E933200 Signals and Systems Fall 2017

Solutions to Midterm Exam II

1. (a) X $\quad -jt\,x(t) \xleftrightarrow{\mathcal{F}} \dfrac{dX(j\omega)}{d\omega}$

(b) X $\quad x_o(t) \xleftrightarrow{\mathcal{F}} j\,\Im\{X(j\omega)\}$

(c) X $\quad$ real and odd $\xleftrightarrow{\mathcal{F}}$ purely imaginary and odd

(d) O $\quad x(t)$ is odd $\Rightarrow X(j\omega)$ is odd $\Rightarrow X(j0) = -X(j0)$

(e) O $\quad \Rightarrow X(j0) = 0$

2. Let $x(t) = \text{sinc}(at)$ and $h(t) = \text{sinc}(t)$

$y(t) = x(t) * h(t) \xrightarrow{\mathcal{F}} Y(j\omega) = \mathcal{F}\{x(t) * h(t)\} = X(j\omega)\cdot H(j\omega)$

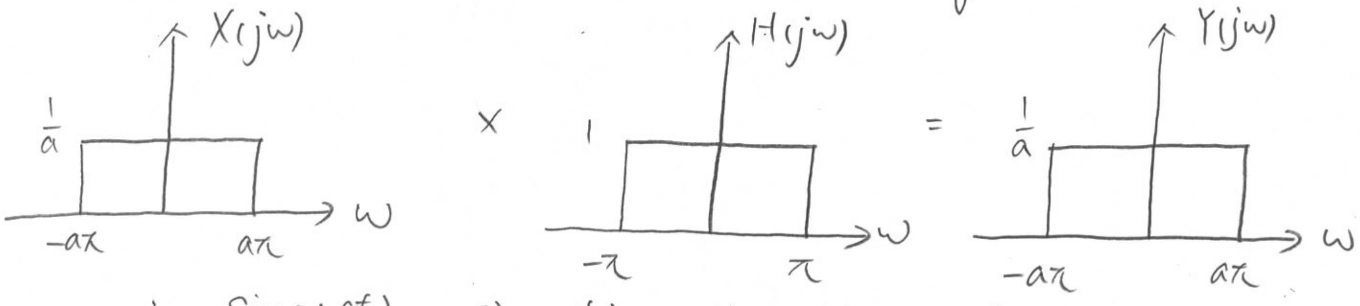
$X(j\omega) = \mathcal{F}\{\text{sinc}(at)\} = \frac{1}{a}\cdot\text{rect}\left(\frac{\omega}{2\pi a}\right)$

$H(j\omega) = \mathcal{F}\{\text{sinc}(t)\} = \text{rect}\left(\frac{\omega}{2\pi}\right)$

$\because 0 < a \le 1$

$\Rightarrow \therefore Y(j\omega) = \frac{1}{a}\,\text{rect}\left(\frac{\omega}{2\pi a}\right)$

$\Rightarrow y(t) = \text{sinc}(at)$



$\therefore \text{sinc}(at) * \text{sinc}(t) = \text{sinc}(at)$ for $0 < a \le 1$

3. $x(t) = \cos\left(\frac{2\pi}{3}t\right) + \sin\left(\frac{7\pi}{3}t\right)$

$= \frac{1}{2}\left[e^{j\left(\frac{2\pi}{3}\right)t} + e^{-j\left(\frac{2\pi}{3}\right)t}\right] + \frac{1}{2j}\left[e^{j\left(\frac{7\pi}{3}\right)t} - e^{-j\left(\frac{7\pi}{3}\right)t}\right]$

$T_0 = \text{lcm}\left(3, \frac{6}{7}\right) = 6$

$\Rightarrow \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{6} = \frac{\pi}{3}$

$\therefore\ x(t) = \frac{1}{2} e^{j\cdot 2\cdot\frac{\pi}{3}\cdot t} + \frac{1}{2} e^{j(-2)\cdot\frac{\pi}{3}\cdot t} + \frac{1}{2j}\cdot e^{j\cdot 7\cdot\frac{\pi}{3}\cdot t} - \frac{1}{2j} e^{j(-7)\cdot\frac{\pi}{3}\cdot t}$

$a_2 = a_{-2} = \frac{1}{2}\ ,\ a_7 = \frac{1}{2j}\ ,\ a_{-7} = -\frac{1}{2j}$

4. $\because\ e^{-t}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{1+j\omega}$

$(-jt)e^{-t}u(t) \xleftrightarrow{\mathcal{F}} \frac{d}{d\omega}\left(\frac{1}{1+j\omega}\right) = \frac{-j}{(1+j\omega)^2}$

$-t^2 e^{-t}u(t) \xleftrightarrow{\mathcal{F}} \frac{d}{d\omega}\left[\frac{-j}{(1+j\omega)^2}\right] = \frac{-2}{(1+j\omega)^3}$

$\therefore\ \mathcal{F}^{-1}\left[\frac{1}{(1+j\omega)^3}\right] = \frac{1}{2} t^2 e^{-t}u(t)$

5.

$a_k = \begin{cases} jk & , |k| < 4 \\ 0 & , \text{otherwise} \end{cases} \qquad \omega = \frac{2\pi}{T} = \frac{2\pi}{6} = \frac{\pi}{6}$

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} = \sum_{k=-3}^{3} (jk) e^{jk\frac{\pi}{3}t}$

$= -3je^{-j\pi t} - 2je^{-j\frac{2\pi}{3}t} - je^{-j\frac{\pi}{3}t} + je^{j\frac{\pi}{3}t}$
$+ 2je^{j\frac{2\pi}{3}t} + 3je^{j\pi t}$

$= j(e^{j\frac{\pi}{3}t} - e^{-j\frac{\pi}{3}t}) + 2j(e^{j\frac{2\pi}{3}t} - e^{-j\frac{2\pi}{3}t})$
$+ 3j(e^{j\pi t} - e^{-j\pi t})$

$\because\ e^{j\omega t} - e^{-j\omega t} = 2j\sin\omega t$

$\Rightarrow x(t) = -2\sin(\frac{\pi}{3}t) - 4\sin(\frac{2\pi}{3}t) - 6\sin(\pi t)$

6. $\because \operatorname{sgn}(t) \xleftrightarrow{\mathcal{F}} \frac{2}{jw}$

according to duality:

$\frac{2}{jt} \xleftrightarrow{\mathcal{F}} 2\pi \operatorname{sgn}(-w) = -2\pi \operatorname{sgn}(w)$

$\Rightarrow \frac{1}{\pi t} \xleftrightarrow{\mathcal{F}} -j \operatorname{sgn}(w)$

$\therefore x(t) = \frac{1}{\pi t} \longleftrightarrow -j \operatorname{sgn}(w)$

7. (a) $X(jw) = \int_{-1}^{0} 10 e^{-jwt} dt + \int_{0}^{1} (-10) e^{-jwt} dt$

$= \frac{10}{-jw} e^{-jwt} \Big|_{-1}^{0} + \frac{10}{jw} e^{-jwt} \Big|_{0}^{1}$

$= \frac{10}{-jw} (1 - e^{jw}) + \frac{10}{jw} (e^{-jw} - 1)$

$= \frac{-20}{jw} + \frac{10}{jw} (e^{jw} + e^{-jw})$

$= \frac{20}{jw} (\cos w - 1)$

(b) duality: $x(t) \xleftrightarrow{\mathcal{F}} X(jw)$

$X(jt) \xleftrightarrow{\mathcal{F}} 2\pi x(-w)$

$x(t) = e^{-|t|} \xleftrightarrow{\mathcal{F}} \frac{2}{1+w^2} = X(jw)$

$X(jt) = \frac{2}{1+t^2} \xleftrightarrow{\mathcal{F}} 2\pi \times e^{-|-w|} = 2\pi e^{-|w|}$

(c) $x(t) = a_0 + \sum_{k=1}^{\infty} (a_k e^{jkw_0 t} + a_{-k} e^{-jkw_0 t})$

$= a_0 + \sum_{k=1}^{\infty} (a_k e^{jkw_0 t} + a_k^* e^{-jkw_0 t})$

$= a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\{a_k e^{jkw_0 t}\}$

$= a_0 + 2\sum_{k=1}^{\infty} [\mathrm{Re}\{a_k\} \cos(kw_0 t) - \mathrm{Im}\{a_k\} \sin(kw_0 t)]$