RBE 595 — Reinforcement Learning
Week #7 Assignment
Temporal Difference Learning

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Problem 1

Between DP (Dynamic Programming), MC (Monte-Carlo) and TD (Temporal Difference), which one of these algorithms use bootstrapping? Explain.

Answer

Bootstrapping is the process of updating the value of a state based on the value of a future state.

- **Dynamic Programming** (DP) uses bootstrapping. This is because DP uses the Bellman equation to update the value of a state based on the value of a future state.
- **Monte-Carlo** (MC) does not use bootstrapping. This is because MC does not use the Bellman equation to update the value of a state based on the value of a future state. Instead, MC uses the actual return value to update the value of a state.
- **Temporal Difference** (TD) uses bootstrapping. This is because TD uses the Bellman equation to update the value of a state based on the value of a future state.

Problem 2

We mentioned that the target value for TD is $[R_{t+1} + \gamma V(s_{t+1})]$. What is the target value for Monte-carlo, Q-learning, SARSA and Expected-SARSA?

Answer

- **Monte-Carlo** (MC) does not use bootstrapping. Therefore, the target value is the actual return value, G_t.
- **Q-Learning** is an off-policy TD control algorithm. Therefore, the target value is $R_{t+1}+\gamma \max_a Q(S_{t+1}, a)$.
- **SARSA** is an on-policy TD control algorithm. Therefore, the target value is $R_{t+1} + \gamma Q(S_{t+1}, A_{t+1})$.
- **Expected-SARSA** is an on-policy TD control algorithm. Therefore, the target value is $R_{t+1} + \gamma \mathbb{E}_\pi [Q(S_{t+1}, A_{t+1}) \mid S_{t+1}]$.

Problem 3

What are the similarities of TD and MC?

Answer

The similarities between TD and MC are as follows:

- Both TD and MC are model-free.
- Both TD and MC are used for prediction and control.

Problem 4

Assume that we have two states x and y with the current value of $V(x) = 10$, $V(y) = 1$. We run an episode of $\{x, 3, y, 0, y, 5, T\}$. What's the new estimate of $V(x)$, $V(y)$ using TD (assume step size $\alpha = 0.1$ and discount rate $\gamma = 0.9$).

Answer

The new estimate of $V(x)$ is as follows:

$$\begin{aligned} V(x) &= V(x) + \alpha\left[R_{t+1} + \gamma V(S_{t+1}) - V(x)\right] \\ &= 10 + 0.1\left[3 + 0.9 \cdot 1 - 10\right] \\ &= 10 + 0.1\left[3.9 - 10\right] \\ &= 10 + 0.1\left[-6.1\right] \\ &= 10 - 0.61 \\ &= 9.39 \end{aligned}$$

However, $V(y)$ gets updated twice in this episode. The first update is as follows:

$$\begin{aligned} V(y) &= V(y) + \alpha\left[R_{t+1} + \gamma V(S_{t+1}) - V(y)\right] \\ &= 1 + 0.1\left[0 + 0.9 \cdot 1 - 1\right] \\ &= 1 + 0.1\left[0.9 - 1\right] \\ &= 1 + 0.1\left[-0.1\right] \\ &= 1 - 0.01 \\ &= 0.99 \end{aligned}$$

The second update is as follows:

$$\begin{aligned} V(y) &= V(y) + \alpha\left[R_{t+1} + \gamma V(S_{t+1}) - V(y)\right] \\ &= 0.99 + 0.1\left[5 + 0.9 \cdot 0 - 0.99\right] \\ &= 0.99 + 0.1\left[5 - 0.99\right] \\ &= 0.99 + 0.1\left[4.01\right] \\ &= 0.99 + 0.401 \\ &= 1.391 \end{aligned}$$

Therefore, the new estimate of $V(x)$ is 9.39 and the new estimate of $V(y)$ is 1.391.

Problem 5

What is the difference between policy and action?

Answer

An *action* is a choice made by the agent at a given state. It is an attempted modification of the environment which leads to a new state or the same state. We give an agent an associated reward for each action.

In contrast, a policy determines how good it is for the agent to perform an action in a given state. Formally, a *policy* is a mapping from states to probabilities of selecting each possible action. It defines a probability distribution over actions for each state.

Problem 6

(Exercise 3.14) The Bellman equation must hold for each state for the value function v_π shown in Figure 3.2 (right-side) of Example 3.5. Show numerically that this equation holds for the center state, valued at +0.7, with respect to its four neighboring states, valued at +2.3, +0.4, -0.4, and +0.7. (These numbers are accurate only to one decimal place.)

Answer

From the textbook, the state-value function for a policy π is defined as,

$$\begin{aligned} v_\pi(s) &\doteq \mathbb{E}_\pi\left[G_t \mid S_t = s\right] \\ &= \sum_a \pi(a \mid s) \sum_{s',r} p(s', r \mid s, a)\left[r + \gamma v_\pi(s')\right] \end{aligned}$$

From Example 3.5, we also know the following given information:

- The action set $A = \{\text{up}, \text{down}, \text{left}, \text{right}\}$ in each state.
- An equiprobable random policy is used. Therefore, $\pi(a \mid s) = 0.25$ for all $a \in A$ and $s \in S$.
- The reward is always 0 for all transitions.
- $\gamma = 0.9$.
- Any action taken deterministically leads to the expected state, so $p = 1$.

Hence, the state-value function for the center state is,

$$\begin{aligned} v_\pi(s_{\text{center}}) &= \sum_a \pi(a \mid s) \sum_{s',r} p(s', r \mid s, a)\left[r + \gamma v_\pi(s')\right] \\ &= \pi(\text{up} \mid s) p(s_{\text{up}}, r \mid s, \text{up})\left[r + \gamma v_\pi(s_{\text{up}})\right] + \pi(\text{down} \mid s) p(s_{\text{down}}, r \mid s, \text{down})\left[r + \gamma v_\pi(s_{\text{down}})\right] \\ &\quad + \pi(\text{left} \mid s) p(s_{\text{left}}, r \mid s, \text{left})\left[r + \gamma v_\pi(s_{\text{left}})\right] + \pi(\text{right} \mid s) p(s_{\text{right}}, r \mid s, \text{right})\left[r + \gamma v_\pi(s_{\text{right}})\right] \\ &= 0.25 \cdot 1 \cdot [0 + 0.9 \cdot 2.3] + 0.25 \cdot 1 \cdot [0 + 0.9 \cdot 0.4] + 0.25 \cdot 1 \cdot [0 + 0.9 \cdot (-0.4)] + 0.25 \cdot 1 \cdot [0 + 0.9 \cdot 0.7] \\ &= 0.25 \cdot 0.9 \cdot [2.3 + 0.4 - 0.4 + 0.7] \\ &= 0.25 \cdot 0.9 \cdot 3.0 \\ &= 0.675 \approx 0.7 \text{ (rounded to one decimal place, as mentioned in prompt)} \end{aligned}$$

Therefore, we see that the Bellman equation holds for the center state, valued at +0.7.

Problem 7

(Exercise 3.17) What is the Bellman equation for action values, that is, for q_π? It must give the action value $q_\pi(s,a)$ in terms of the action values, $q_\pi(s',a')$, of possible successors to the state-action pair (s,a). Hint: the backup diagram below corresponds to this equation. Show the sequence of equations analogous to (3.14), but for action values.

Answer

From the textbook, the action-value function for a policy π is defined as,

$$\begin{aligned}
q_\pi(s,a) &\doteq \mathbb{E}_\pi\left[G_t \mid S_t = s, A_t = a\right] \\
&= \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s, A_t = a\right] \\
&= \mathbb{E}_\pi\left[R_{t+1} + \gamma \sum_{k=0}^{\infty} \gamma^k R_{t+k+2} \mid S_t = s, A_t = a\right] \\
&= \mathbb{E}_\pi\left[R_{t+1} + \gamma G_{t+1} \mid S_t = s, A_t = a\right] \\
&= \mathbb{E}_\pi\left[R_{t+1} \mid S_t = s, A_t = a\right] + \gamma \mathbb{E}_\pi\left[G_{t+1} \mid S_t = s, A_t = a\right]
\end{aligned}$$

Now, let us consider the first and second terms of the above equation separately.

First Term

$$\mathbb{E}_\pi\left[R_{t+1} \mid S_t = s, A_t = a\right] = \sum_{r \in \mathcal{R}} r \cdot p(r \mid s,a) = \sum_{r \in \mathcal{R}} \sum_{s' \in \mathcal{S}} r \cdot p(s', r \mid s, a)$$

Second Term

$$\begin{aligned}
\gamma \mathbb{E}_\pi\left[G_{t+1} \mid S_t = s, A_t = a\right] &= \gamma \sum_{g \in \mathcal{G}} g \cdot p(g \mid s, a) \\
&= \gamma \sum_{g \in \mathcal{G}} \sum_{r \in \mathcal{R}} \sum_{s' \in \mathcal{S}} \sum_{a' \in \mathcal{A}} g \cdot p(g \mid s', a') \cdot p(s', r \mid s, a) \cdot \pi(a' \mid s')
\end{aligned}$$

Where, $\sum_{g \in \mathcal{G}} g \cdot p(g \mid s', a') = \mathbb{E}_\pi\left[G_{t+1} \mid S_{t+1} = s', A_{t+1} = a'\right] = q_\pi(s', a')$

Therefore the second term is,

$$\gamma \mathbb{E}_\pi\left[G_{t+1} \mid S_t = s, A_t = a\right] = \gamma \sum_{r \in \mathcal{R}} \sum_{s' \in \mathcal{S}} \sum_{a' \in \mathcal{A}} q_\pi(s', a') \cdot p(s', r \mid s, a) \cdot \pi(a' \mid s')$$

Now, combining the first and second terms, we get,

Problem 7 continued on next page...

$$q_\pi(s,a) = \sum_{r\in\mathcal{R}}\sum_{s'\in\mathcal{S}} r\cdot p(s',r\mid s,a) + \gamma\sum_{r\in\mathcal{R}}\sum_{s'\in\mathcal{S}}\sum_{a'\in\mathcal{A}} q_\pi(s',a')\cdot p(s',r\mid s,a)\cdot\pi(a'\mid s')$$
$$q_\pi(s,a) = \sum_{s',r} p(s',r\mid s,a)\left[r+\gamma\sum_{a'}\pi(a'\mid s')q_\pi(s',a')\right]$$

Which is the Bellman equation for action values, i.e., for q_π.

Backup Diagram Confirmation

This equation can be verified by looking at the backup diagram given in the prompt. The backup diagram shows that we start with the state-action pair (s,a). To get to the next state, we are subjected to the environment $p(s',r\mid s,a)$. The reward r is added to the discounted return G_{t+1}. This brings us to our new state, s'. At this point, the equation would look as follows,

$$q_\pi(s,a) = \sum_{s',r} p(s',r\mid s,a)\left[r+\gamma v_\pi(s')\right]$$

However we still need to eliminate the $v_\pi(s')$ term. To do this, we go through our policy, π, to get the action a' that we would take in the state s'. Now the equation becomes,

$$q_\pi(s,a) = \sum_{s',r} p(s',r\mid s,a)\left[r+\gamma\sum_{a'}\pi(a'\mid s')q_\pi(s',a')\right]$$

So, the Bellman equation for action values, i.e., for q_π, is confirmed by the backup diagram.

Problem 8

(Exercise 3.22) Consider the continuing MDP shown below. The only decision to be made is that in the top state, where two actions are available, left and right. The numbers show the rewards that are received deterministically after each action. There are exactly two deterministic policies, π_{left} and π_{right}. What policy is optimal if $\gamma = 0$? If $\gamma = 0.9$? If $\gamma = 0.5$?

Answer

The discounted return is defined as,

$$G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \tag{3.8}$$

Case 1: $\gamma = 0$

When $\gamma = 0$, the left policy rewards are calculated as follows,

$$G_{\text{left}} = 1 + 0 + 0 + \cdots = 1$$

Similarly, the right policy rewards are calculated as follows,

$$G_{\text{right}} = 0 + 0 + \cdots = 0$$

In this case, the **left** policy is optimal.

Case 2: $\gamma = 0.9$

When $\gamma = 0.9$, the left policy rewards are calculated as follows,

$$\begin{aligned} G_{\text{left}} &= 1 + 0.9 \cdot 0 + 0.9^2 \cdot 1 + \cdots \\ &= 1 + 0.9^2 + 0.9^4 + \cdots \\ &= \sum_{k=0}^{\infty} 0.9^{2k} \\ &= \sum_{k=0}^{\infty} 0.81^k \\ &= \frac{1}{1-0.81} = \frac{1}{0.19} \\ &= 5.263 \end{aligned}$$

Similarly, the right policy rewards are calculated as follows,

$$\begin{aligned} G_{\text{right}} &= 0 + 0.9 \cdot 2 + 0 + 0.9^3 \cdot 2 + \cdots \\ &= 0.9 \cdot 2 + 0.9^3 \cdot 2 + \cdots \\ &= 2 \cdot \sum_{k=0}^{\infty} 0.9^{2k+1} = 2 \cdot \sum_{k=0}^{\infty} (0.9)(0.81)^k = 2 \cdot \frac{0.9}{1-0.81} \\ &= \frac{1.8}{0.19} = 9.474 \end{aligned}$$

In this case, the **right** policy is optimal.

Case 3: $\gamma = 0.5$

When $\gamma = 0.5$, the left policy rewards are calculated as follows,

$$\begin{aligned}
G_{\text{left}} &= 1 + 0.5 \cdot 0 + 0.5^2 \cdot 1 + \cdots \\
&= 1 + 0.5^2 + 0.5^4 + \cdots \\
&= \sum_{k=0}^{\infty} 0.5^{2k} = \sum_{k=0}^{\infty} 0.25^k \\
&= \frac{1}{1 - 0.25} = \frac{1}{0.75} \\
&= 1.333
\end{aligned}$$

Similarly, the right policy rewards are calculated as follows,

$$\begin{aligned}
G_{\text{right}} &= 0 + 0.5 \cdot 2 + 0 + 0.5^3 \cdot 2 + \cdots \\
&= 0.5 \cdot 2 + 0.5^3 \cdot 2 + \cdots \\
&= 2 \cdot \sum_{k=0}^{\infty} 0.5^{2k+1} = 2 \cdot \sum_{k=0}^{\infty} (0.5)(0.25)^k = 2 \cdot \frac{0.5}{1 - 0.25} \\
&= \frac{1}{0.75} = 1.333
\end{aligned}$$

In this case, both the **left** and **right** policies are optimal.