Modeling DC Motor

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1 Introduction

This document presents the modeling of a DC Motor system using Lagrangian mechanics and co-energy analysis. It also provides a method for including a friction model. A Simulink block diagram is also provided for simulation purposes.

2 DC Motor Description

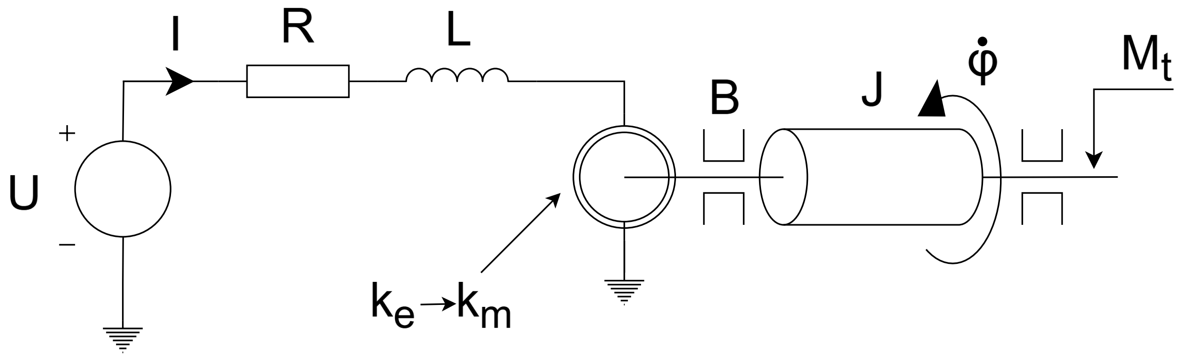

Figure 1: Overview of the DC Motor

DC Motor is considered as a hybrid model that consists of mechanical and electrical submodels. Presented model of DC motor is a simplified version that substitutes magnetic flux with k_m (mechanical) and k_e (electrical) gyrator constants.

3 Lagrangian Formulation - Mechanical Submodel

The kinetic and potential energies of DC Motor's rotor are defined as:

$$E_k = \frac{1}{2}J\omega^2$$
$$E_p = 0$$

The system has frictional forces at shaft bearings which are a dissipative element:

$$D = B\int_0^\omega \omega d\omega = \frac{1}{2}B\omega^2$$

The Lagrangian of mechanical submodel of the DC motor is:

$$\mathcal{L}(\omega) = E_k - E_p = \frac{1}{2}J\omega^2 - 0 = \frac{1}{2}J\dot{\varphi}^2$$

The Rayleigh dissipation function can be written as:

$$\mathcal{R} = D - P = \frac{1}{2}B\omega^2 - 0 = \frac{1}{2}B\dot{\varphi}^2$$

The mechanical gyrator, which in this case is defined as motor torque is a power-conserving interaction between mechanical and electrical energy domains, and is equal:

$$G_T = k_m \cdot i = k_m \cdot \dot{q}$$

Using Lagrange's equation with a non-conservative force, the equation expands into the following form:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right) - \frac{\partial\mathcal{L}}{\varphi} + \frac{\partial\mathcal{R}}{\partial\dot{\varphi}} = G_T - M_T$$

where:

$$M_T - \text{ output torque at shaft [Nm]}$$

This leads to the following differential equation:

$$\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2\right)}{\dot{\varphi}}\right) - \frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2\right)}{\varphi} + \frac{\partial\left(\frac{1}{2}B\dot{\varphi}^2\right)}{\dot{\varphi}} = k_m\dot{q} - M_T$$
$$\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2\right)}{\dot{\varphi}}\right) - 0 + \frac{\partial\left(\frac{1}{2}B\dot{\varphi}^2\right)}{\dot{\varphi}} = k_m\dot{q} - M_T$$
$$\frac{d}{dt}\left(2\cdot\frac{1}{2}J\dot{\varphi}^{2-1}\right) + \left(2\cdot\frac{1}{2}B\dot{\varphi}^{2-1}\right) = k_m\dot{q} - M_T$$
$$\frac{d}{dt}\left(J\dot{\varphi}\right) + B\dot{\varphi} = k_m\dot{q} - M_T$$
$$J\ddot{\varphi} + B\dot{\varphi} = k_m\dot{q} - M_T$$

Sorting yields the equation's final form:

$$J\ddot{\varphi} = -B\dot{\varphi} + k_m\dot{q} - M_T$$

4 Lagrangian Formulation - Electrical Submodel

The kinetic and potential energies of DC Motor's stator are defined as:

$$E_k = \frac{1}{2}Li^2$$
$$E_p = 0$$

The motor's winding has resistance, which is considered a dissipative element:

$$D = \frac{1}{2}Ri^2$$

The Lagrangian of electrical submodel of the DC motor is:

$$\mathcal{L}(\omega) = E_k - E_p = \frac{1}{2}Li^2 - 0 = \frac{1}{2}L\dot{q}^2$$

The Rayleigh dissipation function can be written as:

$$\mathcal{R} = D - P = \frac{1}{2}Ri^2 - Ui = \frac{1}{2}R\dot{q}^2 - U\dot{q}$$

The electrical gyrator, which in this case is defined as back EMF voltage is a power-conserving interaction between electrical and mechanical energy domains, and is equal:

$$G_V = -k_e \cdot \omega = -k_e \cdot \dot{\varphi}$$

Using Lagrange's equation with a non-conservative force (damper), the equation expands into the following form:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathcal{L}}{\varphi} + \frac{\partial \mathcal{R}}{\partial \dot{\varphi}} = G_V$$

This leads to the following differential equation:

$$\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}L\dot{q}^2\right)}{\dot{q}}\right) - \frac{\partial\left(\frac{1}{2}L\dot{q}^2\right)}{q} + \frac{\partial\left(\frac{1}{2}R\dot{q}^2 - U\dot{q}\right)}{\dot{q}} = -k_e\dot{\varphi}$$

$$\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}L\dot{q}^2\right)}{\dot{q}}\right) - 0 + \frac{\partial\left(\frac{1}{2}R\dot{q}^2\right)}{\dot{q}} - \frac{\partial\left(U\dot{q}\right)}{\dot{q}} = -k_e\dot{\varphi}$$

$$\frac{d}{dt}\left(2 \cdot \frac{1}{2}L\dot{q}^{2-1}\right) + \left(2 \cdot \frac{1}{2}R\dot{q}^{2-1}\right) - \left(U\dot{q}^{1-1}\right) = -k_e\dot{\varphi}$$

$$\frac{d}{dt}\left(L\dot{q}\right) + R\dot{q} - U = -k_e\dot{\varphi}$$

$$L\ddot{q} + R\dot{q} - U = -k_e\dot{\varphi}$$

Sorting yields the equation's final form:

$$J\ddot{q} = U - R\dot{q} - k_e\dot{\varphi}$$

5 Co-energy and Energy Expressions - Mechanical Submodel

Momentum of inertia

The general inertia co-energy for rotor is:

$$T = \int_0^v J\omega'\, d\omega' = \frac{1}{2}J\omega^2 = \frac{1}{2}J\dot{\varphi}^2$$

Shaft friction

The shaft bearings dissipate energy as friction:

$$D = \int_0^t B\omega^2(\tau)\, d\tau = \frac{1}{2}B\omega^2 = \frac{1}{2}B\dot{\varphi}^2$$

Co-energy expression

Using the co-energy equation with a non-conservative force, we can write:

$$\frac{d}{dt}\left[\frac{\partial T}{\dot{\varphi}}\right] - \frac{\partial T}{\varphi} + \frac{\partial U}{\varphi} + \frac{\partial D}{\dot{\varphi}} = f_i$$

Adjusting the general co-energy expressions to DC motor specific configuration gives:

$$\begin{aligned}
&T = \frac{1}{2}J\dot{\varphi}^2 \\
&U = 0 \\
&D = \frac{1}{2}B\dot{\varphi}^2 \\
&f_i = k_m\dot{q} - M_T
\end{aligned}$$

By substituting the coefficients into the equation, we obtain the following:

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2\right)}{\dot{\varphi}}\right] - \frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2\right)}{\varphi} + \frac{\partial\left(0\right)}{\varphi} + \frac{\partial\left(\frac{1}{2}B\dot{\varphi}^2\right)}{\dot{\varphi}} = k_m\dot{q} - M_t$$

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}J\dot{\varphi}^2\right)}{\dot{\varphi}}\right] - 0 + 0 + \frac{\partial\left(\frac{1}{2}B\dot{\varphi}^2\right)}{\dot{\varphi}} = k_m\dot{q} - M_t$$

$$\frac{d}{dt}\left[J\dot{\varphi}\right] + B\dot{\varphi} = k_m\dot{q} - M_t$$

$$J\ddot{\varphi} + B\dot{\varphi} = k_m\dot{q} - M_t$$

$$J\ddot{\varphi} = -B\dot{\varphi} + k_m\dot{q} - M_t$$

6 Co-energy and Energy Expressions - Electrical Sub-model

Winding Inductance

The magnetic co-energy for the stator's winding:

$$T = \int_0^i Li' \, di' = \frac{1}{2}Li^2 = \frac{1}{2}L\dot{q}^2$$

Winding Resistance

The winding's resistance dissipate energy as heat:

$$D = \int_0^t Ri^2(\tau) \, d\tau = \frac{1}{2}Ri^2 = \frac{1}{2}R\dot{q}^2$$

Co-energy expression

Using the co-energy equation with a non-conservative force, we can write:

$$\frac{d}{dt}\left[\frac{\partial T}{\dot{q}}\right] - \frac{\partial T}{q} + \frac{\partial U}{q} + \frac{\partial D}{\dot{q}} = f_i$$

Adjusting the general co-energy expressions to DC motor specific configuration gives:

$$\begin{aligned} T &= \frac{1}{2}L\dot{q}^2 \\ U &= 0 \\ D &= \frac{1}{2}R\dot{q}^2 \\ f_i &= U - k_e\dot{\varphi} \end{aligned}$$

The gyrator, as in a DC motor, represents non-conservative, power-preserving coupling between generalized coordinates. So, it should be expressed in the generalized force term.

By substituting the coefficients into the equation, we obtain the following:

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}L\dot{q}^2\right)}{\dot{q}}\right] - \frac{\partial\left(\frac{1}{2}L\dot{q}^2\right)}{q} + \frac{\partial\,(0)}{q} + \frac{\partial\left(\frac{1}{2}R\dot{q}^2 + B_\mu\dot{q}\right)}{\dot{x}} = U - k_e\dot{\varphi}$$

$$\frac{d}{dt}\left[\frac{\partial\left(\frac{1}{2}J\dot{q}^2\right)}{\dot{q}}\right] - 0 + 0 + \frac{\partial\left(\frac{1}{2}R\dot{q}^2\right)}{\dot{q}} = U - k_e\dot{\varphi}$$

$$\frac{d}{dt}\,[L\dot{q}] + R\dot{q} = U - k_e\dot{\varphi}$$

$$L\ddot{q} + R\dot{q} + k_e\dot{\varphi} = U$$

$$L\ddot{q} = U - R\dot{q} - k_e\dot{\varphi}$$

7 Simulink Model for Lagrangian and Co-energy Expression

In order to model the equation properly, we must translate the dynamic equation into a form understandable for Simulink:

$$\begin{cases} J\ddot{\varphi} = -B\dot{\varphi} + k_m\dot{q} - M_t \\ L\ddot{q} = U - R\dot{q} - k_e\dot{\varphi} \end{cases}$$

$$\begin{cases} \ddot{\varphi} = \frac{1}{J}\left(-B\dot{\varphi} + k_m\dot{q} - M_t\right) \\ \ddot{q} = \frac{1}{L}\left(U - R\dot{q} - k_e\dot{\varphi}\right) \end{cases}$$

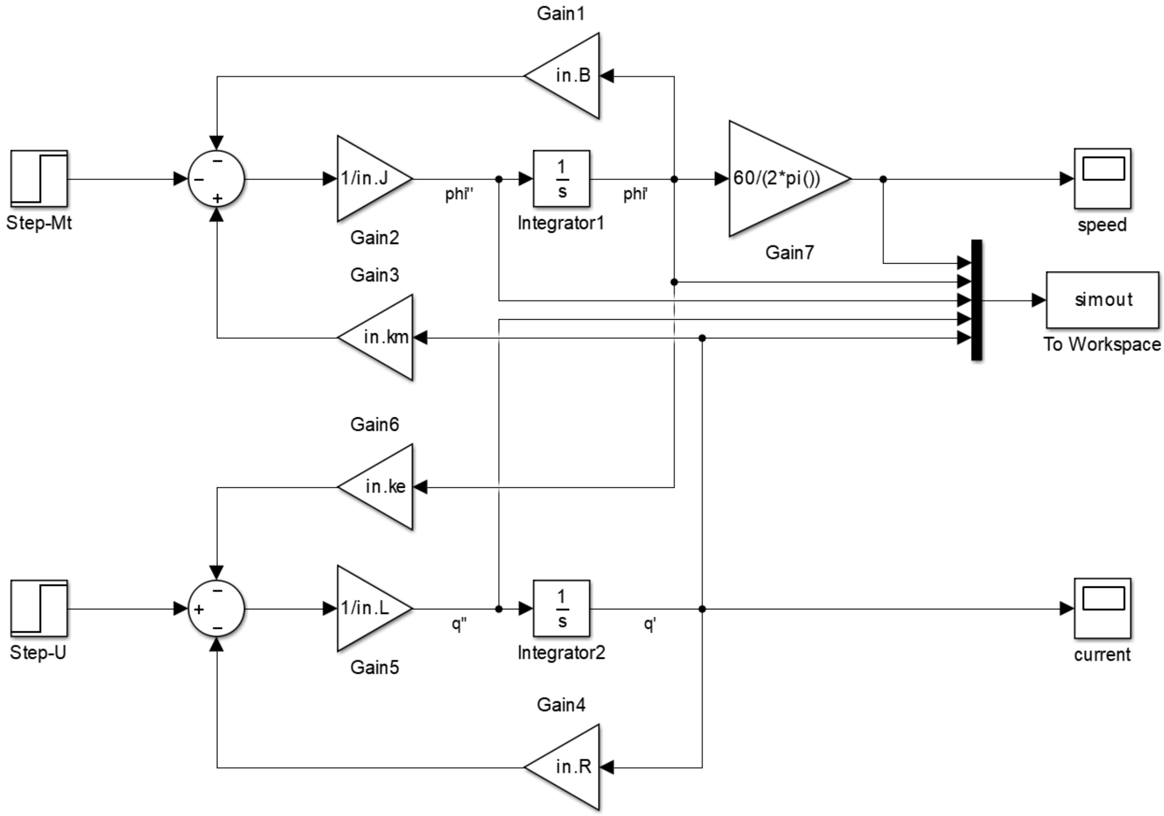

Figure 2: Simulink model of DC motor

8 Matlab and Python script

For script-based system modeling, we must take a slightly different approach to translating mathematical equations into code. Let $\varphi(t) = u(1)$, $\dot{\varphi}(t) = u(2)$, $q(t) = u(3)$, $\dot{q}(t) = u(4)$, in that case the matrix of the DC motor equation will be written as:

$$\frac{d}{dt}\begin{bmatrix} u(1) \\ u(2) \\ u(3) \\ u(4) \end{bmatrix} = \begin{bmatrix} u(2) \\ \left(-B \cdot u(2) + km \cdot u(4) - Mt\right)/J \\ u(4) \\ \left(U - R \cdot x(4) - ke \cdot x(2)\right)/L \end{bmatrix}$$

Now, the equation can be coded into Matlab:

```
    Du = [u(2); (-in.B * u(2) + in.km * u(4) - Mt) / in.J;
         u(4); (Uin - in.R * u(4) - in.ke * u(2)) / in.L];
```

Python counts matrix elements from 0, so the equivalent Python code can be written as:

```python
def motor_dc(u, t, mt_in):
    u_in = U if t > 1 else 0
    mt_val = mt_in if t > 2 else 0

    return u[1], (-B * u[1] + km * u[3] - mt_val) / J, \
           u[3], (u_in - R * u[3] - ke * u[1]) / L
```

9 Simulation Results

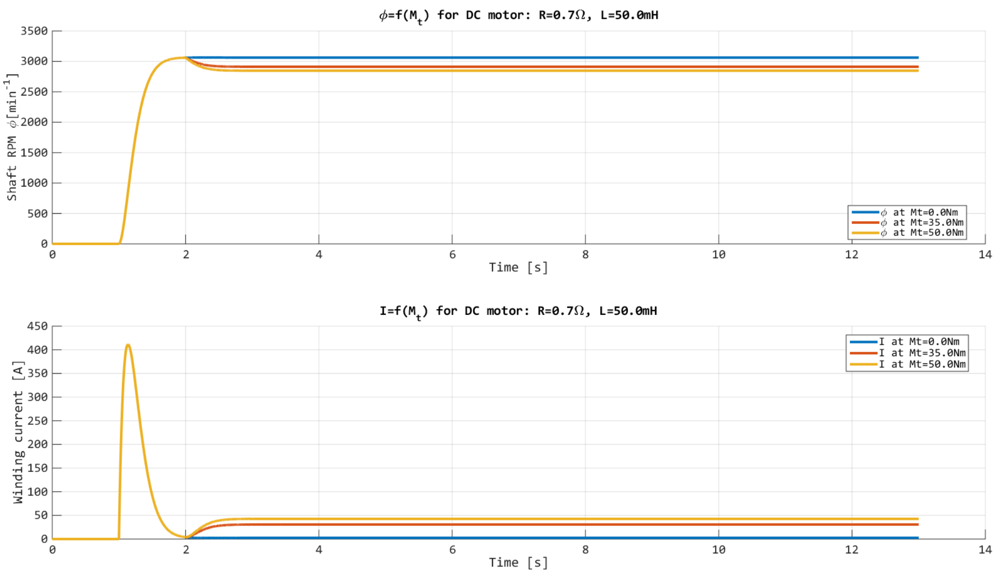

Figure 3: Result of simulation

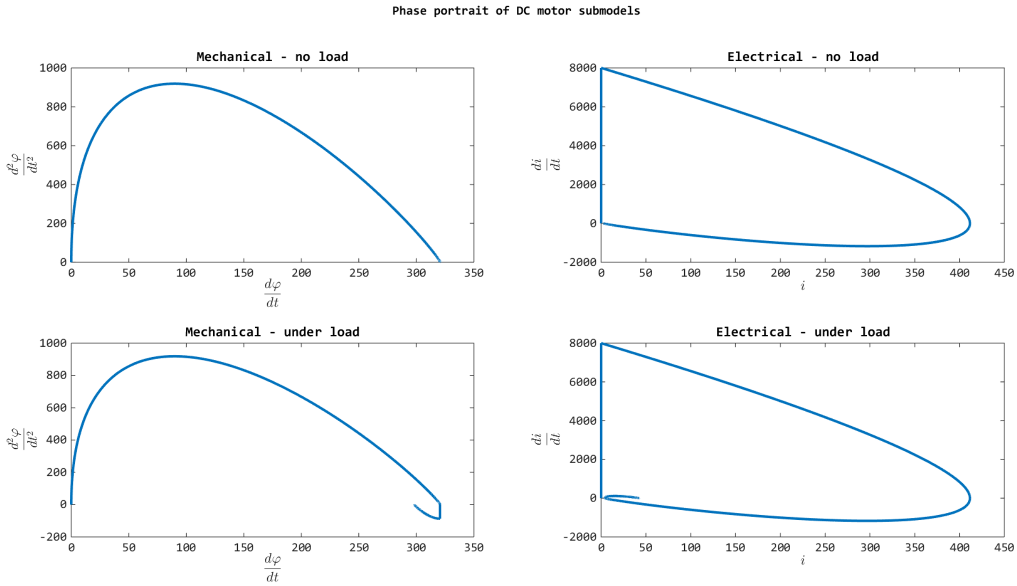

Figure 4: Result of simulation - phase portrait

10 Conclusion

This document demonstrates the Lagrangian and co-energy approach to modeling a DC motor provides a visual block diagram suitable for simulation in Simulink and basic code for scripting.