

# Modeling 2DoF System

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June 19, 2025

## 1 Introduction

This document presents the modeling of a 2 degrees-of-freedom (DoF) system using Lagrangian mechanics and co-energy analysis. A Simulink block diagram is also provided for simulation purposes.

## 2 2DoF System Description

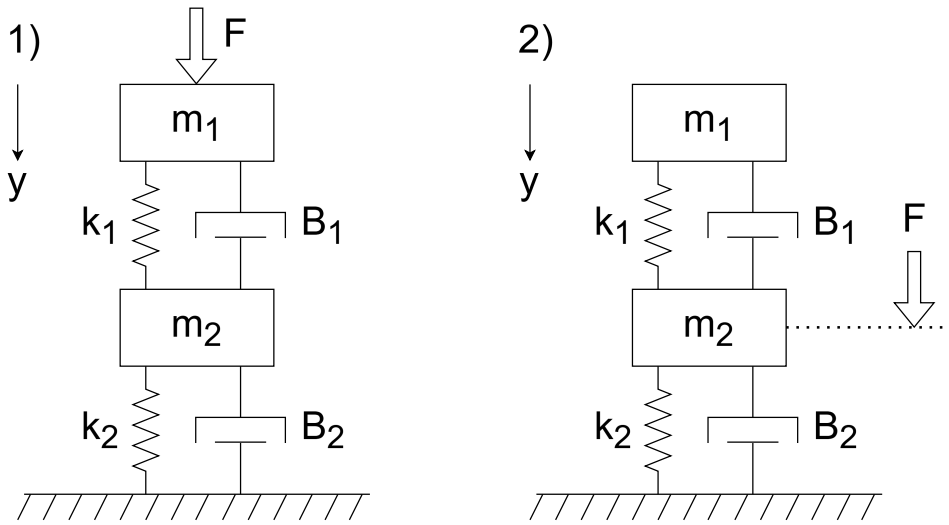


Figure 1: Overview of the 2DoF system

We consider a 2DoF system in 2 cases: first, where force is applied to outer mass  $m_1$ ; second, where force is applied to inner mass  $m_2$ . In this model, forcing can also be applied as displacement, given by initial conditions. In order to perform modeling, the 2DoF system must be uncoupled into 2 separate equivalent 1DoF subsystems.

### 3 Lagrangian Formulation

#### 3.1 Forcing on $m_1$ - subsystem 1a

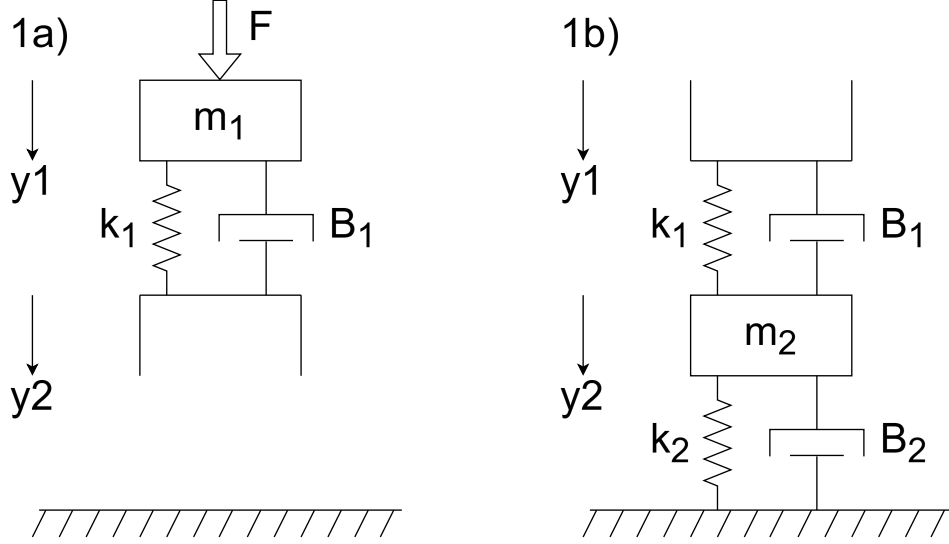


Figure 2: Uncoupled 2DoF system with forcing on  $m_1$

The kinetic and potential energies of subsystem 1a are defined as:

$$T_1 = \frac{1}{2}m_1\dot{y}_1^2$$

$$U_1 = k_1 \int_{y_1}^{y_2} y dy = \frac{1}{2}k_1 y_1^2 - \frac{1}{2}k_1 y_2^2$$

The system also has a damper, which is a dissipative element:

$$D_1 = B_1 \int_{\dot{y}_1}^{\dot{y}_2} \dot{y} d\dot{y} = \frac{1}{2}B_1 \dot{y}_1^2 - \frac{1}{2}B_1 \dot{y}_2^2$$

The Lagrangian of the subsystem 1a is:

$$\mathcal{L}(y, \dot{y}) = T_1 - U_1 = \frac{1}{2}m_1\dot{y}_1^2 - \left( \frac{1}{2}k_1 y_1^2 - \frac{1}{2}k_1 y_2^2 \right)$$

The Rayleigh dissipation function can be written as:

$$\mathcal{D} = D_1 - P_1 = \left( \frac{1}{2}B_1 \dot{y}_1^2 - \frac{1}{2}B_1 \dot{y}_2^2 \right) - F\dot{y}_1$$

Using Lagrange's equation with a non-conservative force (damper), the equation expands into the following form:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{D}}{\partial \dot{q}} = 0$$

This leads to the following differential equation:

$$\frac{d}{dt} \left( \frac{\partial(\frac{1}{2}m_1\dot{y}_1^2 - (\frac{1}{2}k_1y_1^2 - \frac{1}{2}k_1y_2^2))}{\dot{y}} \right) - \frac{\partial(\frac{1}{2}m_1\dot{y}_1^2 - (\frac{1}{2}k_1y_1^2 - \frac{1}{2}k_1y_2^2))}{y} + \frac{\partial((\frac{1}{2}B_1\dot{y}_1^2 - \frac{1}{2}B_1\dot{y}_2^2) - F\dot{y}_1)}{\dot{y}} = 0$$

$$\frac{d}{dt} \left( \frac{\partial(\frac{1}{2}m_1\dot{y}_1^2 - 0)}{\dot{y}} \right) - \frac{\partial(0 - (\frac{1}{2}k_1y_1^2 - \frac{1}{2}k_1y_2^2))}{y} + \frac{\partial((\frac{1}{2}B_1\dot{y}_1^2 - \frac{1}{2}B_1\dot{y}_2^2) - F\dot{y}_1)}{\dot{y}} = 0$$

$$\frac{d}{dt} \left( 2 \cdot \frac{1}{2}m_1\dot{y}_1^{2-1} \right) - \left( - \left( 2 \cdot \frac{1}{2}k_1y_1^{2-1} - 2 \cdot \frac{1}{2}k_1y_2^{2-1} \right) \right) + \left( \left( 2 \cdot \frac{1}{2}B_1\dot{y}_1^{2-1} - 2 \cdot \frac{1}{2}B_1\dot{y}_2^{2-1} \right) - F\dot{y}_1^{1-1} \right) = 0$$

$$\frac{d}{dt} (m_1\dot{y}_1) - (- (k_1y_1 - k_1y_2)) + ((B_1\dot{y}_1 - B_1\dot{y}_2) - F) = 0$$

$$\frac{d}{dt} (m_1\dot{y}_1) - (-k_1y_1 + k_1y_2) + (B_1\dot{y}_1 - B_1\dot{y}_2 - F) = 0$$

$$\frac{d}{dt} (m_1\dot{y}_1) + k_1y_1 - k_1y_2 + B_1\dot{y}_1 - B_1\dot{y}_2 - F = 0$$

$$m_1\ddot{y}_1 + k_1y_1 - k_1y_2 + B_1\dot{y}_1 - B_1\dot{y}_2 - F = 0$$

Sorting yields the equation's final form:

$$m_1\ddot{y}_1 + B_1\dot{y}_1 + k_1y_1 - B_1\dot{y}_2 - k_1y_2 = F$$

### 3.2 Forcing on $m_1$ - subsystem 1b

The kinetic and potential energies of subsystem 1b are defined as:

$$T_2 = \frac{1}{2}m_2\dot{y}_2^2$$

$$U_2 = -k_2 \int_0^{y_2} y dy + k_1 \int_{y_2}^{y_1} y dy = \frac{1}{2}k_2y_2^2 + \frac{1}{2}k_1y_2^2 - \frac{1}{2}k_1y_1^2$$

The system also has a damper, which is a dissipative element:

$$D_2 = -B_2 \int_0^{\dot{y}_2} \dot{y} dy + B_1 \int_{\dot{y}_2}^{\dot{y}_1} \dot{y} dy = \frac{1}{2}B_2\dot{y}_2^2 + \frac{1}{2}B_1\dot{y}_2^2 - \frac{1}{2}B_1\dot{y}_1^2$$

The Lagrangian of the subsystem 1a is:

$$\mathcal{L}(y, \dot{y}) = T_2 - U_2 = \frac{1}{2}m_2\dot{y}_2^2 - \left(\frac{1}{2}k_2y_2^2 + \frac{1}{2}k_1y_2^2 - \frac{1}{2}k_1y_1^2\right)$$

The Rayleigh dissipation function can be written as:

$$\mathcal{D} = D_2 - P_2 = \left(\frac{1}{2}B_2\dot{y}_2^2 + \frac{1}{2}B_1\dot{y}_2^2 - \frac{1}{2}B_1\dot{y}_1^2\right) - 0$$

Using Lagrange's equation with a non-conservative force (damper), the equation expands into the following form:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{D}}{\partial \dot{q}} = 0$$

This leads to the following differential equation:

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial \left( \frac{1}{2}m_2\dot{y}_2^2 - \left( \frac{1}{2}k_2y_2^2 + \frac{1}{2}k_1y_2^2 - \frac{1}{2}k_1y_1^2 \right) \right)}{\dot{y}} \right) - \\ \frac{\partial \left( \frac{1}{2}m_2\dot{y}_2^2 - \left( \frac{1}{2}k_2y_2^2 + \frac{1}{2}k_1y_2^2 - \frac{1}{2}k_1y_1^2 \right) \right)}{y} + \\ \frac{\partial \left( \left( \frac{1}{2}B_2\dot{y}_2^2 + \frac{1}{2}B_1\dot{y}_2^2 - \frac{1}{2}B_1\dot{y}_1^2 \right) - 0 \right)}{\dot{y}} = 0 \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial \left( \frac{1}{2}m_2\dot{y}_2^2 - 0 \right)}{\dot{y}} \right) - \\ \frac{\partial \left( 0 - \left( \frac{1}{2}k_2y_2^2 + \frac{1}{2}k_1y_2^2 - \frac{1}{2}k_1y_1^2 \right) \right)}{y} + \\ \frac{\partial \left( \left( \frac{1}{2}B_2\dot{y}_2^2 + \frac{1}{2}B_1\dot{y}_2^2 - \frac{1}{2}B_1\dot{y}_1^2 \right) - 0 \right)}{\dot{y}} = 0 \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \left( \left( 2 \cdot \frac{1}{2}m_2\dot{y}_2^{2-1} \right) \right) - \\ \left( - \left( 2 \cdot \frac{1}{2}k_2y_2^{2-1} + 2 \cdot \frac{1}{2}k_1y_2^{2-1} - 2 \cdot \frac{1}{2}k_1y_1^{2-1} \right) \right) + \\ \left( \left( 2 \cdot \frac{1}{2}B_2\dot{y}_2^{2-1} + 2 \cdot \frac{1}{2}B_1\dot{y}_2^{2-1} - 2 \cdot \frac{1}{2}B_1\dot{y}_1^{2-1} \right) \right) = 0 \end{aligned}$$

$$\frac{d}{dt} (m_2\dot{y}_2) - (- (k_2y_2 + k_1y_2 - k_1y_1)) + (B_2\dot{y}_2 + B_1\dot{y}_2 - B_1\dot{y}_1) = 0$$

$$\frac{d}{dt}(m_2 \dot{y}_2) + k_2 y_2 + k_1 y_2 - k_1 y_1 + B_2 \dot{y}_2 + B_1 \dot{y}_2 - B_1 \dot{y}_1 = 0$$

$$m_2 \ddot{y}_2 + k_2 y_2 + k_1 y_2 - k_1 y_1 + B_2 \dot{y}_2 + B_1 \dot{y}_2 - B_1 \dot{y}_1 = 0$$

Sorting and grouping yield the final form of the equation:

$$m_2 \ddot{y}_2 + (B_2 + B_1) \dot{y}_2 + (k_2 + k_1) y_2 - B_1 \dot{y}_1 - k_1 y_1 = 0$$

## Equation Results

Ultimately, the equation for the 2DoF system with forcing on  $m_1$  can be written as:

$$\begin{cases} m_1 \ddot{y}_1 + B_1 \dot{y}_1 + k_1 y_1 - B_1 \dot{y}_2 - k_1 y_2 = F \\ m_2 \ddot{y}_2 + (B_2 + B_1) \dot{y}_2 + (k_2 + k_1) y_2 - B_1 \dot{y}_1 - k_1 y_1 = 0 \end{cases}$$

### 3.3 Forcing on $m_2$ - subsystem 2a

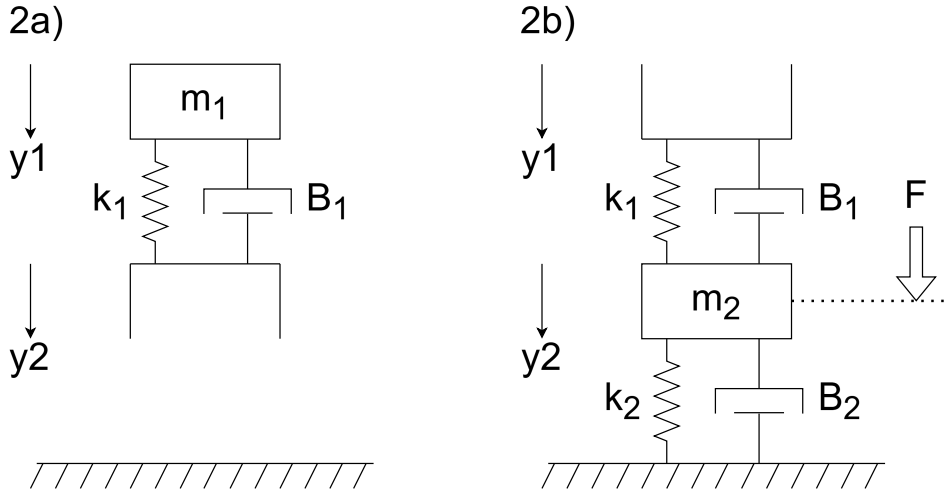


Figure 3: Uncoupled 2DoF system with forcing on  $m_2$

The kinetic and potential energies of subsystem 2a are defined as:

$$T_1 = \frac{1}{2} m_1 \dot{y}_1^2$$

$$U_1 = k_1 \int_{y_1}^{y_2} y dy = \frac{1}{2} k_1 y_1^2 - \frac{1}{2} k_1 y_2^2$$

The system also has a damper, which is a dissipative element:

$$D_1 = B_1 \int_{\dot{y}_1}^{\dot{y}_2} \dot{y} dy = \frac{1}{2} B_1 \dot{y}_1^2 - \frac{1}{2} B_1 \dot{y}_2^2$$

The Lagrangian of subsystem 2a is:

$$\mathcal{L}(y, \dot{y}) = T_1 - U_1 = \frac{1}{2}m_1\dot{y}_1^2 - \left(\frac{1}{2}k_1y_1^2 - \frac{1}{2}k_1y_2^2\right)$$

The Rayleigh dissipation function can be written as:

$$\mathcal{D} = D_1 - P_1 = \left(\frac{1}{2}B_1\dot{y}_1^2 - \frac{1}{2}B_1\dot{y}_2^2\right) - 0$$

Using Lagrange's equation with a non-conservative force (damper), the equation expands into the following form:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{D}}{\partial \dot{q}} = 0$$

The solving process in this model is identical to the previous case. The step-by-step process will not be explained, resulting in the following solution of subsystem 2a:

$$m_1\ddot{y}_1 + B_1\dot{y}_1 + k_1y_1 - B_1\dot{y}_2 - k_1y_2 = 0$$

### 3.4 Forcing on $m_2$ - subsystem 2b

The kinetic and potential energies of subsystem 1b are defined as:

$$T_2 = \frac{1}{2}m_2\dot{y}_2^2$$

$$U_2 = -k_2 \int_0^{y_2} y dy + k_1 \int_{y_2}^{y_1} y dy = \frac{1}{2}k_2y_2^2 + \frac{1}{2}k_1y_2^2 - \frac{1}{2}k_1y_1^2$$

The system also has a damper, which is a dissipative element:

$$D_2 = -B_2 \int_0^{\dot{y}_2} \dot{y} dy + B_1 \int_{\dot{y}_2}^{\dot{y}_1} \dot{y} dy = \frac{1}{2}B_2\dot{y}_2^2 + \frac{1}{2}B_1\dot{y}_2^2 - \frac{1}{2}B_1\dot{y}_1^2$$

The Lagrangian of subsystem 2a is:

$$\mathcal{L}(y, \dot{y}) = T_2 - U_2 = \frac{1}{2}m_2\dot{y}_2^2 - \left(\frac{1}{2}k_2y_2^2 + \frac{1}{2}k_1y_2^2 - \frac{1}{2}k_1y_1^2\right)$$

The Rayleigh dissipation function can be written as:

$$\mathcal{D} = D_2 - P_2 = \left(\frac{1}{2}B_2\dot{y}_2^2 + \frac{1}{2}B_1\dot{y}_2^2 - \frac{1}{2}B_1\dot{y}_1^2\right) - F\dot{y}_2$$

Using Lagrange's equation with a non-conservative force (damper), the equation expands into the following form:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{D}}{\partial \dot{q}} = 0$$

As mentioned above, the step-by-step process will not be explained. The resulting solution of subsystem 2b is:

$$m_2\ddot{y}_2 + (B_2 + B_1)\dot{y}_2 + (k_2 + k_1)y_2 - B_1\dot{y}_1 - k_1y_1 = F$$

## Equation Results

Ultimately, the equation for the 2DoF system with forcing on  $m_2$  can be written as:

$$\begin{cases} m_1 \ddot{y}_1 + B_1 \dot{y}_1 + k_1 y_1 - B_1 \dot{y}_2 - k_1 y_2 = 0 \\ m_2 \ddot{y}_2 + (B_2 + B_1) \dot{y}_2 + (k_2 + k_1) y_2 - B_1 \dot{y}_1 - k_1 y_1 = F \end{cases}$$

## 4 Co-energy and Energy Expressions

### Mass

The general inertia co-energy for mass is:

$$W'_m = W_m = \int_0^v m v' dv' = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{y}^2$$

In linear systems, energy and co-energy are equal.

### Spring

The general energy stored in the spring  $k_1$  is:

$$W'_s = W_s = \int_0^y k y' dy' = \frac{1}{2} k y^2$$

Since the spring is conservative (no energy loss), co-energy equals energy.

### Damper

The general damper dissipates energy as friction:

$$W_d = \int_0^t B v^2(\tau) d\tau = \frac{1}{2} B v^2 = \frac{1}{2} B \dot{y}^2$$

Dampers do not store energy, so they do not have co-energy.

Using the co-energy equation with a non-conservative force, we can write:

$$\frac{d}{dt} \left[ \frac{\partial T}{\dot{y}} \right] - \frac{\partial T}{y} + \frac{\partial U}{y} + \frac{\partial D}{\dot{y}} = f_i$$

### 4.1 Forcing on $m_1$ - subsystem 1a

We need to adjust the general co-energy expressions to 2DoF subsystems specific configuration:

$$\begin{aligned} W_m &= \frac{1}{2} m_1 \dot{y}_1^2 \\ W_s &= \frac{1}{2} k_1 y_1^2 - \frac{1}{2} k_1 y_2^2 \\ W_d &= \frac{1}{2} B_1 \dot{y}_1^2 - \frac{1}{2} B_1 \dot{y}_2^2 \end{aligned}$$

Finally, we can assign following assumptions:

$$\begin{aligned} T_1 &= W_m \\ U_1 &= W_s \\ D_1 &= W_d \\ f_i &= F \end{aligned}$$

By substituting the coefficients into the equation, we obtain the following:

$$\begin{aligned} \frac{d}{dt} \left[ \frac{\partial \left( \frac{1}{2} m_1 \dot{y}_1^2 \right)}{\dot{y}} \right] - \frac{\partial \left( \frac{1}{2} m_1 \dot{y}_1^2 \right)}{y} + \frac{\partial \left( \frac{1}{2} k_1 y_1^2 - \frac{1}{2} k_1 y_2^2 \right)}{y} + \frac{\partial \left( \frac{1}{2} B_1 \dot{y}_1^2 - \frac{1}{2} B_1 \dot{y}_2^2 \right)}{\dot{y}} &= F \\ \frac{d}{dt} \left[ \frac{\partial \left( \frac{1}{2} m_1 \dot{y}_1^2 \right)}{\dot{y}} \right] - 0 + \frac{\partial \left( \frac{1}{2} k_1 y_1^2 - \frac{1}{2} k_1 y_2^2 \right)}{y} + \frac{\partial \left( \frac{1}{2} B_1 \dot{y}_1^2 - \frac{1}{2} B_1 \dot{y}_2^2 \right)}{\dot{y}} &= F \\ \frac{d}{dt} [m_1 \dot{y}_1] + k_1 y_1 - k_1 y_2 + B_1 \dot{y}_1 - B_1 \dot{y}_2 &= F \\ m_1 \ddot{y}_1 + B_1 \dot{y}_1 + k_1 y_1 - B_1 \dot{y}_2 - k_1 y_2 &= F \end{aligned}$$

## 4.2 Forcing on $m_1$ - subsystem 1b

In this case, we also need to adjust the general co-energy expressions to 2DoF 2nd sub-systems specific configuration:

$$\begin{aligned} W_m &= \frac{1}{2} m_2 \dot{y}_2^2 \\ W_s &= \frac{1}{2} k_2 y_2^2 \\ W_d &= \frac{1}{2} B_2 \dot{y}_2^2 \end{aligned}$$

Finally, we can assign following assumptions:

$$\begin{aligned} T_2 &= W_m \\ U_2 &= W_s \\ D_2 &= W_d \\ f_i &= B_1 \Delta \dot{y} + k_1 \Delta y \end{aligned}$$

By substituting the coefficients into the equation, we obtain the following:

$$\frac{d}{dt} \left[ \frac{\partial \left( \frac{1}{2} m_2 \dot{y}_2^2 \right)}{\dot{y}} \right] - \frac{\partial \left( \frac{1}{2} m_2 \dot{y}_2^2 \right)}{y} + \frac{\partial \left( \frac{1}{2} k_2 y_2^2 \right)}{y} + \frac{\partial \left( \frac{1}{2} B_2 \dot{y}_2^2 \right)}{\dot{y}} = B_1 \Delta \dot{y} + k_1 \Delta y$$



$$\frac{d}{dt} \left[ \frac{\partial \left( \frac{1}{2} m_2 \dot{y}_2^2 \right)}{\dot{y}} \right] - 0 + \frac{\partial \left( \frac{1}{2} k_2 y_2^2 \right)}{y} + \frac{\partial \left( \frac{1}{2} B_2 \dot{y}_2^2 \right)}{\dot{y}} = B_1 (\dot{y}_1 - \dot{y}_2) + k_1 (y_1 - y_2)$$

$$m_2 \ddot{y}_2 + k_2 y_2 + B_2 \dot{y}_2 = B_1 \dot{y}_1 - B_1 \dot{y}_2 + k_1 y_1 - k_1 y_2$$

$$m_2 \ddot{y}_2 + k_2 y_2 + B_2 \dot{y}_2 = B_1 \dot{y}_1 - B_1 \dot{y}_2 + k_1 y_1 - k_1 y_2$$

$$m_2 \ddot{y}_2 + (B_2 + B_1) \dot{y}_2 + (k_2 + k_1) y_2 - B_1 \dot{y}_1 - k_1 y_1 = 0$$

## Equation Results

Ultimately, the equation for the 2DoF system with forcing on  $m_1$ , solved via co-energy expression can be written as:

$$\begin{cases} m_1 \ddot{y}_1 + B_1 \dot{y}_1 + k_1 y_1 - B_1 \dot{y}_2 - k_1 y_2 = F \\ m_2 \ddot{y}_2 + (B_2 + B_1) \dot{y}_2 + (k_2 + k_1) y_2 - B_1 \dot{y}_1 - k_1 y_1 = 0 \end{cases}$$

### 4.3 Forcing on $m_2$ - subsystem 2a

We need to adjust the general co-energy expressions to 2DoF subsystems specific configuration:

$$\begin{aligned} W_m &= \frac{1}{2} m_1 \dot{y}_1^2 \\ W_s &= \frac{1}{2} k_1 y_1^2 - \frac{1}{2} k_1 y_2^2 \\ W_d &= \frac{1}{2} B_1 \dot{y}_1^2 - \frac{1}{2} B_1 \dot{y}_2^2 \end{aligned}$$

Finally, we can assign following assumptions:

$$\begin{aligned} T_1 &= W_m \\ U_1 &= W_s \\ D_1 &= W_d \\ f_i &= 0 \end{aligned}$$

By substituting the coefficients into the equation, we obtain the following:

$$\frac{d}{dt} \left[ \frac{\partial \left( \frac{1}{2} m_1 \dot{y}_1^2 \right)}{\dot{y}} \right] - \frac{\partial \left( \frac{1}{2} m_1 \dot{y}_1^2 \right)}{y} + \frac{\partial \left( \frac{1}{2} k_1 y_1^2 - \frac{1}{2} k_1 y_2^2 \right)}{y} + \frac{\partial \left( \frac{1}{2} B_1 \dot{y}_1^2 - \frac{1}{2} B_1 \dot{y}_2^2 \right)}{\dot{y}} = 0$$

$$\frac{d}{dt} \left[ \frac{\partial \left( \frac{1}{2} m_1 \dot{y}_1^2 \right)}{\dot{y}} \right] - 0 + \frac{\partial \left( \frac{1}{2} k_1 y_1^2 - \frac{1}{2} k_1 y_2^2 \right)}{y} + \frac{\partial \left( \frac{1}{2} B_1 \dot{y}_1^2 - \frac{1}{2} B_1 \dot{y}_2^2 \right)}{\dot{y}} = 0$$

$$\frac{d}{dt} [m_1 \dot{y}_1] + k_1 y_1 - k_1 y_2 + B_1 \dot{y}_1 - B_1 \dot{y}_2 = 0$$

$$m_1 \ddot{y}_1 + B_1 \dot{y}_1 + k_1 y_1 - B_1 \dot{y}_2 - k_1 y_2 = 0$$

#### 4.4 Forcing on $m_2$ - subsystem 2b

In this case, we also need to adjust the general co-energy expressions to 2DoF 2nd sub-systems specific configuration:

$$\begin{aligned} W_m &= \frac{1}{2} m_2 \dot{y}_2^2 \\ W_s &= \frac{1}{2} k_2 y_2^2 \\ W_d &= \frac{1}{2} B_2 \dot{y}_2^2 \end{aligned}$$

Finally, we can assign following assumptions:

$$\begin{aligned} T_2 &= W_m \\ U_2 &= W_s \\ D_2 &= W_d \\ f_i &= F + B_1 \Delta \dot{y} + k_1 \Delta y \end{aligned}$$

By substituting the coefficients into the equation, we obtain the following:

$$\frac{d}{dt} \left[ \frac{\partial \left( \frac{1}{2} m_2 \dot{y}_2^2 \right)}{\dot{y}} \right] - \frac{\partial \left( \frac{1}{2} m_2 \dot{y}_2^2 \right)}{y} + \frac{\partial \left( \frac{1}{2} k_2 y_2^2 \right)}{y} + \frac{\partial \left( \frac{1}{2} B_2 \dot{y}_2^2 \right)}{\dot{y}} = F + B_1 \Delta \dot{y} + k_1 \Delta y$$

$$\frac{d}{dt} \left[ \frac{\partial \left( \frac{1}{2} m_2 \dot{y}_2^2 \right)}{\dot{y}} \right] - 0 + \frac{\partial \left( \frac{1}{2} k_2 y_2^2 \right)}{y} + \frac{\partial \left( \frac{1}{2} B_2 \dot{y}_2^2 \right)}{\dot{y}} = F + B_1 (\dot{y}_1 - \dot{y}_2) + k_1 (y_1 - y_2)$$

$$m_2 \ddot{y}_2 + k_2 y_2 + B_2 \dot{y}_2 = F + B_1 \dot{y}_1 - B_1 \dot{y}_2 + k_1 y_1 - k_1 y_2$$

$$m_2 \ddot{y}_2 + k_2 y_2 + B_2 \dot{y}_2 = F + B_1 \dot{y}_1 - B_1 \dot{y}_2 + k_1 y_1 - k_1 y_2$$

$$m_2 \ddot{y}_2 + (B_2 + B_1) \dot{y}_2 + (k_2 + k_1) y_2 - B_1 \dot{y}_1 - k_1 y_1 = F$$

#### Equation Results

Ultimately, the equation for the 2DoF system with forcing on  $m_1$ , solved via co-energy expression can be written as:

$$\begin{cases} m_1 \ddot{y}_1 + B_1 \dot{y}_1 + k_1 y_1 - B_1 \dot{y}_2 - k_1 y_2 = 0 \\ m_2 \ddot{y}_2 + (B_2 + B_1) \dot{y}_2 + (k_2 + k_1) y_2 - B_1 \dot{y}_1 - k_1 y_1 = F \end{cases}$$

## 5 Simulink Model for Lagrangian and Co-energy Expression - Forcing on $m_1$

In order to model the equation properly, we must translate the dynamic equation into a form understandable for Simulink:

$$\begin{cases} m_1 \ddot{y}_1 + B_1 \dot{y}_1 + k_1 y_1 - B_1 \dot{y}_2 - k_1 y_2 = F \\ m_2 \ddot{y}_2 + (B_2 + B_1) \dot{y}_2 + (k_2 + k_1) y_2 - B_1 \dot{y}_1 - k_1 y_1 = 0 \end{cases}$$

$$\begin{cases} \ddot{y}_1 = \frac{1}{m_1} (F - B_1 \dot{y}_1 - k_1 y_1 + B_1 \dot{y}_2 + k_1 y_2) \\ \ddot{y}_2 = \frac{1}{m_2} (-(B_2 + B_1) \dot{y}_2 - (k_2 + k_1) y_2 + B_1 \dot{y}_1 + k_1 y_1) \end{cases}$$

Below, the Simulink model is presented.

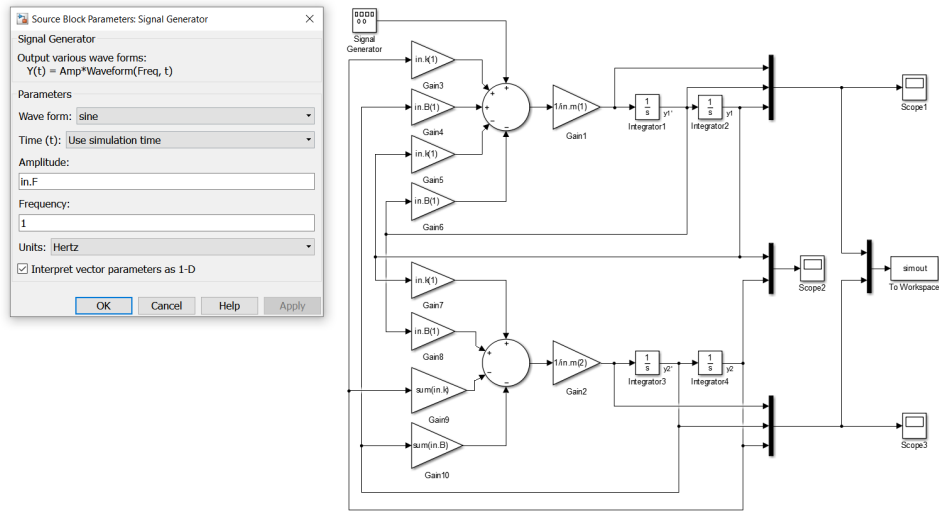


Figure 4: Simulink model of 2DoF system - Forcing on  $m_1$

## 6 Simulink Model for Lagrangian and Co-energy Expression - Forcing on $m_2$

Same as above, we must translate the dynamic equation into a form understandable for Simulink:

$$\begin{cases} m_1 \ddot{y}_1 + B_1 \dot{y}_1 + k_1 y_1 - B_1 \dot{y}_2 - k_1 y_2 = 0 \\ m_2 \ddot{y}_2 + (B_2 + B_1) \dot{y}_2 + (k_2 + k_1) y_2 - B_1 \dot{y}_1 - k_1 y_1 = F \end{cases}$$

$$\begin{cases} \ddot{y}_1 = \frac{1}{m_1} (-B_1 \dot{y}_1 - k_1 y_1 + B_1 \dot{y}_2 + k_1 y_2) \\ \ddot{y}_2 = \frac{1}{m_2} (F - (B_2 + B_1) \dot{y}_2 - (k_2 + k_1) y_2 + B_1 \dot{y}_1 + k_1 y_1) \end{cases}$$

Below, the Simulink model is presented.

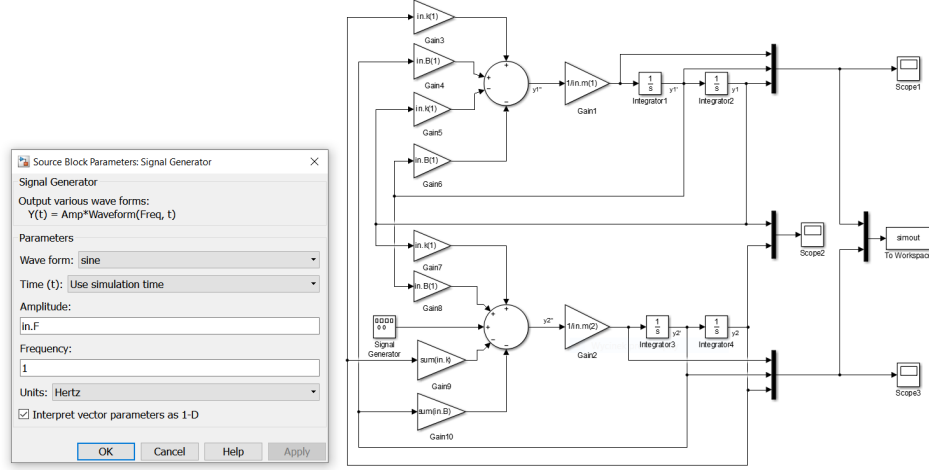


Figure 5: Simulink model of 2DoF system - Forcing on  $m_2$

## 7 Matlab and Python script - forcing on $m_1$

For script-based system modeling we must take a slightly different approach to translating mathematical equation into code. Let  $y_1(t) = y(1)$ ,  $y'_1(t) = y(2)$ ,  $y_2(t) = y(3)$ ,  $y'_2(t) = y(4)$ , in that case the matrix of the 2DoF equation will be written as:

$$\frac{d}{dt} \begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix} = \begin{bmatrix} y(2) \\ (F - B(1) \cdot y(2) - k(1) \cdot y(1) + B(1) \cdot y(4) + k(1) \cdot y(3)) / m(1) \\ y(4) \\ (- (B(2) + B(1)) \cdot y(4) - (k(2) + k(1)) \cdot y(3) + B(1) \cdot y(2) + k(1) \cdot y(1)) / m(2) \end{bmatrix}$$

Now, the equation can be coded into Matlab:

```
1 Dy = [y(2); subsystem(1); y(4); subsystem(2)];
```

where:

```
1 subsystem1 = (F - B(1) * y(2) - k(1) * y(1) + B(1) * y(4) + k(1) * y(3)) / m(1)
2 subsystem2 = (- (sum(B)) * y(4) - (sum(k)) * y(3) + B(1) * y(2) + k(1) * y(1)) / m(2)
```

Python counts matrix elements from 0, so the equivalent Python code can be written as:

```
1 def dual_dof(x, t):
2     subsystem1 = (F - B[0]*x[1] - k[0]*x[0] + B[0]*x[3] + k[0]*x[2]) / m[0]
3     subsystem2 = (0 - sum(B)*x[3] - sum(k)*x[2] + B[0]*x[1] + k[0]*x[0]) / m[1]
4     return x[1], subsystem1, x[3], subsystem2
```

## 8 Matlab and Python script - forcing on $m_2$

For script-based system modeling we must take a slightly different approach to translating mathematical equation into code. Let  $y_1(t) = y(1)$ ,  $y'_1(t) = y(2)$ ,  $y_2(t) = y(3)$ ,  $y'_2(t) =$

$y(4)$ , in that case the matrix of the 2DoF equation will be written as:

$$\frac{d}{dt} \begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix} = \begin{bmatrix} y(2) \\ (-B(1) \cdot y(2) - k(1) \cdot y(1) + B(1) \cdot y(4) + k(1) \cdot y(3)) / m(1) \\ y(4) \\ (F - (B(2) + B(1)) \cdot y(4) - (k(2) + k(1)) \cdot y(3) + B(1) \cdot y(2) + k(1) \cdot y(1)) / m(2) \end{bmatrix}$$

Now, the equation can be coded into Matlab:

```
1 Dy = [y(2); subsystem(1); y(4); subsystem(2)];
```

where:

```
1 subsystem1 = (- B(1) * y(2) - k(1) * y(1) + B(1) * y(4) + k
2   (1) * y(3)) / m(1)
3 subsystem2 = (F - (sum(B)) * y(4) - (sum(k)) * y(3) + B(1) * y
4   (2) + k(1) * y(1)) / m(2)
```

Python counts matrix elements from 0, so the equivalent Python code can be written as:

```
1 def dual_dof(x, t):
2     subsystem1 = (0 - B[0]*x[1] - k[0]*x[0] + B[0]*x[3] + k[0]*x
3       [2]) / m[0]
4     subsystem2 = (F - sum(B)*x[3] - sum(k)*x[2] + B[0]*x[1] + k
5       [0]*x[0]) / m[1]
6     return x[1], subsystem1, x[3], subsystem2
```

## 9 Simulation Results

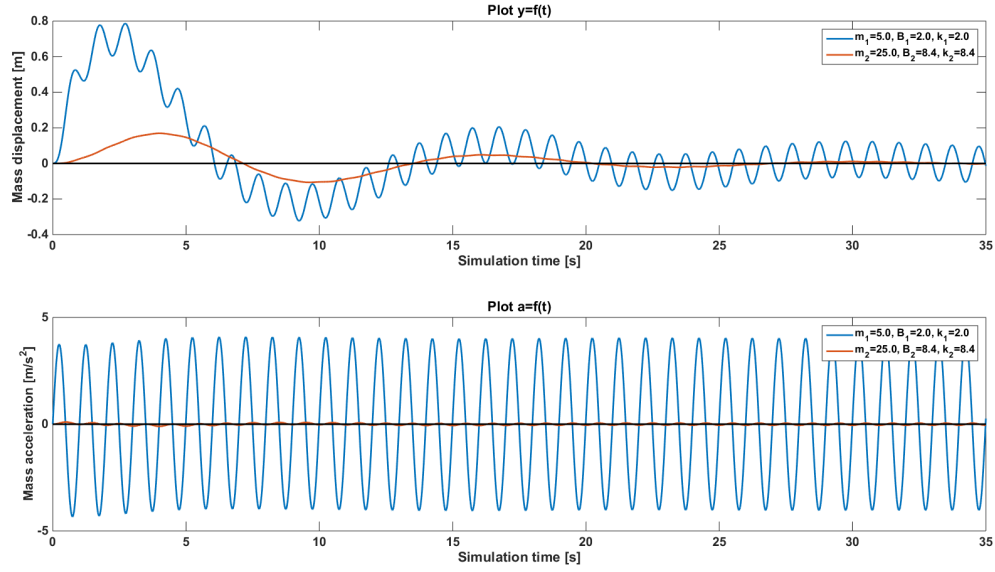


Figure 6: Forcing on  $m_1$

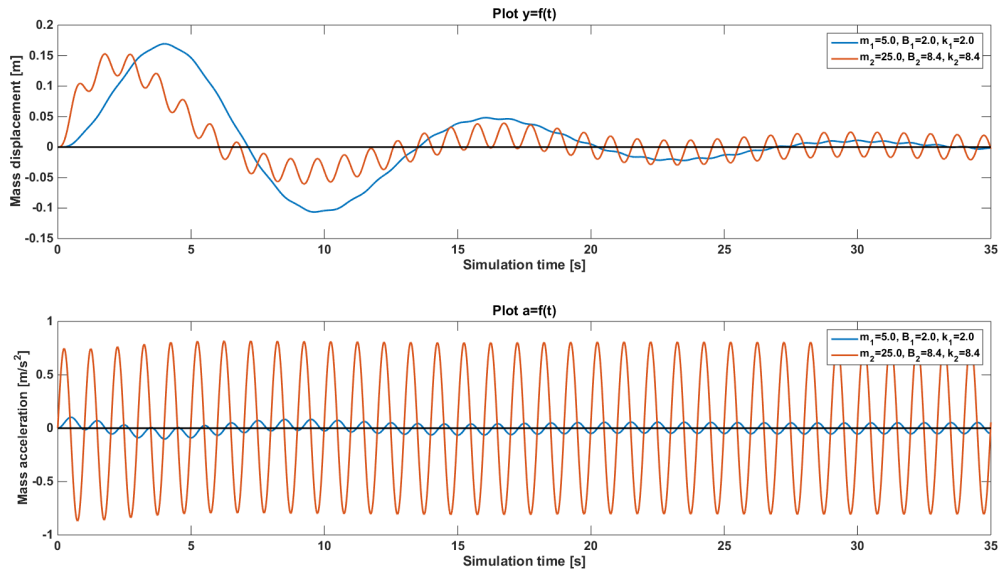


Figure 7: Forcing on  $m_2$

## 10 Conclusion

This document demonstrates the Lagrangian and co-energy approach to modeling a 2DoF system. It also provides a visual block diagram suitable for simulation in Simulink and basic code for scripting.