18.03 Completeness of Fourier Expansion

Jeremy Orloff

Theorem (Completeness theorem)

A continuous periodic function f equals its Fourier series.

For concreteness we will assume f has period 2π.

Notes:

1. This is called completeness because it says the set of functions $\cos(nt)$ and $\sin(nt)$ form a *complete* set of basis functions. That is, you don't need any more functions to express every period 2π function as a linear combination.

2. We will gloss over some analytic issues like convergence, they are not too hard in the current context.

3. This theorem can be proved as a simple consequence of the Stone-Weierstass theorem. We will not resort to that.

Before we start the proof we need some preliminary notions.

1. *Convolution*: Assume f and g have period 2π. We define

$$f * g(t) = \int_{-\pi}^{\pi} f(u)g(t-u)\,du.$$

As with Laplace this is commutitive, i.e. $f * g = g * f$.

2. *Periodic delta function*

$$\tilde{\delta} = \ldots \delta(t+4\pi) + \delta(t+2\pi) + \delta(t) + \delta(t-2\pi) + \ldots$$

Claim: If $f(t)$ is periodic (period 2π) then $\tilde{\delta} * f(t) = f(t)$.

Proof: In the interval $[-\pi, \pi]$ the only non-zero term in the sum defining $\tilde{\delta}$ is $\delta(t)$. $\Rightarrow \tilde{\delta} * f(t) = \int_{-\pi}^{\pi} \tilde{\delta}(u)f(t-u)\,du = \int_{-\pi}^{\pi} \delta(u)f(t-u)\,du = f(t)$.

3. Consider the function $h(t) = \left(\frac{1+\cos(t)}{2}\right)$ on the interval $[-\pi, \pi]$. We have

(i) $h(0) = 1$

(ii) As t goes from 0 to π (or $-\pi$) $h(t)$ decreases to 0.

Consequently for large k the graph of $h(t)^k$ is nearly a spike of unit height at the origin. With this in mind we define

$$h_k(t) = c_k \left(\frac{1+\cos(t)}{2}\right)^k$$

where c_k is chosen so that $\int_{-\pi}^{\pi} h_k(t)\,dt = 1$.

As k grows the graph gets thinner and spikier (in order for this to have area 1 we must have c_k growing larger). Since the area is always 1 this shows

$$\lim_{k\to 0} h_k(t) = \tilde{\delta}(t).$$

Notice that h_k is a linear combination of powers of $\cos(t)$. Also recall powers of $\cos(t)$ can all be written as linear combinations of terms of the form $\sin(nt)$ and $\cos(nt)$ (you can easily use Euler's formula to see this). Combining these statements we have h_k is a linear combination terms of the form $\sin(nt)$ and $\cos(nt)$.

Proof of the completeness theorem:

Denote the Fourier series of f by $f_1(t)$. We know

$$f_1(t) = \frac{a_0}{2} + \sum a_n \cos(nt) + \sum b_n \sin(nt)$$

where

$$a_n = \frac{2}{\pi} \int_{-\pi}^{\pi} f(t)\cos(nt)\,dt, \text{ and } b_n = \frac{2}{\pi} \int_{-\pi}^{\pi} f(t)\sin(nt)\,dt.$$

The orthogonality relations guarantee that f_1 gives the same coefficients. That is

$$a_n = \frac{2}{\pi} \int_{-\pi}^{\pi} f_1(t)\cos(nt)\,dt, \text{ and } b_n = \frac{2}{\pi} \int_{-\pi}^{\pi} f_1(t)\sin(nt)\,dt.$$

Our goal is to show that $f = f_1$, or equivalently $g = f - f_1 = 0$. To do this, first note that

$$\int_{-\pi}^{\pi} g(t)\sin(nt)\,dt = \int_{-\pi}^{\pi} (f(t) - f_1(t))\sin(nt)\,dt = b_n - b_n = 0.$$

Likewise

$$\int_{-\pi}^{\pi} g(t)\cos(nt)\,dt = \int_{-\pi}^{\pi} (f(t) - f_1(t))\cos(nt)\,dt = a_n - a_n = 0.$$

Since $h_k(t)$ is just a sum of sines and cosines this shows

$$h_k * g(t) = \int_{-\pi}^{\pi} h_k(t-u)g(u)\,du = 0.$$

So

$$\lim_{k\to\infty} h_k * g(t) = 0.$$

But, this limit is also $\delta * g(t) = g(t)$. That is, we have shown $g(t) = 0$. QED