**MTH-203 Multivariate Calculus [Monsoon 2022]**
**Midsem**

**Date: 16/10/22** **Max mark: 35**

**Attempt any 5 out of 7 questions, each question weightage 7 marks**

**Q1** The derivative of f(x,y) at $P_o(1,2)$ in the direction of $\hat{i}+\hat{j}$ is $2\sqrt{2}$ and in the direction of $-2\hat{j}$ at $P_o(1,2)$ is -3.What is the derivative of f at $P_o(1,2)$ in the direction of $-\hat{i}-2\hat{j}$ ?

**Q2** Find the maximum and minimum of $f(x,y,z) = 4y-2z$ subject to the constraints $2x-y-z=2$ and $x^2+y^2=1$

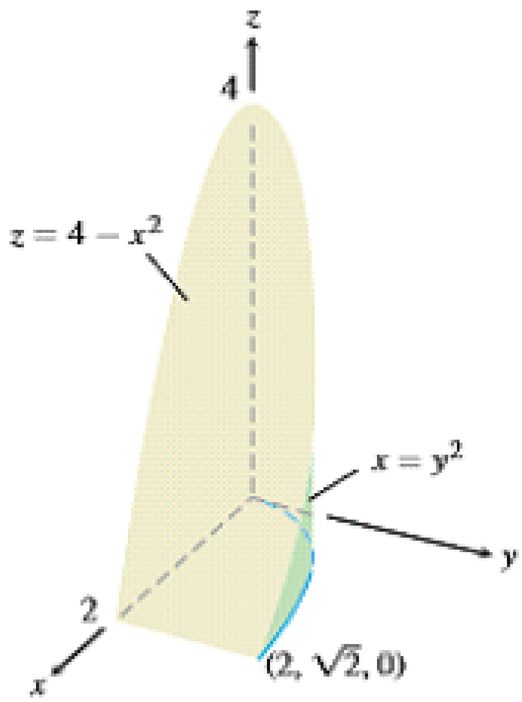
**Q3** Find the area of the region common to the interiors of the cardioids $r=1+\cos\theta$ and $r=1-\cos\theta$

**Q4** Evaluate $\iiint |xyz|\, dxdydz$, over the solid ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$

**Q5** Find a unit vector orthogonal to A in the plane of B and C, if $A = 2\hat{i}-\hat{j}+\hat{k}$, $B=\hat{i}+2\hat{j}+\hat{k}$ and $C=\hat{i}+\hat{j}-2\hat{k}$

**Q6** What angle does the line of intersection of the planes $2x+y-z=0$ and $x+y+2z=0$ make with the positive x-axis ?

**Q7** A solid in the first octant is bounded by the planes $y=0$ and $z=0$ and by the surfaces $z=4-x^2$ and $x=y^2$ (figure given below). Its density function is $\delta(x,y,z)=kxy$, where k is a constant. Find the mass of the solid.



# Midsem Solution

Q1 let $u_1 = \hat{i} + \hat{j} \longrightarrow \hat{u}_1 = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$ ] 0.5

given $u_2 = -2\hat{j} \longrightarrow \hat{u}_2 = -\hat{j}$ ] 0.5

$$D_{u_1}f = \vec{\nabla} f \cdot \hat{u}_1 = 2\sqrt{2}$$
$$D_{u_2}f = \vec{\nabla} f \cdot \hat{u}_2 = -3$$
] 0.5

$$D_{u_1}f = f_x(1,2)\frac{1}{\sqrt{2}} + f_y(1,2)\frac{1}{\sqrt{2}} = 2\sqrt{2}$$

$$\Rightarrow f_x(1,2) + f_y(1,2) = 4 \quad \text{——} \quad (1)$$
] 1.5

Also,

$$D_{u_2}f = f_x(1,2)(0) + f_y(1,2)(-1) = -3$$

$$\Rightarrow \quad f_y(1,2) = 3$$

Then from Eq (1), $f_x(1,2) = 1$ ] 1.5

$$\Rightarrow \nabla f(1,2) = f_x(1,2)\hat{i} + f_y(1,2)\hat{j}$$
$$= \hat{i} + 3\hat{j}$$
] 1

Now let $u = -\hat{i} - 2\hat{j} \Rightarrow \hat{u} = -\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}$ ] 0.5

$$\Rightarrow (D_u f)_{P_0} = \nabla f \cdot u = -\frac{1}{\sqrt{5}} - \frac{6}{\sqrt{5}} = -\frac{7}{\sqrt{5}}$$
] 1

Q2:- $f = 4y - 2z$, $g = 2x - y - z - 2$, $h = x^2 + y^2 - 1$

$\nabla f = 4\hat{j} - 2\hat{k}$ — 0.5

$\nabla g = 2\hat{i} - \hat{j} - \hat{k}$ — 0.5

$\nabla h = 2x\hat{i} + 2y\hat{j}$ → 0.5

$\nabla f = \lambda \nabla g + \mu \nabla h$ → 0.5

$4\hat{j} - 2\hat{k} = \lambda(2\hat{i} - \hat{j} - \hat{k}) + \mu(2x\hat{i} + 2y\hat{j})$

$\Rightarrow 2y\mu - \lambda = 4$

$2\lambda + 2x\mu = 0$

$\boxed{2 = \lambda}$ — 0.5

$\Rightarrow x = -2/\mu$, $y = 3/\mu$

Put in $x^2 + y^2 = 1$, we get

$\mu = \pm\sqrt{13}$ — 0.5

If $\mu = \sqrt{13} \Rightarrow x = -2/\sqrt{13}$, $y = 3/\sqrt{13}$ — 0.5

If $\mu = -\sqrt{13} \Rightarrow x = 2/\sqrt{13}$, $y = -3/\sqrt{13}$ → 0.5

Put ~~x~~ and y in $g(x,y,z)$, we get.

$z = -2 + 7/\sqrt{13}$ $\left(x = \frac{2}{\sqrt{13}}, y = -3/\sqrt{13}\right)$ — 0.5

$z = -2 - 7/\sqrt{13}$ $\left(x = -2/\sqrt{13}, y = 3/\sqrt{13}\right)$ → 0.5

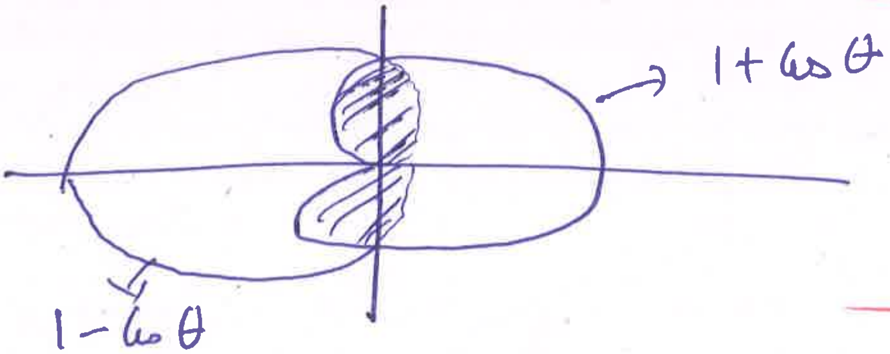
Two points $\left(\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, \frac{-7}{\sqrt{13}}-2\right)$ and
$\left(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}, \frac{7}{\sqrt{13}}-2\right)$

Put in $F(x,y,z)$ to get minimum and maximum value

$$F\left(\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, -\frac{7}{\sqrt{13}}-2\right) = 4+\frac{26}{\sqrt{13}} \quad (\max) \quad 1$$

$$F\left(\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}, -2+\frac{7}{\sqrt{13}}\right) = 4-\frac{26}{\sqrt{13}} \quad (\min) \quad 1$$

Q3 $r = 1+\cos\theta$, $r = 1-\cos\theta$




2

All four region are symmetrical.

$$\text{So } \text{Area} = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{1-\cos\theta} r\,dr\,d\theta \quad 2.5$$

$$= 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{1-\cos\theta} d\theta$$

$$= 2\int_0^{\pi/2}[1+\cos^2\theta-2\cos\theta]\,d\theta$$

$$= 2\int_0^{\pi/2}\left[1+\frac{1+\cos 2\theta}{2}-2\cos\theta\right]d\theta$$

1.5 for calculation

$$\text{Area} = \cancel{2} \int_0^{\pi/2} \left[\frac{3 - 4\cos\theta + \cos 2\theta}{\cancel{2}}\right] d\theta$$

$$= \left[3\theta - 4\sin\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$\boxed{\text{Area} = \frac{3\pi}{2} - 4}$$ (1)

Q4 let $x = au,\ y = bv,\ z = cw$ (1)

$$J(u,v,w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$ (1.5)

$$I = \iiint |xyz|\, dx\, dy\, dz$$

$$= \iiint a^2b^2c^2\, uvw\, dw\, dv\, du$$

(2.5)

$$= 8a^2b^2c^2 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 (\rho \sin\phi \cos\theta) \cdot \left[(\rho \sin\phi \sin\theta)(\rho\cos\phi)(\rho^2 \sin\phi)\right] d\rho\, d\phi\, d\theta$$

$$= \frac{4a^2b^2c^2}{3} \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \cos\theta \sin^3\phi \cos\phi\, d\phi\, d\theta$$ (1)

$$= \frac{a^2b^2c^2}{3} \int_0^{\pi/2} \sin\theta \cos\theta\, d\theta = \frac{a^2b^2c^2}{6}$$ (1)

for correct answer.

Q5 $\vec{B} = \hat{i} + 2\hat{j} + \hat{k}$ , $\vec{C} = \hat{i} + \hat{j} - 2\hat{k}$

B

The vector $\vec{B} \times \vec{C}$ is normal to the plane of B and C
$\Rightarrow$ $\vec{A} \times (\vec{B} \times \vec{C})$ is orthogonal to A and parallel to the plane of B and C (1)

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = -5\hat{i} + 3\hat{j} - k \quad (1.5)$$

$$\text{and } \vec{u} = \vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{i} & j & k \\ 2 & -1 & 1 \\ -5 & 3 & -1 \end{vmatrix} = -2\hat{i} - 3\hat{j} + k \quad (2)$$

$\Rightarrow$

let $|\vec{u}| = |A \times (\vec{B} \times \vec{C})| = \sqrt{4 + 9 + 1} = \sqrt{14}$ (1.5)

$\left[\hat{u} = \dfrac{\vec{u}}{|\vec{u}|} = \dfrac{1}{\sqrt{14}} (-2\hat{i} - 3\hat{j} + k)\right]$ is the desired unit vector. (1)

Q 6 The direction of the intersection of the planes $2x+y-z=0$ and $x+y+2z=0$ is

$$\vec{v} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 1 & +2 \end{vmatrix}$$

$$\vec{v} = 3\hat{i} - 5\hat{j} + \hat{k}$$ (3)

The angle $(\theta)$ between $\vec{v}$ and $x$-axis $(\hat{i})$

$$\cos\theta = \frac{\vec{v}\cdot\hat{i}}{|\vec{v}|\,|\hat{i}|} = \frac{\vec{v}\cdot\hat{i}}{|\vec{v}|}$$ (2)

$$\cos\theta = \frac{(3\hat{i}-5\hat{j}+\hat{k})\cdot\hat{i}}{\sqrt{3^2+5^2+1}}$$ (2)

$$\cos\theta = \frac{3}{\sqrt{35}}$$

$$\boxed{\theta = \cos^{-1}\left(3/\sqrt{35}\right)}$$

Q7 :

$$M = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} k\,xy\,dz\,dy\,dx$$ (3)

$$= k\int_0^2 \int_0^{\sqrt{x}} [z]_0^{4-x^2}\,xy\,dy\,dx$$

$$= k\int_0^2 \int_0^{\sqrt{x}} xy\,(4-x^2)\,dy\,dx$$

(3) for correct calculation

$$= k\int_0^2 \int_0^{\sqrt{x}} (4xy - x^3y)\,dy\,dx$$

$$= k\int_0^2 \left[\frac{4xy^2}{2} - \frac{x^3y^2}{2}\right]_0^{\sqrt{x}} dx$$

$$= k\int_0^2 \left(\frac{4x^2}{2} - \frac{x^4}{2}\right) dx$$

$$= k\left[\frac{2x^3}{3} - \frac{x^5}{10}\right]_0^2$$

$$= k\left[\frac{16}{3} - \frac{2^5}{10}\right]$$

$$= 2^4 k\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$\boxed{M = \frac{32k}{15}}$$ — (1)

Another 2 forms for M, Answer will be same

$$M = \int_0^{\sqrt{2}} \int_{y^2}^{2} \int_0^{4-x^2} kxy\,dz\,dx\,dy$$

$$M = \int_0^4 \int_0^{\sqrt{4-z}} \int_0^{\sqrt{x}} kxy\,dy\,dx\,dz$$