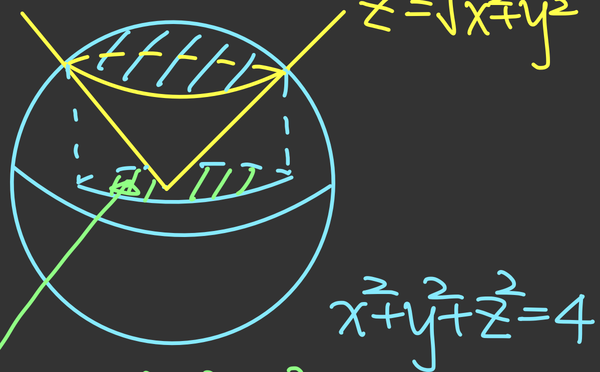
Find a parametrization for the part of the sphere $x^2+y^2+z^2=4$ that lies above the cone $z=\sqrt{x^2+y^2}$



$\Omega=\{(x,y)\mid x^2+y^2\leq 2\}$

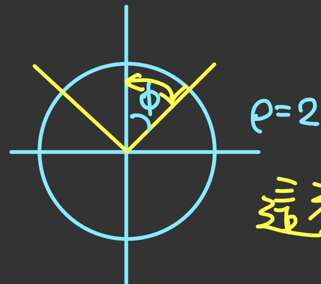
The intersection of $z=\sqrt{x^2+y^2}$ and $x^2+y^2+z^2=4$

is $4=x^2+y^2+(\sqrt{x^2+y^2})^2$

$\Rightarrow x^2+y^2=2$

1° $r(u,v)=\langle u,v,\sqrt{u^2+v^2}\rangle \quad (u,v)\in\Omega$

2° $r(\phi,\theta)=\langle 2\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ 2\cos\phi\rangle$



$(\phi,\theta)\in\Omega=\{(\phi,\theta)\mid 0\leq\phi\leq\frac{\pi}{4},\ 0\leq\theta\leq 2\pi\}$

這裡我們需要去看ϕ角的範圍

第一種是把Ω當底面，z座標當高

第二種是直接用球座標將半徑固定，可以直接得到球面的參數式

Evaluate $\iint_S xyz\,dS$, S is the cone with parametric equations

$x = u\cos v,\ y = u\sin v,\ z = u,\ 0 \le u \le 1,\ 0 \le v \le \frac{\pi}{2}$

$$r(u,v) = \langle u\cos v,\ u\sin v,\ u\rangle,\quad (u,v) \in \Omega := [0,1] \times [0,\tfrac{\pi}{2}]$$

$$r_u = \langle \cos v, \sin v, 1\rangle,\quad r_v = \langle -u\sin v, u\cos v, 0\rangle$$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle -u\cos v, -u\sin v, u\rangle$$

$$|r_u \times r_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2}\,u \quad (\because u \in [0,1])$$

$$\iint_S xyz\,dS = \iint_\Omega \underbrace{(u\sin v)}_{x}\underbrace{(u\cos v)}_{y}\underbrace{u}_{z}\cdot \underbrace{|r_u \times r_v|\,du\,dv}_{dS}$$

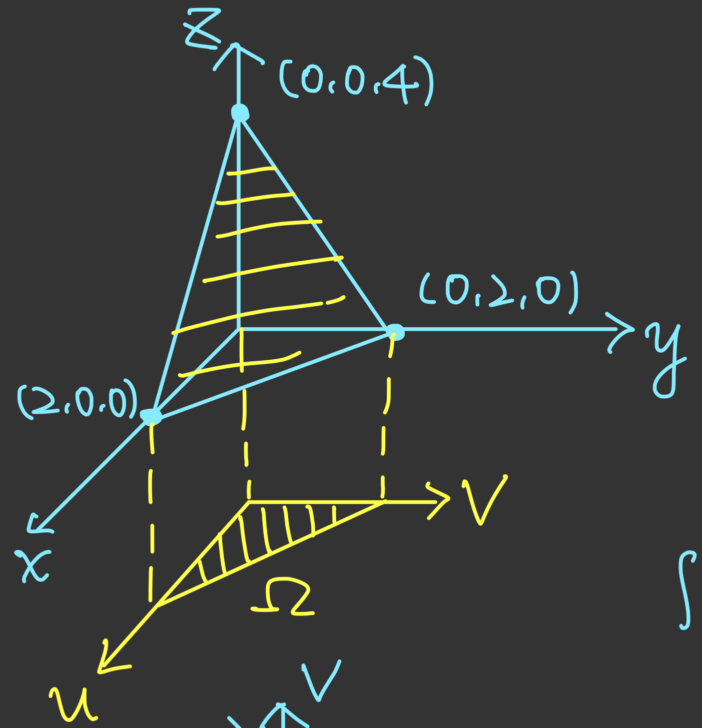
$$= \int_0^{\pi/2}\int_0^1 \sqrt{2}\,u^4 \sin v\cos v\,du\,dv$$

$$= \sqrt{2}\cdot \int_0^1 u^4\,du \cdot \int_0^{\pi/2} \sin v\cos v\,dv$$

$$= \sqrt{2}\cdot \frac{1}{5}u^5\Big|_0^1 \cdot \frac{1}{2}\sin^2 v\Big|_0^{\pi/2}$$

$$= \sqrt{2}\cdot\frac{1}{5}\cdot\frac{1}{2} = \frac{\sqrt{2}}{10}$$

Evaluate $\iint_S xz\,dS$. S is the part of the plane $2x+2y+z=4$ that lies in the first octant.



$$r(u,v)=\langle u, v, 4-2u-2v\rangle,\ (u,v)\in\Omega$$

$$r_u=\langle 1,0,-2\rangle,\ r_v=\langle 0,1,-2\rangle$$

$$r_u\times r_v=\begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{vmatrix}=\langle 2,2,1\rangle$$

$$|r_u\times r_v|=3$$

$$\iint_S xz\,dS=\iint_\Omega \underbrace{u}_{x}\,\underbrace{(4-2u-2v)}_{z}\cdot\underbrace{3\,dA}_{dS}$$

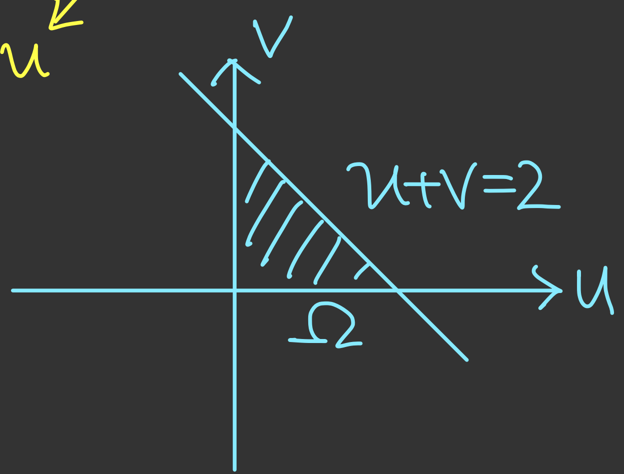
$$=3\int_0^2\int_0^{2-u}(4u-2u^2-2uv)\,dv\,du$$

$$=3\int_0^2 4u(2-u)-2u^2(2-u)-u(2-u)^2\,du$$

$$=3\int_0^2(8u-4u^2-4u^2+2u^3-4u+4u^2-u^3)\,du$$

$$=3\left(\frac{1}{4}u^4-\frac{4}{3}u^3+2u^2\right)\Big|_0^2$$

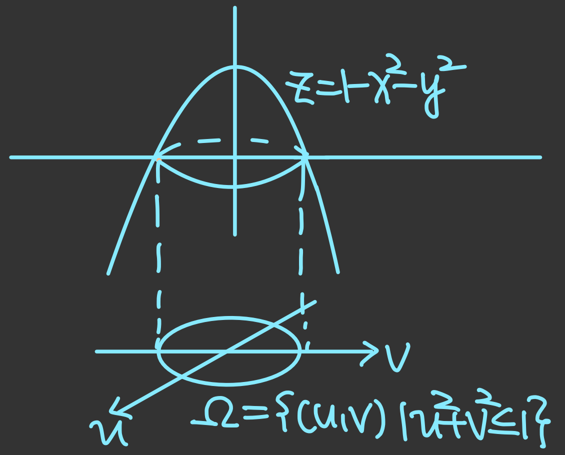
$$=4$$



Let $\vec{F}(x,y,z) = \langle x^2 \sin z, y^2, xy \rangle$. S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy plane, oriented upward.

Evaluate $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$

Method 1 Direct Computation



$$r(u,v) = \langle u, v, 1-u^2-v^2 \rangle, \ (u,v) \in \Omega$$

$$r_u = \langle 1, 0, -2u \rangle, \ r_v = \langle 0, 1, -2v \rangle$$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = \langle 2u, 2v, 1 \rangle$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 \sin z & y^2 & xy \end{vmatrix} = \langle x, -y + x^2 \cos z, 0 \rangle$$

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_\Omega \langle u, -v + u^2 \cos(1-u^2-v^2), 0 \rangle \cdot \langle 2u, 2v, 1 \rangle \, dA$$

$$= \iint_\Omega 2u^2 - 2v^2 + u^2 v \cos(1-u^2-v^2) \, dA$$

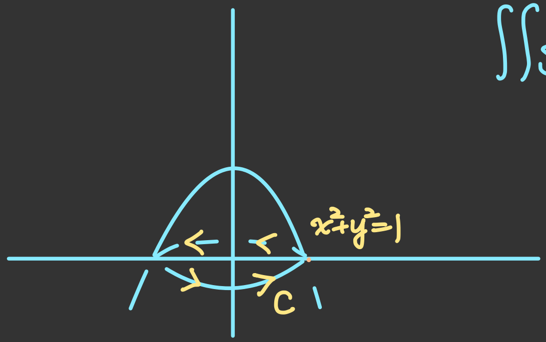
$$= \int_0^{2\pi} \int_0^1 2r^2 \cos^2\theta - 2r^2 \sin^2\theta + r^3 \cos^2\theta \sin\theta \cos(1-r^2) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r^3\cos 2\theta \, dr\, d\theta + r^4\cos(1-r^2)\cos^2\theta\sin\theta \, dr\, d\theta$$

$$= \underbrace{\int_0^{2\pi}\cos 2\theta \, d\theta}_{=0} \cdot \int_0^1 2r^3 \, dr + \underbrace{\int_0^{2\pi}\cos^2\theta\sin\theta \, d\theta}_{=0} \cdot \int_0^1 r^4\cos(1-r^2)\, dr$$

$$= 0$$

Method 2 Use Stoke's theorem



$C = \vec{r}(\theta) = \langle \cos\theta, \sin\theta, 0\rangle$

$\theta \in [0, 2\pi]$

$$\iint_S \text{curl}\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} \langle 0, \sin^2\theta, \cos\theta\sin\theta\rangle \cdot \langle -\sin\theta, \cos\theta, 0\rangle \, d\theta$$

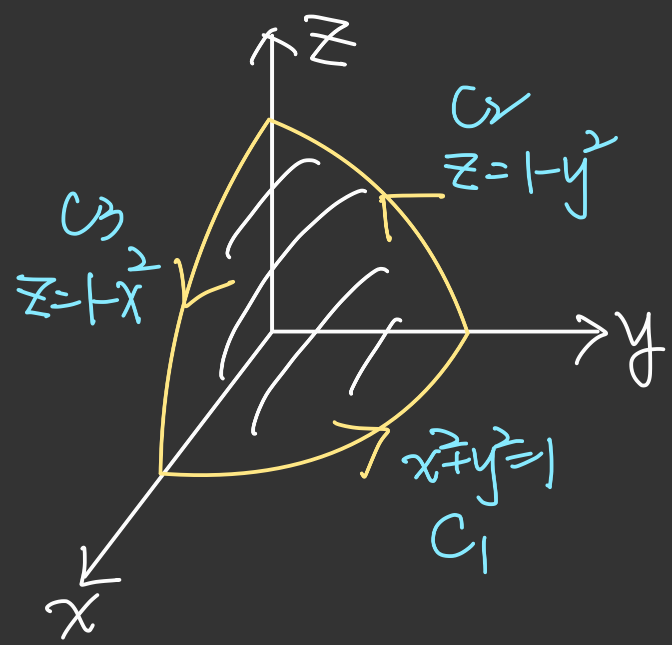
$$= \int_0^{2\pi} \sin^2\theta\cos\theta \, d\theta$$

$$= \frac{1}{3}\sin^3\theta \Big|_0^{2\pi}$$

$$= 0$$

Let $\vec{F}(x,y,z) = \langle xy, yz, zx \rangle$. C is the boundry of the part of the parabolid $z = 1 - x^2 - y^2$ in the first octant.

Evaluate $\int_C \vec{F} \cdot d\vec{r}$.



$$\int_C \vec{F}\cdot d\vec{r} = -\frac{1}{3} - \frac{1}{4} - \frac{2}{3} + \frac{2}{5}$$
$$= -\frac{17}{20}$$

Method 1 Direct computation

$C_1 = r_1(\theta) = \langle \cos\theta, \sin\theta, 0 \rangle, \ \theta \in [0, \frac{\pi}{2}]$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \langle \cos\theta\sin\theta, 0, 0 \rangle \langle -\sin\theta, \cos\theta, 0 \rangle d\theta$$
$$= \int_0^{\frac{\pi}{2}} -\cos\theta\sin^2\theta \, d\theta = -\frac{1}{3}\sin^3\theta \Big|_0^{\pi/2} = -\frac{1}{3}$$

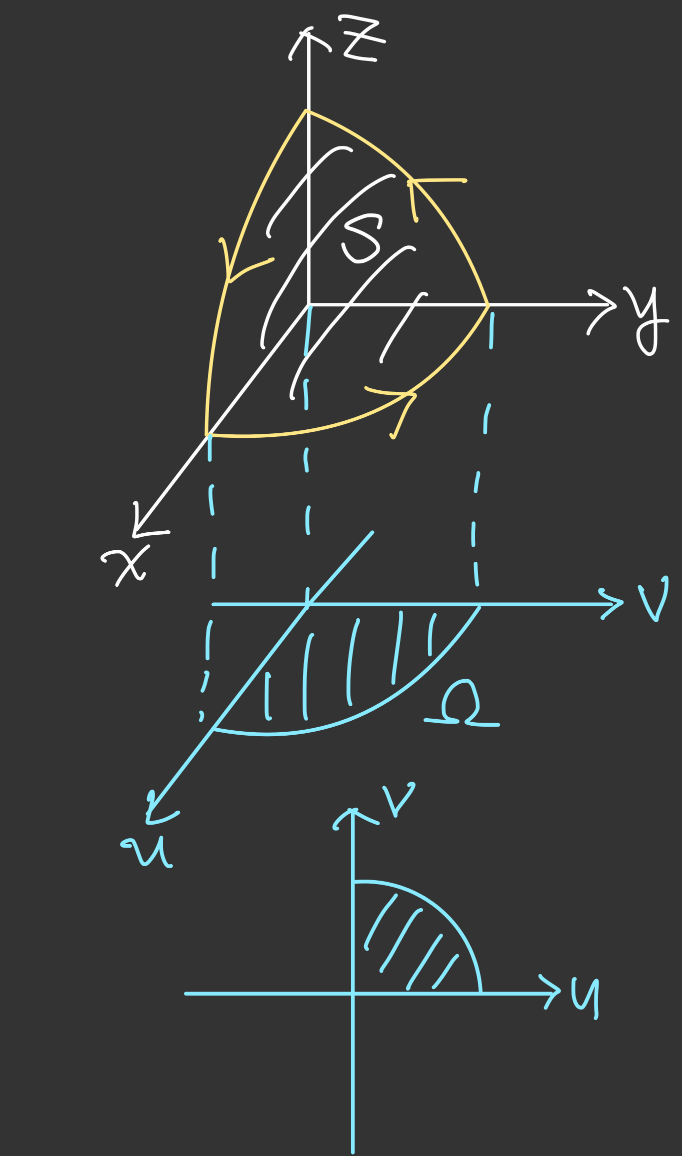
$C_2: r_2(t) = \langle 0, 1-t, 1-(1-t)^2 \rangle, \ t \in [0,1]$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 \langle 0, (1-t)(2t-t^2), 0 \rangle \langle 0, -1, 2-2t \rangle dt$$
$$= \int_0^1 (-t^3 + 3t^2 - 2t)\, dt = \left(-\frac{t^4}{4} + t^3 - t^2\right)\Big|_0^1 = \frac{-1}{4}$$

$C_3 = r_3(t) = \langle t, 0, 1-t^2 \rangle, \ t \in [0,1]$

$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_0^1 \langle 0, 0, t(1-t^2) \rangle \langle 1, 0, -2t \rangle dt$$
$$= \int_0^1 (-2t^2 + 2t^4)\, dt = \left(-\frac{2}{3}t^3 + \frac{2}{5}t^5\right)\Big|_0^1$$
$$= -\frac{2}{3} + \frac{2}{5}$$

Method 2 Use stoke's theorem



$$r(u,v) = \langle u, v, 1-u^2-v^2 \rangle, \ (u,v) \in \Omega$$

$$r_u \times r_v = \langle 2u, 2v, 1 \rangle$$

$$\text{curl}\vec{F} = \langle -y, -z, -x \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{S}$$

$$= \iint_\Omega \langle -v, u^2+v^2-1, -u \rangle \cdot \langle 2u, 2v, 1 \rangle \, dA$$

$$= \iint_\Omega (-2uv + 2u^2v + 2v^3 - 2v - u) \, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 (-2r^2\cos\theta\sin\theta + 2r^3\cos^2\theta\sin\theta + 2r^3\sin\theta - 2r\sin\theta - r\cos\theta) \, r \, dr \, d\theta$$

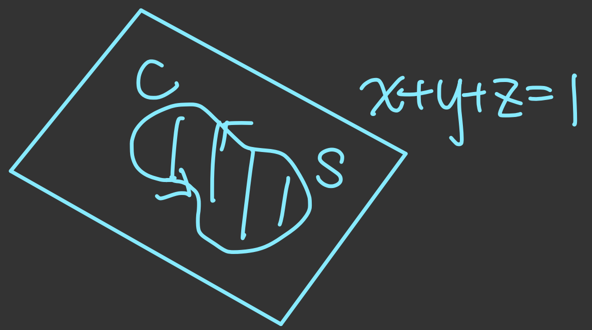
$$= \int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos\theta\sin\theta + \frac{2}{5}\sin\theta - \frac{2}{3}\sin\theta - \frac{1}{3}\cos\theta \, d\theta$$

$$= \left(-\frac{1}{4}\sin^2\theta - \frac{2}{5}\cos\theta + \frac{2}{3}\cos\theta - \frac{1}{3}\sin\theta\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \left(-\frac{1}{4} - \frac{1}{3}\right) - \left(-\frac{2}{5} + \frac{2}{3}\right)$$

$$= \frac{-17}{20}$$

Let C be a simple closed smooth curve that lies in the plane $x+y+z=1$.
Show that the line integral $\int_C z\,dx - 2x\,dy + 3y\,dz$ depends only on the area of the region enclosed by C



Let $\vec{F}(x,y,z) = \langle z, -2x, 3y \rangle$

$\text{curl}\,\vec{F} = \langle 3, 1, -2 \rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$

$$= \iint_S \langle 3,1,-2\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle dS$$

$$= \iint_S \frac{2}{\sqrt{3}}\, dS$$

$$= \frac{2}{\sqrt{3}}\ (\text{surface of area})$$

$r(u,v) = \langle u, v, 1-u-v \rangle$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= \langle 1, 1, 1 \rangle$$

$|r_u \times r_v| = \sqrt{3}$

Evaluate $\int_C (y+\sin x)\,dx + (z^2+\cos y)\,dy + x^3\,dz$,

where C is the curve $r(t) = \langle \sin t, \cos t, \sin 2t\rangle$, $0 \le t \le 2\pi$

如果直接計算 $\int_0^{2\pi} \big((\cos t+\sin(\sin t))\cdot\cos t + (\sin^2 2t+\cos(\cos t))\cdot(-\sin t) + \sin^3 t\cdot 2\cos 2t\big)\,dt$

$$= \int_0^{2\pi} \cos^2 t + \sin(\sin t)\cos t - \sin t(4\sin^2 t\cos^2 t) - \cos(\cos t)\sin t + 2\sin^3 t\cos^2 t - 2\sin^5 t\,dt$$

計算略複雜,但還是可以算. 我們注意到 $z=\sin 2t = 2\sin t\cos t = 2xy$

曲線落在曲面 $z=2xy$ 上

$\text{curl}\,\vec{F} = \langle -2z, -3x^2, -1\rangle$

$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

$r(u,v) = \langle u, v, 2uv\rangle$

Ω

$u^2+v^2 \le 1$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2v \\ 0 & 1 & 2u \end{vmatrix} = \langle -2v, -2u, 1\rangle$$

向上的法向量

$$= \iint_\Omega \langle -4uv, -3u^2, -1\rangle \cdot \langle 2v, 2u, -1\rangle\,dA$$

$$= \iint_\Omega (-4uv^2 - 6u^3 + 1)\,dA$$

$$= \int_0^{2\pi}\int_0^1 (-4r^3\cos\theta\sin^2\theta - 6r^3\cos^3\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(-\frac{4}{5}\cos\theta\sin^2\theta - \frac{6}{5}\cos^3\theta + \frac{1}{2}\right)d\theta$$

$$= \pi$$

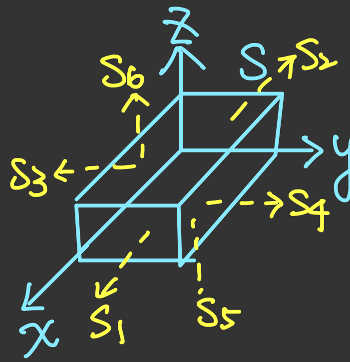
但因為 $\langle \sin\theta, \cos\theta, \sin 2\theta\rangle$ 是順時鐘轉的,

所以使用 stokes 時要取向下的法向量 $\langle 2v, 2u, -1\rangle$

Let $\vec{F}(x,y,z) = \langle xye^{z}, xy^2z^3, -ye^{z} \rangle$

S is the surface of the box bounded by coordinate planes and the plane $x=3$, $y=2$, $z=1$. Evaluate $\iint_S \vec{F}\cdot d\vec{S}$.

Method 1 Direct computation



Consider $S = \bigcup_{k=1}^{6} S_k$

S_1 : $r_1(u,v) = \langle 0, u, v \rangle$, $(u,v) \in D_1 := [0,2] \times [0,1]$

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{D_1} \langle 0, 0, -ue^{v} \rangle \cdot \underbrace{\langle -1, 0, 0 \rangle}_{\text{outward}} dA = 0$$

S_2 : $r_2(u,v) = \langle 3, u, v \rangle$, $(u,v) \in D_1$

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_{D_1} \langle 3ue^{v}, 3u^2v^3, -ue^{v} \rangle \cdot \underbrace{\langle 1, 0, 0 \rangle}_{\text{outward}} dA$$

$$= \iint_{D_1} 3ue^{v} dA = 3\int_0^1 e^{v} dv \cdot \int_0^2 u\,du = 6(e-1)$$

S_3 : $r_3(u,v) = \langle u, 0, v \rangle$, $(u,v) \in D_2 = [0,3] \times [0,1]$

$$\iint_{S_3} \vec{F}\cdot d\vec{S} = \iint_{D_2} \langle 0, 0, 0 \rangle \cdot \langle 0, -1, 0 \rangle = 0$$

$S_4: r_4(u,v) = \langle u, 2, v \rangle,\ (u,v) \in D_2$

$$\iint_{S_4} \vec{F} \cdot d\vec{S} = \iint_{D_2} \langle 2ue^v, 4uv^3, -2e^v \rangle \cdot \langle 0,1,0 \rangle \, dA = \iint_{D_2} 4uv^3 \, dA = \int_0^1 \int_0^3 4uv^3 \, du \, dv = \frac{9}{2}$$

$S_5: r_5(u,v) = \langle u, v, 0 \rangle,\ (u,v) \in D_3 := [0,3] \times [0,2]$

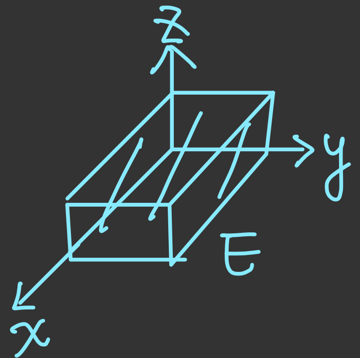
$$\iint_{S_5} \vec{F} \cdot d\vec{S} = \iint_{D_3} \langle uv, 0, -v \rangle \cdot \langle 0,0,-1 \rangle \, dA = \iint_{D_3} v \, dA = \int_0^2 \int_0^3 v \, du \, dv = 6$$

$S_6: r_5(u,v) = \langle u, v, 1 \rangle,\ (u,v) \in D_3$

$$\iint_{S_6} \vec{F} \cdot d\vec{S} = \iint_{D_3} \langle uve, uv^2, -ve \rangle \cdot \langle 0,0,1 \rangle \, dA = \iint_{D_3} -ve \, dA = \int_0^2 \int_0^3 -ve \, du \, dv = -6e$$

$$\iint_S \vec{F} \cdot d\vec{S} = 0 + 6(e-1) + 0 + \frac{9}{2} + 6 - 6e = \frac{9}{2}$$

Method 2 Use divergence theorem



$E = [0,3] \times [0,2] \times [0,1]$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \frac{\partial}{\partial x}(xye^z) + \frac{\partial}{\partial y}(xy^2z^3) + \frac{\partial}{\partial z}(-ye^z) \, dV$$

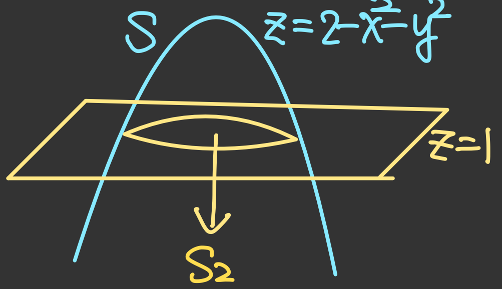
$$= \iiint_E (ye^z + 2xyz^3 - ye^z) \, dV$$

$$= \int_0^1 \int_0^2 \int_0^3 2xyz^3 \, dx \, dy \, dz = \frac{9}{2}$$

Let $\vec{F}(x,y,z) = \langle z\tan^{-1}(y^2), z^3\ln(x^2+1), z \rangle$

Find the flux of $\vec{F}$ across the part of paraboloid $x^2+y^2+z=2$

that lies above the plane $z=1$ and is oriented upward.



Try to direct compute

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_\Omega \langle (2-u^2-v^2)\tan^{-1}(v^2), (2-u^2-v^2)^3\ln(u^2+1), 2-u^2-v^2 \rangle \cdot \langle 2u, 2v, 1\rangle\, dA$$

$S: r(u,v) = \langle u, v, 2-u^2-v^2 \rangle$

$\Omega = \{(u,v) \mid u^2+v^2 \le 1\}$

$r_u \times r_v = \langle 2u, 2v, 1 \rangle$

此計算過於複雜，所以試圖使用 Divergence thm.

但 Divergence 需要封閉的曲面，所以我們考慮 S_2

$S_2: r(u,v) = \langle u, v, 1 \rangle, (u,v) \in \Omega$，$S \cup S_2$ 是封閉曲面

$$\iint_{S\cup S_2} \vec{F}\cdot d\vec{S} = \iiint_E \frac{\partial}{\partial x}(z\tan^{-1}(y^2)) + \frac{\partial}{\partial y}(z^2\ln(x^2+1)) + \frac{\partial}{\partial z}(z)\, dV$$

$$= \iiint_E 1\, dV$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E 1\,dV - \iint_{S_2} \vec{F}\cdot d\vec{S}$$

$$\iiint_E 1\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{2-x^2-y^2} 1\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2-x^2-y^2)\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^1 (2-r^2)\,r\,dr\,d\theta$$

$$= 2\pi\left(r^2 - \frac{1}{4}r^4\right)\Big|_0^1 = \frac{3}{2}\pi$$

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_\Omega \langle \tan^{-1}(v^2), \ln(u^2+1), 1\rangle \cdot \langle 0,0,-1\rangle\,dA$$

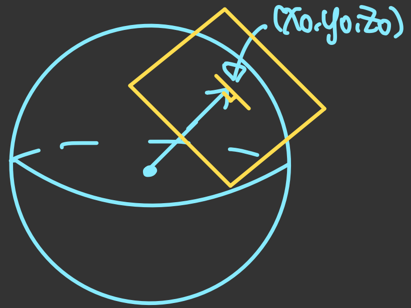
$\langle 0,0,-1\rangle$ outward normal

$$= \iint_\Omega (-1)\,dA$$

$$= \int_0^{2\pi}\int_0^1 (-1)\,r\,dr\,d\theta = -\pi$$

$$\iint_S \vec{F}\cdot d\vec{S} = \frac{3}{2}\pi - (-\pi) = \frac{5}{2}\pi$$

Evaluate $\iint_S (2x+2y+z^2)\,dS$ where S is the sphere $x^2+y^2+z^2=1$



因為 $\langle x_0,y_0,z_0\rangle$ 和

和 (x_0,y_0,z_0) 在球上

的切面垂直.

所以我們可以得知

球面的unit normal

是 $\frac{1}{\sqrt{x^2+y^2+z^2}}\langle x,y,z\rangle$

Unit normal of S outward is $\langle x,y,z\rangle$ ($\because x^2+y^2+z^2=1$)

$$\iint_S (2x+2y+z^2)\,dS = \iint_S \underbrace{\langle 2,2,z\rangle}_{:=\vec{F}(x,y,z)} \cdot \underbrace{\langle x,y,z\rangle dS}_{=d\vec{S}}$$

$$= \iiint_E \frac{\partial}{\partial x}(2)+\frac{\partial}{\partial y}(2)+\frac{\partial}{\partial z}(z)\,dV$$

$$= \iiint_E 1\,dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi} 1\,d\theta \cdot \int_0^{\pi}\sin\phi\,d\phi \cdot \int_0^{1}\rho^2\,d\rho$$

$$= \frac{4}{3}\pi$$