# Chapter 4

# Polytropic stellar models

As mentioned in Sec. 2.2, the equation of hydrostatic equilibrium can be solved if the pressure is a known function of the density, $P = P(\rho)$. In this situation the mechanical structure of the star is completely determined. A special case of such a relation between $P$ and $\rho$ is the *polytropic relation*,

$$P = K\rho^{\gamma} \tag{4.1}$$

where $K$ and $\gamma$ are both constants. The resulting stellar models are known as *polytropic stellar models* or simply polytropes. Polytropic models have played an important role in the historical development of stellar structure theory. Although nowadays their practical use has mostly been superseded by more realistic stellar models, due to their simplicity polytropic models still give useful insight into several important properties of stars. Moreover, in some cases the polytropic relation is a good approximation to the real equation of state. We have encountered a few examples of polytropic equations of state in Chapter 3, e.g. the pressure of degenerate electrons, and the case where pressure and density are related adiabatically.

In this brief chapter – and the accompanying computer practicum – we will derive the analytic theory of polytropes and construct polytropic models, and study to which kind of stars they correspond, at least approximately.

## 4.1 Polytropes and the Lane-Emden equation

If the equation of state can be written in polytropic form, the equations for mass continuity ($\mathrm{d}m/\mathrm{d}r$, eq. 2.3) and for hydrostatic equilibrium ($\mathrm{d}P/\mathrm{d}r$, eq. 2.12) can be combined with eq. (4.1) to give a second-order differential equation for the density:

$$\frac{1}{\rho r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\rho^{\gamma-2}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\frac{4\pi G}{K\gamma} \tag{4.2}$$

The exponent $\gamma$ is often replaced by the so-called polytropic index $n$, which is defined by

$$n = \frac{1}{\gamma - 1} \qquad \text{or} \qquad \gamma = 1 + \frac{1}{n} \tag{4.3}$$

In order to construct a polytropic stellar model we have to solve eq. (4.2), together with two boundary conditions which are set in the centre, $r = 0$:

$$\rho(0) = \rho_c \qquad \text{and} \qquad \left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{r=0} = 0, \tag{4.4}$$

where $\rho_c$ is a parameter to be chosen, or determined from other constraints.

**Table 4.1.** Numerical values for polytropic models with index $n$.

| $n$ | $z_n$ | $\Theta_n$ | $\rho_c/\overline{\rho}$ | $N_n$ | $W_n$ |
|---|---|---|---|---|---|
| 0 | 2.44949 | 4.89898 | 1.00000 | ... | 0.119366 |
| 1 | 3.14159 | 3.14159 | 3.28987 | 0.63662 | 0.392699 |
| 1.5 | 3.65375 | 2.71406 | 5.99071 | 0.42422 | 0.770140 |
| 2 | 4.35287 | 2.41105 | 11.40254 | 0.36475 | 1.638183 |
| 3 | 6.89685 | 2.01824 | 54.1825 | 0.36394 | 11.05068 |
| 4 | 14.97155 | 1.79723 | 622.408 | 0.47720 | 247.559 |
| 4.5 | 31.8365 | 1.73780 | 6189.47 | 0.65798 | 4921.84 |
| 5 | $\infty$ | 1.73205 | $\infty$ | $\infty$ | $\infty$ |

In order to simplify eq. (4.2), we define two new dimensionless variables $w$ (related to the density) and $z$ (related to the radius) by writing

$$\rho = \rho_c w^n, \tag{4.5}$$

$$r = \alpha z, \qquad \text{with} \quad \alpha = \left( \frac{n+1}{4\pi G} K \rho_c^{1/n-1} \right)^{1/2}. \tag{4.6}$$

This choice of $\alpha$ ensures that the constants $K$ and $4\pi G$ are eliminated after substituting $r$ and $\rho$ into eq. (4.2). The resulting second-order differential equation is called the *Lane-Emden equation*:

$$\frac{1}{z^2} \frac{\mathrm{d}}{\mathrm{d}z} \left( z^2 \frac{\mathrm{d}w}{\mathrm{d}z} \right) + w^n = 0. \tag{4.7}$$

A polytropic stellar model can be constructed by integrating this equation outwards from the centre. The boundary conditions (4.4) imply that in the centre ($z = 0$) we have $w = 1$ and $\mathrm{d}w/\mathrm{d}z = 0$. For $n < 5$ the solution $w(z)$ is found to decrease monotonically and to reach zero at finite $z = z_n$, which corresponds to the surface of the model.

No general analytical solution of the Lane-Emden equation exists. The only exceptions are $n = 0$, 1 and 5, for which the solutions are:

$$n = 0: \qquad w(z) = 1 - \frac{z^2}{6} \qquad\qquad z_0 = \sqrt{6}, \tag{4.8}$$

$$n = 1: \qquad w(z) = \frac{\sin z}{z} \qquad\qquad z_1 = \pi, \tag{4.9}$$

$$n = 5: \qquad w(z) = \left( 1 + \frac{z^2}{3} \right)^{-1/2} \qquad z_5 = \infty. \tag{4.10}$$

The case $n = 0$ ($\gamma = \infty$) corresponds to a homogeneous gas sphere with constant density $\rho_c$, following eq. (4.5). The solution for $n = 5$ is peculiar in that it has infinite radius; this is the case for all $n \geq 5$, while for $n < 5$ $z_n$ grows monotonically with $n$. For values of $n$ other than 0, 1 or 5 the solution must be found by numerical integration (this is quite straightforward, see the accompanying computer practicum). Table 4.1 lists the value of $z_n$ for different values of $n$, as well as several other properties of the solution that will be discussed below.

### 4.1.1 Physical properties of the solutions

Once the solution $w(z)$ of the Lane-Emden equation is found, eq. (4.5) fixes the relative density distribution of the model, which is thus uniquely determined by the polytropic index $n$. Given the solution for a certain $n$, the physical properties of a polytropic stellar model, such as its mass and radius, are then determined by the parameters $K$ and $\rho_c$, as follows.

The radius of a polytropic model follows from eq. (4.6):

$$R = \alpha z_n = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_c^{(1-n)/2n}\, z_n. \tag{4.11}$$

The mass $m(z)$ interior to $z$ can be obtained from integrating eq. (2.3), using eqs. (4.5), (4.6) and (4.7):

$$m(z) = \int_0^{\alpha z} 4\pi r^2 \rho\, dr = -4\pi\alpha^3 \rho_c\, z^2 \frac{\mathrm{d}w}{\mathrm{d}z}. \tag{4.12}$$

Hence the total mass of a polytropic model is

$$M = 4\pi\alpha^3 \rho_c\, \Theta_n = 4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n}\, \Theta_n, \tag{4.13}$$

where we have defined $\Theta_n$ as

$$\Theta_n \equiv \left(-z^2 \frac{\mathrm{d}w}{\mathrm{d}z}\right)_{z=z_n}. \tag{4.14}$$

By eliminating $\rho_c$ from eqs. (4.11) and (4.13) we can find a relation between $M$, $R$ and $K$,

$$K = N_n\, G M^{(n-1)/n} R^{(3-n)/n} \qquad \text{with} \qquad N_n = \frac{(4\pi)^{1/n}}{n+1}\, \Theta_n^{(1-n)/n}\, z_n^{(n-3)/n}. \tag{4.15}$$

Numerical values of $\Theta_n$ and $N_n$ are given in Table 4.1. From the expressions above we see that $n = 1$ and $n = 3$ are special cases. For $n = 1$ the radius is independent of the mass, and is uniquely determined by the value of $K$. Conversely, for $n = 3$ the mass is independent of the radius and is uniquely determined by $K$. For a given $K$ there is only one value of $M$ for which hydrostatic equilibrium can be satisfied if $n = 3$.

The average density $\bar{\rho} = M/(\frac{4}{3}\pi R^3)$ of a polytropic star is related to the central density by eqs. (4.11) and (4.13) as

$$\bar{\rho} = \left(-\frac{3}{z}\frac{\mathrm{d}w}{\mathrm{d}z}\right)_{z=z_n} \rho_c = \frac{3\Theta_n}{z_n^3}\, \rho_c \tag{4.16}$$

Hence the ratio $\rho_c/\bar{\rho}$, i.e. the degree of central concentration of a polytrope, only depends on the polytropic index $n$. This dependence is also tabulated in Table 4.1. One may invert this relation to find the central density of a polytropic star of a given mass and radius.

The central pressure of a polytropic star follows from eq. (4.1), which can be written as

$$P_c = K \rho_c^{(n+1)/n}.$$

In combination with (4.15) and (4.16) this gives

$$P_c = W_n \frac{GM^2}{R^4} \qquad \text{with} \qquad W_n = \frac{z_n^4}{4\pi(n+1)\Theta_n^2}. \tag{4.17}$$

Note that in our simple scaling estimate, eq. (2.14), we found the same proportionality $P_c \propto GM^2/R^4$, where the proportionality constant $W_n$ is now determined by the polytropic index $n$ (see Table 4.1). We can eliminate $R$ in favour of $\rho_c$ to obtain the very useful relation

$$P_c = C_n\, GM^{2/3}\rho_c^{4/3} \qquad \text{with} \qquad C_n = \frac{(4\pi)^{1/3}}{n+1}\Theta_n^{-2/3}, \tag{4.18}$$

where you may verify that the constant $C_n$ is only weakly dependent on $n$, unlike $W_n$ in (4.17).

We give without derivation an expression for the gravitational potential energy of a polytrope of index $n$:

$$E_{\rm gr} = -\frac{3}{5-n}\,\frac{GM^2}{R}. \tag{4.19}$$

(The derivation can be found in K&W Sec. 19.9 and Maeder Sec. 24.5.1.)

## 4.2 Application to stars

Eq. (4.15) expresses a relation between the constant $K$ in eq. (4.1) and the mass and radius of a polytropic model. This relation can be interpreted in two very different ways:

- The constant $K$ may be given in terms of physical constants. This is the case, for example, for a star dominated by the pressure of degenerate electrons, in either the non-relativistic limit or the extremely relativistic limit. In that case eq. (4.15) defines a unique relation between the mass and radius of a star.
- In other cases the constant $K$ merely expresses proportionality in eq. (4.1), i.e. $K$ is a free parameter that is constant in a particular star, but may vary from star to star. In this case there are many different possible values of $M$ and $R$. For a star with a given mass and radius, the corresponding value of $K$ for this star can be determined from eq. (4.15).

In this section we briefly discuss examples for each of these two interpretations.

### 4.2.1 White dwarfs and the Chandrasekhar mass

Stars that are so compact and dense that their interior pressure is dominated by degenerate electrons are known observationally as *white dwarfs*. They are the remnants of stellar cores in which hydrogen has been completely converted into helium and, in most cases, also helium has been fused into carbon and oxygen. Since the pressure of a completely degenerate electron gas is a function of density only (Sec. 3.3.5), the mechanical structure if a white dwarf is fixed and is independent of temperature. We can thus understand some of the structural properties of white dwarfs by means of polytropic models.

We start by considering the equation of state for a degenerate, non-relativistic electron gas. From eq. (3.35) this can be described by a polytropic relation with $n = 1.5$. Since the corresponding $K$ is determined by physical constants, eq. (4.15) shows that such a polytrope follows a mass-radius relation of the from

$$R \propto M^{-1/3}. \tag{4.20}$$

More massive white dwarfs are thus more compact, and therefore have a higher density. Above a certain density the electrons will become relativistic as they are pushed up to higher momenta by the Pauli exclusion principle. The degree of relativity increases with density, and therefore with the mass of the white dwarf, until at a certain mass all the electrons become extremely relativistic, i.e., their speed $v_e \to c$. In this limit the equation of state has changed from eq. (3.35) to eq. (3.37), which is also a polytropic relation but with $n = 3$. We have already seen above that an $n = 3$ polytrope is special in the sense that it has a unique mass, which is determined by $K$ and is independent of the radius:

$$M = 4\pi\,\Theta_3 \left(\frac{K}{\pi G}\right)^{3/2}. \tag{4.21}$$

This value corresponds to an upper limit to the mass of a gas sphere in hydrostatic equilibrium that can be supported by degenerate electrons, and thus to the maximum possible mass for a white dwarf. Its existence was first found by Chandrasekhar in 1931, after whom this limiting mass was named. Substituting the proper numerical values into eq. (4.21), with $K$ corresponding to eq. (3.37), we obtain the *Chandrasekhar mass*

$$M_{\rm Ch} = 5.836\,\mu_e^{-2}\,M_\odot. \tag{4.22}$$

White dwarfs are typically formed of helium, carbon or oxygen, for which $\mu_e = 2$ and therefore $M_{\rm Ch} = 1.46\,M_\odot$. Indeed no white dwarf with a mass exceeding this limit is known to exist.

### 4.2.2 Eddington's standard model

As an example of a situation where $K$ is not fixed by physical constants but is essentially a free parameter, we consider a star in which the pressure is given by a mixture of ideal gas pressure and radiation pressure, eq. (3.45). In particular we make the assumption that the ratio $\beta$ of gas pressure to total pressure is constant, i.e. has the same value in each layer of the star. Since $P_{\rm gas} = \beta P$ we can write

$$P = \frac{1}{\beta}\frac{\mathcal{R}}{\mu}\rho T, \tag{4.23}$$

while also

$$1 - \beta = \frac{P_{\rm rad}}{P} = \frac{aT^4}{3P}. \tag{4.24}$$

Thus the assumption of constant $\beta$ means that $T^4 \propto P$ throughout the star. If we substitute the complete expression for $T^4$ into eq. (4.24) we obtain

$$P = \left(\frac{3\mathcal{R}^4}{a\mu^4}\,\frac{1-\beta}{\beta^4}\right)^{1/3}\rho^{4/3}, \tag{4.25}$$

which is a polytropic relation with $n = 3$ for constant $\beta$. Since we are free to choose $\beta$ between 0 and 1, the constant $K$ is indeed a free parameter dependent on $\beta$.

The relation (4.25) was derived by Arthur Eddington in the 1920s for his famous 'standard model'. He found that in regions with a high opacity $\kappa$ (see Ch. 5) the ratio of local luminosity to mass coordinate $l/m$ is usually small, and vice versa. Making the assumption that $\kappa l/m$ is constant throughout the star is equivalent to assuming that $\beta$ is constant (again, see Ch. 5). Indeed, for stars in which radiation is the main energy transport mechanism this turns out to be approximately true, even though it is a very rough approximation to the real situation. Nevertheless, the structure of stars on the main sequence with $M \gtrsim M_\odot$ is reasonably well approximated by that of a $n = 3$ polytrope. Since the mass of a $n = 3$ polytrope is given by eq. (4.21), we see from eq. (4.25) that there is a unique relation between the mass $M$ of a star and $\beta$. The relative contribution of radiation pressure increases with the mass of a star. This was also noted by Eddington, who pointed out that the limited range of known stellar masses corresponds to values of $\beta$ that are significantly different from 0 or 1.

## Suggestions for further reading

Polytropic stellar models are briefly covered in Chapter 24.5 of Maeder and treated more extensively in Chapter 19 of Kippenhahn & Weigert and Chapter 7.2 of Hansen.

## Exercises

### 4.1 The Lane-Emden equation

(a) Derive eq. (4.2) from the stellar structure equations for mass continuity and hydrostatic equilibrium. (Hint: multiply the hydrostatic equation by $r^2/\rho$ and take the derivative with respect to $r$).

(b) What determines the second boundary condition of eq. (4.4), i.e., why does the density gradient have to vanish at the center?

(c) By making the substitutions (4.3), (4.5) and (4.6), derive the Lane-Emden equation (4.7).

(d) Solve the Lane-Emden equation analytically for the cases $n = 0$ and $n = 1$.

### 4.2 Polytropic models

(a) Derive $K$ and $\gamma$ for the equation of state of an ideal gas at a fixed temperature $T$, of a non-relativistic degenerate gas and of a relativistic degenerate gas.

(b) Using the Lane-Emden equation, show that the mass distribution in a polytropic star is given by eq. (4.12), and show that this yields eq. (4.13) for the total mass of a polytrope.

(c) Derive the expressions for the central density $\rho_c$ and the central pressure $P_c$ as function of mass and radius, eqs. (4.16) and (4.17).

(d) Derive eq. (4.18) and compute the constant $C_n$ for several values of $n$.

### 4.3 White dwarfs

To understand some of the properties of white dwarfs (WDs) we start by considering the equation of state for a degenerate, non-relativistic electron gas.

(a) What is the value of $K$ for such a star? Remember to consider an appropriate value of the mean molecular weight per free electron $\mu_e$.

(b) Derive how the central density $\rho_c$ depends on the mass of a non-relativistic WD. Using this with the result of Exercise 4.2(b), derive a radius-mass relation $R = R(M)$. Interpret this physically.

(c) Use the result of (b) to estimate for which WD masses the relativistic effects would become important.

(d) Show that the derivation of a $R = R(M)$ relation for the extreme relativistic case leads to a unique mass, the so-called *Chandrasekhar mass*. Calculate its value, i.e. derive eq. (4.22).

### 4.4 Eddington's standard model

(a) Show that for constant $\beta$ the virial theorem leads to

$$E_{\text{tot}} = \frac{\beta}{2} E_{\text{gr}} = -\frac{\beta}{2-\beta} E_{\text{int}}, \tag{4.26}$$

for a classical, non-relativistic gas. What happens in the limits $\beta \to 1$ and $\beta \to 0$?

(b) Verify eq. (4.25), and show that the corresponding constant $K$ depends on $\beta$ and the mean molecular weight $\mu$ as

$$K = \frac{2.67 \times 10^{15}}{\mu^{4/3}} \left( \frac{1-\beta}{\beta^4} \right)^{1/3} . \tag{4.27}$$

(c) Use the results from above and the fact that the mass of an $n = 3$ polytrope is uniquely determined by $K$, to derive the relation $M = M(\beta, \mu)$. This is useful for numerically solving the amount of radiation pressure for a star with a given mass.