信号与系统课程笔记：Lecture 27

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1 Z 变换的性质

(1) ★ 时移：$f(t \pm t_0) \overset{\mathscr{F}}{\leftrightharpoons} F(\omega)e^{\pm j\omega t_0}$

$f(t-t_0)U(t-t_0) \overset{\mathscr{L}}{\leftrightharpoons} F(s)e^{-st_0}$（只考虑因果信号）

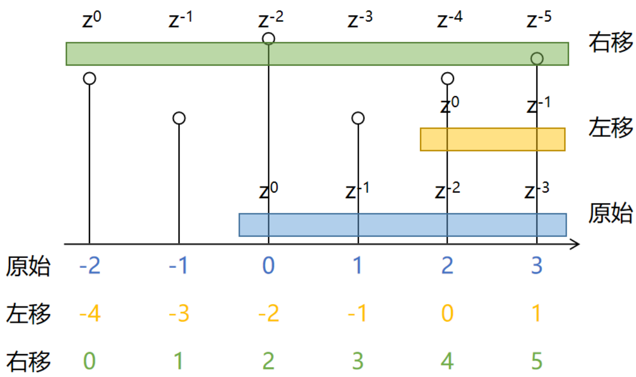
① 因果信号

$f[k-m] \cdot U[k-m] \overset{\mathscr{Z}}{\leftrightharpoons} z^{-m}F(z) \quad k, m \in \mathbb{Z}^+$

② 单边 Z 变换

左移：$f[k-m]U[k] \overset{\mathscr{Z}}{\leftrightharpoons} z^{-m}(F(z) - \sum_{k=0}^{m-1} f[k]z^{-k})$

右移：$f[k-m]U[k] \overset{\mathscr{Z}}{\leftrightharpoons} z^{-m}(F(z) + \sum_{k=-m}^{-1} f[k]z^{-k})$



(2) 部分和：if $f[k] \leftrightharpoons F(z)$

than $\sum_{i=-\infty}^{k} f[i] \rightleftharpoons \frac{z}{z-1}F(z)$

证明：$\sum_{i=-\infty}^{k} f[i] = f[k] * U[k] \rightleftharpoons \frac{z}{z-1}F(z)$

(3) 初值/中值定理：$f[0] = \lim_{z\to+\infty} F(z)$

$f[\infty] = \lim_{z\to 1} \frac{z-1}{z}F(z)$

因果信号：$f[m] = \lim_{z\to+\infty} z^m F(z)$

2 例题

(1) $\sum_{m=0}^{+\infty}\delta[k-mN] \rightleftharpoons \sum_{m=0}^{+\infty} z^{-mN} = \frac{1}{1-z^{-N}} = \frac{z^N}{z^N-1} \quad |z|>1$

(2) $a^kU[k] \rightleftharpoons F(\frac{z}{a}) = \frac{z/a}{z/a-1} = \frac{z}{z-a}$

(3) $k\cdot U[k]$

解：① 微分：$kU[k] \rightleftharpoons (-z)\frac{\mathrm{d}}{\mathrm{d}z}(\frac{z}{z-1}) = (-z)\frac{(z-1)-z}{(z-1)^2} = \frac{z}{(z-1)^2}$

② 卷积：$kU[k] = U[k]*U[k-1] \rightleftharpoons \frac{z}{z-1}\cdot\frac{z}{z-1}\cdot z^{-1} = \frac{z}{(z-1)^2}$

③ 时移：令 $f[k] = k\cdot U[k]$

则 $f[k+1] = (k+1)U[k+1] = kU[k+1] + U[k+1] = f[k] + U[k]$

等式两边进行 Z 变换：$zF(z) - zf[0] = F(z) + \frac{z}{z-1}$

则 $F(z) = \frac{z}{(z-1)^2}$

(4) $\sum_{i=0}^{k} a^i = \sum_{i=-\infty}^{k} a^iU[k] \rightleftharpoons \frac{z}{z-1}\cdot\frac{z}{z-a}$

3 逆 Z 变换

3.1 方法一

$\sum_{i=1}^{M} a_iy[k=-m_i] = \sum_{i=1}^{N} b_if[k=-n_i]$
$F(z) = \frac{b_mz^m+b_{m-1}z^{m-1}+\cdots+b_1z+b_0}{a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0}$

(1) $m>n$，化简成 $F(z) = \beta_1z^{m-n} + \beta_2z^{m-n-1} + \cdots + \beta_{i-1}z + \beta_i + F'(z)$ 的形式。

(2) $m\leqslant n$，$\frac{F(z)}{z} = \sum_{i=1}^{n}\frac{p_i}{z-z_i}$

例 1：$F(z) = \frac{z+2}{2z^2-7z+3}$（单极点）

解：$\frac{F(z)}{z} = \frac{z+2}{2z(z-0.5)(z-3)} = \frac{p_1}{z} + \frac{p_2}{z-0.5} + \frac{p_3}{z-3}$

$p_1 = \frac{F(z)}{z}z\Big|_{z=0} = \frac{2}{3}$

$p_2 = \frac{F(z)}{z}(z-0.5)\Big|_{z=0.5} = \frac{z+2}{2z(z-3)}\Big|_{z=0.5} = -1$

$p_3 = \frac{F(z)}{z}(z-3)\Big|_{z=3} = \frac{z+2}{2z(z-0.5)}\Big|_{z=3} = \frac{1}{3}$

则 $\frac{F(z)}{z} = \frac{2}{3z} + (-1)\frac{1}{z-0.5} + \frac{1}{3}\frac{1}{z-3}$

则 $F(z) = \frac{2}{3} + (-1)\frac{z}{z-0.5} + \frac{1}{3}\frac{z}{z-3}$

两边 Z 变换，得 $f[k] = \frac{2}{3}\delta[k] + (-1)0.5^kU[k] + \frac{1}{3}3^kU[k]$

例 2：$F(z) = \frac{z}{z^2+4}$（共轭双极点）

解：$\frac{F(z)}{z} = \frac{1}{(z+2j)(z-2j)} = \frac{k_1}{z+2j} + \frac{k_2}{z-2j}$

$k_1 = \frac{F(z)}{z}(z+2j)\Big|_{z=-2j} = \frac{1}{4j} = \frac{1}{4}e^{-j\frac{\pi}{2}}$

$k_2 = \frac{1}{4}e^{j\frac{\pi}{2}}$

代入后两边 Z 变换，得 $f[k] = \frac{1}{4}e^{-j\frac{\pi}{2}}(2j)^kU[k] + \frac{1}{4}e^{j\frac{\pi}{2}}(-2j)^kU[k]$

$= \frac{1}{2}2^k\cos[\frac{\pi}{2}k - \frac{\pi}{2}]U[k]$

例 3：$F(z) = \frac{z^3+z^2}{(z-1)^3}$（多重极点）

解：$\frac{F(z)}{z} = \frac{z^2+z}{(z-1)^3} = \frac{k_{11}}{(z-1)^3} + \frac{k_{12}}{(z-1)^2} + \frac{k_{13}}{z-1}$

$k_{11} = \frac{1}{0!}\frac{F(z)}{z}(z-1)^3\big|_{z=1} = z^2+z\big|_{z=1} = 2$

$k_{12} = \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}z}(\frac{F(z)}{z}(z-1)^3)\big|_{z=1} = 2z+1\big|_{z=1} = 3$

$k_{13} = \frac{1}{2!}\frac{\mathrm{d}}{\mathrm{d}z^2}(\frac{F(z)}{z}(z-1)^3)\big|_{z=1} = 1$

则 $F(z) = \frac{2z}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{z}{z-1}$

两边 Z 变换，得 $f[k] = 2k(k-1)U[k] + 3kU[k] + U[k]$

3.2 方法二

$F(z) = f[0]z^0 + f[1]z^{-1} + f[2]z^{-2} + \cdots$

例：$F(z) = \frac{z}{z^2-3z+2} = z^{-1} + 3z^{-2} + 7z^{-3} + \cdots$

$f[k] = \{0, 1, 3, 7, \cdots\}$

$z^{-1}+3z^{-2}+7z^{-3}$

z^2-3z+2) $z+0z^0+0z^{-1}+0z^{-2}$ ......

z-3+$2z^{-1}$

3-$2z^{-1}$

3-$9z^{-1}$+$6z^{-2}$

$7z^{-1}$-$6z^{-2}$

7-21+14

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