CALCULUS I LECTURE 7: GRAPH OF FUNCTIONS AND OPTIMIZATION PROBLEMS

1. Graph and critical points

In the previous lecture, we have learned the criteria for finding the local maximum, local minimum of a function. Namely, one can see whether a point a is a local max/min by checking the sign of its second derivatives. In this lecture, we will use this concept to solve some concrete problems.

Before we start, there are several remarks we should make here.

Remark 1.1. Suppose f is defined on (c, d) with no critical point in this open interval, then f' will not change the sign in this interval. Namely, f must be increasing or decreasing on (c, d).

This gives us the following implication.

Proposition 1.2. For any function f defined on $[c, d]$ with only one local maximum/minimum in this region. Then it is also an absolute maximum/minimum for it.

Also, we can see that, suppose we have the critical points a_i are isolated with $c < a_1 < a_2 < a_3, ... < a_n < d$. Suppose all of them are local maximum/minimum. Then we have

Proposition 1.3. If f attends a local maximum/minimum at a_i, then it attends a local minimum/maximum at a_{i+1}.

Here we didn't analyse the case when an **inflection point** presents. In any event, one should check, case-by-case, the behavior near a critical which fails the second derivative test. For example, if a is a critical point, one should check the sign of derivative $f'(x)$ near a. By this process, one will be able to know either the point a is maximum ($f'(x) > 0$ when $x < a$ and $f'(x) < 0$ when $x > a$, near the point a), minimum ($f'(x) < 0$ when $x < a$ and $f'(x) > 0$ when $x > a$, near the point a) or an inflection point ($f'(x)$ has the same sign near a, not equals a).

In general, no matter a point a is a critical point or not, we always have

Proposition 1.4. If $f''(a) > 0$, then f is convex near a; If $f''(a) < 0$, then f is concave near a.

2. Examples

All these examples come from the textbook.

Example 2.1. A farmer has 1200 m of fencing and wants to fence off a rectangular field that borders a straight river (He needs no fence along the river). What are the dimensions of the fields that has the largest area?

Let us denote by x the width of that area, so the depth of the area will be $\frac{1}{2}(1200 - x)$. Therefore, to maximize the area, we need to find the (absolute) maximum for $\frac{1}{2}(1200 - x)x$. This can be achieved by taking the derivatives for this function

$$f(x) = \frac{1}{2}(1200 - x)x; \tag{2.1}$$

$$f'(x) = \frac{1}{2}(1200 - 2x); \tag{2.2}$$

$$f''(x) = -1. \tag{2.3}$$

We can see that there is only one critical point $x = 600$ for f on $[0, 1200]$, with its second derivative negative. That gives us the local maximum, which is also absolute maximum. So the dimension will be 600×300.

Example 2.2. A cylindrical can is to be made to hold 1 L oil. Find the dimensions that will minimize the cost of metal to manufacture it.

We can suppose that the metal we use is proportional to the surface area of the can. Let the radius of this can be r and the height of the can be h. Then we have

$$\pi r^2 h = 1. \tag{2.4}$$

So $h = \frac{1}{\pi r^2}$. Now, the surface area of this can can be written as

$$V(r) = 2\pi r h + 2\pi r^2 = 2\frac{1}{r} + 2\pi r^2. \tag{2.5}$$

Taking derivatives for this function, one obtains

$$V'(r) = -\frac{2}{r^2} + 2\pi r; \tag{2.6}$$

$$V''(r) = \frac{4}{r^3} + 2\pi. \tag{2.7}$$

Therefore, there is only one critical point $r = \frac{1}{\sqrt[3]{\pi}}$ for V on $[0, \infty)$, where $V''(\frac{1}{\sqrt[3]{\pi}}) > 0$. This implies that $V((\frac{1}{\sqrt[3]{\pi}})$ is the local minimum as well as the absolute minimum for V on $[0, \infty)$.

Example 2.3. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

The distance of a point (x, y) to $(1, 4)$ is

$$\sqrt{(x-1)^2 + (y-4)^2}. \tag{2.8}$$

By the equation, we have

$$\sqrt{(x-1)^2 + (y-4)^2} = \sqrt{(\frac{1}{2}y^2 - 1)^2 + (y-4)^2}. \tag{2.9}$$

Notice that this is a nonnegative function. It is the same to minimize the square of this function to obtain the point minimize the original one. Therefore, one can consider

$$f(y) = (\frac{1}{2}y^2 - 1)^2 + (y-4)^2 = \frac{1}{4}y^4 - y^2 + 1 + y^2 - 8y + 16 \tag{2.10}$$
$$= \frac{1}{4}y^4 - 8y + 17$$

So we have

$$f'(y) = y^3 - 8; \tag{2.11}$$

$$f''(y) = 3y^2. \tag{2.12}$$

We have only one critical point $y = 2$ for f on $\mathbb{R}$, with its second derivative $f''(2) = 12 > 0$. So $f(2) = 5$ is absolute minimum of f. The distance we want, therefore, is $\sqrt{5}$.