CALCULUS I LECTURE 16: TECHNIQUES OF INTEGRATION II

1. Rational functions

Let us denote by $\mathbb{R}[x]$ the space of polynomials with real coefficients. For any $P(x) \in \mathbb{R}[x]$, we define the degree of P to be the integer n such that

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \tag{1.1}$$

with $a_n \neq 0$. We usually denote the degree of P by $deg(P)$.

A **rational function** is a function $f(x) = \frac{P(x)}{Q(x)}$ with $P, Q \in \mathbb{R}[x]$ defined on the set $\{Q(x) \neq 0\}$. We call a rational function f is **proper** if and only if $deg(Q) > deg(P)$, otherwise it is call **improper**. Notice that, and improper rational function $f(x)$ can be written as

$$f(x) = S(x) + \frac{R(x)}{Q(x)} \tag{1.2}$$

with S, R and Q are polynomials, $deg(R) < deg(Q)$. We can also assume the leading coefficient of Q is one.

According to (1.2), one can obtain the formula for the integration of rational functions if one can find the formula for the integration of proper rational functions.

2. Partial fractions of Rational functions

Let $f(x) = \frac{P(x)}{Q(x)}$ be a proper rational function with

$$Q(x) = \Pi_{k=1}^{m} \alpha_k(x)^{r_k} \Pi_{k=1}^{l} \beta_k(x)^{s_k} \tag{2.1}$$

where $\alpha_k(x) = x + b_k$ and $\beta_k(x) = x^2 + f_k x + g_k$ with $f_k^2 - 4e_k g_k < 0$. Then we have the following proposition.

Proposition 2.1. Suppose that $f(x)$ is a proper rational function defined as above. Then

$$f(x) = \sum_{k=1}^{m}\sum_{p=1}^{r_k} \frac{A_{k,p}}{\alpha_k(x)^p} + \sum_{k=1}^{l}\sum_{p=1}^{s_k} \frac{B_{k,p}x + C_{k,p}}{\beta_k(x)^p} \tag{2.2}$$

for some $A_{k,p}$, $B_{k,p}$, $C_{k,p} \in \mathbb{R}$.

The proof of this proposition is not difficult, but it needs some background knowledge of linear algebra. We will explain that in the appendix.

According to Proposition 2.1, to obtain the formula for $\int f(x)dx$, one shall know the formula of the following integrals:

$$\int \frac{A}{(x+b)^p}dx; \tag{2.3}$$

$$\int \frac{Bx+C}{(x^2+fx+g)^p}dx \tag{2.4}$$

with $f^2-4eg<0$.

(2.3) can be computed directly by using the substitution rule: Let $u=x+b$, then

$$\begin{aligned}\int \frac{1}{(x+b)^p}dx &= \int \frac{1}{u^{-p}}\frac{dx}{du}du = \int u^{-p}du\\ &= \frac{1}{(-p+1)}u^{-p+1}+c = \frac{1}{(1-p)}\frac{1}{(x+b)^{p-1}}+c\end{aligned}$$

for some constant c when $p\neq 1$;

$$\int \frac{1}{(x+b)}dx = \ln|x+b|+c$$

for some constant c.

Now we consider the integration (2.4). Here we can assume $B=1$. In this case, we have

$$\begin{aligned}\int \frac{x+C}{(x^2+fx+g)^p}dx &= \int \frac{x+\frac{f}{2}}{((x+\frac{f}{2})^2+(g-\frac{f^2}{4}))^p}dx\\ &\quad+\int \frac{C-\frac{f}{2}}{((x+\frac{f}{2})^2+(g-\frac{f^2}{4}))^p}dx.\end{aligned} \tag{2.5}$$

Taking $u=x+\frac{f}{2}$ and $v=u^2+(g-\frac{f^2}{4})$, the first term on the right hand side of (2.5) will be

$$\begin{aligned}\int \frac{u}{(u^2+(g-\frac{f^2}{4}))^p}\frac{dx}{du}du &= \int \frac{u}{(u^2+(g-\frac{f^2}{4}))^p}du\\ &= \int \frac{1}{2v^p}dv = \frac{1}{2(1-p)v^{p-1}}+c\\ &= \frac{1}{2(1-p)(x^2+fx+g)^{p-1}}+c\end{aligned}$$

for some constant c.

The second term on the right hand side of (2.5), by taking $u=(g-\frac{f^2}{4})^{-\frac{1}{2}}(x+\frac{f}{2})$, will be

$$\Big(C-\frac{f}{2}\Big)\Big(g-\frac{f^2}{4}\Big)^{(p+\frac{1}{2})}\int \frac{1}{(u^2+1)^p}du. \tag{2.6}$$

Now, we can take $u=\tan\theta$. Then by substitution rule, we have

$$\int \frac{1}{(u^2+1)^p}du = \int \cos^{(2p-2)}\theta d\theta \tag{2.7}$$

which is solvable by the technique we learned in the previous lecture.

Example 2.2. Evaluate

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx.$$

By solving

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}, \tag{2.8}$$

we have $A = 1$, $B = 1$, $C = -1$. So

$$\begin{aligned} \int \frac{2x^2 - x + 4}{x^3 + 4x} dx &= \int \frac{1}{x} + \int \frac{x}{x^2 + 4} - \frac{1}{x^2 + 4} \\ &= \ln|x| + \frac{1}{2}\ln|x^2 + 4| - \frac{1}{2}\tan^{-1}(\frac{x}{2}) + C \end{aligned}$$

for some constant $C \in \mathbb{R}$.

3. Appendix: Vector spaces

Here we use a different way to define **vector spaces**. Let $\mathcal{B} = \{v_i\}_{i=1}^n$ be any finite set[1]. A vector space generated by $\mathcal{B}$ is a set V with the following properties:

a. $\mathcal{B} \subset V$.
b. If $x, y \in V$ and $c \in \mathbb{R}$, then $x + cy \in V$.

We call a finite subset of V, say S, is **linearly independent** if and only if

$$\sum_{s_k \in S} a_k s_k = 0 \text{ implies } a_k = 0 \text{ for all } a_k. \tag{3.1}$$

One can show that, for any vector space, there exists a maximum number n such that the number of elements in a linearly independent set will always less than or equal to n. We call this number the **dimension** of V. Denote by $dim(V)$.

Finally, we have two vector spaces V_1 and V_2 with $V_1 \subset V_2$, then $V_1 = V_2$ if and only if $dim(V_1) = dim(V_2)$.

With all these facts in our mind, we can prove Proposition 2.1. First, let V_2 be the collection of proper rational function with Q given. Clearly this is a vector space with dimension $deg(Q)$. Let V_1 be the rational function which can be expressed by (2.2). One can also check that this is a vector space of dimension $deg(Q)$. Since $V_1 \subset V_2$, so they are equal.

[1] Here we only consider finite dimensional vector spaces. In general cases, we can take $\mathcal{B}$ to be an arbitrary set.