6. Parvesal Theorem

$\begin{cases} \text{periodic function} \Rightarrow \frac{1}{2L}\int_{-L}^{L} f^2(x)\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2) \Rightarrow \text{Fourier Series ①} \\ \text{not periodic function} \Rightarrow \int_{-\infty}^{\infty} f^2(x)\,dx = \int_{-\infty}^{\infty} F^2(w)\,dw \Rightarrow \text{Fourier Transform ②} \end{cases}$

① Fourier Series $\Rightarrow f(x) = a_0 + a_1\cos\frac{\pi}{L}x + a_2\cos\frac{2\pi}{L}x \cdots + b_1\sin\frac{\pi}{L}x + b_2\sin\frac{2\pi}{L}x + \cdots$

$f^2(x) = a_0^2 + a_1^2\cos^2\frac{\pi}{L}x + a_2^2\cos^2\frac{2\pi}{L}x + \cdots + b_1^2\sin^2\frac{\pi}{L}x + b_2^2\sin^2\frac{2\pi}{L}x \cdots + 2a_0a_1\cos\frac{\pi}{L}x + \cdots$

$+ 2a_0b_1\sin\frac{\pi}{L}x + 2a_1b_1\cos\frac{\pi}{L}\sin\frac{\pi}{L}x + \cdots$

$\because$ periodic & orthogonal

$\Rightarrow \int_{-L}^{L} f^2(x)\,dx = a_0^2 \cdot 2L + a_1^2 L + a_2^2 L + \cdots b_1^2 L + b_2^2 L + \cdots + 0 \cdots$

$\Rightarrow \frac{1}{2L}\int_{-L}^{L} f^2(x)\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$

② FT $\Rightarrow \begin{cases} f(x) = \int_{-\infty}^{\infty} F(w)\, e^{iwx}\,dw \\ F(w) = \int_{-\infty}^{\infty} f(x)\cdot e^{-iwx}\,dx \end{cases}$, $\overline{F(w)} = \int_{-\infty}^{\infty} \overline{f}(x)\cdot e^{iwx}\,dx$

$\Rightarrow \int_{-\infty}^{\infty} f^2(x)\,dx = \int_{-\infty}^{\infty} f(x)\cdot\overline{f}(x)\,dx = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(w)\,e^{iwx}\,dw\right)\overline{f}(x)\,dx\,dw$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(w)\cdot\overline{f(x)}\,e^{iwx}\,dx\,dw = \int_{-\infty}^{\infty} F(w)\cdot\left(\int_{-\infty}^{\infty}\overline{f(x)}\,e^{iwx}\,dx\right)dw$

$= \int_{-\infty}^{\infty} F(w)\cdot\overline{F(w)}\,dw = \int_{-\infty}^{\infty} F^2(w)\,dw$ ※