Partitioning the sphere with constant area quadrangles

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Abstract

In this paper, we address the problem of partitioning the sphere into $24n^2$ spherical quadrangles which all have exactly *the same area*. In addition, these quadrangles have other properties:

- the angular values are generally close to $\pi/2$.
- the edges have nearly the same length.
- the degree of all vertices is 4, except eight ones which have degree 3.

1 Introduction

The problem that we tackle in this paper is the partitioning of the sphere with spherical quadrangles which have exactly *the same area*. The practical purpose is to compute the mesh of a laser target.

We have found no references on this problem in the literature. However, the five regular polyedra — octahedron, cube, tetrahedron, dodecahedron, icosahedron —, which yield regular meshes by straightforward projection, have been known for a long time (see for example [2, 3]). In the same way, sphere triangulations by *congruent* triangles have long been studied [5, 6, 8]. Some of edge-to-edge triangulations involving isosceles triangles can be used to yield partitions by congruent (hence equal area) quadrilaterals. We cannot use these partitions because, either the number of cells is limited (less than a few hundreds) or some vertices have an enormous degree (proportional to the number of cells). Actually, we want a mesh that satisfies the usual criteria requested for finite elements methods, like conformity, refinability, local regularity (bounded degree of vertices, small variations of areas, and same type for the cells), angular quality, respect of symmetries, easiness of interpolation ... In [4], there are some triangulations satisfying these criteria, but the areas of the cells, although not very different, are never exactly the same.

To build the mesh (with constant area quadrangles), we begin by computing a simple partition (see Figure 1) which solves the problem, but with a small number (24) of faces. Next, we refine this simple mesh. We show through some examples why the refinement is difficult, then we give a refinement method which yields constant area spherical quadrangles (see Figure 5). Finally, some further developments are outlined.

2 Initial partitioning

It seems clever to initialize the partitioning with a regular polyhedron, whose spherical projection gives a straightforward regular mesh. Only the cube is a regular polyhedron with quadrangular faces. We consider a cube whose all vertices lie on the sphere and we project each edge on the sphere with respect to its centre O. This yields a first partitioning of the sphere with 6 spherical quadrangles of equal area.

A first refinement of this mesh is easy. Let $Oxyz$ be a coordinate system with axes orthogonal to the faces of the cube. The 3 planes Oxy, Oyz and Ozx cut each of the 6 previous spherical quadrangles in four cells. This yields the second partitioning of the sphere with 24 constant area spherical quadrangles for symmetry reasons. This is the basic mesh (see Figure 1) which we are going to refine into $24n^2$ cells.

Without loss of generality, we assume in this paper that the sphere has radius one. Note that the area of each of these 24 spherical quadrangles is $\frac{4\pi}{24} = \frac{\pi}{6}$.

3 Refinement of the basic mesh

3.1 Choice of a strategy

Suppose that we cut the spherical quadrangle $< A, B, C, D >$ (see Figure 2) by joining the midpoints of its opposite edges with great circular arcs[1]. This way, the four new quadrangles do not have equal areas.

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[1] A great circular arc is a portion of a great circle. A great circle is a circle on the sphere which has the same centre as the sphere. A vertical great circular arc is a great circular arc which lies on a vertical plane.

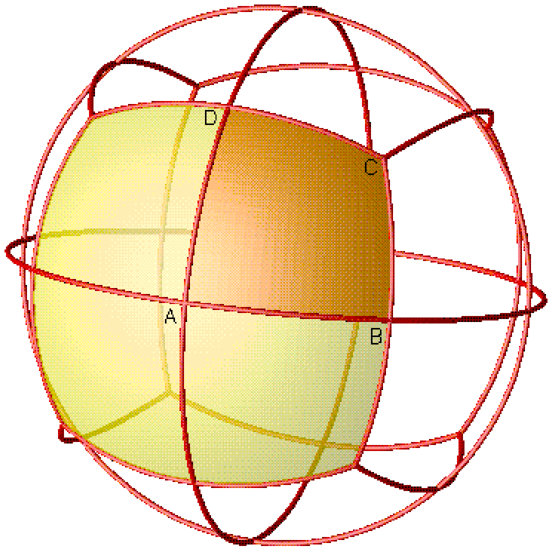

Figure 1: The first partitioning.

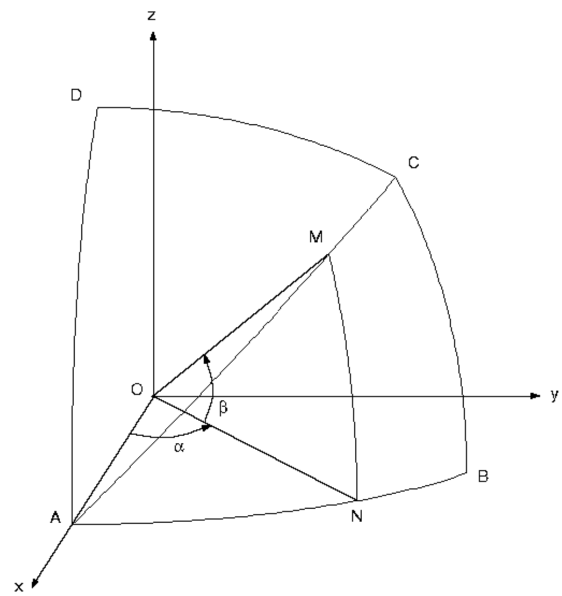

Figure 2: The basic quadrangular cell $< A, B, C, D >$.

Note that $AB = AD$ and $CD = CB$ but $AB \neq CD$. More generally, the spherical quadrangle $< A, B, C, D >$ is symmetrical with respect to (OAC) plane. Consequently, we only compute the partitioning of the spherical triangle $< A, B, C >$. The remainder on $< A, B, C, D >$ is obtained by the symmetry: $M(x, y, z) \longmapsto M'(x' = x, y' = z, z' = y)$.

Nevertheless, there exists an infinity of possible meshes which solve the problem (but not in an easy fashion). So, we need some additional contraints.

In order to simplify the computation, we decide to divide $< A, B, C >$ into n great circular vertical strips of suitable width. The position of the i^{th} vertical great circular arc[2] $\overline{NM}$ (see Figure 3) is such that the area of the spherical triangle $< A, N, M >$ equals $\frac{i^2}{2}$ cell-areas; a cell-area equals $\dfrac{4\pi}{24n^2}$.

Let α and β be the spherical coordinates of M such that $\alpha = \widehat{AON}$ and $\beta = \widehat{NOM}$. Within this system of coordinates, the position of the i^{th} vertical is entirely determined by the angle α.

The next difficulty will be to compute the position of the vertices on these vertical lines so that all the quadrangles have the same area.

3.2 Computation of the vertical arcs in $< A, B, C >$

Suppose we are computing the i^{th} vertical arc $\overline{NM}$.

[2]We note $\overline{NM}$ the (shortest) portion of the great circle between N and M.

The area σ of the spherical triangle $< A, N, M >$ is defined by: $\sigma = \dfrac{\pi}{12}\left(\dfrac{i}{n}\right)^2$.

Computing $\hat{M}$: Let $\hat{M}$, $\hat{N}$, $\hat{A}$ (resp. $\hat{B}$, $\hat{C}$) be the angles[3] in M, N, A (resp. B, C) inside $< M, N, A >$ (resp. $< A, B, C >$). The area of $< A, N, M >$ is equal to the spherical excess [1, 7] of $< A, N, M >$, so:

$$\sigma = \hat{A} + \hat{N} + \hat{M} - \pi = \frac{\pi}{4} + \frac{\pi}{2} + \hat{M} - \pi = \hat{M} - \frac{\pi}{4}$$

$$\Rightarrow \hat{M} = \sigma + \frac{\pi}{4}.$$

Spherical coordinates (α, β) of M: $M(\alpha, \beta)$ lies on the great circle determined by A and C, hence:

$$\det(\overrightarrow{OM}, \overrightarrow{OA}, \overrightarrow{OC}) = 0$$

$$\Rightarrow \begin{vmatrix} \cos\beta\cos\alpha & 1 & \frac{1}{\sqrt{3}} \\ \cos\beta\sin\alpha & 0 & \frac{1}{\sqrt{3}} \\ \sin\beta & 0 & \frac{1}{\sqrt{3}} \end{vmatrix} = 0$$

$$\Rightarrow \frac{\sin\alpha}{\sin\beta} = \frac{1}{\cos\beta}. \qquad (1)$$

The sine theorem ([1, 7]) in $< A, N, M >$ yields:

$$\frac{\sin\overline{NM}}{\sin\hat{A}} = \frac{\sin\overline{AN}}{\sin\hat{M}}$$

[3]The angle between two great circular arcs is measured as that between the tangents to the great circles at the point of intersection or as the angle of intersection of the planes of the great circles.

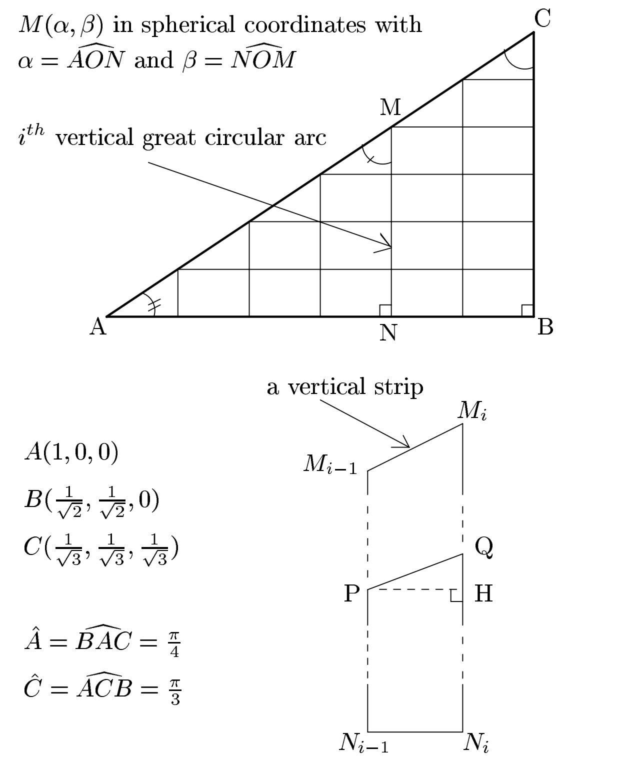

Figure 3: Partitioning spherical triangle $< A, B, C >$.

$$\Rightarrow \frac{\sin\beta}{1/\sqrt{2}} = \frac{\sin\alpha}{\sin\hat{M}}$$

$$\Rightarrow \frac{\sin\alpha}{\sin\beta} = \sqrt{2}\ \sin\hat{M}\ . \qquad (2)$$

$$(1) \quad \text{and} \quad (2) \quad \Rightarrow \cos\beta = \frac{1}{\sqrt{2}\sin\hat{M}}\ .$$

We obtain several expressions of $\sin\beta$:

$$\begin{cases} \sin\beta = \sqrt{1-\cos^2\beta}\ . \\ \sin\beta = \sqrt{1-\dfrac{1}{2\sin^2\hat{M}}} = \dfrac{\sqrt{-\cos(2\hat{M})}}{\sqrt{2}\sin\hat{M}}\ . \\ \sin\beta = \cos\beta\sqrt{-\cos(2\hat{M})}\ . \end{cases}$$

Let us compute α:

$$\begin{cases} (1)\ \Rightarrow \sin\alpha = \dfrac{\sin\beta}{\cos\beta} = \sqrt{-\cos(2\hat{M})}\ . \\ \cos\alpha = \sqrt{1-sin^2\alpha} = \sqrt{1+\cos(2\hat{M})} = \sqrt{2}\cos\hat{M}\ . \end{cases}$$

This yields the position of all the vertical great circular arcs of the triangle $< A, B, C >$, and in particular, the coordinates of the first and the last vertex lying on these arcs.

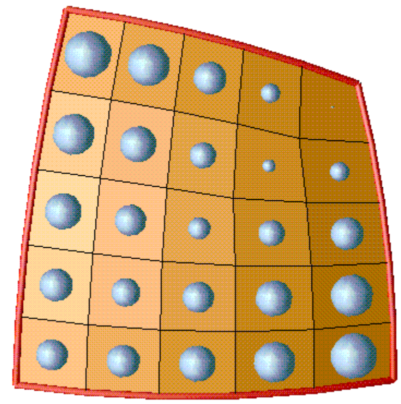

Figure 4: Visualization of the differences between the cell areas.

3.3 Coordinates of the other vertices

Computing these coordinates is not an easy task. Suppose we divided the vertical circular arcs of $< A, B, C >$ into equal parts (the i^{th} arc is divided into i equal parts), and that we linked the vertices of the $(i-1)^{th}$ arc to the corresponding vertices (except for the topmost) of the i^{th} arc. Then the cells in the resulting mesh would not all have the same area. For example, see Figure 4, where the differences between cell areas are represented by "glyphs" proportional to the error. Only the lower left cell has a correct area: the ones in the bottom right and left top corners are too large, and all the others are too small ...

Now, the exact computation. In order to perform the partition, we still have to compute all the vertices lying between N_i and M_i for all the successive arcs $\overline{N_iM_i}$, $i \in [2, n]$.

We suppose that we know all the vertices on the $(i-1)^{th}$ arc $\overline{N_{i-1}M_{i-1}}$ and we compute those lying on the i^{th} arc $\overline{N_iM_i}$. This argument holds because the first arc does not have vertices between N_1 and M_1. From the j^{th} ($j \in [1,i]$) vertex P of $\overline{N_{i-1}M_{i-1}}$, we compute the j^{th} vertex Q of $\overline{N_iM_i}$ such that the area of the spherical quadrangle[4] $< P, N_{i-1}, N_i, Q >$ equals j cell-areas. Let H be the great circular orthogonal projection[5] of P on the vertical great circular arc $\overline{N_iM_i}$. Let (α_0, β_0) (resp. (α, β_1), (α, β)) be the spherical coordinates of P (resp. H, Q). Note that α is known since the vertical arc $\overline{N_iM_i}$ is known whatever $i \in [1, n]$ (see 3.2).

[4]The area of a spherical quadrangle is equal to the sum of its internal spherical angles minus 2π.

[5]This means that the great circle determined by the points P and H is orthogonal to the great circle determined by the points N_i and M_i.

Computation of the coordinates of H: The equation of the great circle through the points P and H is:

$$\begin{vmatrix} \cos\beta_1\cos\alpha & \cos\beta_0\cos\alpha_0 & -\sin\alpha \\ \cos\beta_1\sin\alpha & \cos\beta_0\sin\alpha_0 & \cos\alpha \\ \sin\beta_1 & \sin\beta_0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \tan\beta_1 = \frac{\tan\beta_0}{\sin\alpha\sin\alpha_0 + \cos\alpha\cos\alpha_0}\,.$$

Computation of the area of $< P, H, Q >$: The coordinates of H are known, so we can compute the area of the spherical quadrangle $< P, N_{i-1}, N_i, H >$, hence:

$$\begin{aligned} &area(< P, H, Q >) \\ &\quad= area(< P, N_{i-1}, N_i, Q >) - \\ &\qquad\qquad area(< P, N_{i-1}, N_i, H >) \\ &\quad= \frac{\pi j}{6n^2} - area(< P, N_{i-1}, N_i, H >)\,. \end{aligned}$$

Computation of $\Gamma = \hat{P} + \hat{Q}$: Let $\hat{P}$, $\hat{H}$ and $\hat{Q}$ the spherical angles in P, Q and H inside $< P, H, Q >$.

$$|area(< P, H, Q >)| = \hat{P} + \frac{\pi}{2} + \hat{Q} - \pi$$

$$\Rightarrow \Gamma = |area(< P, H, Q >)| + \frac{\pi}{2}\,.$$

Computation of the angle $\hat{Q}$: Since the spherical triangle $< P, H, Q >$ is right-angled in H, it satisfies (see [7]) the relations:

$$\begin{cases} \cos\hat{Q} = \cos\overline{PH}\sin\hat{P} & \text{(a)} \\ \sin\overline{HQ} = \dfrac{\sin\hat{P}}{\sin\hat{Q}}\sin\overline{PH} & \text{(b)} \end{cases}$$

$$\begin{aligned} \text{(a)} \quad &\Rightarrow \quad \cos\hat{Q} = \cos\overline{PH}\sin(\Gamma - \hat{Q}) \\ &\Rightarrow \quad \cos\hat{Q} = \cos\overline{PH}(\sin\Gamma\cos\hat{Q} - \cos\Gamma\sin\hat{Q}) \\ &\Rightarrow \quad \tan\hat{Q} = \tan\Gamma - \frac{1}{\cos\overline{PH}\cos\Gamma}\,. \end{aligned}$$

Spherical coordinates (α, β) of the point Q: α is known. Let δ be the sign (± 1) of $area(< P, H, Q >)$. Using (b), we can compute β:

$$\begin{aligned} \beta &= \overline{N_i H} + \delta\,\overline{HQ} \\ &= \beta_1 + \delta\,\sin^{-1}\left(\frac{\sin(\Gamma - \hat{Q})}{\sin(\hat{Q})}\sin\overline{PH}\right)\,. \end{aligned}$$

This algorithm is completely outlined in *Appendix A*.

4 Conclusion, remarks and extensions

In this paper, we have presented a method to mesh the sphere with quadrangles which have exactly the same area. The algorithm is simple, fast (linear) and robust.

Nevertheless, we have not studied the problem of numerical precision. Experimentally, it seems that a precision of 10^{-14} is reached with double precision coordinates on the unit sphere.

It would also be interesting to build meshes on different types of conics like ellipsoids, hyperboloids, paraboloids ...

We can use another criterion to compute the quadrangles areas. Instead of the spherical area, we consider the minimum bounding surface (hyperbolic paraboloid) of these skew quadrilaterals (the vertices are linked with line segments instead of great circular arcs). Then, it is a lot more difficult to build a constant area quadrangular meshing because we don't have the additivity property on the surfaces. If we want plane quadrilaterals, the problem is even more difficult.

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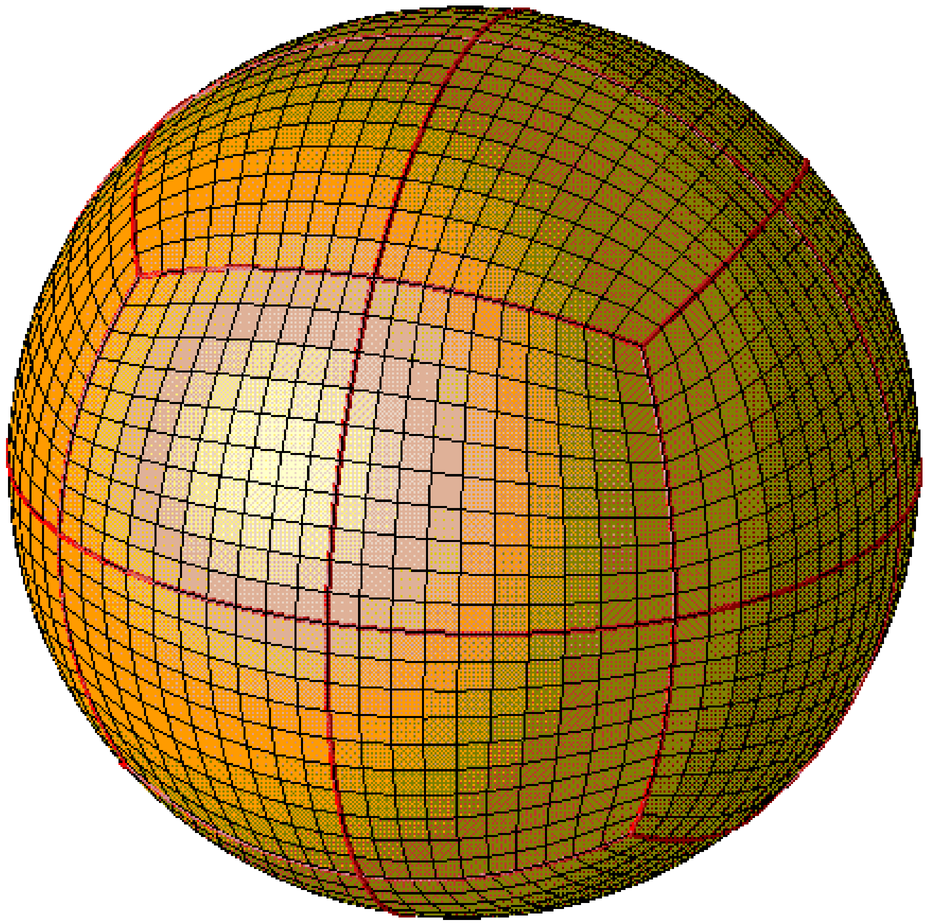

Figure 5: Constant area quadrangular meshing (2400 cells) of the sphere.

Remarks:

– Pictures 1, 4 and 5 were obtained with the visualization package Open Data Explorer: http://www.opendx.org and http://www.research.ibm.com/dx.

– The ratios between the longest and the shortest edge lengths (resp. angle measures) lie in $[1.27, 1.59]$ (resp. $[1.33, 1.74]$) if the number of cells lie in $[24, 1.014 \times 10^9]$. Note that these two ratios seem to converge towards the upper bounds of the previous intervalls when the number of cells tends towards $+\infty$.

Appendix A: ALGORITHM

Meshing the basic quadrangular cell $< A, B, C, D >$

$\alpha_0 = 0; \quad \beta_{0,0} = 0; \quad M_{0,0}(\alpha_0, \beta_{0,0});$ *Coordinates of vertex A*

FOR $i = 1$ to n DO *Filling the line $j = 0$ ($\overline{AB}$) and the diagonal ($\overline{AC}$)*

$$\Lambda_i = \frac{\pi}{12}\left(\frac{i}{n}\right)^2 + \frac{\pi}{4};$$
$$\cos\alpha_i = \sqrt{2}\cos\Lambda_i;$$
$$M_{i,0}(\alpha_i, 0);$$
$$\cos\beta_{i,i} = \frac{1}{\sqrt{2}\sin\Lambda_i};$$
$$M_{i,i}(\alpha_i, \beta_{i,i});$$

END FOR i

FOR $i = 2$ to n DO *Filling the first triangle $< A, B, C >$*

α_{i-1} = first spherical coordinate of $M_{i-1,0}$;
α_i = first spherical coordinate of $M_{i,0}$;
FOR $j = 1$ to $i - 1$ DO
$\beta_{i-1,j}$ = second spherical coordinate of $M_{i-1,j}$;

$$\tan\theta_{i,j} = \frac{\tan\beta_{i-1,j}}{\cos(\alpha_i - \alpha_{i-1})};$$
$$H_{i,j}(\alpha_i, \theta_{i,j});$$
$$\overrightarrow{U_{i,j}} = \overrightarrow{OH_{i,j}} - (\overrightarrow{OM_{i-1,j}} \cdot \overrightarrow{OH_{i,j}})\, \overrightarrow{OM_{i-1,j}};$$
$$\overrightarrow{V_{i,j}} = \overrightarrow{OM_{i-1,0}} - (\overrightarrow{OM_{i-1,j}} \cdot \overrightarrow{OM_{i-1,0}})\, \overrightarrow{OM_{i-1,j}};$$
$$\cos\gamma_{i,j} = \frac{\overrightarrow{U_{i,j}} \cdot \overrightarrow{V_{i,j}}}{||\overrightarrow{U_{i,j}}|| \quad ||\overrightarrow{V_{i,j}}||};$$
$$a_{i,j} = \frac{\pi j}{6n^2} - \gamma_{i,j} + \frac{\pi}{2};$$
$$\delta_{i,j} = sign(a_{i,j});$$
$$\Gamma_{i,j} = |a_{i,j}| + \frac{\pi}{2};$$
$$\tan\varphi_{i,j} = \tan\Gamma_{i,j} - \frac{1}{(\overrightarrow{OM_{i-1,j}} \cdot \overrightarrow{OH_{i,j}})\cos\Gamma_{i,j}};$$
$$\beta_{i,j} = \theta_{i,j} + \delta_{i,j}\arcsin\left[\left(\frac{\sin\Gamma_{i,j}}{\tan\varphi_{i,j}} - \cos\Gamma_{i,j}\right) \; || \overrightarrow{OM_{i-1,j}} \wedge \overrightarrow{OH_{i,j}} || \right];$$
$$M_{i,j}(\alpha_i, \beta_{i,j});$$

END FOR j
END FOR i

FOR $i = 1$ to n DO *Filling the symmetrical triangle $< A, D, C >$*
FOR $j = 0$ to $i - 1$ DO
$M_{j,i}(x) = M_{i,j}(x);$
$M_{j,i}(y) = M_{i,j}(z);$
$M_{j,i}(z) = M_{i,j}(y);$
END FOR j
END FOR i

Remarks:

– *The vertices $M_{i,j}$ ($i, j \in [0, n]$) have regular grid connections ($A = M_{0,0}$, $B = M_{n,0}$, $C = M_{n,n}$ and $D = M_{0,n}$).*
– *$M(\alpha, \beta)$ in spherical coord. is equivalent to $M(x = \cos\beta\cos\alpha,\ y = \cos\beta\sin\alpha,\ z = \sin\beta)$ in cartesian coord.*