CALCULUS I LECTURE 11: NEWTON'S METHOD

1. Idea of the method

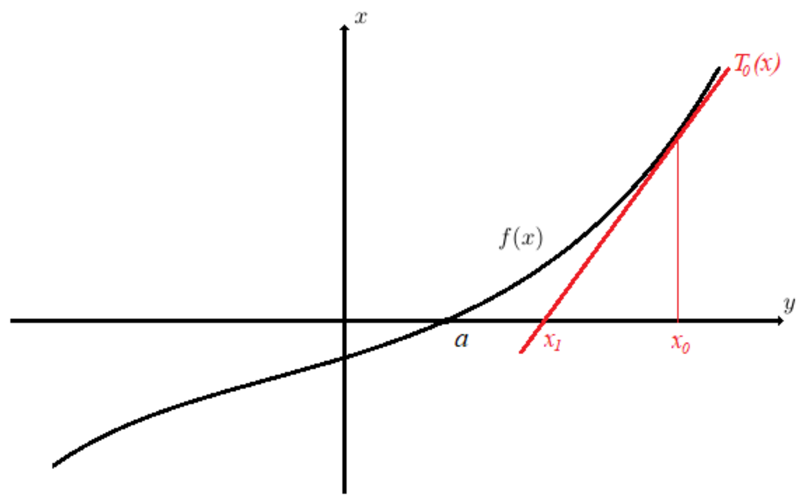
We should start with a function $f : X \to Y$ with $X, Y \subset \mathbb{R}$. Suppose that there exists a value $a \in X$ such that $f(a) = 0$ with $(a - c, a + c) \subset X$ for some $c > 0$. Then **Newton's method** provides us a way to find (approximate) the value a by iteration.

This method works under the following two conditions:
(a). The second derivative of f is continuous and bounded near a.
(b). $f'(a) \neq 0$.
When (one of) these two conditions fail, we cannot guarantee Newton's method works. It may work in some cases, but it is also fail in many other cases.

To begin with, let x_0 be a point near a. Now, we consider the tangent line of f at $(x_0, f(x_0))$:

$$T_0(x) = f(x_0) + f'(x_0)(x - x_0). \tag{1.1}$$

This is also the equation of linear approximation for $f(x)$. Therefore, by solving $T_0(x) = 0$, we should have an approximate value for y. We call this solution x_1.



Pic.1

By (1.1), we can solve $T_0(x_1) = 0$ by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \tag{1.2}$$

Now, we repeat this process, by replacing x_0 by x_1 and replacing T_0 by T_1, the tangent line of f at $(x_1, f(x_1))$, then we have a sequence defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.3}$$

The Newton's method tells us the following theorem:

Theorem 1.1. *Suppose f satisfies* **(a), (b)** *and $f(a) = 0$. For any x_0 sufficiently close to a, the sequence $\{x_n\}$ defined in (1.3) converges to a.*

Proof. Since **(a)** is true, we can assume that $|f''(x)| < M$ for all $x \in X$. Since **(b)** holds, we can assume that $|f'(x)| > K$ near a (say, on $(a - c, a + c)$ for some c).

Now, we call $g_0(x) = f(x) - T_0(x)$. We can see that $g_0(x_0) = g_0'(x_0) = 0$. By mean value theorem (applying twice), we have

$$g(x) = g(x) - g(x_0) = g'(\xi_1)(x - x_0) = g''(\xi_2)(\xi_1 - x_0)(x - x_0). \tag{1.4}$$

So

$$|g(x)| < M|x - x_0|^2. \tag{1.5}$$

Substitute $x = a$, then $g(a) = f(a) - f(x_0) - f'(x_0)(a - x_0)$. By (1.5), we have

$$\Big|\frac{f(x_0)}{f'(x_0)} + (a - x_0)\Big| < \frac{M}{K}|x_0 - a|^2 \tag{1.6}$$

By the definition of x_1 in (1.2), we have

$$|x_1 - a| < \frac{M}{K}|x_0 - a|^2. \tag{1.7}$$

Inductively, one can show that

$$|x_{n+1} - a| < \frac{M}{K}|x_n - a|^2. \tag{1.8}$$

So suppose we have $|x_0 - a| < \frac{K}{2M}$, then we have

$$|x_1 - a| < \frac{1}{2}|x_0 - a| \tag{1.9}$$

and inductively

$$|x_{n+1} - a| < \frac{1}{2}|x_n - a|. \tag{1.10}$$

This implies $\{x_n\}$ converges. One can check $\lim_{n\to\infty} x_n = a$.

$\square$