End-Semester Exam: Math-I (Linear Algebra)

Indraprastha Institute of Information Technology, Delhi

22nd November, 2019

Duration: 120 minutes **Maximum Marks:** 80

Instructions:

1. Commence each answer to a question on a fresh page. If some part of a question is done later, it should also be commenced on a fresh page, and this should be clearly mentioned in the main question.
2. You may use without proof any result covered in the course (either in lecture or tutorial). However, it should be clearly identified. Results taken from other sources must be proved.

Question 1.

(a) (6 marks) Find $col(A), null(A)$ and $row(A)$ for the matrix

$$A = \begin{bmatrix} 5 & 2 & -4 \\ -5 & 2 & 16 \\ 0 & 7 & 21 \end{bmatrix}$$

and a basis for each of the three.

(b) (4 marks) Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for

$$\mathbf{b} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}?$$

Question 2 (10 marks)**.** Let $V = C[0, \pi]$, the vector space of all continuous functions defined on the interval $[0, \pi]$. Equip V with the inner product

$$\langle f, g \rangle := \int_0^\pi f(x)g(x)\,dx\,.$$

Find an orthonormal basis for the subspace

$$W = Span\{1, \sin x, \sin^2 x, \sin^3 x\},$$

using the Gram-Schmidt algorithm.

Question 3. Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x, y, z) = (-x - y + z, x - 4y + z, 2x - 5y).$$

(a) (1 mark) Find the matrix of T with regard to the standard basis $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(b) (2 marks) Find the change of basis matrix $P_{S\to\mathcal{B}}$ where $\mathcal{B}$ is the ordered basis

$$\{\mathbf{u}_1 = (1, 0, 0), \mathbf{u}_2 = (1, 1, 0), \mathbf{u}_3 = (-1, -1, 1)\}.$$

(c) (5 marks) Find the matrix $[T]_\mathcal{B}$ of the operator T with regard to the ordered basis $\mathcal{B}$.

(d) (2 marks) Find $[T\mathbf{b}]_\mathcal{B}$ where $\mathbf{b} = (1, 2, -1)$.

NB: Regard all vectors as column vectors.

Question 4 (10 marks)**.** Find a diagonal matrix D, and an orthogonal matrix P such that $A = PDP^{-1}$, where A is given below.

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}.$$

Question 5. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be any function for which

$$\mathbf{x} \cdot \mathbf{y} = T(\mathbf{x}) \cdot T(\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

(a) (3 marks) Show that the vectors $\{T(\mathbf{e})_\mathbf{1}, \ldots, \mathbf{T}(\mathbf{e_n})\}$ form an orthonormal basis of $\mathbb{R}^n$.

(Recall that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the columns of the identity matrix I_n.)

(b) (7 marks) Show that T is a linear transformation.

(Hint: Consider coordinates with respect to suitable bases.)

Question 6. Let V be a vector space over the field $\mathbb{R}$. A linear transformation $T : V \to \mathbb{R}$ is called a *linear functional*. The vector space of all linear functionals is called the *dual space* of V, denoted by V^*.

(a) (1 marks) Suppose that $V = \mathbb{R}^n$ $(n > 1)$, and let $\mathbf{a}$ be a fixed but arbitrary vector in $\mathbb{R}^n$. Consider the mapping $T : \mathbb{R}^n \to \mathbb{R}$ given by

$$T\mathbf{x} = \mathbf{x} \cdot \mathbf{a}, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Show that T is a linear functional.

For parts (b),(c) and (d), assume that V is a finite-dimensional vector space with $\dim(V) = n$ $(n \geq 2)$.

(b) (2 mark) What is $\dim(V^*)$? Justify.

(c) (3 marks) If T is a non-zero linear functional, what is the nullity of T? Justify.

(d) (4 marks) Prove/disprove: If K is a hyperspace of V, then there exists a linear functional $T : V \to \mathbb{R}$ such that $\ker(T) = K$.

(NB: A hyperspace is subspace of dimension $n - 1$.)

Question 7. Consider the linear operator $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$, ($\mathbb{R}^{2\times 2}$ is also known as $M_{2\times 2}$) given by $T(X) = BX$, for any $X \in \mathbb{R}^{2\times 2}$, where

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

(a) (2 marks) Is T surjective (YES/NO)? Justify.

(b) (1 mark) Is T injective (YES/NO)? Justify.

(c) (2 marks) Find the matrix of T with regard to any suitable ordered basis.

(d) (5 marks) Find a basis for $\ker(T)$ and a basis for $\operatorname{range}(T)$, and determine $\mathit{rank}\, T$ and $\mathit{nullity}\, T$. Justify your answer.

Question 8. Let $\mathcal{B}$ be a set of non-zero polynomials in $\mathbb{R}_n(t)$ (also known as $\mathbb{P}_n$) such that each has a degree distinct from the rest.

(a) (8 marks) Show that $\mathcal{B}$ is a linearly independent set.

(b) (2 marks) Show that $\mathcal{B}$ is a basis for $\mathbb{R}_n(t)$ (aka $\mathbb{P}_n$), if it has exactly $n + 1$ elements.