FMFT: Fully Massive Four-loop Tadpoles

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Abstract

We present FMFT — a package written in FORM that evaluates four-loop fully massive tadpole Feynman diagrams. It is a successor of the MATAD package that has been successfully used to calculate many renormalization group functions at three-loop order in a wide range of quantum field theories especially in the Standard Model. We describe an internal structure of the package and provide some examples of its usage.

Keywords: IBP reduction; loop integrals; four-loop; massive tadpoles;

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PROGRAM SUMMARY

Program Title: FMFT

Licensing provisions: GPLv3

Programming language: FORM

Nature of problem:

Reduction of all kinds of fully massive four-loop vacuum integrals to the small set of master integrals.

Solution method:

Reduction by means of explicit solution for IBP relations for topologies **X**,**H** and **BMW**. Representation of integrals from topology **FG** as convolution of one-loop and two-loop massive propagator-type integrals, each reduced separately using generalized dimensional recurrence-relations.

Additional comments:

Available for download from URL: https://github.com/apik/fmft/

1. Introduction

In the minimal subtraction schemes (MS) it is usually possible to reduce calculation of a divergent part of a given L-loop diagram to calculation of a massless L-loop propagator-type diagram. This is possible because renormalization constants are independent of masses and external momenta of a particular diagram [1]. In practice, an infrared rearrangement (IRR) technique [2] is used to set all but one external momenta and masses to zero, provided that no infrared(IR) divergences appear. If there is no way to route external momentum without introducing an IR divergences, another technique to calculate divergent parts of L-loop integrals is usually used. It is based on insertion of an equal auxiliary mass in all propagators of the diagram and setting all external momenta to zero, hence reducing the problem to calculation of fully massive tadpole integrals [3, 4].

This technique was used in three-loop calculations of Higgs self-coupling beta-function in the Standard Model [5, 6, 7], in which one needs to evaluate the divergent part of four-point Green function with four external scalar legs. At four loops, the method was used to find anomalous dimensions and beta-function in QCD [8, 9, 10], its generalization for the case of extended fermion sector [11, 12] and also in the course of calculations of higher moments of anomalous dimensions of operators of twist-2 in QCD [13] and

$N = 4$ supersymmetric Yang-Mills theory [14]. Finally, fully massive tadpole diagrams had found its application in five-loop calculation of vacuum energy beta-function in scalar theory [15] and recent results for five-loop QCD renormalization constants [16].

All these results require calculation of thousands of fully massive integrals. For systematic solution of such a problem the integration by parts(IBP) [17] relations are usually applied. IBP relations allow one to reduce large number of initial integrals to a small number of master integrals. It is possible to carry out the procedure of initial integrals reduction to set of master integrals in an automatic way and in general case the problem can be solved by the Laporta algorithm [18]. In some cases one can resolve recurrence relations originating from IBP identities explicitly and create a special purpose package for reduction of integrals of special type. Famous examples of such type of solutions are the MINCER [19] package for reduction of three-loop massless propagators and the FORCER [20] package, which extends MINCER to the four-loop level. The problem of reduction of three-loop vacuum-type integrals, not necessary fully massive, can be solved by the package MATAD [21]. Laporta algorithm was successfully used for reduction of four-loop tadpoles in [22, 23]. In this article we present the FMFT package for reduction of fully massive four-loop tadpoles which can find application in calculations of four-loop renormalization group functions in Standard Model and moments of splitting functions in QCD.

Integrals reducible by means of FMFT can be attributed to the single auxiliary topology (1) with ten propagators, i.e.:

$$I_{n_1...n_{10}} = \int \frac{d[k_1]d[k_2]d[k_3]d[k_4]}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_{1;4}^{n_5} D_{2;4}^{n_6} D_{3;4}^{n_7} D_{1;2}^{n_8} D_{1;3}^{n_9} D_{1;23}^{n_{10}}}, \qquad (1)$$

where denominators are defined as

$$D_a = k_a^2 - m^2, \; D_{a;b} = (k_a - k_b)^2 - m^2, \; D_{a;bc} = (k_a - k_b - k_c)^2 - m^2. \quad (2)$$

In (2) all masses m^2 are set to one during integral evaluation. For single scale integrals the mass dependence can be easily reconstructed from dimensional considerations. The integration measure is defined as $d[k] = e^{\varepsilon \gamma_E} \frac{d^d k}{i\pi^{d/2}}$.

2. Internal structure and usage details

Since FMFT is written in FORM [24], its installation reduces to extraction of distribution archive to an appropriate place. For proper operation of the FMFT
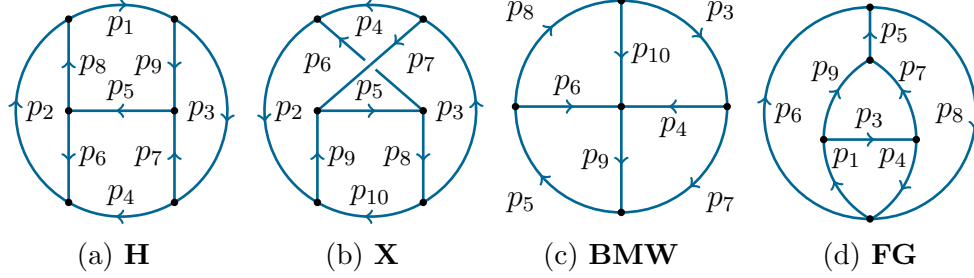
Figure 1: Base topologies with 9 lines (a),(b), and 8 lines (c),(d)

package at least `FORM` version 4 is required. The latter supports `PolyRatFun`, which is used for polynomial division and factorization.

The main steps of operation of the `FMFT` package are explained in the dia. 1. The detailed description of each step can be found in the following sections.

2.1. Reduction of the top-level topologies

We can associate any fully massive four-loop tadpole integral with one of two top-level nine propagator topologies: planar **H** (see fig. 1(a)) and nonplanar **X** (see fig. 1(b)) or their subtopologies. If we shrink one of the lines in a diagram corresponding to topologies **H** or **X**, we get an integral corresponding to either **BMW** (see fig. 1(c)) or **FG** (see fig. 1(d)). All other integrals can be associated with topology **FG** and its subtopologies.

As a rule, in the course of integral reduction the most time-consuming part is not the reduction of integrals of the top-level topologies, e.g., **X**,**H** and **BMW**, but the reduction of integrals with smaller number of lines, which still have large powers of propagator denominators and numerator. All such integrals in the case of four-loop tadpoles can be mapped onto the topology **FG**. Due to this, for topologies **X**, **H** and **BMW** we use recurrence relations obtained from the rules generated by the the `LiteRed` package [25]. Our main goal, however, is to optimize the reduction of the topology **FG**.

2.2. Reduction of topology **FG**

The main idea of the reduction strategy for topology **FG** implemented in `FMFT` is based on the observation that the corresponding integrals can be represented as a convolution of two propagator-type integrals:

4

$$J_{\text{FG}} = \int d[p] \left(\right).\tag{3}$$

One of these integrals is two-loop ($\mathbf{F}$), while another one is one-loop ($\mathbf{G}$). The main difference between the standard IBP reduction and the proposed approach is the application of reduction to each part separately. Both parts are propagator-type diagrams with all propagators having equal masses and arbitrary external momentum. For the reduction of one- and two-loop propagator-type integrals with arbitrary masses and external momenta there exists a closed-form solution as a set of generalized recurrence relations [26]. The latter are also implemented in the form of `Mathematica` package `TARCER` [27].

Possible presence of numerators involving scalar products like $k_1 \cdot k_4$ or $k_2 \cdot k_4$, which connect both integrals and do not let to apply reduction rules immediately. To disentangle integrals we need to apply tensor reduction to one of the integrals first. The easiest way is to express one-loop integral $\mathbf{G}$ in the form:

$$G_{\mu_1\ldots\mu_r}(n; n_1, n_2) = \int d^n k \frac{k_{\mu_1}\ldots k_{\mu_r}}{d_1^{n_1} d_2^{n_2}},\tag{4}$$

$$d_1 = k^2 - m^2 \quad,\quad d_2 = (k - p)^2 - m^2.$$

Then applying the general formula for one-loop tensor integral reduction [28], we can express it as a sum of scalar integrals with shifted space-time dimension. For the one-loop propagator case the general expression is reduced to

$$G_{\mu_1\ldots\mu_r}(d; n_1, n_2) = (-1)^r \sum_{j=0}^{[r/2]} \left(-\frac{1}{2}\right)^j \left\{[g]^j [p]^{r-2j}\right\}_{\mu_1\ldots\mu_r}\tag{5}$$

$$\times \frac{\Gamma(n_1 + r - 2j)}{\Gamma(n_1)} G(d + 2(r - j); n_1 + r - 2j, n_2),$$

where the structure

$$\left\{[g]^a [p]^b\right\}_{\mu_1\ldots\mu_r}$$

5

Diagram 1 Main steps of four-loop tadpoles reduction with `FMFT`

1. Apply the reduction rules for topologies **X**, **H**, **BMW**
2. Map the integrals onto topology **FG** expressible as a convolution of two-loop (**F**) and one-loop (**G**) integrals
3. Reduce tensor one-loop integral corresponding to the **G** part of the integral to a scalar one with shifted space-time dimension
4. Reduce the **F** part of the integral with a numerator:

   (a) cancel scalar products, if possible
   (b) substitute integrals with irreducible scalar products by scalar integrals in shifted space-time dimension (by means of tables)
   (c) apply dimension recurrence relations to convert integrals with shifted space-time dimension to the initial dimension
   (d) reduce the obtained scalar integrals (in original space-time dimension) to a set of master integrals

5. Do partial fractioning in p^2 (the momentum, external to **F** and **G**)
6. Rewrite $(1 - \mathrm{loop}) \otimes (2 - \mathrm{loop})$ as an integral **FG** with a different mass on line with p_5 and arbitrary power n_5
7. Apply recurrence relations to reduce the power n_5 of the topology **FG** to zero or one

is symmetric with respect to $\mu_1 \dots \mu_r$ Lorentz indices and is constructed from a metric tensors $g_{\mu\nu}$ and b momenta p.

The resulting scalar integrals with shifted space-time dimension can be reduced to master integrals in the initial space-time dimension by means of dimension recurrence relations(DRR) from [26]. The latter are also used in the course of two-loop integral reduction (see below).

2.3. Generalized recurrence relations and the reduction of two-loop massive propagator-type integrals

After splitting the topology **FG** into parts and the tensor reduction of one-loop subdiagram via (5), we end up with a convolution of the *scalar* one-loop propagator-type diagram and a two-loop diagram with a numerator.

The most general form of two-loop diagram can be represented as

$$T^{abxyz}_{n_1n_2n_3n_4n_5} = \int d[k_1]d[k_2]\frac{(k_1\cdot p)^a(k_2\cdot p)^b(k_1\cdot k_1)^x(k_2\cdot k_2)^y(k_1\cdot k_2)^z}{C_1^{n_1}C_2^{n_2}C_3^{n_3}C_4^{n_4}C_5^{n_5}}\,, \quad (6)$$

where massive denominators are introduced in accordance with (3):

$$C_1 = k_1^2 - m^2,\, C_2 = k_2^2 - m^2, C_3 = (k_1-p)^2 - m^2,\, C_4 = (k_2-p)^2 - m^2,$$
$$C_5 = (k_1-k_2)^2 - m^2\,. \quad (7)$$

The rules from [26] can be used to cancel some of scalar products in the numerator and denominator of (6), leading to integrals with $x,y,z = 0$. In particular, if we have all $n_i > 0$ we are only left with scalar integrals without numerator. If some of n_i are equal to zero then the irreducible scalar products in the numerator, $(k_1\cdot p)$ and $(k_2\cdot p)$, cannot be canceled. Due to this, the general form of two-loop subintegral we need to reduce can be cast into:

$$T^{ab}_{n_1n_2n_3n_4n_5} = \int d[k_1]d[k_2]\frac{(k_1\cdot p)^a(k_2\cdot p)^b}{C_1^{n_1}C_2^{n_2}C_3^{n_3}C_4^{n_4}C_5^{n_5}}\,, \quad (8)$$

The integrals (8) with irreducible numerator can be reduced to a combination of scalar integrals in shifted space-time dimension using rules from [26][1]. To speed up the calculation we prefer to use tables for such substitutions. The latter were pre-generated in advance, instead of generating them on the fly. The substitution rules stored in the tables distributed with the package should be sufficient for most of practical applications and allow to reduce integrals with $a+b \leq 20$. For higher powers of numerator, RHS of substitution rules stored in tables becomes too long and it is more efficient to implement reduction for integrals (8) with negative powers of denominators instead of performing dimensional shifts on integrals with irreducible numerators.

As a next step we use DRR to connect scalar integrals in shifted space-time dimension with integrals in initial space-time dimension and reduce later to a set of irreducible integrals. For such a purpose we follow along the lines of [26] and implement in the FMFT package a set of DRR connecting integrals with space-time dimension $d + 2$ and d together with recurrence relations

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[1]Alternatively reduction rules for the integrals with negative indices can be used and will be implemented in future versions of the code.

to reduce integrals with fixed space-time dimension to the following master integrals:

$$\begin{bmatrix} \texttt{F[11111], V[1111], J[211], J[111], T2[111],} \\ \texttt{G[11]G[11], G[11]T1[1], T1[1]T1[1]} \end{bmatrix} \otimes \begin{bmatrix} \texttt{G[11]} \\ \texttt{T1[1]} \end{bmatrix} . \qquad (9)$$

In (9) G and T1 are one-loop self-energy and tadpole integrals respectively, T2 is the two-loop tadpole and F,V,J are two-loop integrals with five, four and three lines, respectively, as defined in [26]. All possible one- and two-loop integrals entering convolution (9) are listed in Appendix A.

As a result of application of the above-mentioned relations, the integral in the form (3) can be represented as a sum of different convolutions of one- and two-loop massive propagator-type master integrals (9) and a p^2-dependent function. This p^2-dependent function has the form of a product of scalar propagators with momentum p^2 and different masses, not necessary equal to m^2. The masses different from m^2 arise from coefficients dependent on p^2 and m^2 in front of integrals. Such a coefficients goes to denominator when integral is substituted into other relations. Fortunately only quadratic in p^2 denominators arise during reduction of massive propagator-type integrals and number of these new masses is fixed and the section 2.4 is devoted to the problem of reduction of such tadpole integrals with different masses.

2.4. Recurrence relations for tadpoles with different masses

At the last stage of reduction by applying partial fractioning to p^2- dependent denominators each term of the integrand of (3) can be represented as a convolution of one and two-loop integrals with fixed indices from the set (9) and a single p^2-dependent propagator:

$$J_i(n, m_j) = \int d[p] \frac{F_i(p^2) G_i(p^2)}{\left(p^2 - m_j^2\right)^n} . \qquad (10)$$

Here for each of the integrals J_i the corresponding integrals $F_i(p^2)$ and $G_i(p^2)$ have fixed propagator powers given by combinations from (9) and the mass m_j^2 takes one of the possible values: $m_j^2 = \{0, m^2, 3m^2, 4m^2, 9m^2\}$. The denominator power n can be either positive or negative, whereas for subsequent evaluation we need to reduce it to zero or one.

One of the possible ways to construct recurrence relations connecting integrals in the form of (10) with different propagator powers n is to apply

original Laporta ideas [18] and derive difference equations for the integrals, in which one of the propagator powers is treated symbolically and all others are fixed numbers.

Unfortunately, the application of Laporta reduction algorithm to the integrals with symbolic power of one of the propagators is not a well-developed field and there is no publicly available software tools. Due to this, we decided to use the following trick. Instead of a system of difference equations for integrals $J_i(n, m_j)$, we construct system of differential equations

$$\frac{\partial L_i}{\partial M^2} = A_{ik} L_k \tag{11}$$

for auxiliary integrals $L_i = J_i(1, M)$ in the variable M^2, which was kept as symbol during all the steps.

For further discussion we need to separate two cases: the first one, when integral (10) has $n \geq 0$, and the second one, when p^2- dependence is in the numerator. The second case will be considered later, but now we want to focus on the first case.

We can see that the expansion of scalar M^2-dependent propagator of one of the L_i integrals in a small dimensionless variable $z = \frac{M^2 - m_j^2}{m^2}$ has the following form:

$$\frac{1}{p^2 - M^2} = \frac{1}{p^2 - m_j^2} + \frac{m^2}{\left(p^2 - m_j^2\right)^2} z + \frac{m^4}{\left(p^2 - m_j^2\right)^3} z^2 + \dots \tag{12}$$

If we set m_j^2 in (12) to be equal to one of the values of our interest, we can relate the n-th coefficients of L_i integral expansions in the variable z with the integral $J_i(n + 1, m_j)$. At the same time, we can look for a solution of the system (11) in the form of formal series (13) in small variable z:

$$L_i = \sum_{n=0}^{\infty} c_{i,n} z^n, \quad z = \frac{M^2 - m_j^2}{m^2}. \tag{13}$$

From Eqs. (12) and (13) we can construct the following relation (14):

$$J_i(n + 1, m_j) = \frac{c_{i,n}}{m^{2n}}, \tag{14}$$

connecting the integral J_i having the denominator involving m_j^2 in power n with the coefficients of expansion of the auxiliary integrals L_i in Taylor series

9

in z. For each possible m_j^2 from the set $\{0, m^2, 3m^2, 4m^2, 9m^2\}$ expansion variable z and system of equations (14) are unique.

Substituting the ansatz (13) into the system (11) and equating the coefficients of equal powers in z, we obtain the system of difference equations in variable n for the coefficients $c_{i,n}$ and, hence, for the integrals $J_i(n, m_j)$. The constructed system can be transformed to the triangle form and then used for reduction of the integrals $J_i(n, m_j)$ to a set of master integrals having the form $J_i(1, m_j)$ or $J_i(0, m_j)$. It should be noticed that one needs to construct a separate system of recurrence relations for all possible values of the mass m_j^2.

As an example, we consider recurrence relations for the integral of topology **FG** with the following set of indices: n_1, n_2, n_3, n_6, $n_{10} = 0$ and n_4, n_7, n_8, $n_9 = 1$. This integral is a product of two one-loop tadpoles `T1[1]` with propagators in unit power and a two-loop vacuum integral dependent on n. One-loop integrals do not contribute to difference equations and can be discarded, so it is sufficient to write down a recurrence relation only for the two-loop part (15):

$$
\begin{aligned}
\left(\overset{n}{\ominus}\right) &= \theta(n-3)\frac{n-d}{3(n-1)}\left(\overset{n-2}{\ominus}\right) &&+ \theta(n-2)\frac{d+1-2n}{3(n-1)}\left(\overset{n-1}{\ominus}\right)\\
&+ \theta(n-2)\frac{d-2}{3(n-1)}\left(\bigcirc\bigcirc_{n-1}\right) &&+ \delta(n-2)\frac{2-d}{3}\left(\bigcirc\bigcirc\right)\\
\left(\bigcirc_{n}\right) &= \theta(n-2)\frac{d+2-2n}{2(n-1)}\left(\bigcirc_{n-1}\right), && n > 1.
\end{aligned}
\tag{15}
$$

Here $\theta(n)$ with $n \geq 0$ and $\delta(n)$ with $n = 0$ are equal to one, while for other values of n both functions are equal to zero. We can see that such "one-dimensional" relations affect only single propagator power leading to small number of terms at each reduction step and can be effectively implemented in `FORM`.

In the second case with p^2-dependence in the numerator of the integral (10) it is sufficient to consider $m_j^2 = 0$ and $n \leq 0$. Such integrals with arbitrary power n should also be reduced to the integrals $J_i(0, 0)$.

We can use the same auxiliary integrals $L_i = J_i(1, M)$ dependent on the mass M^2 and the system of differential equations (11) as before, but now expand the scalar propagator with mass M^2 in the opposite limit, i.e., in

small $\bar{z} = \frac{m^2}{M^2}$:

$$\frac{1}{p^2 - M^2} = -\frac{1}{m^2}\bar{z} - \frac{p^2}{m^4}\bar{z}^2 - \frac{p^4}{m^6}\bar{z}^3 + \ldots \tag{16}$$

As before we can construct formal solution of the system (11) in the form of series:

$$L_i = \sum_{n=0}^{\infty} \bar{c}_{i,n}\bar{z}^n, \quad \bar{z} = \frac{m^2}{M^2}. \tag{17}$$

Then the integrals (10) with $n \leq 0$ will be related to expansion coefficients (17) in $\bar{z}$

$$J_i(-n, 0) = -\bar{c}_{i,n}m^{2(n+1)}. \tag{18}$$

As in the case of integrals with denominators we substitute the ansatz (17) into the system (11) and equate the coefficients in front of equal powers of $\bar{z}$. In such a way we obtain a system of difference equations for $\bar{c}_{i,n}$, which means that we can construct the recurrence relations for reduction of the integrals $J_i(-n, 0)$ to the integrals of the type $J_i(0, 0)$.

It is necessary to note that after the application of the recurrence relations to the integrals (10) with $m_j^2 \neq m^2$ the result can involve the integrals like $J_i(1, m_j)$ with two different masses. On the other hand, we know that if one applies traditional IBP reduction to the fully massive four-loop tadpoles that can be expressed in terms of master integrals with only one mass scale [10]. Thanks to this property all integrals $J_i(1, m_j \neq m)$ should cancel in the final answer. Such cancellation is a good check for correctness of the whole four-loop integrals reduction procedure implemented in the FMFT package.

At this step the main reduction part of FMFT is finished and the result is expressed in terms of symbolic expressions corresponding to master integrals from paper [10] and coefficients dependent on d. For the case of four space-time dimensions result can be expanded in ε near $d = 4 - 2\varepsilon$ and actual expansions for master integrals from [10] can be substituted.

3. Comparison with other codes and examples

To estimate the FMFT package performance and illustrate its applicability to reduction of complicated integrals we calculate a nonplanar integral of topology **X** (fig. 1(b))

$$F(n) = I(-n, 1, 1, 1, 1, 1, 1, 1, 1, 1), \tag{19}$$

in which the propagator powers are written in accordance with the auxiliary topology (1). The integral has a nontrivial numerator and we compare the results of calculation for different numerator powers n. The comparison was performed with the C++ version of the FIRE 5 [29] package. Code FIRE is known to be very efficient general-purpose tool for solution of the reduction problem with many successful applications and not restricted to the reduction of fully massive tadpoles. It can be used not only for IBP reduction with the help of Laporta algorithm, but also in combination with the package LiteRed [25]. When used together FIRE 5 acts as efficient tool for application of reduction rules from resolved recurrence relations obtained by means of LiteRed.

$n =$	3	4	5	6	7	8
FMFT	0:00:11	0:00:27	0:01:55	0:07:35	0:25:31	01:30:31
FIRE	0:01:58	0:09:10	0:28:17	2:16:42	9:19:57	46:42:29

Table 1: Comparison to FIRE, time format is hh:mm:ss

Timing results for reduction of a single integral with FMFT and FIRE 5 are present in table 1. For the FMFT reduction we use multithread version TFORM with eight active workers(-w8 option). Similar setup was used for FIRE 5. It was running on eight CPU cores and in memory reduction was used (options #memory and #threads 8). In addition, the reduction rules from LiteRed package were utilized. The main goal of comparison present in table 1 is to illustrate that FMFT can be used for reduction of complicated integrals as can bee seen from the time spent for integral reduction with such efficient tool as FIRE.

In the listing 1 we present simple FORM program to illustrate FMFT usage for reduction of four-loop integrals. Integral with numerator is defined via p_i for scalar products in the numerator and $d_i = p_i^2 - m^2$ for massive denominators in correspondence to one of the top-level topologies fig. 1(a) and fig. 1(b). The main entry point is the fmft routine. The result of its application is the reduction of the initial integral to the set of master integrals identified in the work [10] with coefficients exhibiting exact dependence on the space-time dimension parameter d.

```
#-
* load main library code
#include fmft.hh
```

12

```
* input with numerator
L ex = p2.p3/d1/d2^2/d4/d5/d6/d7/d8/d9;
* call reduction routines
#call fmft
* expand near d=4-2*ep up to ep^1
#call exp4d(1)

b ep;
Print+s;
.end
```

Listing 1: Example program

By means of procedure `exp4d(n)` the result of the reduction can be expanded in ε up to the order ε^n near $d = 4 - 2\varepsilon$ space-time dimensions. Output from the program calculating integral from the listing 1 is the following:

```
    ex =
        + ep^-4 * ( 3/8 )

        + ep^-3 * ( 25/8 )

        + ep^-2 * ( 137/8 + 3/4*z2 - 81/4*S2 + 3/4*z3 )

        + ep^-1 * ( 363/8 - 3/2*T1ep - 1/2*z2 - 81*S2 -
            3/8*z4 + 1/2*D6 - 6*z3)

        + 1/2*PR14ep0 + 1/2*PR15ep0;

        + ep * ( 1/2*Oep(1,PR14) + 1/2*Oep(1,PR15) )
```

Listing 2: Sample output

where `z2,z3,z4` are Riemann zeta functions, `S2,T1ep` are non zeta parts of two-loop and `D6` three-loop terms of tadpole integrals ε-expansion defined in [21]. Finite parts of four-loop integrals `PR14ep0,PR15ep0` are kept as symbols and its numerical values can be substituted from [23]. To denote truncation of ε-expansion series we use common function `Oep` with first argument corresponding to order in ε and second argument containing master integral name.

Acknowledgments

Appendix A. Two-loop massive master integrals

$$\mathtt{F[11111]} = T_{11111} = \qquad \qquad \text{(A.1)}$$

$$\mathtt{V[1111]} = T_{01111} = \qquad \qquad \text{(A.2)}$$

$$\mathtt{J[211]} = T_{20011} = \qquad \qquad \text{(A.3)}$$

$$\mathtt{J[111]} = T_{10011} = \qquad \qquad \text{(A.4)}$$

$$\mathtt{T2[111]} = T_{11001} = \qquad \qquad \text{(A.5)}$$

$$\mathtt{G[11]G[11]} = T_{11110} = \qquad \qquad \text{(A.6)}$$

$$\mathtt{G[11]T1[1]} = T_{11100} = \qquad \qquad \text{(A.7)}$$

$$\mathtt{T1[1]T1[1]} = T_{11000} = \qquad \qquad \text{(A.8)}$$

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