Quantum Phase Estimation

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The Problem

Unitary operator on n qubits

Eigenvector with eigenvalue $\lambda = e^{i2\pi\phi}$ $(0 \leq \phi < 1)$

Find out ϕ

Eigenvalues of unitaries are always of form above

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This problem occurs in diverse tasks

- Shor's algorithm
- Determining $n^{\underline{o}}$ of solutions in unstructured search

A Certain Subroutine

Previous problem uses an important subroutine called

Quantum Fourier Transform (QFT)

Essentially a change-of-basis operation which encodes information of computational basis states in local phases

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QFT: 1 Qubit

$$QFT_1 \, |0\rangle = \frac{1}{\sqrt{2}}\big(\, |0\rangle + 1 \, |1\rangle \, \big) \qquad QFT_1 \, |1\rangle = \frac{1}{\sqrt{2}}\big(\, |0\rangle + (-1) \, |1\rangle \, \big)$$

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$$QFT_1 |0\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + 1 |1\rangle \big) \qquad QFT_1 |1\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + (-1) |1\rangle \big)$$

Hence $QFT_1 = H$. Operation $\underbrace{H^{-1}}_{= H}$ allows to extract information encoded in local phases

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Hence $QFT_1 = H$. Operation $\underbrace{H^{-1}}_{= H}$ allows to extract information encoded in local phases

Exercise

Let $\omega_1 = \underbrace{e^{i2\pi\frac{1}{2}}}_{\text{angle of } \pi \text{ radians}}$. Show that $QFT_1 |x\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + \omega_1^{1\cdot x}\, |1\rangle \Big)$

QFT: 2 Qubits

Let $\omega_2 = e^{i2\pi\frac{1}{4}}$

$$QFT_2 \left|00\right\rangle = \frac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2\cdot 0} \left|1\right\rangle \big) \otimes \frac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{1\cdot 0} \left|1\right\rangle \big)$$

$$QFT_2 \left|01\right\rangle = \frac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2\cdot 1} \left|1\right\rangle \big) \otimes \frac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{1\cdot 1} \left|1\right\rangle \big)$$

$$QFT_2 \left|10\right\rangle = \frac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2\cdot 2} \left|1\right\rangle \big) \otimes \frac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{1\cdot 2} \left|1\right\rangle \big)$$

$$QFT_2 \left|11\right\rangle = \frac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2\cdot 3} \left|1\right\rangle \big) \otimes \frac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{1\cdot 3} \left|1\right\rangle \big)$$

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Exercise

Use Bloch sphere to study $QFT_2\,|x\rangle$. Specifically note that

- previously, info. of $|x\rangle$ encoded by vectors pointing to the poles; now is encoded by vectors in the *xz*-plane
- for every ω_2-rotation on the second qubit there are *two* such rotations on the first qubit

QFT: 2 Qubits

In order to derive a circuit for QFT_2, we calculate

$$
\begin{aligned}
QFT_2 \left| x \right\rangle &= \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2\cdot x} \left|1\right\rangle \big) \otimes \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{1\cdot x} \left|1\right\rangle \big) \\
&= \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2(2x_1+x_2)} \left|1\right\rangle \big) \otimes \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2x_1+x_2} \left|1\right\rangle \big) \\
&= \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{4x_1+2x_2} \left|1\right\rangle \big) \otimes \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2x_1+x_2} \left|1\right\rangle \big) \\
&= \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{4x_1}\omega_2^{2x_2} \left|1\right\rangle \big) \otimes \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2x_1}\omega_2^{x_2} \left|1\right\rangle \big) \\
&= \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2x_2} \left|1\right\rangle \big) \otimes \tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + \omega_2^{2x_1}\omega_2^{x_2} \left|1\right\rangle \big) \\
&= \underbrace{\tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + (-1)^{x_2} \left|1\right\rangle \big)}_{H\left|x2\right\rangle} \otimes \underbrace{\tfrac{1}{\sqrt{2}}\big(\left|0\right\rangle + (-1)^{x_1}\omega_2^{x_2} \left|1\right\rangle \big)}_{\text{some controlled rot. on } H\left|x1\right\rangle}
\end{aligned}
$$

QFT: 2 Qubits

Take $R_2 |0\rangle = |0\rangle$ and $R_2 |1\rangle = \omega |1\rangle$. Intuitively, R_2 rotates a vector in the xz-plane $\frac{\pi}{2}$ radians

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It yields a controlled-R_2 operation defined by $|x\rangle |0\rangle \mapsto |x\rangle |0\rangle$ and $|x\rangle |1\rangle \mapsto R_2 |x\rangle |1\rangle$. Equivalently

$$|0\rangle |x_2\rangle \mapsto |0\rangle |x_2\rangle \qquad |1\rangle |x_2\rangle \mapsto \omega^{x_2} |1\rangle |x_2\rangle$$

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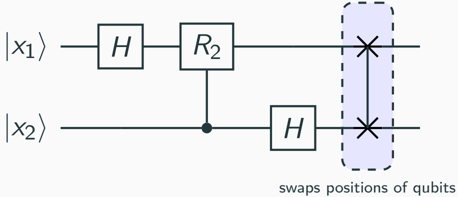
Putting all pieces together we derive the QFT circuit for 2 qubits



QFT: 3 Qubits

Let $\omega_n = e^{i2\pi \cdot \frac{1}{2^n}}$ (division of the unit circle in 2^n slices)

$$QFT_3 \left| x \right\rangle = \left(\left|0\right\rangle + \omega_3^{4\cdot x} \left|1\right\rangle \right) \otimes \left(\left|0\right\rangle + \omega_3^{2\cdot x} \left|1\right\rangle \right) \otimes \left(\left|0\right\rangle + \omega_3^{1\cdot x} \left|1\right\rangle \right)$$

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Actually, it is now easy to extrapolate the general defn. of QFT

$$QFT_n \, |x\rangle = \big(\, |0\rangle + \omega_n^{2^{n-1} \cdot x} \, |1\rangle \, \big) \otimes \cdots \otimes \big(\, |0\rangle + \omega_n^{2^0 \cdot x} \, |1\rangle \, \big)$$

N.B. In both equations above we drop the normalisation factor $\frac{1}{\sqrt{2}}$ in each state to make notation easier on the eyes

QFT: 3 Qubits

In order to derive a circuit for QFT_3, we observe

$$\omega_n^2 = \omega_{n-1} \text{ and thus } \omega_n^{2^{n-1}} = e^{i\pi} = -1$$

and recall that a binary number $x_1 \ldots x_n$ represents the natural number $2^{n-1} \cdot x_1 + \cdots + 2^0 \cdot x_n$.

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and recall that a binary number $x_1 \ldots x_n$ represents the natural number $2^{n-1} \cdot x_1 + \cdots + 2^0 \cdot x_n$. We then calculate

$$\begin{aligned}
& QFT_3 \left| x \right\rangle \\
&= \left(\left|0\right\rangle + \omega_3^{4 \cdot x} \left|1\right\rangle \right) \otimes \left(\left|0\right\rangle + \omega_3^{2 \cdot x} \left|1\right\rangle \right) \otimes \left(\left|0\right\rangle + \omega_3^{1 \cdot x} \left|1\right\rangle \right) \\
&= \left(\left|0\right\rangle + (-1)^{x} \left|1\right\rangle \right) \otimes \left(\left|0\right\rangle + \omega_3^{2 \cdot x} \left|1\right\rangle \right) \otimes \left(\left|0\right\rangle + \omega_3^{1 \cdot x} \left|1\right\rangle \right) \\
&= \left(\left|0\right\rangle + (-1)^{x_3} \left|1\right\rangle \right) \otimes \left(\left|0\right\rangle + \omega_3^{2 \cdot x} \left|1\right\rangle \right) \otimes \left(\left|0\right\rangle + \omega_3^{1 \cdot x} \left|1\right\rangle \right) \\
&= H \left|x_3\right\rangle \otimes \left(\left|0\right\rangle + \omega_3^{2 \cdot (4x_1 + 2x_2 + x_3)} \left|1\right\rangle \right) \otimes \left(\left|0\right\rangle + \omega_3^{1 \cdot x} \left|1\right\rangle \right) \\
&= H \left|x_3\right\rangle \otimes \left(\left|0\right\rangle + \omega_3^{2 \cdot (4x_1 + 2x_2)} \omega_3^{2 \cdot x_3} \left|1\right\rangle \right) \otimes \left(\left|0\right\rangle + \omega_3^{1 \cdot x} \left|1\right\rangle \right)
\end{aligned}$$

QFT: 3 Qubits

$\dots\dots$

$$= H\,|x_3\rangle \otimes \big(\,|0\rangle + \omega_2^{2\cdot(2x_1+x_2)}\omega_2^{x_3}\,|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_3^{4x_1+2x_2+x_3}\,|1\rangle\,\big)$$

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$$= H\,|x_3\rangle \otimes \underbrace{\big(\,|0\rangle + \omega_2^{2\cdot(2x_1+x_2)}\omega_2^{x_3}\,|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_2^{2x_1+x_2}\omega_3^{x_3}\,|1\rangle\,\big)}_{\text{some controlled-rotations on } QFT_2|x_1x_2\rangle}$$

QFT: 3 Qubits

$$\cdots\cdots$$

$$
\begin{aligned}
&= H\,|x_3\rangle \otimes \big(\,|0\rangle + \omega_2^{2\cdot(2x_1+x_2)}\omega_2^{x_3}\,|1\rangle\,\big) \otimes \big(\,|0\rangle + \omega_3^{4x_1+2x_2+x_3}\,|1\rangle\,\big) \\
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\end{aligned}
$$

Calculation easily extends to QFT_n (*in lieu* of QFT_3) which suggests a recursive defn. for the general QFT circuit

QFT: 3 Qubits

Take $R_n |0\rangle = |0\rangle$ and $R_n |1\rangle = \omega_n |1\rangle$. Intuitively, R_n rotates a vector in the xz-plane 'one 2^n-th of the unit circle'

It yields a controlled-R_n operation defined by $|x\rangle |0\rangle \mapsto |x\rangle |0\rangle$ and $|x\rangle |1\rangle \mapsto R_n |x\rangle |1\rangle$. Equivalently

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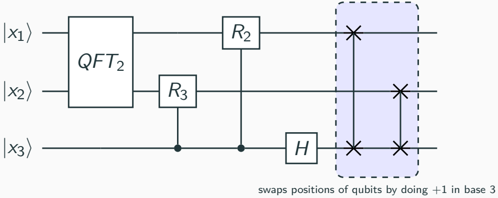
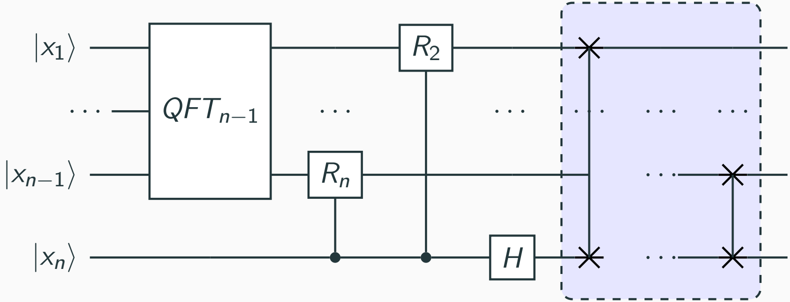


General QFT Circuit



swaps positions of qubits by doing $+1$ in base n

Complexity of QFT

How many gates does the QFT circuit require?

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$$\text{nº gates } QFT_n = \text{nº gates } QFT_{n-1} \; + \; \underbrace{1}_{\text{Hadamard}} \; + \; \underbrace{n-1}_{\text{Rotations } R_n} \; + \; \underbrace{n-1}_{\text{nº of swap gates}}$$

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We then calculate,

$$\begin{aligned}
\text{nº gates } QFT_n &= \text{nº gates } QFT_{n-1} + n + n - 1 \\
&= \textstyle\sum_{i=1}^{n} i + \sum_{i=0}^{n-1} i \\
&= \tfrac{(n+1)n}{2} + \tfrac{n(n-1)}{2} \\
&\approx \tfrac{n^2}{2} + \tfrac{n^2}{2} \\
&= n^2
\end{aligned}$$

Thus complexity of QFT is polynomial

An Equivalent Formulation of QFT

Previously we saw that

$$QFT_n |x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \omega_n^{2^{n-1}\cdot x}|1\rangle) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(|0\rangle + \omega_n^{1\cdot x}|1\rangle)$$

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Equivalent and useful definition given by

$$QFT_n \left| x \right\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} \omega_n^{x \cdot k} \left| k \right\rangle$$

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Examples with $n = 1$ and $n = 2$

$$QFT_1 \left|x\right\rangle = \tfrac{1}{\sqrt{2}}(\left|0\right\rangle + \omega_1^{x}\left|1\right\rangle)$$
$$QFT_2 \left|x\right\rangle = \tfrac{1}{\sqrt{2^2}}(\left|00\right\rangle + \omega_2^{x}\left|01\right\rangle + \omega_2^{2\cdot x}\left|10\right\rangle + \omega_2^{3\cdot x}\left|11\right\rangle)$$

Exercise 1

Show that both definitions of QFT coincide when $n = 2$

Exercise 2

Can you show that both definitions coincide for arbitrary n?

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Eigenvalues of unitaries are always of form above

A Simple Example

Take a unitary U with an eigenvector $|\psi\rangle$ whose eigenvalue is $e^{i2\pi\phi}$

ϕ is equal to one of the values $\{0 \cdot \frac{1}{2}, 1 \cdot \frac{1}{2}\}$. Find out ϕ

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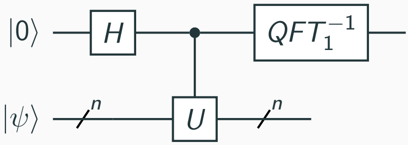
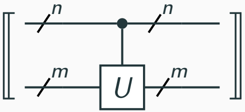
This is obtained via the circuit



Multi-Controlled Operations

Take a unitary U on n qubits

It gives rise to a multi-controlled operation



$$= |x\rangle\,|y\rangle \mapsto |x\rangle\, U^{x}\,|y\rangle$$

decimal representation of x

Intuitively it applies U to $|y\rangle$ a number of times equal to x

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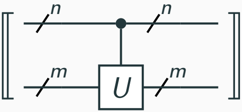

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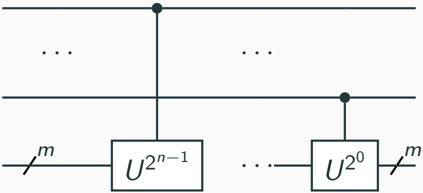
Examples

$|10\rangle\,|y\rangle \mapsto |10\rangle\,(UU\,|y\rangle)$ and $|00\rangle\,|y\rangle \mapsto |00\rangle\,|y\rangle$

Multi-Controlled Operations

Recall that a binary number $x_1 \dots x_n$ corresponds to the natural number $2^{n-1}x_1 + \cdots + 2^0 x_n$

We use this to build the previous multi-controlled operation in terms of simpler operations



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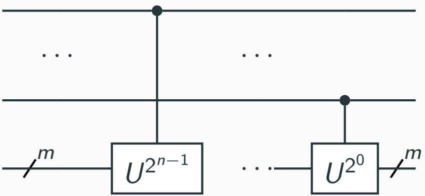

Note that the multi-controlled operation uses n 'simply'-controlled rotations U^{2^i}

Another Example

Take a unitary U with an eigenvector $|\psi\rangle$ whose eigenvalue is $e^{i2\pi\cdot\phi}$

ϕ is equal to one of the following values $\left\{0 \cdot \frac{1}{4}, 1 \cdot \frac{1}{4}, 2 \cdot \frac{1}{4}, 3 \cdot \frac{1}{4}\right\}$

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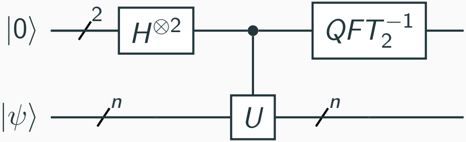
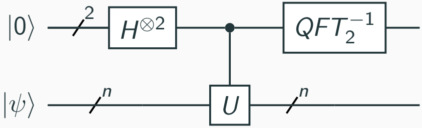
In order to discover ϕ we use the circuit



Another Example



$$
\begin{aligned}
&|0\rangle\,|0\rangle \\
&\overset{H^{\otimes 2}}{\mapsto}\ \frac{1}{\sqrt{2^2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
&\overset{\text{ctrl. } U}{\mapsto}\ \frac{1}{\sqrt{2^2}}(|00\rangle + e^{i2\pi\phi}\,|01\rangle + e^{i2\pi\phi\cdot 2}\,|10\rangle + e^{i2\pi\phi\cdot 3}\,|11\rangle) \\
&= \frac{1}{\sqrt{2^2}}(|00\rangle + e^{i2\pi x\cdot\frac{1}{4}}\,|01\rangle + e^{i2\pi x\cdot\frac{1}{4}\cdot 2}\,|10\rangle + e^{i2\pi x\cdot\frac{1}{4}\cdot 3}\,|11\rangle) \\
&= \frac{1}{\sqrt{2^2}}(|00\rangle + \omega_2^{x}\,|01\rangle + \omega_2^{x\cdot 2}\,|10\rangle + \omega_2^{x\cdot 3}\,|11\rangle) \\
&\overset{QFT_2^{-1}}{\mapsto}\ |x\rangle
\end{aligned}
$$

Yet Another Example

Take a unitary U with eigenvector $|\psi\rangle$ whose eigenvalue is $e^{i2\pi\phi}$

ϕ is equal to one of the following values $\left\{0 \cdot \frac{1}{2^n}, \ldots, 2^{n-1} \cdot \frac{1}{2^n}\right\}$

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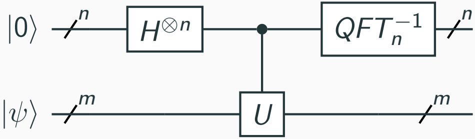
In order to discover ϕ we use the following circuit



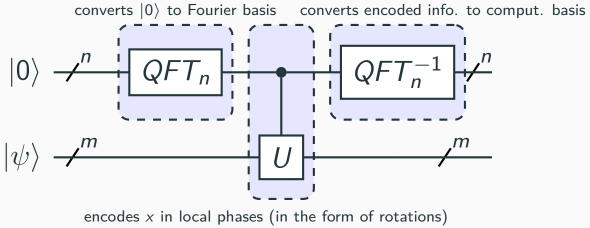
Exercise

Prove that the circuit returns x with $\phi = x \cdot \frac{1}{2^n}$

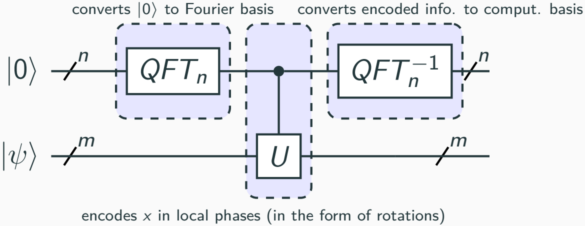
Yet Another Example

Exercise

Show that $QFT_n |0\rangle = H^{\otimes n} |0\rangle$. Note that this allows to rewrite the previous circuit in the one below

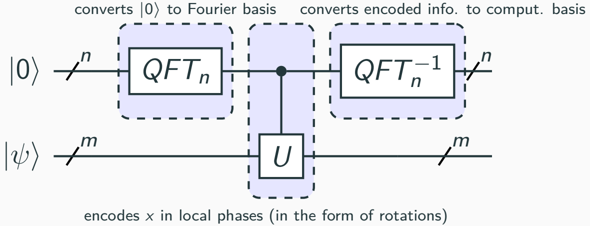


Complexity of Quantum Phase Estimation



How many gates does the circuit above use?

Complexity of Quantum Phase Estimation



How many gates does the circuit above use?

n 'Hadamards' $+$ n 'simply'-controlled gates $+$ n^2 gates for QFT_n^{-1}