot \sum_{j=0}^{\infty} c_j e(t-j-\tau)\right] \\
&= \mathrm{E}\left[\sum_{i,j=0}^{\infty} c_i c_j \cdot e(t-i)e(t-j-\tau)\right] \\
&= \sum_{i,j=0}^{\infty} c_i c_j \cdot \mathrm{E}[e(t-i)e(t-j-\tau)] \\
&= \{\text{non null only when } i=j+\tau\} \\
&= \sum_{i=0}^{\infty} c_{j+\tau} c_j \cdot \lambda^2
\end{aligned}$$

doesn't depend on t.

So if $\sum_{i=0}^{\infty} c_i^2 < \infty$ then the MA(∞) process is well defined and is a S.S.P.

Observation. MA(∞) processes are very general, they almost *cover* the class of stationary stochastic processes (i.e. apart from few exceptions, all S.S.P. can be written as MA(∞)).

However, MA(∞) are difficult to handle since there are infinite coefficients and, moreover, the computation of the covariance function requires the computation of the sum of an infinite series (hard in general).

On the other hand, MA(n) are too limited, that is why we will look into AR and ARMA models.