Quantum Computing @ MEF
Background

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1 Quantum Measurement

In order to render notation more convenient, we will often omit the parentheses in function application and start to denote linear maps by capital letters. Also, we will now use $|0\rangle$ and $|1\rangle$ to denote the elements $(1,0)$ and $(0,1)$ in $\mathbb{C}^2$, respectively. We extend this notation to any space $\mathbb{C}^{2^n}$ by observing that,

$$\mathbb{C}^{2^n} \simeq \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}}$$

and representing $|b_1\rangle \otimes \cdots \otimes |b_n\rangle \in \mathbb{C}^{2^n}$ simply as $|b_1, \ldots, b_n\rangle$. Thus, a vector $v \in \mathbb{C}^2$ is a linear combination $\alpha\,|0\rangle + \beta\,|1\rangle$ and $\|v\| = 1$ entails that the equation $|\alpha|^2 + |\beta|^2 = 1$ holds. Later on we will see that $|\alpha|^2$ is the probability of observing $|0\rangle$ when measuring a qubit in state v and analogously for $|\beta|^2$. Similarly, a vector $v \in \mathbb{C}^4$ is a linear combination $\alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle$ and $\|v\| = 1$ entails that the equation $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$ holds. The component $|\alpha|^2$ is the probability of observing $|00\rangle$ when measuring two qubits at state v, and analogously for the three other components.

In this course, we will heavily use two maps M_0 and M_1 of type $\mathbb{C}^2 \to \mathbb{C}^2$ for measuring qubits. The map M_0 is defined by the equations,

$$M_0\,|0\rangle = |0\rangle \qquad\qquad M_0\,|1\rangle = 0$$

and represents the outcome of the qubit measured being at state $|0\rangle$; the map M_1 arises from an analogous reasoning. For the space $\mathbb{C}^{2^n}$ we represent the outcome of the i-th qubit being at state $|k\rangle$ by the map,

$$\underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{i-1 \text{ times}} \otimes\, M_k \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n-i \text{ times}} : \mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$$

We call ‘measurement maps’ those maps that are built in this way and that arise by composing measurement maps with one another.

Postulate 1 (Quantum measurement)**.** Let $v \in \mathbb{C}^{2^n}$ be a quantum state of n qubits and let us consider a measurement map $M : \mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$. Then the probability of the outcome represented by M is $\langle M\,v, M\,v\rangle$ and the quantum state of the n qubits after the observed outcome is defined by,

$$\frac{M\,v}{\|M\,v\|}$$

(note that we perform a normalisation, which is necessary because measurement maps are not unitary).

Exercise 1. Let $H : \mathbb{C}^2 \to \mathbb{C}^2$ be the unitary map defined by the matrix,

$$\frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

What is the probability of the outcome $|0\rangle$ when measuring $H|0\rangle$?

Exercise 2. Consider the quantum state,

$$\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

What is the probability of the outcome $|0\rangle$ when measuring the leftmost qubit? Let us assume that we indeed observed that the leftmost qubit is at state $|0\rangle$. What is the probability of the outcome $|1\rangle$ when measuring the rightmost qubit?

Exercise 3. Consider the quantum state,

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

What is the probability of the outcome $|0\rangle$ when measuring the leftmost qubit? What is the probability of the outcome $|1\rangle$ when measuring the rightmost qubit? Assume that we indeed observed that the leftmost qubit is at state $|0\rangle$. Then what is the probability of the outcome $|1\rangle$ when measuring the rightmost qubit? [1]

2 Entanglement

Consider two vector spaces V and W. We say that a vector $u \in V \otimes W$ is entangled if it cannot be written as $v \otimes w$ for some $v \in V$ and $w \in W$. In words, the state u (of a composite system) is entangled if it cannot be seen as a mere aggregation v, w of states (of the constituent systems). If the state u is not entangled then we say that is separable.

Exercise 4. Show that the quantum state,

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

is entangled.

The quantum state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ (mentioned in the previous exercise) can be obtained from the unitary map $CX \cdot (H \otimes \mathrm{id})$ and the initial state $|0\rangle \otimes |0\rangle$, where $CX : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ reads as "controlled not" and is defined as,

$$CX|00\rangle = |00\rangle, \qquad CX|01\rangle = |01\rangle, \qquad CX|10\rangle = |11\rangle, \qquad CX|11\rangle = |10\rangle.$$

[1] The quantum state briefly studied in this exercise is one of those that gave rise to the famous phrase 'spooky action at a distance' by A. Einstein.

In a nutshell CX flips the state of the second qubit depending on the state of the first qubit being $|0\rangle$ or $|1\rangle$ – such a behaviour extends to all elements of $\mathbb{C}^2 \otimes \mathbb{C}^2$ by linearity. Actually, any initial state $|i\rangle \otimes |j\rangle$ in the usual basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ and the operator $CX \cdot (H \otimes \mathrm{id})$ yield an entangled quantum state. The four states obtained in this way are usually called Bell states, and are defined as follows:

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \qquad \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle \qquad \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \qquad \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$