Model Shift and Model Risk Management

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Model shift refers to the quantitative change in a model's parameters and outputs resulting from shifts in the input data. It is particularly useful in credit risk modeling for understanding how models react to changes in their underlying assumptions, data, or environment. The concept of model shift enables practitioners to:

- efficiently address "What if. . . ?" scenarios, quantifying how data shifts impact model outputs without needing complete model redevelopment;
- provide a systematic way to assess and respond to model sensitivities and weaknesses, enhancing model validation, monitoring, and risk management.

The following slides describe the framework for quantifying the model shift in one of the most commonly used methods in credit risk: logistic regression with categorical risk factors.

In this framework both data and models can be presented as observed and expected counts in a high-dimensional contingency table. These in turn are converted to points in a Data Space of high dimensional vectors. Here a data point is defined by the proportion of the observed population in each cell, viewed as a vector of real numbers indexed by the cells. Likewise, the model point is defined by the proportion of the expected population in each cell etc. We can keep track separately of the total population size for purposes of inference, but note that the data point and model point do not vary as population size changes. The maximum likelihood construction of logistic regression model from data depends solely on the proportions.

Applications of Model Shift

Model shift in response to data shift is more than an academic exercise. It is clearly closely related to data drift and concept drift, well known concerns in model management, and it underlies modeling techniques such as imputation, bootstrapping and rebalancing. We are interested in it as a method for important analyses in modern model risk management.

The following list outlines some use cases of the model shift:

- *Dynamic Model Reweighting*: Enables real-time updates of model parameters as new data streams in, ensuring agility in model adjustments without waiting for periodic reviews.
- *Prioritizing Validation Investigations*: Quickly computes model shifts for various data shift scenarios, enabling efficient triage and focus on the most impactful concerns.
- *Quantifying Business Impacts*: Links data shifts to business-relevant metrics like Probability of Default (PD) in Risk-Weighted Assets (RWA), ensuring sensitivity analyses are connected to actionable outcomes.
- *Sensitive Data Shift Identification*: Enables the identification of data shifts that have the most significant impact on models, enriching the validation narrative with actionable insights.
- *Bespoke Model Monitoring*: Defines monitoring metrics for sensitive data shifts, creating early warning systems, particularly for population shifts that do not immediately affect model outputs.
- *Automated Validation and Monitoring*: Streamlines validation and monitoring processes, integrating them with dynamic model updates for continuous, real-time risk management.

Practitioners can refer to this document for further details.

Methods for Quantifying Model Shift

- The methods for quantifying model shift are:

  1. matrix multiplication approach;
  2. weighted binomial logistic regression;
  3. weighted quasi-binomial regression (weighted fractional logistic regression).

- Dr. Alan Forrest proposes a first-order approximation using a matrix multiplication approach to quantify changes in model parameters directly.

- Andrija Djurovic introduces two exact alternative methods (weighted binomial and fractional logistic regression) based on re-estimating model parameters, both of which can address the same task.

- The direct but approximate approach is useful where many thousands or millions of shifts are to be compared or summarised. The exact approaches are fully accurate and ideal for testing smaller numbers of sensitivities or scenarios.

- All three approaches are related to the widely used binomial logistic regression method with categorized risk factors commonly employed in developing PD models.

- Similarly, practitioners can extend the proposed framework to Ordinary Least Squares (OLS) regression, a commonly used method for modeling Loss Given Default (LGD) and Exposure at Default (EaD).

Matrix Multiplication Approach

The first-order model shift (Δp) can be explicitly represented as a matrix multiplication of the data shift. The following formulas illustrate the process of approximating parameter changes given the data shifts (Δx^{+} and Δx^{-}):

$$\Delta p = C^{-1} D^{T} \left[(I + Z)^{-1} \Delta x^{+} - (I + Z^{-1})^{-1} \Delta x^{-} \right]$$

where:

- D is the design matrix;
- Y^{+} and Y^{-} are the diagonal matrices of modeled frequencies restricted to binary output 1 and 0, respectively;
- $Z = Y^{+}(Y^{-})^{-1}$ is diagonal matrix of modeled odds ratios;
- I is identity matrix dimensions `nrow(Z) x nrow(Z)`;
- $Y = (I + Z)^{-1}(I + Z^{-1})^{-1}(Y^{+} + Y^{-})$;
- $C = D^{T} Y D$;
- Δx^{+} and Δx^{-} are the shifts in the proportions of input factors for binary outputs 1 and 0, respectively.

Practitioners can refer to this document for further details.

Weighted Binomial Logistic Regression

Another way to quantify changes in the parameters of the logistic regression based on the data shift is to re-estimate the weighted binomial logistic regression.

The following formula presents the log-likelihood function used to estimate the parameters (β) of the weighted logistic regression::

$$\mathcal{L}(\beta) = \sum_{i=1}^{n} w_i \left[y_i \log\left(\frac{1}{1+\exp(-\mathbf{x}_i\beta)}\right) + (1-y_i)\log\left(1 - \frac{1}{1+\exp(-\mathbf{x}_i\beta)}\right)\right]$$

where:

- y_i is the binary response variable for the `i`-th observation (either 0 or 1);
- $\mathbf{x}_i$ is the vector of predictors for the `i`-th observation;
- w_i is the associated weight of the `i`-th observation.

Weighted Quasi-Binomial Regression

The third method for quantifying changes in logistic regression parameters, based on the data shift, is by re-estimating the weighted quasi-binomial regression. Unlike binomial logistic regression, which requires a dichotomous target (0/1), weighted quasi-binomial regression processes fractions between 0 and 1. The weighted fractional logistic regression parameters can be estimated similarly to binomial logistic regression by maximizing the log-likelihood function with an additional term to account for dispersion. Since the additional term affects only the standard error of estimates, the estimated coefficients between weighted binomial and weighted quasi-binomial regression are identical.

The following formula presents the log-likelihood function used to estimate the model parameters ($\boldsymbol{\beta}$), along with the adjustment of the variance-covariance matrix ($\hat{\Sigma}$) based on the dispersion parameter:

$$\mathcal{L}(\beta) = \sum_{i=1}^{n} w_i \left[y_i \log\left(\frac{1}{1+\exp(-\mathbf{x}_i\boldsymbol{\beta})}\right) + (1-y_i)\log\left(1 - \frac{1}{1+\exp(-\mathbf{x}_i\boldsymbol{\beta})}\right)\right]$$

$$\hat{\Sigma} = \hat{\Phi}\hat{V}$$

where:

- y_i is the binary response variable for the i-th observation (either 0 or 1);
- $\mathbf{x}_i$ is the vector of predictors for the i-th observation;
- w_i is the associated weight of the i-th observation;
- $\hat{\Phi}$ is the estimate of the dispersion parameter;
- $\hat{V}$ is the estimated variance-covariance matrix assuming a binomial distribution (the "naive" variance-covariance matrix).

Simulation Study

The following steps outline the simulation framework used to quantify changes in model parameters based on a simulated scenario:

1. Assume a simplified PD model consisting of the target variable `Creditability` and two categorical risk factors: `Account_Balance` and `Maturity`. The simulation dataset is available here.

2. The risk factor `Account_Balance` includes four categories with the following distribution of observations:

```
##  01  02  03  04
## 274 269  63 394
```

3. The risk factor `Maturity` includes five categories with the following distribution of observations:

```
## 01 (-Inf,8)   02 [8,16)  03 [16,36)  04 [36,45) 05 [45,Inf)
##          87         344         399         100          70
```

Simulation Study cont.

4. The final PD model is estimated using binomial logistic regression and dummy encoding in the form: `Creditability ~ Account_Balance + Maturity` with the following estimated coefficients:

```
##                     Estimate Std. Error z value Pr(>|z|)
## (Intercept)          -1.3234     0.3720 -3.5579   0.0004
## Account_Balance02    -0.5064     0.1809 -2.7997   0.0051
## Account_Balance03    -1.0873     0.3332 -3.2629   0.0011
## Account_Balance04    -2.0194     0.2029 -9.9507   0.0000
## Maturity02 [8,16)     0.9783     0.3873  2.5262   0.0115
## Maturity03 [16,36)    1.4282     0.3809  3.7495   0.0002
## Maturity04 [36,45)    1.8817     0.4248  4.4297   0.0000
## Maturity05 [45,Inf)   2.4041     0.4491  5.3532   0.0000
```

5. Assume the following scenario: the portfolio structure changes as the bank plans to increase loan approvals for riskier groups, specifically clients in the `Account_Balance` category 01, by 40%. Simultaneously, loan approvals for clients in category 04 will decrease by the same number. Given this scenario, the new allocation of `Account_Balance` modalities is:

```
##    01    02    03    04
## 383.6 269.0  63.0 284.4
```

   An additional assumption is that the observed default rates remain unchanged.

6. Based on this scenario and the resulting portfolio structure changes, the objective is to quantify the change in the estimated parameters of the final PD model using the three methods presented in the previous slides.

Simulation Results - Matrix Multiplication Approach

Data Points x:

##	Account_Balance	Maturity	1	0
## 1	01	01 (-Inf,8)	0.004	0.018
## 2	01	02 [8,16)	0.033	0.053
## 3	01	03 [16,36)	0.063	0.055
## 4	01	04 [36,45)	0.019	0.010
## 5	01	05 [45,Inf)	0.016	0.003
## 6	02	01 (-Inf,8)	0.004	0.013
## 7	02	02 [8,16)	0.029	0.061
## 8	02	03 [16,36)	0.038	0.064
## 9	02	04 [36,45)	0.014	0.014
## 10	02	05 [45,Inf)	0.020	0.012
## 11	03	01 (-Inf,8)	0.000	0.008
## 12	03	02 [8,16)	0.007	0.021
## 13	03	03 [16,36)	0.006	0.015
## 14	03	04 [36,45)	0.001	0.005
## 15	04	01 (-Inf,8)	0.001	0.039
## 16	04	02 [8,16)	0.011	0.129
## 17	04	03 [16,36)	0.022	0.136
## 18	04	04 [36,45)	0.008	0.029
## 19	04	05 [45,Inf)	0.004	0.015

Model Points y:

##	Account_Balance	Maturity	1	0
## 1	01	01 (-Inf,8)	0.0046	0.0174
## 2	01	02 [8,16)	0.0357	0.0503
## 3	01	03 [16,36)	0.0621	0.0559
## 4	01	04 [36,45)	0.0184	0.0106
## 5	01	05 [45,Inf)	0.0142	0.0048
## 6	02	01 (-Inf,8)	0.0024	0.0146
## 7	02	02 [8,16)	0.0269	0.0631
## 8	02	03 [16,36)	0.0409	0.0611
## 9	02	04 [36,45)	0.0144	0.0136
## 10	02	05 [45,Inf)	0.0205	0.0115
## 11	03	01 (-Inf,8)	0.0007	0.0073
## 12	03	02 [8,16)	0.0054	0.0226
## 13	03	03 [16,36)	0.0057	0.0153
## 14	03	04 [36,45)	0.0022	0.0038
## 15	04	01 (-Inf,8)	0.0014	0.0386
## 16	04	02 [8,16)	0.0120	0.1280
## 17	04	03 [16,36)	0.0203	0.1377
## 18	04	04 [36,45)	0.0070	0.0300
## 19	04	05 [45,Inf)	0.0053	0.0137

Data Shifts Δx^+ and Δx^-:

##	Account_Balance	Maturity	n	dx_plus	dx_minus
## 1	01	01 (-Inf,8)	22	-0.0016	-0.0072
## 2	01	02 [8,16)	86	-0.0132	-0.0212
## 3	01	03 [16,36)	118	-0.0252	-0.0220
## 4	01	04 [36,45)	29	-0.0076	-0.0040
## 5	01	05 [45,Inf)	19	-0.0064	-0.0012
## 6	02	01 (-Inf,8)	17	0.0000	0.0000
## 7	02	02 [8,16)	90	0.0000	0.0000
## 8	02	03 [16,36)	102	0.0000	0.0000
## 9	02	04 [36,45)	28	0.0000	0.0000
## 10	02	05 [45,Inf)	32	0.0000	0.0000
## 11	03	01 (-Inf,8)	8	0.0000	0.0000
## 12	03	02 [8,16)	28	0.0000	0.0000
## 13	03	03 [16,36)	21	0.0000	0.0000
## 14	03	04 [36,45)	6	0.0000	0.0000
## 15	04	01 (-Inf,8)	40	0.0003	0.0108
## 16	04	02 [8,16)	140	0.0031	0.0359
## 17	04	03 [16,36)	158	0.0061	0.0378
## 18	04	04 [36,45)	37	0.0022	0.0081
## 19	04	05 [45,Inf)	19	0.0011	0.0042

C Matrix (`MxM`):

##	(Intercept)	Account_Balance02	Account_Balance03	Account_Balance04	Maturity02 [8,16)	Maturity03 [16,36)	Maturity04 [36,45)	Maturity05 [45,Inf)
## (Intercept)	0.1741	0.0598	0.0105	0.0395	0.0551	0.0758	0.0208	0.0148
## Account_Balance02	0.0598	0.0598	0.0000	0.0000	0.0189	0.0245	0.0070	0.0074
## Account_Balance03	0.0105	0.0000	0.0105	0.0000	0.0044	0.0042	0.0014	0.0000
## Account_Balance04	0.0395	0.0000	0.0000	0.0395	0.0110	0.0177	0.0057	0.0038
## Maturity02 [8,16)	0.0551	0.0189	0.0044	0.0110	0.0551	0.0000	0.0000	0.0000
## Maturity03 [16,36)	0.0758	0.0245	0.0042	0.0177	0.0000	0.0758	0.0000	0.0000
## Maturity04 [36,45)	0.0208	0.0070	0.0014	0.0057	0.0000	0.0000	0.0208	0.0000
## Maturity05 [45,Inf)	0.0148	0.0074	0.0000	0.0038	0.0000	0.0000	0.0000	0.0148

The Estimated Coefficient Changes:

##	(Intercept)	Account_Balance02	Account_Balance03	Account_Balance04	Maturity02 [8,16)	Maturity03 [16,36)	Maturity04 [36,45)	Maturity05 [45,Inf)
##	0.0150	0.0056	-0.0055	0.0041	-0.0027	-0.0160	-0.0158	-0.0920

Simulation Results - Weighted Binomial Logistic Regression

Sample of the Aggregated Dataset with Initial Counts (n):

```
## Account_Balance    Maturity Creditability  n
##              01 01 (-Inf,8)             0 18
##              01 01 (-Inf,8)             1  4
##              01   02 [8,16)             0 53
##              01   02 [8,16)             1 33
##              01  03 [16,36)             0 55
##              01  03 [16,36)             1 63
##              01  04 [36,45)             0 10
##              01  04 [36,45)             1 19
##              01 05 [45,Inf)             0  3
##              01 05 [45,Inf)             1 16
```

Sample of the Aggregated Dataset with Simulated Counts (n_s):

```
## Account_Balance    Maturity Creditability  n_s
##              01 01 (-Inf,8)             0 25.2
##              01 01 (-Inf,8)             1  5.6
##              01   02 [8,16)             0 74.2
##              01   02 [8,16)             1 46.2
##              01  03 [16,36)             0 77.0
##              01  03 [16,36)             1 88.2
##              01  04 [36,45)             0 14.0
##              01  04 [36,45)             1 26.6
##              01 05 [45,Inf)             0  4.2
##              01 05 [45,Inf)             1 22.4
```

The Estimated Coefficient Changes:

```
##        (Intercept)   Account_Balance02   Account_Balance03   Account_Balance04   Maturity02 [8,16)  Maturity03 [16,36)  Maturity04 [36,45) Maturity05 [45,Inf)
##             0.0158              0.0056             -0.0052              0.0042             -0.0048             -0.0165             -0.0150             -0.0923
```

Simulation Results - Weighted Quasi-Binomial Regression

Sample of the Aggregated Dataset with Initial Counts (n):

```
## Account_Balance    Maturity   n      frac
##              01 01 (-Inf,8)  22 0.1818182
##              01   02 [8,16)  86 0.3837209
##              01  03 [16,36) 118 0.5338983
##              01  04 [36,45)  29 0.6551724
##              01 05 [45,Inf)  19 0.8421053
##              02 01 (-Inf,8)  17 0.2352941
##              02   02 [8,16)  90 0.3222222
##              02  03 [16,36) 102 0.3725490
##              02  04 [36,45)  28 0.5000000
##              02 05 [45,Inf)  32 0.6250000
```

Sample of the Aggregated Dataset with Simulated Counts (n_s):

```
## Account_Balance    Maturity   n      frac   n_s
##              01 01 (-Inf,8)  22 0.1818182  30.8
##              01   02 [8,16)  86 0.3837209 120.4
##              01  03 [16,36) 118 0.5338983 165.2
##              01  04 [36,45)  29 0.6551724  40.6
##              01 05 [45,Inf)  19 0.8421053  26.6
##              02 01 (-Inf,8)  17 0.2352941  17.0
##              02   02 [8,16)  90 0.3222222  90.0
##              02  03 [16,36) 102 0.3725490 102.0
##              02  04 [36,45)  28 0.5000000  28.0
##              02 05 [45,Inf)  32 0.6250000  32.0
```

The Estimated Coefficient Changes:

```
##       (Intercept)   Account_Balance02   Account_Balance03   Account_Balance04    Maturity02 [8,16)  Maturity03 [16,36)  Maturity04 [36,45) Maturity05 [45,Inf)
##            0.0158              0.0056             -0.0052              0.0042              -0.0048             -0.0165             -0.0150             -0.0923
```

Simulation Results - Summary

The table below provides a summary and comparison of the model shift simulation results:

```
##         coefficient matrix multiplication weighted logistic weighted quasi-binomial
##         (Intercept)                0.0150            0.0158                  0.0158
##   Account_Balance02                0.0056            0.0056                  0.0056
##   Account_Balance03               -0.0055           -0.0052                 -0.0052
##   Account_Balance04                0.0041            0.0042                  0.0042
##    Maturity02 [8,16)               -0.0027           -0.0048                 -0.0048
##   Maturity03 [16,36)               -0.0160           -0.0165                 -0.0165
##   Maturity04 [36,45)               -0.0158           -0.0150                 -0.0150
##  Maturity05 [45,Inf)               -0.0920           -0.0923                 -0.0923
```