Modeling a Pendulum

ARO (@_art_of_electronics_)

June 20, 2025

1 Introduction

This document presents the modeling of a physical pendulum using Lagrangian mechanics and co-energy analysis. A Simulink block diagram is also provided for simulation purposes.

2 Pendulum Description

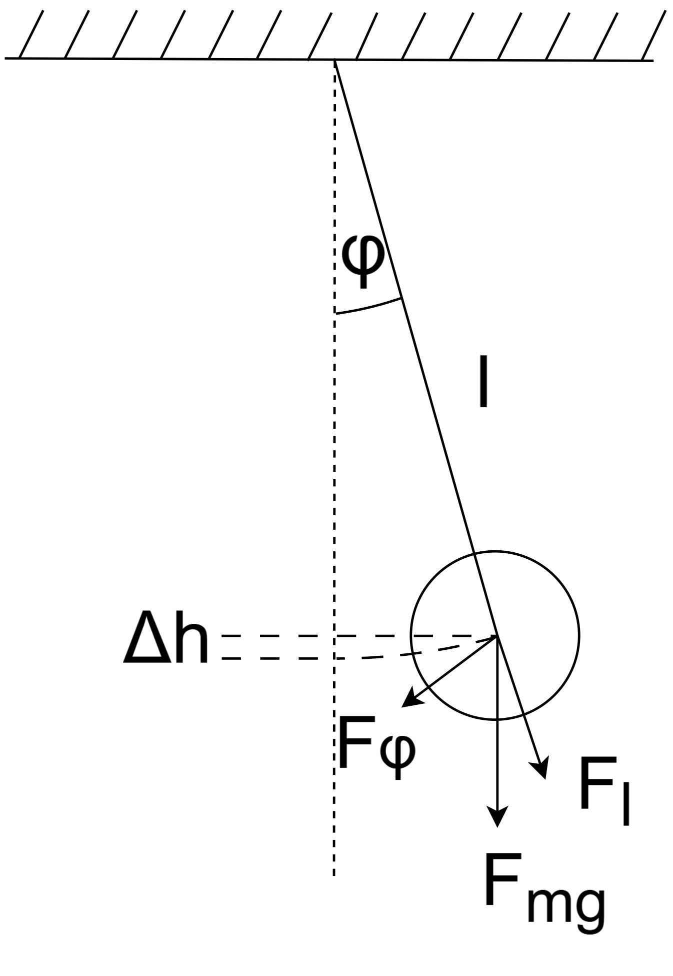

Figure 1: Overview of the pendulum

In this paper, considerations regarding pendulum, in case of modeling and derivation the equation will be performed on physical pendulum, before it will be simplified into a mathematical pendulum. Physical form of a pendulum is a more broadly considered case. Also the model will try to incorporate the resistance factor of the medium in which the pendulum will operate. The model will be presented for more linear form, where linearization around equilibrium meeting $\sin\varphi = \varphi$ will guarantee isochronism of the system.

3 Lagrangian Formulation

The kinetic and potential energies of the pendulum are defined as:

$$E_k = \frac{1}{2}J\dot{\varphi}^2$$
$$E_p = mgh$$

A pendulum swings through space in rotational motion, so instead of mass it is wise to use momentum of inertia, hence the initial consideration is a physical pendulum. In this case, the momentum of inertia is:

$$J = ml^2$$

The height h changes as:

$$h = l \cdot (1 - \cos\varphi)$$

The system does not have a typical damper, however the resistance of the medium can be considered as dissipative element:

$$D = B\int_0^{\dot{\varphi}} J\dot{\varphi}d\varphi = \frac{1}{2}BJ\dot{\varphi}^2$$

The Lagrangian of the pendulum is:

$$\mathcal{L}(x,\dot{x}) = E_k - E_p = \frac{1}{2}J\dot{\varphi}^2 - mgh = \frac{1}{2}J\dot{\varphi}^2 - mgl \cdot (1 - \cos\varphi)$$

The Rayleigh dissipation function can be written as:

$$\mathcal{D} = D - P = \left(\frac{1}{2}BJ\dot{\varphi}^2\right) - 0 = \frac{1}{2}BJ\dot{\varphi}^2$$

Using Lagrange's equation with a non-conservative force, the equation expands into the following form:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) - \frac{\partial\mathcal{L}}{\partial q} + \frac{\partial\mathcal{D}}{\partial\dot{q}} = 0$$

This leads to the following differential equation:

$$\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2 - mgl \cdot (1 - \cos\varphi)\right)}{\dot{\varphi}}\right) - \frac{\partial\left(\frac{1}{2}ml^2\dot{\varphi}^2 - mgl \cdot (1 - \cos\varphi)\right)}{\varphi} + \frac{\partial\left(\frac{1}{2}BJ\dot{\varphi}^2\right)}{\dot{\varphi}} = 0$$

$$\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2 - mgl + mgl\cos\varphi\right)}{\dot{\varphi}}\right) - \frac{\partial\left(\frac{1}{2}ml^2\dot{\varphi}^2 - mgl + mgl\cos\varphi\right)}{\varphi} + \frac{\partial\left(\frac{1}{2}BJ\dot{\varphi}^2\right)}{\dot{\varphi}} = 0$$

$$\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2 - 0 + 0\right)}{\dot{\varphi}}\right) - \frac{\partial(0 - 0 + mgl\cos\varphi)}{\varphi} + \frac{\partial\left(\frac{1}{2}BJ\dot{\varphi}^2\right)}{\dot{\varphi}} = 0$$

$$\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2\right)}{\dot{\varphi}}\right) - \frac{\partial(mgl\cos\varphi)}{\varphi} + \frac{\partial\left(\frac{1}{2}BJ\dot{\varphi}^2\right)}{\dot{\varphi}} = 0$$

$$\frac{d}{dt}\left(2 \cdot \frac{1}{2}J\dot{\varphi}^{2-1}\right) - (-mgl\sin\varphi) + \left(2 \cdot \frac{1}{2}BJ\dot{\varphi}^{2-1}\right) = 0$$

$$\frac{d}{dt}(J\dot{\varphi}) + mgl\sin\varphi + BJ\dot{\varphi} = 0$$

$$J\ddot{\varphi} + BJ\dot{\varphi} + mgl\sin\varphi = 0$$

The above equation is a solution to the general case of a physical pendulum with damping. Now we can substitute $J = ml^2$:

$$ml^2\ddot{\varphi} + Bml^2\dot{\varphi} + mgl\sin\varphi = 0$$

Dividing both sides by $J = ml^2$ finally yields a mathematical pendulum:

$$\ddot{\varphi} + B\dot{\varphi} + \frac{g}{l}\sin\varphi = 0$$

Explicitly undampened form can be written as:

$$\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0$$

4 Co-energy and Energy Expressions

Momentum of inertia

The general momentum of inertia co-energy for mass in rotational motion is:

$$T = \int_0^{\omega} J\omega'\,d\omega' = \frac{1}{2}J\omega^2 = \frac{1}{2}J\dot{\varphi}^2$$

Torque

The hypothetical potential co-energy caused by swinging motion is:

$$U = \tau(\varphi) = F\Delta h(\varphi) = mg(l - l\cos\varphi)$$

Resistance of medium

The dissipation in form of resistance of medium is:

$$D = \int_0^t BJ\omega^2(t)\,dt = \frac{1}{2}BJ\omega^2 = \frac{1}{2}BJ\dot{\varphi}^2$$

Using the co-energy equation with a non-conservative force, we can write:

$$\frac{d}{dt}\left[\frac{\partial T}{\dot{y}}\right] - \frac{\partial T}{y} + \frac{\partial U}{y} + \frac{\partial D}{\dot{y}} = f_i$$

By substituting the coefficients into the equation, we obtain the following:

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2\right)}{\dot{\varphi}}\right] - \frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2\right)}{\varphi} + \frac{\partial\left(mg(l - l\cos\varphi)\right)}{\varphi} + \frac{\partial\left(\frac{1}{2}BJ\dot{\varphi}^2\right)}{\dot{\varphi}} = 0$$

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2\right)}{\dot{\varphi}}\right] - \frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2\right)}{\varphi} + \frac{\partial\left(mgl\right)}{\varphi} + \frac{\partial\left(-mgl\cos\varphi\right)}{\varphi} + \frac{\partial\left(\frac{1}{2}BJ\dot{\varphi}^2\right)}{\dot{\varphi}} = 0$$

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2\right)}{\dot{\varphi}}\right] - 0 + 0 + \frac{\partial\left(-mgl\cos\varphi\right)}{\varphi} + \frac{\partial\left(\frac{1}{2}BJ\dot{\varphi}^2\right)}{\dot{\varphi}} = 0$$

$$\frac{d}{dt}\left[J\dot{\varphi}\right] + mgl\sin\varphi + BJ\dot{\varphi} = 0$$

$$J\ddot{\varphi} + BJ\dot{\varphi} + mgl\sin\varphi = 0$$

Substituting $J = ml^2$ and dividing both sides by ml^2 gives:

$$ml^2\ddot{\varphi} + Bml^2\dot{\varphi} + mgl\sin\varphi = 0$$

$$\ddot{\varphi} + B\dot{\varphi} + \frac{g}{l}\sin\varphi = 0$$

5 Simulink Model for Lagrangian and Co-energy Expression

In order to model the equation properly, we must translate the dynamic equation into a form understandable for Simulink:

$$\ddot{\varphi} + B\dot{\varphi} + \frac{g}{l}\sin\varphi = 0$$

$$\ddot{\varphi} = -B\dot{\varphi} - \frac{g}{l}\sin\varphi$$

In the following, the Simulink model is presented.

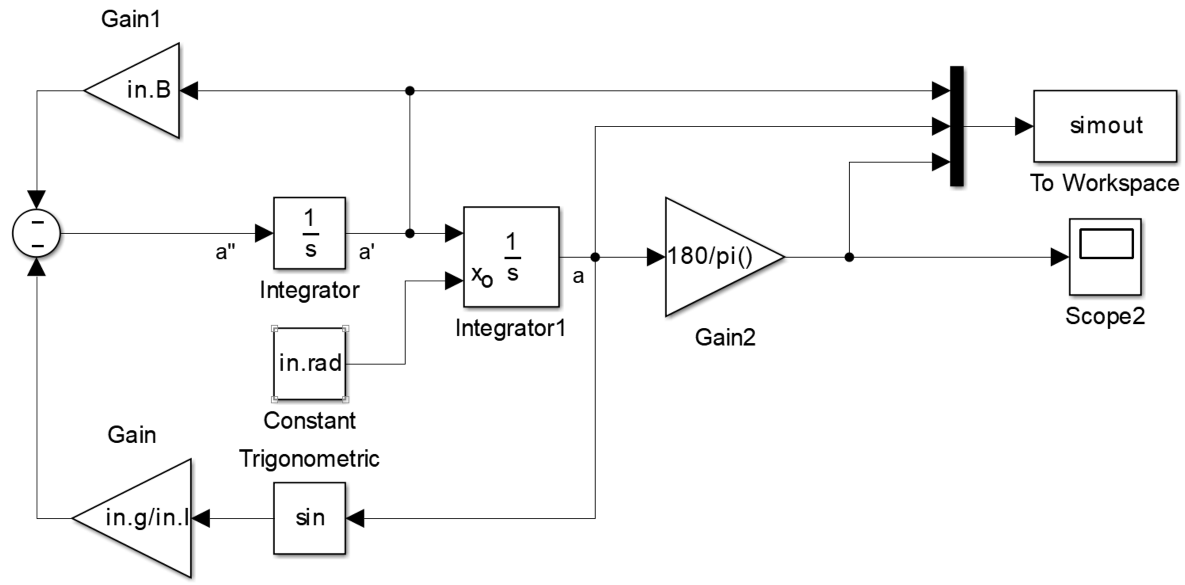

Figure 2: Simulink model of pendulum

6 Matlab and Python script

For script-based system modeling, we must take a slightly different approach to translating mathematical equations into code. Let $\varphi = phi(1)$, $\dot{\varphi}(t) = phi(2)$, in that case the matrix of pendulum equation will be written as:

$$\frac{d}{dt}\begin{bmatrix} phi(1) \\ phi(2) \end{bmatrix} = \begin{bmatrix} phi(2) \\ -B \cdot phi(2) - g/l \cdot \sin(phi(1)) \end{bmatrix}$$

Now, the equation can be coded into Matlab:

```
    Dphi = [phi(2); -in.B * phi(2) - in.g / in.l * sin(phi(1))];
```

Equivalent Python code can be written as:

```
def pendulum(phi, t, B):
    return phi[1], -B * phi[1] - g / l * np.sin(phi[0])
```

7 Simulation Results

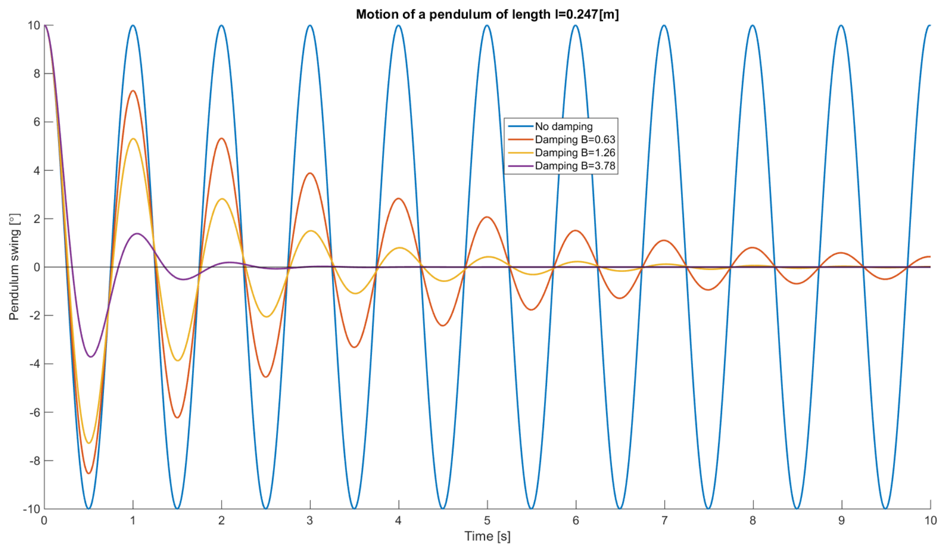

Figure 3: Result of simulation

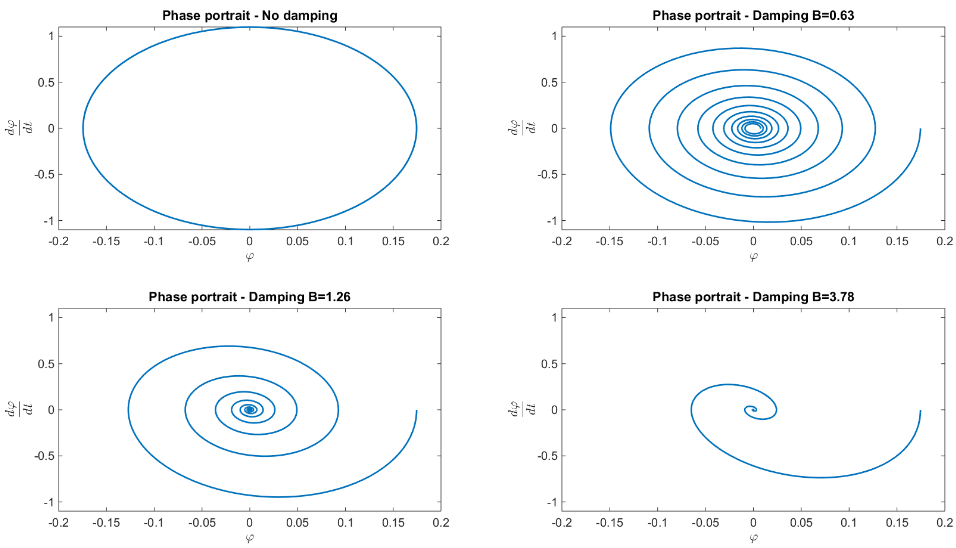

Figure 4: Phase portrait of pendulum

8 Conclusion

This document demonstrates the Lagrangian and co-energy approach to modeling a pendulum. It also provides a visual block diagram suitable for simulation in Simulink and basic code for scripting.