Modeling RLC Circuit

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1 Introduction

This document presents the modeling of a series RLC circuit using Lagrangian mechanics, co-energy analysis, and derives the state-space representation. A Simulink block diagram is also provided for simulation purposes.

2 RLC Circuit Description

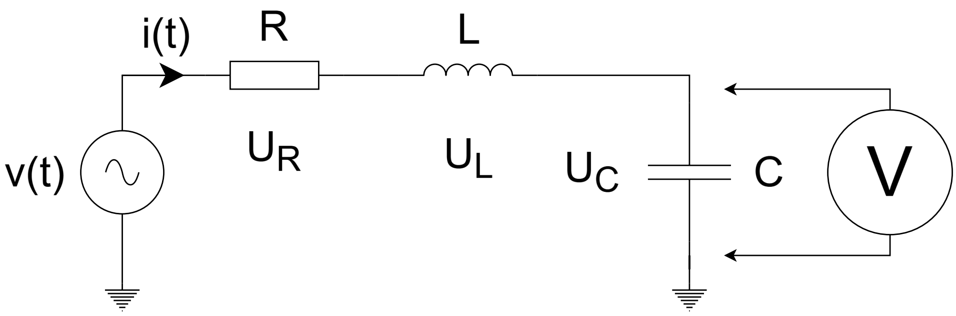

Figure 1: Overview of the RLC circuit

We consider a series RLC circuit with a voltage source $v(t)$, resistor R, inductor L, and capacitor C. The loop current is denoted as $i(t)$, and the charge on the capacitor is $q(t)$.

3 Lagrangian Formulation

The kinetic and potential energies are defined as:

$$T = \frac{1}{2}L\dot{q}^2$$
$$U = \frac{1}{2C}q^2$$

The circuit also has a resistor, which is a dissipative element:

$$D = \frac{1}{2}R\dot{q}^2$$

The Lagrangian of the RLC circuit is:

$$\mathcal{L}(q, \dot{q}) = T - U = \frac{1}{2}L\dot{q}^2 - \frac{1}{2C}q^2$$

Using Lagrange's equation with a non-conservative force (resistor), the equation expands with the Rayleigh dissipation function to:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{D}}{\partial \dot{q}} = v(t)$$

This leads to the following differential equation:

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}L\dot{q}^2 - \frac{1}{2C}q^2\right)}{\dot{q}}\right] - \frac{\partial\left(\frac{1}{2}L\dot{q}^2 - \frac{1}{2C}q^2\right)}{q} + \frac{\partial\left(\frac{1}{2}R\dot{q}^2\right)}{\dot{q}} = v(t)$$

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}L\dot{q}^2 - 0\right)}{\dot{q}}\right] - \frac{\partial\left(0 - \frac{1}{2C}q^2\right)}{q} + \frac{\partial\left(\frac{1}{2}R\dot{q}^2\right)}{\dot{q}} = v(t)$$

$$\frac{d}{dt}\left[\left(2 \cdot \frac{1}{2}L\dot{q}^{2-1}\right)\right] + \left(2 \cdot \frac{1}{2C}q^{2-1}\right) + \left(2 \cdot \frac{1}{2}R\dot{q}^{2-1}\right) = v(t)$$

$$\frac{d}{dt}\left[L\dot{q}\right] + \frac{1}{C}q + R\dot{q} = v(t)$$

$$L\ddot{q} + \frac{1}{C}q + R\dot{q} = v(t)$$

Taking into account that:

$$\dot{q} = i;$$

$$i = C \cdot \frac{d}{dt}u(t) = C \cdot u'(t)$$

We can transform the current form of equation into voltage form:

$$L\frac{d}{dt}\dot{q} + \frac{1}{C}q + R\dot{q} = v(t)$$

$$L\frac{d}{dt}\left(C \cdot u'(t)\right) + \frac{1}{C} \cdot \left(\int_0^t C\frac{du(t)}{dt}\right) + R \cdot C \cdot u'(t) = v(t)$$

$$LCu''(t) + u(t) + RCu'(t) = v(t)$$

Sorting yields the equation's final form:

$$LCu''(t) + RCu'(t) + u(t) = v(t)$$

4 Co-energy and Energy Expressions

Inductor

The magnetic co-energy for the inductor:

$$W'_L = W_L = \int_0^i Li'\,di' = \frac{1}{2}Li^2$$

In linear systems, energy and co-energy are equal.

Capacitor

The energy stored in the capacitor is:

$$W'_C = W_C = \int_0^q \frac{1}{C}q'\,dq' = \frac{1}{2C}q^2$$

Since the capacitor is conservative (no energy loss), co-energy equals energy.

Resistor

The resistor dissipates energy as heat:

$$W_R = \int_0^t Ri^2(\tau)\,d\tau = \frac{1}{2}Ri^2$$

Resistors do not store energy, so they do not have co-energy.

Using the co-energy equation with a non-conservative force, we can write:

$$\frac{d}{dt}\left[\frac{\partial T}{\dot{q}}\right] - \frac{\partial T}{q} + \frac{\partial U}{q} + \frac{\partial D}{\dot{q}} = f_i$$

where:

$$\begin{aligned} T &= W_L \\ U &= W_C \\ D &= W_R \\ i &= \dot{q} \\ f_i &= v(t) \end{aligned}$$

By substituting the coefficients into the equation, we obtain the following:

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}L\dot{q}^2\right)}{\dot{q}}\right] - \frac{\partial\left(\frac{1}{2}L\dot{q}^2\right)}{q} + \frac{\partial\left(\frac{1}{2C}q^2\right)}{q} + \frac{\partial\left(\frac{1}{2}R\dot{q}^2\right)}{\dot{q}} = v(t)$$

$$\frac{d}{dt}\left[2\cdot\frac{1}{2}L\dot{q}^{2-1}\right] - 0 + 2\cdot\frac{1}{2C}q^{2-1} + 2\cdot\frac{1}{2}R\dot{q}^{2-1} = v(t)$$

$$\frac{d}{dt}\left[L\dot{q}\right] + \frac{1}{C}q + R\dot{q} = v(t)$$

$$L\ddot{q} + \frac{1}{C}q + R\dot{q} = v(t)$$

Substituting once again:

$$i = C \cdot \frac{d}{dt}u(t) = C \cdot u'(t)$$

yields:

$$L\frac{d}{dt}\left(C \cdot u'(t)\right) + \frac{1}{C} \cdot \left(\int_0^t C\frac{du(t)}{dt}\right) + R \cdot C \cdot u'(t) = v(t)$$

$$LCu''(t) + u(t) + RCu'(t) = v(t)$$

$$LCu''(t) + RCu'(t) + u(t) = v(t)$$

5 State-Space Representation

From the equation of preservation of momentum we can write:

$$\begin{cases} \dot{p} = -\frac{1}{C}q - \frac{R}{L}p + v(t) \\ \dot{q} = \frac{1}{L}p \end{cases}$$

where:

$$\begin{aligned} p &- \text{momentum} \\ \dot{p} &- \text{derivative of momentum} \\ q &- \text{electrical charge} \\ \dot{q} &- \text{electrical current} \\ R, L, C &- \text{circuit parameters} \\ v(t) &- \text{input voltage} \end{aligned}$$

The general state-space model equation can be written as:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

Let $\dot{x} = \left[\begin{smallmatrix} \dot{p} \\ \dot{q} \end{smallmatrix}\right]$, $x = \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$, and expand the coefficients as follows:

$$\begin{cases} \dot{p} = -\frac{1}{C}q - \frac{R}{L}p + 1 \cdot v(t) \\ \dot{q} = 0q + \frac{1}{L}p + 0 \cdot v(t) \end{cases}$$

We can write the upper part of space-state equation as:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v(t)$$

For the lower part of the space-state equation, in order to activate the only meaningful parameter of the system, let $y = q$. Expanding this assumption into a known schematic, we obtain:

$$y = 1 \cdot q + 0 \cdot p + 0 \cdot v(t)$$

The lower part of state-space equation is:

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} v(t)$$

Eventually, we have obtained the state-space coefficients as:

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$D = \begin{bmatrix} 0 \end{bmatrix}$$

6 Simulink Model for Lagrangian and Co-energy Expression

In order to model the equation properly, we must translate the dynamic equation into a form understandable for Simulink:

$$LCu''(t) = v(t) - RCu'(t) - u(t)$$

$$u''(t) = \frac{1}{LC} \left(v(t) - RCu'(t) - u(t) \right)$$

Below, the Simulink model is presented.

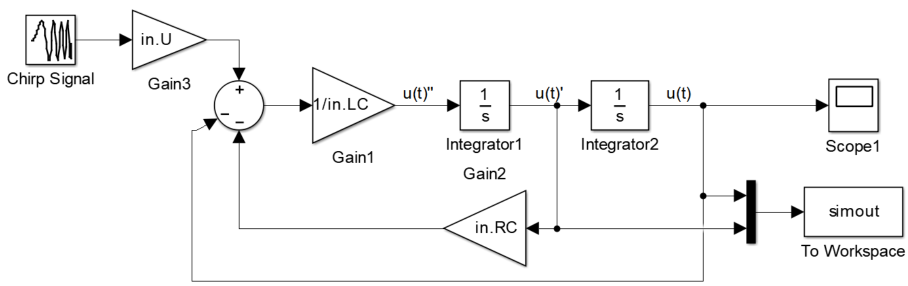

Figure 2: Simulink model of the RLC circuit

7 Simulink Model for State-space

Since the state-space model observes electrical charge flow in the circuit, we must add below listed formulas in order to view voltage and electrical current:

$$u(t) = \frac{1}{C} \cdot q$$
$$i(t) = \left| \frac{d}{dt} q \right|$$

Below, the state-space model is presented.

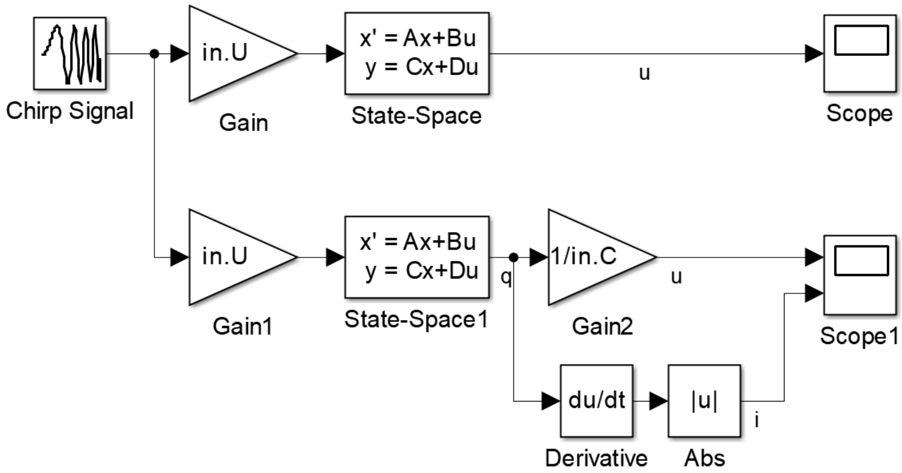

Figure 3: State-space model of the RLC circuit

8 Matlab and Python script

For script-based system modeling we must take a slightly different approach to translating mathematical equation into code. Let $u(t) = u(1)$, $u'(t) = u(2)$, in that case the matrix of the RLC equation will be written as:

$$\frac{d}{dt}\begin{bmatrix} u(1) \\ u(2) \end{bmatrix} = \begin{bmatrix} u(2) \\ \left(v(t) - RCu(2) - u(1)\right)/LC \end{bmatrix}$$

Now, the equation can be coded into Matlab:

```
  Du = [u(2); (ku*Uw-R*C*u(2)-u(1))/(L*C)];
```

Python counts matrix elements from 0, so the equivalent Python code can be written as:

```python
def rlc(u, t, R):
    return u[1], (U * w - R * C * u[1] - u[0]) / (L * C)
```

The only thing left is to describe the chirp signal. In Matlab the chirp signal can be coded as:

```
k = (fmax - fmin) / tmax;
Uw = cos((2 * pi() * (k / 2) .*t .*t) + (2 * pi() * fmin .*t));
```

In Python:

```
k = (fmax - fmin) / max(t)
w = np.cos((2 * np.pi * (k / 2) * t * t) + (2 * np.pi * fmin * t)
    )
```

9 Simulation Results

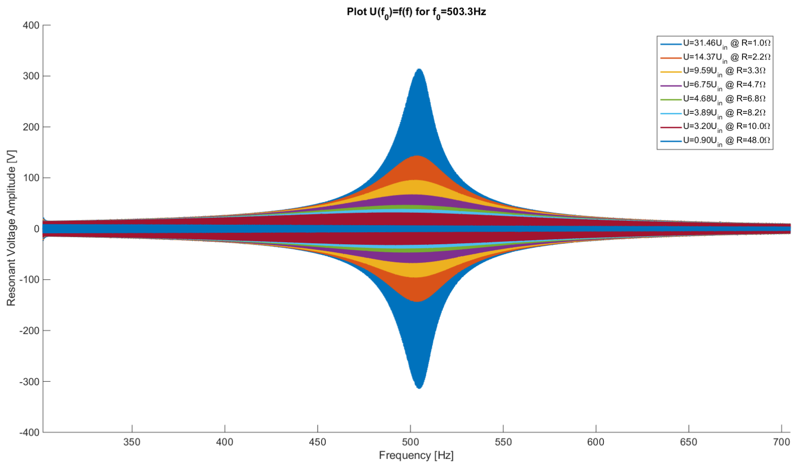

Figure 4: Plot of resonant voltage amplitude

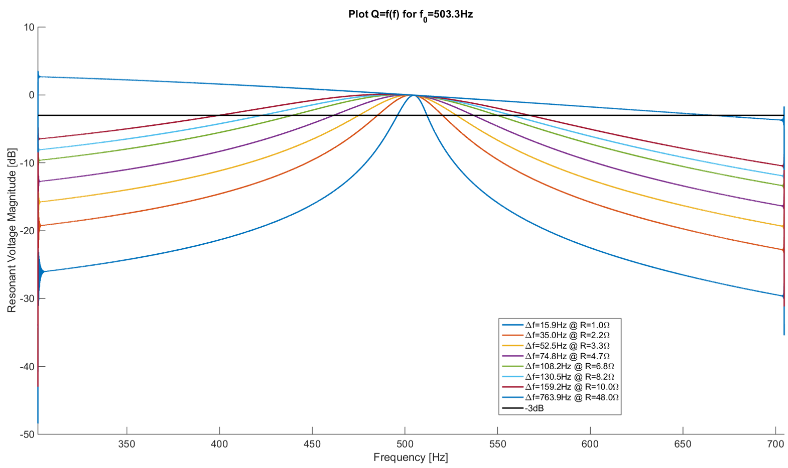

Figure 5: Plot of RLC circuit band frequency

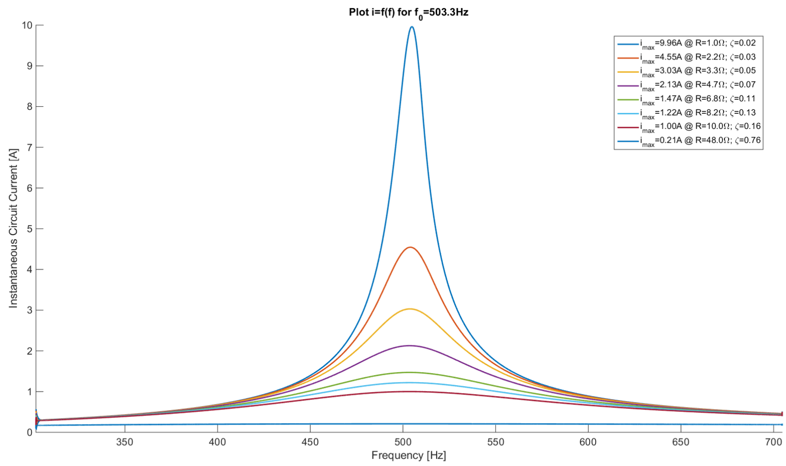

Figure 6: Plot of resonant current amplitude

10 Conclusion

This document demonstrates the Lagrangian, co-energy and state-space approach to modeling a simple RLC circuit. It also provides a visual block diagram suitable for simulation in Simulink and basic code for scripting.