AST1100 Lecture Notes

23-24: Cosmology: models of the universe

1 The FRW-metric

Cosmology is the study of the universe as a whole. In the lectures on cosmology we will look at current theories of how the universe looked in the beginning, how it looks today, how it evolved from the beginning until today and how it will continue in the future. We learned i the lectures on general relativity that how an object moves depends on the geometry of space and time. The question of how the universe evolves is mainly a question about what happens to its 'ingredients' as time goes by. How do the objects in the universe move with time? To answer this question, we thus need to find the geometry of the universe as a whole.

We know that in order to find the geometry of spacetime, we need to specify the content of spacetime in the stress-energy tensor $T_{\mu\nu}$ on the right hand side in the Einstein equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu}.$$

But how can we specify the mass and energy content at every point in spacetime of the whole universe? As always, we need to make some approximations and simplifying assumptions. One main assumption is the *cosmological principle*. The cosmological principle states that the universe is homogeneous and isotropic. The universe is assumed to, on average, have a similar composition and density in all positions and to look the same in all directions. The two conditions, homogeneity and isotropy may, at first look, seem identical. It is easy to convince yourself that they are not: Imagine a universe filled with trees which are all aligned with each other, i.e. the top of the trees all point in the same direction and the roots all point in the same direction. This universe is homogeneous: at every single point in the universe there is a tree, and the density of trees is the same everywhere. But this universe is not isotropic: When you look in one direction you will see a lot of tree tops, when you look in the other direction you will see a lot of roots. This universe does not look the same in all directions and is therefore not isotropic. Now, rotate all the trees in random directions. Then you will see the same number of tree tops and roots no matter which direction you look. This universe filled with randomly oriented trees would be homogeneous and isotropic.

The cosmological principle is an assumption with a philosophical basis: Why should the universe have a very different composition in different places, or why should there be a preferred direction? Data of the universe taken at many different wavelengths seem to support the cosmological principle. This simplifies our task: if we model the universe to have a density ρ which on average is the same in all positions we can find a solution to the Einstein equation. Of course, the density of the universe is not the same in every position when you look at small parts of the universe. Clearly the density in the Earth is not the same as in the empty space around it. But when taking the average density over a large volume of the universe, the density turns out to be the same no matter where you take this volume to be. Thus, our solution to the Einstein equation will be valid on the largest scales in the universe. It will not describe correctly what happens in a particular star or in a particular galaxy, but it will describe the physics of larger scales. The general form of the solution to the Einstein equation under the assumption of homogeneity and isotropy is given by the *Friedmann-Robertson-Walker (FRW) metric* (a 'metric' is just a different word for the line element)

$$\Delta s^2 = \Delta t^2 - R^2(t)\left[\frac{\Delta r^2}{1-kr^2} + r^2\Delta\theta^2 + r^2\sin^2\theta\Delta\phi^2\right]. \tag{1}$$

Here (r, θ, ϕ) are spherical coordinates for any position in the universe. The center $r = 0$ of the spherical coordinate system can be chosen anywhere in the universe. We normally choose it to be here, the position of the observer. The form of the time dependent function $R(t)$ depends on the properties of the matter than fills the universe. We will later discuss the possible functional forms of $R(t)$. The parameter k can have the values $+1$, 0 or -1 and decides the curvature of the spatial geometry of the universe. We will later relate this to the total density of the universe.

Looking at this metric, there is one thing which might be surprising. We will now consider the proper distance between two objects in the universe. One object which is located at $r = 0$ and one at $r = \Delta r$. We have learned that in order to measure proper distances we need to make a measurement such that $\Delta t = 0$. This corresponds to measuring the distance with a meter stick in the same way as we have seen before: take two events on each end of the stick to happen at the same time $\Delta t = 0$. The spacetime distance between these two events is then by definition the proper distance between the events (which corresponds to the physical distance we would measure with a meter stick). We thus have for the proper distance

$$\Delta L = R(t)\frac{\Delta r}{\sqrt{1 - kr^2}},$$

where $\Delta\theta = \Delta\phi = 0$. The most surprising observation here is that the proper distance between two objects changes with time. The function $R(t)$ is called the *scale factor* and decides how the proper distance between objects changes with time. Thus, if we measure the distance between two objects at two different moments, we will measure two different distances and the difference will be given by $R(t)$. If $R(t)$ increases with time, then the proper distance between all objects in the universe also increases with time and we say that the universe is expanding. On the contrary, a universe in which $R(t)$ is decreasing with time is contracting. We will later see that our universe seems to be expanding. But as always in the theory of relativity, reality is different for different observers/coordinates. If we use the coordinate r to measure distances between objects , then the distances between objects do not change. We call r the comoving coordinate, it is the distance measured by the *comoving observer*, an observer who is expanding with the universe. The comoving observer is an observer with meter sticks which are expanding with the expansion of the universe. This observer will always measure the same distance between objects.

We will now look at an **analogy** in order to get a deeper understanding for the meaning of an expanding universe. It is however very important to note that this is an analogy, it is a model which is constructed to make the expansion of the universe easier to understand. But it is only that: an analogy with limitations.

We will for a moment imagine the universe to be one dimensional. We will consider the universe to be the rim of a circle. Since the universe is one dimensional, everything is confined to the rim of the circle. All creatures in this universe are one dimensional, living on the rim of the circle and having an extension only in the direction along the rim of the circle (see figure 1). Everything can be described by a coordinate ϕ, a radial coordinate $\vec{R}$ does not exist for this universe, it is only a parameter describing the size $2\pi R$ of

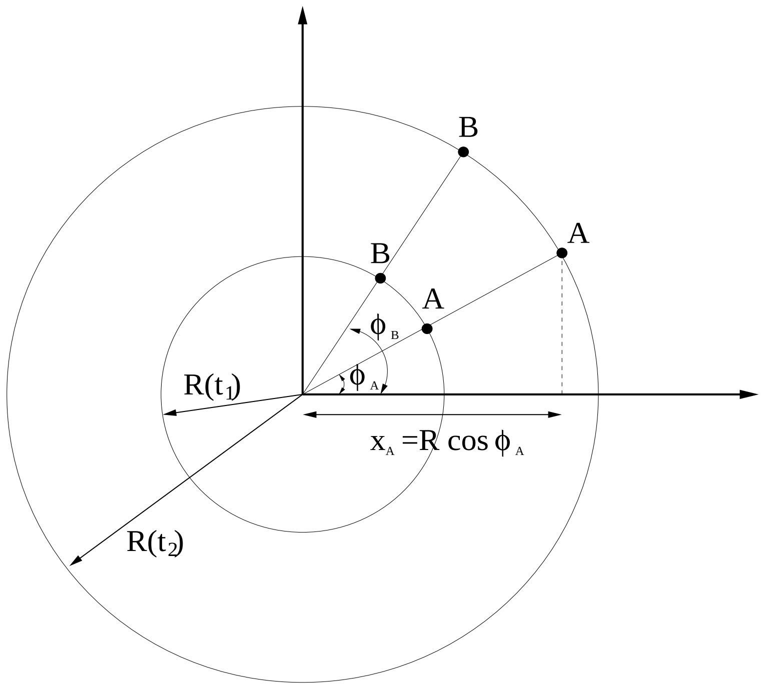

Figure 1: Two objects A and B at rest in the one-dimensional universe. We show the universe at two moments t_1 and t_2. The universe has expanded from t_1 to t_2 and the distance between the two objects has increased even if the objects have not moved. The coordinate ϕ is the only coordinate necessary in order to give the position of an object in the one-dimensional universe.

the universe. There is no radial dimension it is only helping us to understand events on the one dimensional rim-of-the-circle universe. The radius of the circle is given by R and might be a function of time $R(t)$. In the figure we see the universe at to different times t_1 and t_2.

Assuming Euclidean geometry, the spatial part of the line element for this universe is given by

$$\Delta s^2 = \Delta x^2 + \Delta y^2. \tag{2}$$

But this is a two dimensional metric, the coordinates x and y do not exist in the one-dimensional universe. We need to transform this to an expression in terms of one single coordinate which specifies the position in this one-dimensional universe. As a first step, we can use x as this one dimensional coordinate. If we consider only $\phi = [0, \pi]$, then the x coordinate uniquely specifies a position on he rim of the circle. We can get rid of the y coordinate, using that

$$x^2 + y^2 = R^2.$$

Thus, $y = \sqrt{R^2 - x^2}$ and taking the derivative of y we find

$$dy = \frac{-x dx}{\sqrt{R^2 - x^2}}$$

or in terms of intervals $\Delta y = -x\Delta x/\sqrt{R^2 - x^2}$ which we insert in the line element (equation 2) and obtain (after some reorganizing, check that you get the same result!)

$$\Delta s^2 = \frac{R^2 \Delta x^2}{R^2 - x^2}. \tag{3}$$

Now we have expressed the one-dimensional proper distance Δs along the rim of the circle in terms of intervals in the coordinate Δx. As mention above, a more logical way to specify a position on the rim of the circle is simply in terms of the angle ϕ. Here we will use $\cos\phi$ as the position coordinate on the circle and we will call it r. Note that r is the coordinate increasing along the rim of the circle from a point defined to be $r = 0$ on the rim of the circle, it does **not** start from the centre of the circle as the coordinate denoted by symbol r often does. However for a one-dimensional creature living on the rim of the circle this is indeed a measure of distance from the point of observation along the rim. We define

$$r = \cos\phi = \frac{x}{R},$$

the last transition comes from the fact that $x = R\cos\phi$. Substituting x with r in equation 3 we obtain

$$\Delta s^2 = \frac{R^2 dr^2}{1 - r^2}.$$

We see that the spatial part of the line element for this one-dimensional universe is similar to the spatial part of the FRW line element for $k = +1$ (equation 1). If the radius R of this circle is increasing such that $R = R(t)$ the one-dimensional creatures on the circle will experience that proper distances increase with the scale factor $R(t)$. The one-dimensional creatures will not use the term 'radius' about $R(t)$, but they will call it a scale factor as the effect they see of this increasing radius is the increasing distance between points on the circle. An observer will always define himself as being at $r = 0$ corresponding to $\phi = \pi/2$ (see equation 2). Measuring the distance to another object (for instance a galaxy) at $\phi = \Delta\phi$ corresponding to $r = \Delta r$, he finds that the proper distance is just

$$\Delta L = \delta s = R(t)\frac{\Delta r}{\sqrt{1 - r^2}},$$

just as for the FRW metric. Looking at figure 1 we see that the coordinate interval Δr (shown in the figure in terms of $\phi_B - \phi_A$) between the two points A and B on the circle is unchanged when the radius of the circle increases:

$$\Delta r = r_1 - r_0 = \cos\phi_1 - \cos\phi_0.$$

The position r (remember that $r = \cos\phi$) is always the same for an object at rest. The r coordinate, or comoving coordinate, does not change with time even when the radius of the circle increases. Therefore the coordinate distance Δr is also unchanged when the radius increases. What does change is the distance ΔL between points measured on local meter sticks. Thus, the one dimensional universe is expanding when the radius of the circle increases. Or stated in a better way: The expansion of the universe can be modeled as an increasing radius $R(t)$. That the universe expands and the distances between all objects increases with time, does not mean that the objects are really moving away from each other. Looking at the circle with increasing radius we see that the objects on the circle are at rest while the space between them expands. This is what we see in our universe: the distance to all distant objects is increasing as a consequence of the expansion of the universe. We measure that distant objects are moving away from us, but we see from the

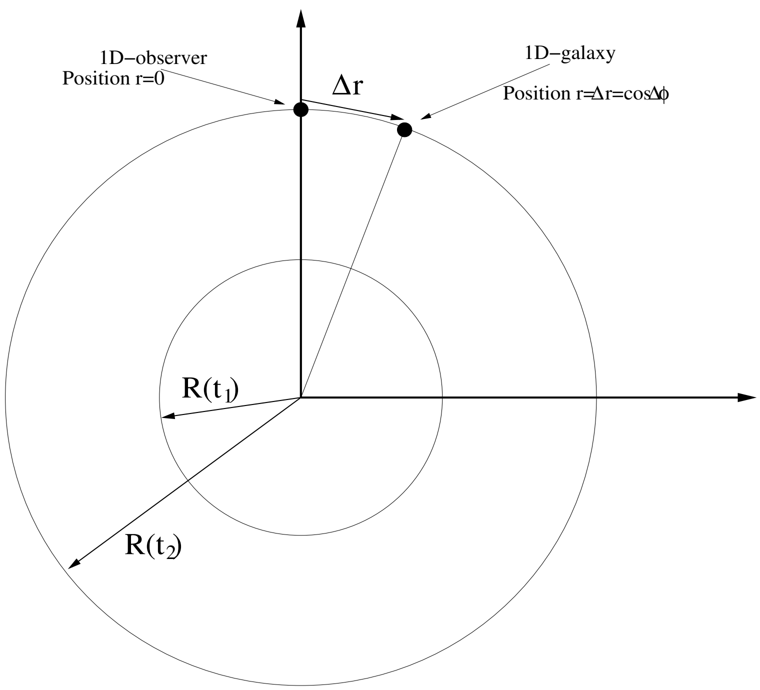
1D−observer
Position r=0
Δr
1D−galaxy
Position r=Δr=cosΔφ
R(t1)
R(t2)

Figure 2:

previous example that the objects are really at rest, it is the space between the objects which expands. It is the line element, or the way of measuring distances, which changes.

If the universe expands today, we would expect the distances between objects in the universe to have been much smaller in the past. As the radius of the circle gets smaller and smaller, the proper distances between objects is getting smaller and smaller. Going far back in time, the radius $R(t)$ was very small and therefore the distances between objects in the universe were very small. This is what we call the Big Bang. The distances between objects in the universe were very small, meaning that the density and therefore the temperature in the universe was extremely high. But note that we are NOT talking about the Big Bang as an explosion. Looking at the circle you see that all objects at all positions in the universe were close to each other and therefore extremely dense and extremely hot at very early times. One of the most common misconceptions about the Big Bang is that it was an explosion in one point and that the universe started expanding out from this point. We see from the example with the circle that this is absolutely not true: The Big Bang happened at all points on the circle at the same time. All positions r (or ϕ) on the circle had a very high density when the radius of the circle R was very small. But all these points never collapse to a single point: No matter how small you make the radius R, the position r of the objects and the comoving distance Δr between objects on the circle does not change. All the points in the universe never become one single point. They can always be characterized with a position r. It is only the measure of the proper distance between objects which is changing and which is very small at early times. So the Big Bang is nothing else than a very dense and hot universe which is expanding. The Big Bang happened at every single point in the universe and is not an explosion from one single point.

We can carry this analogy over to our three dimensional space. In the exercises you will study the case of a two-dimensional universe: in that case, the universe is the surface of a sphere which is expanding out in a third dimension (and the universe is only the surface, the third dimension does not exist it only serves as a model to help the understanding). You will find in the same manner as for the circle that the two-dimensional creatures living on the surface of a sphere will note the increasing radius of the sphere as an expanding two-dimensional universe. We will now take this even further: We will look at a four-dimensional sphere expanding in a four-dimensional space. Your brain is not capable of imagining a four-dimensional sphere in a four-dimensional space, but we can use the example with the circle to help us. This fourth dimension is not time but a fourth spatial dimension. This fourth dimension does not exist: as in the above examples this is an analogy to help us understand the FRW metric and the expanding space. We will look at the three spatial dimensions of our universe as the three-dimensional surface of a four-dimensional sphere expanding in four dimensions, but the whole (spatial part of the) universe is within the three dimensions of the surface and it is meaningless to make references to points outside the three dimensional surface (exactly as the whole one-dimensional universe was within the rim of the circle, the radial distance $\vec{R}$ had no meaning since the radial dimension did not exist).

Assuming Euclidean geometry in the four-dimensional space we can write the spatial line element in this four-dimensional space as

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 + \Delta w^2, \tag{4}$$

where w is the coordinate in the imaginary fourth dimension (again, we are here only looking at spatial dimensions, forgetting about the time dimension for the moment). Thus for a given coordinate difference Δx, Δy, Δz and Δw between two points in this four dimensional space, the distance between these two points is given by Δs. The radius of a four-dimensional Euclidean sphere is given by

$$x^2 + y^2 + z^2 + w^2 = R^2.$$

We will now measure three-dimensional distance along the three-dimensional surface using coordinates r, θ, ϕ instead of x,y,z and w. It can be shown (and you will show this for a three-dimensional sphere in the exercises) that we can define the coordinate r such that $\Delta x = R\Delta r$ (exactly as in the one-dimensional case above, check!), $\Delta y = Rr\Delta\theta$ and $\Delta z = Rr\sin\theta\Delta\phi$ where (r, θ, ϕ) are coordinates along the three-dimensional surface. Our aim is now to rewrite the expression for Δs^2 in terms of r, θ, ϕ and we are now ready to do this for the first three terms:

$$\Delta x^2 + \Delta y^2 + \Delta z^2 = R^2\Delta r^2 + R^2r^2\Delta\theta^2 + R^2r^2\sin^2\theta\Delta\phi^2. \tag{5}$$

We now use the expression for the radius of the three-dimensional sphere to eliminate w in the spatial line elements Δs^2 (exactly as we did to eliminate y in the one-dimensional case):

$$w = \sqrt{R^2 - x^2 - y^2 - z^2} = \sqrt{R^2 - R^2r^2},$$

where the last transition comes from equation 5 by taking Δx, Δy, Δz and Δr from the origin where $x = y = z = r = 0$ such that $\Delta\theta = \Delta\phi = 0$. Taking the derivative of w with respect to r we find

$$dw^2 = \frac{r^2 R^2 dr^2}{1 - r^2}.$$

Writing this in terms of intervals Δw and Δr we can insert this in equation 4 to obtain

$$\Delta s^2 = R^2 \left[\frac{\Delta r^2}{1 - r^2} + r^2 \Delta\theta^2 + r^2 \sin^2\theta \Delta\phi^2 \right]$$

which is exactly the spatial part of the FRW metric using $k = 1$ and $R = R(t)$ for an expanding four-dimensional sphere. Again, the model of our three-dimensional universe as an expanding three-dimensional surface embedded in four-dimensional space is an analogy which gives the correct spatial metric and thereby a model which can be used to understand several aspects of the geometry of the universe. But there are no observations which indicate that there is a large fourth dimension in which our four-dimensional sphere is expanding. In modern string-theory, the universe is viewed as 11-dimensional, but in most theories these extra dimensions are much smaller than the normal three spatial dimensions.

The lesson to be learned here is again: If we shrink this three-dimensional sphere to a very small radius $R(t)$, the three-dimensional coordinates (r, θ, ϕ) of objects do not change, and the three-dimensional comoving distance Δr is always unchanged no matter how small you make $R(t)$. The universe did not start from one single point. When the radius $R(t)$ was very small, all proper distances between objects in the three-dimensional universe were very small and the density was very high. The universe was **everywhere** very hot and very dense. This expanding universe with a hot and dense gas is called the Big Bang. But it is no explosion, and the universe is not expanding into anything. Expansion, as we have seen, just means that one particular measure of distances between objects, as seen from one particular group of observers, is increasing. The comoving observer does not see any expansion and measures all distances to be constant.

Finally some words about the parameter k. In our examples with one, two and three-dimensional spheres the parameter k was always $k = +1$. This is a signature of positive curvature: a sphere has positive curvature. On the three-dimensional surface of a four-dimensional sphere, space has positive

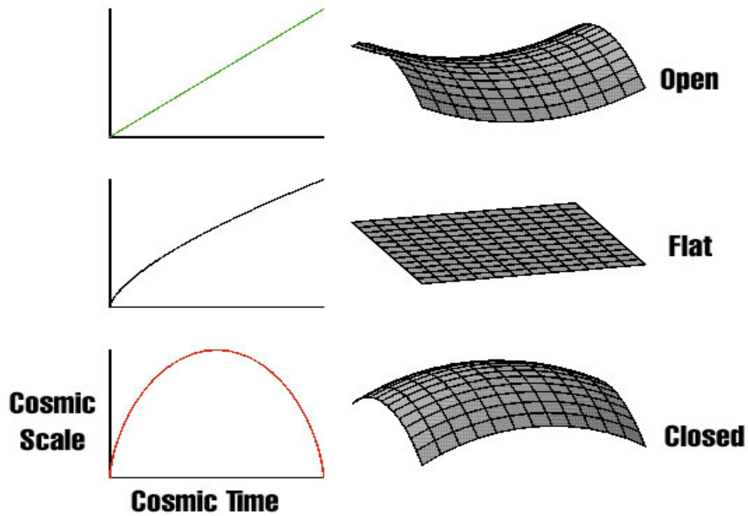

Figure 3: Three possible geometries of a surface (or of the universe), hyperbolic, flat or spherical. In each case the matrix Δs^2, the way we measure distances along the surface, is different. We have studied the lower case in detail. Figure is borrowed from the NASA homepage.

curvature. If $k = 0$ we see that the spatial part of the metric is, apart from the factor $R(t)$, just the metric for Euclidean geometry. Thus $k = 0$ for flat space, space with zero curvature. Finally, if $k = -1$ we would need to use a hyperbola in our one-dimensional universe instead of a circle. The three-dimensional universe needs in this case to be viewed as a three-dimensional hyperbolic surface embedded in a four-dimensional space. In a hyperbolic space, $k = -1$ and space has negative curvature. In order to understand the meaning of space with positive, negative or zero curvature, it is easier to use the analogy with a sphere, plane or hyperbolic surface expanding in a three-dimensional space (see figure 3).

2 Hubble's law and cosmological redshift

Einstein realized that the general theory of relativity predicted an expanding or contracting universe. At the time it was not known that the universe was expanding and Einstein introduced a constant, called the cosmological constant Λ, in his equations in order to make a static universe. In 1929 Edward Hubble found that distant galaxies always moved away from us and that the further away they were, the faster away from us they moved. He formulated Hubble's law

$$v = H_0 d,$$

where v is the velocity of a distant galaxy, d is the distance and H_0 is a constant known as Hubble's constant. After Hubble's discovery, Einstein realized that it was a mistake to introduce the cosmological constant Λ in order to make the universe static. His theory was correctly predicting that the universe was expanding. The Hubble constant is thus simply the rate of expansion of the universe. Hubble's law can be used to make a very crude estimate of the age of the universe. If we write the Hubble law as

$$d = v\frac{1}{H_0},$$

which we recognize as distance equals velocity times time, $s = vt$. If we assume that the expansion velocity v has been the same throughout the whole lifetime of the universe then $1/H_0$ represents the time when the distant galaxy with velocity v was at the same position as we are $d = 0$ (remember that they were never really at the same position, but very close seen from one special coordinate system). During the time $1/H_0$ it has moved with a constant velocity v and is therefore at the moment at a distance $d = v/H_0$. The crude age estimate $1/H_0$ is called the *Hubble time* $t_H = 1/H_0$.

We can easily derive Hubble's law by looking at the FRW-metric. The proper distance d to a galaxy, i.e. the actual measured distance, is found by taking $\Delta t = 0$ in the metric. We found above that this is just

$$d(t) = \frac{R(t)r}{\sqrt{1-kr^2}},$$

where we have set $r = 0$ at the observer's position so that $\Delta r = r - 0 = r$ is the comoving distance to the galaxy. If we make a measurement of the distance at time t we find

$$d(t) = \frac{R(t)r}{\sqrt{1-kr^2}}.$$

Then at a later time Δt we make a new measurement of the distance and find

$$d(t+\Delta t) = \frac{R(t+\Delta t)r}{\sqrt{1-kr^2}} = \frac{R(t+\Delta t)}{R(t)}d(t).$$

The velocity with which we measure the distant galaxy to move away is simply $(d(t+\Delta t) - d(t))/\Delta t$. We thus have

$$v = d(t)\frac{\frac{R(t+\Delta t)}{R(t)} - 1}{\Delta t} = d(t)\frac{1}{R(t)}\frac{\Delta R}{\Delta t} = d(t)\frac{\dot{R}}{R(t)} \equiv H(t)d(t),$$

where the dot denotes time derivative. We have derived Hubble's law and found that the Hubble's constant is just the rate of change of the scale factor

$$H(t) = \frac{1}{R(t)}\frac{dR(t)}{dt}.$$

This is what we expected: Hubble's constant is just the expansion rate of the universe which is the rate with which the scale factor $R(t)$ changes with time. We see that $H(t)$ is time dependent and we therefore call it the *Hubble parameter* whereas the Hubble constant is the value of the Hubble parameter today. It is common to use the index 0 to denote functions of time taken at the current time t_0. In this way $H_0 = H(t_0)$.

If proper distances between objects increases with time, one can also conclude that the proper distance between two peaks in a wave will increase with time. Thus, the wavelength of electromagnetic radiation will change with time. Consider an electromagnetic wave where the proper distance $\lambda(t_1)$ between two peaks in the wave is measured at time t_1 such that

$$\lambda(t_1) = \frac{R(t_1)\Delta r}{\sqrt{1-kr^2}},$$

where Δr is the coordinate distance (comoving distance) between the two peaks in the wave. When the same wave is measured at a later time t_2, the distance between the same two peaks is found to be

$$\lambda(t_2) = \frac{R(t_2)\Delta r}{\sqrt{1-kr^2}} = \frac{R(t_2)}{R(t_1)}\lambda(t_1).$$

The comoving distance Δr does not change and is therefore the same at the two different times. Thus, we obtain a change in wavelength z (the redshift is usually denoted $z = \Delta\lambda/\lambda$)

$$z \equiv \frac{\Delta\lambda}{\lambda} = \frac{\lambda(t_2) - \lambda(t_1)}{\lambda(t_1)} = \frac{R(t_2)}{R(t_1)} - 1. \tag{6}$$

When the universe is expanding $R(t_2) > R(t_1)$ and the wavelength of electromagnetic waves are getting larger, the radiation is redshifted. Thus, we observe light from distant galaxies to be redshifted since the light from these galaxies was emitted a long time ago at a time t_1 different from the time of observation t_2. When Hubble observed distant galaxies to move away from us he made this conclusion on the basis of the redshift of the galaxies. He used the normal Doppler formula for redshifts. Now we know that we are not looking at normal velocities: these galaxies are just at rest (apart from smaller peculiar velocities) at a fixed coordinate r. It is the space between us and the galaxies which is expanding, stretching the electromagnetic waves and thus causing a redshift. But since we observe that the proper distance to the galaxies is changing we can also interpret this as a Doppler effect and to first order we therefore have

$$z = \frac{\Delta\lambda}{\lambda} = v,$$

and we can write Hubble's law as

$$z = H_0 d.$$

In the lectures on distance measurements we used this to measure distances to distant galaxies: Measure the redshift z, use the Hubble constant H_0 obtained from other observations and get the distance d to the galaxy.

The redshift z is often used as a time parameter. From Hubble's law we see that the further away a galaxy is, the higher is the redshift. But the further away the galaxy is, the longer is the distance that light has traveled to reach us. Thus the further away the galaxy is, the further back in time we are looking. The redshift z tells us the distance. But since we know the speed of light we can also find how long time ago the light was emitted and thus at which epoch in the history of the universe we are looking when we look at a specific galaxy. Therefore, the redshift z of an observed object also tells us the time epoch at which we are observing this object. We usually normalize the scale factor such that $R(t_0) = 1$ today. We can therefore rewrite equation 6 for the redshift as

$$z = \frac{1}{R(t)} - 1, \tag{7}$$

where t is the age of the universe at the time when light was emitted from the observed object.

3 The Friedmann equations

In order to understand how the universe evolves, we need to obtain an expression for $R(t)$. We also need to find k to know the curvature of the universe. Inserting the FRW-metric on the left side in the Einstein equation and the homogeneous and isotropic matter content on the right side, the Einstein equation reduces to two equations called the Friedmann equations,

$$
\begin{aligned}
(\left(\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho(t)R^2(t)) &= -k & (8)\\
\frac{d^2R(t)}{dt^2} &= -\frac{4}{3}\pi G(\rho(t) + 3P(t))R(t). & (9)
\end{aligned}
$$

Remember that $H(t) = 1/R(t)dR(t)/dt$. Note that the density is only a function of time $\rho = \rho(t)$ and not of space: we have assumed a homogeneous universe and the density is therefore the same in all positions. Note further that ρ is the **total energy density**, not just mass density. We usually divide the total energy density into two parts, the energy density from mass (mass density) ρ_m and the energy density from radiation (photons) ρ_r. In the lectures on cosmology we will measure mass in kilograms and we therefore need to specify the gravitational constant G. We do however, measure distance and time in the same units so $c = 1$. Finally note that the pressure P enters in this equation. This is the pressure of the material of which the universe consists. At early times, the universe was filled with a dense gas and the pressure of this gas needs to be specified in the Friedman equation. At later times, the universe consists mainly of what we call 'dust', pressureless matter. Thus today the overall pressure in the universe is $P = 0$.

We will start by looking at the first Friedmann equation (equation 8). If the universe has a flat geometry, then $k = 0$ (we found this by looking at the metric, remember ?) and we have

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G},$$

where the subscript c refers to 'critical'. We will soon see why. Note that the density of the universe is a function of time. This should not be surprising: due to the expansion of the universe, the volume increases but the matter inside this volume remains the same. Thus, the density needs to decrease with time. We will now use this definition of ρ_c and insert it in the Friedman equation. Inserting the defined quantity ρ_c in equation 8 (now NOT assuming $k = 0$) we can write the first Friedmann equation as

$$k = \frac{8}{3}\pi G R(t)^2(\rho(t) - \rho_c(t)).$$

We see from this expression that if the density $\rho(t)$ of the universe is larger than the critical density, then $k > 0$ (actually $k = 1$ since k can only have the values +1, -1 and 0) and the universe has positive curvature. In this case we can use the model of the four-dimensional sphere expanding in a four-dimensional space to describe the universe. On the other hand if the density is lower than the critical density then $k < 0$ (again, $k = -1$) and the curvature of the universe is negative meaning that the universe can be modeled as a three-dimensional hyperbolic hypersurface expanding in a four-dimensional Euclidean space. If the density is exactly equal to the critical density then the spatial geometry of the universe is flat (and $k = 0$). We will later study which consequences the curvature of the universe has for the fate of the universe.

We now define the *density parameter*

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)},$$

which is simply the ratio of the density to the critical density. If $\Omega = 1$ then the universe is flat, if $\Omega < 1$ the curvature is negative (hyperbolic three-dimensional geometry) and if $\Omega > 1$ the curvature is positive (spherical three-dimensional geometry).

Now we know how the density of the universe decides the curvature k of the spatial geometry. The next step is to find out how the universe evolves in these three different geometries. The evolution of the universe is governed by the function $R(t)$. We will start by a simplified case, the pressureless universe.

4 The evolution of the pressureless universe

We will start by studying a universe which contains pressureless dust. This applies to the present universe, but does not apply to the earliest phases of the universe when the universe contained a dense gas with a non-zero pressure. Nevertheless this example will tell us the main properties of the three different geometries, spherical, hyperbolic and flat.

In the dust dominated universe, most of the energy density is made up of matter and not radiation. We therefore have $\rho \approx \rho_m$. Before solving for $R(t)$ we will need to find a general property of a pressureless universe. To find this property we start be taking the time derivative of the first Friedmann equation 8:

$$2\dot{R}\ddot{R} = \frac{8\pi G}{3}\frac{d}{dt}(\rho R^2).$$

We eliminate $\ddot{R}$ by using the second Friedmann equation (equation 9), remembering that $P = 0$

$$-\dot{R}R\frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\frac{d}{dt}(\rho R^2).$$

Now we note that

$$\frac{d}{dt}(\rho R^3) = R\frac{d}{dt}(\rho R^2) + \dot{R}(\rho R^2).$$

Inserting this in the previous equation we find

$$\frac{d}{dt}(\rho R^3) = 0,$$

or ρR^3 is a constant. This is not very surprising: R^3 is proportional to an expanding volume element so ρR^3 is simply the total mass within a volume element. This equation simply tells us that mass is conserved. If we use 0 as subscript for quantities taken today (so that $H_0 = H(t_0)$, $R_0 = R(t_0)$, $\Omega_0 = \Omega(t_0)$ etc.) we can thus write

$$\rho(t)R(t)^3 = \rho_0 R_0^3,$$

since ρR^3 has the same value at all times. We thus have

$$\rho_m(t) = \rho_{m0}\left(\frac{R_0}{R(t)}\right)^3, \tag{10}$$

where I have added the subscript m in this equation only to highlight that this equation is true in general for the matter component of the universe, but not for the radiation component. Remember that since we are in a dust dominated universe, the total energy density and the matter density are almost equal $\rho \approx \rho_m$. We normally normalize the scale factor $R(t)$ in such a manner that the scale factor today is one, $R_0 = 1$. This gives

$$\rho(t) = \frac{\rho_0}{R^3(t)}. \tag{11}$$

In the following we will use this equation together with the Friedmann equations to find an expression for $R(t)$.

We insert equation 11 in the first Friedmann equation to obtain

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G\rho_0}{3R} = -k. \tag{12}$$

We will first find the form of $R(t)$ for a flat universe, $k = 0$. Reorganizing this equation we find

$$dR\sqrt{R} = \sqrt{\frac{8\pi G\rho_0}{3}}dt$$

which we integrate from the beginning of the universe $t = 0$ and $R = 0$ until time t

$$\int_0^R dR\sqrt{R} = \int_0^t \sqrt{\frac{8\pi G\rho_0}{3}}dt,$$

giving

$$R_{\mathrm{k=0}}(t) = (6\pi G\rho_0)^{1/3}t^{2/3}$$

being valid for a flat universe. Thus in a flat universe the scale factor will increase for ever. We can also find the expansion rate by taking the time derivative

$$\frac{dR_{\mathrm{k=0}}}{dt} = \frac{2}{3}(6\pi G\rho_0)^{1/3}t^{-1/3}.$$

As time goes, the expansion rate slows down and as $t \to \infty$, the expansion rate goes to zero. A pressureless universe with flat geometry will expand for ever with a decreasing expansion rate reaching zero after an infinite amount of time.

This integral is much harder to do in universes with non-zero curvature. In the exercises you will show that the results can be written on a parameterized form as

$$\begin{aligned} R_{\mathrm{k=1}}(x) &= \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1 - \cos x), \\ t_{\mathrm{k=1}}(x) &= \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(x - \sin x), \end{aligned}$$

for a positively curved universe. Here $x > 0$ is the parameter used to evolve time and scale factor. Insert any number x in these two equations and you will find the scale factor $R(t)$ for a time t in a pressureless positively curved universe.

In the exercises you will make plots of the scale factor as a function of time. Here we will do some simple considerations. For $x = 0$, we have $t = 0$ and $R = 0$ which is the Big Bang. When x increases, also $R(t)$ increases so the universe expands. But as x reaches π we see that the scale factor reaches its maximum value. When $\cos x = -1$ no value of x can make the scale factor larger. When x continues to increase, the scale factor starts decreasing. The universe is contracting. When $x \to 2\pi$, the scale factor $R \to 0$ and the whole universe again reaches infinite density and stops in a Big Crunch. Thus a positively curved universe starts expanding after the Big Bang. After reaching maximum expansion, the universe starts contracting and ends in a Big Crunch. Some theories predict that this universe will that start again in a Big Bang which again will end in a Big Crunch and so on. This is called the *cyclic universe*. Models of the universe with positive curvature $k = +1$ are for obvious reasons called *closed universes*.

Similarly for the universe with negative curvature we have

$$\begin{aligned} R_{\mathrm{k}=-1}(x) &= \frac{1}{2}\frac{\Omega_0}{1-\Omega_0}(\cosh x - 1), \\ t_{\mathrm{k}=-1}(x) &= \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh x - x). \end{aligned}$$

Recall that for $x = 0$, $\cosh x = 1$ and therefore $R = 0$ which represents the Big Bang. As x increases also $\cosh x$ increases. Remember that as $x \to \infty$ we have that $\cosh x \to \infty$. Thus, the universe with negative curvature will expand forever and is therefore called an *open universe*.

Clearly, the open and closed universes will have very different fates and the flat universe is just at the limit: The expansion will stop when $R(t)$ reaches infinity. One can understand this in the following manner: A closed universe is a universe which has enough mass ($\rho > \rho_c$) such that gravitation will stop the expansion and make the universe collapse back to itself. An open universe is a universe with low density: there is not enough mass in the universe for gravitation to stop the expansion ($\rho < \rho_c$). In all cases, the expansion is slowed down with time due to the presence of gravitating mass, but only in the closed universes is the density high enough to stop the expansion and make the universe contract. The critical density ρ_c of a flat universe is the limiting case: if the density of the universe is larger than the critical density, the universe ends in a Big Crunch. If the density is smaller than the critical density, the expansion of the universe will gradually slow down but never enough to stop the expansion.

We can use these equation to find the age of the universe at a certain redshift z. Combining the above equations for $R(x)$ and $t(x)$ with equation 7 for the redshift as a function of scale factor, we obtain (you will show this in the exercises)

$$\frac{t(z)_{\mathrm{k=0}}}{t_H} = \frac{2}{3}\frac{1}{(1+z)^{3/2}}$$

for flat universes. For closed universes we have

$$\frac{t(z)_{\mathrm{k=1}}}{t_H} = \frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\left[\cos^{-1}\left(\frac{\Omega_0 z-\Omega_0+2}{\Omega_0 z+\Omega_0}\right)-\frac{2\sqrt{(\Omega_0-1)(\Omega_0 z+1)}}{\Omega_0(1+z)}\right]$$

and for open universes

$$\frac{t(z)_{\mathrm{k=-1}}}{t_H} = \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\left[-\cosh^{-1}\left(\frac{\Omega_0 z-\Omega_0+2}{\Omega_0 z+\Omega_0}\right)-\frac{2\sqrt{(1-\Omega_0)(\Omega_0 z+1)}}{\Omega_0(1+z)}\right]$$

Remember that t_H is the Hubble time $t_H = 1/H_0$. When you observe an object at a redshift z you can use these expressions to find the age of the universe when the light from the object was emitted and thus the epoch of the universe at which you observe this object. This is important for understanding the evolution of the universe and to study how the universe looked different in different time epochs.

5 Two components: matter and radiation

There is one big problem with the previous equations for the age of the universe. We used the expansion rate which is valid only for a pressureless universe. In the early universe, a high density of radiation made the radiation pressure important and changed the rate of expansion of the universe. For the moment we will use a general expression for the equation of state

$$P = w\rho,$$

where w is a dimensionless constant. For a dust dominated universe, this constant is $w \approx 0$ leading to $P = 0$. Going back to the Early universe, the universe consisted of a dense photon gas. The radiation pressure was dominating. We have previously learned that the energy density of a photon gas is $\rho_r = aT^4$ and radiation pressure $P_r = (1/3)aT^4$ such that

$$P_r = \frac{1}{3}\rho_r,$$

and therefore $w = (1/3)$ for a photon gas. We will later look at one more example of the equation of state.

Above, we found that the energy density in the universe decreases as $\rho \propto R^{-3}$ (equation 10) when dust is the dominating species in the universe ($w = 0$). We will now try to find out the general relation (general w) for how the energy density evolves with the scale factor $R(t)$ of the universe. In the exercises you will use the same approach as for the pressureless universe to show that

$$\frac{d}{dt}(\rho R^{3(1+w)}) = 0,$$

giving

$$\rho(t)R^{3(1+w)}(t) = \rho_0 R_0^{3(1+w)}.$$

Thus we may write the density as

$$\rho(t) = \rho_0 \left(\frac{R_0}{R(t)}\right)^{3(1+w)}. \tag{13}$$

For radiation $w = (1/3)$, the density therefore goes as

$$\rho_r(t) = \rho_{r0} \left(\frac{1}{R(t_0)}\right)^4,$$

where we have again set $R_0 = 1$.

In general the energy density of the universe can be written as a sum of a radiation part and a matter part

$$\rho(t) = \rho_m(t) + \rho_r(t).$$

or in terms of the densities relative to the critical density

$$\Omega(t) = \Omega_m(t) + \Omega_r(t),$$

where $\Omega_m(t) = \rho_m(t)/\rho_C(t)$ and $\Omega_r(t) = \rho_r(t)/\rho_C(t)$. In the very early universe, radiation was the dominant species and we can can write $\rho \approx \rho_r$ and $w \approx (1/3)$. In the recent universe, matter is dominating and we can write $\rho \approx \rho_m$ and $w = 0$. The period when the two species have similar densities is more difficult and we will not treat this in detail here. It is a good approximation for many purposes to assume that we had two eras with a sudden transition.

With the more general equation of state, you will show in the exercises that for a flat universe

$$R(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}. \tag{14}$$

Thus, the expansion rate of the universe depends on the equation of state of its content. A radiation dominated universe ($w = (1/3)$ giving $R(t) \propto t^{1/2}$) expands slower than a matter dominated universe.

The big question now is when the universe was radiation dominated. For which time period can we assume $w = 0$ and for which time period can we use $w = (1/3)$? We need to find the *time of matter-radiation equality*, the time when the matter and radiation densities are equal denoted t_{eq}. Before t_{eq} the universe was radiation dominated and after t_{eq} the universe was matter dominated. So by definition at $t = t_{eq}$,

$$\rho_m(t_{eq}) = \rho_r(t_{eq})$$

giving

$$\rho_{m0}\frac{1}{R_{eq}^3} = \rho_{r0}\frac{1}{R_{eq}^4},$$

where we have set $R_0 = 1$ and $R(t_{eq}) = R_{eq}$. Dividing by ρ_{c0} on both sides we have

$$R_{eq} = \frac{\Omega_{r0}}{\Omega_{m0}} \approx \frac{1}{3570},$$

where we used the current observed ratio between matter and radiation density in the universe. Thus, the period of matter-radiation equality happened when the size of the universe was a factor roughly 3570 times smaller than today. We can use formula 6 to find the redshift z_{eq} of matter-radiation equality

$$z_{eq} \approx 1/R_{eq} \approx 3570.$$

We can further find the time after the Big Bang when matter started to be the dominating species in the universe. Using equation 14 we find

$$R_{eq} = \left(\frac{t_{eq}}{t_0}\right)^{2/3},$$

giving

$$t_{eq} \approx R_{eq}^{3/2} t_0 \approx 60000\text{years},$$

using the current age of the universe $t_0 = 13.2 \times 10^9$ years. Thus, the universe started out as radiation dominated, then after about 60 000 years, the density in matter was larger than the density in radiation.

Knowing the expressions for the expansion parameter, we can also find an expression for the age of the universe. For a flat universe, we can rewrite the first Friedmann equations as (check that you understand why)

$$H^2 = \frac{8}{3}\pi G\rho.$$

Now we use equation 13 with $R_0 = 1$

$$H = \sqrt{\frac{8\pi G\rho_0}{3}}(R(t))^{3(1+w)/2}.$$

Since the universe is flat, the density is always equal to the critical density. Using the expression above for the critical density at time t_0 we have

$$\frac{1}{R(t)}\frac{dR(t)}{dt} = H_0(R(t))^{3(1+w)/2}$$

Reorganizing, we have

$$R^{(1+3w)/2}dR = H_0 dt.$$

Integrating this expression from $t = 0$ to $t = t_0$ and from $R = 0$ to $R = 1$, we have

$$\frac{2}{3(1+w)} = H_0 t_0,$$

or

$$t_0 = \frac{2}{3(1+w)H_0}.$$

The measured expansion rate today indicate a Hubble constant of $H_0 \approx 71$km/s/Mpc. The units of the Hubble constant is usually given as km/s/Mpc:

It is the velocity in km/s with which distant galaxies are moving away away from us per distance in Mpc. This means for instance that a galaxy at the distance of 1 Mpc will be observed to move away from us at the speed of 71 km/s. Assuming a pressureless universe $w = 0$ this expression then gives

$$t_0 \approx 9.1 \times 10^9 \text{years}.$$

This is lower than the current estimate of the age of the universe of $t_0 = 13.2 \times 10^9$years obtained with a more detailed analysis.

We have already seen in the above models that the rate of expansion of the universe decelerates. The gravitational forces between all particles in the universe are working against the expansion and decelerating it. This deceleration is measured with the deceleration parameter q defined as

$$q(t) = -\frac{1}{R(t)H^2(t)}\frac{d^2R(t)}{dt^2}.$$

The reason for this complicated expression for the deceleration (just the second derivative of $R(t)$ would suffice as a measure of deceleration) is to obtain a dimensionless measure. Note the minus sign: the deceleration parameter is defined to be positive when the expansion of the universe decelerates. We can use the second Friedmann equation to find an expression for q in a pressureless universe (note that we do not assume anything about the geometry of the universe this time, it can be flat, open or closed):

$$\frac{1}{R(t)}\frac{d^2R(t)}{dt^2} = -\frac{4}{3}\pi G\rho(t).$$

We use the definition of $\Omega(t)$ to write the density in terms of the critical density $\rho_c(t)$ and $\Omega(t)$. Using the expression for the critical density we have

$$\frac{1}{R(t)}\frac{d^2R(t)}{dt^2} = -\frac{4}{3}\pi G\rho_c(t)\Omega(t).$$

Using the expression above for the critical density as well as the definition of $q(t)$ we have

$$q(t) = \frac{1}{2}\Omega(t).$$

Not unexpectedly, the deceleration parameter depends on the density of the universe. High density means high deceleration, low density means low deceleration just as we anticipated. There is also another consequence of this equation: we can now try to measure the deceleration parameter q and thereby obtain Ω in order to find out whether the geometry of the universe is flat, open or closed. To measure the deceleration parameter, we would need to measure the expansion rate of the universe over a time period to see how it changes. This is not a problem: observing objects at different redshifts z means studying the universe at a different epoch. We will now see how one can use supernovae of type Ia to measure the deceleration parameter and thereby the density parameter today Ω_0.

6 Supernovae as a cosmological prove

We remember from the lectures on cosmic distance measurements that supernova of type Ia are used as standard candles as their luminosity is roughly the same for all supernoave of this kind. We will now see that we can use this to measure distances to supernovae exploding at different epochs and thereby find out how the expansion rate of the universe is changing with time. Knowing how the expansion rate of the universe changes with time, we can find the deceleration parameter and thereby the geometry of the universe as described in the previous section.

We will try to find the apparent magnitude m with which we observe a supernova or any other object with luminosity L. Luminosity is the energy ΔE emitted per unit of time Δt at the source,

$$L = \frac{\Delta E}{\Delta t},$$

We receive this radiation as a flux, energy $\Delta E'$ per unit of time $\Delta t'$ per unit area A at the observer's position,

$$F = \frac{\Delta E'}{\Delta A \Delta t'}. \tag{15}$$

We will now try to express this flux in terms of the known luminosity L of the supernova as well as the distance to the supernova.

We know that the energy is transmitted by photons with energy $e = h\nu$. We start by finding a relation between the emitted energy per photon $e = h/\lambda$ and the received energy per photon $e' = h/\lambda'$ per photon. We have previously found (go back and check how!) that photons are redshifted according to

$$\frac{\lambda'}{\lambda} = 1 + z$$

such that

$$e' = \frac{e}{(1+z)}.$$

and therefore

$$\Delta E' = \frac{\Delta E}{(1+z)},$$

where z is the redshift og the supernova.

To find the surface area A we need to consider the area over which the photons are distributed at a given moment in time. Thus, we need to find the surface area of a sphere which is frozen in time, $\Delta t = 0$, at the moment when we receive the photons. We can use the FRW-metric to study the geometry on this sphere. If we are confined to the sphere, $\Delta r = 0$. Setting $\Delta t = 0$ and $\Delta r = 0$ in the FRW line element we have

$$\Delta s^2 = R^2(t_0) r^2 (\Delta\theta^2 + \sin\theta \Delta\phi^2), .$$

where t_0 appears because the area is measured today (at the time when we receive the photons) at $t = t_0$. We see that this line element is the same as the line element for the geometry of an Euclidean sphere of radius $rR(t_0)$. We know that an Euclidean sphere with this radius has surface area $A = 4\pi r^2 R^2(t_0)$. Again, we use $R(t_0) = R_0 = 1$ so we will write $A = 4\pi r^2$ in the following.

Finally we need to relate the time Δt at the time of emission to the time $\Delta t'$ at the time of reception of the photons. In figure 4 we see two peaks of an electromagnetic wave. For an observer at rest, the time it takes from one peak passes to the second has passed is given by $\Delta t = \lambda/c$. Again we have

$$\frac{\lambda'}{\lambda} = 1 + z$$

giving

$$\frac{\Delta t'}{\Delta t} = 1 + z$$

Inserting all these relation in the expression for the flux (equation 15) we get

$$F = \frac{L}{4\pi r^2 (1+z)^2}.$$

This expression gives the flux F with which we observe an object with luminosity L at coordinate distance r corresponding to redshift z. We can use

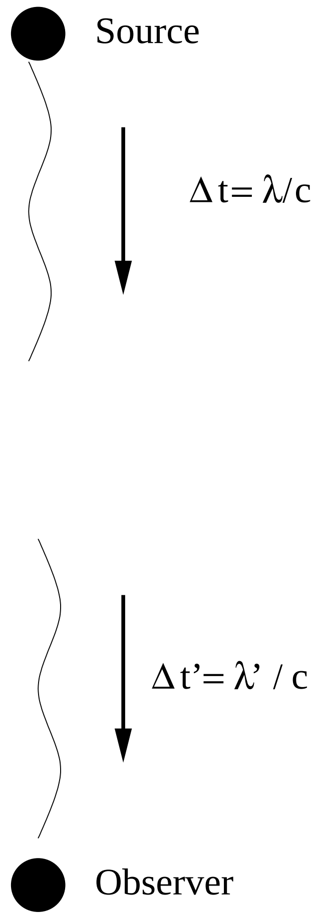

Figure 4: The energy ΔE is received within a time interval $\Delta t \propto \lambda$ at the source, for the same photons an energy $\Delta E'$ is received within a time interval $\Delta t' \propto \lambda'$ at the observation.

this expression to make a practical definition of distance in a universe with FRW geometry. We have discussed two different definitions of distance, the coordinate or comoving distance r and the proper distance. None of these are easy to measure in a practical way. As we discussed in the lectures on cosmic distances, we usually measure distances based on the emitted and observed fluxes of the objects. We will use this to **define** the *luminosity distance* d_L as

$$F = \frac{L}{4\pi d_L^2}.$$

This is the relation between received flux, luminosity and distance in Euclidean geometry. In FRW geometry, the distance d_L is not equal to the proper distance, it is the luminosity distance, a third and practical way of defining distance. Combining the previous two expressions for the flux, we find that the luminosity distance is given by

$$d_L = r(1 + z).$$

We need the coordinate distance r to the object in order to find a value for the luminosity distance. In the exercises you will rewrite r in this expression to obtain

$$d_L = \frac{1}{H_0 q_0^2}[q_0 z + (q_0 - 1)(\sqrt{1 + 2zq_0} - 1)].$$

The luminosity distance depends on the redshift of the object as well as on the current expansion rate and deceleration parameter. This makes sense: the size of the area A of a sphere should depend on the geometry of the universe. We have seen that the geometry of the universe depends on the density parameter Ω and thereby the acceleration parameter q.

Now we are almost ready to measure the distance to a supernova. We use the relation between apparent and absolute magnitude

$$m - M = 5 \log \frac{d_L}{10\text{pc}}.$$

Note that it is the luminosity distance that enters here. Why? Remember that this expression was obtained using Euclidean geometry $F = L/(4\pi d^2)$ (go back an check!), taking general relativity into account we have now seen that this is only correct if we use the luminosity distance d_L (if we had used coordinate distance r we should have included $(1+z)$ as we have seen above).

Inserting the expression for d_L we have

$$m - M = 5\log(q_0 z + (q_0 - 1)(\sqrt{1 + 2zq_0} - 1)) - 10\log(q_0) - 5\log(H_0 \times 10\text{pc})$$

We have a function combining the difference in apparent and absolute magnitude $m - M$, the redshift of the supernova z and the expansion and deceleration parameters today H_0 and q_0.

We remember that supernovae of type Ia are standard candles which means that we easily can find their luminosity and thereby the absolute magnitude M. By studying the spectrum of a supernova we can find z. We can find H_0 by other kinds of observations. The only unknown in this equation is thus q_0. We can therefore find the deceleration parameter by studying supernovae. We also know that $q_0 = \Omega_0/2$ so we can also find the geometry of the universe. There are uncertainties in the measurement of the luminosity and thereby M of a supernova, but by measuring many supernovae at different distances, these uncertainties can be reduced. By using many supernovae one can even find the best fitting H_0 and q_0 by using methods similar to the least square method. In the exercises coming next week you will try this method on a set of simulated supernova data.

The first results for q_0 using observations of a large amount of supernovae came in 1998. The surprising results shocked the whole astronomical community. It turns out that $q_0 \approx -0.6$. The deceleration parameter was designed such that it is positive for a decelerating expansion. But the result shows that the deceleration parameter is **negative** meaning that **the expansion of the universe is accelerating**. The classical models for the universe all predict that the expansion of the universe should decelerate due to the attractive gravitational forces. But the opposite is true.

7 The cosmological constant and dark energy

We will see that a possible solution to the problem is the cosmological constant, a constant which appears in the derivation of the Einstein equation. Einstein gave this constant a value different from zero in order to make the universe static, but after the discovery of the expansion of the universe, it was set to zero. There is no physical reasons why this constant should be zero. The Friedmann equations including the cosmological constant look like this

$$\begin{aligned}
\dot{R}^2(t) - \frac{8}{3}\pi G\rho(t)R^2(t) - \frac{\Lambda}{3}R(t)^2 &= -k \\
\frac{d^2R(t)}{dt^2} &= -\frac{4}{3}\pi G(\rho(t) + 3P)R(t) + \frac{\Lambda}{3}R(t).
\end{aligned}$$

Here Λ is the cosmological constant. To understand the physical effect of dark energy, notice that if we define

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \tag{16}$$

and

$$P_\Lambda = -\frac{\Lambda}{8\pi G} \tag{17}$$

then we can write the Friedmann equations in the classical form as

$$\begin{aligned}
(\dot{R}^2(t) - \frac{8}{3}\pi G\rho'(t)R^2(t)) &= -k \qquad (18) \\
\frac{d^2R(t)}{dt^2} &= -\frac{4}{3}\pi G(\rho'(t) + 3P'(t))R(t). \qquad (19)
\end{aligned}$$

exactly in the same form as before (check!) but with

$$\rho' = \rho + \rho_\Lambda = \rho_m + \rho_r + \rho_\Lambda$$

and

$$P' = P + P_\Lambda.$$

We see that the cosmological constant plays the same role as an additional component contributing to the energy density of the universe. From the above equations we see that (using equation 16 and 17)

$$P_\Lambda = -\rho_\Lambda$$

is the equation of state of this cosmological constant related component which is called *dark energy*. Thus, for dark energy $w = -1$. This means that dark energy has negative pressure. Positive pressure means that energy needs to be supplied to reduce the volume of a gas. That space has negative pressure means that energy has to be supplied to increase the volume. In a way one could say that a rubber band has negative pressure: one needs to add energy in order to stretch it.

We can now look at the consequences that dark energy has on the expansion of the universe. First we note that the energy density ρ_Λ of dark energy is a constant since Λ is a constant. This is in contrast to the other types of energy densities that we have in the universe: the density of matter and radiation decreases with time since the volume increases. The dark energy density however is constant even with an increasing volume. Thus, as time passes by the dominating species in the universe will be the dark energy. Actually, this has already happened. Today we measure that $\Omega_{m0} \approx 0.3$ but we also measure that $\Omega_0 = 1$. Thus, the universe is flat, but the total energy density in matter is $\Omega_{m0} \approx 0.3$. The remaining contribution to the energy density of the universe is the dark energy. As much as 70% of the energy in the universe today is dark energy. Thus, the expansion of the universe today cannot be described by the above functions for $R(t)$. We need to find the form of $R(t)$ in a universe dominated by the cosmological constant.

We cannot insert $w = -1$ in the equations that we have derived for $R(t)$ since this would lead to infinities. We need to start with the Friedmann equation. Assuming the universe is flat ($\Omega = 1, k = 0$) and dominated by the cosmological constant the first Friedmann equation reads

$$\frac{dR}{dt} = \sqrt{\frac{8}{3}\pi G \rho_\Lambda} R$$

Since ρ_Λ is a constant, this can easily by integrated

$$\sqrt{\frac{8}{3}\pi G \rho_\Lambda} \int_0^t dt = \int_0^R \frac{dR}{R}$$

giving

$$\ln R = \sqrt{\frac{8}{3}\pi G \rho_\Lambda} t$$

or

$$R(t) = e^{\sqrt{\frac{8}{3}\pi G \rho_\Lambda} t}.$$

We see that a universe dominated by dark energy is expanding exponentially with time. Thus, the expansion of the universe accelerates and the universe will eventually become extremely big and empty. In the distant future we will not be able to see other galaxies around us (but long before this happens, the Sun has already become a white dwarf). The results from the supernova data showed that this process has started: the universe has started an accelerated expansion.

8 Problems

Problem 1 We have deduced from the FRW-metric that all points in the universe appear to move away from each other. We know that this is because of the expansion of the space between these objects. According to the FRW-metric, the distance between all objects in the universe increases with the scale factor $R(t)$.

1. The Hubble constant is about 71 km/s/Mpc. If we assume that what the FRW metric tells us is true, with what speed v does the Sun move away from Earth due to the expansion of the universe?

2. Assume that this velocity is roughly constant in time. How much smaller was the Sun-Earth distance at the moment when the solar system was formed about 4.6 billion years ago? (give the answer in AU)

3. If we go back in time, this means that not so long ago, the Earth must have been much closer to the Sun which we know from geological data is not the case. Can you explain why ? Why doesn't the distance between the Earth and the Sun increase because of the expansion of the universe ?

Problem 2

We will now show that the metric of two-dimensional beings living on the two-dimensional surface of an expanding three-dimensional sphere is similar to the FRW-metric in two dimensions. Study the example with a one dimensional universe in the text before doing this exercise. The sphere is expanding in three-dimensional Euclidean space with metric

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2.$$

We will now try to find the metric for the two-dimensional universe confined to the surface of the sphere. The universe is only the surface of the sphere, there is nothing, not even space inside or outside this surface. Since we are looking for the metric of a two-dimensional surface, we need only two coordinates instead of the coordinates x, y and z.

1. We can easily get rid of the z coordinate. Show that

$$z = \sqrt{R^2 - x^2 - y^2}$$

2. Thus, we can specify the location of any point on the surface of the sphere by giving a position (x, y) (we will now look only at the upper hemisphere $z > 0$). We will now shift to coordinates which are natural for a two-dimensional being living in this two-dimensional universe. For simplicity, we will look at a person, 2D-John, living on the north pole of the sphere such that $x = y = 0$ and $z = R$ are his three-dimensional coordinates. 2D-John thinks that coordinates r and ϕ are reasonable coordinates to use. He defined $r = 0$ at his position. In figure 5 we see the sphere from above with 2D-John in the middle (on the north pole) and two 2D-galaxies called A and B located on the sphere at positions $(r = r_A, \phi_A = 0)$ and $(r = r_B, \phi_B = \Delta\phi_{AB})$ in 2D-John's coordinates. The coordinate distances between these two galaxies in John's coordinate system is Δr_{AB} and $\Delta\phi_{AB}$. We will now look at the proper distance Δs_{AB} between galaxy A and galaxy B along the surface of the sphere.

   We need to write this distance in terms of 2D-John's coordinates, Δr_{AB} and $\Delta\phi_{AB}$ instead of the three-dimensinal coordinates Δx_{AB}, Δy_{AB} and Δz_{AB} which do not have any meaning for 2D-John. We have

   $$\Delta s_{AB}^2 = \Delta x_{AB}^2 + \Delta y_{AB}^2 + \Delta z_{AB}^2$$

   Using the expression above we can now get rid of the z coordinate by constraining the distance to be along the sphere. We will do this in the end. First we will rewrite distances Δx and Δy in terms of Δr and $\Delta\phi$. The r coordinate measured from the north pole (2D-John's position), is similar to the θ coordinate used in normal three-dimensional polar coordinates. We let galaxy A be at angle θ and galaxy B at angle $\theta + \Delta\theta$. We will use a definition of r similar to the definition we used in the one-dimensional case

   $$r = \sin\theta.$$

   Show that

   $$\Delta r = \cos\theta\Delta\theta.$$

3. Make a drawing of the situation from the side showing the projection of the galaxies A and B in the x-y plane. Use the geometry on the figure to show that the distance Δx_{AB} can be written as

   $$\Delta x_{AB} = R\sin\theta - R\sin(\theta + \Delta\theta)\cos\Delta\phi_{AB} \approx R\sin\theta - R\sin(\theta + \Delta\theta),$$

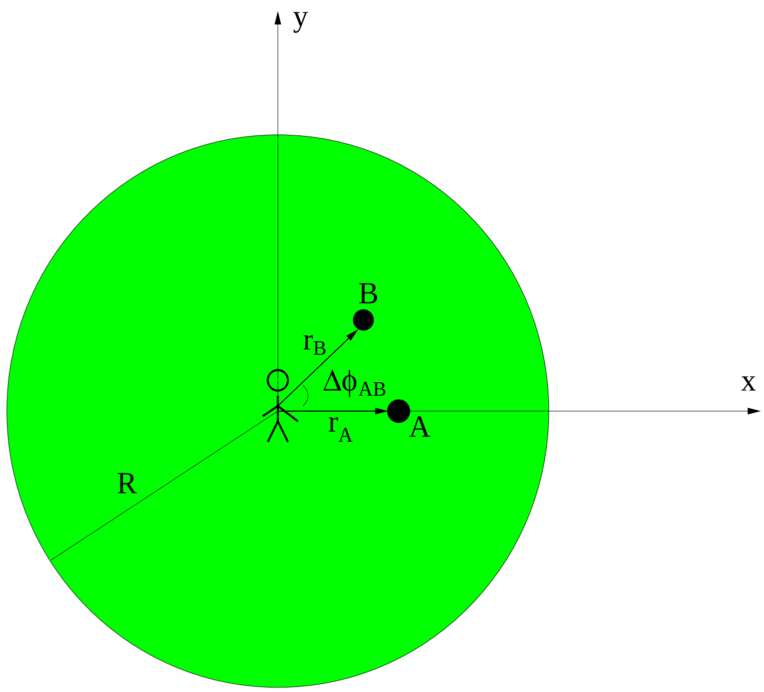

Figure 5: 2D-John living on the north pole in the two-dimensional surface-of-sphere-universe. Here we see the universe from above as well as x and y axis of a three dimensional coordinate system. 2D-John defines a coordinate system (r, ϕ) measured along the surface (in his 2D universe) with $r = 0$ at the north pole.

where θ is still the normal three-dimensional polar coordinate.

4. Show that for small $\Delta\theta$, we can write (a Taylor expansion of the last term could be a good idea)

$$\Delta x_{AB} \approx R\cos\theta\Delta\theta_{AB} = R\Delta r_{AB}$$

5. Now turn to your drawing with the projection of A and B in the x-y plane. Use the geometry to find that

$$\Delta y_{AB} = R\sin\theta\sin\Delta\phi_{AB} \approx Rr\Delta\phi_{AB},$$

   where ϕ is the normal ϕ coordinate in a three-dimensional polar coordinate system, but this is identical to the ϕ coordinate which 2D-John is using.

6. Show that

$$\Delta x_{AB}^2 + \Delta y_{AB}^2 = R^2(\Delta r_{AB}^2 + r^2\Delta\phi_{AB}^2).$$

7. Now we need to find an expression for Δz. Show that the z-position for an object at coordinate r in this universe is given by

$$z = R\sqrt{1-r^2}.$$

   hint: We have that $x = y = r = 0$ at John's position. We measure the coordinates x, y and r from this point. Thus an object at position r has a distance Δr from John's position. This also holds for the x and y coordinate.

8. Show that

$$\Delta z_{AB}^2 = \frac{R^2r^2\Delta r^2}{1-r^2}$$

9. Now you have all the information you need in order to find the metric for John's geometry, the geometry on the 2D universe on the surface of the sphere. Show that it can be written as

$$\Delta s^2 = R^2\left(\frac{\Delta r^2}{1-r^2} + r^2\Delta\phi^2\right).$$

   which is similar to the spatial part of the FRW metric which we have in our 3D space.

Problem 3 We will study the parameterized solutions for $R(t)$ in the case of an open and closed geometry.

1. Choose either the $k = 1$ or the $k = -1$ case and show that the parameterized solutions $R(t)$ of the Friedmann equations are correct: insert the expressions for $R(x)$ and $t(x)$ into the first Friedmann equation and show that this is indeed a solution to the equations. **Some hints:** Use the first Friedmann equation taken at $t = t_0$ to show that $H_0^2 = k/(\Omega_0 - 1)$. Show that this for instance gives $R(x) = (1/2)\alpha(1 - \cos(x))$, $t(x) = (1/2)\alpha(x - \sin(x))$ where α is a constant for $k = +1$. Show also that the first Friedmann equation can be written as $\dot{R}^2 - \alpha/R = -k$ when using equation 12.

2. Use Python/Matlab (or whatever) to plot $R(t)$ as a function of t for an open, flat and closed universe together on the same plot. Take $\Omega_0 = 0.9$ for the open universe and $\Omega_0 = 1.1$ for the closed universe.

3. Make a plot of $R(t)$ as a function of t for a pressureless universe with $\Omega_0 = 1.5$, how many years does it take from Big Bang to Big Crunch in this universe?

4. In the text we have seen expressions for the time $t(z)$ after the Big Bang as a function of redshift z. Use the expression for $R(t)$ in a flat universe as well as equation 7 to show that the expression for $t(z)/t_H$ is correct for flat universes.

5. Now, choose either the open or the closed universe and repeat the previous exercise for this universe using the parameterized expressions for $R(x)$ and $t(x)$.

Problem 4 In the text you show that

$$\rho_m = \rho_{m0} \left(\frac{R_0}{R(t)} \right)^3 ,$$

for a pressureless universe. Study this derivation and make sure that you understand every step in detail. Then you're ready for this exercise: here we will deduce a similar relation in a universe with the general equation of state $p = w\rho$. We will do the derivation step by step

1. Use the same approach as in the text to show that

$$\frac{d}{dt}(\rho R^{3(1+w)}) = 0.$$

   You might need write this derivative in terms of $d/dt(\rho R^2)$ and then identify the terms you have from the Friedmann equation in a similar way as in the text.

2. Use the previous result to show that

$$\rho(t) = \rho_0 \left(\frac{R_0}{R(t)} \right)^{3(1+w)}$$

problem 5 In the text we deduced how the scale factor $R(t)$ changes with time t in a pressureless flat universe. We found that

$$R_{k=0}(t) = (6\pi G\rho_0)^{1/3} t^{2/3}.$$

Here we will still look at flat universes, but with a general equation of state $p = w\rho$. We will skip the subscript $k = 0$ as all quantities in this exercise are for the flat universe.

1. First show that we can rewrite the previous equation for the pressureless universe as

$$R(t) = \left(\frac{t}{t_0} \right)^{2/3},$$

   when we define $R_0 = 1$ and t_0 is the time today.

2. Now go back to the text and study in detail how we arrived at this equation starting with the Friedmann equations. Then repeat exactly the same procedure, but now use equation 13 to show that

$$R(t) = \left(\frac{t}{t_0} \right)^{\frac{2}{3(1+w)}}$$

problem 6

We used in the text that the luminosity distance can be written as

$$d_L = \frac{1}{H_0 q_0^2}[q_0 z + (q_0 - 1)(\sqrt{1 + 2zq_0} - 1)].$$

We will now show this for the flat universe.

1. In a flat universe the value of q_0 is known. Show that the previous expression can be rewritten as

$$d_L = \frac{2}{H_0}[1 + z - \sqrt{1+z}].$$

for a flat universe.

2. If we use the FRW metric, show that for a photon traveling from a source at coordinate r at time t and received at Earth at coordinate $r = 0$ at time t_0 we have

$$0 = dt^2 + R^2(t)\frac{dr^2}{1 - kr^2}$$

3. Show the following equation for the coordinate distance r to the object

$$\int_t^{t_0} \frac{dt'}{R(t')} = \int_0^r \frac{dr'}{\sqrt{1 - k(r')^2}}.$$

4. Assume a flat pressureless universe (all objects that we observe emitted light in the era of the pressureless universe) and use the form of $R(t)$ in such a universe. Use the previous equation for a light beam to show that the coordinate distance r that light emitted at time t and received at time t_0 has traveled is

$$r = \frac{3t_0}{R(t_0)}\left[1 - \left(\frac{t}{t_0}\right)^{1/3}\right]$$

for a flat pressureless universe.

5. Now use the expression relating $R(t_0)/R(t)$ to the redshift z and deduce the above expression for the luminosity distance in a flat pressureless universe.

6. Insert the expression for the luminosity distance (the general expression including q_0) into the relation between apparent and absolute magnitude and show that

$$m - M = 5\log(q_0 z + (q_0 - 1)(\sqrt{1 + 2zq_0} - 1)) - 10\log(q_0) - 5\log(H_0 \times 10\text{pc})$$

7. We will now make a plot with $m - M$ on the y-axis and redshift z on the x-axis. Include values of redshift up to $z = 2$. Assume the current value of the Hubble constant $H_0 = 71$ km/s/Mpc. Plot three models of the universe on the same plot, $\Omega_0 = 0.3$, $\Omega_0 = 1$ and $\Omega_0 = 1.5$. Constrain the range on the y-axis of the plot for values of $m - M$ between 37 and 46. Explain how you can use this plot, combined with observations of supernovae to find the geometry of the universe. Then, use the plot to answer the following two questions: (a) in order to find the geometry of the universe, would you observe nearby or distant supernovae ? (b) Roughly what minimum redshift should the supernovae that you observe have in order for you to easily find the geometry of the universe ?

8. In problem 3 of the lecture notes on the 'end state of stars', you used a simplified model of a supernova to find the absolute magnitude and thereby the distance of the supernova. Study this exercises once more since we will now use the same assumptions about the supernovae to find the geometry of the universe. In that problem, we observed the velocity v of the expanding shell and the temperature T of the shell a time Δt after the explotion. We will in the following use these observations, as well as the observed redshift z of supernovae to estimate the geometry of the universe.

9. Using this model of the expanding shell, show that you can write $m-M$ for the supernova as

$$m - M = m - 86.7 + 10\log_{10}(T) + 5\log_{10}(v) + 5\log_{10}(\Delta t),$$

where m is apparent magnitude and M is absolute magnitude (not observed directly), the temperature T is measured in Kelvin, the shell velocity v is measured in m/s and the time delay Δt is measured in seconds.

10. You will now look at three files. Each file contains a list of 1000 observed (simulated) supernovae. Each file is for observations made by different creatures living in universes with different values of Ω. The files can be found here:

$$http://folk.uio.no/frodekh/AST1100/lecture25/supernovadata\_?.txt$$

There are five columns in each file: first column is the redshift (the supernovae are ordered by increasing redshift), the second column is the apparent magnitude, the third column is the measured surface temperature of the expanding shell, the fourth column is the shell velocity in km/s and the last column is the time of observation measured in days after the supernova exploded. Load these data and construct an array which gives $m - M$ as a function of redshift. Make a plot with $m - M$ on the y-axis and redshift z on the x-axis for one of the universes. Use the same range on the y-axis for $m - M$ as indicated above. On top of this plot, plot the same three lines for the same three models that you plotted above. Can you use this plot to tell if the universe you are looking at is open, flat or closed? Repeat for the other two universes. Which of these three set of observations do you think were made by us (human beings) in our universe?

11. You have now seen how you can use observations of supernovae to get an idea of the geometry of the universe. Now we will see how wrong you were and how easy it is to fool yourself if you don't make a thorough analysis of the data. We will now make a better analysis by the now well known least square fitting. The function on which we will make least square fitting is $(m-M)$. You have models of $m-M$ for different values of q_0 (and thereby Ω_0) and you have a set of observations of $m - M$. So we will again try to find the model, that is, the value of q_0, which gives the smallest possible difference between the model and the data. Show that you can write the above expression for the model in terms of q_0 as

$$m-M = 5\log(q_0 z+(q_0-1)(\sqrt{1+2zq_0}-1))-5\log(q_0^2)-5\log(H_0\times 10\mathrm{pc})$$

It is very important that you use exactly this expression in your computer code otherwise you may run into some numerical problems.

12. We can now construct the function to minimize, i.e. the sum of the square of the model minus the data, summed over all redshifts. This is given by

$$\Delta(q_0) \quad = \quad \sum_{z=0}^{2}[(m-M)_{\mathrm{data}} - (m-M)_{\mathrm{model}}]^2$$

$$
\begin{aligned}
= \quad & \sum_{z=0}^{2}[(m-M)_{\mathrm{data}} - (5\log(q_0 z + (q_0-1)(\sqrt{1+2zq_0}-1)) \\
& -5\log(q_0^2) - 5\log(H_0 \times 10\mathrm{pc}))]^2.
\end{aligned}
$$

You can now construct a code in exactly the same way as you did before with least square minimization: make an array for $\Delta(q_0)$. Then make a loop over a set of possible values of q_0 and calculate $\Delta(q_0)$ each time. You should choose the range of q_0 to go from $q_0 = -0.25$ to $q_0 = 2$. Use about 1000 different values for q_0 in this range. Then find the value of q_0 which minimizes $\Delta(q_0)$. Which Ω_0 does this value correspond to? You should get a surprise in one of these models. You will find in that model that the universe accelerates and you get a negative value for Ω_0. For that model, Ω_0 is clearly not negative, it is the relation between q_0 and Ω_0 which is wrong in this case. You will not learn the correct relation in this corse, but explain the physical reasons why we may get a negative q_0. What is happening?

13. Now we will return to the question above and see if you have changed your mind: which of these three sets of supernova observations do you think is most similar to observations made in our universe?

14. We will finally look into a problem which we have with this way of measuring the geometry of the universe. Make the least square fitting above for the model with negative q_0, but this time you only include the supernovae up to redshift $z = 0.2$ Which value of q_0 do you get? Why did you get a different value this time? Supernovae in the nearby universe $z < 0.2$ are much easier to find and observe than distant supernovae. We therefore have much more data for supernovae with low redshift. This can make measurements of q_0 uncertain. We have not included error bars in our analysis, but you have already got an idea of the size of the errors if we only include nearby supernovae.

15. We have used a very simple model of the supernova. A more sophisticated model is used in the real analysis. But these models are based on observations in the nearby universe. One of the big debates in cosmology novadays is whether we can trust that the same model for the supernova is true for the distant supernovae. Clearly if the model is wrong, also the q_0 obtained with least square fitting is wrong. Can you find a good reason why a model for supernovae which is found to

be correct for nearby supernova is not necessarily correct for distant supernovae? (think back to the lectures on stellar evolution)