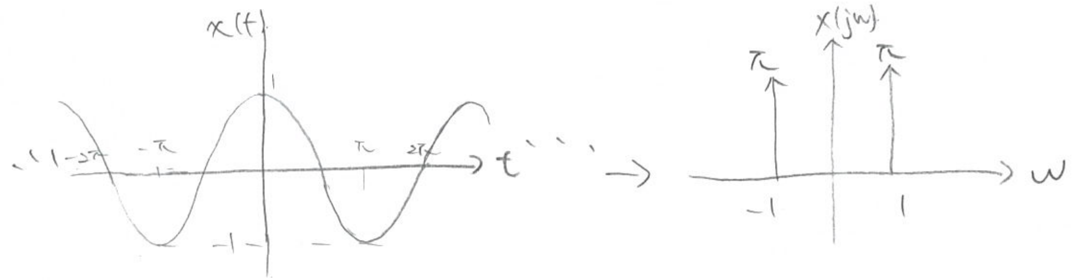
E 933200 signals and systems Fall 2016

Solutions to Midterm Exam II

1.

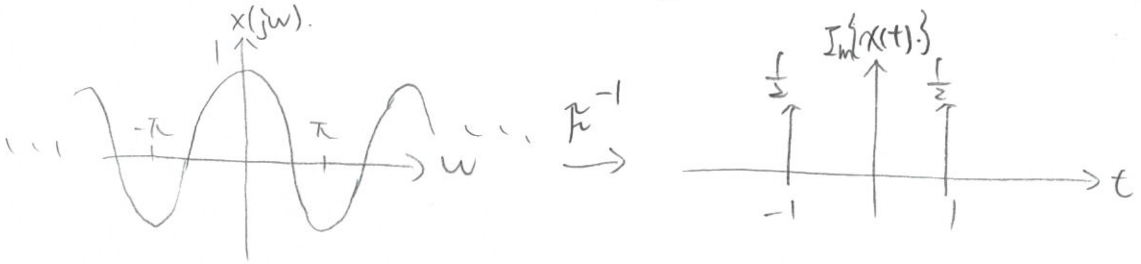
(a) $x(t) = \cos(t)$.



$X(jw) = \pi\,\delta(w-1) + \pi\,\delta(w+1)$

(b) $X(jw) = j\cos(w)$

$x(t) = \mathcal{F}^{-1}\{j\cos(w)\} = j\mathcal{F}^{-1}\{\cos(w)\} = \frac{j}{2}(\delta(t-1) + \delta(t+1))$



2.

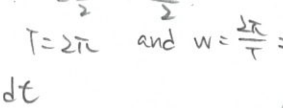
(a)

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T x(t)x^*(t)\,dt = \frac{1}{T}\int_T x(t)\sum_{k=-\infty}^{\infty} a_k^* e^{-jkwt}\,dt$$

$$= \sum_{k=-\infty}^{\infty} a_k^* \cdot \underbrace{\frac{1}{T}\int_T x(t)e^{-jkwt}\,dt}_{= a_k} = \sum_{k=-\infty}^{\infty} a_k^* a_k = \sum_{k=-\infty}^{\infty} |a_k|^2$$

(b) Let the periodic square wave defined over one period as

$$x(t) = \begin{cases} 1 & |t| < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |t| < \pi \end{cases}$$



where the period $T = 2\pi$ and $w = \frac{2\pi}{T} = 1$

$a_k = \frac{1}{T}\int_T x(t)\,e^{-jkwt}\,dt$

$a_0 = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\cdot dt = \frac{\pi}{2\pi} = \frac{1}{2}$

$k \neq 0 \quad a_k = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-jwkt}\,dt$

$= \frac{1}{2\pi}\cdot\frac{-1}{jwk}\left(e^{-jwk\cdot\frac{\pi}{2}} - e^{+jwk\cdot\frac{\pi}{2}}\right)$

$= \frac{1}{2\pi jwk}\cdot 2j\sin\frac{wk\pi}{2} = \frac{\sin\left(\frac{k\pi}{2}\right)}{\pi k}$

$$a_h^2 = \begin{cases} 0 & h = 2k \\ \frac{1}{h^2\pi^2} & h = 2k-1 \end{cases} \qquad k \in N$$

$\frac{1}{T}\int_T x^2(t)\,dt = \frac{1}{2}$

By parseval's relation, we have

$\frac{1}{2} = \sum_{k=-\infty}^{\infty} |a_k|^2 = 2\sum_{k=1}^{\infty} a_k^2 + \left(\frac{1}{2}\right)^2$

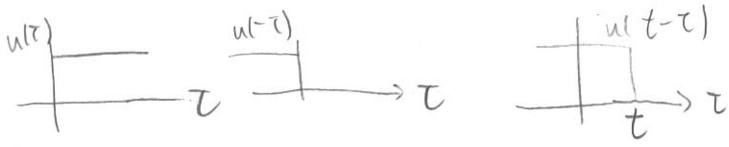
$\left(\frac{1}{2} - \frac{1}{4}\right)/2 = \sum_{k=1}^{\infty} \frac{1}{\pi^2}\frac{1}{(2k-1)^2}$

$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$

3.

(a)

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau) u(t-\tau) d\tau = \int_{-\infty}^{t} x(\tau) d\tau$$



$\because (u(t-\tau) = 0$ for $\tau > t)$

(b) $$U(jw) = \mathcal{F}\left\{ \frac{1 + sgn(t)}{2} \right\}$$

$$= \frac{1}{2}\left[2\pi\delta(w) + \frac{2}{jw}\right]$$

$$= \pi\delta(w) + \frac{1}{jw}$$

(c) $$\mathcal{F}\left\{ \int_{-\infty}^{t} x(\tau) d\tau \right\}$$

$$= \mathcal{F}\{ x(t) * u(t) \} = X(jw) \cdot U(jw) = X(jw)\left(\pi\delta(w) + \frac{1}{jw}\right)$$

$$= \frac{X(jw)}{jw} + \pi X(jw) \cdot \delta(w)$$

$$= \frac{X(jw)}{jw} + \pi X(0)\delta(w).$$

4. $X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$

$$D(j\omega) = \frac{dX(j\omega)}{d\omega} = \frac{d}{d\omega}\left(\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right)$$

$$= \int_{-\infty}^{\infty} x(t) \left(\frac{d}{d\omega} \cdot e^{-j\omega t} \right) dt$$

$$= \int_{-\infty}^{\infty} (x(t) \cdot -jt) e^{-j\omega t} dt$$

$$d(t) = \mathcal{F}^{-1}\{ D(j\omega) \} = -jt\, x(t)$$

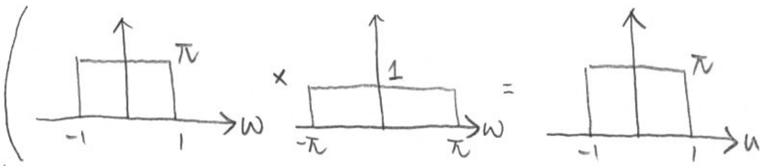
5. $y(t) = x(t) * h(t) \Rightarrow Y(j\omega) = \mathcal{F}\{x(t) * h(t)\} = X(j\omega) \cdot H(j\omega)$

$$X(j\omega) = \mathcal{F}\{\mathrm{sinc}(\tfrac{t}{\pi})\} = \pi \cdot \mathrm{rect}(\tfrac{\omega}{2})$$

$$\left(\mathcal{F}\{\mathrm{sinc}(\tfrac{Wt}{\pi})\} = \tfrac{\pi}{W} \cdot \mathrm{rect}(\tfrac{\omega}{2W}) \text{ , 令 } W = 1 \text{ , } \mathcal{F}\{\mathrm{sinc}(\tfrac{t}{\pi})\} = \pi\, \mathrm{rect}(\tfrac{\omega}{2}) \right)$$

$$H(j\omega) = \mathcal{F}\{\mathrm{sinc}(t)\} = \mathrm{rect}(\tfrac{\omega}{2\pi}) \quad \text{, 令 } W = \pi$$

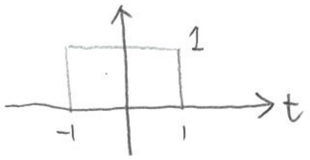
$$Y(j\omega) = X(j\omega) \cdot H(j\omega) = \pi \cdot \mathrm{rect}(\tfrac{\omega}{2})$$



$$\Rightarrow y(t) = \mathcal{F}^{-1}\{\pi \cdot \mathrm{rect}(\tfrac{\omega}{2})\} = \mathrm{sinc}(\tfrac{t}{\pi})$$

6.

$$x(t) = rect\left(\frac{t}{2}\right) = \begin{cases} 1 & , |t| \le 1 \\ 0 & , |t| > 1 \end{cases}$$



(a) $X(jw) = \int_{-\infty}^{\infty} rect\left(\frac{t}{2}\right) \cdot e^{-jwt}\, dt$

$$= \int_{-1}^{1} e^{-jwt}\, dt = \frac{1}{-jw} e^{-jwt} \Big|_{-1}^{1}$$

$$= \frac{1}{-jw}\left(e^{-jw} - e^{jw}\right) = \frac{1}{jw}\left(e^{jw} - e^{-jw}\right)$$

$$= \frac{1}{jw}\left(2j\, \sin(w)\right) = \frac{2\sin w}{w}$$

(b) $\int_{0}^{\infty} \frac{\sin w}{w}\, dw = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin w}{w}\, dw$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw) e^{jwt}\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin w}{w} e^{jwt}\, dw$$

Letting $t=0$,

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(w)}{w}\, dw$$

$$\Rightarrow 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin w}{w}\, dw$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin w}{w}\, dw = \frac{\pi}{2}$$