Angular distribution between two vectors

Antonio Martinez-Sanchez

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Given two arbitrary unit vectors $\vec{v}_0 = (x_0, y_0, z_0)$ and $\vec{v}_1 = (x_1, y_1, z_1)$ ($||\vec{v}_0|| = ||\vec{v}_1|| = 1$), the angle between them is

$$\alpha = \arccos\left(\vec{v}_0 \cdot \vec{v}_1\right)$$

where $\cdot$ is the dot-product.

Given a rotation $R(\vec{v}_0) = \vec{v}_0^r = \mathbf{z}$ where $\mathbf{z}$ is the vector collinear to Z-axis in the rotated space, then $R(\vec{v}_1) = \vec{v}_1^r$ and $\alpha = \alpha_r = \arccos(\mathbf{z} \cdot \vec{v}_1^r)$. The angle is invariant to rations on the vectors so the analysis in the rotated space is valid.

In the rotated space the angle between vectors, α_r, correspond with the polar angle, ϕ, in spherical coordinates. On the other, hand to obtain points such that any small area on the sphere is expected to contain the same number of points, choose u and w to be random variates on $(0, 1)$. Then the azimuthal and polar angles becomes

$$\begin{aligned} \theta &= 2\pi u \\ \phi &= \arccos\left(2w - 1\right) \end{aligned} \tag{1}$$

for a set of points which are uniformly distributed over the unit sphere.

Since the differential element of solid angle is given by

$$d\Omega = \sin\phi d\theta d\phi = -d\theta d(\cos\phi) \tag{2}$$

The distribution P of polar angles can be found from

$$P_\phi = P_w \left|\frac{dw}{d\phi}\right| d\phi \tag{3}$$

By operating on Eq. (3) the distribution becomes

$$P_\phi = \frac{1}{2}\sin\phi \tag{4}$$

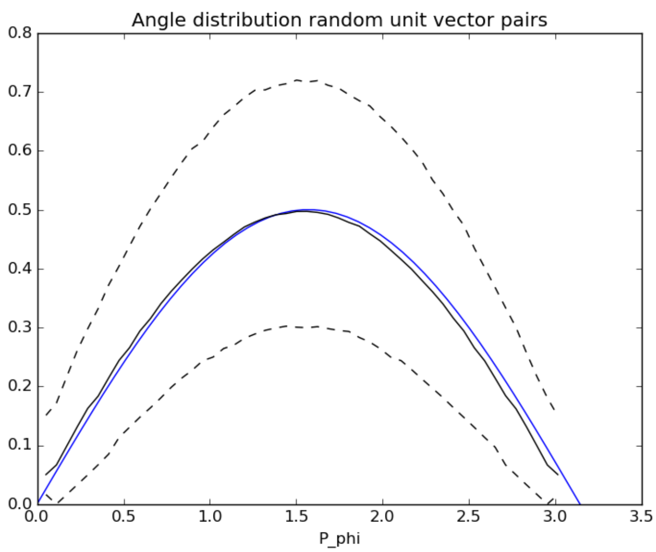

Figure 1: Angle distribution between two arbitrary unit vectors. In blue analytic solution, in black the results of 20000 random simulation of 1000 pairs of unit vectors with an interval of confidence of 99%.