Chapter 11: An Introduction to Graph Theory

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Outline

- ***<u>Definitions and Examples</u>***
- Subgraphs, Complements, and Graph Isomorphism
- Vertex Degree: Euler Trails and Circuits
- Planar Graphs
- Hamilton Paths and Cycles
- Graph Coloring and Chromatic Polynomials

Directed / Undirected Graph (1/2)

- **Definition 11.1:** Let V be a finite nonempty set, and let $E \subseteq V \times V$. The pair (V, E) is then called a *directed graph* (on V), or *digraph* (on V), where V is the set of *vertices*, or *nodes*, and E is its set of (*directed*) *edges* or *arcs*. We write $G = (V, E)$ to denote such a graph.

When there is no concern about the direction of any edge, we still write $G = (V, E)$. But now E is a set of unordered pairs of elements taken from V, and G is called an *undirected graph*.

Directed / Undirected Graph (1/2)

- **Definition 11.1 (cont.):**

  Whether $G = (V, E)$ is directed or undirected, we often call V the *vertex set* of G and E the *edge set* of G.

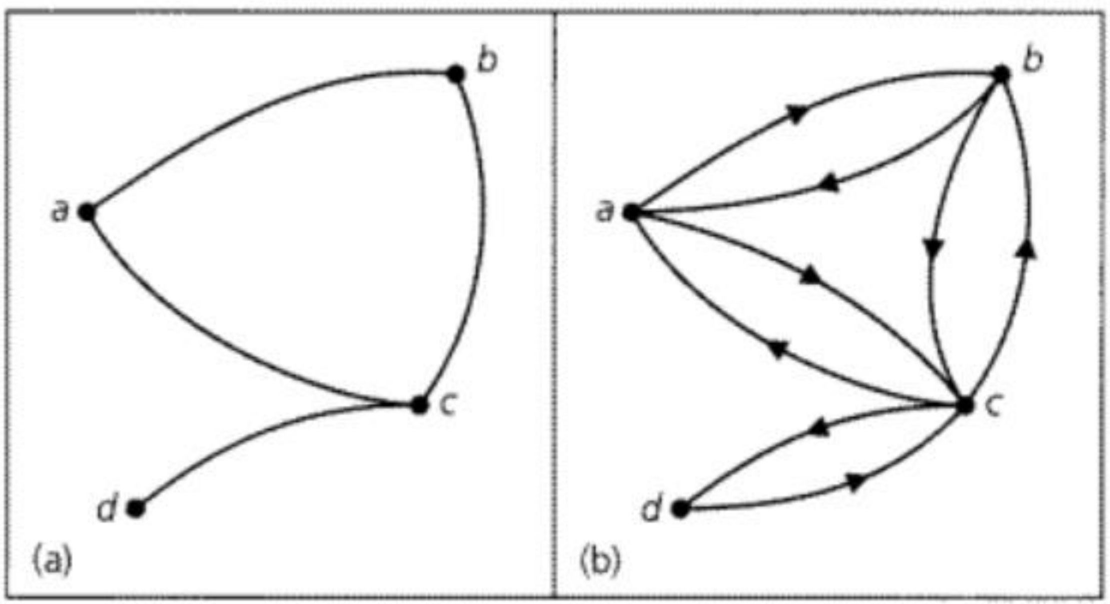

Figure 11.3

Walk (1/6)

- **Definition 11.2:** Let x, y be (not necessarily distinct) vertices in an undirected graph *G* = (*V*, *E*). An x-y *walk* in *G* is a (loop-free) finite alternating sequence

  $$x = x_0, e_1, x_1, e_2, x_2, e_3, \ldots, e_{n-1}, x_{n-1}, e_n, x_n = y$$

  of vertices and edges from G, starting at vertex x and ending at vertex y and involving the n edges $e_i = \{x_{i-1}, x_i\}$, where $1 \leq i \leq n$.

Walk (2/6)

- **Definition 11.2 (cont.):**

  The *length* of this walk is n, the number of edges in the walk. (When $n = 0$, there are no edges, $x = y$, and the walk is called *trivial*. These walks are not considered very much in our work.)

  Any x-y walk where $x = y$ (and $n > 1$) is called a *closed walk*. Otherwise the walk is called *open*.

Walk (3/6)

- **Example 11.1:** For the graph in Fig.11.4 we find, for example, the following three open walks. We can list the edges only or the vertices only (if the other is clearly implied).

1) {*a*, *b*}, {*b*, *d*}, {*d*, *c*}, {*c*, *e*}, {*e*, *d*}, {*d*, *b*}: This is an *a*-*b* walk of length 6 in which we find the vertices *d* and *b* repeated, as well as the edge {*b*, *d*) (= {*d*, *b*}).

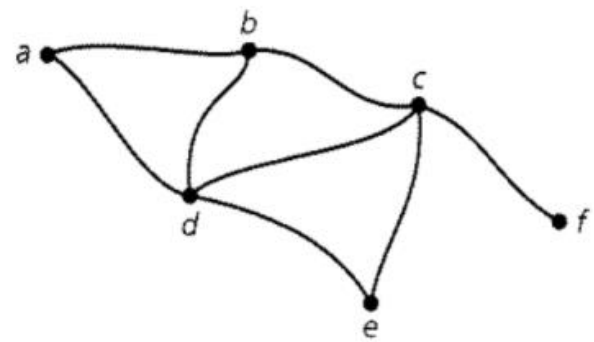

Figure 11.4

Walk (4/6)

- **Example 11.1 (cont.):**

  2) $b \to c \to d \to e \to c \to f$: Here we have a b-f walk where the length is 5 and the vertex c is repeated, but no edge appears more than once.

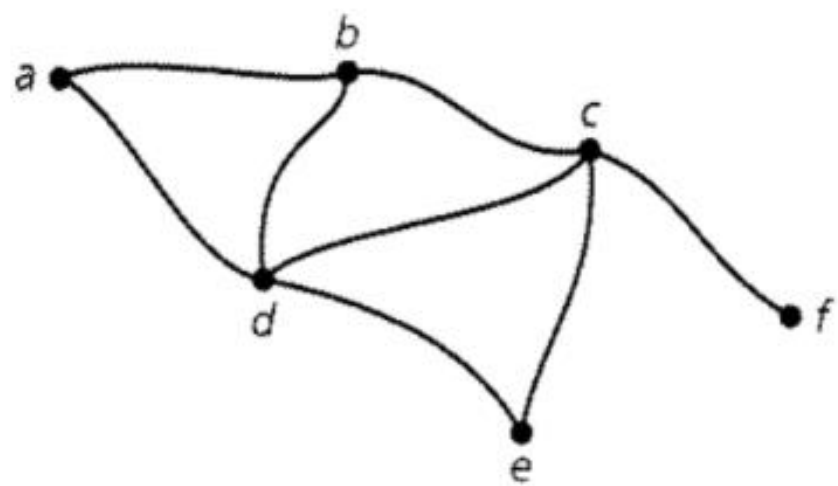

Figure 11.4

Walk (5/6)

- **Example 11.1 (cont.):**
  3) {*f*, *c*}, {*c*, *e*}, {*e*, *d*}, {*d*, *a*}: In this case the given *f*-*a* walk has length 4 with no repetition of either vertices or edges.

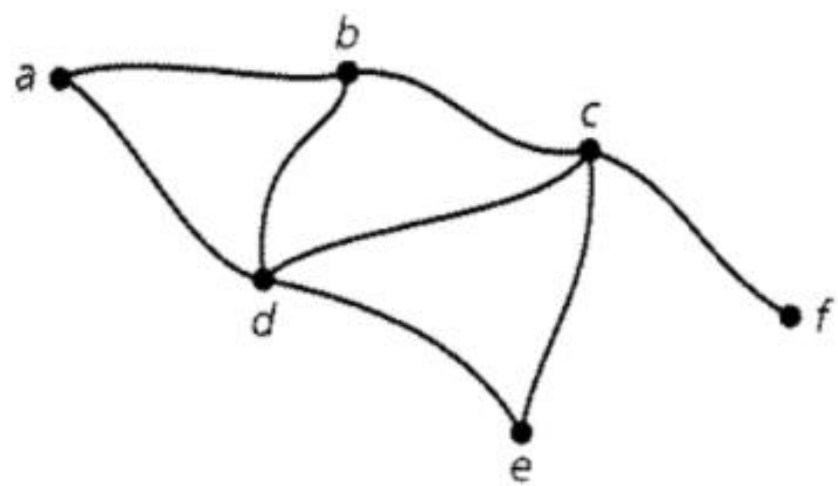

Figure 11.4

Walk (6/6)

- **Example 11.1 (cont.):**
  Since the graph of Fig. 11.4 is undirected, the *a*-*b* walk in part (1) is also a *b*-*a* walk (we read the edges, if necessary, as {*b*, *d*}, {*d*, *e*}, {*e*, *c*}, {*c*, *d*}, {*d*, *b*}, and {*b*, *a*}). Similar remarks hold for the walks in parts (2) and (3).
  Finally, the edges {*b*, *c*}, {*c*, *d*), and {*d*, *b*) provide a *b*-*b* (closed) walk. These edges (ordered appropriately) also define (closed) *c*-*c* and *d*-*d* walks.

Trail, Circuit, Path, Cycle (1/7)

- **Definition 11.3:** Consider any x-y walk in an undirected graph $G = (V, E)$.

  a) If no edge in the x-y walk is repeated, then the walk is called an x-y *trail*. A closed x-x trail is called a *circuit*.

  b) If no vertex of the x-y walk occurs more than once, then the walk is called an x-y *path*. When $x = y$, the term *cycle* is used to describe such a closed path.

Trail, Circuit, Path, Cycle (2/7)

- **Convention**: In dealing with circuits, we shall always understand the presence of at least one edge. When there is only one edge, then the circuit is a loop (and the graph is no longer loop-free). Circuits with two edges arise in multigraphs, a concept we shall define shortly.

  The term *cycle* will always imply the presence of at least three distinct edges (from the graph).

Trail, Circuit, Path, Cycle (3/7)

- **Example 11.2:**

  a) The *b*-*f* walk in part (2) of Example 11.1 is a *b*-*f* trail, but it is not a *b*-*f* path because of the repetition of vertex *c*. However, the *f*-*a* walk in part (3) of that example is both an *f*-*a* trail (of length 4) and an *f*-*a* path (of length 4).

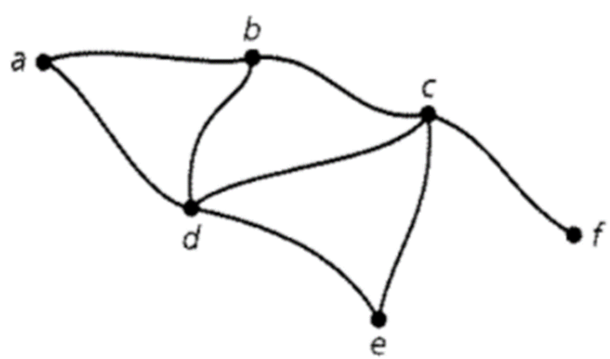

Figure 11.4

- $b \rightarrow c \rightarrow d \rightarrow e \rightarrow c \rightarrow f$

- $\{f, c\}, \{c, e\}, \{e, d\}, \{d, a\}$

Trail, Circuit, Path, Cycle (4/7)

b) In Fig. 11.4, the edges {*a*, *b*}, {*b*, *d*}, {*d*, *c*}, {*c*, *e*}, {*e*, *d*}, and {*d*, *a*} provide an *a*-*a* circuit. The vertex *d* is repeated, so the edges do not give us an *a*-*a* cycle.

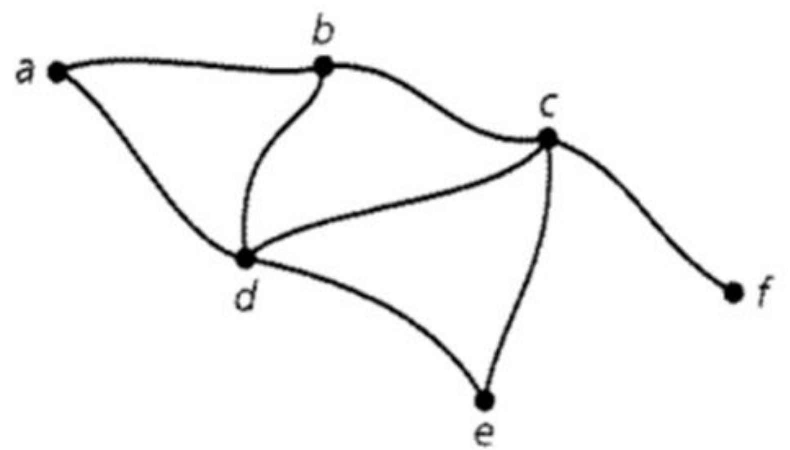

Figure 11.4

Trail, Circuit, Path, Cycle (5/7)

- **Example 11.2 (cont.):**
  c) The edges {*a*, *b*}, {*b*, *c*}, {*c*, *d*}, and {*d*, *a*} provide an *a*-*a* cycle (of length 4) in Fig. 11.4. When ordered appropriately these same edges may also define a *b*-*b*, *c*-*c*, or *d*-*d* cycle. Each of these cycles is also a circuit.

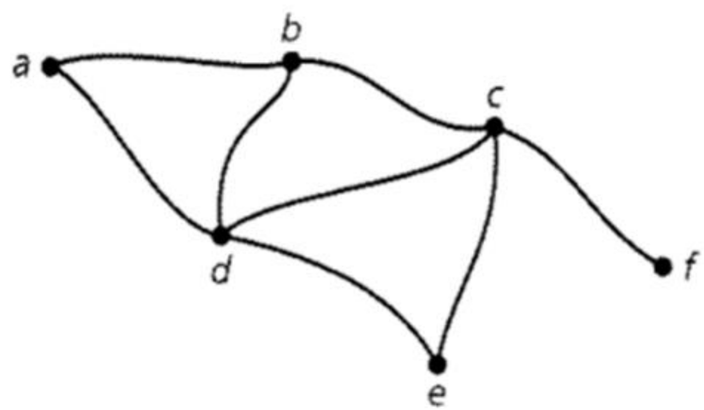

Figure 11.4

Trail, Circuit, Path, Cycle (6/7)

- For a directed graph we shall use the adjective *directed*, as in, for example, *directed walks*, *directed paths*, and *directed cycles*.

- Before continuing, we summarize (in Table 11.1) for future reference the results of Definitions 11.2 and 11.3. Each occurrence of “Yes” in the first two columns here should be interpreted as “Yes, possibly." Table 11.1 reflects the fact that a path is a trail, which in turn is an open walk. Furthermore, every cycle is a circuit, and every circuit (with at least two edges) is a closed walk.

Trail, Circuit, Path, Cycle (7/7)

Table 11.1

Repeated Vertex (Vertices)	Repeated Edge(s)	Open	Closed	Name
Yes	Yes	Yes		Walk (open)
Yes	Yes		Yes	Walk (closed)
Yes	No	Yes		Trail
Yes	No		Yes	Circuit
No	No	Yes		Path
No	No		Yes	Cycle

Connected and Disconnected (1/6)

- **Theorem 11.1:** Let *G* = (*V*, *E*) be an undirected graph, with *a*, *b* ∈*V*, *a*≠*b*. If there exists a trail (in *G*) from *a* to *b*, then there is a path (in *G*) from *a* to *b*.

  Proof: Since there is a trail from *a* to *b*, we select one of shortest length, say {*a*, x_1}, {x_1, x_2}, ..., {x_n, *b*}. If this trail is not a path, we have the situation {*a*, x_1}, {x_1, x_2},..., (x_{k-1}, x_k}, {x_k, x_{k+1}}, {x_{k+1}, x_{k+2}}, ..., {x_{m-1}, x_m}, {x_m, x_{m+1}},..., {x_n, *b*}, where *k* < *m* and x_k = x_m, possibly with *k* = 0 and a (= x_0) = x_m, or *m* = *n* + 1 and x_k = *b*(= x_{n+1}).

Connected and Disconnected (2/6)

- **Proof (cont.)**:

  But then we have a contradiction because {*a*, x_1}, {x_1, x_2}, …, {x_{k-1}, x_k}, {x_m, x_{m+1}},…, {x_n, *b*} is a shorter trail from *a* to *b*.
- **Definition 11.4:** Let *G* = (*V*, *E*) be an undirected graph. We call *G* *connected* if there is a path between any two distinct vertices of *G*.

Connected and Disconnected (3/6)

- **Definition 11.4 (cont.):**

Let $G = (V, E)$ be a directed graph. Its associated undirected graph is the graph obtained from G by ignoring the directions on the edges. If more than one undirected edge results for a pair of distinct vertices in G, then only one of these edges is drawn in the associated undirected graph.

When this associated graph is connected, we consider G *connected*. A graph that is not connected is called *disconnected*.

Connected and Disconnected (4/6)

- **Example 11.3:** In Fig. 11.5 we have an undirected graph on $V = \{a, b, c, d, e, f, g\}$. This graph is not connected because, for example, there is no path from a to e.

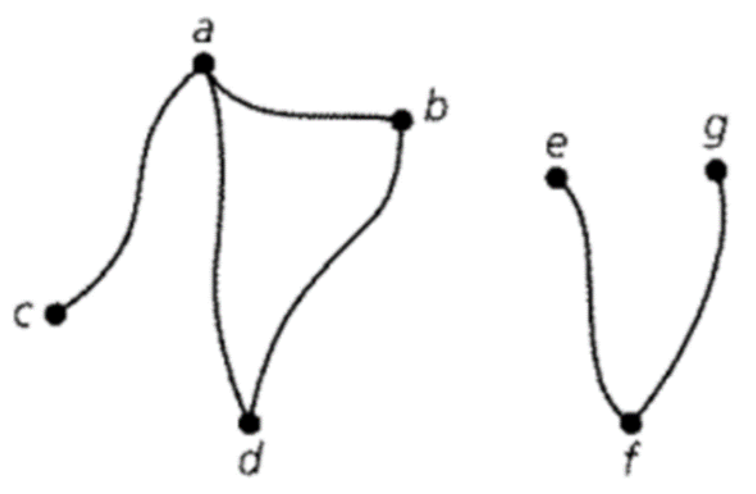

Figure 11.5

Connected and Disconnected (5/6)

- **Example 11.3 (cont.):** The graph is composed of pieces (with vertex sets $V_1 = \{a, b, c, d\}$, $V_2 = \{e, f, g\}$, and edge sets $E_1 = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}\}$, $E_2 = \{\{e, f\}, \{f, g\}\}$) that are themselves connected, and these pieces are called the (connected) components of the graph.
- An undirected graph $G = (V, E)$ is disconnected if and only if V can be partitioned into at least two subsets V_1, V_2 such that there is no edge in E of the form $\{x, y\}$,

where $x \in V_1$, and $y \in V_2$. A graph is connected if and only if it has only one component.

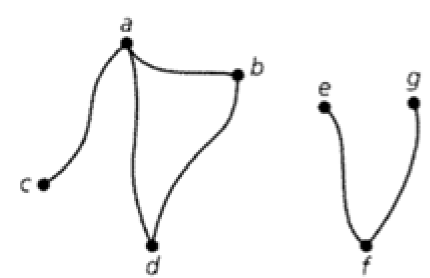

Figure 11.5

Connected and Disconnected (6/6)

- **Definition 11.5:** For any graph $G = (V, E)$, the number of components of G is denoted by $\kappa(G)$.

- **Example 11.4:** For the graphs in Figs. 11.1, 11.3, and 11.4, $\kappa(G) = 1$ because these graphs are connected; $\kappa(G) = 2$ for the graphs in Figs. 11.2 and 11.5.

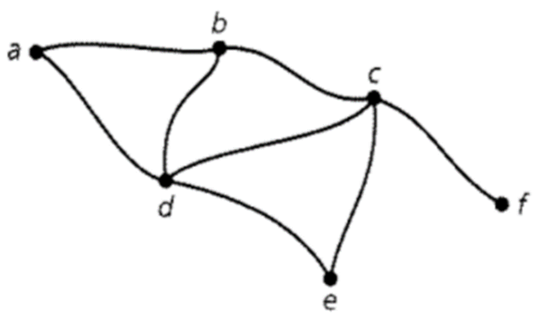

Figure 11.4

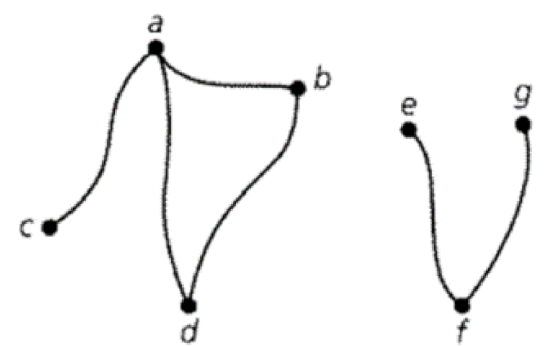

Figure 11.5

Multigraph (1/3)

- **Definition 11.6:** Let V be a finite nonempty set. We say that the pair (V, E) determines a *multigraph* G with vertex set V and edge set E if, for some $x, y \in V$, there are two or more edges in E of the form (*a*) (x, y) (for a directed multigraph), or (*b*) $\{x, y\}$ (for an undirected multigraph).

- In either case, we write $G = (V, E)$ to designate the multigraph, just as we did for graphs.

Multigraph (2/3)

- Figure 11.6 shows an example of a directed multigraph. There are three edges from *a* to *b*, so we say that the edge (*a*, *b*) has multiplicity 3. The edges (*b*, *c*) and (*d*, *e*) both have multiplicity 2. Also, the edge (*e*, *d*) and either one of the edges (*d*, *e*) form a (directed) circuit of length 2 in the multigraph.

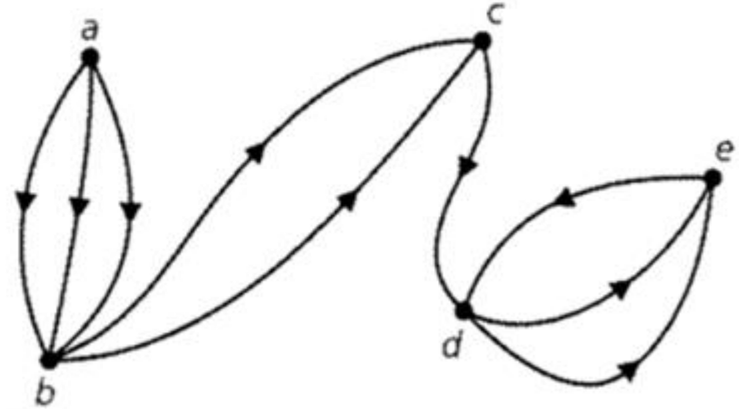

Figure 11.6

Multigraph (3/3)

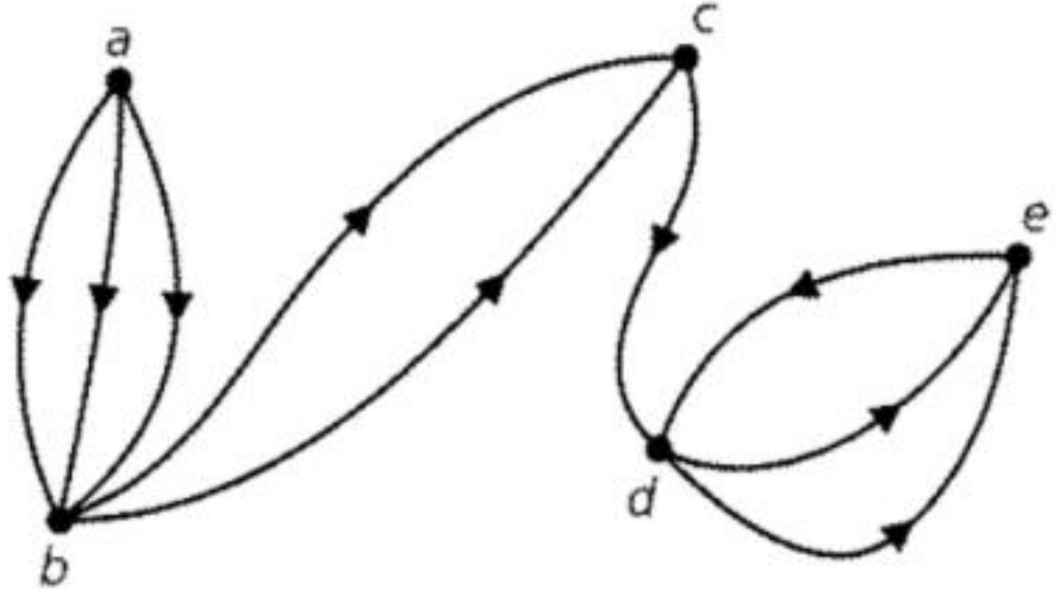

Figure 11.6

Outline

- Definitions and Examples
- ***<u>Subgraphs, Complements, and Graph Isomorphism</u>***
- Vertex Degree: Euler Trails and Circuits
- Planar Graphs
- Hamilton Paths and Cycles
- Graph Coloring and Chromatic Polynomials

Subgraphs, Complements, and Graph Isomorphism

Definition 11.7:
If $G = (V, E)$ is a graph (directed or undirected), then $\mathrm{G}_1 = (V_1, E_1)$is called a $subgraph$ of G if $\emptyset \neq V_1 \subseteq \mathrm{V}$ and $E_1 \subseteq \mathrm{E}$, where each edge in E_1 is incident with vertices in V_1.

Subgraphs, Complements, and Graph Isomorphism

Figure 11.14(a) provides us with an undirected graph G and two of its subgraphs, G_1 and G_2. The vertices a, b are isolated in subgraph G_1. Part (b) of the figure provides a directed example. Here vertex w is isolated in the subgraph G'.

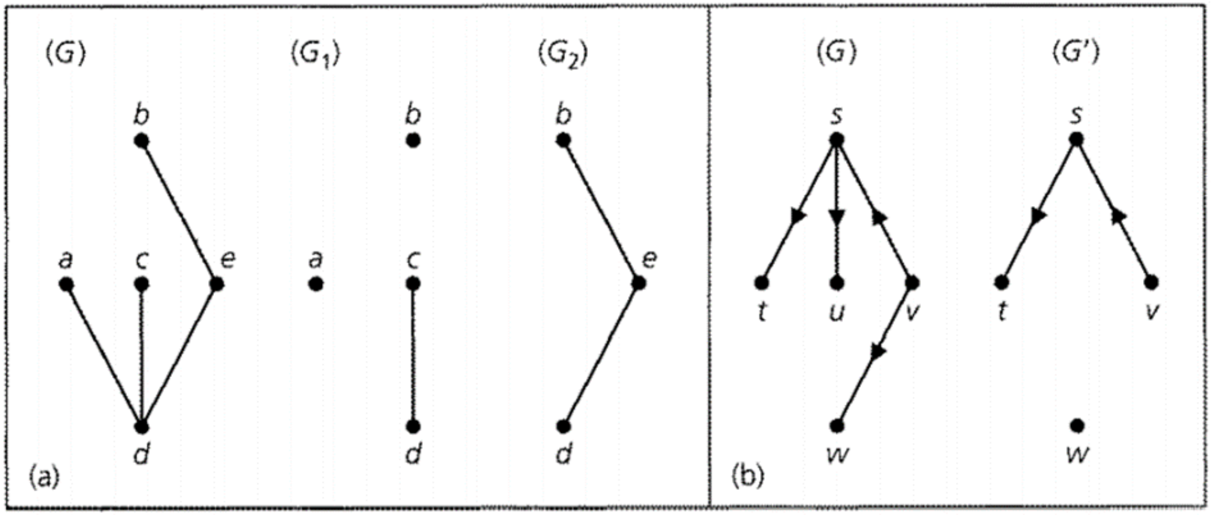

Figure 11.14

Subgraphs, Complements, and Graph Isomorphism

Definition 11.8:

Given a (directed or undirected) graph $G = (V, E)$, let G_1 = $(V_1, E_{1,})$ be a subgraph of G. If $V_1 = V$ then G_1 is called a $spanning\ subgraph$ of G.

Subgraphs, Complements, and Graph Isomorphism

In part (a) of Fig. 11.14 neither G_1: nor G_2 is a spanning subgraph of G.

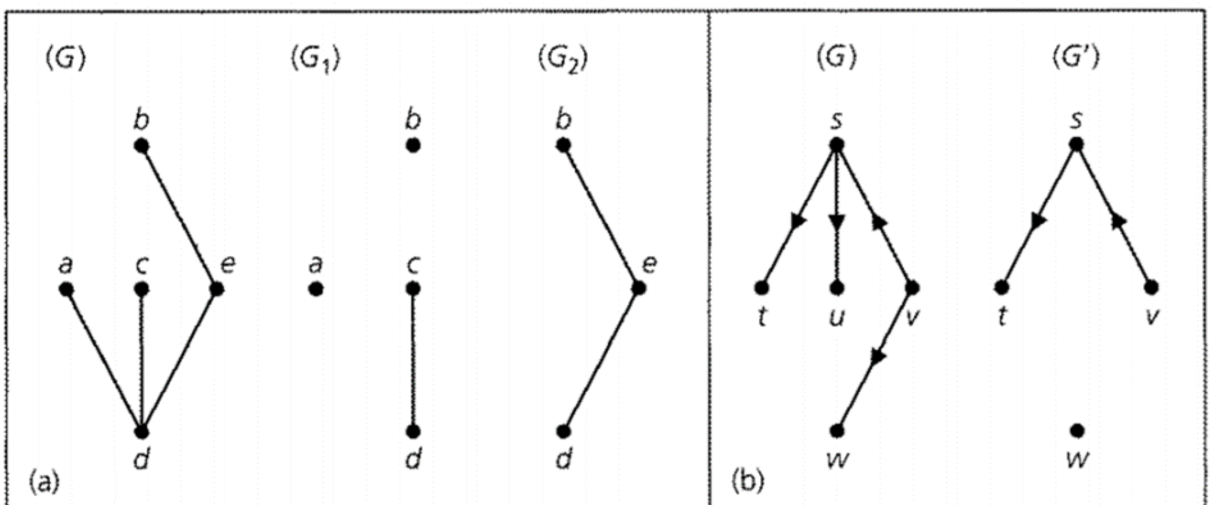

Figure 11.14

The subgraphs G_3 and G_4 - shown in part (a) of Fig. 11.15 --are both spanning subgraphs of G.

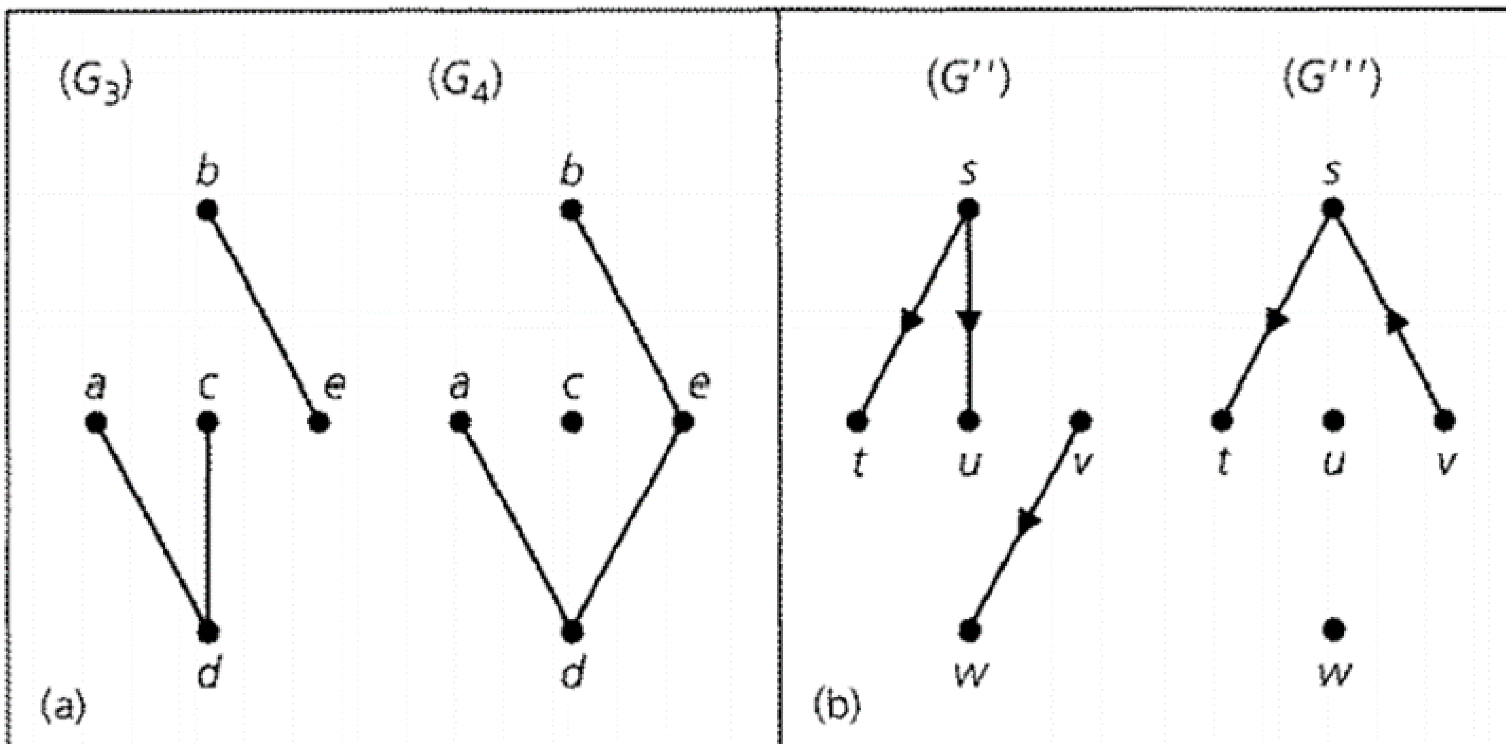

Figure 11.15

The directed graph G' in part (b) of Fig, 11.14 is a subgraph, but *not* a spanning subgraph, of the directed graph G given in that part of the figure.

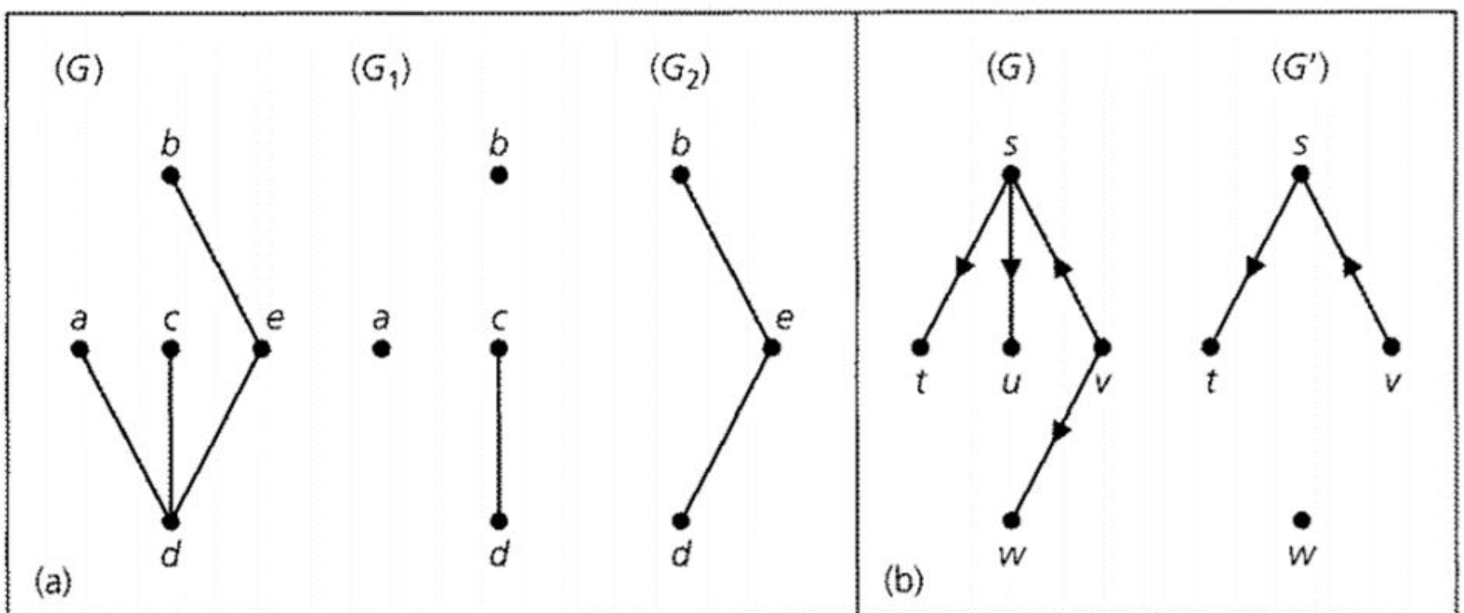

Figure 11.14

Subgraphs, Complements, and Graph Isomorphism

In part (b) of Fig. 11.15 the directed graphs $G''$and G''' are two of the 2^4= 16 possible spanning subgraphs.

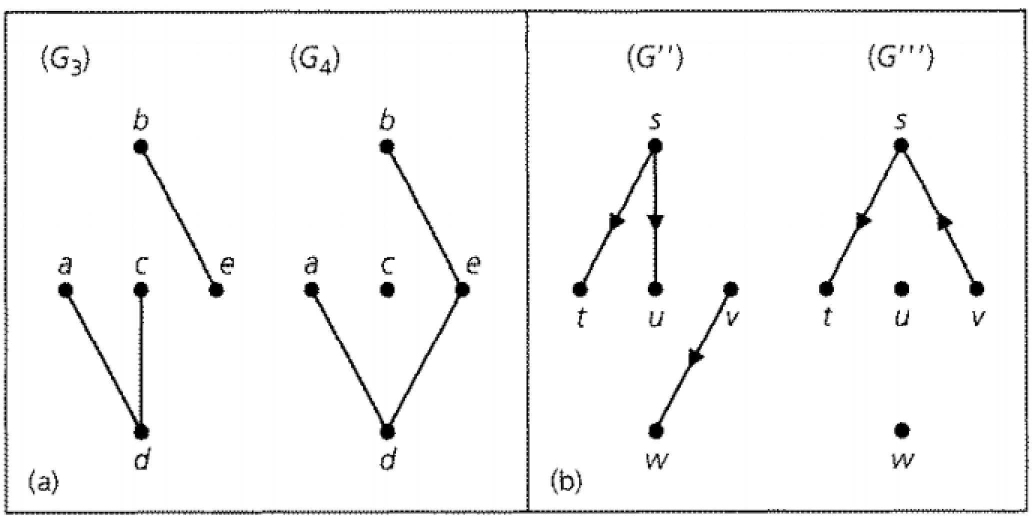

Figure 11.15

Subgraphs, Complements, and Graph Isomorphism

Definition 11.9:

Let $G = (V, E)$ be a graph (directed or undirected). If $\emptyset \neq U \subseteq V$, the **subgraph** of G **induced** by U is the subgraph whose vertex set is U and which contains all edges (from G) of either the form (a) (x, y), for $x, y \in U$ (when G is directed), or (b) $\{x, y\}$, for $x, y \in U$ (when G is undirected). We denote this subgraph by (U).

A subgraph G' of a graph $G = (V, E)$ is called an **induced subgraph** if there exists $\emptyset \neq U \subseteq V$ where $G' = \langle U \rangle$.

Subgraphs, Complements, and Graph Isomorphism

Example 11.5:

Let $G = (V, E)$ denote the graph in Fig. 11.16(a). The subgraphs in parts (b) and (c) of the figure are induced subgraphs of G.

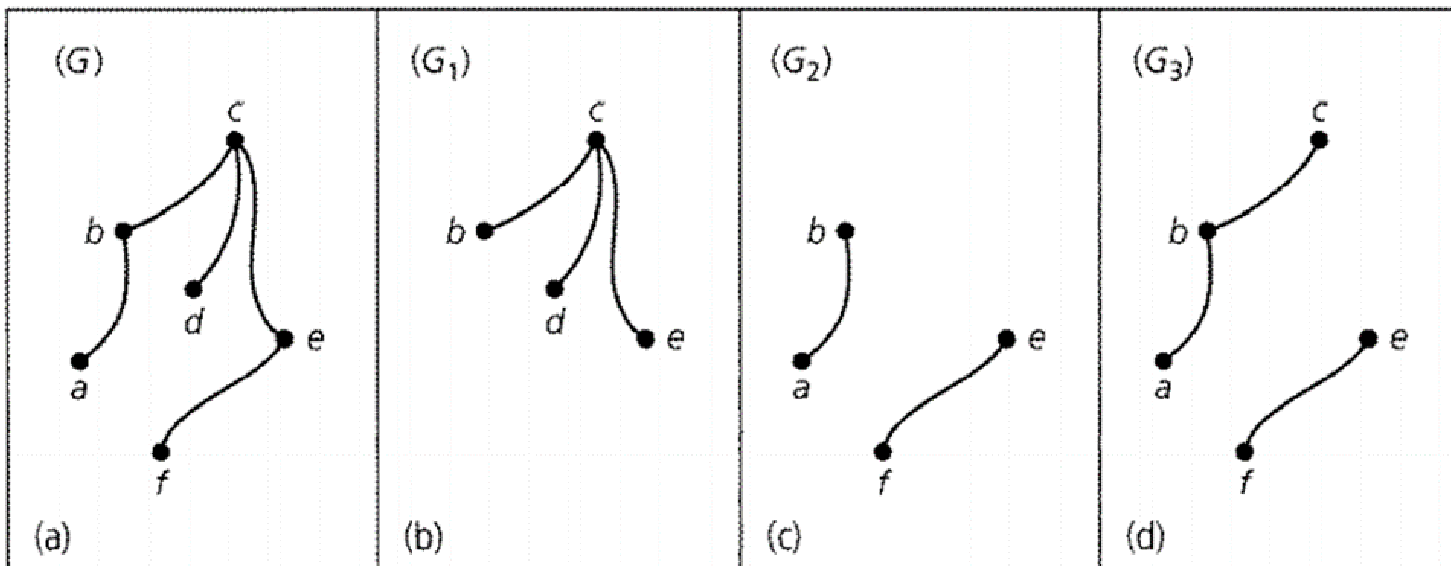

Figure 11.16

For the connected subgraph in part (b), $G_1 = \langle U_1 \rangle$ for $U_1 = \{b, c, d, e\}$.
The disconnected subgraph in part (c) is $G_2 = \langle U_2 \rangle$ for $U_2 = \{a, b, e, f\}$, Finally, G_3 in part (d) of Fig. 11.16 is a subgraph of G. But it is not an induced subgraph; the vertices c, e are in G_3, but the edge $\{c,\ e\}$(of G) is not present.

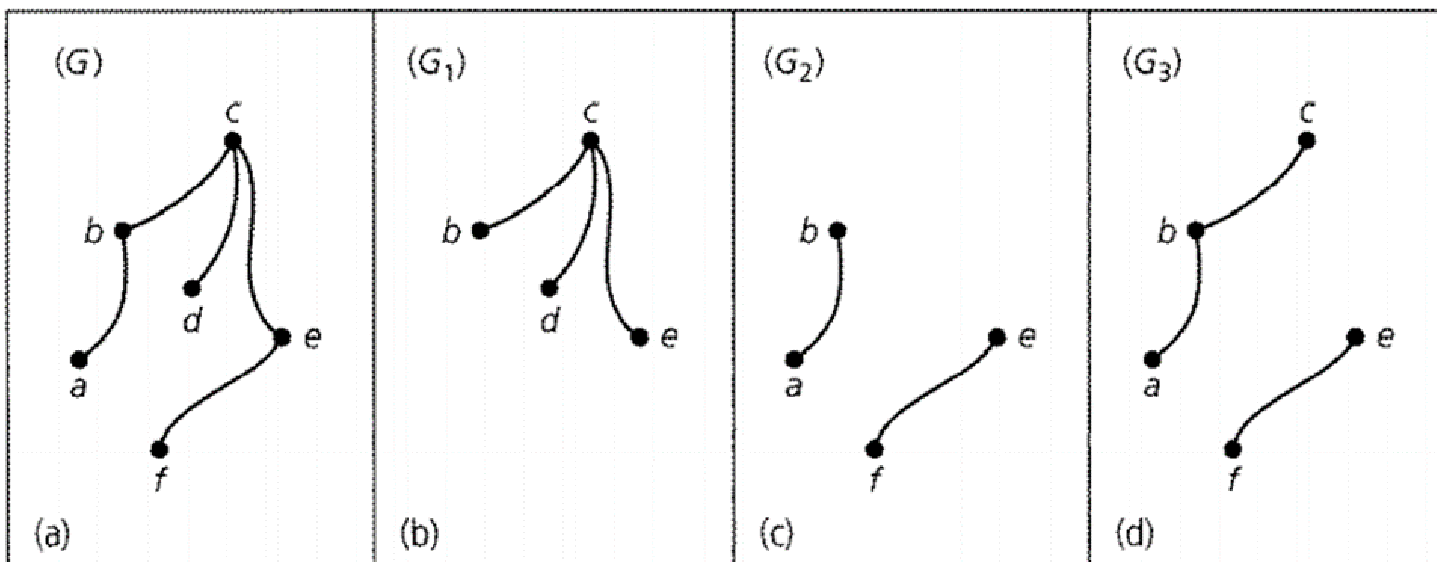

Figure 11.16

Subgraphs, Complements, and Graph Isomorphism

Definition 11.10:

Let v be a vertex in a directed or an undirected graph $G = (V, E)$. The subgraph of G denoted by $G - v$ has the vertex set $V_1 = V - \{v\}$ and the edge set $E_1 \subseteq \mathrm{E}$, where E_1 contains all the edges in E except for those that are incident with the vertex v. (Hence $G - v$ is the subgraph of G induced by V_1.)

Subgraphs, Complements, and Graph Isomorphism

If e is an edge of a directed or an undirected graph $G = (V, E)$, we obtain the subgraph $G - e = (V_1, E_1)$ of G, where the set of edges $E_1 = E - \{e\}$, and the vertex set is unchanged (that is, $V_1 = V$).

Subgraphs, Complements, and Graph Isomorphism

Example 11.6:

Let $G = (V, E)$ be the undirected graph in Fig. 11.17(a). Part (b) of this figure is the subgraph G_1(of G), where $G_1 = G - c$. It is also the subgraph of G induced by the set of vertices $U_1 = \{a, b, d, f, g, h\}$, so $G_1 = \langle V - \{c\}\rangle = \langle U_1 \rangle$. In part (c) of Fig. 11.17 we find the subgraph G_2 of G, where $G_2 = G - e$ for e the edge $\{c, d\}$.

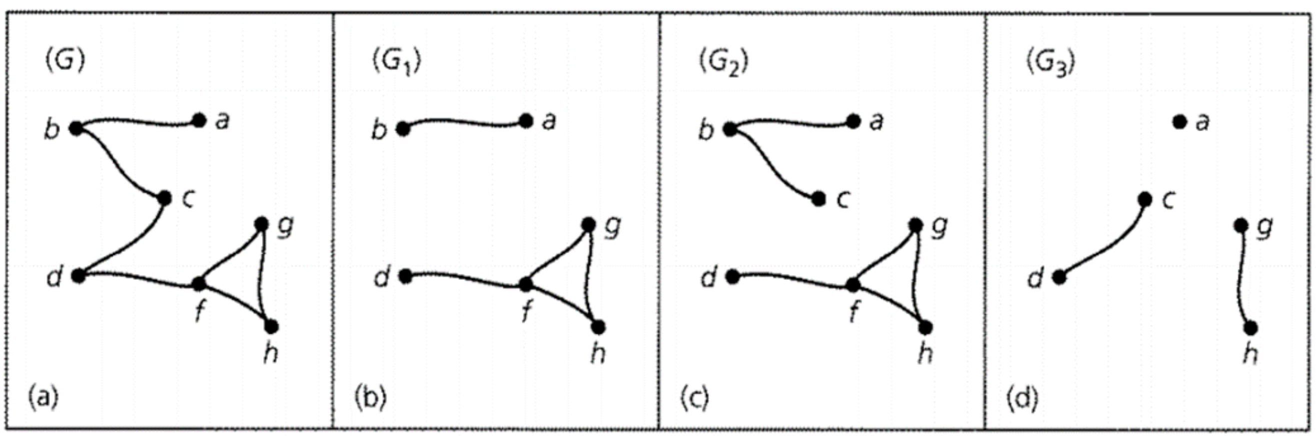

Figure 11.17

Subgraphs, Complements, and Graph Isomorphism

Fig. 11.17(d) shows how the ideas in Definition 11.10 can be extended to the deletion of more than one vertex (edge). We may represent this subgraph of G as

$$G_3 = (G - b) - f = (G - f) - b = G - \{b, f\} = \langle U_3 \rangle,$$

for $U_3 = \{a, c, d, g, h\}$

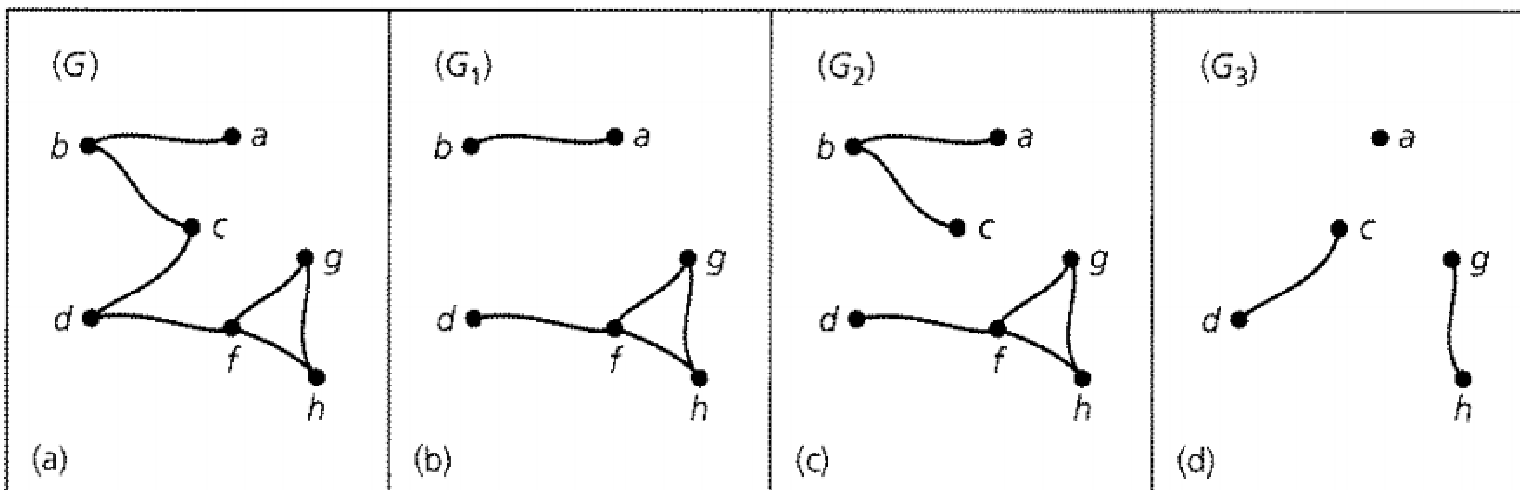

Figure 11.17

Subgraphs, Complements, and Graph Isomorphism

Definition 11.11:

Let V be a set of n vertices. The $complete\ graph$ on V, denoted K_n, is a loop-free undirected graph, where for all $a, b \in V$, $a \neq b$, there is an edge $\{a, b\}$.

Subgraphs, Complements, and Graph Isomorphism

Figure 11.18 provides the complete graphs K_n, for $1 \leq n \leq 4$. We shall realize, when we examine the idea of graph isomorphism, that these are the only possible complete graphs for the given number of vertices.

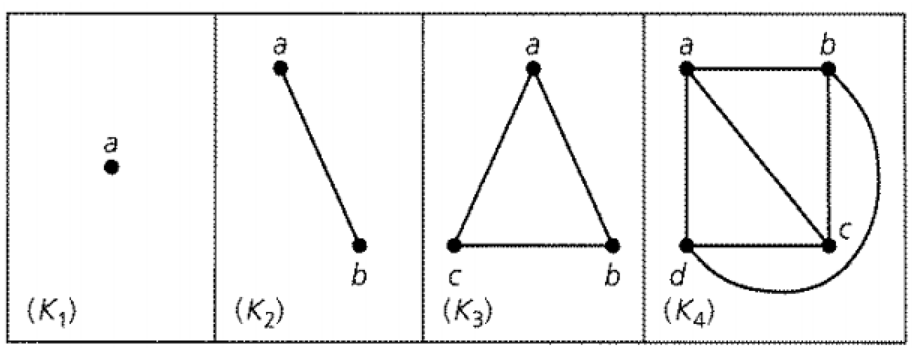

Figure 11.18

Subgraphs, Complements, and Graph Isomorphism

Definition 11.12:

Let G be a loop-free undirected graph on n vertices. The $complement$ of G, denoted $\bar{G}$, is the subgraph of $K_n$consisting of the n vertices in G and all edges that are not in G. (If $G = K_n$, $\bar{G}$ is a graph consisting of n vertices and no edges. Such a graph is called a $null$ graph.)

Subgraphs, Complements, and Graph Isomorphism

Figure 11.19(a) shows an undirected graph on fou vertices.
Its complement is shown in part (b) of the figure.
In the complement, vertex a is isolated.

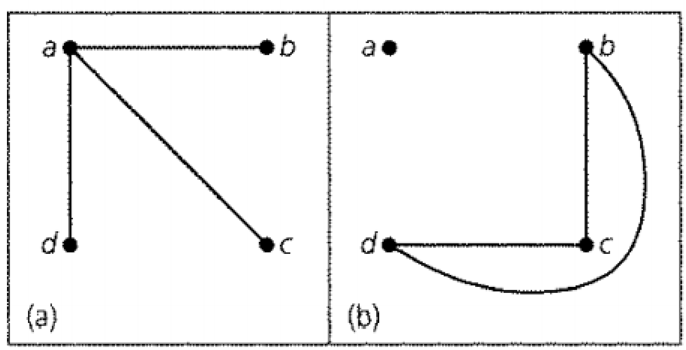

Figure 11.19

Subgraphs, Complements, and Graph Isomorphism

Definition 11.13:

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs.
A function $f: V_1 \to V_2$ is called a $graph\ isomorphism$ if (a) f is one-to-one and onto, and (b) for all $a, b \in V_1$, $\{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$. When such a function exists, G_1 and G_2 are called $isomorphic\ graphs$.

Subgraphs, Complements, and Graph Isomorphism

Example 11.8:

In Fig. 11.25 we have two graphs, each on ten vertices. Unlike the graphs in Fig. 11.24, it is not immediately apparent whether or not these graphs are isomorphic.

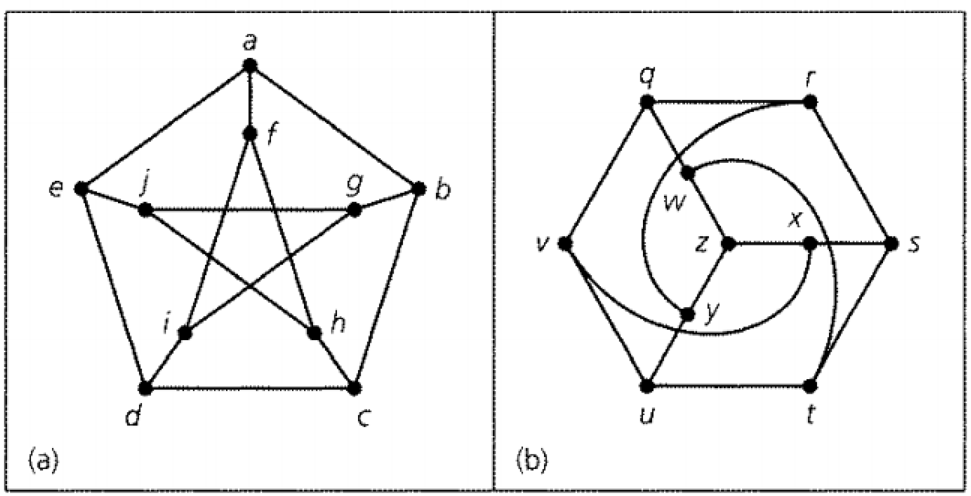

Figure 11.25

Subgraphs, Complements, and Graph Isomorphism

One finds that the correspondence given by

$$\begin{array}{ccccc} a \to q & c \to u & e \to r & g \to x & i \to z \\ b \to v & d \to y & f \to w & h \to t & j \to s \end{array}$$

preserves all adjacencies. For example, $\{f, h\}$, is an edge in graph (a) with $\{w, t\}$ the corresponding edge in graph (b).

But how did we come up with the correspondence?

Subgraphs, Complements, and Graph Isomorphism

We note that because an isomorphism preserves adjacencies, it preserves graph substructures such as paths and cycles. In graph (a) the edges $\{a, f\}, \{f, i\}, \{i, d\}, \{d, e\}$, and $\{e, a\}$ constitute a cycle of length 5. Hence we must preserve this as we try to find an isomorphism.

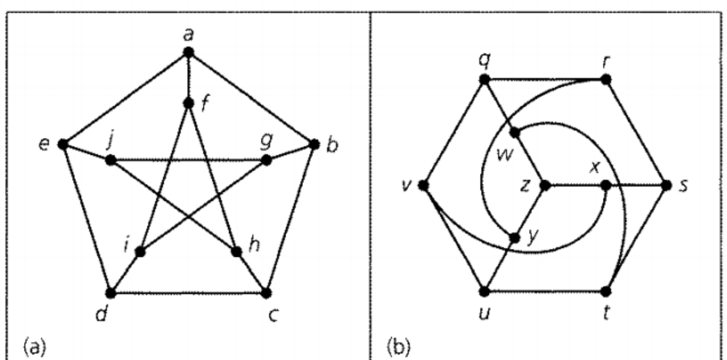

Figure 11.25

Subgraphs, Complements, and Graph Isomorphism

One possibility for the corresponding edges in graph (b) is $\{q\ ,w\}, \{w, z\}, \{z, y\}, \{y,\ r\}$, and $\{r, q\}$ which also provides a cycle of length 5. (A second possible choice is given by the edges in the cycle $y \rightarrow r \rightarrow$ s $\rightarrow$ t $\rightarrow$ u $\rightarrow$y.)

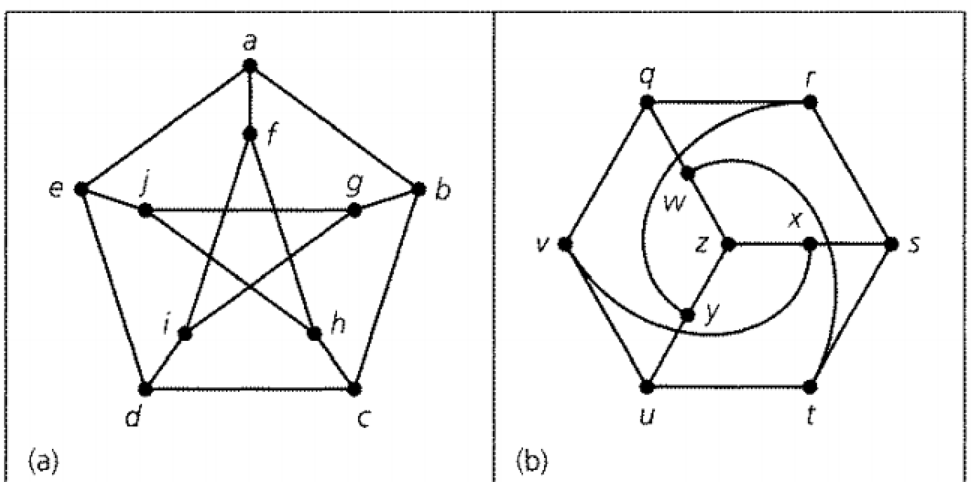

Figure 11.25

Subgraphs, Complements, and Graph Isomorphism

In addition, starting at vertex a in graph (a), we find a path that will "visit' each vertex only once. We express this path by $\mathrm{a} \to f \to h \to c \to b \to g \to j \to e \to d \to i$. For the graphs to be isomorphic there must be a corresponding path in graph (b). Here the path described by $q \to w \to t \to u \to v \to x \to s \to r \to y \to z$ is the counterpart.

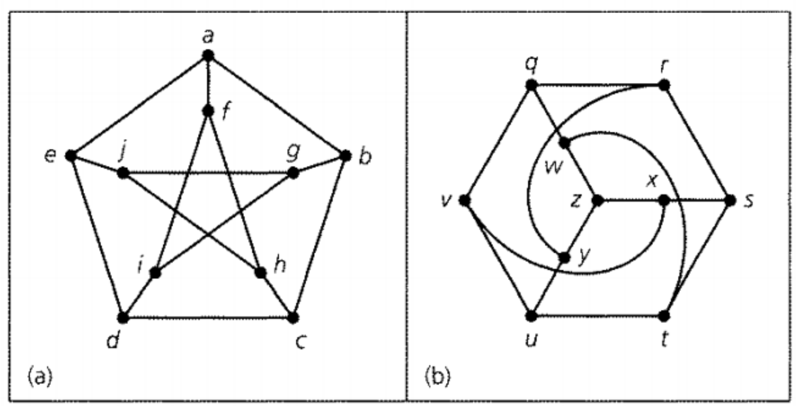

Figure 11.25

Outline

- Definitions and Examples
- Subgraphs, Complements, and Graph Isomorphism
- ***<u>Vertex Degree: Euler Trails and Circuits</u>***
- Planar Graphs
- Hamilton Paths and Cycles
- Graph Coloring and Chromatic Polynomials

Degree

Definition 11.14: Let G be an undirected graph or multigraph. For each vertex v of G, the *degree* of v, written $\deg(v)$, is the number of edges in G that are incident with v. Here a loop at a vertex v is considered as two incident edges for v.

Degree

EXAMPLE 11.10: For the graph in Fig. 11.32, $\deg(b) = \deg(d) = \deg(f) = \deg(g) = 2, \deg(c) = 4, \deg(e) = 0$, and $\deg(h) = 1$. For vertex a we have $\deg(a) = 3$ because we count a loop twice. Since h has degree 1, it is called a pendant vertex.

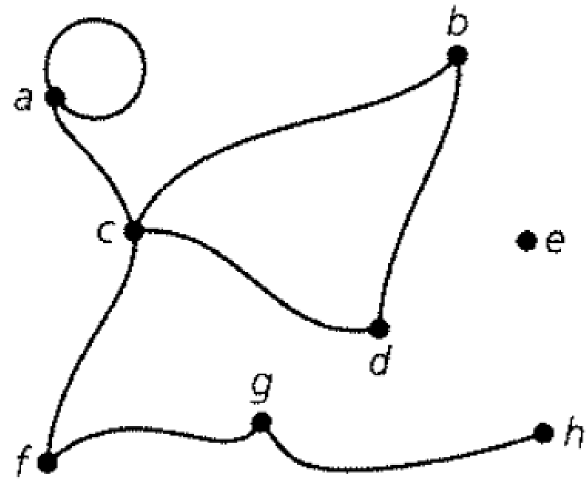

Figure 11.32

Degree

THEOREM 11.2: If $G = (V, E)$ is an undirected graph or multigraph, then $\sum_{v \in V} \deg(v) = 2|E|$.

Proof: As we consider each edge (a, b) in graph G, we find that the edge contributes a count of 1 to each of $\deg(a), \deg(b)$, and consequently a count of 2 to $\sum_{v \in V} \deg(v)$. Thus $2|E|$ accounts for $\deg(v)$, for all $v \in V$, and $\sum_{v \in V} \deg(v) = 2|E|$.

EXAMPLE 11.11 (1/2)

EXAMPLE 11.11:

An undirected graph (or multigraph) where each vertex has the same degree is called a regular graph. If $\deg(v) = k$ for all vertices v, then the graph is called k-regular. Is it possible to have a 4-regular graph with 10 edges?

From Theorem 11.2, $\sum_{v \in V} \deg(v) = 2|E|$, so we have five vertices of degree 4. Figure 11.33 provides two nonisomorphic examples that satisfy the requirements.

EXAMPLE 11.11 (2/2)

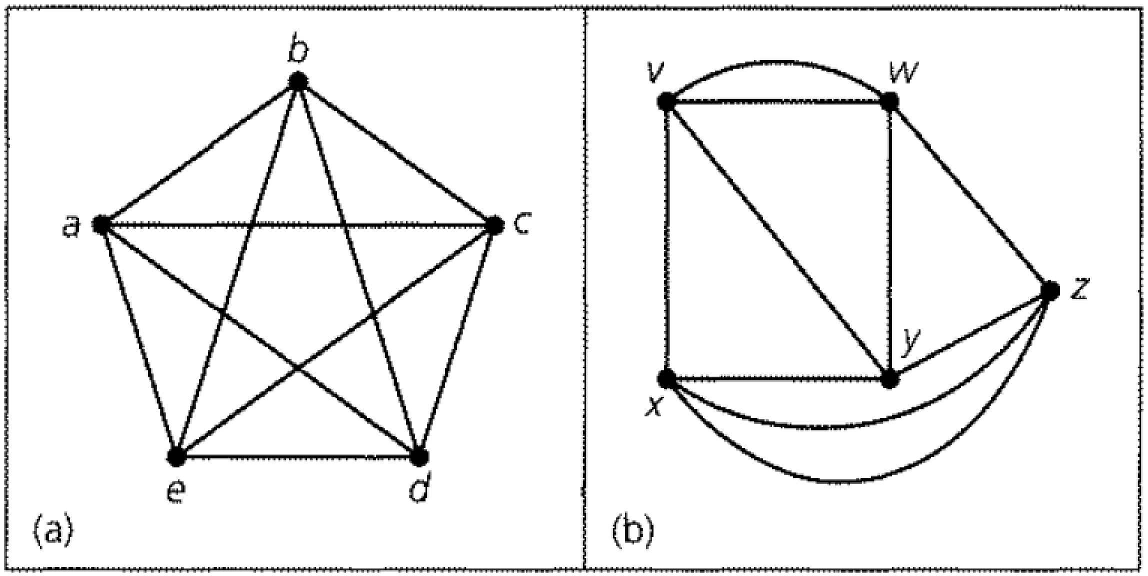

Figure 11.33

Euler Circuit

Definition 11.15: Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G is said to have an *Euler circuit* if there is a circuit in G that traverses every edge of the graph exactly once. If there is an open trail from a to b in G and this trail traverses each edge in G exactly once, the trail is called an Euler trail.

EXAMPLE 11.13 (1/2)

EXAMPLE 11.13: The Seven Bridges of Königsberg. During the eighteenth century, the city of Königsberg (in East Prussia) was divided into four sections (including the island of Kneiphof) by the Pregel River. Seven bridges connected these regions, as shown in Fig. 11.37(a). It was said that residents spent their Sunday walks trying to find a way to walk about the city so as to cross each bridge exactly once and then return to the starting point.

EXAMPLE 11.13 (2/2)

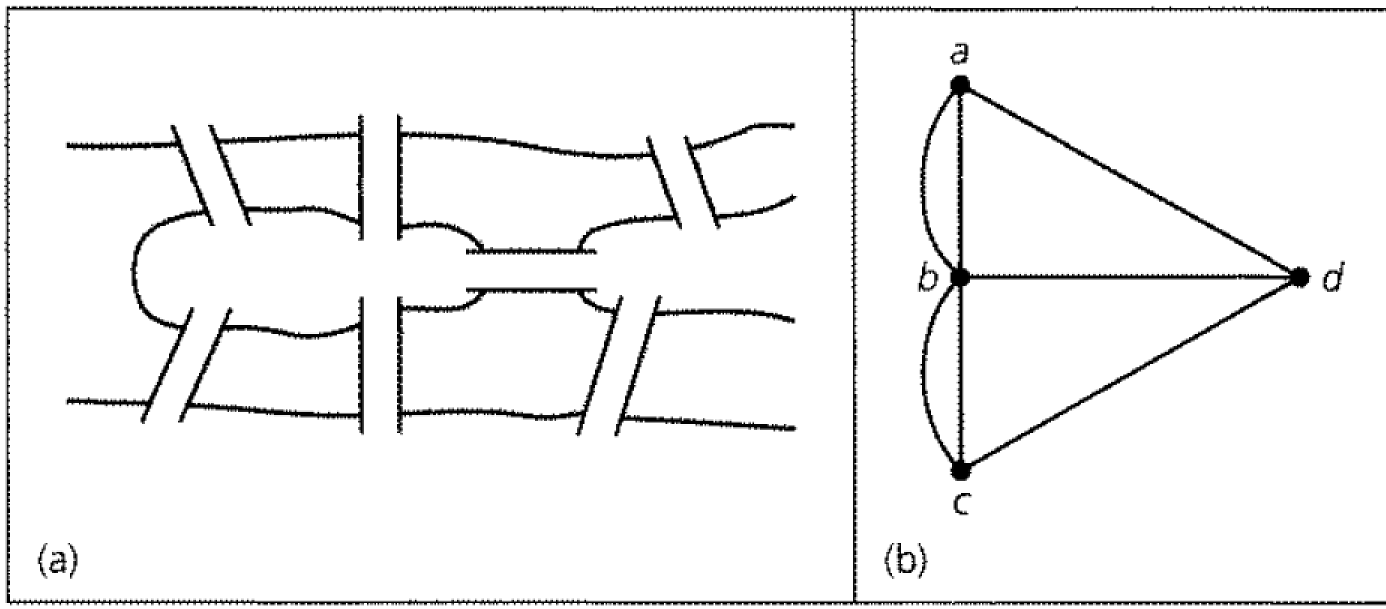

Figure 11.37

THEOREM 11.3 (1/5)

THEOREM 11.3: Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G has an Euler circuit if and only if G is connected and every vertex in G has even degree.

$Proof$: If G has an Euler circuit, then for all $a, b \in V$ there is a trail from a to b ---- namely, that part of the circuit that starts at a and terminates at b. Therefore, it follows from Theorem 11.1 that G is connected.

Let s be the starting vertex of the Euler circuit. For any other vertex v of G, each time the circuit comes to v it then departs from the vertex. Thus the circuit has traversed either two (new) edges that are incident with v or a (new) loop at v. In either case a count of 2 is contributed to $\deg(v)$.

THEOREM 11.3 (2/5)

Since v is not the starting point and each edge incident to v is traversed only once, a count of 2 is obtained each time the circuit passes through v, so $\deg(v)$ is even. As for the starting vertex s, the first edge of the circuit must be distinct from the last edge, and because any other visit to s results in a count of 2 for $\deg(s)$, we have $\deg(s)$ even.

THEOREM 11.3 (3/5)

Conversely, let G be connected with every vertex of even degree. If the number of edges in G is 1 or 2, then G must be as shown in Fig. 11.38. Euler circuits are immediate in these cases.

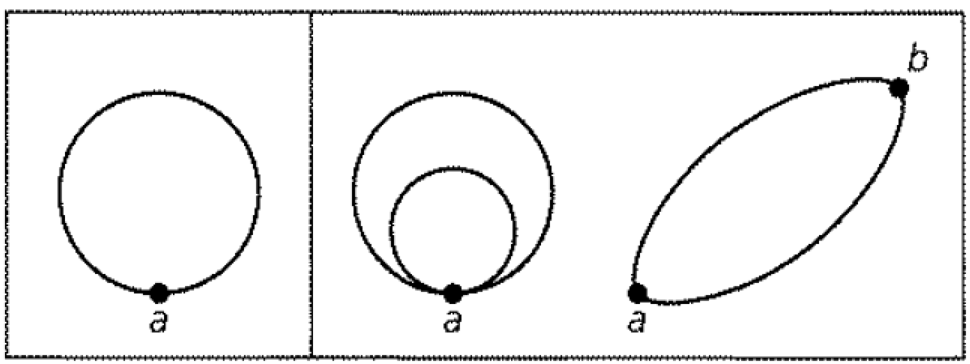

Figure 11.38

THEOREM 11.3 (4/5)

We proceed now by induction and assume the result true for all situations where there are fewer than n edges. If G has n edges, select a vertex s in G as a starting point to build an Euler circuit.

The graph (or multigraph) G is connected and each vertex has even degree, so we can at least construct a circuit C containing s. (Verify this by considering the longest trail in G that starts at s.) Should the circuit contain every edge of G, we are finished. If not, remove the edges of the circuit from G, making sure to remove any vertex that would become isolated.

THEOREM 11.3 (5/5)

The remaining subgraph K has all vertices of even degree, but it may not be connected. However, each component of K is connected and will have an Euler circuit. In addition, each of these Euler circuits has a vertex that is on C.

Consequently, starting at s we travel on C until we arrive at a vertex s_i that is on the Euler circuit of a component C_1 of K. Then we traverse this Euler circuit and, returning to s_1, continue on C until we reach a vertex s_2 that is on the Euler circuit of component C_2 of K. Since the graph G is finite, as we continue this process we construct an Euler circuit for G.

COROLLARY 11.2 (1/2)

COROLLARY 11.2: If G is an undirected graph or multigraph with no isolated vertices, then we can construct an Euler trail in G if and only if G is connected and has exactly two vertices of odd degree.

Proof: If G is connected and a and b are the vertices of G that have odd degree, add an additional edge (a, b) to G. We now have a graph G, that is connected and has every vertex of even degree.

COROLLARY 11.2 (2/2)

Hence G has an Euler circuit C, and when the edge {a,b) is removed from C, we obtain an Euler trail for G. (Thus the Euler trail starts at one of the vertices of odd degree and terminates at the other odd vertex.) We leave the details of the converse for the reader.

Degree for directed graph

Definition 11.16: Let $G = (V, E)$ be a directed graph or multigraph. For each $v \in V$,

a) The *incoming*, or in-degree of v is the number of edges in G that are incident into v, and this is denoted by $id(v)$.

b) The *outgoing*, or out-degree of v is the number of edges in G that are incident from v, and this is denoted by $od(v)$.

For the case where the directed graph or multigraph contains one or more loops, each loop at a given vertex v contributes a count of 1 to each of $id(v)$ and $od(v)$.

Degree for directed graph

THEOREM 11.4: Let $G = (V, E)$ be a directed graph or multigraph with no isolated vertices. The graph G has a directed Euler circuit if and only if G is connected and $id(v) = od(v)$ for all $v \in V$.

Proof: The proof of this theorem is left for the reader.

Outline

- Definitions and Examples
- Subgraphs, Complements, and Graph Isomorphism
- Vertex Degree: Euler Trails and Circuits
- ***Planar Graphs***
- Hamilton Paths and Cycles
- Graph Coloring and Chromatic Polynomials

Planar Graphs(1/24)

Definition 11.17:

A graph (or multigraph) G is called $planar$ if G can be drawn in the plane with its edges intersecting only at vertices of G. Such a drawing of G is called an $embedding$ of G in the plane.

Planar Graphs (2/24)

EXAMPLE 11.15:

The graphs in Fig. 11.47 are planar. The first is a 3-regular graph, because each vertexhas degree 3; it is planar because no edgesintersect except at the vertices. In graph (b) it appears that we have a $nonplanar$ graph;
the edges $\{x, z\}$ and $\{w, y\}$ overlap at a point other than a vertex. However, we can redraw this graph as shown in part (C) of the figure. Consequently,K_4 is planar.

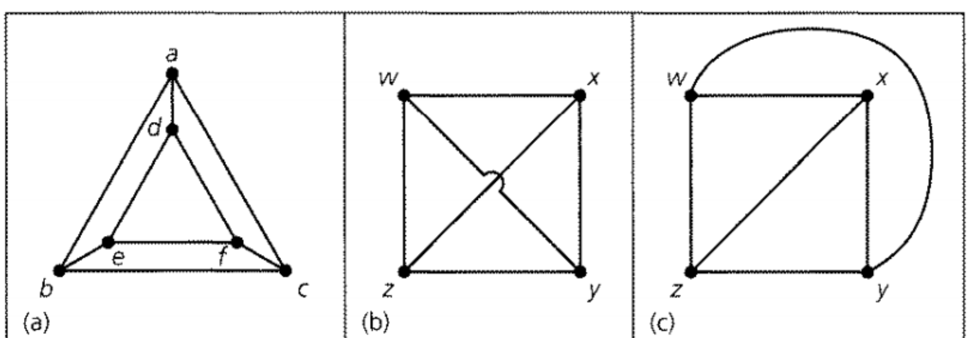

Figure 11.47

Planar Graphs (3/24)

Definition 11.18:

A graph $G = (V, E)$ is called $bipartite$ if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$, and every edge of G is of the form $\{a, b\}$ with $a \in V_1$ and $b \in V_2$. If each vertex in V_1 is joined with every vertex in V_2 , we have a $complete\ bipartite$ graph. In this case, if $|V_1| = m$, $|V_2| = n$, the graph is denoted by $K_{m,n}$

Planar Graphs (5/24)

Definition 11.19:

Let $G = (V, E)$ be a loop-free undirected graph, where $E \neq \emptyset$. An $elementary\ subdivision$ of G results when an edge $e = \{u, w\}$, is removed from G and then the edges $\{u, v\}, \{v, w\}$ are added to $G - e$, where $u \notin \mathrm{V}$.

The loop-free undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called $homeomorphic$ if they are isomorphic or if they can both be obtained from the same loop-free undirected graph H by a sequence of elementary subdivisions.

Planar Graphs (6/24)

EXAMPLE 11.18:

a) Let $G = (V, E)$ be a loop-free undirected graph with $|E| \geq 1$. If $G'$is obtained from G by an elementary subdivision, then the graph

$$G' = (V' , E')$$

satisfies $|V'| = |\mathrm{V}| + 1$ and $|\mathrm{E}'| = |\mathrm{E}| + 1$

b) Consider the graphs $G, G_1, G_2,$ and G_3 in Fig, 11.51

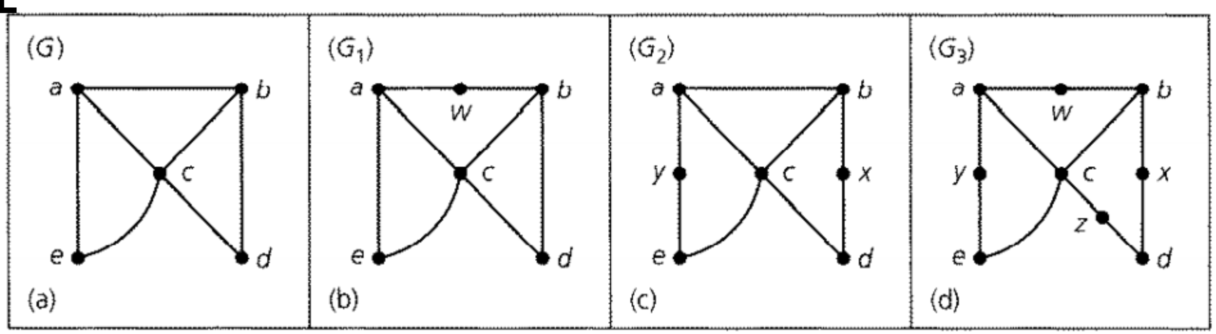

Figure 11.51

Planar Graphs (7/24)

Here G_1 is obtained from G by means of one elementary subdivision: Delete edge $\{a, b\}$ from G and then add the edges $\{a, w\}$ and $\{w\ , b\}$. The graph G_2 is obtained from G by two elementary subdivisions.

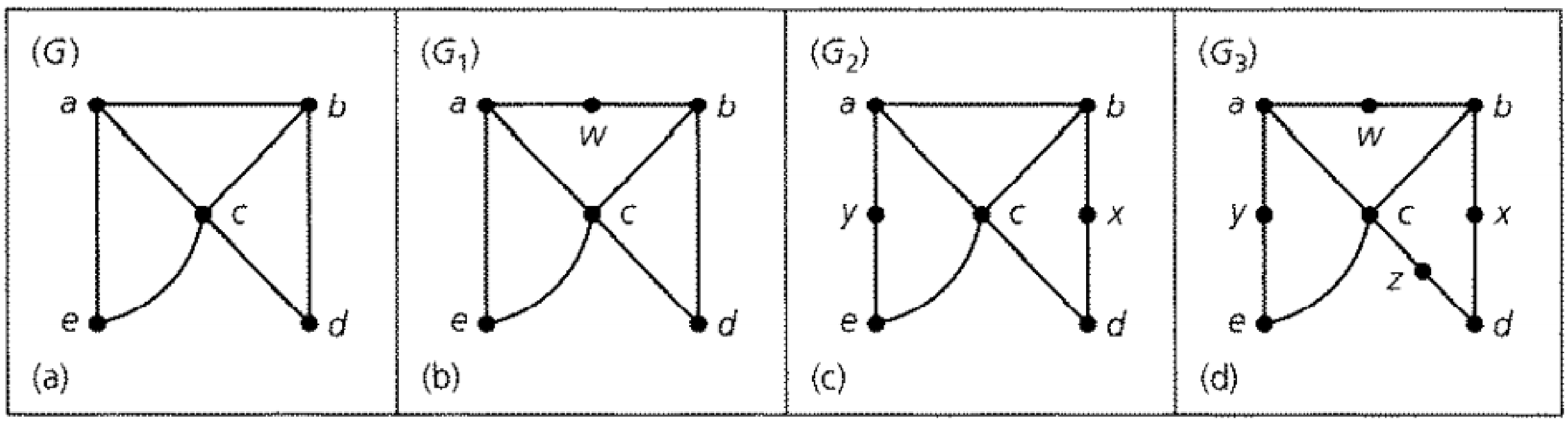

Figure 11.51

Planar Graphs

Hence G_1 and G_2 are homeomorphic. Also, G_3 can be obtained from G by four elementary subdivisions so G_3 is homeomorphic to both G_1 and G_2.

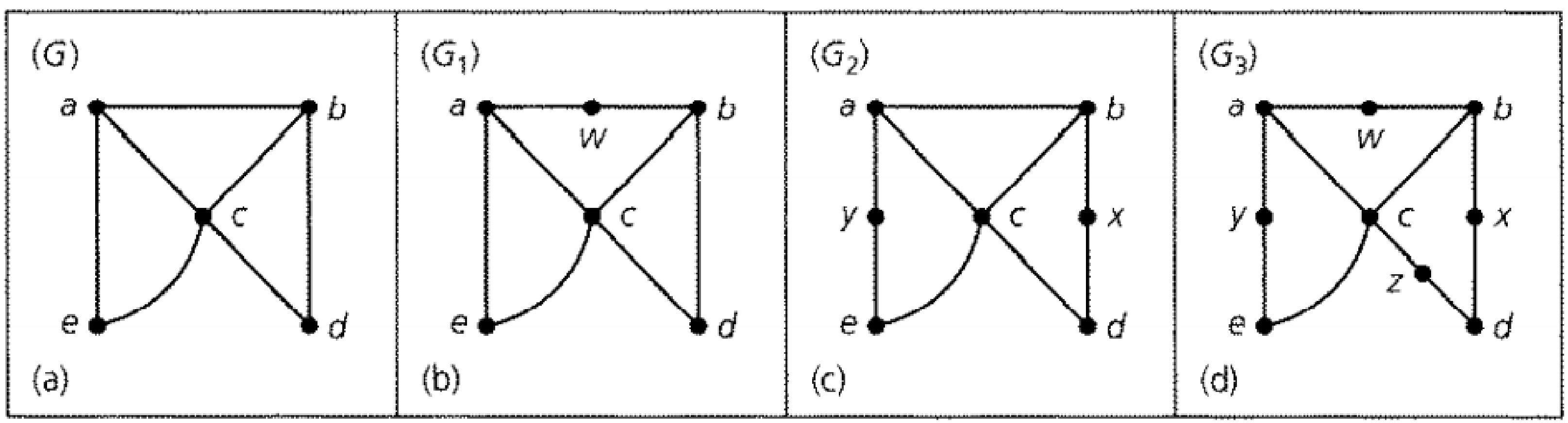

Figure 11.51

Planar Graphs (8/24)

THEOREM 11.5:

$Kuratowski's$ Theorem. A graph is nonplanar if and only if it contains a subgraph that is homeomorphic to either K_5 or $K_{3,3}$.

Planar Graphs (9/24)

EXAMPLE 11.19:

a) Figure 11.52(a) is a familiar graph called the $Petersen\ graph$. Part (b) of the figure provides a subgraph of the Petersen graph that is homeomorphic to $K_{3,3}$.

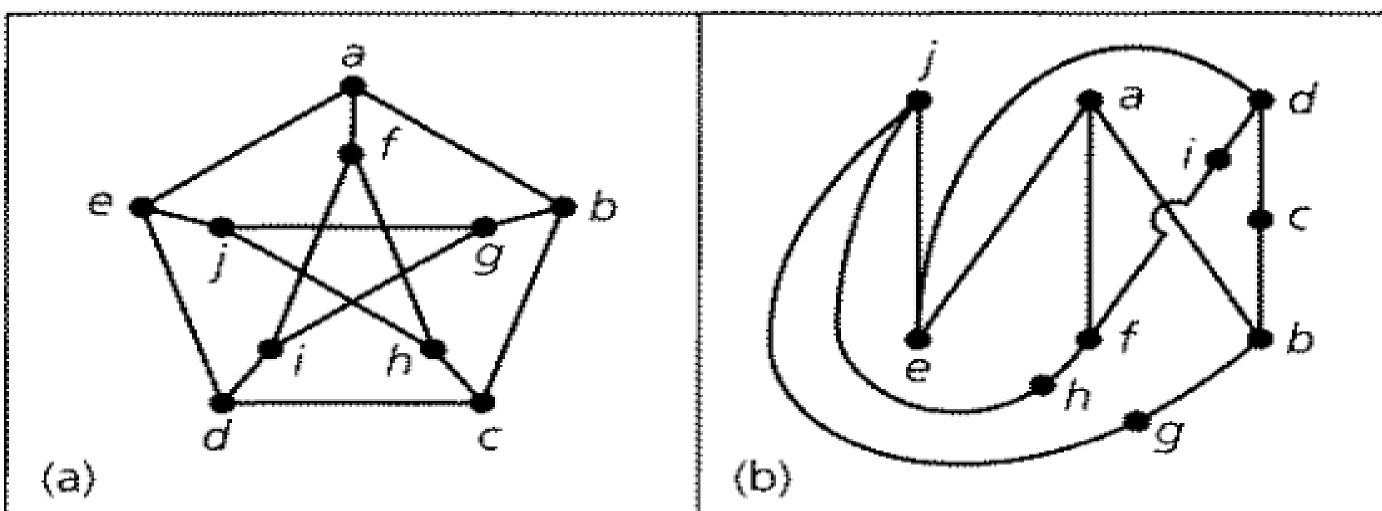

Figure 11.52

Planar Graphs (12/24)

(Figure 11.53 shows how
the subgraph is obtained from $K_{3,3}$ by a sequence of four elementary subdivisions.)
Hence the Petersen graph is nonplanar.

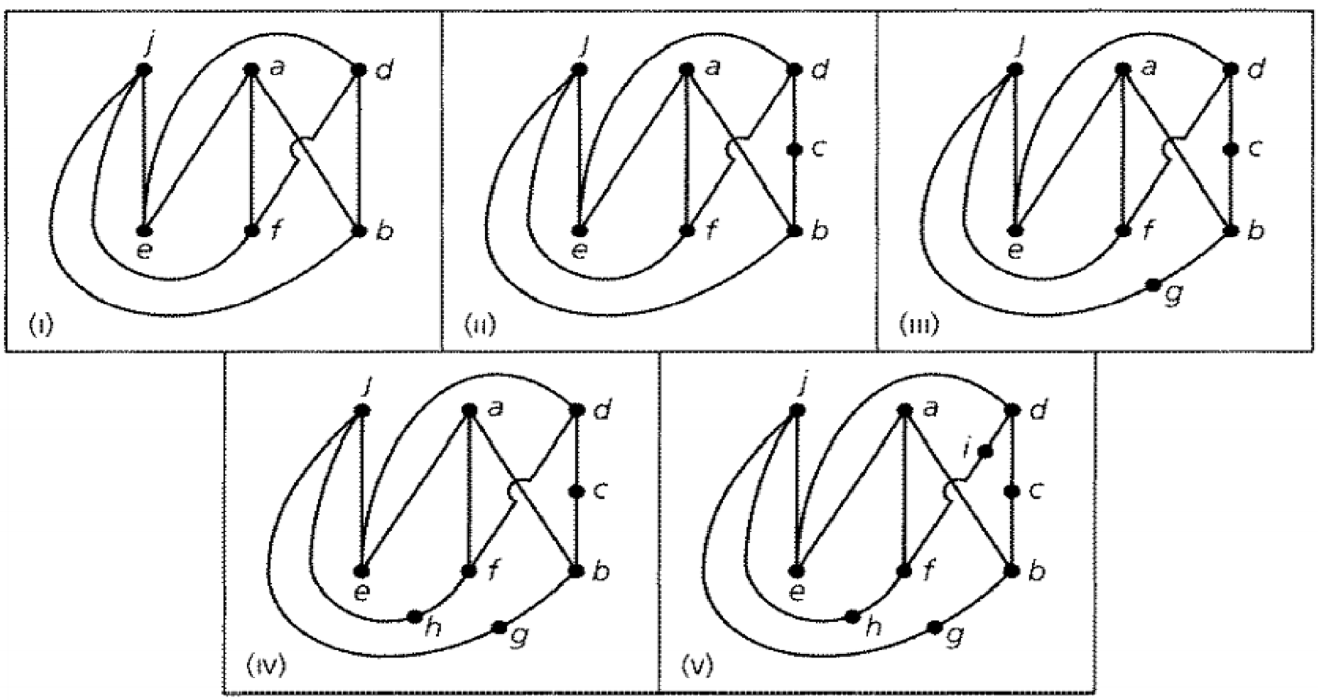

Figure 11.53

Planar Graphs (10/24)

b) In part (a) of Fig. 11.54 we find the 3-regular graph G, which is isomorphic to the 3-dimensional hypercube Q_3.

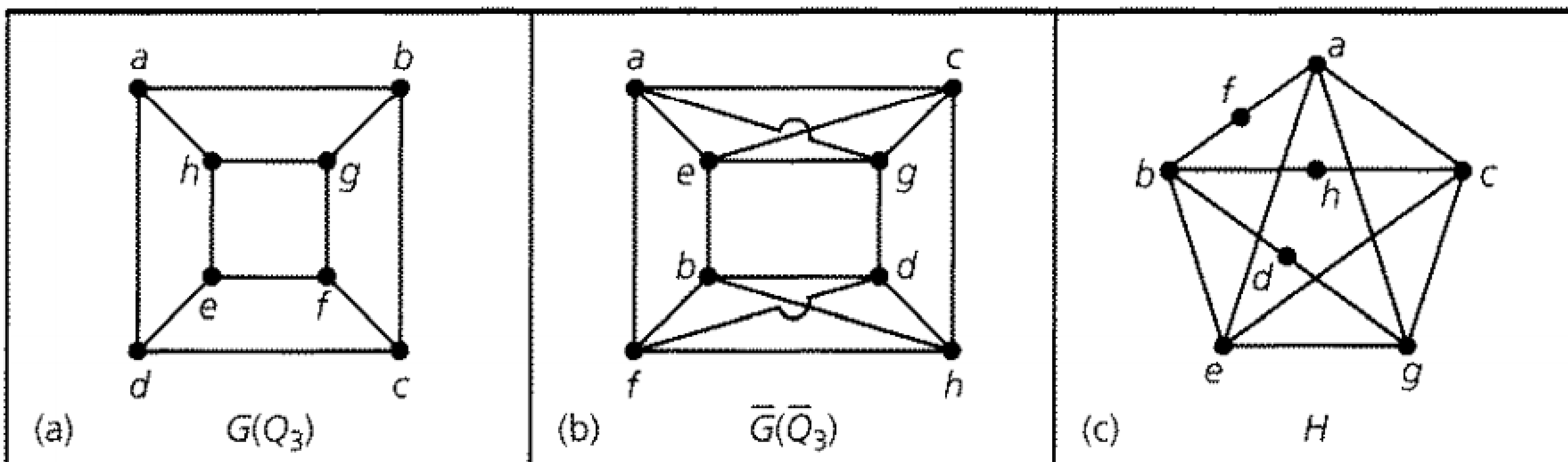

Figure 11.54

Planar Graphs (13/24)

The 4-regular complement of G is shown in Fig. 11.54(b), where the edges $\{a, g\}$; and $\{d, f\}$ suggest that G *may be* nonplanar. Figure 11.54(c) depicts a subgraph H of $\bar{G}$ that is homeomorphic to K_5 , so by Kuratowski's Theorem it follows that $\bar{G}$ is nonplanar.

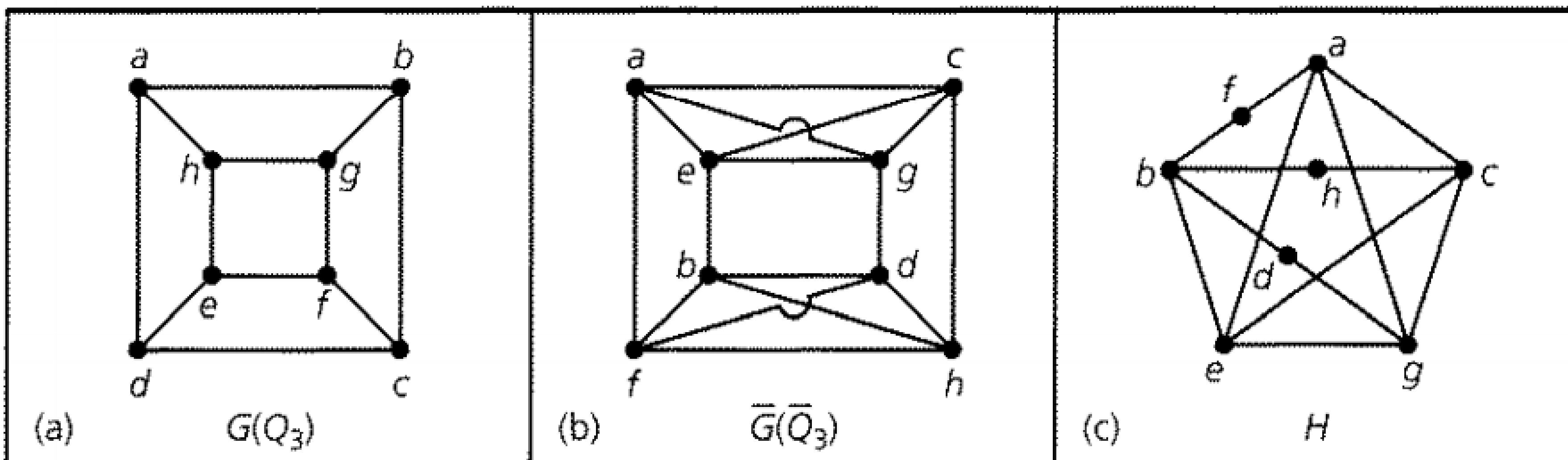

Figure 11.54

Planar Graphs (14/24)

THEOREM 11.6:

Let $G = (V, E)$ be a connected planar graph or multigraph with $|V| = v$ and $|E| = e$. Let r be the number of regions in the plane determined by a planar embedding (or, depiction) of G; one of these regions has infinite area and is called the $infinite\ region$. Then $v - e + r = 2$.

Planar Graphs (15/24)

Proof: The proof is by induction on e. If $e\ = 0\ or\ I$, then G is isomorphic to one of the graphs in Fig. 11.56. The graph in part (a) has $v = 1$, $e = 0, and\ r = 1$; $so, v - e + r = 1 - 0 + 1 = 2$. For graph (b),$v = 1, e = 1, and\ r = 2$. Graph (c) has $v = 2, e = 1, and\ r = 1$. In both cases, $v - e + r = 2$

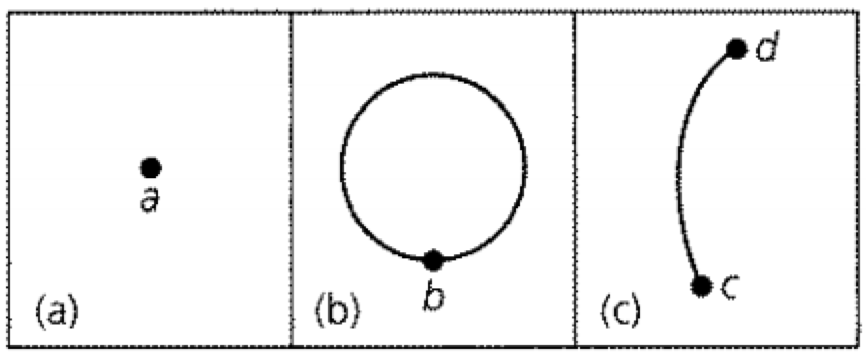

Figure 11.56

Planar Graphs (16/24)

Now let $k \in N$ and assume that the result is true for every connected planar graph or multigraph with e edges, where $0 \leq e \leq k$. If $G = (V, E)$ is a connected planar graph or multigraph with v vertices, r regions, and $e = k + 1$ edges, let $a, b \in V$ with $\{a, b\} \in E$. Consider the subgraph H of G obtained by deleting the edge $\{a , b\}$ from G. (If G is a multigraph and $\{a, b\}$ is one of a set of edges between a and b, then we remove it only once.) Consequently, we may write $H = G - \{a, b\}$ or $G = H + \{a, b\}$. We consider the following two cases, depending on whether H is connected or disconnected.

Planar Graphs (17/24)

Case 1:

The results in parts (a), (b), (c), and (d) of Fig. 11.57 show us how a graph G may be obtained from a connected graph H when the (new) loop $\{a, a\}$ is drawn as in parts (a) and (b) or when the (new) edge $\{a, b\}$ joins two distinct vertices in H as in parts (c) and (d). In all of these situations, H has v vertices, k edges, and $r-1$ regions because one of the regions for H is split into two regions for G. The induction hypothesis applied to graph H tells us that $v - k + (r - 1) = 2$, and from this it follows that $2 = v-(k+1)+r = v-e+r,$ So Euler's Theorem is true for G in this case.

Planar Graphs (18/24)

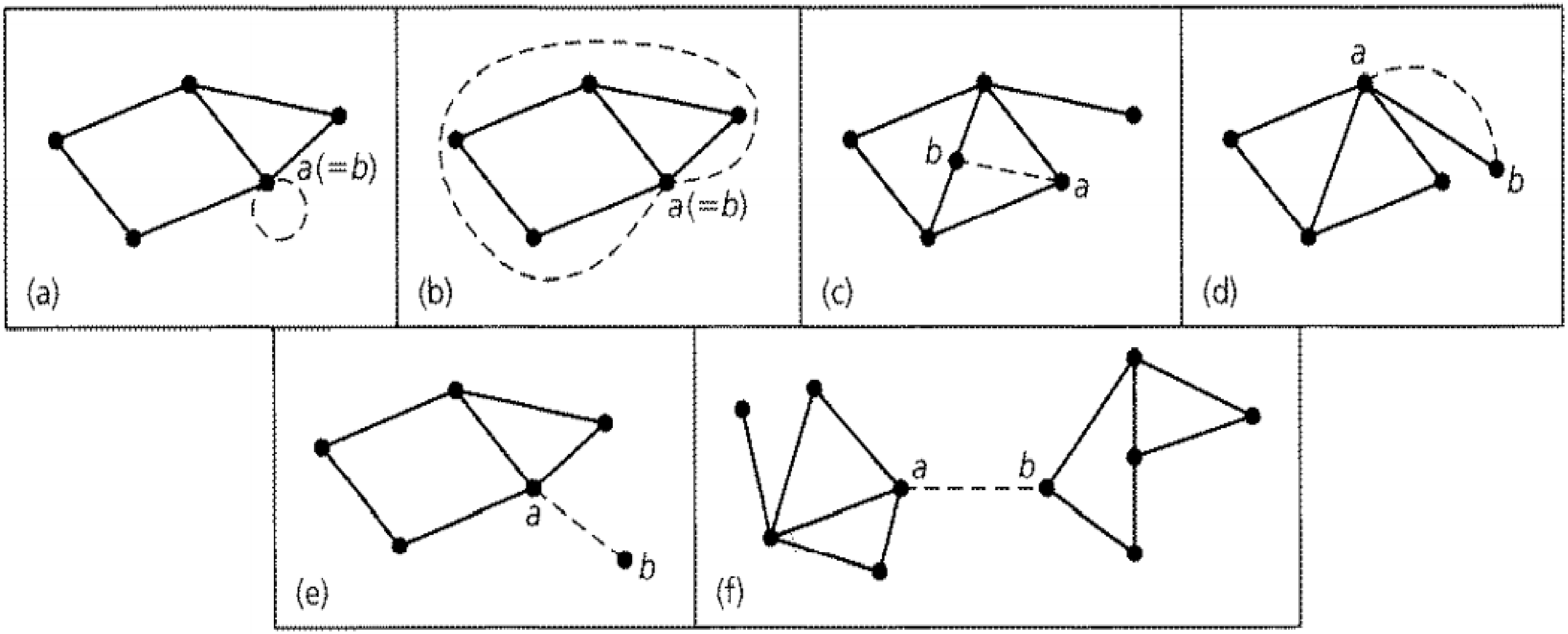

Figure 11.57

Planar Graphs (19/24)

Case 2:

Now we consider the case where $G - \{a, b\} = H$ is a disconnected graph [as demonstrated in Fig. 11.57(e) and (f)]. Here H has v vertices, k edges, and r regions. Also, H has two components H_1 and H_2 , where H_i has v_i vertices, e_i edges, and r_i regions, for $i \ = \ 1, 2$. [Part (e) of Fig, 11.57 indicates that one component could consist of just an isolated vertex.] Furthermore, $v_1 + v_2 = v$, $e_1 + e_2 = k (= e - 1)$, and $r_1 + r_2 = r + 1$ because each of H_1 and H_2 determines an infinite region.

Planar Graphs (20/24)

When we apply the induction hypothesis to each of H_1 and H_2 we learn that

$$v_1 - e_1 + r_1 = 2 \text{ and } v_2 - e_2 + r_2 = 2$$

Consequently,$(v_1 + v_2) - (e_1 + e_2) + (r_1 + r_2) = v - (e - 1) + (r + 1) = 4$.and from this it follows that $v - e + r = 2$, thus establishing Euler's Theorem for G in this case.

Planar Graphs (21/24)

COROLLARY 11.3:

Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v$, $|E| = e > 2$, and r regions. Then $3r \leq 2e$ and $e \leq 3v - 6$.

Proof: Since G is loop-free and is not a multigraph, the boundary of each region (including the infinite region) contains at least three edges ---- hence, each region has degree ≥ 3.

Planar Graphs (22/24)

Consequently, $2e = 2|E| =$ the sum of the degrees of the r regions determined by G and $2e \geq 3r$. From Euler's Theorem,$2 = v - e + r \leq v - e + \left(\frac{2}{3}\right)e = v - \left(\frac{1}{3}\right)e$,so $6 \leq 3v - e$, or $e \leq 3v - 6$.

Planar Graphs (23/24)

EXAMPLE 11.20:

The graph K_5 is loop-free and connected with ten edges and five vertices. Consequently,

$3v - 6 = 15 - 6 = 9 < 10 = e$. Therefore, by Corollary 11.3, we find that K_5 is nonplanar.

Planar Graphs (24/24)

Definition 11.20:

Let $G = (V, E)$ be an undirected graph or multigraph. A subset E' of E is called a $cut - set$ of G if by removing the edges (but not the vertices) in E' from G, we have $\kappa(G) < \kappa(G')$, where $G' = (V, E - E')$; but when we remove (from E) any proper subset E" of E', we have $\kappa(G) = \kappa(G'')$, for $G'' = (\mathrm{V}, \mathrm{E} - E'')$

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Hamilton cycle and Hamilton path (1/33)

- **Definition 11.21:** If $G = (V, E)$ is a graph or multigraph with $|V| \geq 3$, we say that G has a *Hamilton cycle* if there is a cycle in G that contains every vertex in V. A *Hamilton path* is a path (and not a cycle) in G that contains each vertex.

Hamilton cycle and Hamilton path (2/33)

- **Example 11.26:** Referring back to the hypercubes in Fig. 11.35 we find in Q_2 the cycle

$$00 \rightarrow 10 \rightarrow 11 \rightarrow 01 \rightarrow 00$$

and in Q_3 the cycle

$$000 \rightarrow 100 \rightarrow 110 \rightarrow 010 \rightarrow 011 \rightarrow 111 \rightarrow 101 \rightarrow 001 \rightarrow 000.$$

Hence Q_2 and Q_3 have Hamilton cycles (and paths).

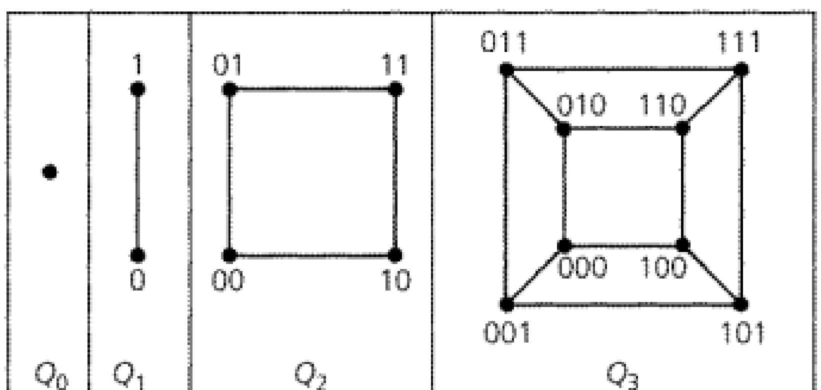


Figure 11.35

Hamilton cycle and Hamilton path (3/33)

- **Example 11.26 (cont.):**

  In fact, for all $n \geq 2$, we find that Q_n has a Hamilton cycle. (The reader is asked to establish this in the Section Exercises.) [Note, in addition, that the listings: 00, 10, 11, 01 and 000, 100, 110, 010, 011, 111, 101, 001 are examples of Gray codes (which were introduced in Example 3.9).]

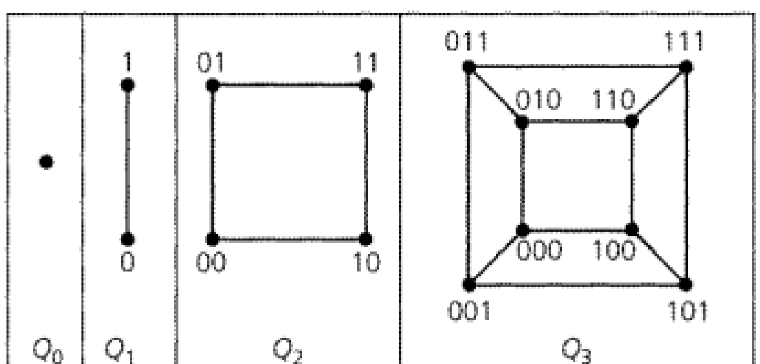


Figure 11.35

Hamilton cycle and Hamilton path (4/33)

- **Example 11.27:** If *G* is the graph in Fig. 11.78, the edges {*a*, *b*}, {*b*, *c*}, {*c*, *f*}, {*f*, *e*}, {*e*, *d*}, {*d*, *g*},{*g*, *h*}, {*h*, *i*} yield a Hamilton path for *G*. But does *G* have a Hamilton cycle?

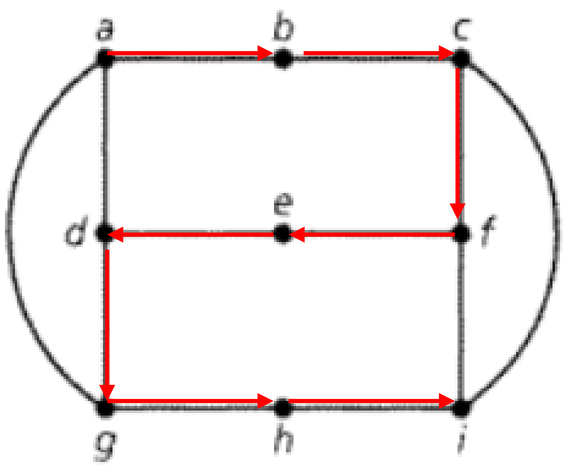


Figure 11.78

Hamilton cycle and Hamilton path (5/33)

- **Example 11.27 (cont.):**

  Since *G* has nine vertices, if there is a Hamilton cycle in *G* it must contain nine edges. Let us start at vertex *b* and try to build a Hamilton cycle. Because of the symmetry in the graph, it doesn't matter whether we go from *b* to *c* or to *a*. We'll go to *c*. At c we can go either to *f* or to *i*. Using symmetry again, we go to *f*.

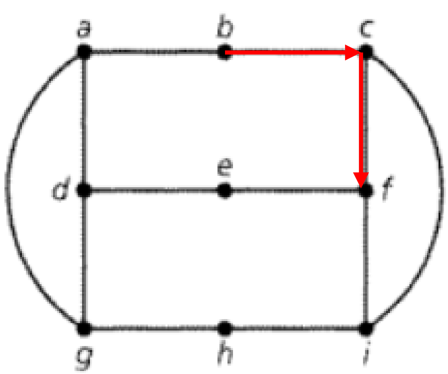


Figure 11.78

Hamilton cycle and Hamilton path (6/33)

- **Example 11.27 (cont.):**

  Then we delete edge {*c*, *i*} from further consideration because we cannot return to vertex *c*. In order to include vertex *i* in our cycle, we must now go from *f* to *i* (to *h* to *g*). With edges {*c*, *f*} and {*f*, *i*} in the cycle, we cannot have edge {*e*, *f*} in the cycle.

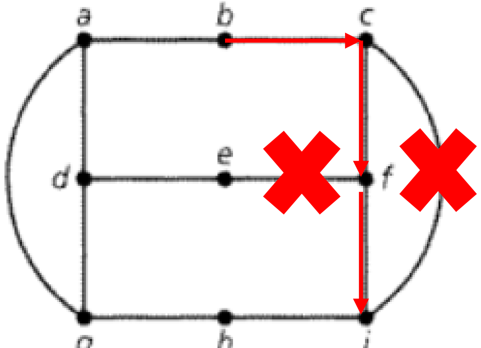


Figure 11.78

Hamilton cycle and Hamilton path (7/33)

- **Example 11.27 (cont.):**

  [Otherwise, in the cycle we would have deg(*f*) > 2.] But then once we get to *e* we are stuck.

  Hence there is no Hamilton cycle for the graph.

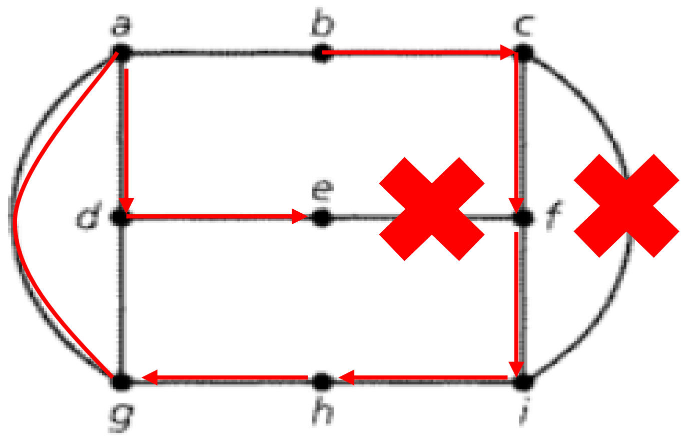


Figure 11.78

Hamilton cycle and Hamilton path (8/33)

- **Example 11.27** indicates a few helpful hints for trying to find a Hamilton cycle in a graph G=(V, E).

1) If G has a Hamilton cycle, then for all $v \in V$, deg(v) $\geq$ 2.

2) If $a \in V$ and deg(a) = 2, then the two edges incident with vertex a must appear in every Hamilton cycle for G.

Hamilton cycle and Hamilton path (9/33)

3) If $a \in V$ and deg(a) > 2, then as we try to build a Hamilton cycle, once we pass through vertex a, any unused edges incident with a are deleted from further consideration.

4) In building a Hamilton cycle for G, we cannot obtain a cycle for a subgraph of G unless it contains all the vertices of G.

Hamilton cycle and Hamilton path (10/33)

- **Example 11.28:** In Fig. 11.79(a) we have a connected graph *G*, and we wish to know whether *G* contains a Hamilton path. Part (b) of the figure provides the same graph with *a* set of labels *x*, *y*. This labeling is accomplished as follows: First we label vertex *a* with the letter *x*. Those vertices adjacent to *a* (namely, *b*, *c*, and *d*) are then labeled with the letter *v*.

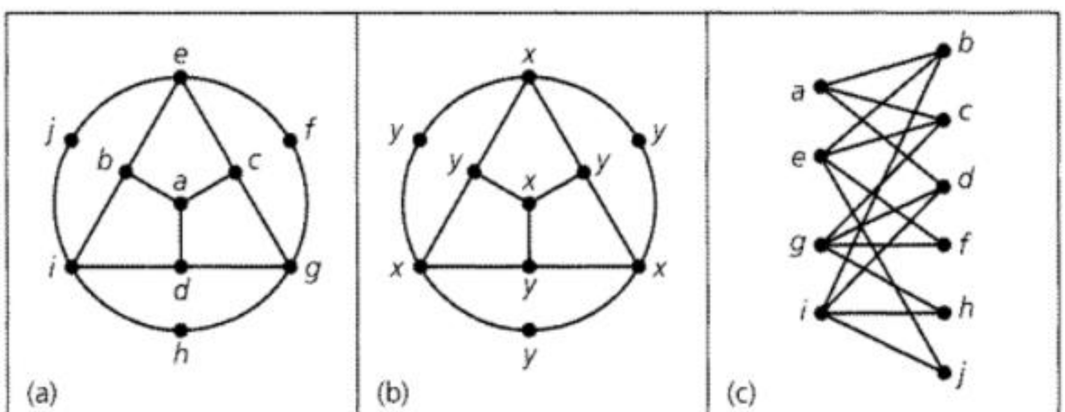


Figure 11.79

Hamilton cycle and Hamilton path (11/33)

- **Example 11.28 (cont.):**

  Then we label the unlabeled vertices adjacent to *b*, *c*, or *d* with *x*. This results in the label *x* on the vertices *e*, *g*, and *i*. Finally, we label the unlabeled vertices adjacent to *e*, *g*, or *i* with the label *y*. At this point, all the vertices in *G* are labeled.

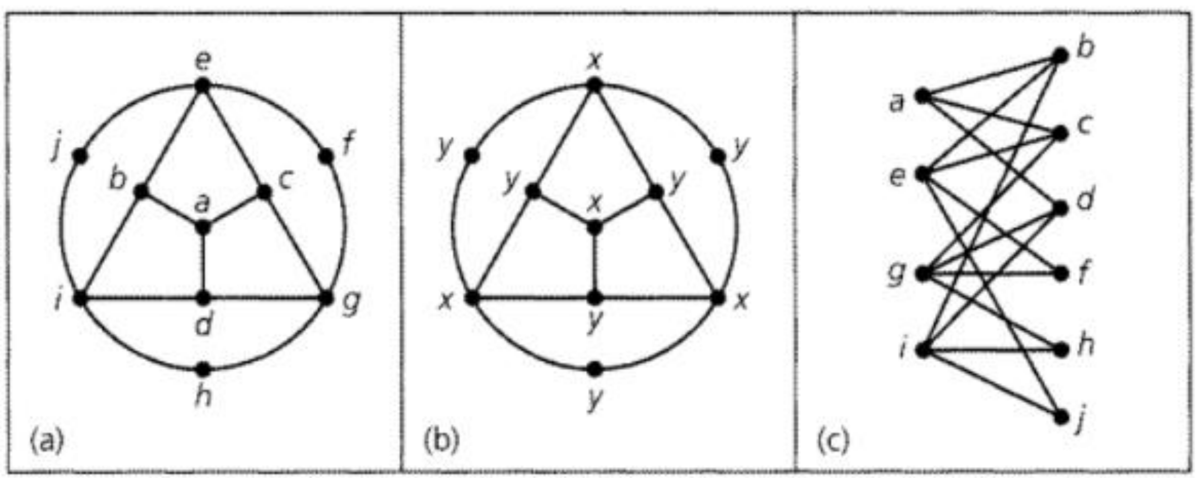


Figure 11.79

Hamilton cycle and Hamilton path (12/33)

- **Example 11.28 (cont.):**

  Now, since |*V*| - 10, if *G* is to have a Hamilton path there must be an alternating sequence of five *x*'s and five *y*'s. Only four vertices are labeled with *x*, so this is impossible. Hence *G* has no Hamilton path (or cycle).

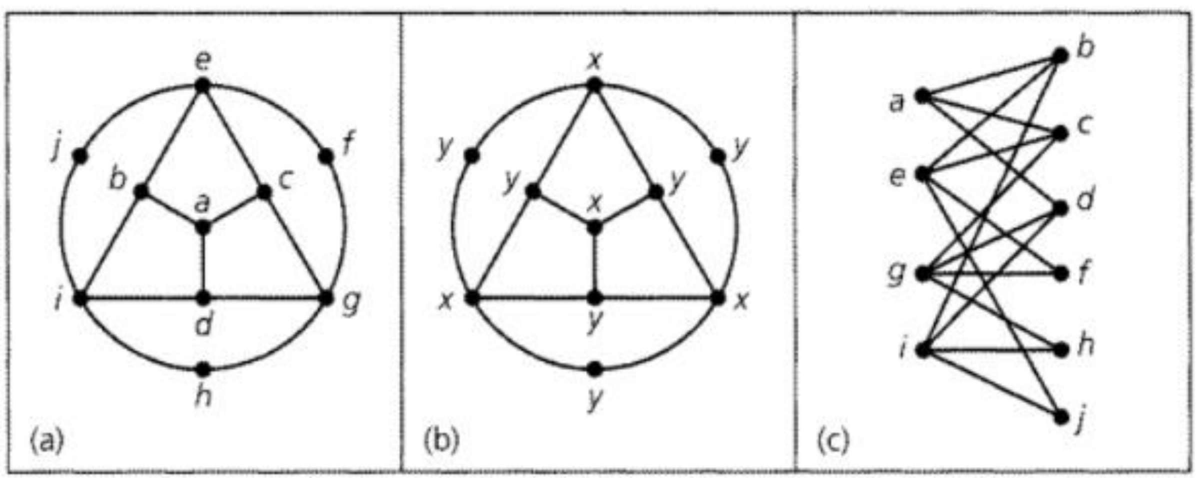


Figure 11.79

Hamilton cycle and Hamilton path (13/33)

- **Example 11.28 (cont.):**

  But why does this argument work here? In part (c) of Fig. 11.79 we have redrawn the given graph, and we see that it is bipartite. From Exercise 10 in the previous section we know that a bipartite graph cannot have a cycle of odd length. It is also true that if a graph has no cycle of odd length, then it is bipartite.

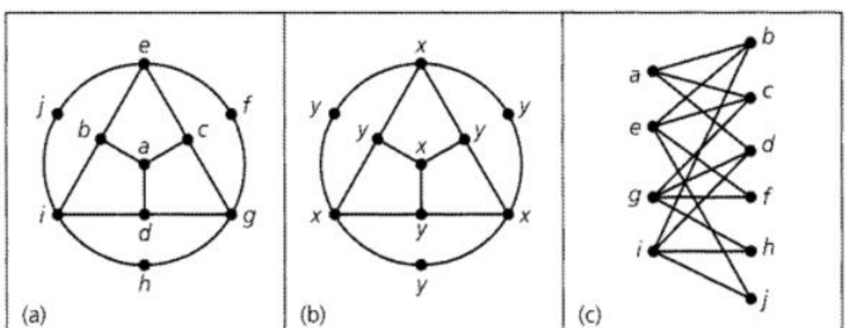


Figure 11.79

Hamilton cycle and Hamilton path (14/33)

- **Example 11.28 (cont.):**

  (The proof is requested of the reader in Exercise 9 of this section.) Consequently, whenever a connected graph has no odd cycle (and is bipartite), the method described above may be helpful in determining when the graph does not have a Hamilton path. (Exercise 10 in this section examines this idea further.)

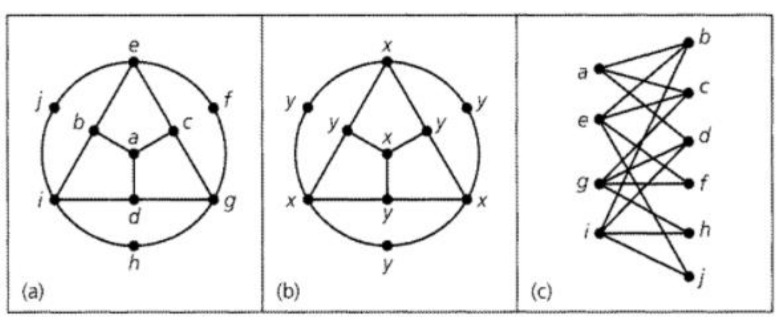


Figure 11.79

Hamilton cycle and Hamilton path (15/33)

- **Example 11.29:** At Professor Alfred‘s science camp, 17 students have lunch together each day at a circular table. They are trying to get to know one another better, so they make an effort to sit next to two different colleagues each afternoon. For how many afternoons can they do this? How can they arrange themselves on these occasions?

To solve this problem we consider the graph K_n, where $n \geq 3$ and is odd.

Hamilton cycle and Hamilton path (16/33)

- **Example 11.29 (cont.):**
  This graph has n vertices (one for each student) and $\binom{n}{2}$ = $n(n$ - 1)/2 edges. A Hamilton cycle in K_n corresponds to a seating arrangement. Each of these cycles has n edges, so we can have at most (1/n)$\binom{n}{2}$ = (n - 1)/2 Hamilton cycles with no two having an edge in common.

Hamilton cycle and Hamilton path (17/33)

- **Example 11.29 (cont.):**

  Consider the circle in Fig. 11.80 and the subgraph of K_n, consisting of the n vertices and the n edges {1, 2}, {2, 3}, ... {n-1, n}, {n, 1}. Keep the vertices on the circumference fixed and rotate this Hamilton cycle clockwise through the angle $[1/(n - 1)](2\pi)$.

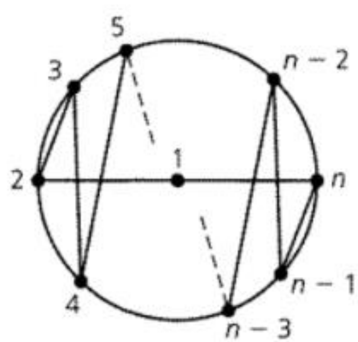


Figure 11.80

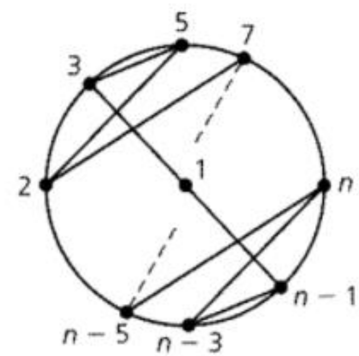


Figure 11.81

Hamilton cycle and Hamilton path (18/33)

- **Example 11.29 (cont.):**
  This gives us the Hamilton cycle (Fig. 11.81) made up of edges (1, 3), (3, 5), (5, 2), (2, 7), ..., {n, n - 3}, {n - 3, n - 1}, {n - 1, 1}. This Hamilton cycle has no edge in common with the first cycle.

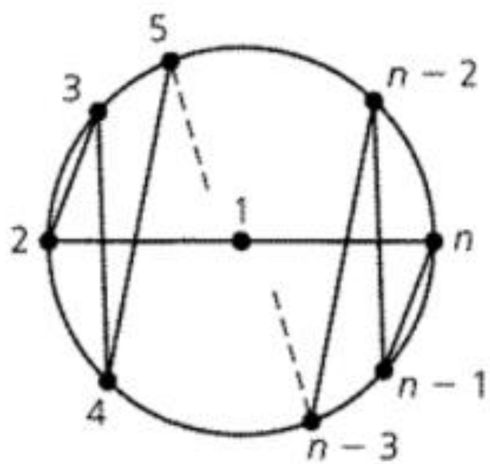


Figure 11.80

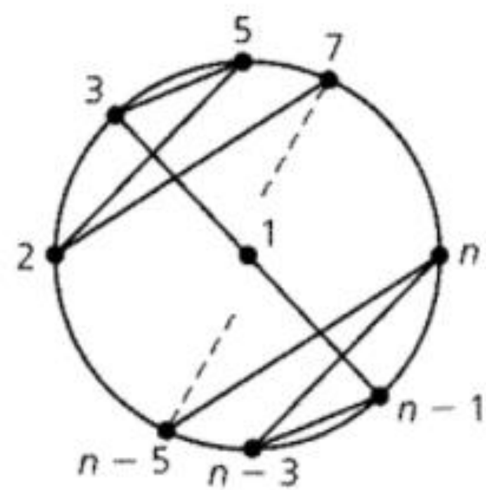


Figure 11.81

Hamilton cycle and Hamilton path (19/33)

- **Example 11.29 (cont.):**
  When n $\geq$ 7 and we continue to rotate the cycle in Fig. 11.80 in this way through angles $[k/(n - 1)](2\pi)$, where $2 \leq k \leq (n - 3)/2$, we obtain a total of $(n - 1)/2$ Hamilton cycles, no two of which have an edge in common.

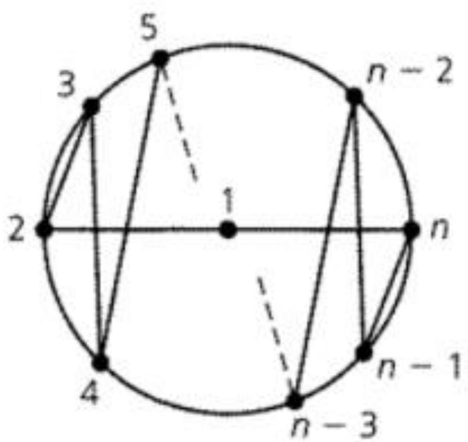


Figure 11.80

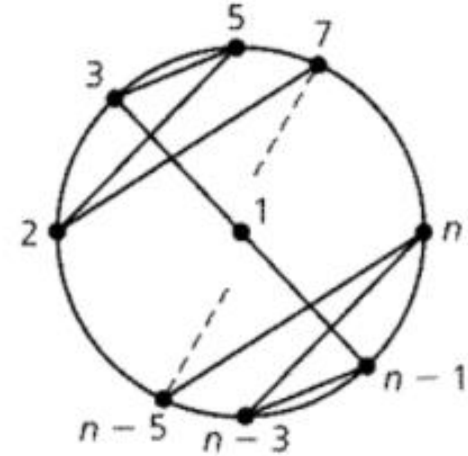


Figure 11.81

Hamilton cycle and Hamilton path (20/33)

- **Theorem 11.7:** Let K_n^* be a complete directed graph - that is, K_n^* has *n* vertices and for each distinct pair *x*, *y* of vertices, exactly one of the edges (*x*, *y*) or (*y*, *x*) is in K_n^*. Such a graph (called a *tournament*) always contains a (directed) Hamilton path.

  Proof: Let $m \geq 2$ with p_m a path containing the m - l edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{m-1}, v_m)$. If $m=n$, we're finished. If not, let *v* be a vertex that doesn't appear in p_m.

Hamilton cycle and Hamilton path (21/33)

- **Theorem 11.7 (cont.):**
  If (v, v_1) is an edge in K_n^*, we can extend p_m by adjoining this edge. If not, then (v_1, v) must be an edge. Now suppose that (v, v_2) is in the graph. Then we have the larger path: (v_1, v), (v, v_2). (v_2, v_3), ..., (v_{m-1}, v_m). If (v, v_2) is not an edge in K_n^*, then (v_2, v) must be. As we continue this process there are only two possibilities:

Hamilton cycle and Hamilton path (22/33)

- **Theorem 11.7 (cont.):**

  (a) For some $1 \leq k \leq m - 1$ the edges (v_k, v), (v, v_{k+1}) are in K_n^*, and we replace (v_k, v_{k+1}) with this pair of edges.

  (b) (v_m, v) is in K_n^* and we add this edge to p_m.

  Either case results in a path p_{m+1} that includes $m + 1$ vertices and has m edges. This process can be repeated until we have such a path p_n on n vertices.

Hamilton cycle and Hamilton path (23/33)

- **Example 11.30:** In a round-robin tournament each player plays every other player exactly once. We want to somehow rank the players according to the results of the tournament. Since we could have players *a*, *b*, and *c* where *a* beats *b* and *b* beats *c*, but *c* beats *a*, it is not always possible to have a ranking where a player in a certain position has beaten all of the opponents in later positions.

Hamilton cycle and Hamilton path (24/33)

- **Example 11.30 (cont.):**

  Representing the players by vertices, construct a directed graph *G* on these vertices by drawing edge (x, y) if x beats y. Then by **Theorem 11.7**, it is possible to list the players such that each has beaten the next player on the list.

Hamilton cycle and Hamilton path (25/33)

- **Theorem 11.8:** Let $G = (V, E)$ be a loop-free graph with $|V| = n \geq 2$. If $\deg(x) + \deg(y) \geq n - 1$ for all $x, y \in V$, $x \neq y$, then G has a Hamilton path.

Proof: First we prove that G is connected. If not, let C_1, C_2 be two components of G and let x, $y \in$V with x a vertex in C_1 and y a vertex in C_2. Let C_i have n_i vertices, $i = 1, 2$. Then $\deg(x) \leq n_1 -1$, $\deg(y) \leq n_2 - 1$, and $\deg(x) + \deg(y) \leq (n_1 + n_2) - 2 \leq n - 2$, contradicting the condition given in the theorem. Consequently, G is connected.

Hamilton cycle and Hamilton path (26/33)

- **Proof (cont.):**

  Now we build a Hamilton path for G. For m $\geq$ 2, let p_m be the path $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{m-1}, v_m\}$ of length m - 1. (We relabel vertices if necessary.) Such a path exists, because for $m = 2$ all that is needed is one edge. If v_1 is adjacent to any vertex v other than $v_2, v_3, \ldots, v_\mathrm{m}$, we add the edge $\{v, v_1\}$ to p_m to get P_{m+1}. The same type of procedure is carried out if v_m is adjacent to a vertex other than $v_1, v_2, \ldots, v_{m-1}$.

Hamilton cycle and Hamilton path (27/33)

- **Proof (cont.):**

  If we are able to enlarge p_m to p_n in this way, we get a Hamilton path. Otherwise the path p_m: $\{v_1, v_2\}, \ldots, \{v_{m-1}, v_\mathrm{m}\}$ has $v_{1,}$ v_m adjacent only to vertices in p_m, and $m < n$. When this happens we claim that G contains a cycle on these vertices. If v_1 and v_m are adjacent, then the cycle is $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{m-1}, v_m), (v_m, v_1\}$.

Hamilton cycle and Hamilton path (28/33)

- **Proof (cont.):**

  If v_1 and v_m are not adjacent, then v_1 is adjacent to a subset S of the vertices in $\{v_2, v_3, ..., v_{m-1}\}$. If there is a vertex $v_t \in S$ such that v_m is adjacent to v_{t-1}, then we can get the cycle by adding $\{v_1, v_t\}$, $\{v_{t-1}. v_m\}$ to p_m and deleting $(v_{t-1}. v_t\}$ as shown in Fig. 11.82.

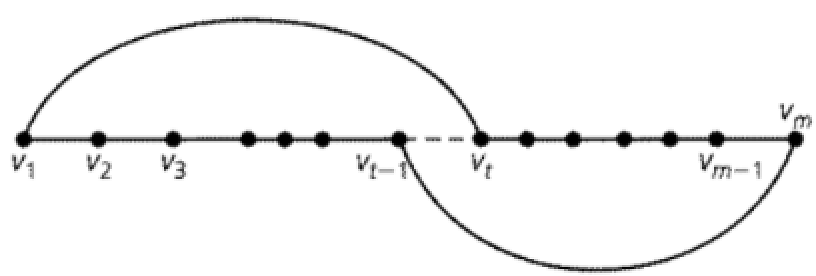


Figure 11.82

Hamilton cycle and Hamilton path (29/33)

- **Proof (cont.):**

  If not, let $|S| = k < m - 1$. Then $\deg(v_1) = k$ and $\deg(v_m) \leq (m - 1) - k$, and we have the contradiction $\deg(v_1) + \deg(v_m) \leq m - 1 < n - 1$. Hence there is a cycle connecting $v_1, v_2, \ldots, v_m$.

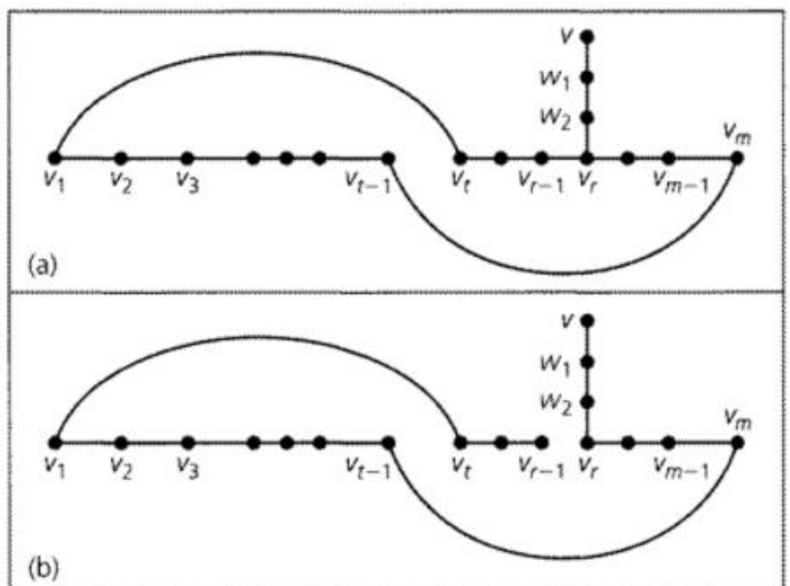


Figure 11.83

Hamilton cycle and Hamilton path (30/33)

- **Proof (cont.):**

  Now consider a vertex $v \in V$ that is not found on this cycle. The graph G is connected, so there is a path from v to a first vertex v_r in the cycle, as shown in Fig. 11.83(a). Removing the edge $\{v_{r\text{-}1}, v_r\}$(or $\{v_1, v_t\}$ if $r = t$), we get the path (longer than the original p_m) shown in Fig. 11.83(b).

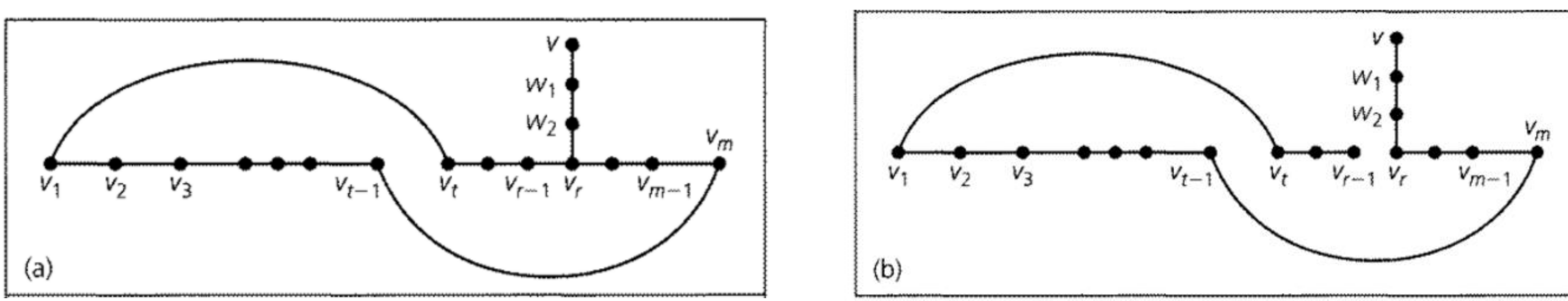


Figure 11.83

Hamilton cycle and Hamilton path (31/33)

- **Proof (cont.):**

  Repeating this process (applied to p_m) for the path in Fig. 11.83(b), we continue to increase the length of the path until it includes every vertex of G.

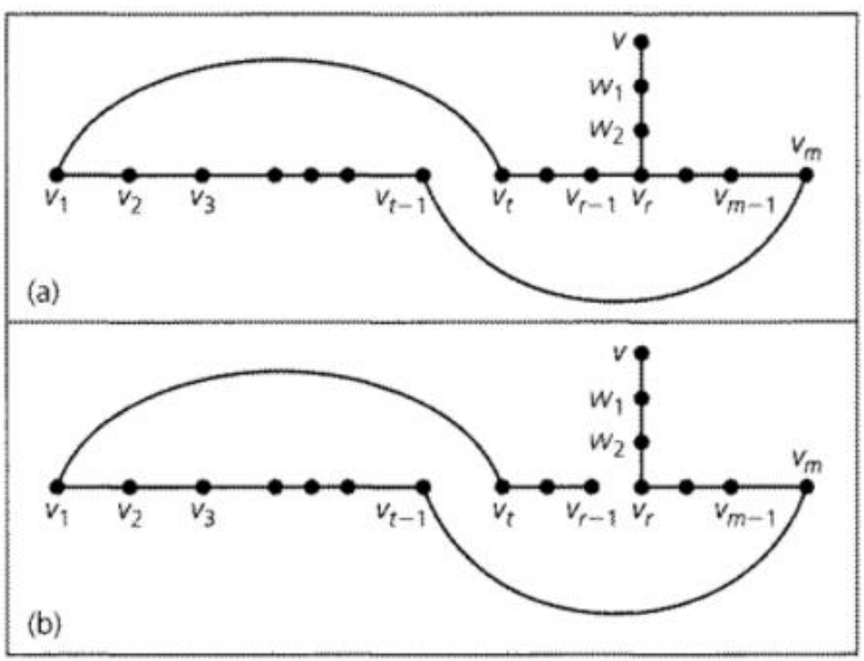


Figure 11.83

Hamilton cycle and Hamilton path (32/33)

- **Corollary 11.4:** Let $G = (V, E)$ be a loop-free graph with n (≥ 2) vertices. If $\deg(v) \geq (n - 1)/2$ for all $v \in V$, then G has a Hamilton path.

- **Theorem 11.9:** Let $G = (V, E)$ be a loop-free undirected graph with $|V| = \mathrm{n} \geq 3$. If $\deg(x) + \deg(y) \geq n$ for all nonadjacent $x, y \in V$, then G contains a Hamilton cycle.

Hamilton cycle and Hamilton path (33/33)

- **Corollary 11.5:** If $G = (V, E)$ is a loop-free undirected graph with $|V| = n \geq 3$, and if $\deg(v) > n/2$ for all $v \in V$, then G has a Hamilton cycle.

- **Corollary 11.6:** If $G = (V, E)$ is a loop-free undirected graph with $|V| = n \geq 3$, and if $|E| \geq \binom{n-1}{2} + 2$, then G has a Hamilton cycle.

Outline

- Definitions and Examples
- Subgraphs, Complements, and Graph Isomorphism
- Vertex Degree: Euler Trails and Circuits
- Planar Graphs
- Hamilton Paths and Cycles
- ***Graph Coloring and Chromatic Polynomials***

Chromatic number(1/4)

Definition 11.22:

If $G = (V, E)$ is an undirected graph, a *proper coloring* of G occurs when we color the vertices of G so that if $\{a, b\}$ is an edge in G, then a and b are colored with different colors. (Hence adjacent vertices have different colors.) The minimum number of colors needed to properly color G is called the *chromatic number* of G and is written $\chi(G)$.

Chromatic number(2/4)

EXAMPLE 11.31.

For the graph G in Fig. 11.87, we start at vertex a and next to each vertex write the number of a color needed to properly color the vertices of G that have been considered up to that point. Going to vertex b, the 2 indicates the need for a second color because vertices a and b a

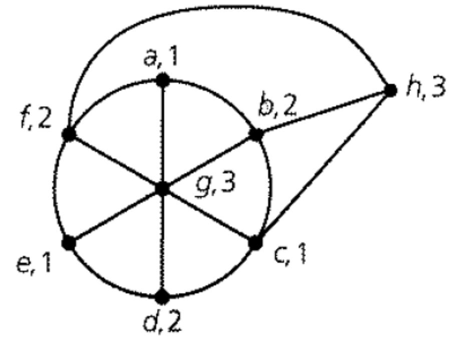


Figure 11.87

Chromatic number(3/4)

EXAMPLE 11.31 (cont).

Proceeding alphabetically to f, we find that two colors are needed to properly color $\{a, b, c, d, e, f\}$. For vertex g a third color is needed; this third color can also be used for vertex h because $\{g, h\}$ is not an edge in G. Thus this sequential coloring (labeling) method gives us a proper coloring for G, so $\chi(G) \leq 3$. Since K_3 is a subgraph of G, we have $\chi(G) \geq 3$, so $\chi(G) = 3$.

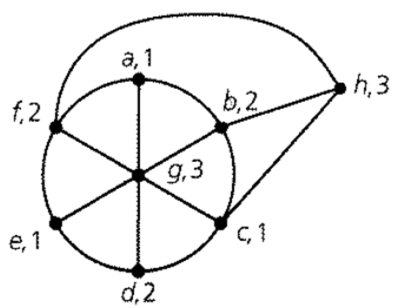


Figure 11.87

Chromatic number(4/4)

EXAMPLE 11.32.

a) For all $n \geq 1$, $\chi(K_n) = n$.

b) The chromatic number of the Herschel graph (Fig. 11.86) is 2.

c) If G is the Petersen graph, then x(G) = 3.

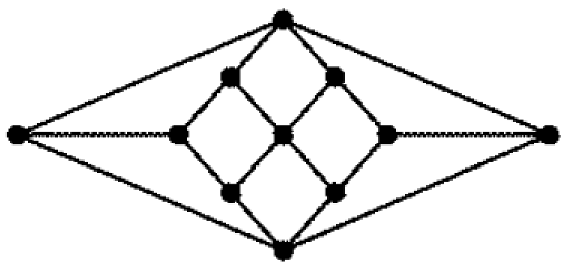


Figure 11.86

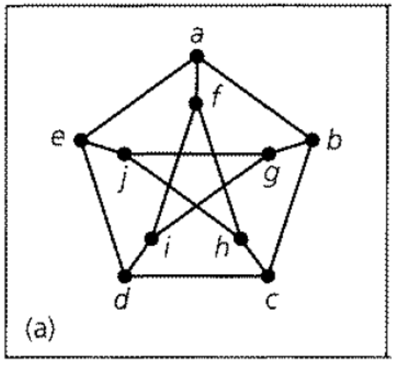


Figure 11.52

Chromatic Polynomial (1/5)

Let G be an undirected graph, and let λ be the number of colors that we have available for properly coloring the vertices of G. Our objective is to find a polynomial function $P(G, A)$, in the variable λ , called the *chromatic polynomial* of G, that will tell us in how many different ways we can properly color the vertices of G, using at most λ colors.

Chromatic Polynomial (2/5)

The vertices in an undirected graph $G\ (V, E)$ are distinguished by labels. Consequently, two proper colorings of such a graph will be considered different in the following sense: A proper coloring (of the vertices of G) that uses at most λ colors is a function f, with domain V and codomain $\{1, 2, 3, \ldots, \lambda\}$, where $f(u) \neq f(v)$, for adjacent vertices u, $v\ \in V$. Proper colorings are then different in the same way that these functions are different.

Chromatic Polynomial (3/5)

EXAMPLE 11.34.

a) If $G = (V, E)$ with $|V| = n$ and $E = \phi$, then G consists of n isolated points, and by the rule of product, $P(G, \lambda) = \lambda^n$.

b) If $G = K_n$, then at least n colors must be available for us to color G properly. Here, by the rule of product, $P(G, \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - n + 1)$, which we denote by $\lambda^{(n)}$. For $\lambda < n$, $P(G, \lambda) = 0$ and there are no ways to properly color K_n. $P(G, \lambda) > 0$ for the first time when $\lambda = n = \chi(G)$.

Chromatic Polynomial (4/5)

EXAMPLE 11.34 (cont).

c) For each path in Fig. 11.89, we consider the number of choices (of the λ colors) at each successive vertex. Proceeding alphabetically, we find that $P(G_1, \lambda) = \lambda(\lambda - 1)^3$ and $P(G_2, \lambda) = \lambda(\lambda - 1)^4$. Since $P(G_1, 1) = 0 = P(G_2, 1)$, but $P(G_1, 2) = 2 = P(G_2, 2)$, it follows that $\chi(G1) = \chi(G2) = 2$. If five colors are available we can properly color G_1, in $5(4)^3 = 320$ ways; G_2 can be so colored in $5(4)^4 = 1280$ ways.

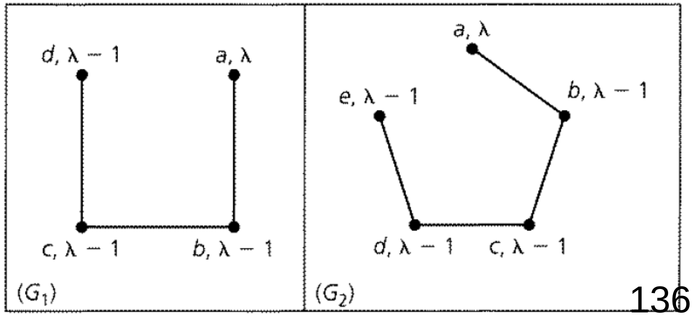


Figure 11.89

Chromatic Polynomial (5/5)

EXAMPLE 11.34 (cont).

d) If G is made up of components $G_1, G_2, \ldots, G_k$, then again by the rule of product, it follows that $P(G, \lambda) = P(G_1, \lambda)P(G_2, \lambda) \cdots P(G_k, \lambda)$.

Let $G = (V, E)$ be an undirected graph. For $e = (a, b) \in E$, let G_e denote the subgraph of G obtained by deleting e from G, without removing vertices a and b; that is, $G_e = G - e$ as defined in Section 11.2. From G_e a second subgraph of G is obtained by coalescing (or, identifying) the vertices a and b. This second subgraph is denoted by $G_e{}'$.

EXAMPLE 11.35.

Figure 11.90 shows G_e and $G_e{}'$, for graph G with the edge e as specified. Note how the coalescing of a and b in G results in the coalescing of the two pairs of edges $\{d, b\}, \{d, a\}$ and $\{a, c\}, \{b, c\}$.

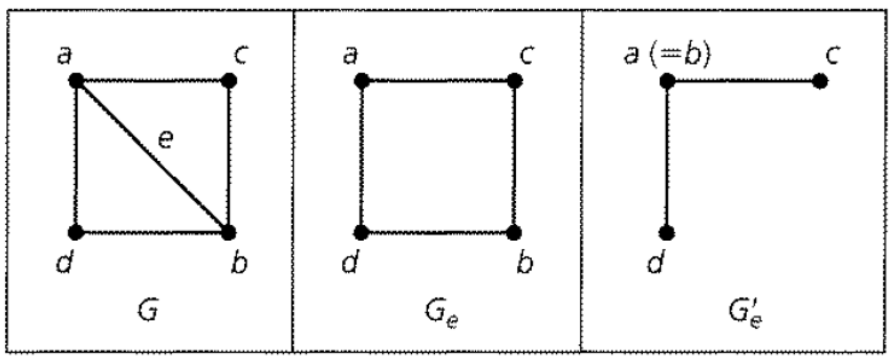


Figure 11.90

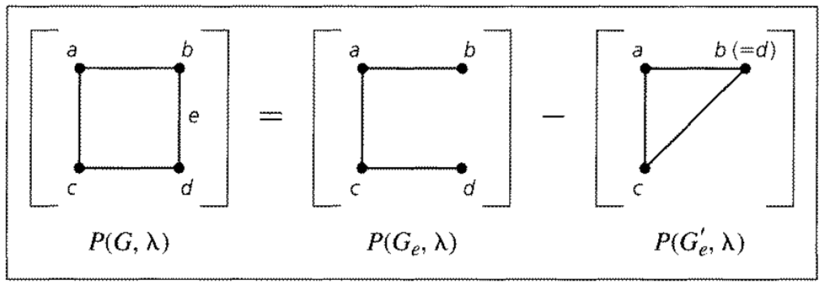
Decomposition Theorem

THEOREM 11.10 (Decomposition Theorem for Chromatic Polynomials). If $G = (V, E)$ is a connected graph and $e \in E$, then

$$P(G_e, \lambda) = P(G, \lambda) + P(G_e{}', \lambda).$$

Proof: Let $e = \{a, b\}$. The number of ways to properly color the vertices in G_e with (at most) λ colors is $P(G_e, \lambda)$. Those colorings where a and b have different colors are proper colorings of G. The colorings of G_e that are not proper colorings of G occur when a and b have the same color. But each of these colorings corresponds with a proper coloring for $G_e{}'$. This partition of the $P(G_e, \lambda)$ proper colorings of G_e into two disjoint subsets results in the formula $P(G_e, \lambda) = P(G, \lambda) + P(G_e{}', \lambda)$.

EXAMPLE 11.36. The following calculations yield $P(G, \lambda)$ for G a cycle of length 4.



From Example 11.34(c) it follows that $P(G_e, \lambda) = \lambda(\lambda - 1)^3$. With $G_e' = K_3$ we have $P(G_e{}', \lambda) = \lambda^{(3)}$. Therefore,

$$P(G, \lambda) = \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2)$$
$$= \lambda(\lambda-1)[(\lambda-1)^2 - (\lambda-2)]$$
$$= \lambda(\lambda-1)(\lambda^2 - 3\lambda + 3) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda$$

Since $P(G, 1) = 0$ while $P(G, 2) = 2 > 0$, we know that $\chi(G) = 2$.

THEOREM 11.11.

For each graph G, the constant term in $P(G, \lambda)$ is 0.

***Proof*:** For each graph G, $x(G) > 0$ because $V \neq \phi$. If $P(G, 1)$ has constant term a, then $P(G, 0) \ = a \ \neq 0$. This implies that there are a ways to color G properly with 0 colors, a contradiction.

THEOREM 11.12.

Let $G\ (V, E)$ with $|E| > 0$. Then the sum of the coefficients in $P(G, \lambda)$ is 0.

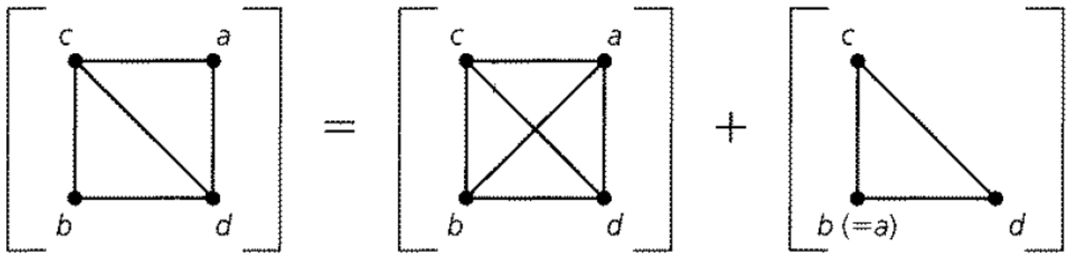
***Proof*:** Since $|E| \geq 1$, we have $\chi(G) \geq 2$, so we cannot properly color G with only one color. Consequently, $P(G, 1) = 0 =$ the sum of the coefficients in $P(G, \lambda)$.

THEOREM 11.13.

Let $G = (V, E)$ with $a, b \in V$ but $\{a, b\} = e \notin E$. We write G_e^+ for the graph we obtain from G by adding the edge $e = \{a, b\}$. Coalescing the vertices a and b in G gives us the subgraph G_e^{++} of G. Under these circumstances $P(G, \lambda) = P(G_e^+, \lambda) + P(G_e^{++}, \lambda)$

Proof**:** This result follows as in Theorem 11.10 because $P(G_e^+, \lambda) = P(G, \lambda) - P(G_e^{++}, \lambda)$.

EXAMPLE 11.38. Let us now apply Theorem 11.13.



$P(G, \lambda)$ $P(G_e^+, \lambda)$ $P(G_e^{++}, \lambda)$

Here $P(G,\lambda) = \lambda^{(4)} + \lambda^{(3)} = \lambda(\lambda-1)(\lambda-2)^2$, so $\chi(G) = 3$. In addition, if six colors are available, the vertices in G can be properly colored in $6(5)(4)^2 = 480$ ways.

For all graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.

i) the union of G_1 and G_2, denoted $G_1 \cup G_2$, is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$; and

ii) when $V_1 \cap V_2 \neq \phi$, the intersection of G_1 and G_2, denoted $G_1 \cap G_2$, is the graph with vertex set $V_1 \cap V_2$ and edge set $E_1 \cap E_2$.

THEOREM 11.14.

Let G be an undirected graph with subgraphs G_1, G_2. If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = K_n$, for some $n \in \mathbb{Z}^+$, then

$$P(G,\lambda) = \frac{P(G_1,\lambda)\cdot P(G_2,\lambda)}{\lambda^{(n)}}.$$

Proof: Since $G_1 \cap G_2 = K_n$, it follows that K_n is a subgraph of both G_1 and G_2 and that $\chi(G_1), \chi(G_2) \geq n$. Given λ colors, there are $\lambda^{(n)}$ proper colorings of K_n. For each of these $\lambda^{(n)}$ colorings there are $P(G_1,\lambda)/\lambda^{(n)}$ ways to properly color the remaining vertices in G_1. Likewise, there are $P(G_2,\lambda)/\lambda^{(n)}$ ways to properly color the remaining vertices in G_2. By the rule of product,

$$P(G,\lambda) = P(K_n,\lambda)\frac{P(G_1,\lambda)}{\lambda^{(n)}}\frac{P(G_2,\lambda)}{\lambda^{(n)}} = \frac{P(G_1,\lambda)P(G_2,\lambda)}{\lambda^{(n)}}$$

EXAMPLE 11.39.

Consider the following graph. Let G_1 be the subgraph induced by the vertices w, x, y, z. Let G_2 be the complete graph K_3 —with vertices v, w, and x. Then $G_1 \cap G_2$ is the edge $\{w, x\}$, so $G_1 \cap G_2 = K_2$. Therefore

$$P(G,\lambda) = \frac{P(G_1,\lambda)P(G_2,\lambda)}{\lambda^{(2)}} = \frac{\lambda^{(4)}\lambda^{(3)}}{\lambda^{(2)}}$$

$$= \frac{\lambda^2(\lambda-1)^2(\lambda-2)^2(\lambda-3)}{\lambda(\lambda-1)}$$

$$= \lambda(\lambda-1)(\lambda-2)^2(\lambda-3)$$