# Entanglement and Teleportation

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# Entanglement Enters the Stage

## The Problem

Two secure labs and in one of these a qubit

Terrain between the two labs full of entities that wish to access the qubit's state

How to transfer this quantum state from one lab to the other?

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Quantumly, we can do better thanks to entanglement

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# Mathematical Notion of Entanglement

**Definition**

A vector $u \in V \otimes W$ is entangled if it cannot be written as a tensor $v \otimes w$ such that $v \in V$ and $w \in W$

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**Example**

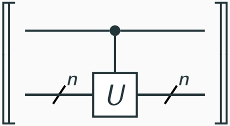
All four states below are entangled

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

They form a basis of $\mathbb{C}^4$, which is often called the Bell basis

# An Important Ingredient for Building Bell States and Beyond

Every quantum operation  gives rise to a 'controlled' quantum operation




$$= \begin{cases} cU\,|0\rangle\,|b\rangle = |0\rangle\,|b\rangle \\ cU\,|1\rangle\,|b\rangle = |1\rangle\,U\,|b\rangle \end{cases}$$

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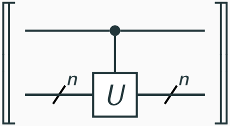
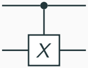
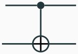
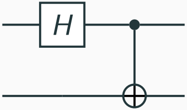



$$= \begin{cases} cU\,|0\rangle\,|b\rangle = |0\rangle\,|b\rangle \\ cU\,|1\rangle\,|b\rangle = |1\rangle\,U\,|b\rangle \end{cases}$$

**N.B.** The circuit  is often denoted as 

# Building Bell States




Every vector in the computational basis of $\mathbb{C}^4$ when fed to the circuit above yields a Bell state

# Postulates of Measurement

Maps $M_0$ and $M_1$ of type $\mathbb{C}^2 \to \mathbb{C}^2$ for measuring a qubit

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

A map $M_k$, $k \in \{0, 1\}$ possibly tensored with identities $\mathrm{id} : \mathbb{C}^2 \to \mathbb{C}^2$ called a measurement

**Postulates**

For a state $v \in \mathbb{C}^{2^n}$ and measurement $M : \mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$

- probability of outcome represented by $M$ is $\langle Mv, Mv \rangle$
- state after the observed outcome is $\frac{1}{\|Mv\|} Mv$

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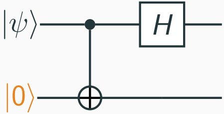
# Quantum Teleportation Intra-Gate pt. I

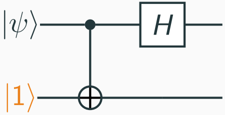
We transfer the top wire qubit's state to the bottom wire




$$
\begin{aligned}
&(H \otimes I)cX(\alpha |0\rangle + \beta |1\rangle) |0\rangle \\
&= (H \otimes I)cX(\alpha |00\rangle + \beta |10\rangle) \\
&= (H \otimes I)(\alpha |00\rangle + \beta |11\rangle) \\
&= |+\rangle \alpha |0\rangle + |-\rangle \beta |1\rangle \\
&= \tfrac{1}{\sqrt{2}}(|0\rangle \alpha |0\rangle + |1\rangle \alpha |0\rangle + |0\rangle \beta |1\rangle - |1\rangle \beta |1\rangle) \\
&= \tfrac{1}{\sqrt{2}}\Big( |0\rangle (\alpha |0\rangle + \beta |1\rangle) + |1\rangle (\alpha |0\rangle - \beta |1\rangle)\Big)
\end{aligned}
$$

# Quantum Teleportation Intra-Gate pt. II

We transfer the top wire qubit's state to the bottom wire




$$(H \otimes I)cX(\alpha |0\rangle + \beta |1\rangle) |1\rangle$$

$$= \ldots$$

$$= \frac{1}{\sqrt{2}}\Big( |0\rangle (\alpha |1\rangle + \beta |0\rangle) + |1\rangle (\alpha |1\rangle - \beta |0\rangle)\Big)$$

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We can transfer a qubit's state, but both qubits need to be connected by a gate

Fortunately we can do better. We use entanglement to establish a secure 'communication channel' and proceed in the following manner . . .

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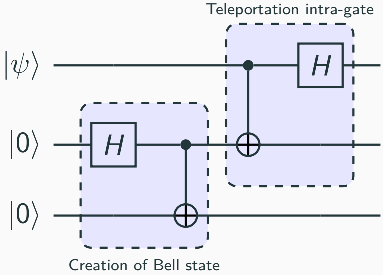
# Quantum Teleportation pt. I




Bottom qubits become entangled and thus connected, even if they are far away from each other later on

# Quantum Teleportation pt. II

$$((H \otimes I) \otimes I)(cX \otimes I)\Big((\alpha |0\rangle + \beta |1\rangle) \otimes \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\Big)$$

$$= \tfrac{1}{\sqrt{2}}((H \otimes I) \otimes I)(cX \otimes I)\Big(\alpha |000\rangle + \alpha |011\rangle + \beta |100\rangle + \beta |111\rangle\Big)$$

$$= \tfrac{1}{\sqrt{2}}((H \otimes I) \otimes I)\Big(\alpha |000\rangle + \alpha |011\rangle + \beta |110\rangle + \beta |101\rangle\Big)$$

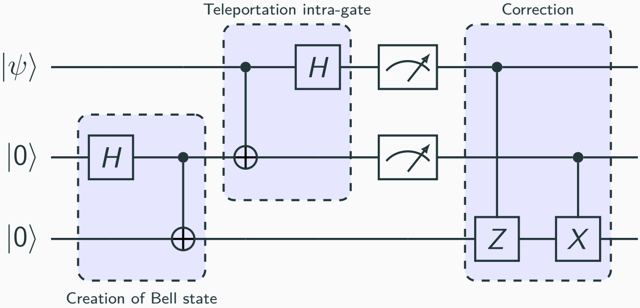
$$= \tfrac{1}{\sqrt{2}}((H \otimes I) \otimes I)\Big(|0\rangle (\alpha |00\rangle + \alpha |11\rangle) + |1\rangle (\beta |10\rangle + \beta |01\rangle)\Big)$$

$$= \tfrac{1}{2}\Big((|0\rangle + |1\rangle)(\alpha |00\rangle + \alpha |11\rangle) + (|0\rangle - |1\rangle)(\beta |10\rangle + \beta |01\rangle)\Big)$$

$$= \Big(|00\rangle (\alpha |0\rangle + \beta |1\rangle) + |01\rangle (\alpha |1\rangle + \beta |0\rangle) + |10\rangle (\alpha |0\rangle - \beta |1\rangle) \ \ldots$$

$$\cdots + |11\rangle (\alpha |1\rangle - \beta |0\rangle)\Big)$$

# Quantum Teleportation pt. III



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**No-cloning**

Did not end up with two copies of $|\psi\rangle$, because the state of the top qubit was destroyed by the measurement

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**FTL communication**

Did not communicate faster than light, because teleportation required us to send two classical bits

## What's Next?

First glimpse of applications of quantum phenomena to algorithmics and communication. Namely

- superposition & interference
- entanglement

Next we will overview more sophisticated applications of these phenomena