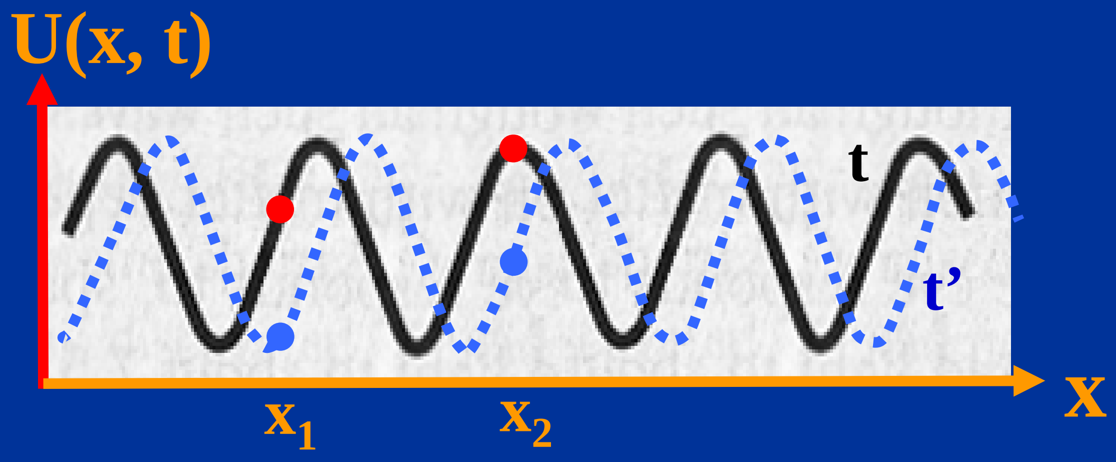
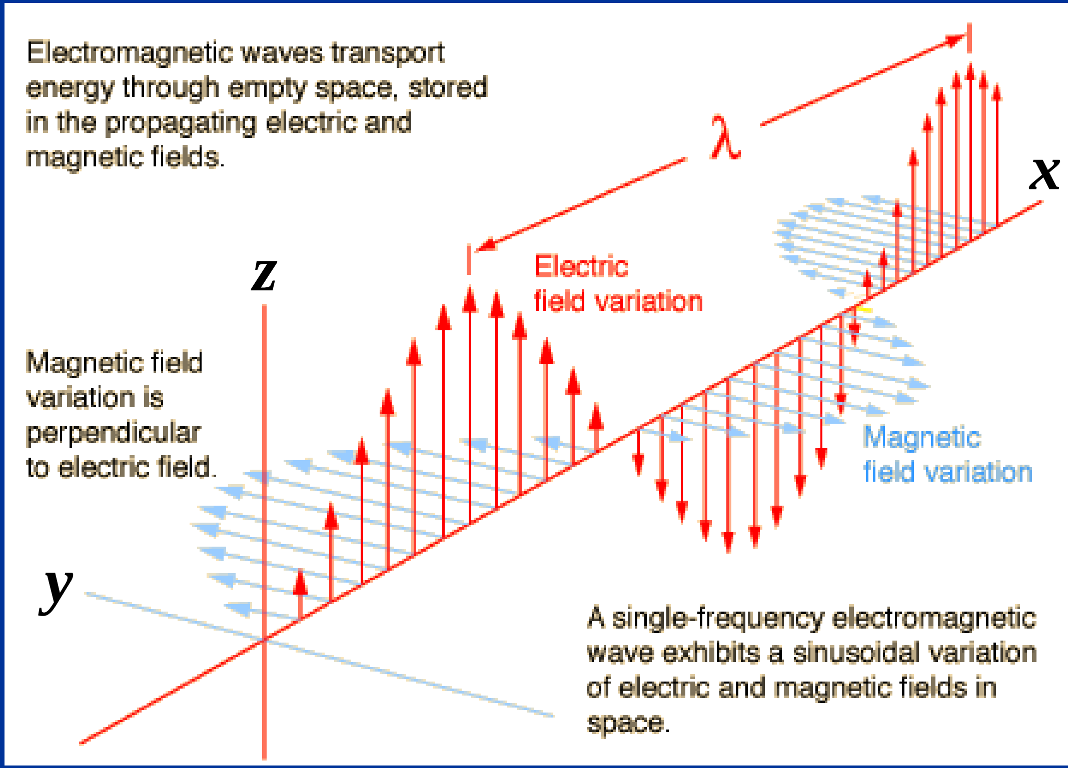
CH2 Particle Properties of Waves

2.1 Electromagnetic waves

Wave — disturbance (e.g. vibration) propagates in space. It is a function of space (location) and time, i.e. U=U(x, t)



Electromagnetic waves



$$E_z = E_{z0} \sin(kx + \omega t)$$

$$B_y = B_{y0} \cdot \sin(kx + \omega t)$$

Electromagnetic waves are lights — James J Maxwell (~1856)

Why did he think so?

Maxwell's Equations in vacuum (Differential form)

I. Gauss' law for electricity $\nabla \cdot E = 0$

II. Gauss' law for magnetism $\nabla \cdot B = 0$

III. Faraday's law of induction $\nabla x\, E = -\frac{\partial B}{\partial t}$

IV. Ampere's law $\nabla\ x\ B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

Do curl to both sides of III:

$$\nabla\ x\ (\nabla\ x\ E) = -\frac{\partial(\nabla\ x\ B)}{\partial t}$$

Substitute with IV:

$$\nabla\ x\ (\nabla\ x\ E) = -\ \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

(Here $\varepsilon_0 = 8.854\times10^{-12}$, or $\varepsilon_0 = 1/(4\pi\cdot9.00\times10^9)\ C^2/(N\cdot m^2)$ is the permittivity of free space, $\mu_0 = 4\pi\times10^{-7}\ T\cdot m/A$ is the permeability of free space)

In one dimensional case, i.e. electrical field along z-axis, propagation along x-axis:

$$\nabla\ x\ (\nabla\ x\ E) = -\frac{\partial^2 E_Z}{\partial x^2}$$

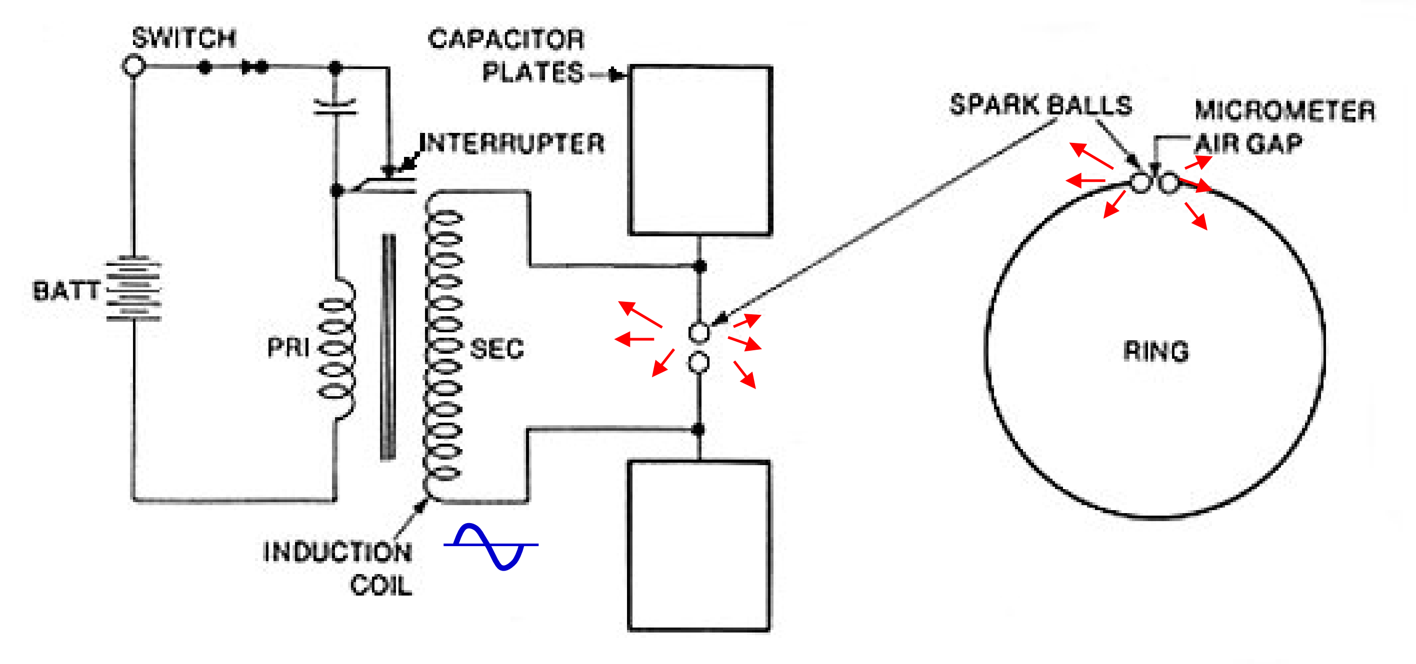
Therefore: $$\frac{\partial^2 E_Z}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_Z}{\partial t^2}$$

Compare to general wave equation $$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 U(x,t)}{\partial t^2}$$

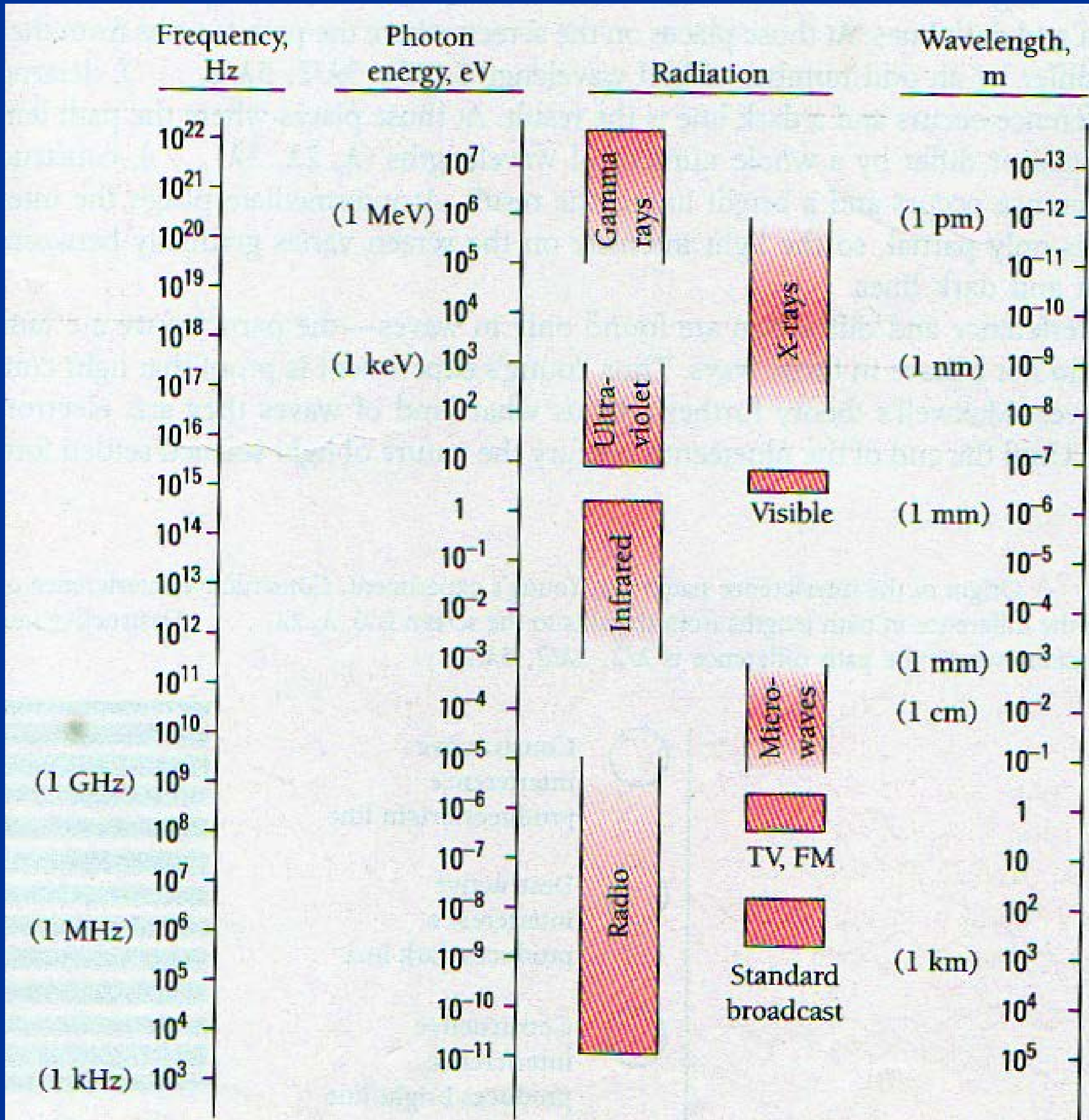
here V is the speed of wave propagation

So, electromagnetic waves travel at speed $$\frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3.00\times10^8 = c$$

In 1888, Hertz designed a brilliant set of experiments tested Maxwell's hypothesis.



Hertz's results showed that electromagnetic waves indeed exist and behave exactly as Maxwell had predicted !



Spectrum of Electromagnetic wave

EM waves visible to eye:

$f = (4.3 — 7.5)\times 10^{14}$ Hz

Wavelengths, $\lambda = (1/f)\cdot c$
698 — 400 nm

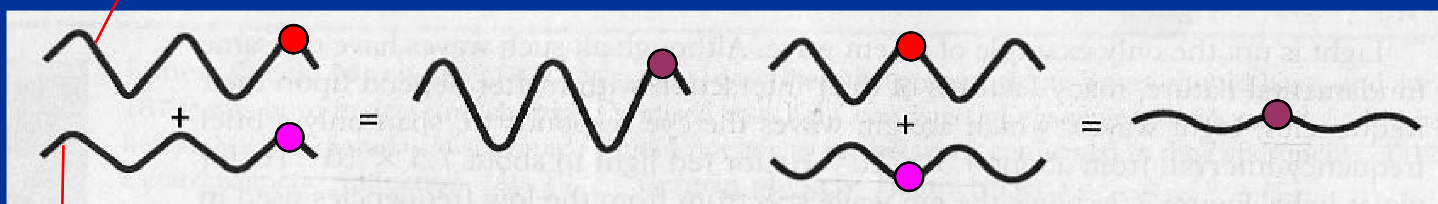
(Sometimes, waves are also described by angular frequency $\omega = 2\pi f$ and wave vector $\mathbf{k} = 2\pi/\lambda$.

So, $c = \lambda f = \omega/\mathbf{k}$)

A characteristic property of all waves is that they obey the **principle of superposition:**

When two or more waves of the same nature travel past a point at the same time, the instantaneous amplitude there is the sum of the instantaneous amplitudes of the individual waves.

$E=E_0 \sin(kx+\omega t)$



$E'=E_0' \sin(k'x+\omega' t)$

$kx+\omega t = \pi/2 \quad E= E_0$

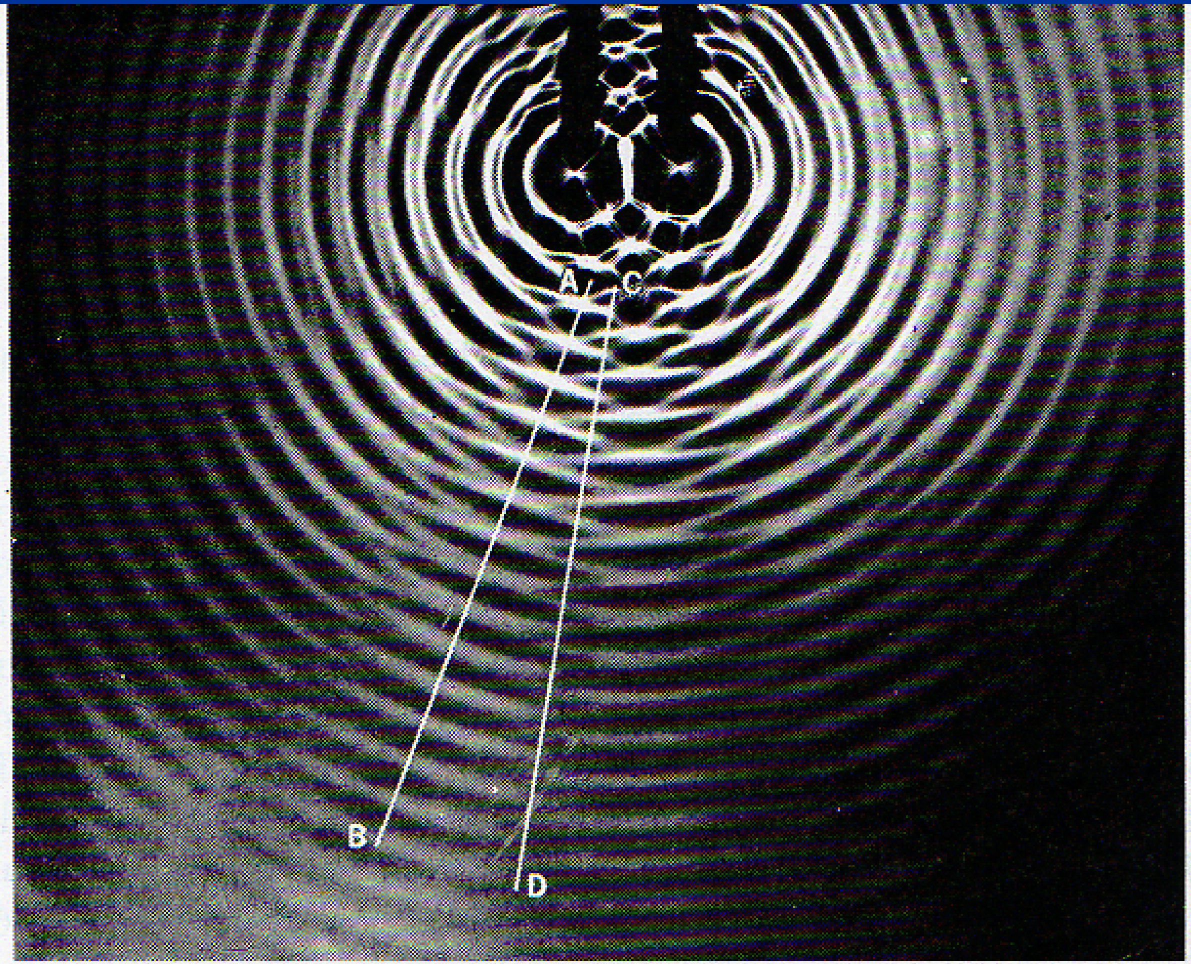
$k'x+\omega' t= 2\pi+\pi/2 \quad E'= E_0'$

$E + E'= E_0+ E_0'$

$kx+\omega t = \pi/2 \quad E= E_0$

$k'x+\omega' t= \pi+\pi/2 \quad E'= -E_0'$

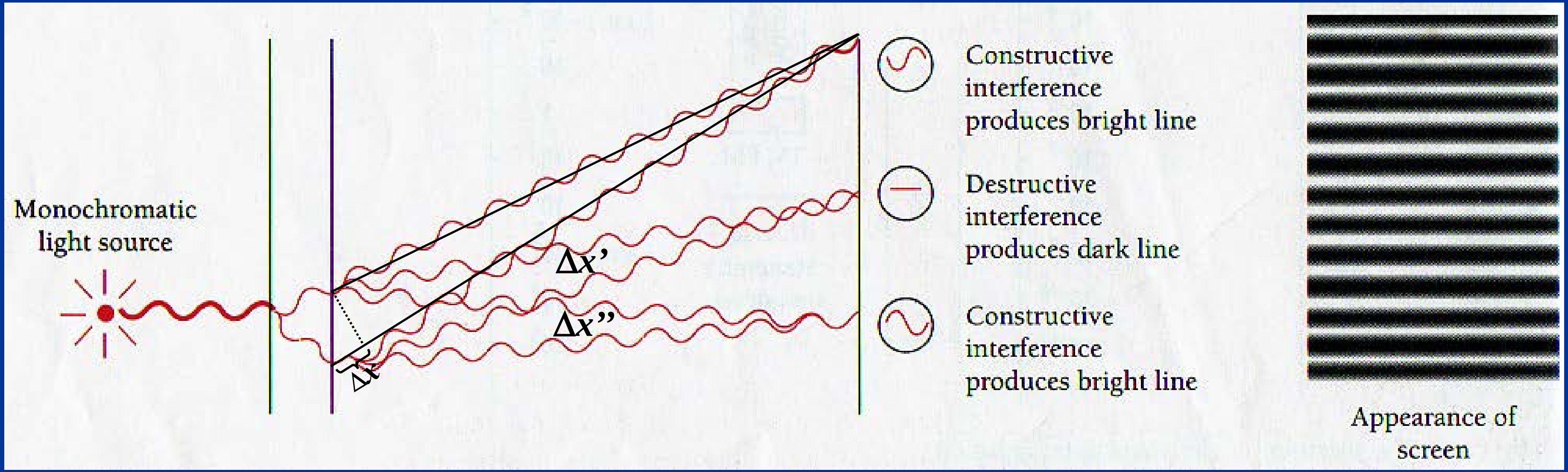
$E + E'= E_0 - E_0'$



The interference of water waves. Constructive interference occurs along the line *AB* and destructive interference occurs along the line *CD*.

Experiments of light interference

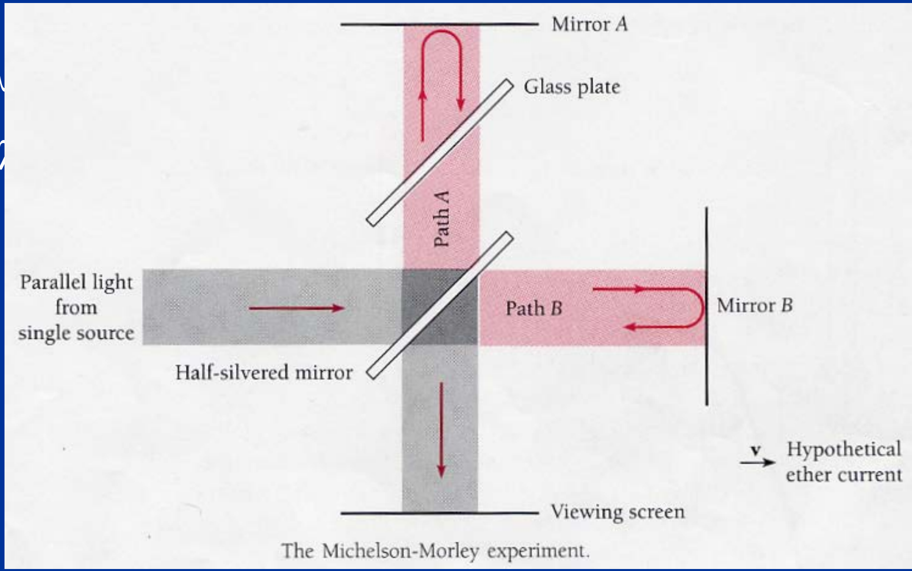
$$E=E_0 \sin(kx+\omega t)$$



Constructive: when light path difference = 0, λ

Destructive: when light path difference = 0, λ/2, 3λ

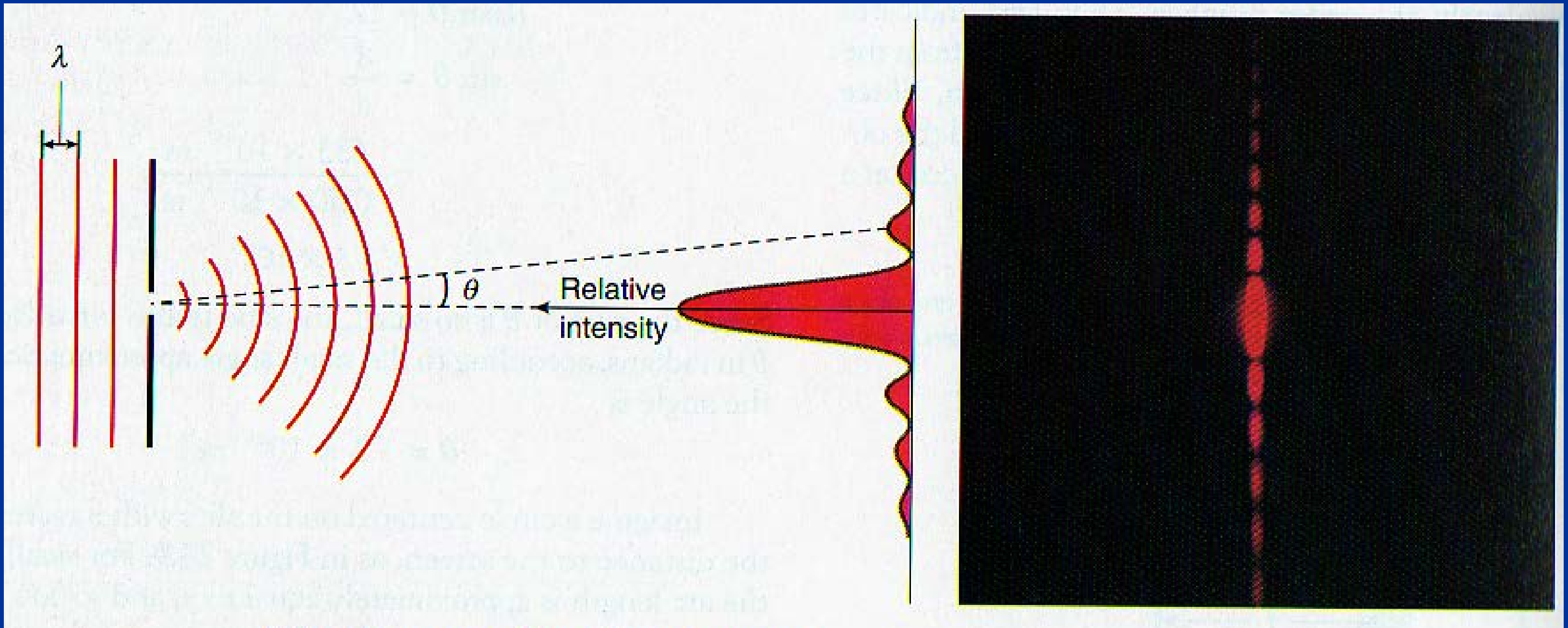
The **Michelson-Morley experiment** we showed in last Chapter is based on the same principle.



The Michelson-Morley experiment.

In addition to interference, there is another characteristic of waves called **diffraction**.

It happens when waves pass through a small hole whose diameter is comparable to the wavelength.



Diffraction pattern formed by a single slit.

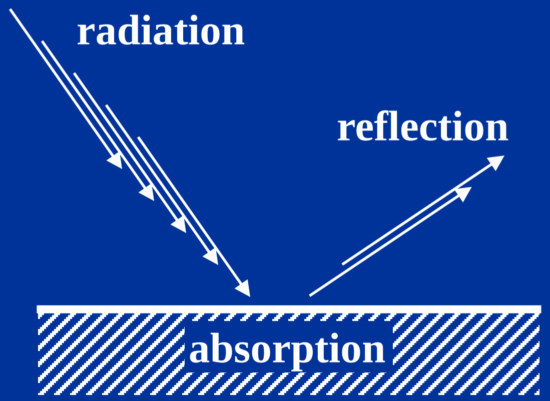
The interference and diffraction experiments confirmed the wave nature of light.

Thermal radiation

We are all familiar with the glow of a hot piece of metal, which gives off visible light whose colour various with the temperature of the metal, going from red to yellow to white as it becomes hotter and hotter.

Thermal radiation is the heat transfer by the emission of electromagnetic waves, which carry energy away from the emitting object

Blackbody Radiation

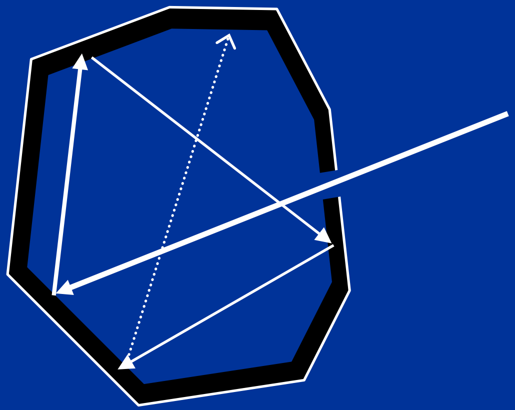


Blackbody: 100% absorption

Under thermal equilibrium:

Absorption = Emission

Hence, blackbody also has maximum emission when being heated up (most efficient emitter)



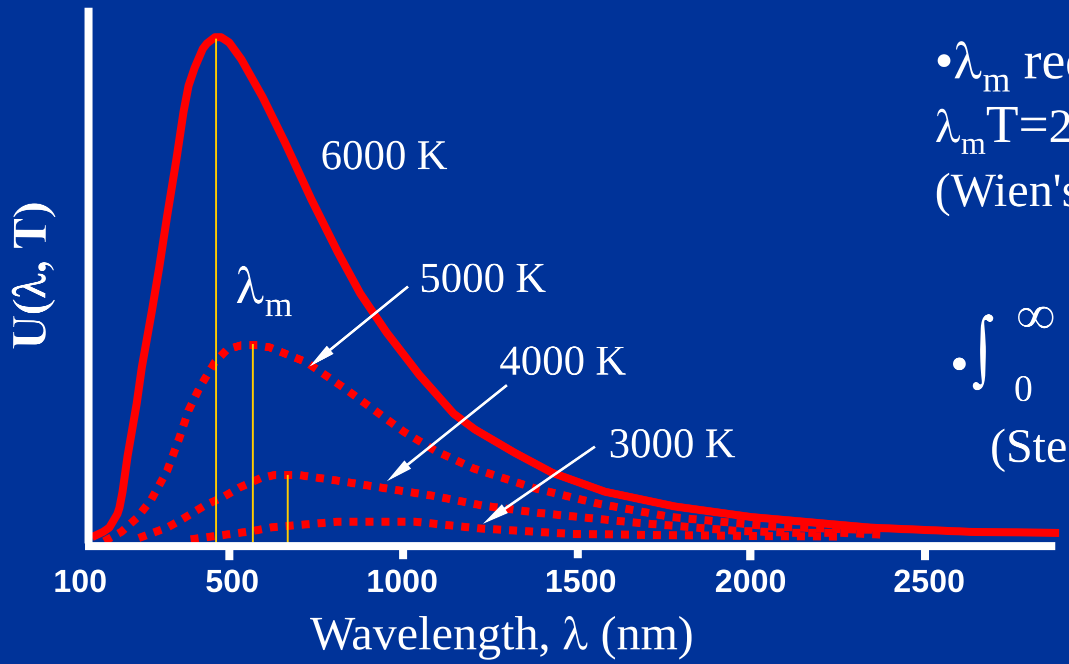
In practical experiments, black-body conditions are met by a cavity with black internal wall and a small opening

Blackbody-radiation or cavity-radiation refers to an object or system which absorbs all radiation incident upon it and re-radiates energy which is characteristic of this radiating system only.

At temperature T, when the cavity reaches thermal equilibrium, the energy density of cavity radiation, **U(**ν), is independent of the shape of cavity and the materials of its wall.

i.e. $U(\nu)=U(\nu, T)$

Experimental observations:



• λ_m reduces as T increases,
$\lambda_m T = 2.898\times10^{-3}$ (m·K)
(Wien's displacement law)

• $\int_0^\infty U(\lambda, T)d\lambda \propto T^4$
(Stefan-Boltzmann law)

Wien's displacement law can be used to estimate the surface temperature of stars:

$$T = \frac{2.898\times10^{-3}}{\lambda_m}$$

e.g. Sun: λ_m=500 nm
T=5800 K

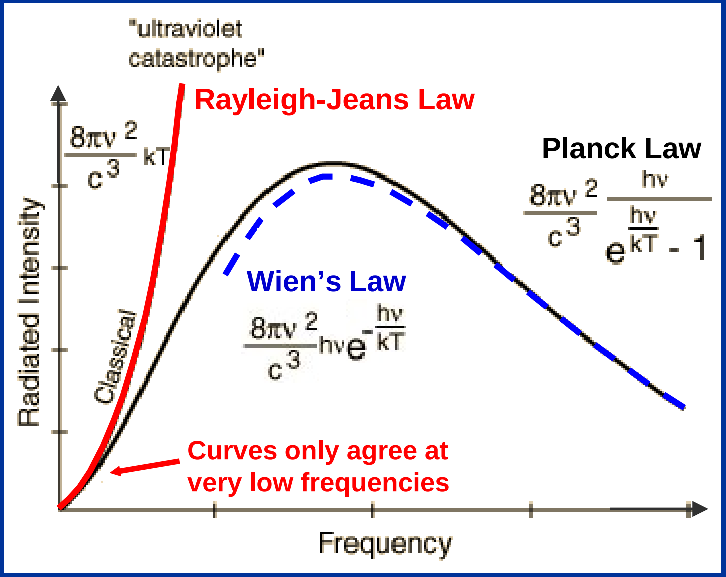
Cosmology — 3K Background Radiation (evidence of “Big-Bang Theory”)

A background radiation in microwave region is observed in every part of the sky. It has the same intensity and distribution of frequencies in all direction, independent of celestial objects. It shows the wavelength dependence of a blackbody at 3 K, which is interpreted as the electromagnetic remnant of the primitive fireball, stretched to long wavelengths by the expansion of the universe.

In classic physics, the blackbody emission was described by two contradictory theorems:

— Rayleigh-Jeans radiation law, valid in the low frequency region

— Wien’s law, valid in the high frequency region



Classic theory of cavity radiation

(Chapter 9, Section 9.5 in Beiser's book, 5th ed.)

Some starting points:

•Radiation is heat transfer by the emission of electromagnetic waves which carry energy away from the emitting object .

•It can be shown based on the thermodynamic argument that cavity radiation is isotropic, homogeneous, independent of the shape of cavity and the materials of its wall.

•It is not necessary to study the behaviour of the electrons in the walls in detail, instead attention can be focused on the behaviour of electromagnetic waves in the interior of the cavity.

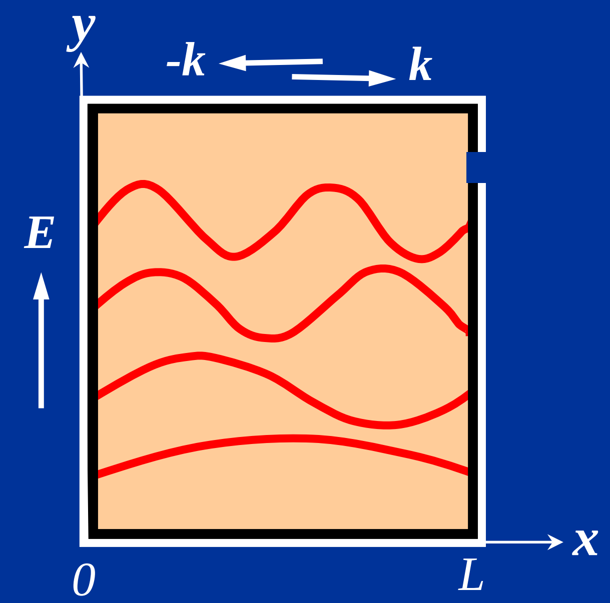
General procedure for calculation of U(ν, T)

1. Classic electromagnetic theory is used to show the radiation inside the cavity must be exist in the form of standing waves with the nodes on the surface of wall.
2. Work out the number, ***N***, of the standing waves at frequency ν (i.e. density of modes)
3. Work out the average energy, $\bar{u}$, of the waves at a given temperature
4. The number of modes times average energy, divided by the volume of cavity, gives the energy density: $\boxed{U(\nu, T) = \boldsymbol{N} \times \bar{u}}$

Think a cavity made of metal

<link>

E=0 at x=0, L; otherwise there is a current flowing on the wall !



Wave equation for one-dimensional electromagnetic waves:

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

The solutions have the plane-wave form:

$E=E_0\sin(kx)\cdot\sin(\omega t)$ with $c=\omega/k$

(here $\omega=2\pi\nu$ is the angular frequency, $k=2\pi/\lambda$ the wave vector)

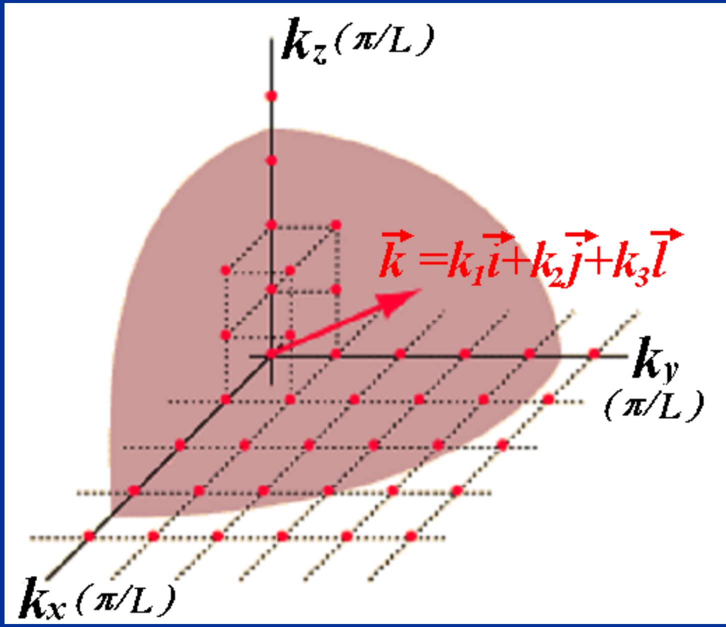
When $x=L$, $E=E_0\sin(kL)\cdot\sin(\omega t)=0$ for any t, hence $kL=n\pi$ ($n=1, 2, 3, \ldots$),

i.e. $\boxed{\mathbf{k=n\pi/L}}$

So, for the standing waves: $\mathbf{E=E_0\sin(n\pi x/L)\cdot\sin(\omega t)}$

Similarly, for electromagnetic waves in a 3-dimensional cubic cavity:

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$



and the solutions for the standing waves are:

$$\mathbf{E = E_0 \sin(k_1 x)\cdot\sin(k_2 y)\cdot\sin(k_3 z)\cdot\sin(\omega t)}$$

where, $\mathbf{k_1 = n_1\pi/L,\ \ k_2 = n_2\pi/L,\ k_3 = n_3\pi/L}$

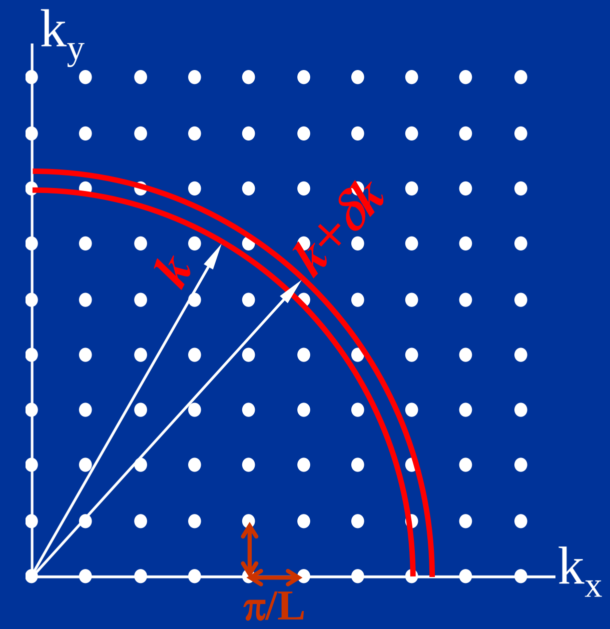
($\mathbf{n_1}$, $\mathbf{n_2}$, $\mathbf{n_3}$ are integers: 1, 2, 3, …)

and $\mathbf{c = \omega/k}$ **with (** $\mathbf{k^2 = k_1^2 + k_2^2 + k_3^2}$ **)**

Let $\vec{\mathbf{k}} = \mathbf{k_1}\vec{\mathbf{i}} + \mathbf{k_2}\vec{\mathbf{j}} + \mathbf{k_3}\vec{\mathbf{l}}$, which is the 3-dimensional wave-vector.

For each of **k**, there are many combinations (modes) of $\mathbf{k_1}$, $\mathbf{k_2}$, $\mathbf{k_3}$ to meet $\mathbf{k^2 = k_1^2 + k_2^2 + k_3^2}$ and all of them are on the first octant of a sphere with radius of **k.**

All the modes with k between k and k+δk fall inside the shell, whose internal radius is k and external radius k+δk.



Every mode (i.e. a spot in the picture) take a space of (π/L)×(π/L)×(π/L) on average

Hence the number of modes inside the shell is:

$$\delta N = \frac{1}{8}\times\frac{\frac{4}{3}\pi(k+\delta k)^3-\frac{4}{3}\pi k^3}{(\pi/L)\cdot(\pi/L)\cdot(\pi/L)} = \frac{L^3}{6\pi^2}\left[(k+\delta k)^3-k^3\right]$$

ignore $(\delta k)^2$ and higher orders, $(k+\delta k)^3 \approx k^3+3k^2\delta k$

So $\delta N = \frac{L^3}{2\pi^2}k^2\delta k$, replace k with $k = \omega/c = 2\pi\nu/c$, then $\delta N(\nu) = \frac{4\pi L^3}{c^3}\nu^2\delta\nu$

When $\delta\nu \to 0$, we have the density of modes: $\frac{dN(\nu)}{d\nu} = \frac{4\pi L^3}{c^3}\nu^2$ (i.e. the number of standing waves at a given frequency ν)

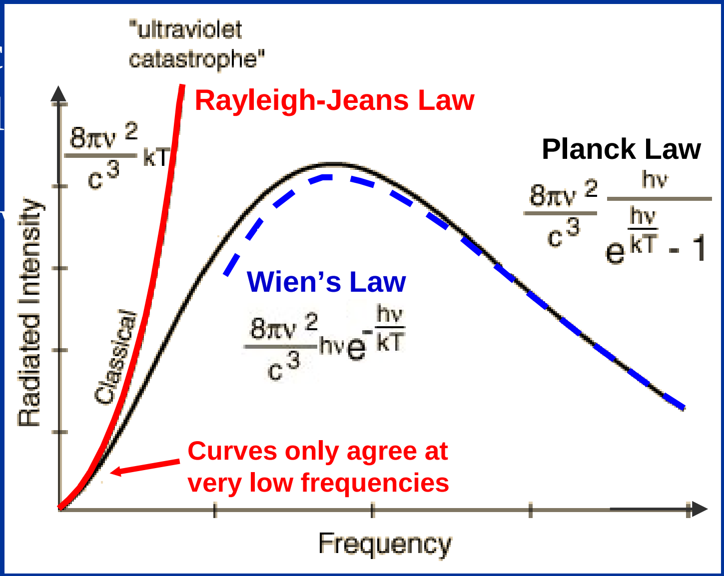
If the averaged energy for the standing waves is $\bar{u}$, then the energy density of blackbody radiation, i.e. energy per unit volume per frequency, is:

$$\boxed{U(\nu,T) = \frac{dN(\nu)}{d\nu}\times\bar{u}\times 2\times\frac{1}{L^3} = \frac{8\pi\nu^2}{c^3}\bar{u}}$$

(the factor of 2 is because each electro-magnetic wave has two polarisations)

Now we need to work out average energy of the radiation system — $\bar{u}$

According to classic theory, the energy of each elec
u, is the square of its amplitude and can be any val



The large numbers of the standing electromagnetic
cavity are the entities of a same kind, which follow

distribution: $e^{-\frac{u}{kT}}$

Hence the average energy is: $\bar{u} = \dfrac{\int_0^\infty u e^{-\frac{u}{kT}} du}{\int_0^\infty e^{-\frac{u}{kT}} du}$

Let $x=-u/kT$ the denominator: $\int_0^\infty e^{-\frac{u}{kT}} du = kT\int_0^\infty e^{-x}dx = -kT e^{-x}\Big|_0^\infty = kT$

the numerator: $\int_0^\infty u e^{-\frac{u}{kT}} du = (kT)^2\int_0^\infty x e^{-x}dx = (kT)^2\left(x e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}dx\right) = (kT)^2$

So, $\bar{u} = \dfrac{(kT)^2}{kT} = kT$,

$$\boxed{U(\nu,T) = \frac{8\pi\nu^2}{c^3}\bar{u} = \frac{8\pi\nu^2}{c^3}kT}$$

Rayleigh-Jeans radiation law

Planck's Assumption (1901):

• The energy exchange between the electrons in the cavity wall and electromagnetic waves only occur in discrete amounts.

• The minimum unit of energy exchange, which is called as "quantum of energy", is ***hv***. Here, ***h***=6.626×10^{-34} (J·s) is the Plank's constant, ***v*** the frequency of electromagnetic waves.

i.e. $\boxed{u = nh\nu,\ \ n=0, 1, 2, 3, \ldots}$

So, the average energy is now: $$\bar{u} = \frac{\sum_{n=0}^{\infty} nh\nu e^{-\frac{nh\nu}{kT}}}{\sum_{n=0}^{\infty} e^{-\frac{nh\nu}{kT}}}$$

compare to classic

$$\bar{u} = \frac{\int_0^{\infty} ue^{-\frac{u}{kT}}du}{\int_0^{\infty} e^{-\frac{u}{kT}}du}$$

Let $Y=\sum_{n}^{\infty} e^{-nh\nu x}$, $x=1/kT$, then

$$\bar{u} = -\frac{d}{dx}\ln Y$$

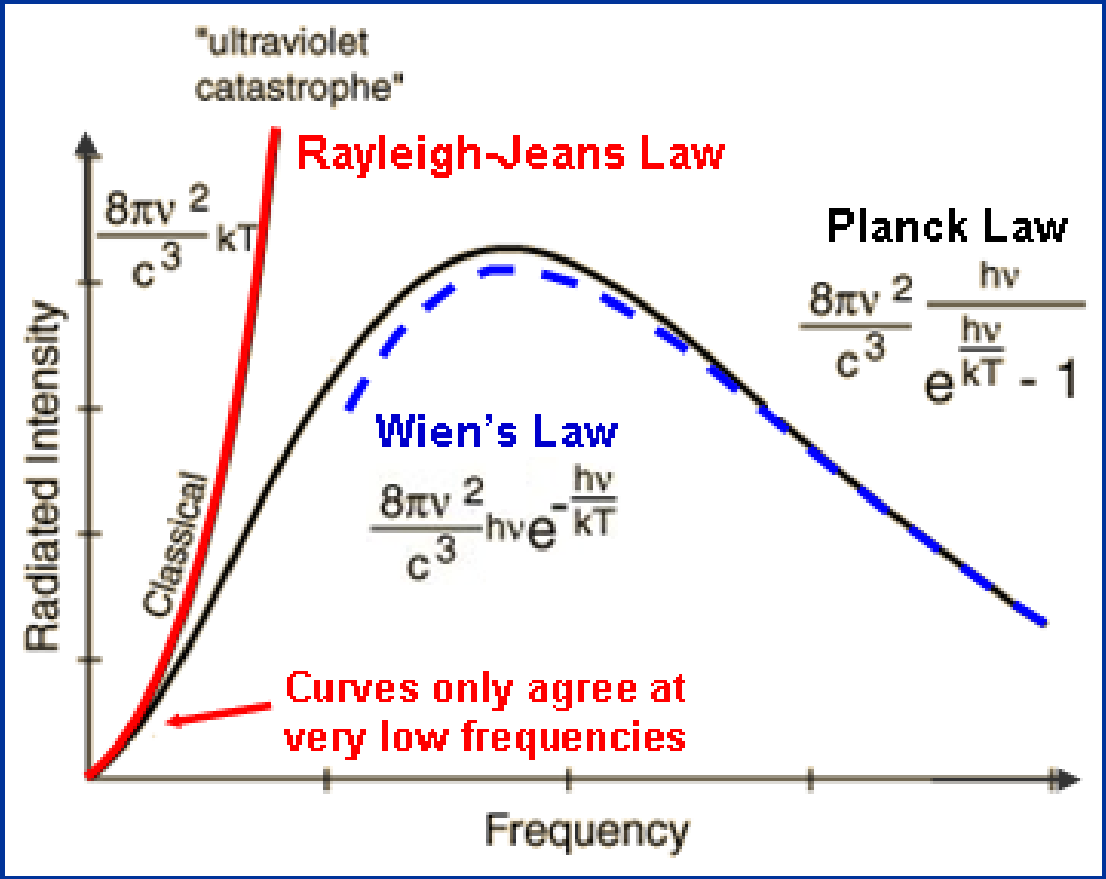
$$Y=\sum_{n}^{\infty} e^{(-h\nu x)n} = \lim_{n\to\infty} \frac{1-e^{(-h\nu x)(n+1)}}{1-e^{(-h\nu x)}} = \frac{1}{1-e^{(-h\nu x)}}$$

$$\bar{u} = -\frac{d}{dx}\ln\left[\frac{1}{1-e^{-h\nu x}}\right] = \frac{d}{dx}\ln(1-e^{-h\nu x}) = \frac{h\nu e^{-h\nu x}}{1-e^{-h\nu x}} = \frac{h\nu}{e^{h\nu x}-1}$$

$x=1/kT$, $\bar{u} = \dfrac{h\nu}{e^{\frac{h\nu}{kT}}-1}$,

$$\boxed{U(\nu,T) = \frac{8\pi\nu^2}{c^3}\bar{u} = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\frac{h\nu}{kT}}-1}}$$

Planck Law



$$\boxed{u = nh\nu,\quad n = 0, 1, 2, 3, \ldots}$$

Plank's law $$U(\nu,T)=\frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\frac{h\nu}{kT}}-1}$$

Rayleigh-Jeans and Wien's laws are special cases of Plank's law

We know: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

If x is small, $e^x \approx 1 + x$, and so for $h\nu/kT \ll 1$ we have

$$\frac{1}{e^{h\nu/kT}-1} \approx \frac{1}{1+\frac{h\nu}{kT}-1} \approx \frac{kT}{h\nu} \qquad h\nu \ll kT$$

Thus at low frequencies Planck's formula becomes

$$U(\nu,\mathrm{T}) \approx \frac{8\pi h}{c^3}\nu^3\left(\frac{kT}{h\nu}\right)d\nu \approx \frac{8\pi kT}{c^3}\nu^2$$

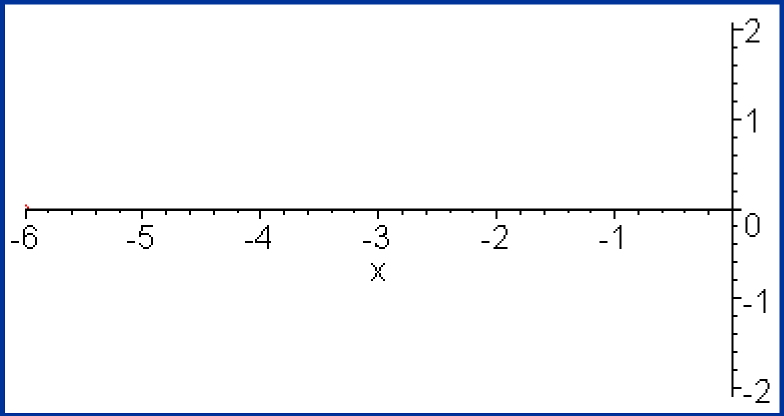
which is the Rayleigh-Jeans formula.

When $h\nu/kT \gg 1$, then

$$e^{h\nu/kT} \gg 1 \qquad e^{h\nu/kT}-1 \approx e^{h\nu/kT}$$

$$\frac{1}{e^{h\nu/kT}-1} \approx e^{-h\nu/kT} \qquad U(\nu,\mathrm{T}) \approx \frac{8\pi\nu^2}{c^3}h\nu e^{-\frac{h\nu}{kT}}$$

which is Wien's law.



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