Setting an Exponential Separation between Quantum and Classical Computation

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Previously. . .

The Problem

Take a function $f : \{0,1\} \to \{0,1\}$

Either $f(0) = f(1)$ or $f(0) \neq f(1)$

Tell us whether the first or second case hold

Classically, need to run f twice. Quantumly, once is enough

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Can we have more impressive differences in complexity?

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Global Phase Factor

Definition

Let $v, u \in \mathbb{C}^{2^n}$ be vectors. If $u = e^{i\theta} v$ we say that it is equal to v up to global phase factor $e^{i\theta}$

Theorem

$e^{i\theta} v$ and v are indistinguishable in the world of quantum mechanics

Proof sketch

Show that equality up to global phase is preserved by operators and normalisation + show that probability outcomes associated with v and $e^{i\theta} v$ are the same

Relative Phase Factor

Definition

We say that vectors $\sum_{x \in 2^n} \alpha_x |x\rangle$ and $\sum_{x \in 2^n} \beta_x |x\rangle$ differ by a relative phase factor if for all $x \in 2^n$

$$\alpha_x = e^{i\theta_x} \beta_x \qquad \text{(for some angle } \theta_x\text{)}$$

Example

Vectors $|0\rangle + |1\rangle$ and $|0\rangle - |1\rangle$ differ by a relative phase factor

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Vectors $|0\rangle + |1\rangle$ and $|0\rangle - |1\rangle$ differ by a relative phase factor

Vectors that differ by a relative phase factor are distinguishable

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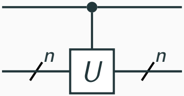
The Phase Kickback Effect pt. I

Recall that every quantum operation  gives rise to a controlled quantum operation, which is depicted below



Let v be an eigenvector of U (*i.e.* $Uv = e^{i\theta}v$) and calculate

$$
\begin{aligned}
& cU\Big((\alpha\,|0\rangle + \beta\,|1\rangle) \otimes v\Big) \\
&= cU(\alpha\,|0\rangle \otimes v + \beta\,|1\rangle \otimes v) \\
&= \alpha\,|0\rangle \otimes v + \beta\,|1\rangle \otimes e^{i\theta} v \\
&= (\alpha\,|0\rangle + e^{i\theta}\beta\,|1\rangle) \otimes v
\end{aligned}
$$

The Phase Kickback Effect pt. II

What just happened?

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What just happened?

- Global phase $e^{i\theta}$ (introduced to v) was 'kickedback' as a relative phase in the control qubit

The Phase Kickback Effect pt. II

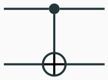
What just happened?

- Global phase $e^{i\theta}$ (introduced to v) was 'kickedback' as a relative phase in the control qubit
- Some information of U is now encoded in the control qubit

In general kickingback such phases causes interference patterns that give away information about U

The Phase Kickback Effect pt. III

Consider the controlled-not operation

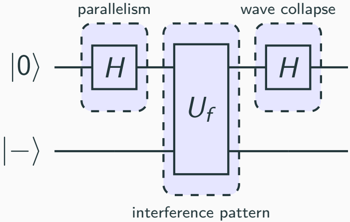


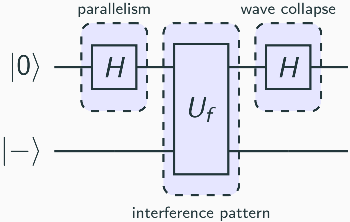
X has $|-\rangle$ as eigenvector with associated eigenstate -1. It thus yields the equation

$$cX\,|b\rangle\,|-\rangle = (-1)^b\,|b\rangle\,|-\rangle$$

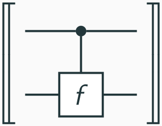
with $|b\rangle$ an element of the computational basis

Back to Deutsch's Problem





U_f can be seen as a generalised controlled not-operation



$$= |x\rangle\,|y\rangle \mapsto \begin{cases} |x\rangle\,|y\rangle & \text{if } f(x) = 0 \\ |x\rangle \neg |y\rangle & \text{if } f(x) = 1 \end{cases}$$

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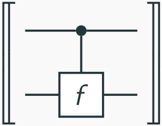
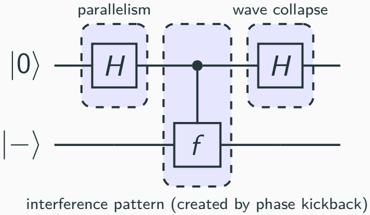


$$= |x\rangle\,|y\rangle \mapsto \begin{cases} |x\rangle\,|y\rangle & \text{if } f(x) = 0 \\ |y\rangle\, \neg\, |y\rangle & \text{if } f(x) = 1 \end{cases}$$

Recall that $|-\rangle$ is an eigenvector of X with eigenstate -1. Thus analogously to before we deduce

$$U_f\,|x\rangle\,|-\rangle = (-1)^{f(x)}\,|x\rangle\,|-\rangle$$

Back to Deutsch's Problem pt. III



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Going Beyond the Current Separation

Albeit looking almost magical how we handled Deutsch's problem, the corresponding complexity difference between quantum and classical is unimpressive

Can we come up with a more impressive separation?

Setting the Stage

Lemma

For $a, b \in \{0,1\}$ the equation $(-1)^a(-1)^b = (-1)^{a \oplus b}$ holds

Prook sketch

Build a truth table for each case and compare the corresponding contents

Definition

Given two bit-strings $x, y \in \{0,1\}^n$ we define their product $x \cdot y \in \{0,1\}$ as $x \cdot y = (x_1 \wedge y_1) \oplus \cdots \oplus (x_n \wedge y_n)$

Setting the Stage

Lemma

For any three binary strings $x, a, b \in \{0,1\}^n$ *the equation* $(x \cdot a) \oplus (x \cdot b) = x \cdot (a \oplus b)$ *holds*

Proof sketch

Follows from the fact that for any three bits $a, b, c \in \{0,1\}$ the equation $(a \wedge b) \oplus (a \wedge c) = a \wedge (b \oplus c)$ holds

Setting the Stage

Lemma

For any element $|b\rangle$ *in the computational basis of* $\mathbb{C}^2$ *we have* $H|b\rangle = \frac{1}{\sqrt{2}} \sum_{z \in 2} (-1)^{b \wedge z} |z\rangle$

Proof sketch

Build a truth table and compare the corresponding contents

Theorem

For any element $|b\rangle$ *in the computational basis of* $\mathbb{C}^{2^n}$ *we have* $H^{\otimes n}|b\rangle = \frac{1}{\sqrt{2^n}} \sum_{z \in 2^n} (-1)^{b \cdot z} |z\rangle$

Proof sketch

Follows from induction on the size of n

The Problem

Take a function $f : \{0,1\}^n \to \{0,1\}$

You are promised that $f(x) = s \cdot x$ for some fixed bit-string s

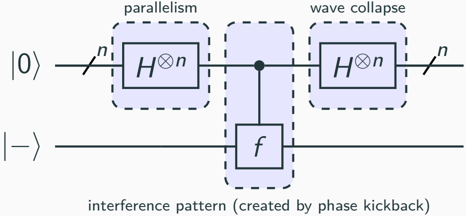
Find s

Classically, we run f n-times by computing

$$f(1\dots 0) = (s_1 \wedge 1) \oplus \cdots \oplus (s_n \wedge 0) = s_1$$
$$\vdots$$
$$f(0\dots 1) = (s_1 \wedge 0) \oplus \cdots \oplus (s_n \wedge 1) = s_n$$

Quantumly, we discover s by running f only once

The Circuit


parallelism
wave collapse
$|0\rangle$
$|-\rangle$
n
$H^{\otimes n}$
f
$H^{\otimes n}$
n
interference pattern (created by phase kickback)

N.B. In order to not overburden notation we omit $|-\rangle$

$$
\begin{aligned}
& H^{\otimes n} |0\rangle \\
& = \tfrac{1}{\sqrt{2^n}} \textstyle\sum_{z \in 2^n} |z\rangle && \{\text{Theorem slide 18}\} \\
& \overset{U_f}{\mapsto} \tfrac{1}{\sqrt{2^n}} \textstyle\sum_{z \in 2^n} (-1)^{f(z)} |z\rangle && \{\text{Definition slide 12}\} \\
& \overset{H^{\otimes n}}{\mapsto} \tfrac{1}{2^n} \textstyle\sum_{z \in 2^n} (-1)^{f(z)} \Big(\sum_{z' \in 2^n} (-1)^{z \cdot z'} |z'\rangle \Big) && \{\text{Theorem slide 18}\} \\
& = \tfrac{1}{2^n} \textstyle\sum_{z \in 2^n} \sum_{z' \in 2^n} (-1)^{(z \cdot s) \oplus (z \cdot z')} |z'\rangle && \{\text{Lemma slide 16}\} \\
& = \tfrac{1}{2^n} \textstyle\sum_{z \in 2^n} \sum_{z' \in 2^n} (-1)^{z \cdot (s \oplus z')} |z'\rangle && \{\text{Lemma slide 17}\}
\end{aligned}
$$

The Computation pt. II

Probability of measuring s at the end given by

$$\begin{aligned}
&\Big|\tfrac{1}{2^n}\textstyle\sum_{z\in 2^n}(-1)^{z\cdot(s\oplus s)}\,|s\rangle\,\Big|^2 \\
&= \Big|\tfrac{1}{2^n}\textstyle\sum_{z\in 2^n}(-1)^{z\cdot 0}\,|s\rangle\,\Big|^2 \\
&= \Big|\tfrac{1}{2^n}\textstyle\sum_{z\in 2^n} 1\,|s\rangle\,\Big|^2 \\
&= \Big|\tfrac{2^n}{2^n}\Big|^2 \\
&= 1
\end{aligned}$$

This means that somehow all values yielding wrong answers were completely cancelled

T.P.C. Show exactly how all the wrong answers were cancelled

Going Even Further Beyond

We went from running f n times to running just once

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Still not very impressive (at least for the Computer Scientist :-))

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Can we do even better?

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Deutsch-Josza

The Problem

Take a function $f : \{0,1\}^n \to \{0,1\}$

You are promised that f is either constant or balanced

Find out which case holds

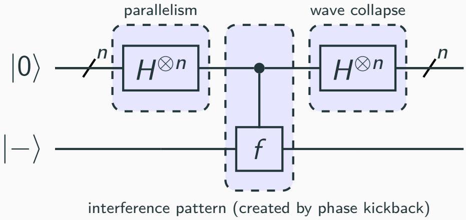
Classically, we evaluate half of the inputs ($\frac{2^n}{2} = 2^{n-1}$), evaluate one more and run the decision procedure,

- output always the same $\implies$ constant
- otherwise $\implies$ balanced

which requires running f $2^{n-1} + 1$ times

Quantumly, we know the answer by running f only once

The Circuit


parallelism
wave collapse
$|0\rangle$
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& \overset{H^{\otimes n}}{\mapsto} \tfrac{1}{2^n} \textstyle\sum_{z \in 2^n} (-1)^{f(z)} \Big(\sum_{z' \in 2^n} (-1)^{z \cdot z'} |z'\rangle \Big) && \{\text{Theorem slide 18}\}
\end{aligned}
$$

We then proceed by case distinction. Assume that f is constant

$$
\begin{aligned}
& \tfrac{1}{2^n} \textstyle\sum_{z \in 2^n} (-1)^{f(z)} \Big(\sum_{z' \in 2^n} (-1)^{z \cdot z'} |z'\rangle \Big) \\
& = \tfrac{1}{2^n} (\pm 1) \textstyle\sum_{z \in 2^n} \Big(\sum_{z' \in 2^n} (-1)^{z \cdot z'} |z'\rangle \Big)
\end{aligned}
$$

The Computation pt. II

Probability of measuring $|0\rangle$ at the end given by

$$
\begin{aligned}
&\Big|\tfrac{1}{2^n}(\pm 1)\textstyle\sum_{z\in 2^n}(-1)^{z\cdot 0}\,|0\rangle\,\Big|^2 \\
&= \Big|\tfrac{1}{2^n}(\pm 1)\textstyle\sum_{z\in 2^n} 1\,|0\rangle\,\Big|^2 \\
&= \Big|\tfrac{2^n}{2^n}\Big|^2 \\
&= 1
\end{aligned}
$$

So if f is constant we measure $|0\rangle$ with probability 1. Now if f is balanced...

The Computation pt. III

$$\frac{1}{2^n}\sum_{z\in 2^n}(-1)^{f(z)}\Big(\sum_{z'\in 2^n}(-1)^{z\cdot z'}\,|z'\rangle\Big)$$

$$= \frac{1}{2^n}\Bigg(\sum_{z\in 2^n, f(z)=0}(-1)^{f(z)}\Big(\sum_{z'\in 2^n}(-1)^{z\cdot z'}\,|z'\rangle\Big)$$

$$+ \sum_{z\in 2^n, f(z)=1}(-1)^{f(z)}\Big(\sum_{z'\in 2^n}(-1)^{z\cdot z'}\,|z'\rangle\Big)\Bigg)$$

$$= \frac{1}{2^n}\Bigg(\sum_{z\in 2^n, f(z)=0}\Big(\sum_{z'\in 2^n}(-1)^{z\cdot z'}\,|z'\rangle\Big)$$

$$+ \sum_{z\in 2^n, f(z)=1}(-1)\Big(\sum_{z'\in 2^n}(-1)^{z\cdot z'}\,|z'\rangle\Big)\Bigg)$$

The Computation pt. IV

Probability of measuring $|0\rangle$ at the end given by

$$\left|\tfrac{1}{2^n}\Big(\textstyle\sum_{z\in 2^n, f(z)=0}(-1)^{z\cdot 0}\,|0\rangle + \textstyle\sum_{z\in 2^n, f(z)=1}(-1)(-1)^{z\cdot 0}\,|0\rangle\Big)\right|^2$$
$$= \left|\tfrac{1}{2^n}\Big(\textstyle\sum_{z\in 2^n, f(z)=0}|0\rangle + \textstyle\sum_{z\in 2^n, f(z)=1}(-1)\,|0\rangle\Big)\right|^2$$
$$= \left|\tfrac{1}{2^n}\Big(\textstyle\sum_{z\in 2^n, f(z)=0}|0\rangle - \textstyle\sum_{z\in 2^n, f(z)=1}|0\rangle\Big)\right|^2$$
$$= 0$$

So if f is balanced we measure $|0\rangle$ with probability 0