E 933200 Signals and Systems Fall 2017

Solutions to Final Exam

1. (a) Y (b) N (c) N (d) Y (e) Y (f) Y (g) N (h) N (i) N (j) N

2. Let $y[n] = \sum_{m=-\infty}^{\infty} x[m] = x[n] * u[n]$

$\because u[n] = \frac{1}{2}(1 + \mathrm{sgn}[n]) \xleftrightarrow{\mathcal{F}} U(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\, \delta(\omega - 2\pi k)$

We have the Fourier transform of $y[n]$ given by

$$
\begin{aligned}
X(e^{j\omega})\,U(e^{j\omega}) &= \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi \sum_{k=-\infty}^{\infty} X(e^{j\omega})\, \delta(\omega - 2\pi k) \\
&= \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi \sum_{k=-\infty}^{\infty} X(e^{j2\pi k})\, \delta(\omega - 2\pi k) \\
&= \underline{\frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)} \quad \#
\end{aligned}
$$

3.

$x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-j\omega})$

$x^*[-n] \xleftrightarrow{\mathcal{F}} X^*(e^{j\omega})$

$$
\begin{aligned}
y[n] = (x[n] + x^*[-n])/2 \xleftrightarrow{\mathcal{F}} & \left[X(e^{j\omega}) + X^*(e^{j\omega})\right]/2 \\
&= \mathrm{Re}\left[X(e^{j\omega})\right]
\end{aligned}
$$

4. $x(t) = e^{-t^2}$

$$X(jw) = \int_{-\infty}^{\infty} e^{-t^2} e^{-jwt}\, dt$$

$$X^2(jw) = \left(\int_{-\infty}^{\infty} e^{-t_1^2} e^{-jwt_1}\, dt_1\right)\left(\int_{-\infty}^{\infty} e^{-t_2^2} \cdot e^{-jwt_2}\, dt_2\right)$$

$$= \int_{-\infty}^{\infty} e^{-(t_1^2+jwt_1)}\, dt_1 \int_{-\infty}^{\infty} e^{-(t_2^2+jwt_2)}\, dt_2$$

$$= \int_{-\infty}^{\infty} e^{-(t_1+\frac{jw}{2})^2-\frac{w^2}{4}}\, dt_1 \int_{-\infty}^{\infty} e^{-(t_2+\frac{jw}{2})^2-\frac{w^2}{4}}\, dt_2$$

$$= \left(e^{-\frac{w^2}{4}}\right)^2 \left(\int_{-\infty}^{\infty} e^{-(t_1+\frac{jw}{2})^2}\, dt_1 \int_{-\infty}^{\infty} e^{-(t_2+\frac{jw}{2})^2}\, dt_2\right)$$

$$= e^{-\frac{w^2}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(t_1+\frac{jw}{2})^2-(t_2+\frac{jw}{2})^2}\, dt_1\, dt_2$$

令 $t_1+\frac{jw}{2} = r\cos\theta$ 代入
$t_2+\frac{jw}{2} = r\sin\theta$

$$X^2(jw) = e^{-\frac{w^2}{2}} \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

$$= e^{-\frac{w^2}{2}} \int_0^{2\pi} \left.\frac{-1}{2} e^{-r^2}\right|_0^{\infty} d\theta$$

$$= e^{-\frac{w^2}{2}} \int_0^{2\pi} \frac{1}{2}\, d\theta$$

$$= \pi\, e^{-\frac{w^2}{2}}$$

則 $X(jw) = \pm\sqrt{\pi}\, e^{-\frac{w^2}{4}}$ (取正)

$$= \sqrt{\pi}\, e^{-\frac{w^2}{4}}$$ #

5. $g[n] = x_{(3)}[n] \xleftrightarrow{F} G(e^{j\omega}) = X(e^{j3\omega})$

$X(e^{j3\omega}) = G(e^{j\omega}) = G(e^{j(\omega-\alpha)}) = X(e^{j3(\omega-\alpha)})$

∴ 3α 必為 2π 的整數倍，又因 $0<\alpha<\pi$

故 $\underline{\alpha = \frac{2\pi}{3}}$ #

6.

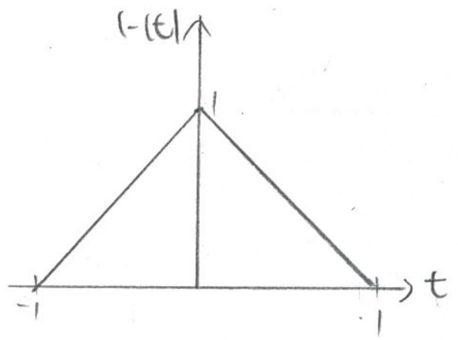
$$X(e^{j\omega}) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

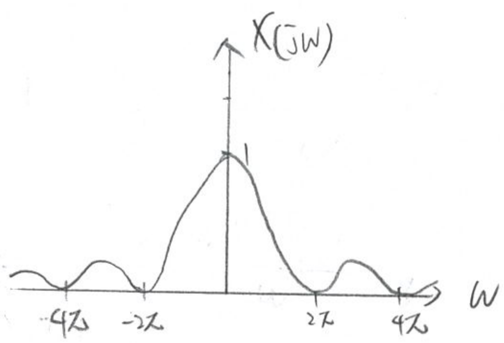
$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{1}{-(a+j\omega)} e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

$$= \underline{\frac{1}{a+j\omega}} \ \#$$

7. $x(t) = rect(t) * rect(t) \xleftrightarrow{F} X(e^{j\omega}) = sinc^2\left(\frac{\omega}{2\pi}\right)$



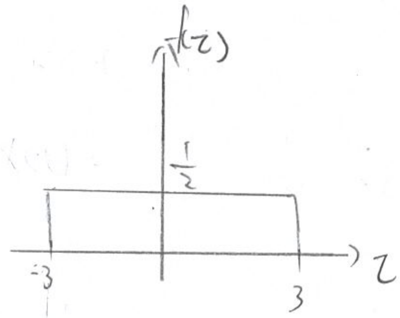


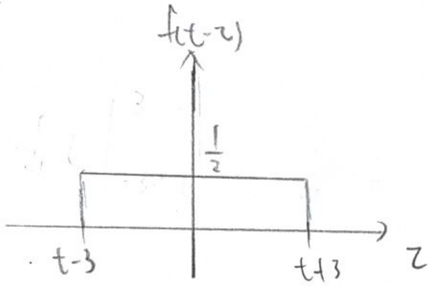
8.

令 $f(t) = \begin{cases} \frac{1}{2} & , |t| < 3 \\ 0 & , t = 0 \end{cases} \xleftrightarrow{F} 3 \cdot \text{sinc}\left(\frac{3w}{\pi}\right)$

$$X(jw) = 3 \cdot \frac{\sin(3w)}{3w} \cdot 3 \cdot \frac{\sin(3w)}{3w} = \left(3\,\text{sinc}\left(\frac{3w}{\pi}\right)\right) \cdot \left(3\,\text{sinc}\left(\frac{3w}{\pi}\right)\right)$$

$$\Rightarrow x(t) = f(t) * f(t) = \int_{-\infty}^{\infty} f(\tau) f(t-\tau)\, d\tau$$





① If $t+3 < -3 \Rightarrow t < -6$

$x(t) = 0$

② If $-3 \leq t+3 < 3 \Rightarrow -6 \leq t < 0$

$x(t) = \int_{-3}^{t+3} \frac{1}{4}\, d\tau = \frac{1}{4}(6+t)$

③ If $-3 \leq t-3 < 3 \Rightarrow 0 \leq t < 6$

$3 \leq t+3$

$x(t) = \int_{t-3}^{3} \frac{1}{4}\, d\tau = \frac{1}{4}(6-t)$

④ If $3 \leq t-3 \Rightarrow t \geq 6$

$x(t) = 0$

$$\Rightarrow x(t) = \begin{cases} \frac{1}{4}(6+t) & , -6 \leq t < 0 \\ \frac{1}{4}(6-t) & , 0 \leq t < 6 \\ 0 & , \text{otherwise} \end{cases}$$