Chapter 15

Network Scheduling by PERT/CPM

15.1 INTRODUCTION

Network scheduling is a technique used for planning and scheduling large projects, in the fields of construction, maintenance, fabrication and purchasing of computer systems, etc. It is a method of minimizing the trouble spots such as production, delays and interruptions, by determining critical factors and co-ordinating various parts of the overall job.

There are two basic planning and controlling techniques that utilize a network to complete a predetermined project or schedule. These are Programme Evaluation Review Technique (PERT) and Critical Path Method (CPM).

A project is defined as a combination of interrelated activities, all of which must be executed in a certain order for its completion.

The work involved in a project can be divided into three phases, corresponding to the management functions of planning, scheduling and controlling.

Planning: This phase involves setting the objectives of the project as well as the assumptions to be made. It also involves the listing of tasks or jobs that must be performed in order to complete a project under consideration. In this phase, in addition to the estimates of costs and duration of the various activities, the manpower, machines and materials required for the project are also determined.

Scheduling: This consists of laying the activities according to their order of precedence and determining the following:

(*i*) The start and finish times for each activity.

(*ii*) The critical path on which the activities require special attention.

(*iii*) The slack and float for the non-critical paths.

Controlling: This phase is exercised after the planning and scheduling. It involves the following:

(*i*) Making periodical progress reports

(*ii*) Reviewing the progress

(*iii*) Analyzing the status of the project

(*iv*) Making management decisions regarding updating, crashing and resource allocation, etc.

15.2 BASIC TERMS

To understand the network techniques, one should be familiar with a few basic terms of which both CPM and PERT are special applications.

Network: It is the graphic representation of logically and sequentially connected arrows and nodes, representing activities and events in a project. Networks are also called *arrow diagrams.*

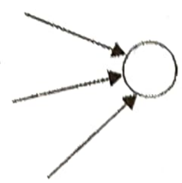
Activity: An activity represents some action and is a time consuming effort necessary to complete a particular part of the overall project. Thus, each and every activity has a point of time where it begins and a point where it ends.

It is represented in the network by an arrow,

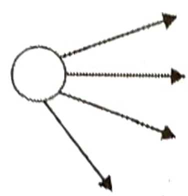
Here *A* is called the *activity*.

Event The beginning and end points of an activity are called *events or nodes*. Event is a point in time and does not consume any resources. It is represented by a numbered circle. The head event called the *jth event* always has a number higher than the tail event, which is also called the *ith event*.

Merge and burst events It is not necessary for an event to be the ending event of only one activity as it can be the ending event of two or more activities. Such an event is defined as a *merge event.*



If the event happens to be the beginning event of two or more activities, it is defined as a *burst event.*



Preceding, succeeding and concurrent activities Activities that must be accomplished before a given event can occur, are termed as *preceding activities.*

Activities that cannot be accomplished until an event has occurred, are termed as *succeeding activities.*

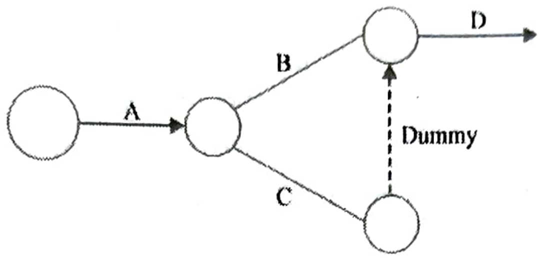
Activities that can be accomplished concurrently i.e., activities taking place at the same time or in the same location, are known as *concurrent activities.*

This classification is relative, which means that one activity can be preceding to a certain event, and the same activity can be succeeding to some other event or it may be a concurrent activity with one or more activities.

Dummy activity Certain activities, which neither consume time nor resources but are used simply to represent a connection or a link between the events are known as *dummies*. It is shown in the network by a dotted line. The purpose of introducing dummy activity is:

(*i*) to maintain uniqueness in the numbering system, as every activity may have a distinct set of events by which the activity can be identified.

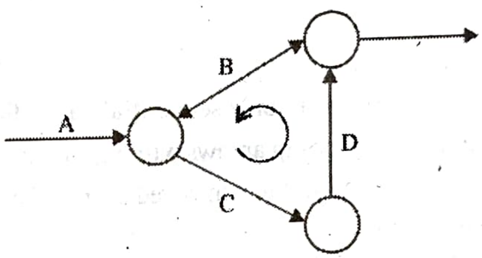
(ii) to maintain a proper logic in the network.



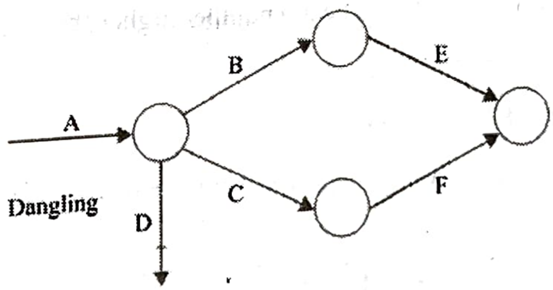
15.3 COMMON ERRORS

Following are the three common errors in a network construction:

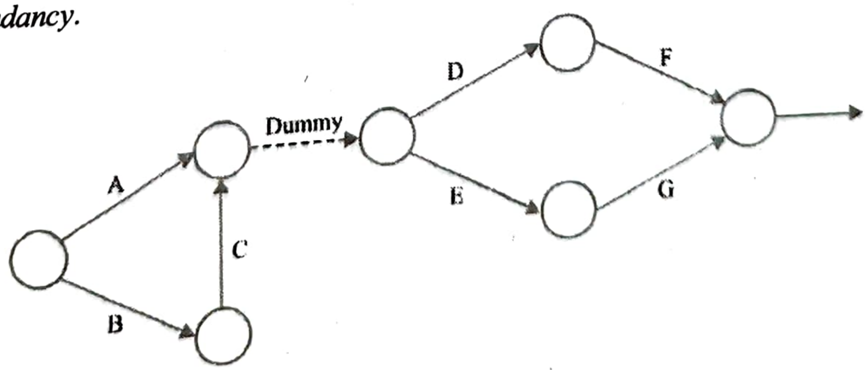
Looping (cycling) In a network diagram, a looping error is also known as *cycling error*. Drawing an endless loop in a network is known as *error of looping*. A loop can be formed, if an activity is represented as going back in time.



Dangling To disconnect an activity before the completion of all the activities in a network diagram, is known as *dangling*.



Redundancy If a dummy activity is the only activity emanating from an event and can be eliminated, it is known as *redundancy*.



15.4 RULES OF NETWORK CONSTRUCTION

There are a number of rules in connection with the handling of events and activities of a project network that should be followed.

(i) Try to avoid the arrows that cross each other.
(ii) Use straight arrows.
(iii) No event can occur until every activity preceding it has been completed.
(iv) An event cannot occur twice, i.e., there must be no loops.
(v) An activity succeeding an event cannot be started until that event has occurred.
(vi) Use arrows from left to right. Avoid mixing two directions, vertical and standing arrows may be used, if necessary.
(vii) Dummies should be introduced only, if it is extremely necessary.
(viii) The network has only one entry point called the *start event* and one point of emergence called the *end or terminal event*.

15.5 NUMBERING THE EVENTS (FULKERSON'S RULE)

After the network is drawn in a logical sequence, every event is assigned a unique number. The number sequence must be such so as to reflect the flow of the network. In numbering the events, the following rules should be observed.

(i) Event numbers should be unique.
(ii) Event numbering should be carried out on a sequential basis, from left to right.
(iii) The initial event, which has all outgoing arrows with no incoming arrow is numbered as 1.
(iv) Delete all the arrows emerging from all the numbered events. This will create at least one new start event, out of the preceding events.
(v) Number all new start events 2, 3 and so on. Repeat this process until the terminal event without any successor activity is reached. Number the terminal node suitably.

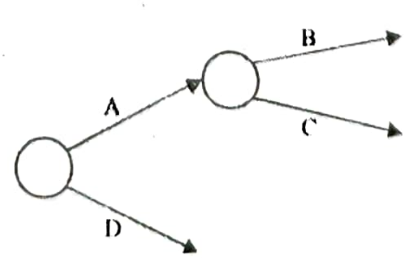
Note: The head of an arrow should always bear a number higher than the one assigned to the tail of the arrow.

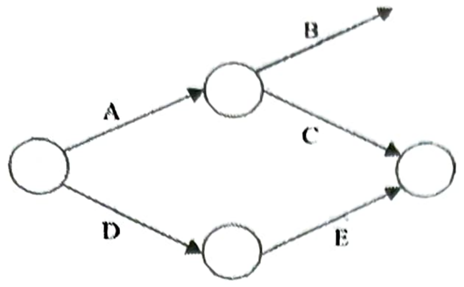
15.5.1 Construction of Network

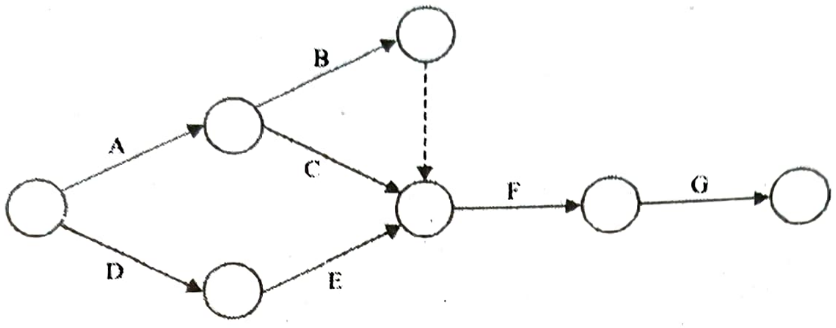
Example 15.1 Construct a network for the project whose activities and precedence relationships are as given below:

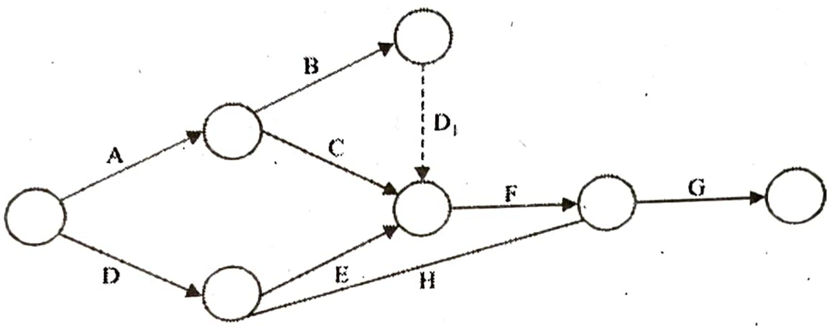
Activities	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Immediate Predecessor	–	*A*	*A*	–	*D*	*B*, *C*, *E*	*F*	*D*	*G*, *H*

Solution From the given constraints, it is clear that *A* and *D* are the starting activities and *I* the terminal activity. *B* and *C* are starting with the same event and are both the predecessors of the activity *F*. Also, *E* has to be the predecessor of both *F* and *H*. Hence, we have to introduce a dummy activity.



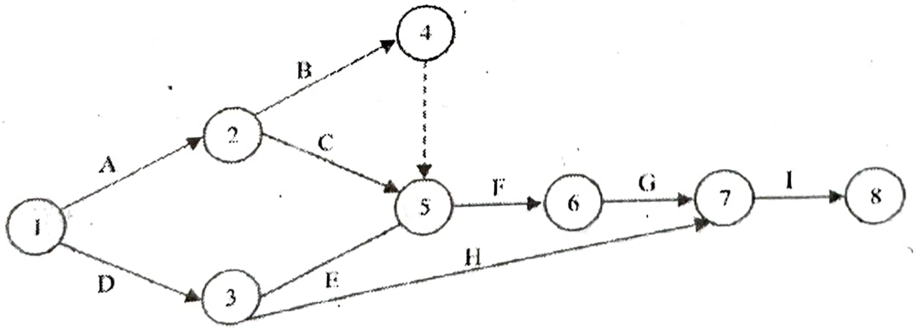






D_1 is the dummy activity.

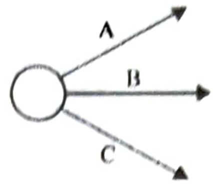
Finally, we have the following network.

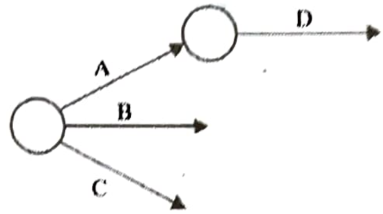


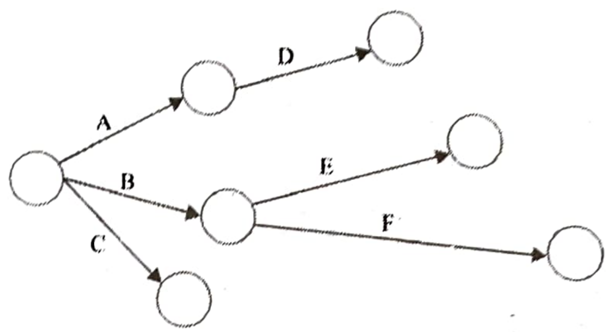
Example 15.2 Construct a network for each of the projects whose activities and their precedence relationships are given below.

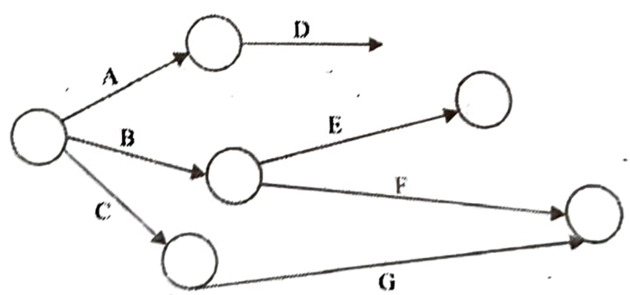
Activity	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
Predecessor	–	–	–	*A*	*B*	*B*	*C*	*D*	*E*	*H*, *I*	*F*, *G*

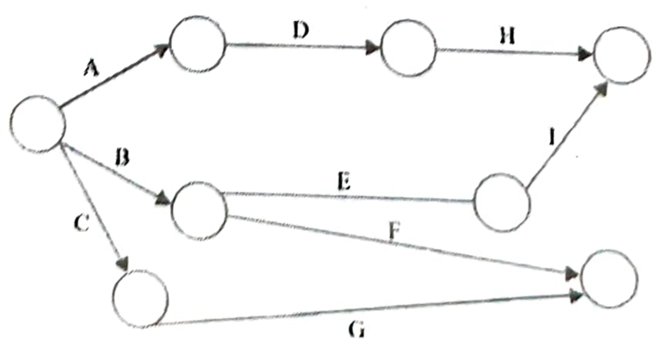
Solution A, B and C are the concurrent activities as they start simultaneously. B becomes the predecessor of activities E and F. Since the activities J and K have two preceding activities, a dummy may be introduced (if possible).

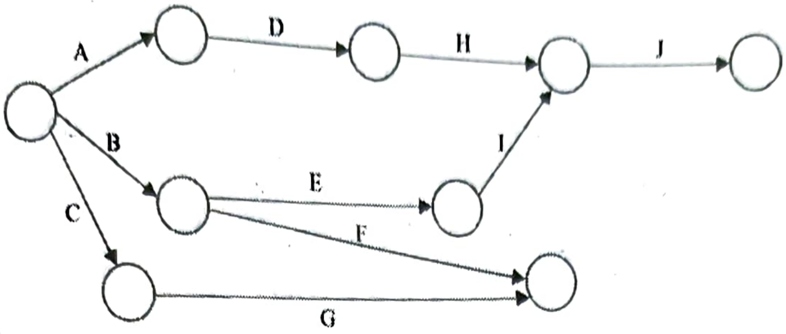












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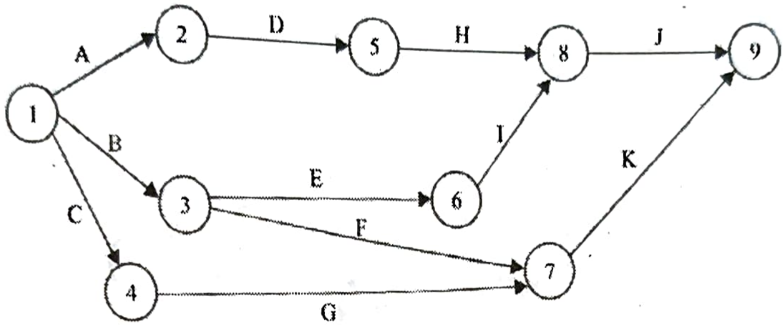
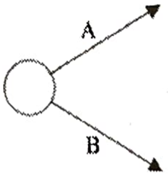


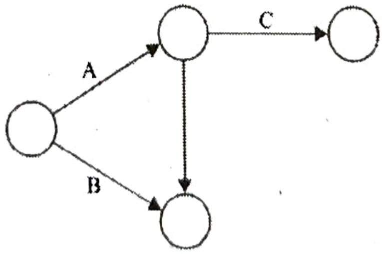
Example 15.3 $A < C, D, I; B < G, F; D < G, F; F < H, K; G, H < J; I, J, K < E$

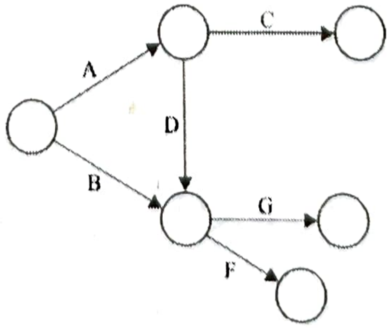
Solution Given $A < C$, which means that C cannot be started until A is completed. That is, A is the preceding activity to C. The above constraints can be given in the following table.

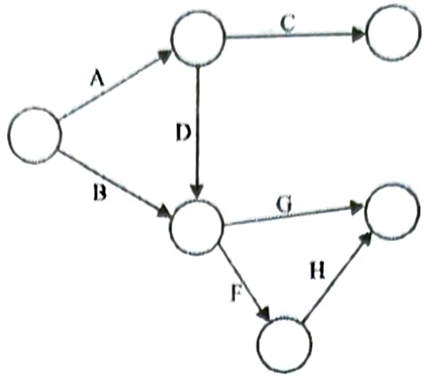
Activity	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
Predecessor	–	–	*A*	*A*	*I, J, K*	*B, D*	*B, D*	*F*	*A*	*G, H*	*F*

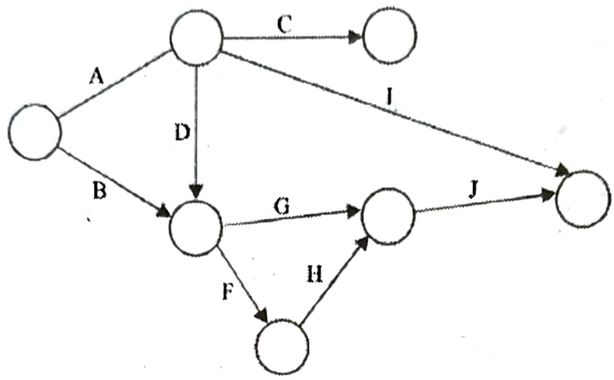
A and *B* are the starting activities, and *E* is the terminal activity.

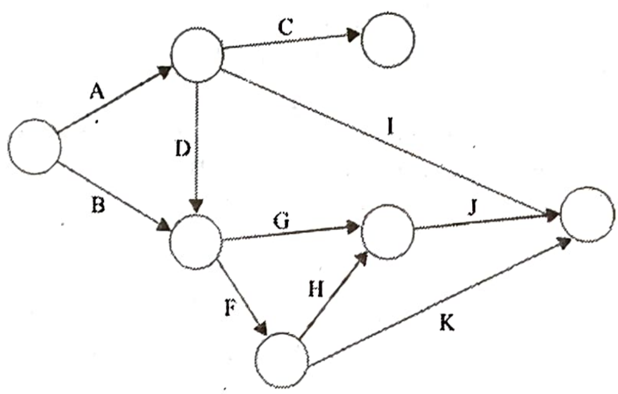




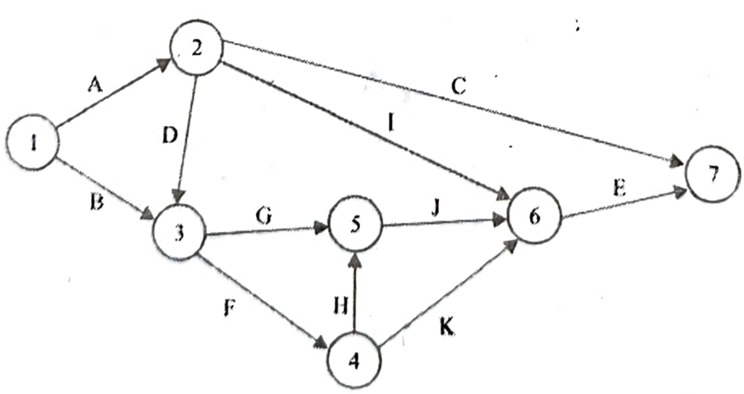








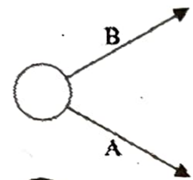
Finally, we have,

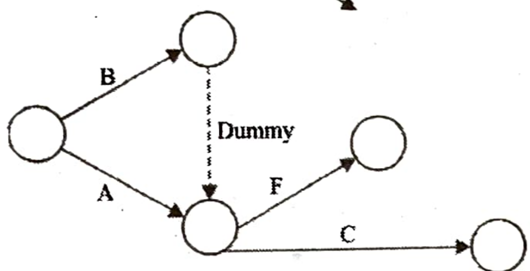


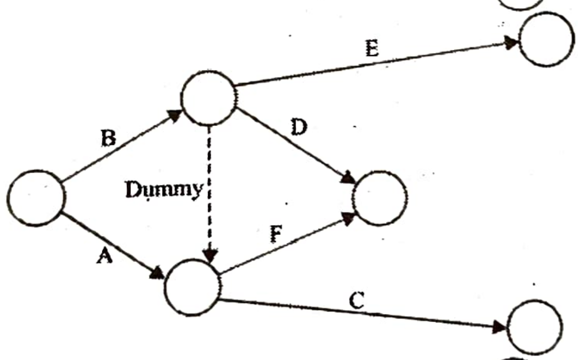
Example 15.4

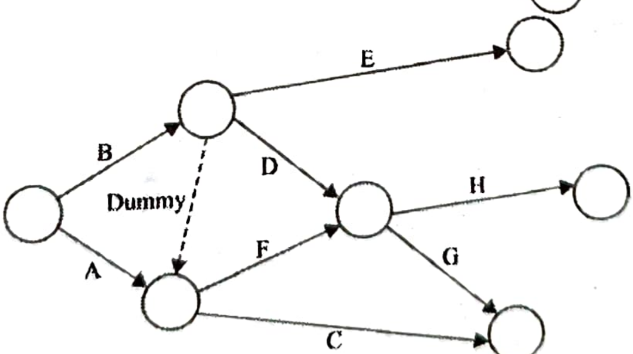
Activities	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Immediate Predecessor	–	–	*A*, *B*	*B*	*B*	*A*, *B*	*F*, *D*	*F*, *D*	*C*, *G*

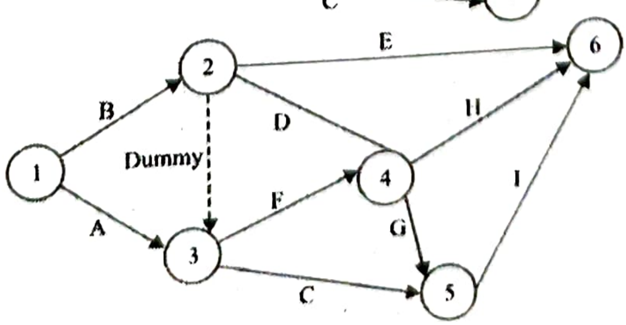
Solution *A* and *B* are concurrent activities as they start simultaneously. *I* is the terminal activity. Since the activities *C* and *F* are coming from both activities *A* and *B*, we need to introduce a dummy activity.









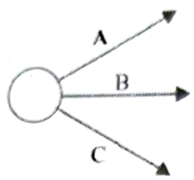


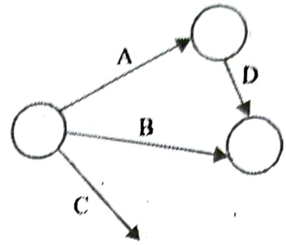
Example 15.5 A, B and C can start simultaneously

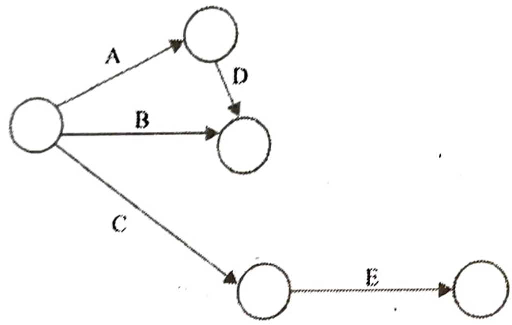
$A < D, I;\ B < G, F;\ D < G, F;\ C < E;\ E < H, K;\ F < H, K;\ G, H < J.$

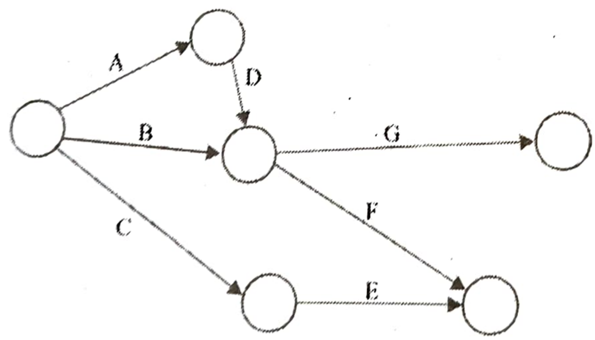
Solution The above constraints can be formatted into a table.

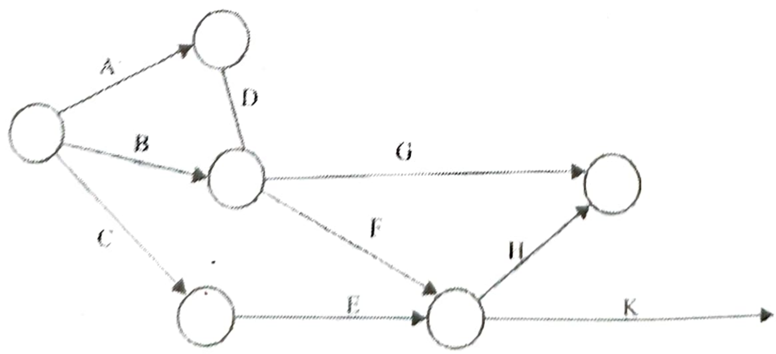
Activity	A	B	C	D	E	F	G	H	I	J	K
Predecessor Activity	–	–	–	A	C	B, D	B, D	E, F	A	G,	E, F

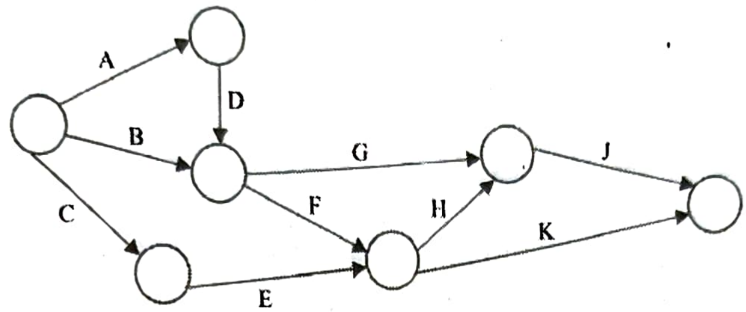


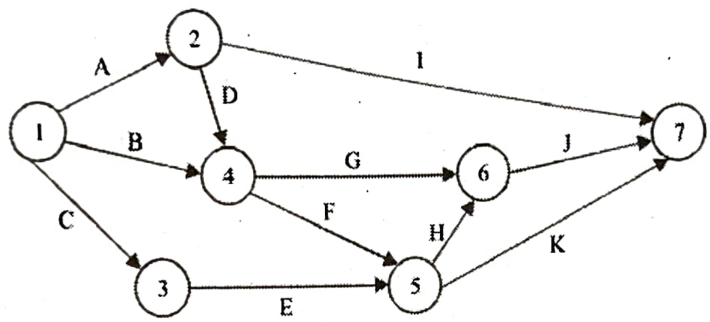












15.6 TIME ANALYSIS

Once the network of a project is constructed, the time analysis of the network becomes essential for planning various activities of the project. Activity time is a forecast of the time for an activity which is expected to take from its starting point to its completion (under normal conditions).

We shall use the following notation for basic scheduling computations.

(i,j) = Activity (i, j) with tail event i and head event j
T_{ij} = Estimated completion time of activity (i, j)
ES_{ij} = Earliest starting time of activity (i, j)
EF_{ij} = Earliest finishing time of activity (i, j)
LS_{ij} = Latest starting time of activity (i, j)
LF_{ij} = Latest finishing time of activity (i, j).

The basic scheduling computation can be put under the following three groups.

15.6.1 Forward Pass Computations (For Earliest Event Time)

Before starting computations, the occurrence time of the initial network event is fixed. The forward pass computation yields the earliest start and the earliest finish time for each activity (i, j) and indirectly the earliest occurrence time for each event namely E_i. This consists of the following three steps:

Step 1 The computations begin from the start node and move towards the end node. Let zero be the starting time for the project.

Step 2 Earliest starting time $(ES)_{ij} = E_i$ is the earliest possible time when an activity can begin, assuming that all of the predecessors are also started at their earliest starting time. Earliest finish time of activity (i,j) is the earliest starting time + the activity time.

$$(EF)_{ij} = (ES)_{ij} + t_{ij}$$

Step 3 Earliest event time for event j is the maximum of the earliest finish time of all the activities, ending at that event.

$$E_j = \operatorname*{Max}_i \, (E_i + t_{ij})$$

The computed 'E' values are put over the respective rectangles representing each event.

15.6.2 Backward Pass Computations (For Latest Allowable Time)

The latest event time (L) indicates the time by which all activities entering into that event must be completed without delaying the completion of the project. These can be calculated by reversing the method of calculations used for the earliest event time. This is done in the following steps.

Step 1 For ending event assume, $E = L$.

Step 2 Latest finish time for activity (i, j) is the target time for completing the project,

$$(LF_{ij}) = L_j$$

Step 3 Latest starting time of the activity (i, j) = latest completion time of (i, j) – the activity time

$$LS_{ij} = LF_{ij} - t_{ij}$$
$$= L_j - t_{ij}$$

Step 4 Latest event time for event i is the minimum of the latest start time of all activities originating from the event.

$$\boxed{L_i = \operatorname*{Min}_j (L_j - t_{ij})}$$

The computed 'L' values are put over the respective triangles representing each event.

15.6.3 Determination of Floats and Slack Times

Float is defined as the difference between the latest and the earliest activity time.

Slack is defined as the difference between the latest and the earliest event time.

Hence, the basic difference between the slack and float is that slack is used for events only; whereas float is used for activities.

There are mainly three kinds of floats as given below.

Total float It refers to the amount of time by which the completion of an activity could be delayed beyond the earliest expected completion time, without affecting the overall project duration time.

Mathematically, the total float of an activity (i, j) is the difference between the latest start time and the earliest start time of that activity.

Hence, the total float for an activity (i, j) denoted by $(TF)_{ij}$ is calculated by the formula,
$(TF)_{ij}$ = (Latest start – Earliest start) for activity (i, j)

$$\boxed{\begin{aligned} &\text{i.e., } (TF)_{ij} = (LS)_{ij} - (ES)_{ij} \\ &\text{or } (TF)_{ij} = (L_j - E_i) - t_{ij} \end{aligned}}$$

where, E_i and L_j are the earliest time and latest time for the tail event i and head event j and t_{ij} is the normal time for the activity (i, j). This is the most important type of float as it concerns the overall project duration.

Free float The time by which the completion of an activity can be delayed beyond the earliest finish time, without affecting the earliest start of a subsequent succeeding activity.

Mathematically, the free float for activity (i, j) denoted by $(FF)_{ij}$ can be calculated by the formula,

$$FF_{ij} = (E_j - E_i) - t_{ij}$$
$$(FF)_{ij} = \text{Total float} - \text{Head event slack}$$

Head event slack $= L_j - E_j$

This float is concerned with the commencement of the subsequent activity.

The free float can take values from zero up to total float, but it cannot exceed total float. This float is very useful for rescheduling an activity with minimum disruption in earlier plans.

Independent float The amount of time by which the start of an activity can be delayed, without affecting the earliest start time of any immediately following activities, assuming that the preceding activity has finished at its latest finish time.

Mathematically, independent float of an activity (i, j) denoted by $(IF)_{ij}$ can be calculated by the formula,

$$IF_{ij} = (E_j - L_i) - t_{ij}$$
or
$$(IF)_{ij} = \text{Free Float} - \text{Tail event slack}$$

where tail event slack is given by,

$$\text{Tail event slack} = L_i - E_i$$

The negative independent float is always taken as zero. This float is concerned with prior and subsequent activities.

$$IF_{ij} \le FF_{ij} \le TF_{ij}$$

Notes: (*i*) If the total float TF_{ij} for any activity (i, j) is zero, then such an activity is called *critical activity*.

(*ii*) The float can be used to reduce project duration. While doing this, the float of not only that activity, but that of other activities will also change.

Critical activity An activity is said to be critical, if a delay in its start cause a further delay in the completion of the entire project.

Critical path The sequence of critical activities in a network which determines the duration of a project is called the critical path. It is the longest path in the network, from the starting event to the ending event and defines the minimum time required to complete the project. In the network it is denoted by a double line and identifies all the critical activities of the project. Hence, for the activities (i, j) to lie on the critical path, following conditions must be satisfied.

(*a*) $ES_i = LF_i$
(*b*) $ES_j = LF_j$
(*c*) $ES_j - ES_i = LF_j - LF_i = t_{ij}$

ES_i and ES_j are the earliest start and finish time of the events i and j.
LF_i and LF_j are the latest start and finish time of the events i and j.

15.7 CRITICAL PATH METHOD (CPM)

The critical path method (CPM) is a step-by-step procedure for scheduling the activities in a project. It is an important tool related to effective project management. The iterative procedure of determining the critical path is as follows:

Step 1 List all the jobs and then draw an arrow (network) diagram. Each job is indicated by an arrow with the direction of the arrow showing the sequence of jobs. The length of the arrows has no significance. The arrows are placed based on the predecessor, successor and concurrent relation within the job.

Step 2 Indicate the normal time (t_{ij}) for each activity (i, j) above the arrow, which is deterministic.

Step 3 Calculate the earliest start time and the earliest finish time for each event and write the earliest time E_i for each event i in the $\square$. Also calculate the latest finish and latest start time. From this we calculate the latest time L_j for each event j and put it in the Δ.

Step 4 Tabulate the various times, namely, normal time, earliest time and latest time on the arrow diagram.

Step 5 Determine the total float for each activity by taking the difference between the earliest start and the latest start time.

Step 6 Identify the critical activities and connect them with the beginning and the ending events in the network diagram by double line arrows. This gives the critical path.

Step 7 Calculate the total project duration.

Note: The earliest start and finish time of an activity, as well as the latest start and finish time of an activity are shown in the table. These are calculated by using the following hints.

To find the earliest time, we consider the tail event of the activity. Let the starting time of the project namely $ES_i = 0$. Add the normal time with the starting time, to get the earliest finish time. The earliest starting time for the tail event of the next activity is given by the maximum of the earliest finish time for the head event of the previous activity.

Similarly, to get the latest time, we consider the head event of the activity.

The latest finish time of the head event of the final activity is given by the target time of the project. The latest start time can be obtained by subtracting the normal time of that activity. The latest finish time for the head event of the next activity is given by the minimum of the latest start time for the tail event of the previous activity.

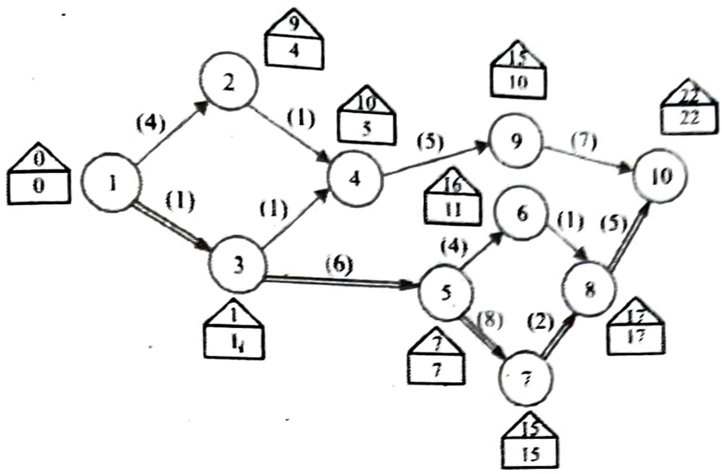
Example 15.6 A project schedule has the following characteristics.

Activity	*1–2*	*1–3*	*2–4*	*3–4*	*3–5*	*4–9*	*5–6*	*5–7*	*6–8*	*7–8*	*8–10*	*9–10*
Time (days)	4	1	1	1	6	5	4	8	1	2	5	7

From the above information, you are required to:

1. Construct a network diagram.
2. Compute the earliest event time and latest event time.
3. Determine the critical path and total project duration.
4. Compute total and free float for each activity.

Solution First we construct the network with the given constraints (here we get it by just connecting the event numbers).



The following table gives the critical path as well as total and free floats calculation.

Activity	Normal time	Earliest		Latest		TF	FF
		Start	Finish	Start	Finish		
1 – 2	4	0	4	5	9	5	5 – 5 = 0
1 – 3	1	0	1	0	1	0	0
2 – 4	1	4	5	9	10	5	0
3 – 4	1	1	2	9	10	8	3
3 – 5	6	1	7	1	7	0	0
4 – 9	5	5	10	10	15	5	0
5 – 6	4	7	11	12	16	5	0
5 – 7	8	7	15	7	15	0	0
6 – 8	1	11	12	16	17	5	0
7 – 8	2	15	17	15	17	0	0
8 – 10	5	17	22	17	22	0	0
9 – 10	7	10	17	15	22	5	5

The earliest and latest calculations are shown below.

Forward pass calculation In this we estimate the earliest start (Es_i) and finish times (Es_j). The earliest time for the event i is given by,

$$E_i = \underset{i}{\text{Max}}\,(ES_i + t_{ij})$$

$$ES_1 = 0 = E_1 = 0$$

$$E_2 = ES_2 = ES_1 + t_{12} = 0 + 4 = 4$$

$$E_3 = ES_3 = ES_1 + t_{13} = 0 + 1 = 1$$

$$E_4 = ES_4 = \text{Max}\,(ES_3 + t_{34}, ES_2 + t_{24})$$

$$= \text{Max}\,(1 + 1, 4 + 1) = 5$$

$$E_5 = (E_3 + t_{35}) = 1 + 6 = 7$$

$$E_6 = E_5 + t_{56} = 7 + 4 = 11$$

$$E_7 = E_5 + t_{57} = 7 + 8 = 15$$

$$E_8 = \text{Max}\,(E_6 + t_{68}, E_7 + t_{78})$$
$$= \text{Max}\,(11 + 1, 15 + 2) = 17$$
$$E_9 = E_4 + t_{49} = 5 + 5 = 10$$
$$E_{10} = \text{Max}\,(E_9 + t_{9,10}, E_8 + t_{8,10})$$
$$= \text{Max}\,(10 + 7, 17 + 5) = 22.$$

Backward pass calculation In this we calculate the latest finish and the latest start time. The latest time L for an event i is given by $L_i = \underset{j}{\text{Min}}\ (LF_j - t_{ij})$, where, LF_j is the latest finish time for the event j, t_{ij} is the normal time of the activity.

$$L_{10} = 22$$
$$L_9 = L_{10} - t_{9,10} = 22 - 7 = 15$$
$$L_8 = L_{10} - t_{8,10} = 22 - 5 = 17$$
$$L_7 = L_8 - t_{7,8} = 17 - 2 = 15$$
$$L_6 = L_8 - t_{6,8} = 17 - 1 = 16$$
$$L_5 = \text{Min}\,(L_6 - t_{5,6}, L_7 - t_{5,7})$$
$$= \text{Min}\,(16 - 4, 15 - 8) = 7$$
$$L_4 = L_9 - t_{4,9} = 15 - 5 = 10$$
$$L_3 = \text{Min}\,(L_4 - t_{3,4}, L_5 - t_{3,5})$$
$$= \text{Min}\,(10 - 1, 7 - 6) = 1$$
$$L_2 = L_4 - t_{2,4} = 10 - 1 = 9$$
$$L_1 = \text{Min}\,(L_2 - t_{12}, L_3 - t_{13}) = \text{Min}\,(9 - 4, 1 - 1) = 0.$$

These calculations are shown in the above table.

To find the TF (Total Float) Considering the activity 1 – 2, *TF* of (1 – 2) = Latest start–Earliest start

$$= 5 - 0 = 5$$

Similarly $TF\,(2 - 4) = LS - ES$

$$= 9 - 4 = 5$$

Free float = *TF* – Head event slack.

Consider the activity 1 – 2

FF of (1 – 2) = *TF* of (1 – 2) – Slack for the head event 2

= 5 – (9 – 4) (from the figure for event 2)

= 5 – 5 = 0

FF of (2 – 4) = *TF* of (2 – 4) – Slack for the head event 4

= 5 – (10 – 5) = 5 – 5 = 0

Like this we calculate the *TF* and *FF* for the remaining activities.

From the above table we observe that the activities 1 – 3, 3 – 5, 5 – 7, 7 – 9, 8 – 10 are the critical activities as their total float is 0.

Hence, we have the following critical path.

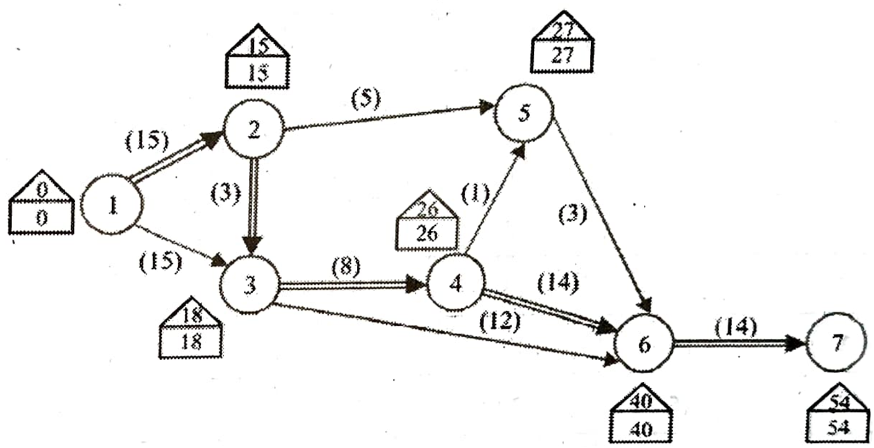
1 – 3 – 5 – 7 – 8 – 10, with the total project duration of 22 days.

Example 15.7 A small maintenance project consists of the following jobs, whose precedence relationships are given below.

Job	*1–2*	*1–3*	*2–3*	*2–5*	*3–4*	*3–6*	*4–5*	*4–6*	*5–6*	*6–7*
Duration (days)	15	15	3	5	8	12	1	14	3	14

1. Draw an arrow diagram representing the project.
2. Find the total float for each activity.
3. Find the critical path and the total project duration.

Solution



Forward pass calculation In this we estimate the earliest start and the earliest finish time ES_j given by,

$ES_j = \underset{i}{\text{Max}} \, (ES_i + t_{ij})$ where, ES_i is the earliest start time and t_{ij} is the normal time for the activity (i, j).

$$ES_1 = 0$$
$$ES_2 = ES_1 + t_{15} = 0 + 15 = 15$$
$$ES_3 = \text{Max}\,(ES_2 + t_{23}, ES_1 + t_{13})$$
$$= \text{Max}\,(15 + 3, 0 + 15) = 18$$
$$ES_4 = ES_3 + t_{34} = 18 + 8 = 26$$
$$ES_5 = \text{Max}\,(ES_2 + t_{25}, ES_4 + t_{45})$$
$$= \text{Max}\,(15 + 5, 26 + 1) = 27$$
$$ES_6 = \text{Max}\,(ES_3 + t_{36}, ES_4 + t_{46}, ES_5 + t_{56})$$
$$= \text{Max}\,(18 + 12, 26 + 14, 27 + 3)$$
$$= 40$$
$$ES_7 = ES_6 + t_{67} = 40 + 14 = 54.$$

Backward pass calculation In this we calculate the latest finish and latest start time LF_i, given by $LF_i = \underset{i}{\text{Min}} \, (LF_j - t_{ij})$ where, LF_j is the latest finish time for the event j

$$LF_7 = 54$$
$$LF_6 = LF_7 - t_{67} = 54 - 14 = 40$$

$$LF_5 = LS_6 - t_{56} = 40 - 3 = 37$$

$$LF_4 = \text{Min}\,(LS_5 - t_{45}, LS_6 - t_{46})$$
$$= \text{Min}\,(37 - 1, 40 - 14) = 26$$

$$LF_3 = \text{Min}\,(LF_4 - t_{34}, LF_6 - t_{36})$$
$$= \text{Min}\,(26 - 8, 40 - 12) = 18$$

$$LF_2 = \text{Min}\,(LF_5 - t_{25}, LF_3 - t_{23})$$
$$= \text{Min}\,(37 - 5, 18 - 3) = 15$$

$$LF_1 = \text{Min}\,(LF_3 - t_{13}, LF_2 - t_{12})$$
$$= \text{Min}\,(18 - 15, 15 - 15) = 0$$

The following table gives the calculations for critical path and total float.

Activity	Normal time	Earliest		Latest		Total float
		Start	Finish	Start	Finish	$LF_j - ES_j$ or $LF_i - ES_i$
		ES_i	ES_j	LF_i	LF_j	
1 – 2	15	0	15	0	15	0
1 – 3	15	0	15	3	18	3
2 – 3	3	15	18	15	18	0
2 – 5	5	15	20	32	37	17
3 – 4	8	18	26	18	26	0
3 – 6	12	18	30	28	40	10
4 – 5	1	26	27	36	37	10
4 – 6	14	26	40	26	40	0
5 – 6	3	27	30	37	40	10
6 – 7	14	40	54	40	54	0

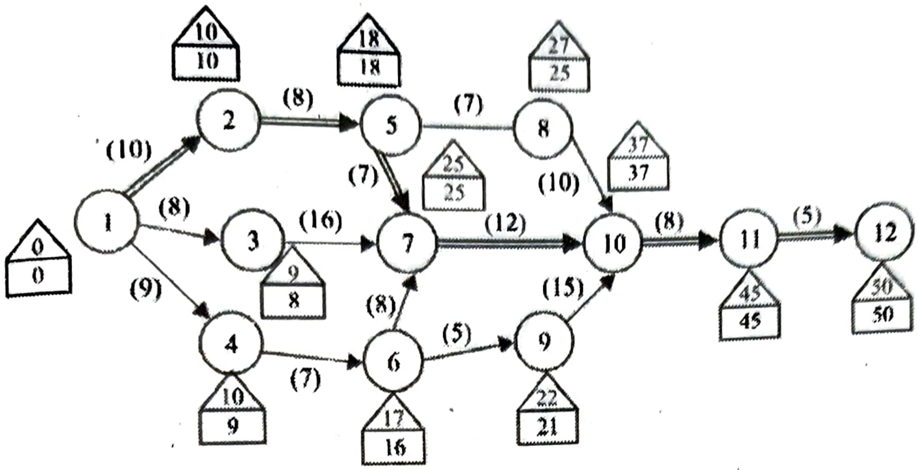
From the above table, we observe that the activities 1—2, 2—3, 3—4, 4—6, 6—7 are the critical activities and the critical path is given by, 1—2—3—4—6—7.

The total time taken for project completion is 54 days.

Example 15.8 The following table shows the jobs of a project with their duration in days. Draw the network and determine the critical path. Also calculate all the floats.

Jobs	1-2	1-3	1-4	2-5	3-7	4-6	5-7	5-8	6-7	6-9	7-10	8-10	9-10	10-11	11-12
Duration	10	8	9	8	16	7	7	7	8	5	12	10	15	8	5

Solution First we construct the network as shown below:



Forward pass calculation In this we calculate the earliest start and the earliest finish time for the activity and the earliest time for each event.

$$
\begin{aligned}
ES_1 &= 0 \\
ES_2 &= ES_1 + t_{12} = 0 + 10 = 10 \\
ES_3 &= ES_1 + t_{13} = 0 + 8 = 8 \\
E_4 &= E_1 + t_{14} \\
&= 0 + 9 = 9 \\
E_5 &= E_2 + t_{25} = 10 + 8 = 18 \\
E_6 &= E_4 + t_{46} = 9 + 7 = 16 \\
E_7 &= \text{Max}\,(E_3 + t_{37}, E_5 + t_{57}, E_6 + t_{67}) \\
&= \text{Max}\,(8 + 16, 16 + 8, 18 + 7) = 25 \\
E_8 &= E_5 + t_{58} = 18 + 7 = 25 \\
E_9 &= E_6 + t_{69} = 16 + 5 = 21 \\
E_{10} &= \text{Max}\,(E_7 + t_{7,10}, E_8 + t_{8,10}, E_9 + t_{9,10} \\
&= \text{Max}\,(25 + 12, 25 + 10, 21 + 15) = 37 \\
E_{11} &= E_{10} + t_{10,11} = 37 + 8 = 45 \\
E_{12} &= E_{11} + t_{11,12} = 45 + 5 = 50
\end{aligned}
$$

Backward pass calculation In this we calculate the latest finish and the latest start time LF_i, given by,

$$
\begin{aligned}
LF_i &= \text{Min}\,(LF_j - t_{ij}) \\
L_{12} &= E_{12} = 50 \text{ (the target completion time)} \\
L_{11} &= L_{12} - t_{11,12} = 50 - 5 = 45 \\
L_{10} &= L_{11} - t_{10,11} = 45 - 8 = 37 \\
L_9 &= L_{10} - t_{9.10} = 37 - 15 = 22 \\
L_8 &= L_{10} - t_{8.10} = 37 - 10 = 27 \\
L_7 &= L_{10} - t_{7.10} = 37 - 12 = 25 \\
L_6 &= \text{Min}\,(L_9 - t_{69}, L_7 - t_7 - t_{67}) \\
&= \text{Min}\,(22 - 5, 25 - 8) = 17 \\
L_5 &= \text{Min}\,(L_8 - t_{58}, L_7 - t_{57}) \\
&= \text{Min}\,(27 - 7, 25 - 7) = 18
\end{aligned}
$$

$$L_4 = L_6 - t_{46} = 17 - 7 = 10$$
$$L_3 = L_7 - t_{37} = 25 - 16 = 9$$
$$L_2 = L_5 - t_{25} = 18 - 10 = 10$$
$$L_1 = \text{Min}\,(L_4 - t_{14}, L_3 - t_{13}, L_2 - t_{12})$$
$$= \text{Min}\,(10 - 9, 9 - 8, 10 - 10) = 0.$$

Computations of the critical path and all the floats are given in the following table:

Activity	Normal time	Earliest		Latest		Floats		
		Start	Finish	Start	Finish	TF	FF	IF
1–2	10	0	10	0	10	0	0	0
1–3	8	0	8	1	9	1	0	0
1–4	9	0	9	1	10	1	0	0
2–5	8	10	18	10	18	0	0	0
3–7	16	8	24	9	25	1	1	0
4–6	7	9	16	10	17	1	0	−1=0
5–7	7	18	25	18	25	0	0	0
5–8	7	18	25	20	27	2	0	0
6–7	8	16	24	17	25	1	1	0
6–9	5	16	21	17	22	1	0	−1=0
7–10	12	25	37	25	37	0	0	0
8–10	10	25	35	27	37	2	2	0
9–10	15	21	36	22	37	1	1	0
10–11	8	37	45	37	45	0	0	0
11–12	5	45	50	45	50	0	0	0

TF = Total float = $LS - ES$ or $LF - EF$

FF = Free float = TF – Head event slack = $TF - (L_j - E_j)$

IF = Independent float = FF – Tail event slack = $FF - (L_i - E_i)$

From the above calculation, we observe that the activity 1–2, 2–5, 5–7, 7–10, 10–11, 11–12 are the critical activities as their total float is 0. Hence, we have the critical path 1–2–5–7–10–11–12, with the total project duration as 50 days.

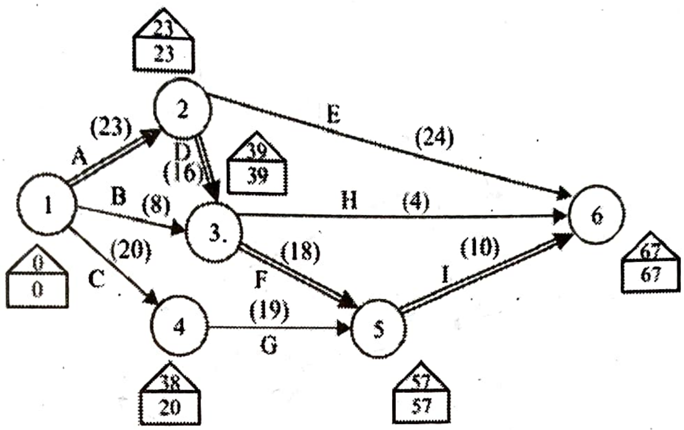
Example 15.9 A project consists of a series of tasks labelled $A, B... H, I$ with the following constraints, $A < D, E; B, D < F; C < G; B < H; F, G < I; W < X, Y$ means X and Y cannot start until W is completed. You are required to construct a network using this notation. Also find the minimum time of completion of the project when the time of completion of each task is given as follows:

Task	A	B	C	D	E	F	G	H	I
Time (days)	23	8	20	16	24	18	19	4	10

Solution The constraints can be given as in the following table:

Activity	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Preceding Activity	–	–	–	*A*	*A*	*B*, *D*	*C*	*B*	*F*, *G*

To determine the minimum time of completion of the project, we compute ES_{ij} and LF_{ij} for each of the tasks (i, j) of the project. The critical path calculations are as follows.



Task	*Normal time* t_{ij}	*Earliest*		*Latest*		*Total floats*
		Start	*Finish*	*Start*	*Finish*	
A 1–2	23	0	23	0	23	0
B 1–3	8	0	8	31	39	8
C 1–4	20	0	20	18	38	18
D 2–3	16	23	39	23	39	0
E 2–6	24	23	47	43	67	20
F 3–5	18	39	57	39	57	0
H 3–6	4	39	43	63	67	24
G 4–5	19	20	39	38	57	18
I 5–6	10	57	67	57	67	0

The above table shows that the critical activities are 1–2, 2–3, 3–5, 5–6 as their total float is zero. Hence, we have the critical path, 1–2–3–5–6, with the total project duration (the least possible time to complete the entire project) as 67 days.

Example 15.10 Tasks $A, B, \ldots H, I$ constitute a project. The notation $X < Y$ means that the task X must be completed before Y is started. With the notation,

$$A<D, A<E, B<F, D<F, C<G, C<H, F<I, G<I$$

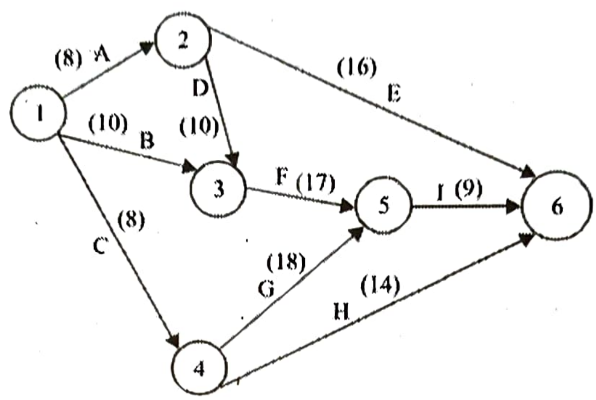
Draw a graph to represent the sequence of tasks and find the minimum time of completion of the project, when the time (in days) of completion of each task is as follows.

The above constraints can be given as in the following table:

Task	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Time (days)	8	10	8	10	16	17	18	14	9

Solution The above constraints are given in the following table:

Activity	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Preceding Activity	–	–	–	*A*	*A*	*B*, *D*	*C*	*C*	*F*, *G*



Time calculation Using forward and backward pass calculation, we first estimate the earliest and the latest time for each event.

$$
\begin{aligned}
ES_1 &= E_1 = 0\\
E_2 &= E_1 + t_{12} = 0 + 8 = 8\\
E_3 &= \text{Max}\,(E_1 + E_{13}, E_2 + t_{23})\\
&= \text{Max}\,(0 + 10, 8 + 10) = 18\\
E_4 &= E_1 + t_{14} = 0 + 8 = 8\\
E_5 &= \text{Max}\,(E_3 + t_{35}, E_4 + t_{45})\\
&= \text{Max}\,(18 + 17, 8 + 18) = 35\\
E_6 &= \text{Max}\,(E_2 + t_{26}, E_4 + t_{46}, E_5 + t_{56})\\
&= \text{Max}\,(8 + 16, 8 + 14, 35 + 9) = 44
\end{aligned}
$$

The value of the latest time can now be obtained.

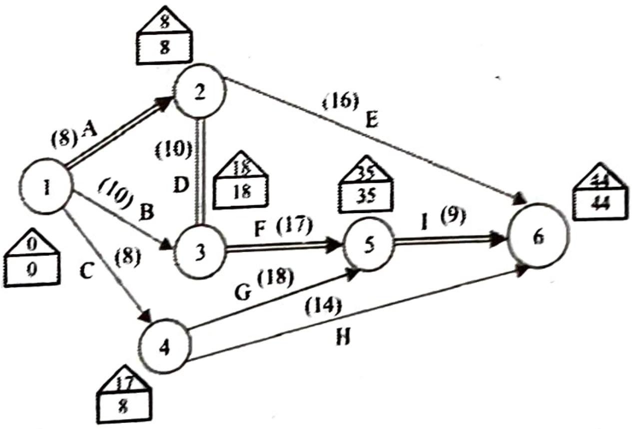
$L_6 = E_6 = 44$ (Target completion time for the project)

$$
\begin{aligned}
L_5 &= L_6 - t_{56} = 44 - 9 = 35\\
L_4 &= \text{Min}\,(L_6 - t_{46}, L_5 - t_{45})\\
&= \text{Min}\,(44 - 14, 35 - 18) = 17\\
L_3 &= L_5 - t_{35} = 35 - 17 = 18\\
L_2 &= \text{Min}\,(L_6 - t_{26}, L_3 - t_{23})\\
&= \text{Min}\,(44 - 16, 18 - 10 = 8\\
L_1 &= \text{Min}\,(L_4 - t_{14}, L_3 - t_{13}, L_2 - t_{12})\\
&= \text{Min}\,(17 - 8, 18 - 10, 8 - 8) = 0.
\end{aligned}
$$

To evaluate the critical events, all these calculations are put in the following table.

Task	Normal Time/days	Earliest		Latest		Floats		
		Start	Finish	Start	Finish	TF	FP	IF
A 1-2	8	0	8	0	8	0	$0-0=0$	$0-0=0$
B 1-3	10	0	10	8	18	8	$8-0=8$	$8-0=8$
C 1-4	8	0	8	9	17	9	$9-9=0$	$0-0=0$
D 2-3	10	8	18	8	18	0	$0-0=0$	$0-0=0$
E 2-6	16	8	24	28	44	20	$20-0=20$	$20-0=20$
F 3-5	17	18	35	18	35	0	$0-0=0$	$0-0=0$
G 4-5	18	8	26	17	35	9	$9-0=9$	$9-9=0$
H 4-6	14	8	22	30	44	22	$22-0=22$	$22-9=13$
I 5-6	9	35	44	35	44	0	$0-0=0$	$0-0=0$

The above table shows that the critical events are the tasks $1-2, 2-3, 3-5, 5-6$ as their total float is zero.



The critical path is given by 1–2–3–5–6 or *A–D–F–I*, with the total project duration as 44 days.

EXERCISES

1. The following table gives the activities and duration of a construction project.

Activity	1–2	1–3	2–3	2–4	3–4	4–5
Duration (days)	20	25	10	12	6	10

(*i*) Draw the network for the project

(*ii*) Find the critical path. [Ans. CPM: 1-2-3-4-5]

2. A small project consists of 11 activities $A, B, C \ldots K$. According to the precedence relationship A and B can start simultaneously, given $A < C, D, I$; $B < G, F$; $D < G, F$; $F < H, K$; $G, H < J$; $I, J, K < E$. The duration of the activities are as follows.

Activity	A	B	C	D	E	F	G	H	I	J	K
Duration (days)	5	3	10	2	8	4	5	6	12	8	9

Draw the network of the project. Summarise the CPM calculations in a tabular form computing the total and free floats of activities as well as determine the critical path. [**Ans.** Critical path A–D–F–H–J–E Project duration 33 days]

3. Draw the network and determine the critical path for the given data. Also calculate all the floats involved in CPM.

Jobs	1–2	1–3	2–4	3–4	3–5	4–5	4–6	5–6
Duration	6	5	10	3	4	6	2	9

[**Ans.** Critical path 1–2–4–5–6 Project duration 31 days]

4. A small maintenance project consists of the following 12 jobs.

Jobs	1–2	2–3	2–4	3–4	3–5	4–6	5–8	6–7	6–10	7–9	8–9	9–10
Duration (days)	2	7	3	3	5	3	5	8	4	4	1	7

Draw the arrow network of the project. Summarise CPM calculations in a tabular form, calculating the three types of floats and hence determine the critical path. [**Ans.** Critical path 1–2–3–4–5–6–7–9–10]

5. Consider the following data for activities in a given project.

Activity	A	B	C	D	E	F
Predecessor	–	A	–	B, C	C	D, E
Time (days)	5	4	7	3	4	2

Draw an arrow diagram for the project. Compute the earliest and the latest event times. What is the minimum project completion time? List the activities on the critical path.

[**Ans.** $A \rightarrow B \rightarrow E \rightarrow$ dummy $\rightarrow F$; minimum completion time 15 days]

6. For the following project, determine the critical path and its duration.

Activity	A	B	C	D	E	F	G	H
Predecessors	–	A	A	B	B	D, E	D	C, F, G
Time (days)	2	4	8	3	2	3	4	8

[**Ans.** 1–2–3–4–6–7; Project duration 21 days]

7. A project has the following time schedule.

Activity	1–2	1–3	1–4	2–5	3–6	3–7	4–6	5–8	6–9	7–8	8–9
Duration (months)	2	2	1	4	8	5	3	1	5	4	3

Construct the network and compute,

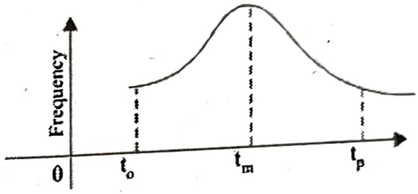
(*i*) Total float for each activity.

(*ii*) Critical path and its duration. [**Ans.** 1–3–6–9; Project duration 15 months]

15.8 PROGRAMME EVALUATION AND REVIEW TECHNIQUE (PERT)

The network methods discussed so far may be termed as deterministic, since estimated activity times are assumed to be known with certainty. However, in the research project or design of a gear box or a new machine, various activities are based on judgement. It is difficult to obtain a reliable time estimate due to the changing technology since time values are subject to chance variations. For such cases, where the activities are non-deterministic in nature, PERT was developed. Hence, PERT is a probabilistic method, where the activity times are represented by a probability distribution. This distribution of activity times is based on three different time estimates made for each activity, which are as follows:

(*i*) Optimistic time estimate (*ii*) Most likely time estimate
(*iii*) Pessimistic time estimate



Time distribution curve

Optimistic time estimate It is the smallest time taken to complete the activity, if everything goes well. There is very little chance that an activity can be completed in a time less than the optimistic time. It is denoted by t_o or a.

Most likely time estimate It refers to the estimate of the normal time the activity would take. This assumes normal delays. It is the mode of the probability distribution. It is denoted by t_m or m.

Pessimistic time estimate It is the longest time that an activity would take, if everything goes wrong. It is denoted by t_p or b. These three time values are shown in the following figure.

From these three time estimates, we have to calculate the expected time of an activity. It is given by the weighted average of the three time estimates,

$$\boxed{t_e = \frac{t_o + 4t_m + t_p}{6}}$$

[β distribution with weights of 1, 4 and 1, for t_o, t_m and t_p estimates respectively.]

Variance of the activity is given by,

$$\sigma^2 = \left[\frac{t_p - t_o}{6}\right]^2$$

The expected length (duration), denoted by T_c of the entire project is the length of the critical path, i.e., the sum of the t_e's of all the activities along the critical path.

The main objective of the analysis through PERT is to find the completion date for a particular event within the specified date T_s, given by $P(Z \le D)$ where,

$$D = \frac{\text{Due date} - \text{Expected date of completion}}{\sqrt{\text{Project variance}}}$$

Here, Z stands for standard normal variable.

15.8.1 PERT Procedure

Step 1 Draw the project network.

Step 2 Compute the expected duration of each activity using the formula,

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

Also calculate the expected variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$ of each activity.

Step 3 Compute the earliest start, earliest finish, latest start, latest finish time and total float of each activity.

Step 4 Find the critical path and identify the critical activities.

Step 5 Compute the project length variance σ^2, which is the sum of the variance of all the critical activities and hence, find the standard deviation of the project length σ.

Step 6 Calculate the standard normal variable $Z = \dfrac{T_s - T_e}{\sigma}$, where T_s is the scheduled time to complete the project.

T_e = Normal expected project length duration.

σ = Expected standard deviation of the project length.

Using the normal curve, we can estimate the probability of completing the project within a specified time.

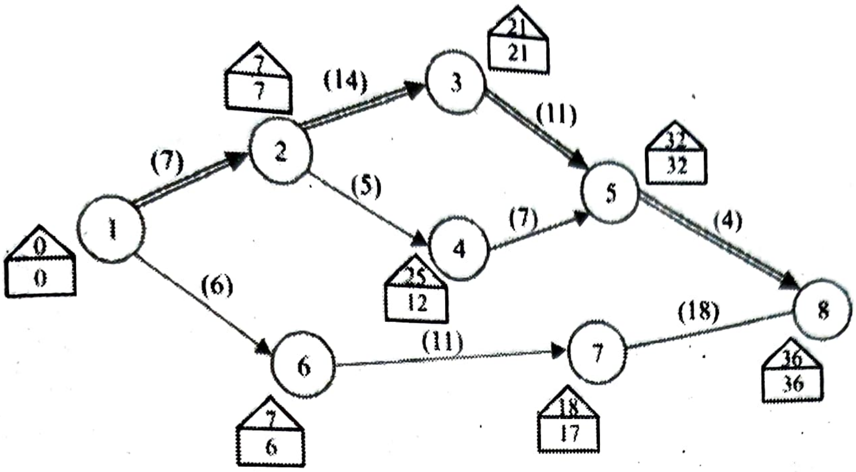
Example 15.11 The following table shows the jobs of a network along with their time estimates.

Job	*1–2*	*1–6*	*2–3*	*2–4*	*3–5*	*4–5*	*6–7*	*5–8*	*7–8*
a (days)	1	2	2	2	7	5	5	3	8
m (days)	7	5	14	5	10	5	8	3	17
b (days)	13	14	26	8	19	17	29	9	32

Draw the project network and find the probability of the project completing in 40 days.

Solution First we calculate the expected time and standard deviation for each activity.

Activity	$t_e = \dfrac{t_o + 4t_m + t_p}{6}$	$\sigma^2 = \left(\dfrac{t_p - t_o}{6}\right)^2$
1–2	$\dfrac{1 + (4 \times 7) + 13}{6} = 7$	$\left(\dfrac{13-1}{6}\right)^2 = 4$
1–6	$\dfrac{2 + (4 \times 5) + 14}{6} = 6$	$\left(\dfrac{14-2}{6}\right)^2 = 4$
2–3	$\dfrac{2 + (4 \times 14) + 26}{6} = 14$	$\left(\dfrac{26-2}{6}\right)^2 = 16$
2–4	$\dfrac{2 + (5 \times 4) + 8}{6} = 5$	$\left(\dfrac{8-2}{6}\right)^2 = 1$
3–5	$\dfrac{7 + (4 \times 10) + 19}{6} = 11$	$\left(\dfrac{19-7}{6}\right)^2 = 4$
4–5	$\dfrac{5 + (5 \times 4) + 17}{6} = 7$	$\left(\dfrac{17-5}{6}\right)^2 = 4$
6–7	$\dfrac{5 + (8 \times 4) + 29}{6} = 11$	$\left(\dfrac{29-5}{6}\right)^2 = 16$
5–8	$\dfrac{3 + (3 \times 4) + 9}{6} = 4$	$\left(\dfrac{9-3}{6}\right) = 1$
7–8	$\dfrac{8 + (4 \times 17) + 32}{6} = 18$	$\left(\dfrac{32-8}{6}\right)^2 = 16$



Expected project duration = 36 days

Critical path 1–2–3–5–8

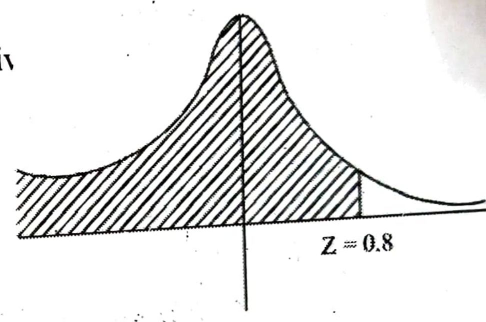
Project length variance, $\sigma^2 = 4 + 16 + 4 + 1 = 25$

Std. Deviation $\sigma = 5$

The probability that the project will be completed in 40 days is giv

$$P(Z \leq D)$$

$$D = \frac{T_s - T_e}{\sigma} = \frac{40 - 36}{5} = \frac{4}{5} = 0.8$$



Area under the normal curve for the region $Z \leq 0.8$

$$P(Z \leq 0.8)$$

$$= 0.5 + \phi(0.8)$$

$$= 0.5 + 0.2881 = 0.7881$$

$$= 78.81\%$$

[$\phi(8) = 0.2881$ (from table)]

Conclusion If the project is performed 100 times, under the same conditions, there will be 78.81 occasions for this job to be completed in 40 days.

Example 15.12 A small project is composed of seven activities, whose time estimates are listed in the table as follows:

Activity	*Estimated duration (weeks)*		
	Optimistic (a)	*Most likely (m)*	*Pessimistic (b)*
1–2	1	1	7
1–3	1	4	7
2–4	2	2	8
2–5	1	1	1
3–5	2	5	14
4–6	2	5	8
5–6	3	6	15

You are required to:

1. Draw the project network.
2. Find the expected duration and variance of each activity.
3. Calculate the earliest and latest occurrence for each event and the expected project length.
4. Calculate the variance and standard deviations of project length.
5. What is the probability that the project will be completed,
   - (*i*) 4 weeks earlier than expected?
   - (*ii*) Not more than 4 weeks later than expected?
   - (*iii*) If the project's due date is 19 weeks, what is the probability of meeting the due date?

Solution The expected time and variance of each activity is computed as shown in the table below:

Activity	*a*	*m*	*b*	$t_e = \frac{a+4m+b}{6}$	$\sigma^2 = \left(\frac{b-a}{6}\right)^2$
1–2	1	1	7	2	1
1–3	1	4	7	4	1
2–4	2	2	8	3	1
2–5	1	1	1	1	0
3–5	2	5	14	6	4
4–6	2	5	8	5	1
5–6	3	6	15	7	4

The earliest and the latest occurrence time for each is calculated as below:

$$E_1 = 0;$$

$$E_2 = 0 + 2 + 2$$

$$E_3 = 0 + 4 = 4$$

$$E_4 = 0 + 3 = 3$$

$$E_5 = \text{Max}\,(2 + 1, 4 + 6) = 10$$

$$E_6 = \text{Max}\,(10 + 7, 3 + 5) = 17.$$

To determine the latest expected time, we start with E_6 being the last event and move backwards subtracting t_e from each activity. Hence, we have,

$$L_6 = E_6 = 17$$

$$L_5 = L_6 - 7 = 17 - 7 = 10$$

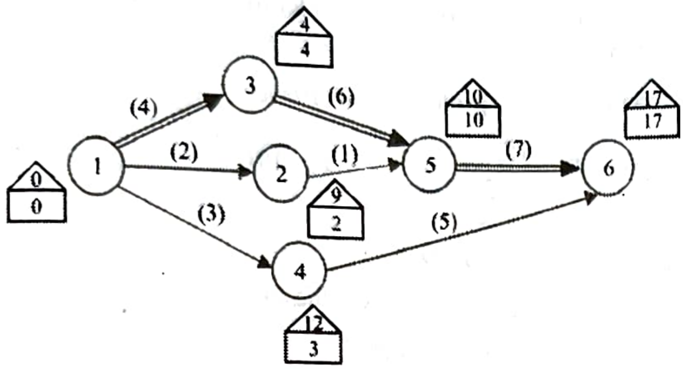
$$L_4 = 17 - 5 = 12$$

$$L_3 = 10 - 6 = 4$$

$$L_2 = 10 - 1 = 9$$

$$L_1 = \text{Min}\,(9 - 2, 4 - 4, 12 - 3) = 0$$

Using the above information we get the following network, where the critical path is shown by the double-line arrows.



We observe the critical path of the above network as $1 \to 3 \to 5 \to 6$.

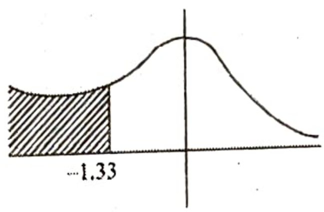
The expected project duration is 17 weeks, i.e., $T_e = 17$ weeks.

The variance of the project length is given by,

$$\sigma^2 = 1 + 4 + 4 = 9.$$

(i) The probability of completing the project within 4 weeks earlier than expected is given by,

$$P(Z \le D), \text{ where } D = \frac{T_s - T_e}{\sigma}$$



$$D = \frac{\text{Due date} - \text{Expected date of completion}}{\sqrt{\text{Project variance}}}$$

$$D = \frac{17 - 4 - 17}{3} = \frac{-21+17}{3} = \frac{-4}{3}$$

$$= -1.33$$

$$\therefore \quad P(Z \le -1.33) = 0.5 - \phi(1.33)$$

$$= 0.5 - 0.4082 \text{ (from the table)}$$

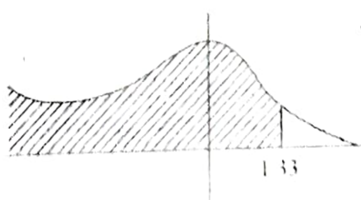
$$= .0918 = 9.18\%.$$

Conclusion If the project is performed 100 times under the same conditions, then there will be 9 occasions for this job to be completed in 4 weeks earlier than expected.

(ii) The probability of completing the project not more than 4 weeks later than expected is given by,

$$P(Z \le D), \text{ where}$$

$$D = \frac{T_s - T_e}{\sigma}$$



Here, $T_s = 17 + 4 = 21$

$$0 = \frac{21 - 17}{3} = \frac{4}{3} = 1.33$$

$$P(Z \le 1.33)$$

$$= 0.5 + \phi(1.33)$$

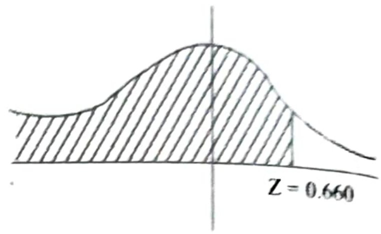
$$= 0.5 + 0.4082 \text{ (from the table)}$$

$$= 0.9082 = 90.82\%.$$

Conclusion If the project is performed 100 times under the same conditions, then there will be 90.82 occasions when this job will be completed not more than 4 weeks later than expected.

(*iii*) The probability of completing the project within 19 weeks, is given by,

$$P(Z \le D), \text{ where } D = \frac{19-17}{3} = \frac{2}{3} \quad [\because T_s = 19]$$
$$= 0.666.$$
$$\text{i.e., } P(Z \le 0.666) = 0.5 + \phi(0.666)$$
$$= 0.5 + 0.2514 \text{ (from the table)}$$
$$= 0.7514 = 75.14\%.$$



Conclusion If the project is performed 100 times, under the same conditions, then there will be 75.14 occasions for this job to be completed in 19 weeks.

Example 15.13 Consider the following project.

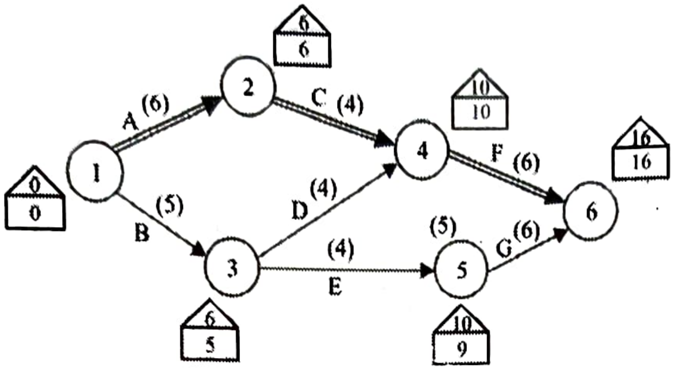
Activity	*Time estimate in weeks*			*Predecessor*
	t_o	t_m	t_p	
A	3	6	9	None
B	2	5	8	None
C	2	4	6	*A*
D	2	3	10	*B*
E	1	3	11	*B*
F	4	6	8	*C, D*
G	1	5	15	*E*

Find the path and standard deviation. Also find the probability of completing the project by 18 weeks.

Solution First we calculate the expected time and variance of each activity as in the following table.

Activity	t_o	t_m	t_p	$t_e = \frac{t_o + 4t_m + t_p}{6}$	$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
A	3	6	9	$\frac{3+(4\times 6)+9}{6} = 6$	$\left(\frac{9-3}{6}\right)^2 = 1$
B	2	5	8	$\frac{2+(4\times 5)+8}{6} = 5$	$\left(\frac{8-2}{6}\right)^2 = 1$
C	2	4	6	$\frac{2+(4\times 4)+6}{6} = 4$	$\left(\frac{6-2}{6}\right)^2 = 0.444$
D	2	3	10	$\frac{2+(4\times 3)+10}{6} = 4$	$\left(\frac{10-2}{6}\right)^2 = 1.777$
E	1	3	11	$\frac{1+(4\times 3)+11}{6} = 4$	$\left(\frac{11-1}{6}\right)^2 = 2.777$
F	4	6	8	$\frac{4+(4\times 6)+8}{6} = 6$	$\left(\frac{8-4}{6}\right)^2 = 0.444$
G	1	5	15	$\frac{1+(4\times 5)+15}{6} = 6$	$\left(\frac{15-1}{6}\right)^2 = 5.444$

We construct the network with the help of the predecessor relation given in the data.



Critical path is 1–2–4–6 or A–C–F.

The project length = 18 weeks.

Project length variance, $\sigma^2 = 1 + 0.444 + 0.444 = 1.888$

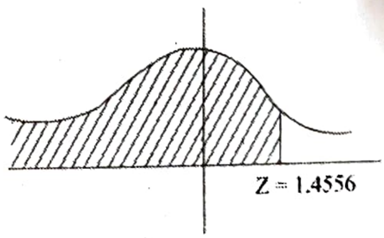
Standard deviation, $\sigma = 1.374$

The probability of completing the project in 18 weeks is given by,

$P(Z \leq D)$, where $D = \dfrac{T_s - T_e}{\sigma}$

$T_s = 18;\ T_e = 16;\ \sigma = 1.374$

$$D = \frac{18-16}{1.374} = 1.4556$$



$$P(Z \leq D) = P(Z \leq 1.4556) = 0.5 + \phi(1.4456)$$
$$= 0.5 + 0.4265 = 0.9265 = 92.65\%.$$

Conclusion If the project is performed 100 times under the same conditions, then there will be 92.65 occasions when this job will be completed by 18 weeks.

Example 15.14 The following table shows the jobs of a network along with their time estimates. The time estimates are in days.

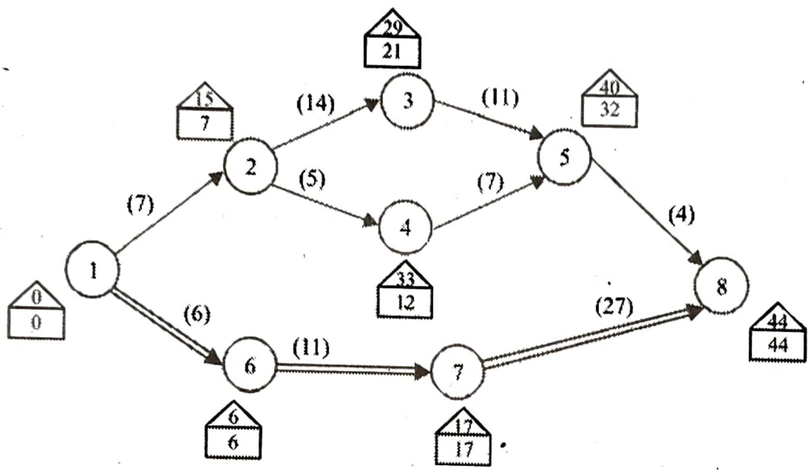
Job	*1–2*	*1–6*	*2–3*	*2–4*	*3–5*	*4–5*	*5–8*	*6–7*	*7–8*
a	3	2	6	2	5	3	1	3	4
m	6	5	12	5	11	6	4	9	19
b	15	14	30	8	17	15	7	27	28

(*i*) Draw the project network.
(*ii*) Find the critical path.
(*iii*) Find the probability of the project being completed in 31 days.

Solution First we calculate the expected time and variance for each activity as shown in the table below.

Activity	a	m	b	$t_e = \frac{a+4m+b}{6}$	$\sigma^2 = \left[\frac{(b-a)}{6}\right]^2$
1–2	3	6	15	7	4
1–6	2	5	14	6	4
2–3	6	12	30	14	16
2–4	2	5	8	5	1
3–5	5	11	17	11	4
4–5	3	6	15	7	4
5–8	1	4	7	4	1
6–7	3	9	27	11	16
7–8	4	19	28	27	16

(*i*) ***Project network***



(*ii*) The critical path is given by, 1–6–7–8 and the project length is given by 44 days.
The project length variance, $\sigma^2 = 4 + 16 + 16 = 36$

Standard deviation, $\sigma = \sqrt{36} = 6$.

(*iii*) The probability of completing the project within 31 days is given by,

$$P(Z \le D), \text{ where } D = \frac{T_s - T_e}{\sigma}$$

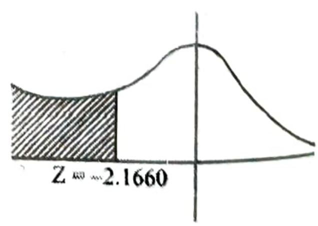
$$= \frac{31-44}{6} = -2.1666$$

i.e., $$P(Z \le -2.1666) = 0.5 - \phi(2.166)$$

$$= 0.5 - 0.4886$$

$$= 0.0114$$

$$= 1.14\%.$$



Conclusion If the project is performed 100 times, under the same conditions, then there will be 1.14 occasions when this job will be completed in 31 days.

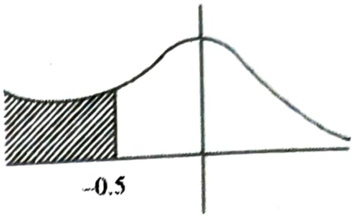
Example 15.15 Using the data given in example 15.14, find the probability of completing the jobs on the critical path in 41 days.

Solution Refer to example 15.14 to find the critical path, project length and project length variance.

The probability of completing the project within 41 days is given by,

$$P(Z \leq D), \text{ where } D = \frac{T_s - T_e}{\sigma}$$

$$= \frac{41-44}{6} = \frac{-3}{6} = -0.5$$



$$P(Z \leq -0.5) = 0.5 - \phi(0.5)$$

$$= 0.5 - 0.1915 \text{ (from the table)}$$

$$= 0.3085 = 30.85\%.$$

Conclusion If the project in performed 100 times under the same conditions, then there will be 30.85 occasions when the project will be completed in 41 days.

Example 15.16 Assuming that the expected times are normally distributed, find the probability of meeting the scheduled time as given for the network.

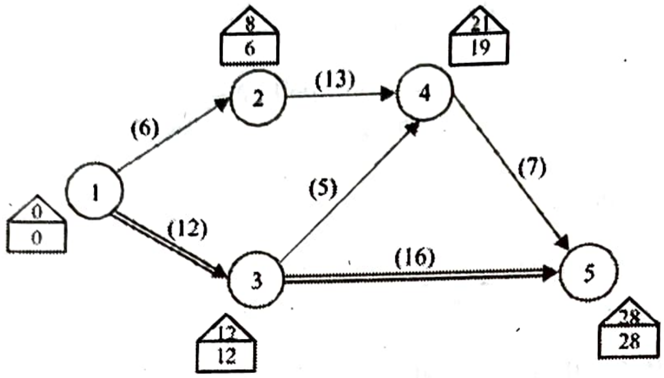
Activity		*Days*	
$(i-j)$	*Optimistic* a	*Most likely* m	*Pessimistic* b
1–2	2	5	14
1–3	9	12	15
2–4	5	14	17
3–4	2	5	12
4–5	6	6	12
3–5	8	17	20

Scheduled project completion time is 30 days. Also find the date on which the project manager can complete the project with a probability of 0.90.

Solution The expected time t_e and variance for each activity is calculated in the following table:

Activity	$t_e = \dfrac{(a+4m+b)}{6}$	$\sigma^2 = \left(\dfrac{b-a}{6}\right)^2$
1–2	6	4
1–3	12	1
2–4	13	4
3–4	5	1
3–5	16	4
4–5	7	1

To determine the critical path, the earliest expected time and the latest allowable time, we draw the project network.



The critical path is given by, 1–3–5 and the project duration is given by 28 days. Project length variance, $\sigma^2 = 1+4=5$. Standard deviation, $\sigma = 2.236$.

The probability of completing the project within 30 days is given by,

$$P(Z \leq D), \text{ where } D = \frac{T_s - T_e}{\sigma} = \frac{30-28}{2.236} = 0.8944$$

$$\begin{aligned} P(Z \leq 0.8944) &= 0.5 + 0.8944 \\ &= 0.5 + 0.3133 \text{ (From table)} \\ &= 0.8133 \\ &= 81.33\%. \end{aligned}$$

Conclusion If the project is performed 100 times under the same conditions, then there will be 81.33 occasions when the project will be completed in 30 days.

If the probability for the completion of the project is 0.90 then the corresponding value of $Z = 1.29$.

$$Z = \frac{T_s - T_e}{S.D} = 1.29$$

i.e., $$\frac{T_s - 28}{2.236} = 1.29.$$

$\therefore$ $$T_s = (1.29)(2.236) + 28$$

$\therefore$ $$T_s = 30.88 \text{ weeks.}$$

EXERCISES

1. The data for a small PERT project is as given below, where *a* represents optimistic time, *m* the most likely time and *b* the pessimistic time. Estimates (in days) of the activities *A*, *B* ... *J*, *K* are given in the table.

Activity	A	B	C	D	E	F	G	H	I	J	K
a	3	2	6	2	5	3	3	1	4	1	2
m	6	5	12	5	11	6	9	4	19	2	4
b	5	14	30	8	17	15	27	7	28	9	12

A, *B* and *C* can start simultaneously; $A < D, I$; $B < G, F$; $D < G, F$; $C < E$; $E < H, K$; $F < H, K$; $G, H < J$.

(*i*) Draw the arrow network of the project.

(*ii*) Calculate the earliest and the latest expected times to each event and find the critical path.

(*iii*) What is the probability that the project will be completed 2 days later than expected?

[**Ans.** Critical path *A*–*D*–*F*–*H*–*J*; Required probability 62.93%]

2. The three estimates for the activities of a project are given below:

Activity	Estimate duration (days)		
	a	m	b
1–2	5	6	7
1–3	1	1	7
1–4	2	4	12
2–5	3	6	15
3–5	1	1	1
4–6	2	2	8
5–6	1	4	7

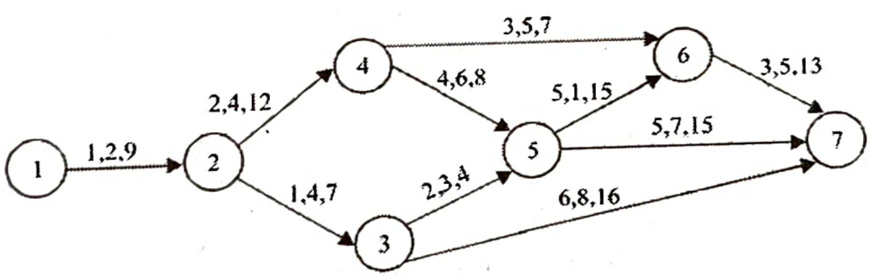
Draw the project network. Find out the critical path and duration of the project. What is the probability that the project will be completed at least 5 days earlier than expected?

What is the probability that the project will be completed by 22 days?

[**Ans.** Critical path 1–2–5– 6; Project duration 17 days. Required probability = 1.36%; Probability that the project will be completed in 22 days = 98.64%]

3. Consider the network shown in the figure below. The estimate t_o, t_m and t_p are shown in this order for each of the activities, on top of the arcs denoting the respective activities.

Find the probability of completing the project in 25 days. [**Ans.** 85.99%]



4. A project is represented by the network shown below and has the following table:

Task	A	B	C	D	E	F	G	H	I
Least time	5	18	26	16	15	6	7	7	3
Greatest time	10	22	40	20	25	12	12	9	5
Most likely time	8	20	33	18	20	9	10	8	4

Determine the following:

(*i*) Expected time of tasks and their variance.

(*ii*) The earliest and the latest expected time to reach each mode.

(*iii*) The critical path.

(*iv*) The probability of completing the project within 41.5 weeks.

[**Ans.** Critical path 1 → 4 → 7; Project duration = 42.8 weeks; Probability of completing the project within 41.5 weeks = 0.30]

5. Consider a project having the following activities and their time estimates. Draw an arrow diagram for the project. Identify the critical path and compute the expected completion time. What is the probability that the project will require at least 75 days?

Activity	Predecessor	t_o	t_m	t_p
			days	
A	–	2	4	6
B	A	8	12	16
C	A	14	16	30
D	B	4	10	16
E	C, B	6	12	18
F	E	6	8	22
G	D	18	18	30
H	F, G	8	14	32

[**Ans.** $A \rightarrow D \rightarrow G \rightarrow H$; Expected completion time = 62 days;
Probability of requiring at least 75 days = 0.9944]

15.9 COST CONSIDERATION IN PERT/CPM

15.9.1 Project Cost

In order to include the cost factors in project scheduling, we must first define the cost duration relationships for various activities in the project. The total cost of any project comprises of direct and indirect costs.

Direct cost This cost is directly dependent upon the amount of resources available in the execution of individual activities e.g. manpower loading, materials consumed, etc. *The direct cost increases if the activity duration is to be reduced.*

Indirect cost This cost is associated with overhead expenses such as managerial services, indirect supplies, general administration, etc. The indirect cost is computed on per day, per week, or per month basis. *The indirect cost decreases if the activity duration is to be reduced.*

Network diagram can be used to identify the activities whose duration should be shortened, so that the completion time of the project can be shortened in the most economic manner. The process of reducing the activity duration by putting on extra effort is called *crashing the activity.*

The crash time (T_C) represents the minimum activity duration time that is possible and any attempts to further crash would only raise the activity cost without reducing the time. The activity cost corresponding to the crash time is called the *crash cost* (C_C), which is the minimum direct cost required to achieve the crash performance time.

The *normal cost* (C_N) is equal to the absolute minimum of the direct cost required to perform an activity. Normal cost is the cost associated when the project is completed with normal time. The corresponding time duration taken by an activity is known as the *normal time* (T_N).

15.9.2 Cost Slope

The cost slope, indicating the increase in cost per unit reduction in time is defined as,

$$\text{Cost slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}} = \frac{C_C - C_N}{T_N - T_C}$$

i.e., it represents the rate of increase in the cost of performing the activity per unit reduction in time and is called *cost/time trade off*. It varies from activity to activity. The total project cost is the sum total of the project's direct and indirect costs.

15.9.3 Time-Cost Optimization Algorithm

Following are the steps involved in project crashing.

Step 1 Find the normal critical path and identify the critical activities.

$$\text{Cost slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}$$

Step 2 Calculate the cost slope for the different activities by using the above formula.

Step 3 Rank the activities. The activity whose cost slope in minimum is to be ranked 1, the next minimum as rank 2 and so on, i.e., the ranking takes place in ascending order of cost slope.

Step 4 By crashing the activities on the critical path, other paths also become critical and are called *parallel paths.*

In such cases, the project duration can be reduced by crashing activities simultaneously on the parallel critical path.

Step 5 Find the total cost of the project at each step.

Step 6 Continue the process until all the critical activities are fully crashed or no further crashing is possible.

In the case of indirect cost, the process of crashing is repeated until the total cost is minimum, beyond which it may increase.

This minimum cost is called the *optimum project cost* and the corresponding time, the *optimum project time.*

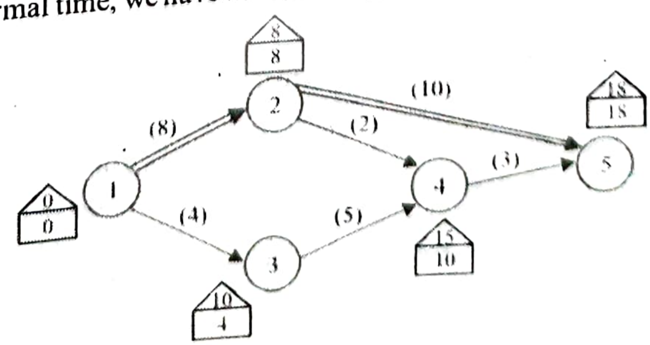
Example 15.17 Determine the optimum project duration and cost for the following data.

Activity	*Normal*		*Crash*	
	Time (days)	*Cost (₹)*	*Time (days)*	*Cost (₹)*
1–2	8	100	6	200
1–3	4	150	2	350
2–4	2	50	1	90
2–5	10	100	5	400
3–4	5	100	1	200
4–5	3	80	1	100

Indirect cost is ₹ 70 per day.

Solution Since the overhead of the project is given, the cost is indirect. Hence, the project duration can be reduced with the total cost.

Making use of the normal time, we have the following network.



Critical path 1–2–5

Project normal duration = 18 days

Cost of the project = Normal cost of all the activities + Indirect cost

$$= 580 + (70 \times 18) = ₹\,1{,}840.$$

We can reduce the project duration from 18 days, by crashing the activity on the critical path. The cost slope and the number of days to be crashed are given in Table (A).

Table (A)

Activity	*Cost slope* $= \dfrac{C_C - C_N}{T_N - T_C}$	
	$\dfrac{200-100}{8-6} =$	
1–2	100(2) (VI)	The number given in the bracket corresponds to the days allowed for crashing and is the difference between T_N and T_C
1–3	50(2) (IV)	
2–4	40(1) (III)	
2–5	60(5) (V)	
3–4	25(4) (II)	
4–5	10(2) (I)	

From the network, the other paths are given by,

$\underline{1-2-5|}$ $\cancel{18}$ $\cancel{17}$ $\cancel{16}$ $\cancel{15}$ $\cancel{14}$ $\cancel{13}$ $\cancel{12}$ $\cancel{11}$

$\underline{1-2-4-5|}$ $\cancel{13}$ $\cancel{12}$ 11

$\underline{1-3-4-5|}$ $\cancel{12}$ 11

We form a table (B) to calculate the optimum project duration and its cost.

Table (B)

Normal duration	*Crash activity*	*Crash cost*	*Indirect cost*	*Total cost*
18	–	–	18 × 70 = 1260	1260 + 580 = 1840
17	1–2 (1)	1 × 50 = 50	17 × 70 = 1190	1820
16	1–2 (2)	50 + (1 × 50) = 100	16 × 70 = 1120	1800
15	2–5 (1)	100 + (60 × 1) = 160	15 × 70 = 1050	1790
14	2–5 (2)	160 + (1 × 60) = 220	14 × 70 = 980	1780
13	2–5 (3)	220 + (1 × 60) = 280	13 × 70 = 910	1770
12	2–5 (4)	280 + (1 × 60) = 340	12 × 70 = 840	1760
11	2–5 (1–2–5) 4–5 (1–3–4–5)	340 + (1 × 60) + (1 × 10) = 410	11 × 70 = 770	1760

We rank the activities in ascending order of cost slope as given in the above table. First we crash the activity 1–2. It is the activity lying on the critical path and with minimum cost slope. This crashing is shown in the other paths also. As the activity 1–2 can be crashed for two days, next we have the activity 2–5 on the critical path. After crashing to 12 days, we get the parallel path namely 1–2–5 and 1–3–4–5.

As there is no common activity between these 2 paths, we crash the activity 2–5 on the path 1–2–5 and 4–5 on the path 1–3–4–5 as it is the activity having the minimum rank. No more crashing is possible as all the activities in the path 1–2–5 are in crash time, even though there are activities available in other parallel paths.

Hence, the optimum project duration is 11 days with total cost of ₹ 1,760.

Example 15.18 The following table gives the activities of a construction project along with other relevant information.

(*i*) What is the normal project length and the minimum project length?

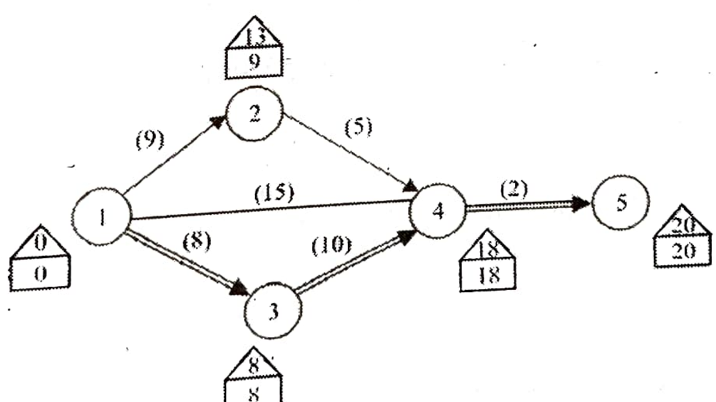
(*ii*) Determine the minimum crashing costs of schedule, ranging from normal length down to and including the minimum length schedule.

Activity i–j	*Normal duration (days)*	*Crash duration (days)*	*Cost of crashing (₹ per day)*
1–2	9	6	20
1–3	8	5	25
1–4	15	10	30
2–4	5	3	10
3–4	10	6	15
4–5	2	1	40

(*iii*) What is the optimal length schedule duration of each job for your solution?

Overhead of the project is ₹ 60 per day.

Solution Since the overhead cost is given, the cost is indirect. The project duration can be reduced by reducing the total cost associated with it.



The critical path comprises the activities 1–3, 3–4 and 4–5 with the normal duration as 20 days.

The total cost associated with the project is 20 × 60 = ₹ 1,200.

Activity	*Cost of crashing*
1–2	20 (3) III
1–3	25 (3) IV
1–4	30 (5) V
2–4	10 (2) I
3–4	15 (4) II
4–5	40 (1) VI

The cost slope is given in the data. We reduce the project duration by crashing the activities lying on the critical path.

The critical activity 3–4 has the minimum cost slope. It is ranked I, so this activity is crashed first for 4 days.

The other paths of the network are:

$$\underline{1-3-4-5|}\ \cancel{20}\ \cancel{19}\ \cancel{18}\ \cancel{17}\ \cancel{16}\ \cancel{15}\ \cancel{14}\ 13$$

$$\underline{1-4-5|}\ \cancel{17}\ \cancel{16}\ \cancel{15}\ \cancel{14}\ 13$$

$$\underline{1-2-4-5|}\ \cancel{16}\ \cancel{15}\ \cancel{14}\ 13$$

Normal duration	***Crash activity***	***Crash cost***	***Overhead cost***	***Total cost***
20	–	–	20 × 60 = 1200	1200
19	3–4 (1)	1 × 15 = 15	19 × 60 = 1140	1155
18	3–4 (2)	15 + (1 × 15) = 30	18 × 60 = 1080	1110
17	3–4 (3)	30 + (1 × 15) = 45	17 × 60 = 1020	1065
16	4–5 (1–3–4–5–1–4–5)	45 + (1 × 40) = 85	16 × 60 = 960	1045
15	3–4 (1–3–4–5) 1–4 (1–4–5) 2–4 (1–2–4–5)	85 + 15 + 30 + 10 = 140	15 × 60 = 900	1040
14	1–3 (1–3–4–5) 1–4 (1–4–5) 2–4 (1–2–4–5)	140 + 25 + 30 + 10 = 205	14 × 60 = 840	1045
13	1–3 (1–3–4–5) 1–4 (1–4–5) 1–2 (1–2–4–5)	205 + 25 + 30 + 20 = 280	13 × 60 = 780	1060

After 17 days, we get parallel paths namely, 1–3–4–5 and 1–4–5. We crash the activity 4–5 for one day at the rate of ₹ 40 per day.

After 16 days, we get all the critical paths. As there is no common activity between them, we crash the activity 3–4 for the path 1–3–4–5, 1–4 for the path 1–4–5 and 2–4 for the path 1–2–4–5.

As the activity 3–4 in the path 1–3–4–5 is in crash time, we crash the activity 1–3 for 1 day at the rate of ₹ 25 per day. The optimum duration is 15 days as it gives the minimum cost of ₹ 1,040.

To find the minimum duration we crash further.

Normal duration	***Crash activity***	***Crash cost***	***Overhead cost***	***Total cost***
12	1–3 (1–3–4–5) 1–4 (1–4–5) 1–2 (1–2–4–5)	280 + 25 + 30 + 20 = 355	12 × 60 = 720	1075

The minimum duration is 12 days and no more crashing is possible as all the activities in the path 1–3–4–5 are in crash time.

Optimum duration = 15 days with total cost associated as ₹ 1,040

Minimum duration = 12 days with total cost associated as ₹ 1,075.

Note: From the above problem, we observe that the optimum duration and minimum duration are not the same.

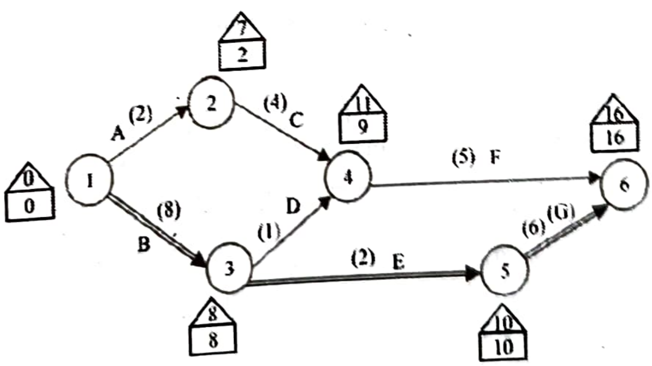
Optimum duration refers to the duration that yields the minimum total cost, whereas minimum duration is the one in which no more crashing is possible beyond that duration.

Example 15.19 The table below provides costs and estimates for a seven-activity project.

Activity	*Time estimate (weeks)*		*Direct cost estimate (₹ 1,000)*	
	Normal	*Crash*	*Normal*	*Crash*
A 1–2	2	1	10	15
B 1–3	8	5	15	21
C 2–4	4	3	20	24
D 3–4	1	1	7	7
E 3–5	2	1	8	15
F 4–6	5	3	10	16
G 5–6	6	2	12	36

(*i*) Draw the project network corresponding to normal time.
(*ii*) Determine the critical path, normal duration and cost of the project.
(*iii*) Crash the activities so that the project completion time reduces to 9 weeks.

Solution As the problem involves *direct cost*, we expect that the project duration can be reduced with an increase in total cost. First we draw the network.



Critical path 1–3–5–6; Normal duration = 16 weeks

Total cost = ₹ 82,000

The calculations for cost slope and crashing number of days are shown in the table below:

Activity	Slope (₹ 1000)
1–2	5 (1) IV
1–3	2 (3) I
2–4	4 (1) III
3–4	0
3–5	7 (1) VI
4–6	3 (2) II
5–6	6 (4) V

The different paths of the network are,

1–3–5–6 | ~~16~~ ~~13~~ ~~11~~ 9

1–3–4–6 | ~~14~~ ~~11~~ 9

1–2–4–6 | ~~11~~ 9

First crashing We crash the activity 1–3, as it is the critical activity with minimum rank. We crash it for 3 weeks at the rate of ₹ 2 (1000) per day.

Project duration reduced to $16 - 3 = 13$ weeks

Total cost $= 82 + (3 \times 2) = 88$ (₹ 1,000)

Second crashing Next, we crash the activity 5–6 for 2 weeks at the rate of ₹ 6 (1,000) per week, as this activity has the next minimum rank.

Project duration reduced to $13 - 2 = 11$ weeks

Total cost $= 88 + (2 \times 6) = 100$ (₹ 1,000).

Third crashing After 11 days, we get all the paths that are critical. As there is no activity in common, we crash the activity 5–6 for 2 weeks in the path 1–3–5–6 and crash the activity 4–6 for 2 weeks, which is common to the paths 1–3–4–6 and 1–2–4–6.

Project duration reduces to $11 - 2 = 9$ weeks.

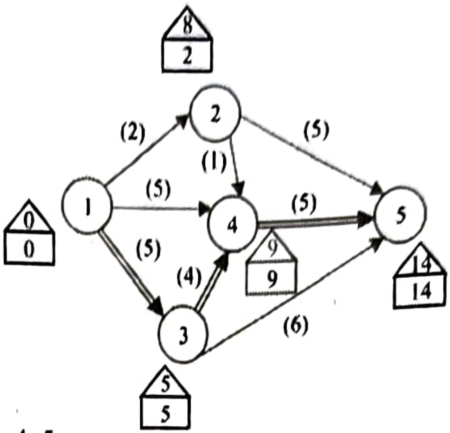
Total cost $= 100 + 2 \times 6 + 2 \times 3 = 118$ (₹ 1,000).

The project duration cannot be reduced beyond 9 weeks as all the activities are in crash time. Hence, the optimum duration is 9 weeks with the total cost associated as ₹ 118 (1,000).

Example 15.20 The following time-cost table (time in weeks and cost in rupees) applies to a project. Use it to arrive at the network associated with completing the project in minimum time with minimum cost.

Activity	Normal		Crash	
	Time (Weeks)	Cost (₹)	Time (Weeks)	Cost (₹)
1–2	2	800	1	1,400
1–3	5	1,000	2	2,000
1–4	5	1,000	3	1,800
2–4	1	500	1	500
2–5	5	1,500	3	2,100
3–4	4	2,000	3	3,000
3–5	6	1,200	4	1,600
4–5	5	900	3	1,600

Solution As the direct cost is given, we reduce the project duration by an increase in total cost. First we construct the network.



Critical path is given by, 1–3–4–5

Project duration = 5 + 4 + 5 = 14 weeks

Total cost associated = ₹ 8,900

We calculate the cost slope for each activity as given in the following table:

Activity	*Cost slope* $= \dfrac{C_C - C_N}{T_N - T_C}$
1–2	600 (1) VI
1–3	333.333 (3) III
1–4	400 (2) V
2–4	–
2–5	300 (2) II
3–4	1,000 (1) VII
3–5	200 (2) I
4–5	350 (2) IV

The project duration is reduced by crashing the activities. First, the activity that lies on the critical path and ranks the minimum is crashed.

First crashing We crash the activity 1–3 for 3 weeks at the rate of ₹ 333.33 per week. Project duration reduced to 14 – 3 = 11 weeks.

$$\begin{aligned}\text{Total cost} &= 8{,}900 + 3\,(333.33)\\ &= ₹\,9{,}899.99\\ &= ₹\,9{,}900\end{aligned}$$

Second crashing Next, crash the activity 4–5 as this activity lies on the critical path with next minimum rank. Crash 4–5 for 2 weeks at the rate of ₹ 350 per week. Project duration is reduced to 11 – 2 = 9 weeks.

$$\text{Total cost} = 9{,}900 + (2 \times 350) = ₹\,10{,}600$$

Third crashing Next, we crash the activity 3–4 for one week, at the rate of ₹ 1,000 per week. Project duration reduced to 9 – 1 = 8 weeks.

$$\text{Total cost} = 10{,}600 + 1{,}000 = ₹\,11{,}600$$

As all the activities on the path 1–3–4–5 are in crash time, no more crashing is possible beyond this.

∴ Optimum and minimum project duration is given by 8 weeks with total cost as ₹ 11,600.

EXERCISES

1. A maintenance foreman has given the following estimates of time and cost of jobs in a maintenance project.

Job	Predecessor	Normal		Crash	
		Time (days)	Cost (₹)	Time (days)	Cost (₹)
A	–	8	80	6	100
B	A	7	40	4	94
C	A	12	100	5	184
D	A	9	70	5	102
E	B, C, D	6	50	6	50

Overhead cost is ₹ 25 per day.

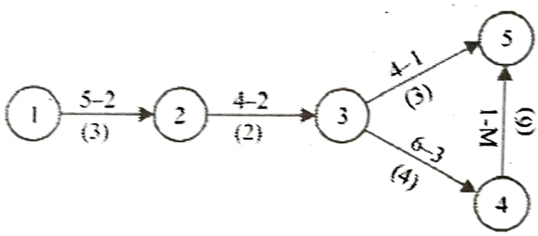
Find,

(*i*) The normal duration and associated cost of the project.

(*ii*) The optimum duration and associated cost of the project.

[**Ans.** Critical path 1–2–5–6; Normal duration = 26 days
Cost = ₹ 990; Optimum duration = 17 days; Cost = ₹ 937]

2. Determine the least cost schedule for the following project, using CPM technique.



Overhead cost per day is ₹ 6. The numbers above and below the activities have the usual meaning.

[**Ans.** Optimum duration = 9 days; Least cost = ₹ 88]

3. Find the optimum schedule for the given project. The overhead cost is ₹ 75 per day.

Activity	Predecessor	Duration (days)		Increase in cost for crashing by one day (₹)
		Normal time	Crash time	
A	–	3	2	150
B	–	4	3	100
C	A	5	4	200
D	A	7	5	300
E	B, C	3	3	0
F	B, C, D	6	2	75

(a) Draw the project, using normal duration.

(b) Find the path and the project duration for the above case.

(c) Find the optimal schedule and optimal project duration.

[**Ans.** CPM *A*–*D*–*F*; Project duration = 16 days;
Optimum duration 12 days; Least cost = ₹ 1,200]

4. The following table gives the activities in a construction project along with other relevant information.

Activity	Normal time (days)	Crash time (days)	Normal cost (₹)	Crash cost (₹)
1–2	20	17	600	720
1–3	25	25	200	200
2–3	10	8	300	440
2–4	12	6	400	700
3–4	5	2	300	420
4–5	10	5	300	600
4–6	5	3	600	900
5–7	10	5	500	800
6–7	8	3	400	700

(a) Draw the activity network of the project.

(b) Find the free and total float for each activity.

(c) Using the above information, crash the activity step-by-step, until all the paths are critical.

[**Ans.** Critical path 1–2–3–4–5–7;
Normal duration 55 days with cost ₹ 3,600;
Reduced duration = 37 days with total cost ₹ 4,860]

SUMMARY

Network Scheduling

Network It is the graphic representation of logically and sequentially connected arrows and nodes representing activities and events in a project.

Activity An activity represents some action and is a time consuming effort, necessary to complete a particular part of the overall project.

Event The beginning and end points of an activity are called events or nodes.

Dummy activity Certain activities, which neither consume time nor resources but are used simply to represent a connection or a link between the events, are known as dummies.

Time analysis We shall use the following notation for basic scheduling computations.

$$(i,j) = \text{Activity } (i, j) \text{ with tail event } i \text{ and head event } j$$

$$T_{ij} = \text{Estimated completion time of activity } (i, j)$$

$$ES_{ij} = \text{Earliest starting time of activity } (i, j)$$

$$EF_{ij} = \text{Earliest finishing time of activity } (i, j)$$

$$LS_{ij} = \text{Latest starting time of activity } (i, j)$$

$$LF_{ij} = \text{Latest finishing time of activity } (i, j).$$

Forward pass computations (for earliest event time)

Step 1 The computations begin from the start node and move towards the end node. Let zero be the starting time for the project.

Step 2 Earliest starting time $(ES_{ij}) = E_i$, is the earliest possible time when an activity can begin assuming that all the predecessors also started at their earliest starting time. Earliest finishing time of activity (i, j) is the earliest starting time + the activity time.

$$(EF)_{ij} = (ES)_{ij} + t$$

Step 3 Earliest event time for event j is the maximum of the earliest finish time of all the activities ending at that event.

$$E_j = \underset{i}{\text{Max}}\,(E_i + t_{ij})$$

Backward Pass Computations (for latest allowable time)

Step 1 For ending event assume $E = L$

Step 2 Latest finish time for activity (i, j) is the target time for completing the project.

$$(LF_{ij}) = L_j$$

Step 3 Latest starting time of the activity (i, j)

= Latest completion time of (ij) – the activity time

$= LS_{ij} = LF_{ij} - t_{ij} = L_j - t_{ij}$

Step 4 Latest event time for event i is the minimum of the latest start time of all activities originating from the event.

$$L_i = \underset{j}{\text{Max}}\,(E_j + t_{ij})$$

Float is defined as the difference between the latest and the earliest activity time.

Slack is defined as the difference between the latest and the earliest event time.

Total float:

$(TF)_{ij}$ = (Latest start – Earliest start) for activity (i, j)

i.e., $(TF)_{ij} = (LS)_{ij} - (ES)_{ij}$

or, $(TF)_{ij} = (L_j - E_i) - t_{ij}$

Free float:

$FF_{ij} = (E_j - E_i) - t_{ij}$

FF_{ij} = Total float – head event slack.

Independent float: $IF_{ij} = (E_j - L_i) - t_{ij}$

or, IF_{ij} = Free float – tail event slack.

Critical activity An activity is said to be critical if a delay in its start will cause a further delay in the completion of the entire project.

Critical path The sequence of critical activities in a network is called the critical path.

PERT

Optimistic time estimate (t_o) It is the smallest time taken to complete the activity if everything goes well.

Most likely time estimate (t_m) It refers to the estimate of the normal time the activity would take.

Pessimistic time estimate (t_p) It is the longest time that an activity would take if everything goes wrong.

PERT Procedure

Step 1 Draw the project network.

Step 2 Compute the expected duration of each activity using the formula,

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

Also calculate the expected variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$ of each activity.

Step 3 Compute the earliest start, earliest finish, latest start, latest finish and total float of each activity.

Step 4 Find the critical path and identify the critical activities.

Step 5 Compute the project length variance σ^2, which is the sum of the variance of all the critical activities and hence, find the standard deviation of the project length σ.

Step 6 Calculate the standard normal variable $Z = \frac{T_s - T_e}{\sigma}$, where T_s is the scheduled time to complete the project.

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