CALCULUS I LECTURE 5: DIFFERENTIATION, INVERSE FUNCTIONS AND IMPLICIT FUNCTIONS

1. Differentiation of inverse functions

Last time we obtained the derivative of $\log(x)$. We obtained this formula basing on facts that $\log(x)$ is the inverse function of e^x, and the chain rule. By the same token, we can derive the general formula for the derivatives of inverse functions.

Definition 1.1. Let $f : X \to Y$ and $g : Y \to X$. We call g inverse function of f if and only if $f \circ g(y) = y$ for all $y \in Y$ and $g \circ f(x) = x$ for all $x \in X$.

Let g be the inverse of f. We have

$$1 = \frac{d(f \circ g)}{dy} = \frac{df}{dx}(g(y))\frac{dg}{dy}. \tag{1.1}$$

Therefore, we obtain the formula

$$\frac{dg}{dy} = \frac{1}{\frac{df}{dx}(g(y))}. \tag{1.2}$$

Example 1.2. By (1.2), we have

$$\frac{d\sin^{-1}}{dx} = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1-\sin^2(\sin^{-1}(x))}} = \frac{1}{\sqrt{1-x^2}}. \tag{1.3}$$

Similarly,

$$\frac{d\cos^{-1}}{dx} = \frac{1}{-\sin(\cos^{-1}(x))} = \frac{1}{-\sqrt{1-x^2}}.$$

One can use formula (1.2) to obtain the derivatives of inverse functions once we have the formula for the derivatives of the original functions.

Exercise 1.3. Find the derivative of $\tan^{-1}$.

2. Implicit functions

It is not always the case that we can write y as a function of x. For example, when we consider the graph of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{2.1}$$

with $a, b > 0$, we cannot write y as a function of x. However, one can expect that, for some portion of this graph, we can write down the function $y = f(x)$. For (2.1), one can use $y = b\sqrt{1 - \frac{x^2}{a^2}}$ to describe the part of this ellipse on $\{y > 0\}$.

In general, this can be described as a problem of implicit functions. An **implicit function** is defined by a relationship $F(x, y) = 0$, without asking y to be a function of x. Any function $y = f(x)$ is an implicit function (by taking $F(x, y) = y - f(x)$).

Suppose that y can be written as a function of x for a portion of the graph determined by $F(x, y) = 0$. Let us denote this function by $y(x)$. Then we have

$$F(x, y(x)) = 0.$$

We can differentiate this equation on both side (with respect to x)

$$0 = \frac{dF(x, y(x))}{dx}. \tag{2.2}$$

Then if one can solve (2.2), one can obtain $\frac{dy}{dx}$.

Let us use (2.1) as our first example. We have

$$\frac{x^2}{a^2} + \frac{y^2(x)}{b^2} - 1 = F(x, y).$$

So

$$0 = \frac{2x}{a^2} + \frac{2y(x)}{b^2}\frac{dy}{dx}.$$

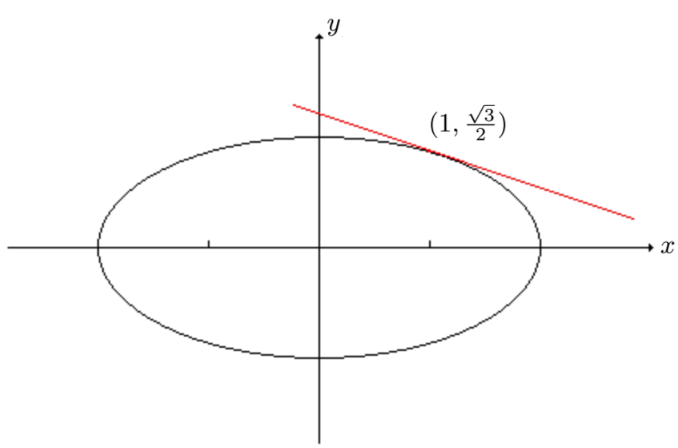
This implies

$$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y(x)}. \tag{2.3}$$

Example 2.1. Let $F(x, y) = \frac{x^2}{4} + y^2 - 1$. $F(x, y) = 0$ is an ellipse with $(1, \frac{\sqrt{3}}{2})$ on it. By (2.3), we have

$$\frac{dy}{dx} = -\frac{1}{4}\frac{x}{y}.$$

When we plug in $(x, y) = (1, \frac{\sqrt{3}}{2})$, we will obtain the slope of the tangent line (the red line in the picture below) for this ellipse at $(1, \frac{\sqrt{3}}{2})$. The value will be $-\frac{1}{2\sqrt{3}}$.



Pic.1

So the equation for the tangent line will be

$$y = -\frac{1}{2\sqrt{3}}(x-1) + \frac{\sqrt{3}}{2}.$$

It is not always a good idea to write an implicit function as several branch of functions. For example, when we have an ellipse, we can actually describe the graph by one variable t where

$$x = a\cos(t), \tag{2.4}$$

$$y = b\sin(t), \tag{2.5}$$

$t \in (0, 2\pi)$. In general, we write both x, y as functions of t. This variable t is called a **parameter** of the curve. In some easier case, we can differentiate

$$F(x(t), y(t)) = 0 \tag{2.6}$$

with respect to t. This gives us a relationship between $\frac{dy}{dt}$ and $\frac{dx}{dt}$ (with also $x(t)$, $y(t)$ involve).

Example 2.2. We consider the ellipse defined by (2.1) with $x = x(t), y = y(t)$. Then we have

$$\frac{1}{a^2}\frac{dx}{dt}x + \frac{1}{b^2}\frac{dy}{dt}y = 0. \tag{2.7}$$

When $x = a\cos(t), y = b\sin(t)$, (2.7) holds because $\sin^2 + \cos^2 = 1$. In general cases, (2.7) determines the ratio of $\frac{dy}{dt}$ and $\frac{dx}{dt}$ when x, y are given.