Modeling 2DoF System

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1 Introduction

This document presents the modeling of a 2 degrees-of-freedom (DoF) system using Lagrangian mechanics and co-energy analysis. A Simulink block diagram is also provided for simulation purposes.

2 2DoF System Description

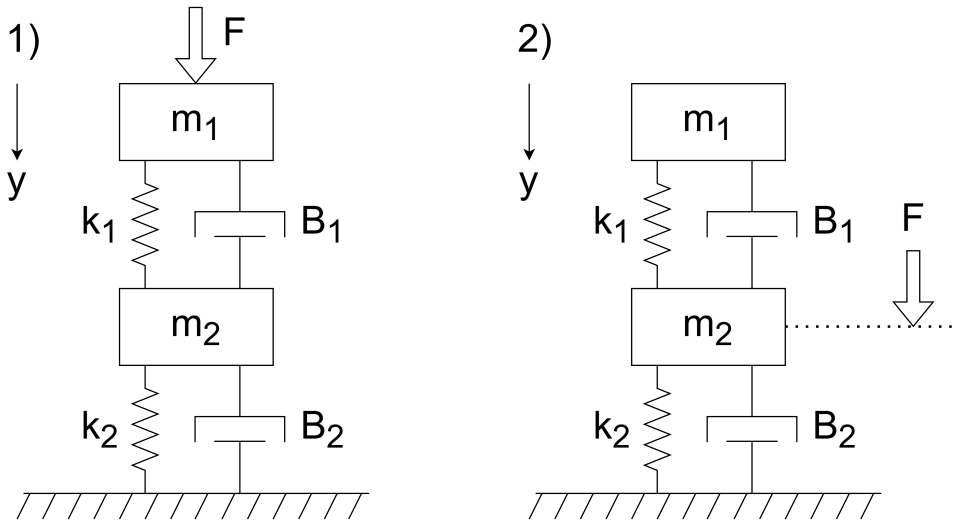

Figure 1: Overview of the 2DoF system

We consider a 2DoF system in 2 cases: first, where force is applied to outer mass m_1; second, where force is applied to inner mass m_2. In this model, forcing can also be applied as displacement, given by initial conditions. In order to perform modeling, the 2DoF system must be uncoupled into 2 separate equivalent 1DoF subsystems.

3 Lagrangian Formulation

3.1 Forcing on m_1 - subsystem 1a

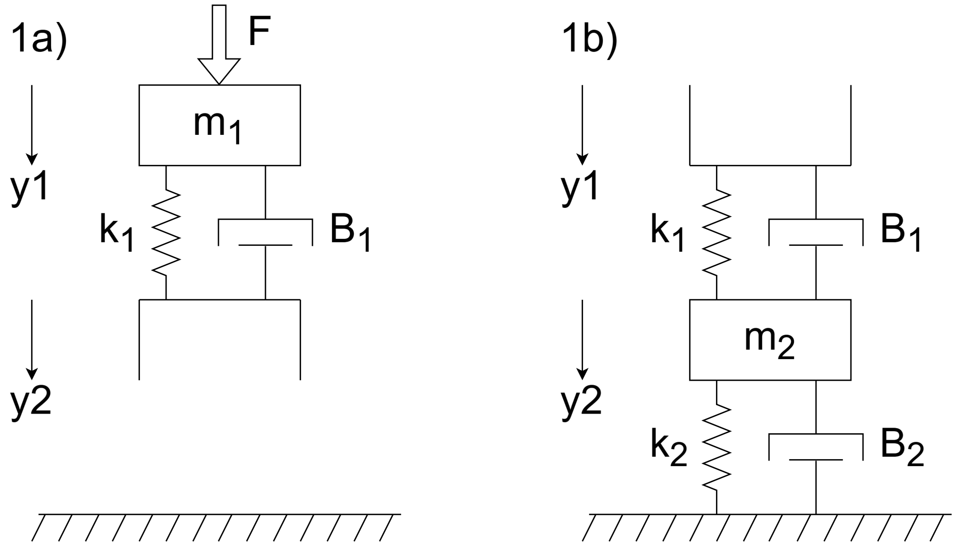

Figure 2: Uncoupled 2DoF system with forcing on m_1

The kinetic and potential energies of subsystem 1a are defined as:

$$
\begin{aligned}
E_{k1} &= \frac{1}{2} m_1 \dot{y_1}^2 \\
E_{p1} &= k_1 \int_{y_1}^{y_2} y dy = \frac{1}{2} k_1 y_1^2 - \frac{1}{2} k_1 y_2^2
\end{aligned}
$$

The system also has a damper, which is a dissipative element:

$$
D_1 = B_1 \int_{\dot{y_1}}^{\dot{y_2}} \dot{y} dy = \frac{1}{2} B_1 \dot{y}_1^2 - \frac{1}{2} B_1 \dot{y_2}^2
$$

The Lagrangian of the subsystem 1a is:

$$
\mathcal{L}(y, \dot{y}) = E_{k1} - E_{p1} = \frac{1}{2} m_1 \dot{y_1}^2 - \left(\frac{1}{2} k_1 y_1^2 - \frac{1}{2} k_1 y_2^2 \right)
$$

The Rayleigh dissipation function can be written as:

$$
\mathcal{D} = D_1 - P_1 = \left(\frac{1}{2} B_1 \dot{y}_1^2 - \frac{1}{2} B_1 \dot{y_2}^2 \right) - F \dot{y}_1
$$

Using Lagrange's equation with a non-conservative force (damper), the equation expands into the following form:

$$
\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{D}}{\partial \dot{q}} = 0
$$

This leads to the following differential equation:

$$\frac{d}{dt}\left(\frac{\partial\big(\frac{1}{2}m_1\dot{y_1}^2-\big(\frac{1}{2}k_1y_1^2-\frac{1}{2}k_1y_2^2\big)\big)}{\dot{y}}\right)-$$

$$\frac{\partial\big(\frac{1}{2}m_1\dot{y_1}^2-\big(\frac{1}{2}k_1y_1^2-\frac{1}{2}k_1y_2^2\big)\big)}{y}+$$

$$\frac{\partial\left(\big(\frac{1}{2}B_1\dot{y}_1^2-\frac{1}{2}B_1\dot{y_2}^2\big)-F\dot{y}_1\right)}{\dot{y}}=0$$

$$\frac{d}{dt}\left(\frac{\partial\big(\frac{1}{2}m_1\dot{y_1}^2-0\big)}{\dot{y}}\right)-\frac{\partial\big(0-\big(\frac{1}{2}k_1y_1^2-\frac{1}{2}k_1y_2^2\big)\big)}{y}+\frac{\partial\left(\big(\frac{1}{2}B_1\dot{y}_1^2-\frac{1}{2}B_1\dot{y_2}^2\big)-F\dot{y}_1\right)}{\dot{y}}=0$$

$$\frac{d}{dt}\left(2\cdot\frac{1}{2}m_1\dot{y_1}^{2-1}\right)-$$

$$\left(-\left(2\cdot\frac{1}{2}k_1y_1^{2-1}-2\cdot\frac{1}{2}k_1y_2^{2-1}\right)\right)+$$

$$\left(\left(2\cdot\frac{1}{2}B_1\dot{y}_1^{2-1}-2\cdot\frac{1}{2}B_1\dot{y_2}^{2-1}\right)-F\dot{y_1}^{1-1}\right)=0$$

$$\frac{d}{dt}\left(m_1\dot{y}_1\right)-\left(-\left(k_1y_1-k_1y_2\right)\right)+\left(\left(B_1\dot{y}_1-B_1\dot{y}_2\right)-F\right)=0$$

$$\frac{d}{dt}\left(m_1\dot{y}_1\right)-\left(-k_1y_1+k_1y_2\right)+\left(B_1\dot{y}_1-B_1\dot{y}_2-F\right)=0$$

$$\frac{d}{dt}\left(m_1\dot{y}_1\right)+k_1y_1-k_1y_2+B_1\dot{y}_1-B_1\dot{y}_2-F=0$$

$$m_1\ddot{y}_1+k_1y_1-k_1y_2+B_1\dot{y}_1-B_1\dot{y}_2-F=0$$

Sorting yields the equation's final form:

$$m_1\ddot{y}_1+B_1\dot{y}_1+k_1y_1-B_1\dot{y}_2-k_1y_2=F$$

3.2 Forcing on m_1 - subsystem 1b

The kinetic and potential energies of subsystem 1b are defined as:

$$E_{k2}=\frac{1}{2}m_2\dot{y_2}^2$$
$$E_{p2}=-k_2\int_0^{y_2}y\,dy+k_1\int_{y_2}^{y_1}y\,dy=\frac{1}{2}k_2y_2^2+\frac{1}{2}k_1y_2^2-\frac{1}{2}k_1y_1^2$$

The system also has a damper, which is a dissipative element:

$$D_2=-B_2\int_0^{\dot{y}_2}\dot{y}\,dy+B_1\int_{\dot{y}_2}^{\dot{y}_1}\dot{y}\,dy=\frac{1}{2}B_2\dot{y}_2^2+\frac{1}{2}B_1\dot{y}_2^2-\frac{1}{2}B_1\dot{y}_1^2$$

The Lagrangian of the subsystem 1a is:

$$\mathcal{L}(y,\dot{y}) = E_{k2} - E_{p2} = \frac{1}{2}m_2\dot{y_2^2} - \left(\frac{1}{2}k_2y_2^2 + \frac{1}{2}k_1y_2^2 - \frac{1}{2}k_1y_1^2\right)$$

The Rayleigh dissipation function can be written as:

$$\mathcal{D} = D_2 - P_2 = \left(\frac{1}{2}B_2\dot{y_2^2} + \frac{1}{2}B_1\dot{y_2^2} - \frac{1}{2}B_1\dot{y_1^2}\right) - 0$$

Using Lagrange's equation with a non-conservative force (damper), the equation expands into the following form:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) - \frac{\partial\mathcal{L}}{\partial q} + \frac{\partial\mathcal{D}}{\partial\dot{q}} = 0$$

This leads to the following differential equation:

$$\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}m_2\dot{y_2^2} - \left(\frac{1}{2}k_2y_2^2 + \frac{1}{2}k_1y_2^2 - \frac{1}{2}k_1y_1^2\right)\right)}{\dot{y}}\right) - \frac{\partial\left(\frac{1}{2}m_2\dot{y_2^2} - \left(\frac{1}{2}k_2y_2^2 + \frac{1}{2}k_1y_2^2 - \frac{1}{2}k_1y_1^2\right)\right)}{y} + \frac{\partial\left(\left(\frac{1}{2}B_2\dot{y_2^2} + \frac{1}{2}B_1\dot{y_2^2} - \frac{1}{2}B_1\dot{y_1^2}\right) - 0\right)}{\dot{y}} = 0$$

$$\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}m_2\dot{y_2^2} - 0\right)}{\dot{y}}\right) - \frac{\partial\left(0 - \left(\frac{1}{2}k_2y_2^2 + \frac{1}{2}k_1y_2^2 - \frac{1}{2}k_1y_1^2\right)\right)}{y} + \frac{\partial\left(\left(\frac{1}{2}B_2\dot{y_2^2} + \frac{1}{2}B_1\dot{y_2^2} - \frac{1}{2}B_1\dot{y_1^2}\right) - 0\right)}{\dot{y}} = 0$$

$$\frac{d}{dt}\left(\left(2\cdot\frac{1}{2}m_2\dot{y_2}^{2-1}\right)\right) - \left(-\left(2\cdot\frac{1}{2}k_2y_2^{2-1} + 2\cdot\frac{1}{2}k_1y_2^{2-1} - 2\cdot\frac{1}{2}k_1y_1^{2-1}\right)\right) + \left(\left(2\cdot\frac{1}{2}B_2\dot{y_2}^{2-1} + 2\cdot\frac{1}{2}B_1\dot{y_2}^{2-1} - 2\cdot\frac{1}{2}B_1\dot{y_1}^{2-1}\right)\right) = 0$$

$$\frac{d}{dt}(m_2\dot{y_2}) - (-(k_2y_2 + k_1y_2 - k_1y_1)) + (B_2\dot{y_2} + B_1\dot{y_2} - B_1\dot{y_1}) = 0$$

$$\frac{d}{dt}\left(m_2\dot{y}_2\right) + k_2 y_2 + k_1 y_2 - k_1 y_1 + B_2\dot{y}_2 + B_1\dot{y}_2 - B_1\dot{y}_1 = 0$$

$$m_2\ddot{y}_2 + k_2 y_2 + k_1 y_2 - k_1 y_1 + B_2\dot{y}_2 + B_1\dot{y}_2 - B_1\dot{y}_1 = 0$$

Sorting and grouping yield the final form of the equation:

$$m_2\ddot{y}_2 + \left(B_2 + B_1\right)\dot{y}_2 + \left(k_2 + k_1\right) y_2 - B_1\dot{y}_1 - k_1 y_1 = 0$$

Equation Results

Ultimately, the equation for the 2DoF system with forcing on m_1 can be written as:

$$\begin{cases} m_1\ddot{y}_1 + B_1\dot{y}_1 + k_1 y_1 - B_1\dot{y}_2 - k_1 y_2 = F \\ m_2\ddot{y}_2 + \left(B_2 + B_1\right)\dot{y}_2 + \left(k_2 + k_1\right) y_2 - B_1\dot{y}_1 - k_1 y_1 = 0 \end{cases}$$

3.3 Forcing on m_2 - subsystem 2a

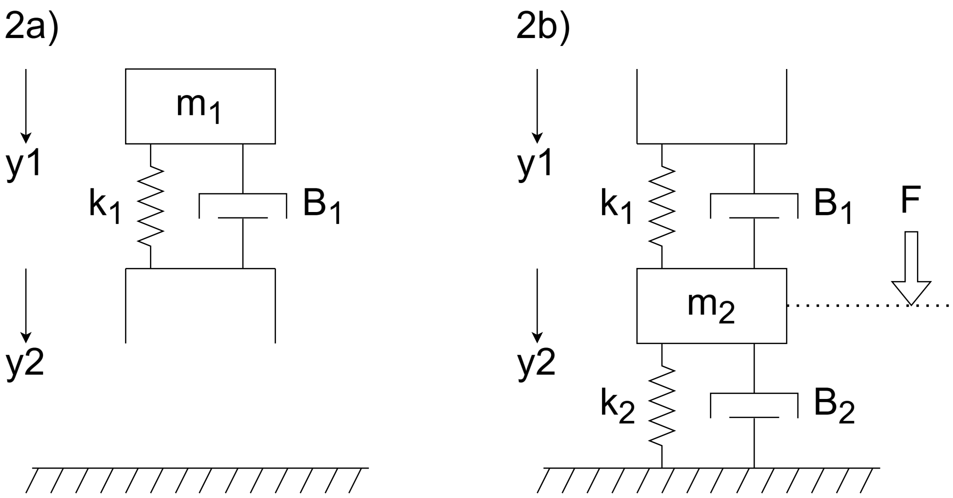

Figure 3: Uncoupled 2DoF system with forcing on m_2

The kinetic and potential energies of subsystem 2a are defined as:

$$E_{k1} = \frac{1}{2} m_1 \dot{y_1}^2$$
$$E_{p1} = k_1 \int_{y_1}^{y_2} y dy = \frac{1}{2} k_1 y_1^2 - \frac{1}{2} k_1 y_2^2$$

The system also has a damper, which is a dissipative element:

$$D_1 = B_1 \int_{\dot{y}_1}^{\dot{y}_2} \dot{y} dy = \frac{1}{2} B_1 \dot{y}_1^2 - \frac{1}{2} B_1 \dot{y_2}^2$$

The Lagrangian of subsystem 2a is:

$$\mathcal{L}(y,\dot{y}) = E_{k1} - E_{p1} = \frac{1}{2}m_1\dot{y_1}^2 - \left(\frac{1}{2}k_1y_1^2 - \frac{1}{2}k_1y_2^2\right)$$

The Rayleigh dissipation function can be written as:

$$\mathcal{D} = D_1 - P_1 = \left(\frac{1}{2}B_1\dot{y}_1^2 - \frac{1}{2}B_1\dot{y_2}^2\right) - 0$$

Using Lagrange's equation with a non-conservative force (damper), the equation expands into the following form:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) - \frac{\partial\mathcal{L}}{\partial q} + \frac{\partial\mathcal{D}}{\partial\dot{q}} = 0$$

The solving process in this model is identical to the previous case. The step-by-step process will not be explained, resulting in the following solution of subsystem 2a:

$$m_1\ddot{y}_1 + B_1\dot{y}_1 + k_1y_1 - B_1\dot{y}_2 - k_1y_2 = 0$$

3.4 Forcing on m_2 - subsystem 2b

The kinetic and potential energies of subsystem 1b are defined as:

$$E_{k2} = \frac{1}{2}m_2\dot{y_2}^2$$
$$E_{p2} = -k_2\int_0^{y_2} y\,dy + k_1\int_{y_2}^{y_1} y\,dy = \frac{1}{2}k_2y_2^2 + \frac{1}{2}k_1y_2^2 - \frac{1}{2}k_1y_1^2$$

The system also has a damper, which is a dissipative element:

$$D_2 = -B_2\int_0^{\dot{y}_2} \dot{y}\,dy + B_1\int_{\dot{y}_2}^{\dot{y}_1} \dot{y}\,dy = \frac{1}{2}B_2\dot{y}_2^2 + \frac{1}{2}B_1\dot{y}_2^2 - \frac{1}{2}B_1\dot{y}_1^2$$

The Lagrangian of subsystem 2a is:

$$\mathcal{L}(y,\dot{y}) = E_{k2} - E_{p2} = \frac{1}{2}m_2\dot{y}_2^2 - \left(\frac{1}{2}k_2y_2^2 + \frac{1}{2}k_1y_2^2 - \frac{1}{2}k_1y_1^2\right)$$

The Rayleigh dissipation function can be written as:

$$\mathcal{D} = D_2 - P_2 = \left(\frac{1}{2}B_2\dot{y}_2^2 + \frac{1}{2}B_1\dot{y}_2^2 - \frac{1}{2}B_1\dot{y}_1^2\right) - F\dot{y}_2$$

Using Lagrange's equation with a non-conservative force (damper), the equation expands into the following form:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) - \frac{\partial\mathcal{L}}{\partial q} + \frac{\partial\mathcal{D}}{\partial\dot{q}} = 0$$

As mentioned above, the step-by-step process will not be explained. The resulting solution of subsystem 2b is:

$$m_2\ddot{y}_2 + (B_2 + B_1)\,\dot{y}_2 + (k_2 + k_1)\,y_2 - B_1\dot{y}_1 - k_1y_1 = F$$

Equation Results

Ultimately, the equation for the 2DoF system with forcing on m_2 can be written as:

$$\begin{cases} m_1\ddot{y}_1 + B_1\dot{y}_1 + k_1 y_1 - B_1\dot{y}_2 - k_1 y_2 = 0 \\ m_2\ddot{y}_2 + (B_2 + B_1)\,\dot{y}_2 + (k_2 + k_1)\,y_2 - B_1\dot{y}_1 - k_1 y_1 = F \end{cases}$$

4 Co-energy and Energy Expressions

Mass

The general inertia co-energy for mass is:

$$T = \int_0^v mv'\,dv' = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{y}^2$$

In linear systems, energy and co-energy are equal.

Spring

The general energy stored in the spring k_1 is:

$$U = \int_0^y ky'\,dy' = \frac{1}{2}ky^2$$

Since the spring is conservative (no energy loss), co-energy equals energy.

Damper

The general damper dissipates energy as friction:

$$D = \int_0^t Bv^2(\tau)\,d\tau = \frac{1}{2}Bv^2 = \frac{1}{2}B\dot{y}^2$$

Dampers do not store energy, so they do not have co-energy.

Using the co-energy equation with a non-conservative force, we can write:

$$\frac{d}{dt}\left[\frac{\partial T}{\dot{y}}\right] - \frac{\partial T}{y} + \frac{\partial U}{y} + \frac{\partial D}{\dot{y}} = f_i$$

4.1 Forcing on m_1 - subsystem 1a

We need to adjust the general co-energy expressions to 2DoF subsystems speciffic configuration:

$$\begin{aligned} T_1 &= \frac{1}{2}m_1\dot{y_1}^2 \\ U_1 &= \frac{1}{2}k_1 y_1^2 - \frac{1}{2}k_1 y_2^2 \\ D_1 &= \frac{1}{2}B_1\dot{y}_1^2 - \frac{1}{2}B_1\dot{y_2}^2 \\ f_i &= F \end{aligned}$$

By substituting the coefficients into the equation, we obtain the following:

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}m_1\dot{y_1}^2\right)}{\dot{y}}\right]-\frac{\partial\left(\frac{1}{2}m_1\dot{y_1}^2\right)}{y}+\frac{\partial\left(\frac{1}{2}k_1y_1^2-\frac{1}{2}k_1y_2^2\right)}{y}+\frac{\partial\left(\frac{1}{2}B_1\dot{y}_1^2-\frac{1}{2}B_1\dot{y_2}^2\right)}{\dot{y}}=F$$

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}m_1\dot{y_1}^2\right)}{\dot{y}}\right]-0+\frac{\partial\left(\frac{1}{2}k_1y_1^2-\frac{1}{2}k_1y_2^2\right)}{y}+\frac{\partial\left(\frac{1}{2}B_1\dot{y}_1^2-\frac{1}{2}B_1\dot{y_2}^2\right)}{\dot{y}}=F$$

$$\frac{d}{dt}\left[m_1\dot{y_1}\right]+k_1y_1-k_1y_2+B_1\dot{y}_1-B_1\dot{y_2}=F$$

$$m_1\ddot{y_1}+B_1\dot{y}_1+k_1y_1-B_1\dot{y_2}-k_1y_2=F$$

4.2 Forcing on m_1 - subsystem 1b

In this case, we also need to adjust the general co-energy expressions to 2DoF 2nd subsystems speciffic configuration:

$$\begin{aligned}T_2&=\frac{1}{2}m_2\dot{y_2}^2\\U_2&=\frac{1}{2}k_2y_2^2\\D_2&=\frac{1}{2}B_2\dot{y}_2^2\\f_i&=B_1\Delta\dot{y}+k_1\Delta y\end{aligned}$$

By substituting the coefficients into the equation, we obtain the following:

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}m_2\dot{y_2}^2\right)}{\dot{y}}\right]-\frac{\partial\left(\frac{1}{2}m_2\dot{y_2}^2\right)}{y}+\frac{\partial\left(\frac{1}{2}k_2y_2^2\right)}{y}+\frac{\partial\left(\frac{1}{2}B_2\dot{y}_2^2\right)}{\dot{y}}=B_1\Delta\dot{y}+k_1\Delta y$$

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}m_2\dot{y_2}^2\right)}{\dot{y}}\right]-0+\frac{\partial\left(\frac{1}{2}k_2y_2^2\right)}{y}+\frac{\partial\left(\frac{1}{2}B_2\dot{y}_2^2\right)}{\dot{y}}=B_1(\dot{y_1}-\dot{y_2})+k_1(y_1-y_2)$$

$$m_2\ddot{y_2}+k_2y_2+B_2\dot{y_2}=B_1\dot{y_1}-B_1\dot{y_2}+k_1y_1-k_1y_2$$

$$m_2\ddot{y_2}+k_2y_2+B_2\dot{y_2}=B_1\dot{y_1}-B_1\dot{y_2}+k_1y_1-k_1y_2$$

$$m_2\ddot{y_2}+\left(B_2+B_1\right)\dot{y_2}+\left(k_2+k_1\right)y_2-B_1\dot{y_1}-k_1y_1=0$$

Equation Results

Ultimately, the equation for the 2DoF system with forcing on m_1, solved via co-energy expression can be written as:

$$\begin{cases} m_1\ddot{y}_1 + B_1\dot{y}_1 + k_1y_1 - B_1\dot{y}_2 - k_1y_2 = F \\ m_2\ddot{y}_2 + (B_2 + B_1)\,\dot{y}_2 + (k_2 + k_1)\,y_2 - B_1\dot{y}_1 - k_1y_1 = 0 \end{cases}$$

4.3 Forcing on m_2 - subsystem 2a

We need to adjust the general co-energy expressions to 2DoF subsystems speciffic configuration:

$$\begin{aligned} T_1 &= \frac{1}{2}m_1\dot{y_1}^2 \\ U_1 &= \frac{1}{2}k_1y_1^2 - \frac{1}{2}k_1y_2^2 \\ D_1 &= \frac{1}{2}B_1\dot{y}_1^2 - \frac{1}{2}B_1\dot{y_2}^2 \\ f_i &= 0 \end{aligned}$$

By substituting the coefficients into the equation, we obtain the following:

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}m_1\dot{y_1}^2\right)}{\dot{y}}\right] - \frac{\partial\left(\frac{1}{2}m_1\dot{y_1}^2\right)}{y} + \frac{\partial\left(\frac{1}{2}k_1y_1^2 - \frac{1}{2}k_1y_2^2\right)}{y} + \frac{\partial\left(\frac{1}{2}B_1\dot{y}_1^2 - \frac{1}{2}B_1\dot{y_2}^2\right)}{\dot{y}} = 0$$

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}m_1\dot{y_1}^2\right)}{\dot{y}}\right] - 0 + \frac{\partial\left(\frac{1}{2}k_1y_1^2 - \frac{1}{2}k_1y_2^2\right)}{y} + \frac{\partial\left(\frac{1}{2}B_1\dot{y}_1^2 - \frac{1}{2}B_1\dot{y_2}^2\right)}{\dot{y}} = 0$$

$$\frac{d}{dt}\left[m_1\dot{y}_1\right] + k_1y_1 - k_1y_2 + B_1\dot{y}_1 - B_1\dot{y}_2 = 0$$

$$m_1\ddot{y}_1 + B_1\dot{y}_1 + k_1y_1 - B_1\dot{y}_2 - k_1y_2 = 0$$

4.4 Forcing on m_2 - subsystem 2b

In this case, we also need to adjust the general co-energy expressions to 2DoF 2nd subsystems speciffic configuration:

$$\begin{aligned} T_2 &= \frac{1}{2}m_2\dot{y_2}^2 \\ U_2 &= \frac{1}{2}k_2y_2^2 \\ D_2 &= \frac{1}{2}B_2\dot{y}_2^2 \\ f_i &= F + B_1\Delta\dot{y} + k_1\Delta y \end{aligned}$$

By substituting the coefficients into the equation, we obtain the following:

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}m_2\dot{y_2}^2\right)}{\dot{y}}\right]-\frac{\partial\left(\frac{1}{2}m_2\dot{y_2}^2\right)}{y}+\frac{\partial\left(\frac{1}{2}k_2y_2^2\right)}{y}+\frac{\partial\left(\frac{1}{2}B_2\dot{y}_2^2\right)}{\dot{y}}=F+B_1\Delta\dot{y}+k_1\Delta y$$

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}m_2\dot{y_2}^2\right)}{\dot{y}}\right]-0+\frac{\partial\left(\frac{1}{2}k_2y_2^2\right)}{y}+\frac{\partial\left(\frac{1}{2}B_2\dot{y}_2^2\right)}{\dot{y}}=F+B_1(\dot{y_1}-\dot{y_2})+k_1(y_1-y_2)$$

$$m_2\ddot{y_2}+k_2y_2+B_2\dot{y_2}=F+B_1\dot{y_1}-B_1\dot{y_2}+k_1y_1-k_1y_2$$

$$m_2\ddot{y_2}+k_2y_2+B_2\dot{y_2}=F+B_1\dot{y_1}-B_1\dot{y_2}+k_1y_1-k_1y_2$$

$$m_2\ddot{y_2}+\left(B_2+B_1\right)\dot{y_2}+\left(k_2+k_1\right)y_2-B_1\dot{y_1}-k_1y_1=F$$

Equation Results

Ultimately, the equation for the 2DoF system with forcing on m_1, solved via co-energy expression can be written as:

$$\begin{cases} m_1\ddot{y_1}+B_1\dot{y_1}+k_1y_1-B_1\dot{y_2}-k_1y_2=0 \\ m_2\ddot{y_2}+\left(B_2+B_1\right)\dot{y_2}+\left(k_2+k_1\right)y_2-B_1\dot{y_1}-k_1y_1=F \end{cases}$$

5 Simulink Model for Lagrangian and Co-energy Expression - Forcing on m_1

In order to model the equation properly, we must translate the dynamic equation into a form understandable for Simulink:

$$\begin{cases} m_1\ddot{y_1}+B_1\dot{y_1}+k_1y_1-B_1\dot{y_2}-k_1y_2=F \\ m_2\ddot{y_2}+\left(B_2+B_1\right)\dot{y_2}+\left(k_2+k_1\right)y_2-B_1\dot{y_1}-k_1y_1=0 \end{cases}$$

$$\begin{cases} \ddot{y_1}=\frac{1}{m_1}\left(F-B_1\dot{y_1}-k_1y_1+B_1\dot{y_2}+k_1y_2\right) \\ \ddot{y_2}=\frac{1}{m_2}\left(-\left(B_2+B_1\right)\dot{y_2}-\left(k_2+k_1\right)y_2+B_1\dot{y_1}+k_1y_1\right) \end{cases}$$

Below, the Simulink model is presented.

6 Simulink Model for Lagrangian and Co-energy Expression - Forcing on m_2

Same as above, we must translate the dynamic equation into a form understandable for Simulink:

$$\begin{cases} m_1\ddot{y_1}+B_1\dot{y_1}+k_1y_1-B_1\dot{y_2}-k_1y_2=0 \\ m_2\ddot{y_2}+\left(B_2+B_1\right)\dot{y_2}+\left(k_2+k_1\right)y_2-B_1\dot{y_1}-k_1y_1=F \end{cases}$$

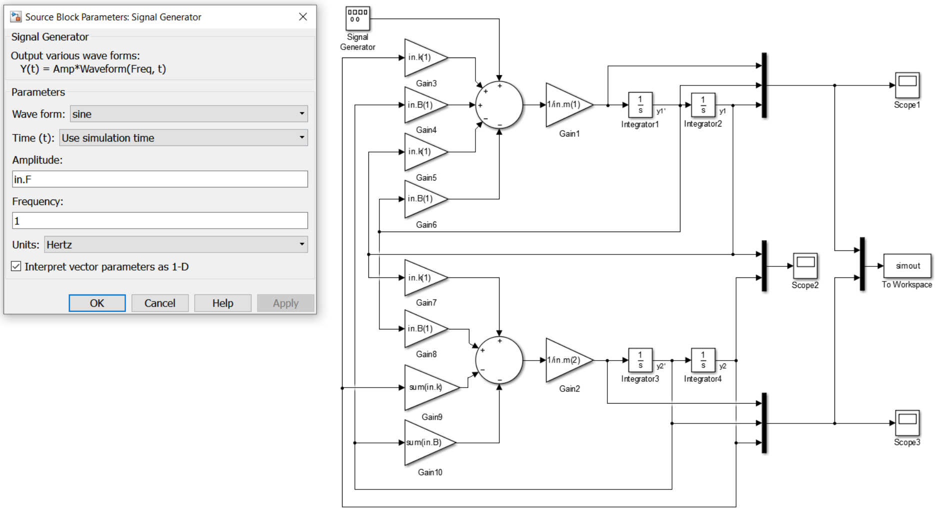

Figure 4: Simulink model of 2DoF system - Forcing on m_1

$$\begin{cases} \ddot{y}_1 = \frac{1}{m_1}\left(-B_1\dot{y}_1 - k_1 y_1 + B_1\dot{y}_2 + k_1 y_2\right) \\ \ddot{y}_2 = \frac{1}{m_2}\left(F - \left(B_2 + B_1\right)\dot{y}_2 - \left(k_2 + k_1\right)y_2 + B_1\dot{y}_1 + k_1 y_1\right) \end{cases}$$

Below, the Simulink model is presented.

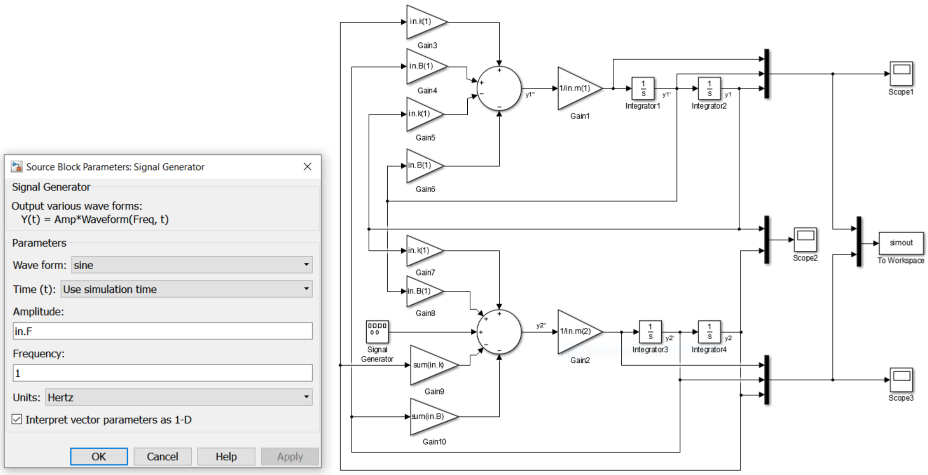

Figure 5: Simulink model of 2DoF system - Forcing on m_2

7 Matlab and Python script - forcing on m_1

For script-based system modeling we must take a slightly different approach to translating mathematical equation into code. Let $y_1(t) = y(1)$, $y_1'(t) = y(2)$, $y_2(t) = y(3)$, $y_2'(t) = y(4)$,in that case the matrix of the 2DoF equation will be written as:

$$\frac{d}{dt}\begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix} = \begin{bmatrix} y(2) \\ \left(F - B(1)\cdot y(2) - k(1)\cdot y(1) + B(1)\cdot y(4) + k(1)\cdot y(3)\right)/m(1) \\ y(4) \\ \left(-\left(B(2)+B(1)\right)\cdot y(4) - \left(k(2)+k(1)\right)\cdot y(3) + B(1)\cdot y(2) + k(1)\cdot y(1)\right)/m(2) \end{bmatrix}$$

Now, the equation can be coded into Matlab:

```
  Dy = [y(2); subsystem(1); y(4); subsystem(2)];
```

where:

```
subsystem(1) = (F - in.B(1) * y(2) - in.k(1) * y(1) + in.B(1) * y
    (4) + in.k(1) * y(3)) / in.m(1);
subsystem(2) = (-( sum(in.B)) * y(4) - (sum(in.k)) * y(3) + in.B
    (1) * y(2) + in.k(1) * y(1)) / in.m(2);
```

Python counts matrix elements from 0, so the equivalent Python code can be written as:

```
def dual_dof(x, t):
    subsystem1 = (F - B[0]*x[1] - k[0]*x[0] + B[0]*x[3] + k[0]*x
        [2]) / m[0]
    subsystem2 = (0 - sum(B)*x[3] - sum(k)*x[2] + B[0]*x[1] + k
        [0]*x[0]) / m[1]
    return x[1], subsystem1, x[3], subsystem2
```

8 Matlab and Python script - forcing on m_2

For script-based system modeling we must take a slightly different approach to translating mathematical equation into code. Let $y_1(t) = y(1)$, $y_1'(t) = y(2)$, $y_2(t) = y(3)$, $y_2'(t) = y(4)$,in that case the matrix of the 2DoF equation will be written as:

$$\frac{d}{dt}\begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix} = \begin{bmatrix} y(2) \\ \left(-B(1)\cdot y(2) - k(1)\cdot y(1) + B(1)\cdot y(4) + k(1)\cdot y(3)\right)/m(1) \\ y(4) \\ \left(F - (B(2)+B(1))\cdot y(4) - (k(2)+k(1))\cdot y(3) + B(1)\cdot y(2) + k(1)\cdot y(1)\right)/m(2) \end{bmatrix}$$

Now, the equation can be coded into Matlab:

```
  Dy = [y(2); subsystem(1); y(4); subsystem(2)];
```

where:

```
subsystem(1) = (-in.B(1) * y(2) - in.k(1) * y(1) + in.B (1) * y
    (4) + in.k(1) * y(3)) / in.m(1);
subsystem(2) = (F -(sum(in.B)) * y(4) - (sum(in.k)) * y(3) + in.B
    (1) * y(2) + in.k(1) * y(1)) / in.m(2);
```

Python counts matrix elements from 0, so the equivalent Python code can be written as:

```
def dual_dof(x, t):
    subsystem1 = (0 - B[0]*x[1] - k[0]*x[0] + B[0]*x[3] + k[0]*x
        [2]) / m[0]
    subsystem2 = (F - sum(B)*x[3] - sum(k)*x[2] + B[0]*x[1] + k
        [0]*x[0]) / m[1]
    return x[1], subsystem1, x[3], subsystem2
```

9 Simulation Results

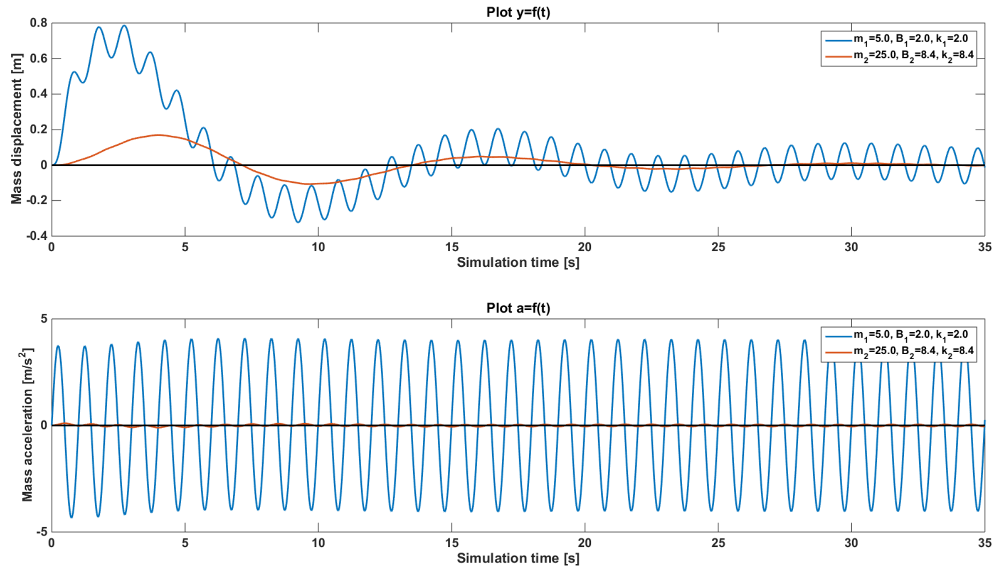

Figure 6: Forcing on m_1

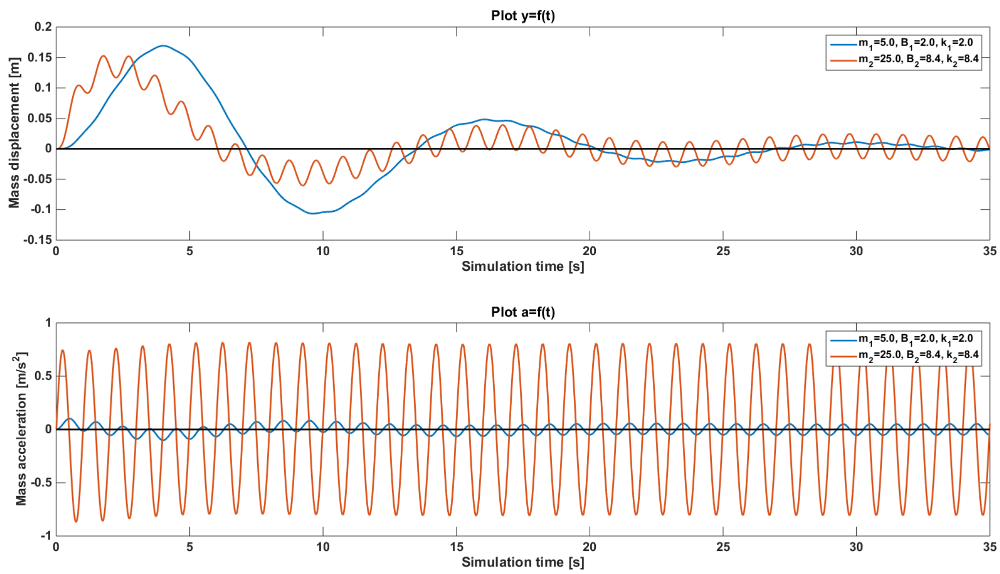

Figure 7: Forcing on m_2

10 Conclusion

This document demonstrates the Lagrangian and co-energy approach to modeling a 2DoF system. It also provides a visual block diagram suitable for simulation in Simulink and basic code for scripting.