1.

2. $\mathrm{T(n)} = 2\mathrm{T}\left(\left\lceil \frac{n}{2} \right\rceil + 2\right) + \mathrm{n}$, show that $\mathrm{T(n)} = O(\log_2 n)$.

Let $\mathrm{T(n)} = an\log_2 n - b\log_2 n - c,\ \mathrm{a} > 0, \mathrm{b} > 0, \mathrm{c} > 0$
Induction hypothesis: $\mathrm{m} < \mathrm{n},\ \mathrm{T(m)} \le am\log_2 m - b\log_2 m - c$;

$$\mathrm{T(n)} = 2\mathrm{T}\left(\left\lceil \frac{n}{2} \right\rceil + 2\right) + \mathrm{n} \le 2\mathrm{T}\left(\frac{n}{2} + 3\right) + n \le 2\left(a\left(\frac{n}{2} + 3\right)\log_2\left(\frac{n}{2} + 3\right) - b\log_2\left(\frac{n}{2} + 3\right) - c\right) + n$$

$$\le an\left(\log_2\left(\frac{n}{2} + 3\right) + \frac{1}{a}\right) + (6a - 2b)\log_2\left(\frac{n}{2} + 3\right) - 2c$$

$$\le an\log_2(\left(\frac{n}{2} + 3\right) * 2^{\frac{1}{a}}) + (6a - 2b)\log_2\left(\frac{n}{2} + 3\right) - 2c)$$

$$\le an\log_2 n - b\log_2 n - c$$

(1) $n \ge \frac{n}{2} + 3,\ \mathrm{n} \ge 6$

(2) $\mathrm{n} \ge \left(\frac{n}{2} + 3\right) * 2^{1/a}$, let $a = 2,\ \mathrm{n} \ge \frac{\mathrm{n}}{\sqrt{2}} + 3\sqrt{2},\ n\left(1 - \frac{1}{\sqrt{2}}\right) \ge 3\sqrt{2},\ n \ge 6/(\sqrt{2} - 1) \approx 14.49,$

$\therefore \mathrm{n} \ge 15.$

(3) $6a - 2b \le -b,\ \mathrm{a} \le \frac{1}{6}b,\ \therefore \mathrm{b} \ge 12,\ a = 2$

(4) $-2c \le -c,\ \mathrm{c} > 0$

3. Use a recursion tree to determine a good upper bound on the recurrence $\mathrm{T(n)} = 3\mathrm{T}(\lfloor n/3 \rfloor) + \mathrm{n}$.

$\mathrm{T(n)} = 3\mathrm{T}(\lfloor n/3 \rfloor) + \mathrm{n} \le 3\mathrm{T}\left(\frac{\mathrm{n}}{3}\right) + \mathrm{n},$

n
n/3 n/3 n/3
n/9 n/9 n/9 n/9 n/9 n/9 n/9 n/9 n/9
…………………………………………………………………
T(1)………………………………………………………… T(1)

The recursion tree height $h = \log_3 n$
The number of leaves is $3^h = 3^{\log_3 n} = n$
Total cost is $\approx n\log_3 n$

Prove the bound $\mathrm{O}(n\log_3 n)$ by induction.
Induction hypothesis: $\mathrm{m} < \mathrm{n},\ \mathrm{T(m)} \le m\log_3 m$

$$\mathrm{T(n)} = 3\mathrm{T}(\lfloor n/3 \rfloor) + \mathrm{n} \le \mathrm{T(n)} = 3\mathrm{T}\left(\frac{\mathrm{n}}{3}\right) + \mathrm{n} \le 3\left(\frac{\mathrm{n}}{3}\right)\log_3\frac{n}{3} + n \le n\log_3 n - n\log_3 3 + n \le n\log_3 n$$

4. Use the master method

(a) $\mathrm{T(n)} = 4\mathrm{T}\left(\frac{\mathrm{n}}{2}\right) + \mathrm{n}$

$\mathrm{a} = 4, \mathrm{b} = 2, f(n) = n,$
$n^{\log_2 4} = n^2,\ f(n) = O(n^{\log_2 4 - \epsilon}),\ \epsilon = 1\ \ \mathrm{T(n)} = \Theta(n^2)$

(b) $\mathrm{T(n)} = 4\mathrm{T}\left(\frac{\mathrm{n}}{2}\right) + n^2$

$\mathrm{a} = 4, \mathrm{b} = 2, f(n) = n^2,\ n^{\log_2 4} = n^2,\ f(n) = O\left(n^{\log_2 4}\right),\ \mathrm{T(n)} = \Theta(n^2\log_2 n)$

(c) $\mathrm{T(n)} = 4\mathrm{T}\left(\frac{\mathrm{n}}{2}\right) + n^3$

$\mathrm{a} = 4, \mathrm{b} = 2, f(n) = n^3,\ n^{\log_2 4} = n^2,\ f(n) = O(n^{2+\epsilon}),\ \epsilon = 1,\ af\left(\frac{n}{b}\right) = cf(n)$, for some constant $\mathrm{c} < 1$. $4\mathrm{f}\left(\frac{\mathrm{n}}{2}\right) = 4 * \frac{n^3}{8} = \frac{n^3}{2} = cf(n)$. $\mathrm{c} = 1/2$.

$\mathrm{T(n)} = \Theta(n^3)$.