End-Semester Exam: Math-I (Linear Algebra)

Indraprastha Institute of Information Technology, Delhi

22nd November, 2019

Duration: 120 minutes **Maximum Marks:** 80

Instructions:

1. Commence each answer to a question on a fresh page. If some part of a question is done later, it should also be commenced on a fresh page, and this should be clearly mentioned in the main question.
2. You may use without proof any result covered in the course (either in lecture or tutorial). However, it should be clearly identified. Results taken from other sources must be proved.

Question 1.

(a) (6 marks) Find $col(A), null(A)$ and $row(A)$ for the matrix

$$A = \begin{bmatrix} 5 & 2 & -4 \\ -5 & 2 & 16 \\ 0 & 7 & 21 \end{bmatrix}$$

and a basis for each of the three.

(b) (4 marks) Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for

$$\mathbf{b} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}?$$

Answer 1.

First Solution:

(a) The cross product of the row vectors $(5, 2, -4)$ and $(-5, 2, 16)$ is $(40, -60, 20)$. As this is orthogonal to the row vector $(0, 7, 21)$, it follows that the nullspace of A is non-trivial, therefore

$$3 > \dim row(A) = \dim null(A).$$

As the first two rows of A are not linear multiples of each other, a basis of $row(A)$ is

$$\{(5, 2, -4), (-5, 2, 16)\}.$$

As the first two columns of A are not linear multiples of each other, a basis of $col(A)$ is

$$\{(5, -5, 0), (2, 2, 7)\}.$$

Thus the nullspace of A is 1-dimensional and a basis for $null(A)$ is

$$(40, -60, 20).$$

(b) We solve the equation

$$c_1(5, -5, 0) + c_2(2, 2, 7) = (-4, 4, 2)$$

for c_1 and c_2. As this equation has no solution, the vector $(-4, 4, 2)$ is not in $col(A)$.

Rubric: 1 mark for finding the cross product, 2 marks for showing that it is orthogonal to the third row. 1 mark each for bases of fundamental subspaces. 4 marks for solving the system of equations mentioned in part (b).

Second Solution:

We compute the RREF of $[A \quad \mathbf{b}]$:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Since the pivot columns of A are the first two columns, a basis of $col(A)$ is

$$\{(5, -5, 0), (2, 2, 7)\}.$$

The rows of the RREF of A which contain pivots are a basis of $row(A)$, namely

$$\{(1, 0, -2), (0, 1, 3)\}.$$

We solve the system of equations

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned}$$

to obtain the basis

$$\{(2, -3, 1)\}$$

for $null(A)$.

(b) The equation $A\mathbf{x} = \mathbf{b}$ has no solution because the augmented column of $[A \quad \mathbf{b}]$ is a pivot column.

Rubric: 2 marks for the RREF. 1 mark each for writing the correct bases of fundamental subspaces. 2 marks for solving the system of equations obtained from the RREF to calculate the basis vector for the nullspace. 3 marks for using the RREF of the augemented matrix to correctly determine whether the system $A\mathbf{x} = \mathbf{b}$ has a solution.

Question 2 (10 marks)**.** Let $V = C[0, \pi]$, the vector space of all continuous functions defined on the interval $[0, \pi]$. Equip V with the inner product

$$\langle f, g\rangle := \int_0^\pi f(x)g(x)\,dx\,.$$

Find an orthonormal basis for the subspace

$$W = Span\{1, \sin x, \sin^2 x, \sin^3 x\},$$

using the Gram-Schmidt algorithm.

Answer 2. Let

$$\begin{aligned}
\mathbf{v}_1 &= 1\\
\mathbf{v}_2 &= \sin x\\
\mathbf{v}_3 &= \sin^2 x\\
\mathbf{v}_4 &= \sin^3 x.
\end{aligned}$$

Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_4\}$ denote the vectors obtained by using the Gram Schmidt algorithm and let

$$\hat{\mathbf{w}}_j = \frac{\mathbf{w}_j}{\|\mathbf{w}_j\|} \qquad \text{for } j = 1, \ldots, 4$$

be the normalized vectors. Then

$$\begin{aligned}
\mathbf{w}_1 &= 1\\
\hat{\mathbf{w}}_1 &= \frac{1}{\sqrt{\pi}}\\
\mathbf{w}_2 &= \sin x - \frac{2}{\pi}\\
\hat{\mathbf{w}}_2 &= \frac{2\pi}{\pi^2 - 8}\left(\sin x - \frac{2}{\pi}\right)\\
\mathbf{w}_3 &= \sin^2 x - \frac{2\pi}{3(\pi^2 - 8)}\left(\sin x - \frac{2}{\pi}\right) - \frac{1}{2}\\
\hat{\mathbf{w}}_3 &= \frac{72(\pi^2 - 8)}{\pi(9\pi^2 - 88)}\left(\sin^2 x - \frac{2\pi}{3(\pi^2 - 8)}\left(\sin x - \frac{2}{\pi}\right) - \frac{1}{2}\right)\\
\mathbf{w}_4 &= \sin^3 x - \frac{45\pi^2 - 436}{45\pi^2 - 440}\left(\sin^2 x - \frac{2\pi}{3(\pi^2 - 8)}\left(\sin x - \frac{2}{\pi}\right)\right) - \frac{19\pi}{\pi^2 - 8}\left(\sin x - \frac{2}{\pi}\right) - \frac{3}{8}
\end{aligned}$$

$\hat{\mathbf{w}}_4$ was not computed by anyone, so DIY.

Rubric: 1 mark for formula and $\mathbf{w}_1$. 5 marks for $\mathbf{w}_2$. 2 marks for $\mathbf{w}_3$. 1 mark for $\mathbf{w}_4$. 1 mark for normalization.

The rest will be typed up later.