

Modeling RLC Circuit

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1 Introduction

This document presents the modeling of a series RLC circuit using Lagrangian mechanics, co-energy analysis, and derives the state-space representation. A Simulink block diagram is also provided for simulation purposes.

2 RLC Circuit Description

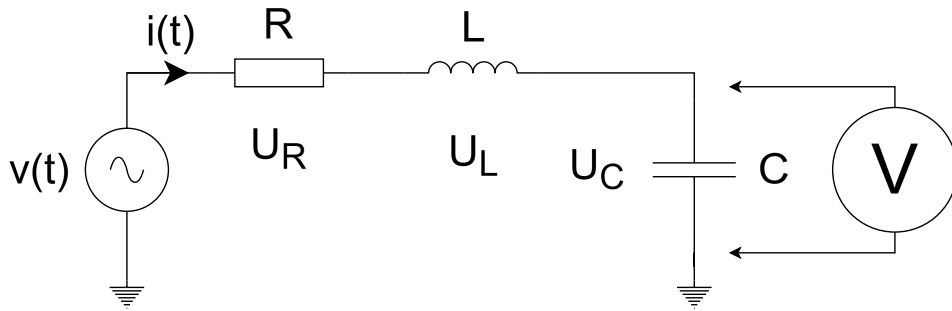


Figure 1: Overview of the RLC circuit

We consider a series RLC circuit with a voltage source $v(t)$, resistor R , inductor L , and capacitor C . The loop current is denoted as $i(t)$, and the charge on the capacitor is $q(t)$.

3 Lagrangian Formulation

The kinetic and potential energies are defined as:

$$T = \frac{1}{2}L\dot{q}^2$$
$$U = \frac{1}{2C}q^2$$

The circuit also has a resistor, which is a dissipative element:

$$D = \frac{1}{2}R\dot{q}^2$$

The Lagrangian of the RLC circuit is:

$$\mathcal{L}(q, \dot{q}) = T - U = \frac{1}{2}L\dot{q}^2 - \frac{1}{2C}q^2$$

The Rayleigh dissipation function can be written as:

$$\mathcal{D} = D - P = \frac{1}{2}R\dot{q}^2 - v(t)\dot{q}$$

Using Lagrange's equation with a non-conservative force (resistor), the equation expands to:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{D}}{\partial \dot{q}} = 0$$

This leads to the following differential equation:

$$\begin{aligned} \frac{d}{dt} \left[\frac{\partial \left(\frac{1}{2}L\dot{q}^2 - \frac{1}{2C}q^2 \right)}{\dot{q}} \right] - \frac{\partial \left(\frac{1}{2}L\dot{q}^2 - \frac{1}{2C}q^2 \right)}{q} + \frac{\partial \left(\frac{1}{2}R\dot{q}^2 - v(t)\dot{q} \right)}{\dot{q}} &= 0 \\ \frac{d}{dt} \left[\frac{\partial \left(\frac{1}{2}L\dot{q}^2 - 0 \right)}{\dot{q}} \right] - \frac{\partial \left(0 - \frac{1}{2C}q^2 \right)}{q} + \frac{\partial \left(\frac{1}{2}R\dot{q}^2 - v(t)\dot{q} \right)}{\dot{q}} &= 0 \\ \frac{d}{dt} \left[\left(2 \cdot \frac{1}{2}L\dot{q}^{2-1} \right) \right] + \left(2 \cdot \frac{1}{2C}q^{2-1} \right) + \left(2 \cdot \frac{1}{2}R\dot{q}^{2-1} - v(t)\dot{q}^{1-1} \right) &= 0 \\ \frac{d}{dt} [L\dot{q}] + \frac{1}{C}q + R\dot{q} - v(t) &= 0 \\ L\ddot{q} + \frac{1}{C}q + R\dot{q} &= v(t) \end{aligned}$$

Taking into account that:

$$\dot{q} = i;$$

$$i = C \cdot \frac{d}{dt}u(t) = C \cdot u'(t)$$

We can transform the current form of equation into voltage form:

$$\begin{aligned} L \frac{d}{dt}\dot{q} + \frac{1}{C}q + R\dot{q} &= v(t) \\ L \frac{d}{dt} (C \cdot u'(t)) + \frac{1}{C} \cdot \left(\int_0^t C \frac{du(t)}{dt} \right) + R \cdot C \cdot u'(t) &= v(t) \\ LCu''(t) + u(t) + RCu'(t) &= v(t) \end{aligned}$$

Sorting yields the equation's final form:

$$LCu''(t) + RCu'(t) + u(t) = v(t)$$

4 Co-energy and Energy Expressions

Inductor

The magnetic co-energy for the inductor:

$$W'_L = W_L = \int_0^i Li' di' = \frac{1}{2}Li^2$$

In linear systems, energy and co-energy are equal.

Capacitor

The energy stored in the capacitor is:

$$W'_C = W_C = \int_0^q \frac{1}{C}q' dq' = \frac{1}{2C}q^2$$

Since the capacitor is conservative (no energy loss), co-energy equals energy.

Resistor

The resistor dissipates energy as heat:

$$W_R = \int_0^t Ri^2(\tau) d\tau = \frac{1}{2}Ri^2$$

Resistors do not store energy, so they do not have co-energy.

Using the co-energy equation with a non-conservative force, we can write:

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}} \right] - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} + \frac{\partial D}{\partial \dot{q}} = f_i$$

where:

$$T = W_L$$

$$U = W_C$$

$$D = W_R$$

$$i = \dot{q}$$

$$f_i = v(t)$$

By substituting the coefficients into the equation, we obtain the following:

$$\frac{d}{dt} \left[\frac{\partial \left(\frac{1}{2}L\dot{q}^2 \right)}{\partial \dot{q}} \right] - \frac{\partial \left(\frac{1}{2}L\dot{q}^2 \right)}{\partial q} + \frac{\partial \left(\frac{1}{2C}q^2 \right)}{\partial q} + \frac{\partial \left(\frac{1}{2}R\dot{q}^2 \right)}{\partial \dot{q}} = v(t)$$

$$\frac{d}{dt} \left[2 \cdot \frac{1}{2}L\dot{q}^{2-1} \right] - 0 + 2 \cdot \frac{1}{2C}q^{2-1} + 2 \cdot \frac{1}{2}R\dot{q}^{2-1} = v(t)$$

$$\frac{d}{dt} [L\dot{q}] + \frac{1}{C}q + R\dot{q} = v(t)$$

$$L\ddot{q} + \frac{1}{C}q + R\dot{q} = v(t)$$

Substituting once again:

$$i = C \cdot \frac{d}{dt}u(t) = C \cdot u'(t)$$

yields:

$$L \frac{d}{dt} (C \cdot u'(t)) + \frac{1}{C} \cdot \left(\int_0^t C \frac{du(t)}{dt} \right) + R \cdot C \cdot u'(t) = v(t)$$

$$LCu''(t) + u(t) + RCu'(t) = v(t)$$

$$LCu''(t) + RCu'(t) + u(t) = v(t)$$

5 State-Space Representation

From the equation of preservation of momentum we can write:

$$\begin{cases} \dot{p} = -\frac{1}{C}q - \frac{R}{L}p + v(t) \\ \dot{q} = \frac{1}{L}p \end{cases}$$

where:

p – momentum

\dot{p} – derivative of momentum

q – electrical charge

\dot{q} – electrical current

R, L, C – circuit parameters

$v(t)$ – input voltage

The general state-space model equation can be written as:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Let $\dot{x} = \begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix}$, $x = \begin{bmatrix} p \\ q \end{bmatrix}$, and expand the coefficients as follows:

$$\begin{cases} \dot{p} = -\frac{1}{C}q - \frac{R}{L}p + 1 \cdot v(t) \\ \dot{q} = 0q + \frac{1}{L}p + 0 \cdot v(t) \end{cases}$$

We can write the upper part of space-state equation as:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v(t)$$

For the lower part of the space-state equation, in order to activate the only meaningful parameter of the system, let $y = q$. Expanding this assumption into a known schematic, we obtain:

$$y = 1 \cdot q + 0 \cdot p + 0 \cdot v(t)$$

The lower part of state-space equation is:

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} v(t)$$

Eventually, we have obtained the state-space coefficients as:

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \end{bmatrix}$$

6 Simulink Model for Lagrangian and Co-energy Expression

In order to model the equation properly, we must translate the dynamic equation into a form understandable for Simulink:

$$LCu''(t) = v(t) - RCu'(t) - u(t)$$

$$u''(t) = \frac{1}{LC} (v(t) - RCu'(t) - u(t))$$

Below, the Simulink model is presented.

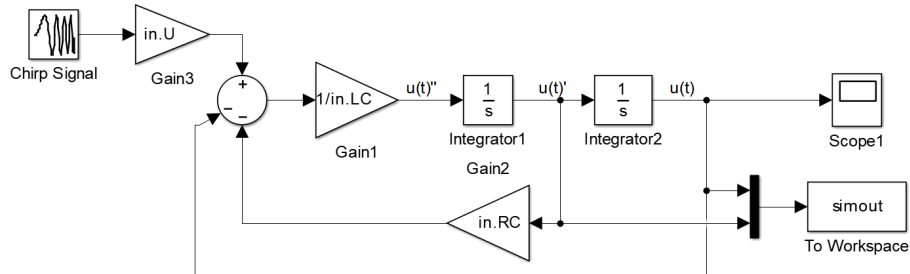


Figure 2: Simulink model of the RLC circuit

7 Simulink Model for State-space

Since the state-space model observes electrical charge flow in the circuit, we must add below listed formulas in order to view voltage and electrical current:

$$u(t) = \frac{1}{C} \cdot q$$

$$i(t) = \left| \frac{d}{dt} q \right|$$

Below, the state-space model is presented.

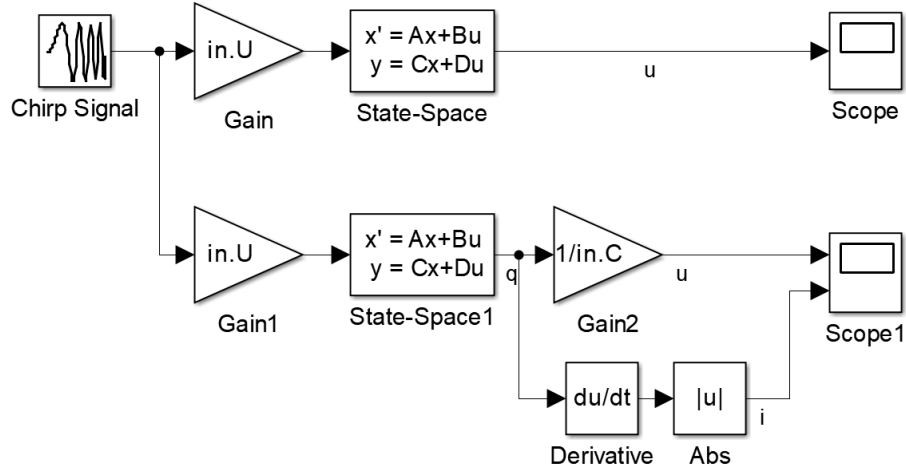


Figure 3: State-space model of the RLC circuit

8 Matlab and Python script

For script-based system modeling we must take a slightly different approach to translating mathematical equation into code. Let $u(t) = u(1)$, $u'(t) = u(2)$, in that case the matrix of the RLC equation will be written as:

$$\frac{d}{dt} \begin{bmatrix} u(1) \\ u(2) \end{bmatrix} = \begin{bmatrix} u(2) \\ (v(t) - RCu(2) - u(1)) / LC \end{bmatrix}$$

Now, the equation can be coded into Matlab:

```
1 Du = [u(2); (ku*Uw-R*C*u(2)-u(1))/(L*C)];
```

Python counts matrix elements from 0, so the equivalent Python code can be written as:

```
1 def rlc(u, t, R):
2     return u[1], (U * w - R * C * u[1] - u[0]) / (L * C)
```

The only thing left is to describe the chirp signal. In Matlab the chirp signal can be coded as:

```

1 k = (fmax - fmin) / tmax;
2 Uw = cos((2 * pi() * (k / 2) .*t .*t) + (2 * pi() * fmin .*t));

```

In Python:

```

1 k = (fmax - fmin) / max(t)
2 w = np.cos((2 * np.pi * (k / 2) * t * t) + (2 * np.pi * fmin * t)
)

```

9 Simulation Results

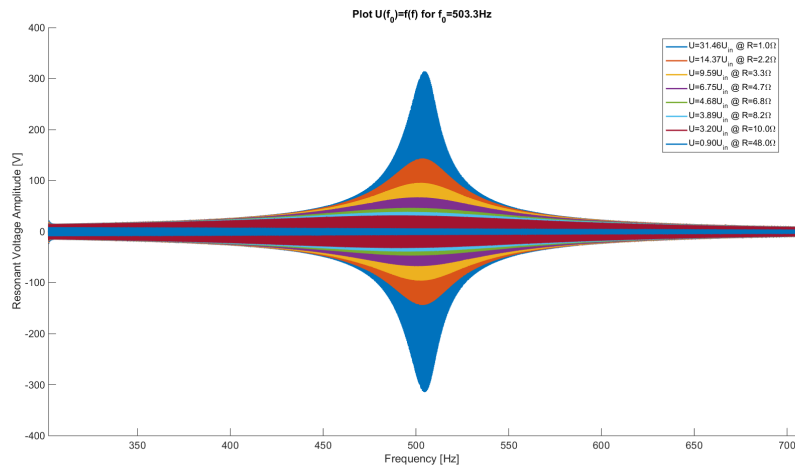


Figure 4: Plot of resonant voltage amplitude

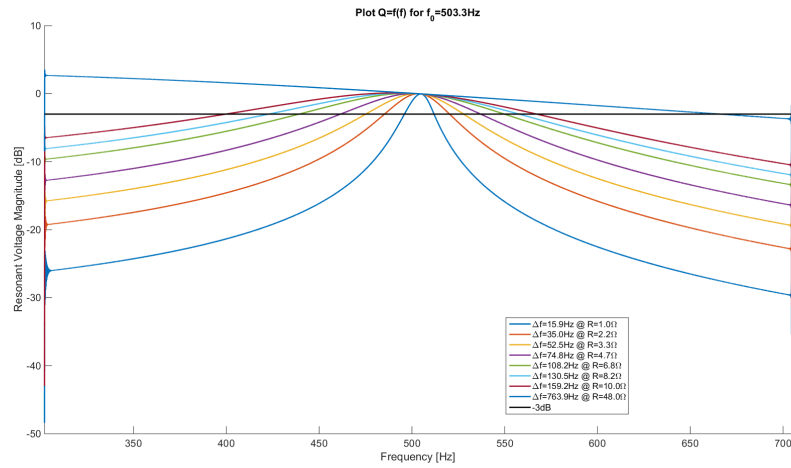


Figure 5: Plot of RLC circuit band frequency

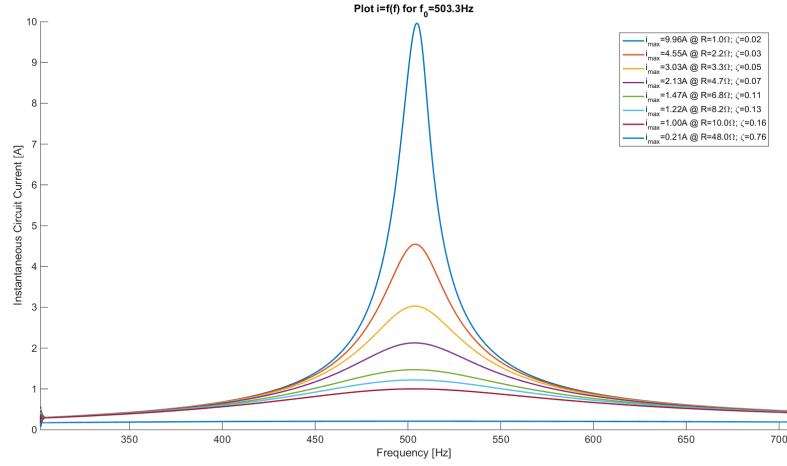


Figure 6: Plot of resonant current amplitude

10 Conclusion

This document demonstrates the Lagrangian, co-energy and state-space approach to modeling a simple RLC circuit. It also provides a visual block diagram suitable for simulation in Simulink and basic code for scripting.