Condensed Matter Emulation

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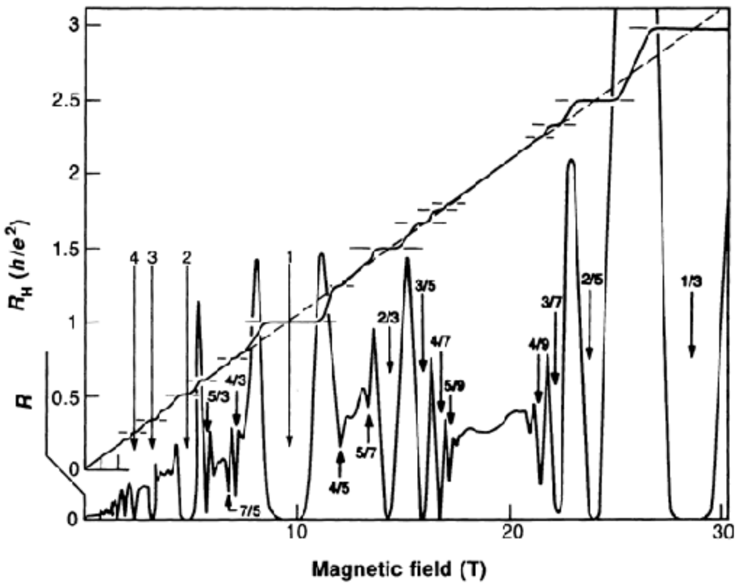
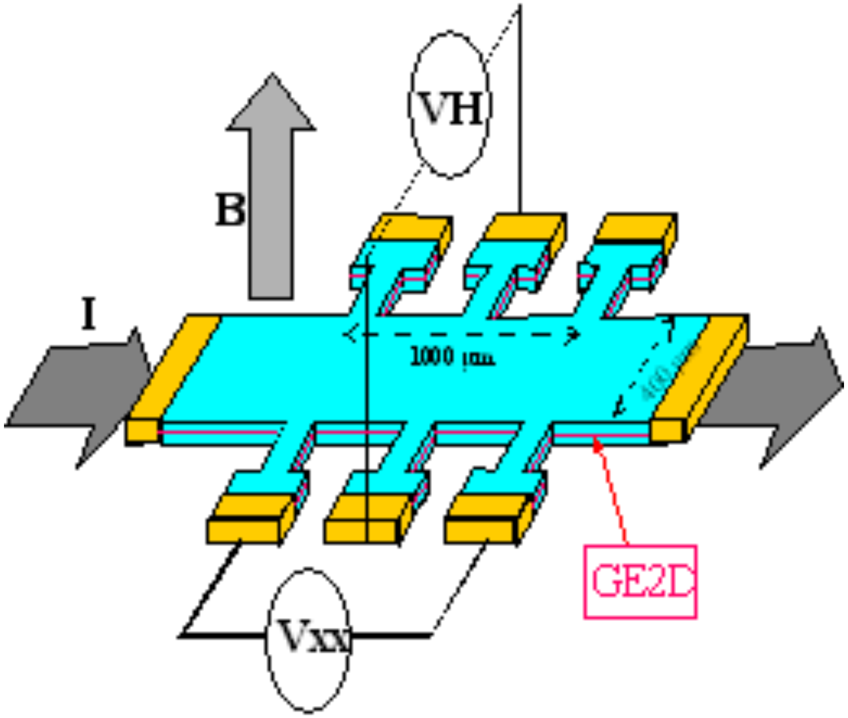
Quantum Hall Physics with BECs
WHY?





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Reference Material

- Condensed matter emulation
  - *Ultracold atomic gases in optical lattices: mimicking condensed matter physics and beyond*, M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen and U. Sen, Advances in Physics **56**, 243 (2007)
  - *Many-body physics with ultracold gases*, I. Bloch, J. Dalibard and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008)
- Rotating BECs
  - *Rotating trapped Bose-Einstein condensates*, A. L. Fetter, Rev. Mod. Phys. **81**, 647 (2009)

Outline

- LLL one body states
  - Harmonic oscillator
  - Rotation
  - Landau Levels
  - Rotation $\rightarrow$ Magnetic field
- Rotational Interlude
  - Vortex lattice formation
  - TF description of rotating condensate
- LLL meanfield description
  - LLL wavefunction
  - Energy minimization $\rightarrow$ conditions for meanfield LLL regime
- Highly correlated states
  - Laughlin wavefunction $\rightarrow$ conditions for HCS

LLL One Body States ($\Omega = 0$)

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Harmonic oscillator

$$H_0 = \hbar\omega_\perp \left(a_+^\dagger a_+ + a_-^\dagger a_- + 1 \right)$$

$$a_\pm = \frac{a_x \mp i a_y}{\sqrt{2}}$$

$$a_\pm^\dagger = \frac{a_x^\dagger \pm i a_y^\dagger}{\sqrt{2}}$$

$$a_x = \frac{1}{\sqrt{2}} \left(\frac{x}{d_\perp} + i \frac{p_x d_\perp}{\hbar} \right) \quad a_y = \frac{1}{\sqrt{2}} \left(\frac{y}{d_\perp} + i \frac{p_y d_\perp}{\hbar} \right)$$

$$a_x^\dagger = \frac{1}{\sqrt{2}} \left(\frac{x}{d_\perp} - i \frac{p_x d_\perp}{\hbar} \right) \quad a_y^\dagger = \frac{1}{\sqrt{2}} \left(\frac{y}{d_\perp} - i \frac{p_y d_\perp}{\hbar} \right)$$

Angular momentum

$$L_z = x p_y - y p_x = \hbar \left(a_+^\dagger a_+ - a_-^\dagger a_- \right)$$

Create and destroy one quantum with positive (negative) circular polarization and one unit of positive (negative) angular momentum

LLL One Body States ($\Omega \neq 0$)

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Rotating system

$$H_0' = H_0 - \Omega L_z = \hbar\omega_\perp + \hbar\left(\omega_\perp - \Omega\right) a_+^\dagger a_+ + \hbar\left(\omega_\perp + \Omega\right) a_-^\dagger a_-$$

Eigenvalues

$$\epsilon\left(n_+, n_-\right) = n_+\hbar\left(\omega_\perp - \Omega\right) + \hbar n_-\left(\omega_\perp + \Omega\right)$$

Landau Levels

$\Omega \to \omega_\perp$ $\longrightarrow$ Eigenvalues are essentially independent of n_+

$\downarrow$

n_- becomes the Landau Level index

LLL One Body States (Ω)

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$$n = n_+ + n_- \qquad\qquad m = n_+ - n_-$$

$$\epsilon\left(\frac{1}{2}(n+m), \frac{1}{2}(n-m)\right) = n\hbar\omega_\perp - m\hbar\Omega$$

- The excitation energy is independent of *m* forming an inverted pyramid of states. For each non-negative integer *n* there are *n +1* degenerate angular momentum states (-*n* … *n,* in steps of 2)
- The degeneracy is lifted
- States become nearly degenerate again, forming essentially horizontal rows.

LLL One Body States ($\Omega \approx 1$)

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LLL Physics appropriate when

$$\Omega/\omega_\perp \approx 1$$

Energy scales

Gap $\longrightarrow$ $2\hbar\omega_\perp$

Interaction energy $\longrightarrow$ $gn(0) = \mu$

$\Longrightarrow$ $\mu / (2\hbar\omega_\perp) \ll 1$

Eigenfunctions of LLL

$$\psi_m \propto r^m e^{i\phi m} e^{-r^2/(2d_\perp^2)}$$

- *m* =*0* represents the vacuum for both circularly polarized modes
- The higher states (*m* >*0*) can be written as

$$\psi_m \propto \zeta^m e^{-r^2/(2d_\perp^2)}, \quad \text{where } \zeta = (x+iy)/d_\perp$$

Rotation and Magnetic field

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$$H_0' = \frac{p^2}{2M} + \frac{1}{2}M\omega_\perp^2 r^2 - \mathbf{\Omega}\cdot\mathbf{r}\times\mathbf{p}$$

$$H_0' = \frac{(p - M\Omega\times\mathbf{r})^2}{2M} + \frac{1}{2}M\left(\omega_\perp^2 - \Omega^2\right)r^2$$

Synthetic vector potential

$$q\mathbf{A} \rightarrow M\Omega\times\mathbf{r}$$

$$\Omega = \hat{k}\Omega_z$$

$$q\mathbf{A} = M\Omega_z\left(-y\hat{i} + x\hat{j}\right)$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = 2\Omega_z M/q$$

Rotational Interlude

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Methods

- condense rotating thermal cloud
- stir with narrow obstacle (laser)
- **deform trap elliptically and rotate**

Rotation Frequency Ω

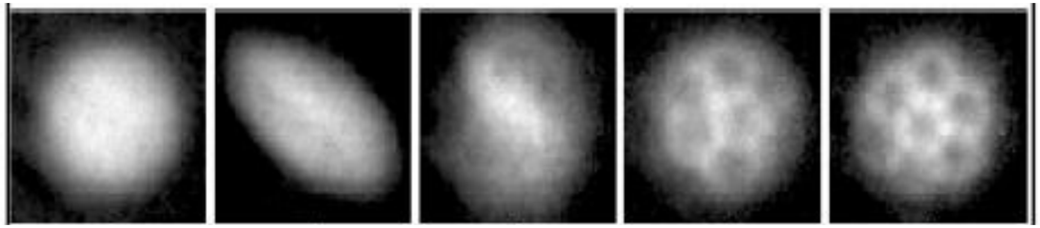
- Low Ω → Long-lived quadrupole oscillations
- $\Omega \sim 0.7\omega_\perp$ → Vortex lattice formation

  - Dynamical instability of quadrupole mode

    [A. Ricati *et al.*, PRL **86**, 377 (2001); Sinha and Y. Castin, PRL **87**, 190402 (2001)]

  - Crystallisation insensitive to temperature

    [Abo-Shaeer *et al.*, PRL **88**, 070409 (2002)]



ENS, Paris

Typical Movie (Increasing Ω)

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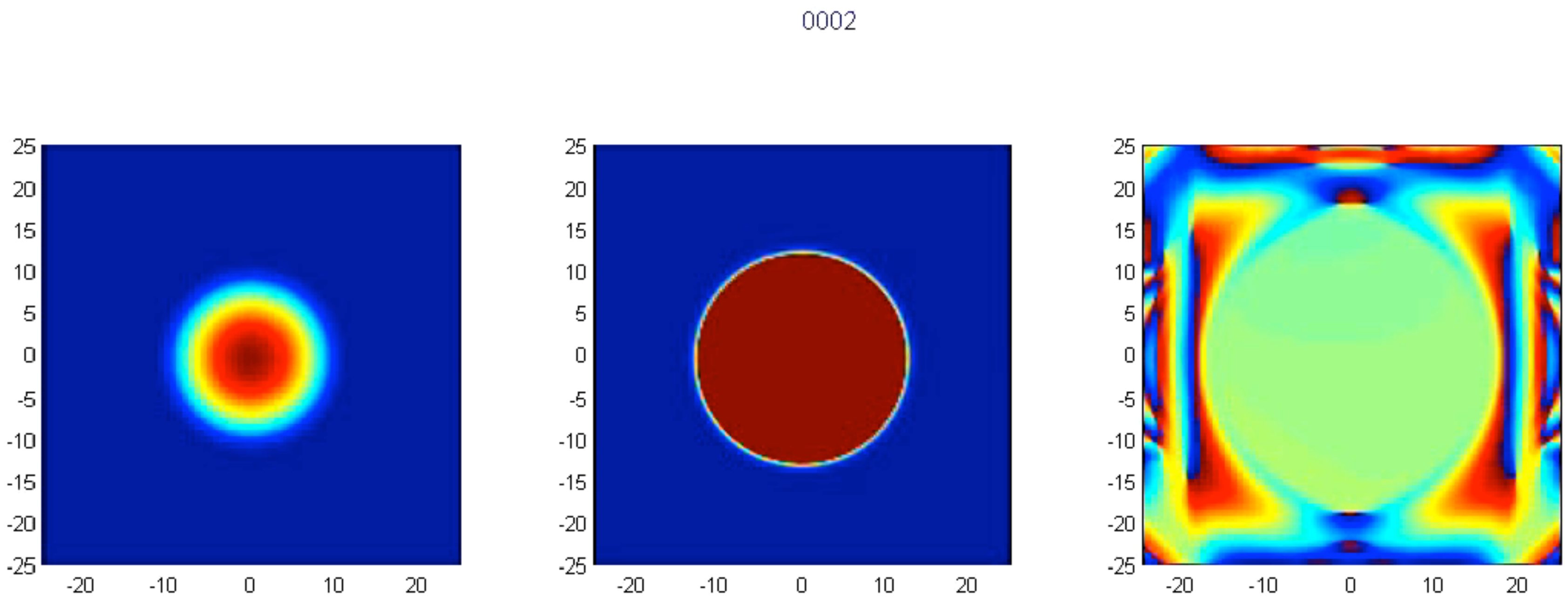
Gross-Pitaevskii Simulation



Hydrodynamical Model

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Gross Pitaevskii Equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + U_T(\mathbf{r}) + g\,|\Psi(\mathbf{r},t)|^2 - \Omega\hat{L}_z\right)\Psi(\mathbf{r},t)$$

$$U_T = \frac{m}{2}\left[(1-\varepsilon)\omega_\perp^2 x^2 + (1+\varepsilon)\omega_\perp^2 y^2 + \gamma^2 z^2\right]$$

Mandelung Transformation

$$\Psi(\mathbf{r},t) = \sqrt{\rho(\mathbf{r},t)}\,\exp[iS(\mathbf{r},t]$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left[\rho\left(\mathbf{v} - \boldsymbol{\Omega}\times\mathbf{r}\right)\right] = 0$$

$$m\frac{\partial\mathbf{v}}{\partial t} + \nabla\left[\frac{1}{2}m\mathbf{v}\cdot\mathbf{v} + U_T + g\rho - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} - m\mathbf{v}\cdot(\boldsymbol{\Omega}\times\mathbf{r})\right] = 0$$

Quantum Pressure = 0

Thomas-Fermi Approximation

$$\mathbf{v} = \frac{\hbar}{m}\nabla S$$

$$\rho(\mathbf{r}) = \rho_0\left(1 - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} - \frac{z^2}{R_z^2}\right)$$

Stable Solutions

Irrotational Velocity Field

$$\mathbf{v} = \alpha\nabla(xy)$$

Using Hydrodynamical Equations

$$\alpha^3 + (1 - 2\Omega^2)\alpha - \varepsilon\Omega = 0$$

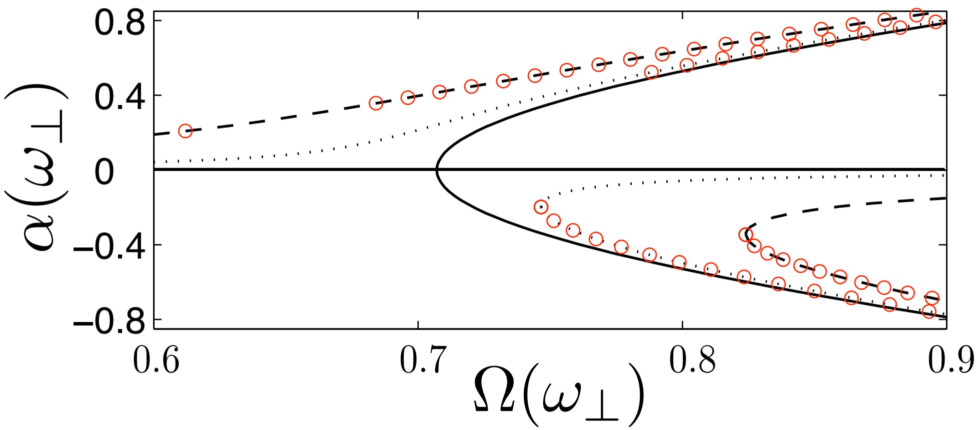
- α quantifies the deformation of the BEC in the rotating frame
- α solutions may not be stable:

$$\frac{\partial}{\partial t}\begin{bmatrix} \delta S \\ \delta\rho \end{bmatrix} = -\begin{bmatrix} \mathbf{v_c}\cdot\nabla & g/m \\ \nabla\cdot\rho_0\nabla & [(\nabla\cdot\mathbf{v}) + \mathbf{v_c}\cdot\nabla] \end{bmatrix}\begin{bmatrix} \delta S \\ \delta\rho \end{bmatrix}$$

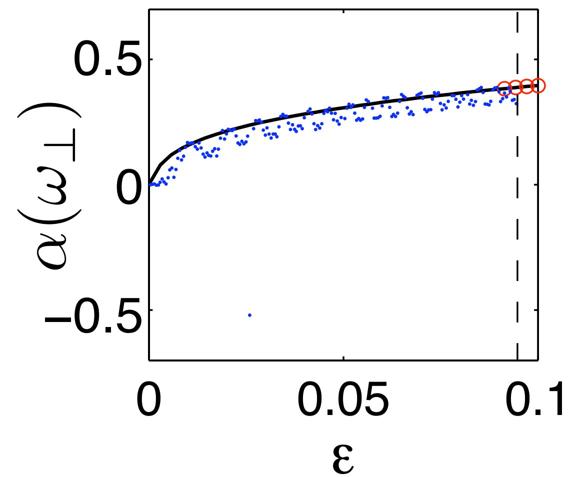
$$\mathbf{v_c} = \mathbf{v} + \Omega\times\mathbf{r}$$

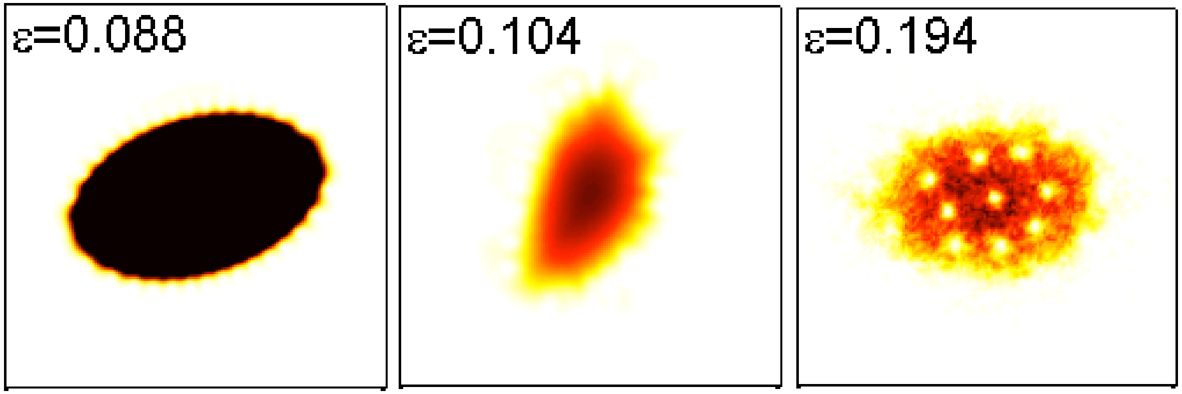
- Perturbations are polynomials
- Positive eigenvalues unstable

(I) Ripple Instability



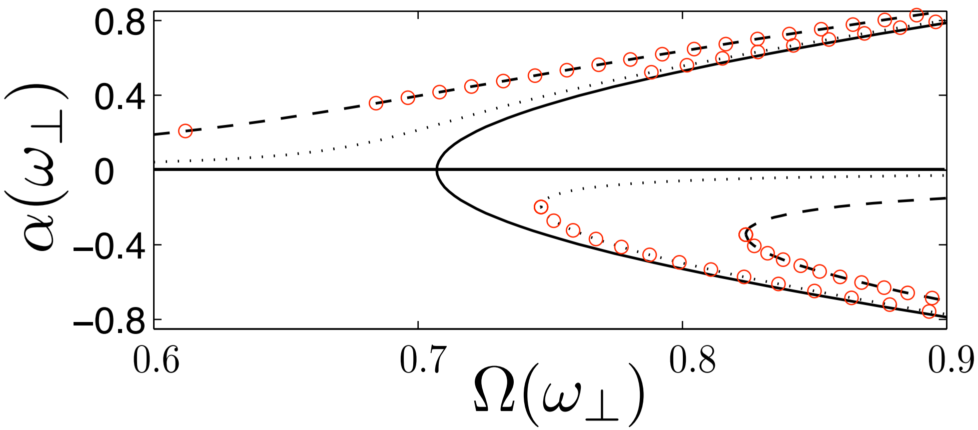
$\Omega = 0.7\omega_\perp$



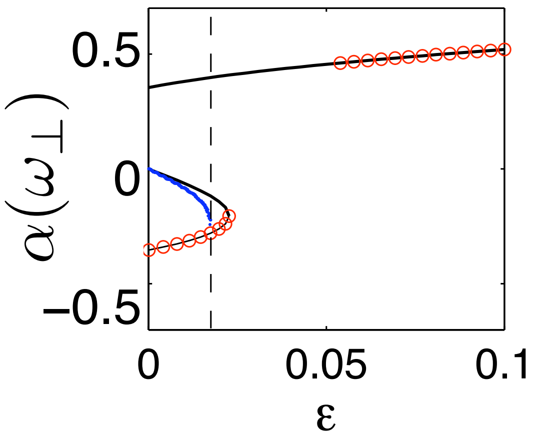


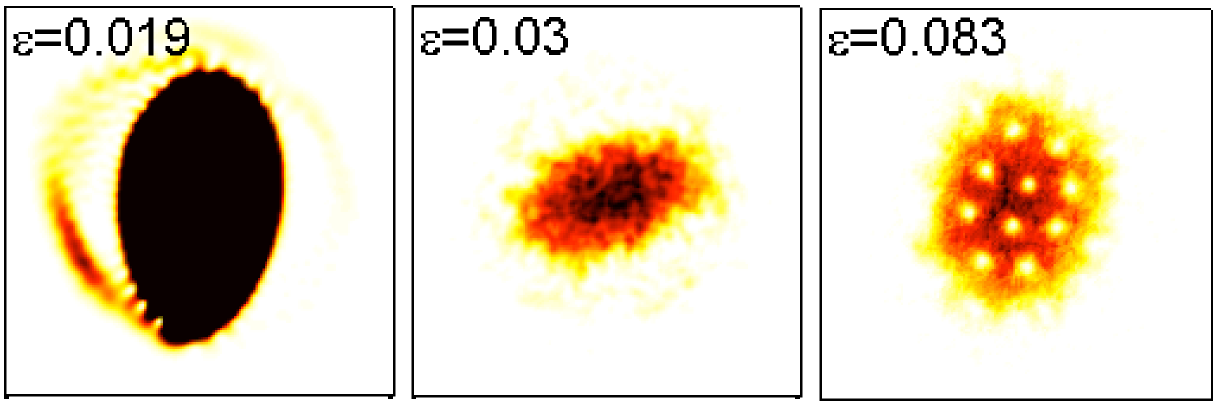
- On upper branch until dynamically unstable
- Density ripples form
- Ripples grow and become unstable

(I) Interbranch Instability



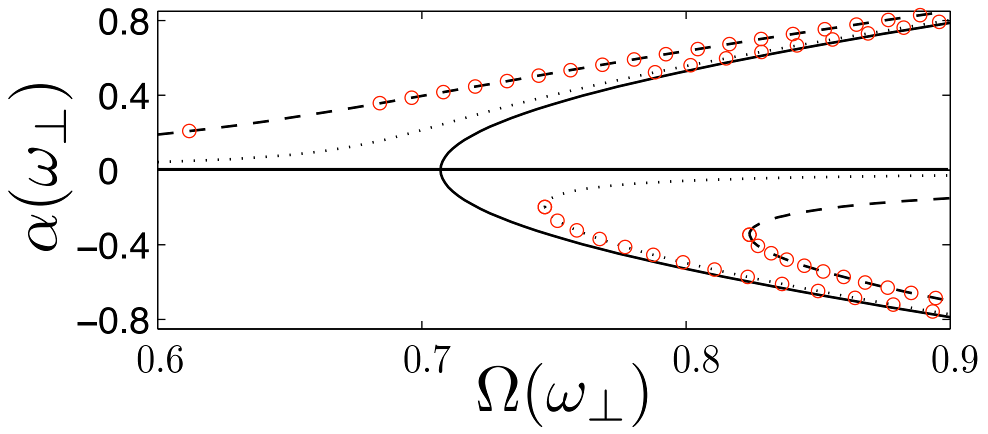
$\Omega = 0.75\omega_\perp$



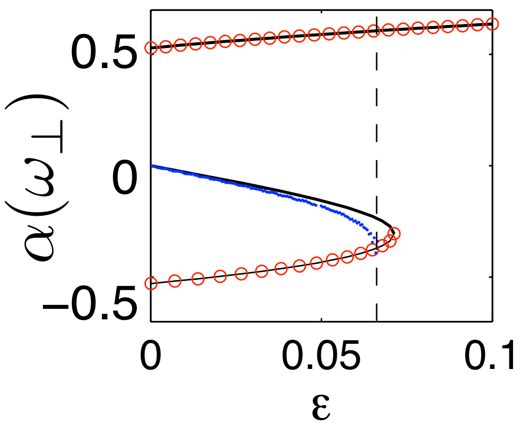


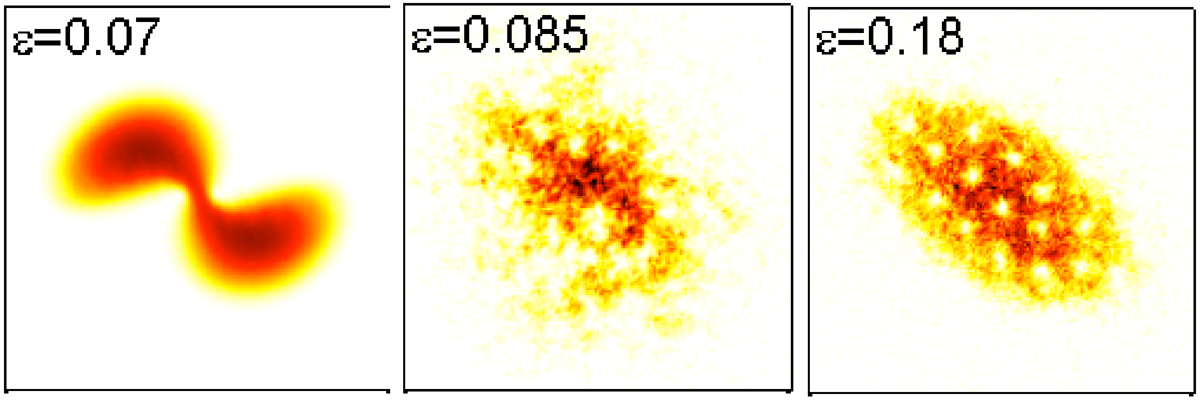
- On lower branch until no longer a solution
- Tries to deform to stable upper branch
- Unstable quadrupole shape oscillations

(I) Catastrophic Instability



$\Omega = 0.8\omega_\perp$





- On lower branch until no longer a solution
- Upper branch dynamically unstable
- Rapid, catastrophic shape instability

Experimental Comparison

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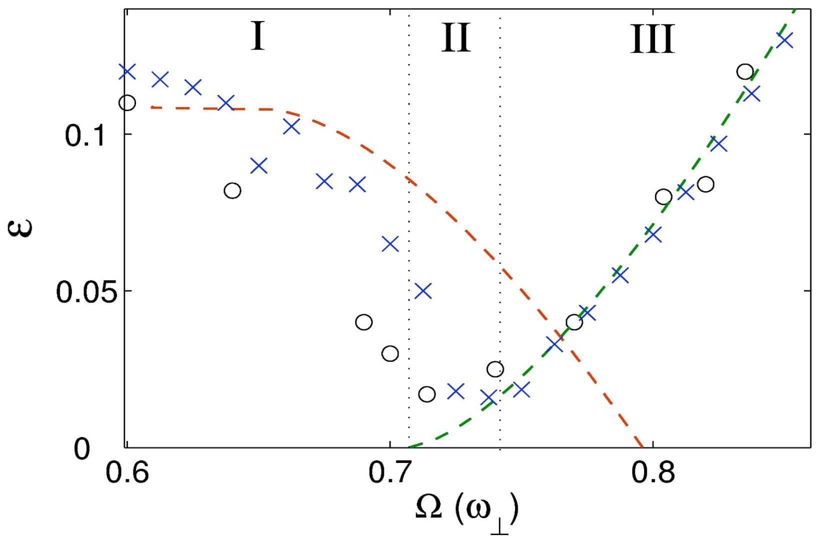

o o o Experimental data of Hodby *et al.*

Lower bound of lower branch

Lower bound of dynamically unstable region

× × × Numerical simulations – point where elliptical condensate breaks down

- From static solutions of hydrodynamic equations can determine regimes of vortex lattice formation
- Using GPE can see three distinct regimes
- Vortex lattice formation is a two dimensional zero temperature effect
- However symmetry must be broken

Instabilities leading to vortex lattice formation in rotating Bose-Einstein condensates, N. G. Parker, R. M. W. van Bijnen and A. M. Martin, PRA **73**, 061603(R) (2006)

An Alternative to Rotation

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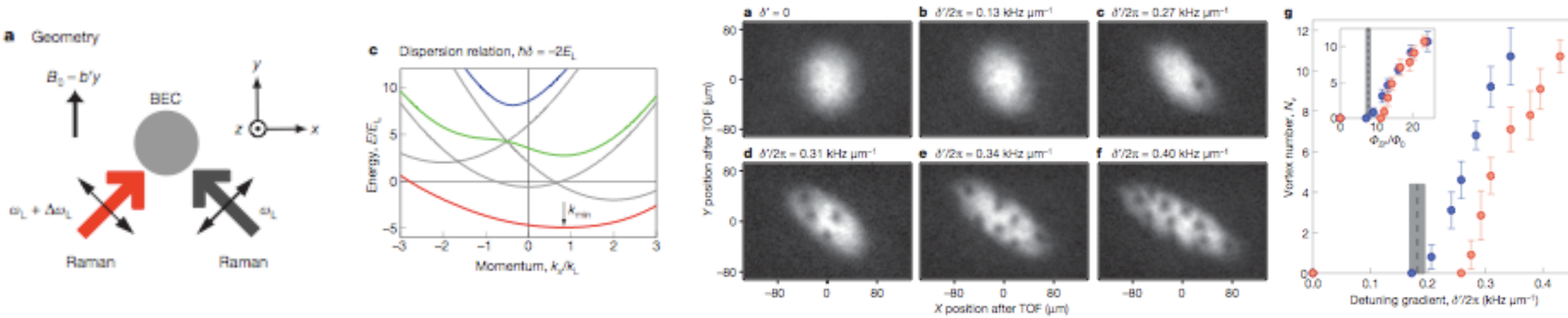
Synthetic gauges in BECs



$$i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{\left(-\hbar\nabla - q^*\mathbf{A}^*\right)}{2m} + V(\mathbf{r},t) + g|\psi|^2\right]\psi$$

$$\frac{\partial}{\partial t}\begin{bmatrix}\delta S \\ \delta n\end{bmatrix} = \begin{bmatrix}\nu\cdot\nabla & \frac{g}{\hbar} \\ \nabla\cdot n(\mathbf{r})\nabla & \left[(\nabla\cdot\nu)+\nu\cdot\nabla\right]\end{bmatrix}\begin{bmatrix}\delta S \\ \delta n\end{bmatrix}$$

$$\frac{\Phi_{B_z^*}}{\Phi_0} = 11.1$$

Synthetic magnetic fields for ultracold neutral atoms, Y.-J. Lin *et al.*, Nature **462**, 628 (2009).

Synthetic magneto-hydrodynamics in Bose-Einstein condensates and routes to vortex nucleation, L.B. Taylor *et al.*, Physical Review A **84**, 021604(R) (2011).

LLL Condensate Wavefunction

$$\psi_{LLL} = \sum_{m \geq 0} c_m \psi_m = f(\zeta) e^{-r^2/(2d_\perp^2)}$$

$$f(\zeta) \propto \prod_j (\zeta - \zeta_j)$$

- $f(\zeta)$ vanishes at each of the points ζ_j which are the positions of the nodes of the condensate wave-function
- The phase of this wave-function increases by 2π whenever ζ moves in the positive sense around any of these zeros
- Thus the points ζ_j are precisely the positions of the vortices in the trial state and minimization with respect to the constants c_m is effectively the same as minimization with respect to the position of the vortices

Energy Minimization

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$$E\left[\psi\right] = \int d^2 r \psi^* \left(\frac{p^2}{2M} + \frac{1}{2} M \omega_\perp^2 r^2 - \Omega L_z + \frac{1}{2} g_{2D} |\psi|^2 \right) \psi$$

$$E\left[\psi_{LLL}\right] = \hbar\Omega + \int d^2 r \left[M\omega_\perp^2 \left(1 - \frac{\Omega}{\omega_\perp}\right) r^2 \left|\psi_{LLL}\right|^2 + \frac{1}{2} g_{2D} |\psi_{LLL}|^4 \right]$$

Unrestricted minimization

$$\left|\psi_{min}\right|^2 = n_{min}(0) \left(1 - \frac{r^2}{R_0^2}\right) = \frac{\mu_{min}}{g_{2D}} \left(1 - \frac{r^2 M \omega_\perp^2 (1 - \tilde{\Omega})}{\mu_{\min}}\right)$$

$$\Downarrow$$

$$\mu_{\min} = \sqrt{\frac{8aN(1-\tilde{\Omega})}{Z}}, \quad \text{where } Z = 2\pi d_z$$

LLL Condition (Unrestricted)

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$$\mu_{\min} \leq 2\hbar\omega_{\perp}$$

$$1 - \tilde{\Omega} \leq \frac{Z}{2Na}$$

Unrestricted minimization!!

What about vortices?

Highly Correlated States (ν)

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Mean field LLL regime:

$$1-\tilde{\Omega} \leq \frac{Z}{2N\beta a}, \quad \text{where } \beta = 1.1596$$

- At higher rotation frequencies the meanfield LLL regime should eventually disappear through a quantum phase transition, leading to a different, highly correlated, manybody ground state.
- For meanfield LLL regime

$$N_v \approx \frac{R_0^2}{d_\perp^2} = \sqrt{\frac{8Na\beta}{Z(1-\tilde{\Omega})}}$$

$$\nu = \frac{N}{N_v} = \sqrt{\frac{Z(1-\tilde{\Omega})N}{8a\beta}}$$

Exact Diagonalization ($v \geq v_c$)

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- The equilibrium state in the meanfield LLL regime is a vortex array that breaks the rotational symmetry and is not an eigenstate of L_z
- Could use exact diagonalisation to study the ground state for increasing N_v
- Studies have investigated different filling fractions, v, from 0.5 to 9.
- Comparison between the meanfield LLL energy and exact diagonalization show that the meanfield vortex lattice is a ground state for $v \geq v_c$ ($v_c = 6$)
- Hence the meanfield LLL regime is valid for ($v_c = 1$)

$$1 - \frac{Z}{2N\beta a} \leq \tilde{\Omega} \leq 1 - \frac{8\beta a}{ZN}$$

Exact Diagonalization ($\nu < \nu_c$)

- The groundstates are rotationally symmetric incompressible vortex liquids that are eigenstates of L_z
- They have close similarities to the bosonic analogs of the Jain sequence of fractional quantum Hall states
- The simplest of these many body ground states is the bosonic Laughlin state

$$\Psi_{Laughlin}\left(\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_N}\right) \propto \prod_{n<n'}^{N} \left(z_n - z_{n'}\right)^2 e^{-\frac{1}{4}\sum_j |z_j|^2}$$

- No off-diagonal long range order and hence no BEC
- The Laughlin state vanishes whenever two particles come together, enforcing the many-body correlations
- The short range two body potential has zero expectation value in this correlated state
- Strong overlap between exact diagonalization and the Laughlin state ($\nu = 1/2$)

Physics of Transition

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- Consider N bosonic particles in a plane, with $2N$ degrees of freedom
- Vortices appear as the system rotates and the corresponding vortex coordinates provide N_v collective degrees of freedom
- For slowly rotating systems the $2N$ particle coordinates provide a convenient description
- In principle, the N_v collective vortex degrees of freedom should reduce the original total $2N$ degrees of freedom to $2N - N_v$, but this is unimportant as long as $N_v << N$
- When N_v becomes comparable with N the depletion of the particle degrees of freedom becomes crucial
- This depletion on the particle degrees of freedom drives the phase transition to a wholly new ground state
- Hence when $\nu = N/N_v$ is small a transition is expected