End-Semester Exam: Math-I (Linear Algebra)

Indraprastha Institute of Information Technology, Delhi

11th April, 2021

Duration: 120 minutes **Maximum Marks:** 80

Instructions:

(i) Please DO NOT substitute approximate decimal values for π in any of your answers.

(ii) Section A students may assume that $F = \mathbb{R}$, in all questions in which a field F is mentioned.

(iii) Start each question on a fresh page, and indicate the page number if part of a question is answered elsewhere in the answer-book.

(iv) Marks will depend on the clarity and completeness of answers.

Question 1 (10 marks)**.** Suppose that V be a vector space over a field F. Let $T \in \mathcal{L}(V, V)$ (i.e. T is a linear transformation from V to V).

Let $v \in V$ be a vector such that $T^m v \neq 0$ but $T^{m+1}(v) = 0$ for some $m \geq 0$.

Show that $v, Tv, \ldots, T^m v$ are linearly independent.

Question 2 (10 marks)**.** Let $V = F^n$ and consider the operator $T : V \to V$ given by

$$T(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i, \ldots, \sum_{i=1}^{n} x_i \right)$$

(a) (1 mark) Construct the matrix A of T relative to any suitable basis of V.

(b) (6 marks) Determine the eigenvalues and corresponding eigenvectors of T.

(c) (3 marks) Is T diagonalizable (YES/NO)? Justify your answer briefly, by referring to a suitable result.

Question 3. Let $(V, \langle ., . \rangle)$ be a finite dimensional inner product space, and let U and W be subspaces of V.

(a) (5 marks) Let W be a subspace of V. Show that

$$(U + W)^{\perp} = U^{\perp} \cap W^{\perp}$$

(b) (5 marks) Show that

$$(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$$

Question 4 (10 marks)**.** Let $a, b \in \mathbb{R}$, and let $b \neq 0$. Orthogonally diagonalize

$$A = \begin{bmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{bmatrix}$$

Question 5 (10 marks)**.** Let

$$A = \begin{bmatrix} -49\pi & 20\pi \\ -136\pi & 55\pi \end{bmatrix}$$

Find an invertible matrix P and a matrix B of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that

$$A = PBP^{-1}.$$

Question 6.

(a) (4 marks) Find an LU factorization of the following matrix:

$$A = \begin{bmatrix} 2 & -4 & 2\pi^2 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$$

(b) (6 marks) Solve the equation $A\mathbf{x} = \mathbf{b}$ by using the LU-factorization of A that you obtained in part (a). **Do not use any other method.**

$$\mathbf{b} = \begin{bmatrix} -4\pi^2 \\ -12\pi^2 + 7 \\ 8 - 4\pi^2 \end{bmatrix}$$

Question 7. Let $V = C^1[-\pi, \pi]$, the set of all continuously differentiable functions defined on the interval $[-\pi, \pi]$.

(A function f is said to be continuously differentiable on $[-\pi, \pi]$, if f is differentiable at every point in $[-\pi, \pi]$ and its derivative f' is continuous on $[-\pi, \pi]$.)

(a) (2 marks) Show that V is a vector space over $\mathbb{R}$, under the usual operations of pointwise addition of functions and pointwise multiplication of a function by a scalar.

(You may assume without proof that every differentiable function is continuous, i.e. $C^1[-\pi, \pi] \subset C[-\pi, \pi]$.)

(b) (2 marks) Show that the mapping $\langle ., . \rangle : V \times V \to \mathbb{R}$, defined by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t + \int_{-\pi}^{\pi} f'(t)g'(t)\,\mathrm{d}t$$

is an inner product on V.

(c) (6 marks) Find an orthogonal basis for the subspace

$$W = \mathrm{Span}\{1, \cos t, \sin t, \cos^2 t\},$$

of V, with respect to the inner product defined in part (c).

Question 8 (10 marks)**.**

Let $n \geq 2$. Let $V = \mathbb{P}_n$, the vector space of polynomials of degree at most n, with real coefficients.

Let $\{p_1, p_2, p_3\}$ be a linearly independent subset of V. Let $A = (a_{ij})$ be a 3×3 matrix having real entries. Let

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Show that the $\{q_1, q_2, q_3\}$ is a linearly independent subset of V if and only if A is invertible.