MTH-203 Multivariate Calculus [Monsoon 2022]
Endsem

Time : 2 hour **Max.marks: 35**
Date: 15/12/22

Instruction :
1) Attempt any 7 question out of 9, each question weighted 5 marks
2) First 7 attempted question will be checked
3) Please justify your answer with appropriate mathematical justification. An answer without justification may fetch zero marks.
4) Use of electronic gadget(laptop, tab,calculator, mobile phone)/ cheat sheet is NOT allowed

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Q1 Graph the curve

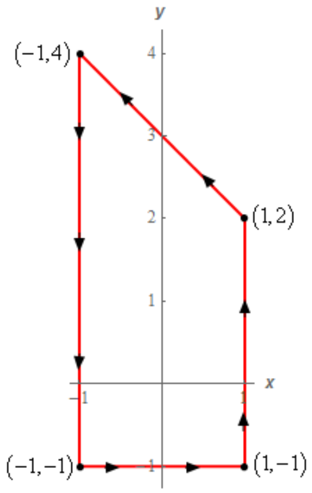
$$r(t) = (4\cos t)\hat{i} + (\sqrt{2}\sin t)\hat{j}$$

and sketch their velocity and acceleration vectors at the given values of t. Then write **a** in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ without finding **T** and **N** and find the value of κ at t = 0 and $t = \frac{\pi}{4}$

Q2 Use Green's Theorem to evaluate

$$\int_C (6y - 9x)dy - (yx - x^3)dx$$

where C is shown below



Q3 Integrate $g(x, y, z) = x\sqrt{y^2 + 4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes x = 0, x = 1 and z = 0

Q4 Use the surface intergral in Stokes theorem to calculate the flux of the curl of the field

$$\mathbf{F} = 2z\hat{i} + 3x\hat{j} + 5y\hat{k}$$

across the surface

$$S : \Gamma(r, \theta) = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j} + (4 - r^2)\hat{k}, \qquad 0 \le r \le 2, 0 \le \theta \le 2\pi$$

in the direction of the outward unit normal **n**

Q5 Find the harmonic conjucate of $e^x \cos y + e^y \cos x + xy$

Q6 Find z for the following equation

(a) $e^{\frac{1}{z}} = 1 - \iota$

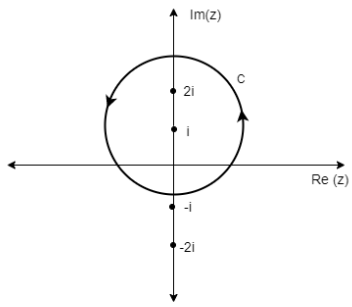
(b) $e^{e^z} = \iota$

Q7 Find the maximum modulus for $sin(z)$ on the square $[0, 2\pi] \times [0, 2\pi]$ and at what points

Q8 Evaluate

$$\int_C \frac{1}{(z^2+4)^2} dz$$

over the contour as shown in the figure below



Q9 The given series is convergent or divergent, justify your answer

$$\frac{1}{2^2+1} + \frac{\sqrt{2}}{3^2+1} + \frac{\sqrt{3}}{4^2+1} + ...$$

Endsem Solution

Q1 (5 marks)

$r(t) = (4\cos t)\hat{i} + (\sqrt{2}\sin t)\hat{j}$

$\Rightarrow x = 4\cos t, \quad y = \sqrt{2}\sin t$

$\Rightarrow \dfrac{x^2}{16} + \dfrac{y^2}{2} = 1$ 1/2

$v = \dfrac{dr}{dt} = (-4\sin t)\hat{i} + (\sqrt{2}\cos t)\hat{j}$ 1/2

$a = \dfrac{dv}{dt} = (-4\cos t)\hat{i} + (\sqrt{2}\sin t)\hat{j}$ 1/2

$r(0) = 4\hat{i}, \quad v(0) = \sqrt{2}\hat{j}, \quad a(0) = -4\hat{i}$

1/2 $r(\pi/4) = 2\sqrt{2}\hat{i} + \hat{j}, \quad v(\pi/4) = -2\sqrt{2}\hat{i} + \hat{j}, \quad a\left(\frac{\pi}{4}\right) = -2\sqrt{2}\hat{i} - \hat{j}$

$|v| = \sqrt{16\sin^2 t + 2\cos^2 t}$

$\Rightarrow a_T = \dfrac{d}{dt}|v| = \dfrac{14\sin t\cos t}{\sqrt{16\sin^2 t + 2\cos^2 t}}$ 1/2

Or for general a_T 1/2

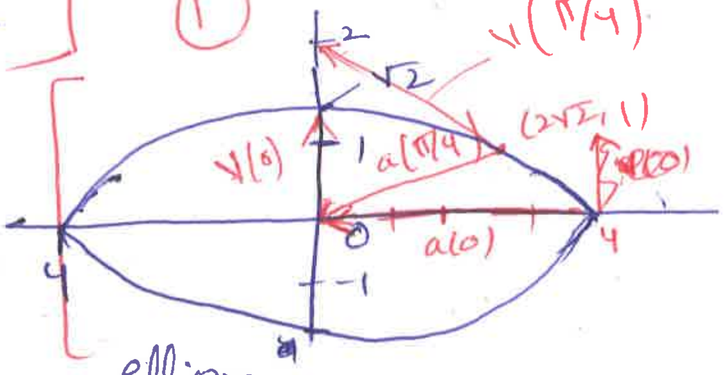
At $t = 0$, $a_T = 0$, At $t = \pi/4$, $a_T = 7/3$

for $t = 0$ $a_N = \sqrt{|a|^2 - a_T} = 4$ | for $t = \pi/4$ $a_N = \dfrac{4\sqrt{2}}{3}$ 1/2

For $t=0$, $K = \dfrac{a_N}{|v|^2} = \dfrac{4}{2} = 2$

For $t = \pi/4$, $K = \dfrac{a_n}{|v|^2} = \dfrac{4\sqrt{2}}{27}$ 1

1



ellipse

Q2 To Evaluate $\int_C (6y-9x)\,dy-(yx-x^3)\,dx$

5 marks

From the integral, we have

$$P = -(yx-x^3) = x^3 - yx$$

$$Q = 6y - 9x$$

Using Green's Th, the line integral becomes,

$$I \quad \int_C (6y-9x)\,dy-(yx-x^3)\,dx$$

$\frac{\partial Q}{\partial x}$ $\quad$ $\frac{\partial P}{\partial y}$

$$I = \iint_D -9-(-x)\,dA = \iint_D (x-9)\,dA$$

1.5

D is the region enclosed by the curve.

$$-1 \le x \le 1$$

$$-1 \le y \le 3-x$$

1

$$\Rightarrow I = \int_{-1}^{1}\int_{-1}^{3-x} (x-9)\,dy\,dx$$

$$= \int_{-1}^{1} \Big[(x-9)\,y\Big]_{-1}^{3-x} dx$$

2

$$= \int_{-1}^{1} (x-9)(4-x)\,dx$$

$$= \int_{-1}^{1} (-x^2+13x-36)\,dx$$

$$= \left[-\frac{1}{3}x^3+\frac{13}{2}x^2-36x\right]_{-1}^{1}$$

$$= \frac{-218}{3}$$

0.5

Q3 marks

$f(x,y,z) = y^2 + 4z = 16$

$\Rightarrow \left[\nabla f = 2y\hat{j} + 4k \Rightarrow |\nabla f| = \sqrt{4y^2+16} = 2\sqrt{y^2+4} \right]$ (1)

and $p = k \Rightarrow |\nabla f \cdot p| = 4$ — (0.5)

$\Rightarrow \left[d\sigma = \dfrac{2\sqrt{y^2+4}}{4}\, dx\, dy \right]$ (1)

$d\sigma = \dfrac{|\nabla f|\, dx dy}{|\nabla f \cdot p|}$

(1) $\Rightarrow \left[\iint_S g\, d\sigma = \int_{-4}^{4}\int_0^1 \left(x\sqrt{y^2+4}\right)\left(\dfrac{\sqrt{y^2+4}}{2}\right) dx\, dy \right]$

(1) $\left[= \int_{-4}^{4}\int_0^1 \dfrac{x(y^2+4)}{2}\, dx\, dy \right.$

$\left. = \int_{-4}^{4} \frac{1}{4}(y^2+4)\, dy = \frac{1}{2}\left[\frac{y^3}{3} + 4y\right]_0^4 \right]$

$= \frac{1}{2}\left[\frac{64}{3} + 16\right] = \left[\frac{56}{3}\right]$ (0.5)

Q4 marks

$F = 2z\,\hat{i} + 3x\,\hat{j} + 5y\,\hat{k}$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5\hat{i} + 2\hat{j} + 3\hat{k}$$ 0.5

$r_r = (\cos\theta)\,\hat{i} + (\sin\theta)\,\hat{j} - 2r k$ 0.5

$r_\theta = (-r\sin\theta)\,\hat{i} + (r\cos\theta)\,\hat{j}$ 0.5

$$\Rightarrow r_r \times r_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= (2r^2\cos\theta)\,\hat{i} + (2r^2\sin\theta)\,\hat{j} + r k$$ 1

$$\hat{n} = \frac{r_r \times r_\theta}{|r_r \times r_\theta|} \quad \text{and} \quad d\sigma = |r_r \times r_\theta|\,dr\,d\theta$$

0.5 $\Rightarrow \nabla \times F \cdot n\,d\sigma = (\nabla \times F)\cdot(r_r \times r_\theta)\,dr\,d\theta$

$= (10r^2\cos\theta + 4r^2\sin\theta + 3r)\,dr\,d\theta$

1 $$\Rightarrow \iint_S \nabla \times F \cdot n\,d\sigma = \int_0^{2\pi}\int_0^{2} [10r^2\cos\theta + 4r^2\sin\theta + 3r]\,dr\,d\theta$$

0.5 $$= \int_0^{2\pi} \left[\frac{10}{3} r^3\cos\theta + \frac{4}{3} r^3\sin\theta + \frac{3}{2} r^2\right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left(\frac{80}{3}\cos\theta + \frac{32}{3}\sin\theta + 6\right) d\theta = 6(2\pi)$$

0.5 $= \underline{12\pi}$

Q5 let $u(x,y) = e^x \cos y + e^y \cos x + xy$

5 marks

We need to find a function $v(x,y)$ such that $f = u + iv$ is analytic, ie

$$u_x = v_y \text{ and } u_y = -v_x$$

(1)

we have

$$u_x = e^x \cos y - e^y \sin x + y = v_y$$

(0.5)

Integrating u_x with respect to y, we get

(1) $$v(x,y) = e^x \sin y - e^y \sin x + \frac{y^2}{2} + A(x) \quad \text{—(1)}$$

where $A(x)$ is an arbitary function of x.

On the other hand,

(0.5) $$u_y = -e^x \sin y + e^y \cos x + x = -v_x$$

Integrate above w.r.t. x

(1) $$v(x,y) = e^x \sin y - e^y \sin x - \frac{x^2}{2} + B(y) \quad \text{—(2)}$$

$B(y)$ is an arbitary f^n of y

Combining the two Expressions, we conclude (Eq (1) & Eq (2)) that

(1)

$$v(x,y) = e^x \sin y - e^y \sin x - \frac{x^2}{2} + \frac{y^2}{2}$$

Satisfies $u_x = v_y$ and $u_y = -v_x$

Q6 (a) $e^{1/z} = 1 - i$, Find z

2.5 marks

Defination If $e^{w} = z$ then $w = \log z$

0.5

$\Rightarrow \left[\frac{1}{z} = \log(1-i) \right] \quad \left[\log z = \log|z| + i \arg(z) \right]$

$\left[\frac{1}{z} = \log|1-i| + i \arg(1-i) \right]$ 1

$\arg(1-i) = -\frac{\pi}{4} + 2n\pi$

$\Rightarrow \left[\underline{\frac{1}{z} = \log\sqrt{2} + i\left[-\frac{\pi}{4} + 2n\pi\right]} \right]$ 1

(b) $e^{e^{z}} = i$

2.5 marks

$\Rightarrow \left[e^{z} = \log i = i\left[\frac{\pi}{2} + 2n\pi\right] \right]$ 0.5

[By some concept as part (a)]

0.5 $\Rightarrow \left[e^{z} = ia, \quad a = \frac{\pi}{2} + 2n\pi \text{ (say)} \right]$

$\left[\begin{aligned} z &= \log(ia) \\ &= \log|ia| + i \arg(ia) \end{aligned} \right]$ 0.5

1 $\left[z = \begin{cases} \log|a| + i\left(\frac{\pi}{2} + 2k\pi\right), & a > 0 \\ \log|a| + i\left(-\frac{\pi}{2} + 2k\pi\right), & a < 0 \end{cases} \right]$

Q1 We use the formula,

5 marks

$$\left[\sin z = \sin x \cosh y + i \cos x \sinh y\right]$$ 0.5

Now

$$|\sin(z)|^2 = \sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x$$

$$= \sin^2 x \cosh^2 y + (1 - \sin^2 x)\sinh^2 y$$

1.5

$$= \sin^2 x (\cosh^2 y - \sinh^2 y) + \sinh^2 y$$

1

$$= \sin^2 x + \sinh^2 y$$

We know the maximum of $\sin^2 x$ is at $x = \pi/2$ and $x = 3\pi/2$

1.5

$x \in [0, 2\pi]$

The max of $\sinh^2 y$ is at $y = 2\pi$

So max modulus is

$$\sqrt{1 + \sinh^2(2\pi)} = \sqrt{\cosh^2(2\pi)} = \cosh(2\pi)$$ 1

This occurs at the points

$$z = x + iy = \frac{\pi}{2} + 2\pi i \text{ and } 3\frac{\pi}{2} + 2\pi i$$

0.5

Both these points are on the boundary of the region.

Q8 $I = \int_C \frac{1}{(z^2+4)^2} dz$

5 marks

① $\left[\frac{1}{(z^2+4)^2} = \frac{1}{(z^2-(2i)^2)^2} = \frac{1}{(z-2i)^2(z+2i)^2} \right]$

Let us consider $f(z) = 1/(z+2i)^2$

① So, we can say that $f(z)$ is an analytic function inside C

Thus, by cauchy's integral formula, we can write it as

① $\left[I = \oint_C \frac{1}{(z^2+4)^2} dz = \int_C \frac{f(z)}{(z-2i)^2} = 2\pi i \, f'(2i) \right]$

② $f'(z) = -2/(z+2i)^3$

$I = 2\pi i \left[\frac{-2}{(2i+2i)^3} \right] = 2\pi i \times \frac{-2}{(4i)^3}$

$I = \pi/16$

Q9 $S = \frac{1}{2^2+1} + \frac{\sqrt{2}}{3^2+1} + \frac{\sqrt{3}}{4^2+1} + \cdots$

$$\Rightarrow S = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n+1)^2+1}$$ 1

$$\Rightarrow a_n = \frac{\sqrt{n}}{(n+1)^2+1}$$

2 1/2

for any $n \in \mathbb{N}$

$$\frac{\sqrt{n}}{(n+1)^2+1} < \frac{\sqrt{n}}{(n+1)^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

Since $\sum \frac{1}{n^{3/2}}$ converges, By comparison Test

1.5

$\sum \frac{n}{(n+1)^2+1}$ is convergent.