Subject: Plane curvilinon motion

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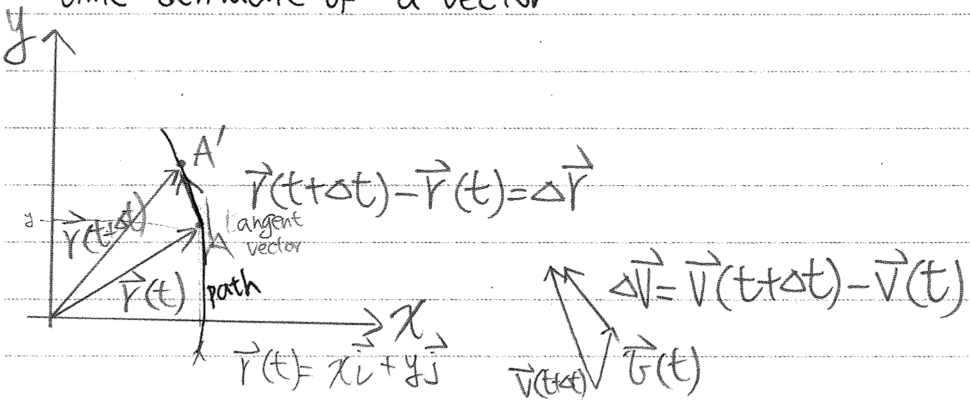
- motion of a particle along a current path on plane x-y
- When $z=0$ in polan coordinate $\Rightarrow (r, \theta)$
  $\phi = 0$ in spherical coordinates $\Rightarrow R \to r$
- time derivative of a vector



Velocity

average velocity $\vec{V}_{av} = \dfrac{\Delta \vec{r}}{\Delta t}$ 向量

average speed $\dfrac{\Delta S}{\Delta t}$ ← 曲線上

Intantaneous velocity $\vec{v} = \dfrac{d\vec{r}}{dt} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta \vec{r}}{\Delta t} = \dot{\vec{r}}$

$$\left(\begin{aligned} \vec{r}(t) &= x(t)\vec{i} + y(t)\vec{j} \\ \dot{\vec{r}}(t) &= \dot{x}(t)\vec{i} + \dot{y}(t)\vec{j} \end{aligned} \right) \quad |\vec{r}| = \sqrt{x^2 + y^2}$$

note: time derivative of Vector $\vec{r}$ is vector $\vec{v}$
$\vec{v}$ has magnitude and direction

about magnitude of $\vec{v}$ — direction of $\vec{v}$: unit tangent vector at A

$v = |\vec{v}| = \dfrac{ds}{dt} = \dot{s}$

note: $\left|\dfrac{d\vec{r}}{dt}\right| \neq \dfrac{d|\vec{r}|}{dt}$ $\left(\left|\dfrac{d\vec{r}}{dt}\right| = \dot{s} = |\vec{v}| = v, \ \dfrac{d|\vec{r}|}{dt} = \dfrac{dr}{dt} = \dot{r} \right)$ ($|\vec{r}| = r$)

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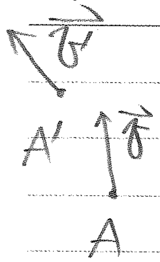
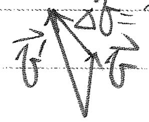


$\vec{v}' = \vec{v}(t+\Delta t)$

$\Delta\vec{v} = \vec{v}(t+\Delta t) - \vec{v}(t)$



Accelation

ave.accel $\vec{a}_{av} = \dfrac{\Delta \vec{v}}{\Delta t}$

inst.accel $\vec{a} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta \vec{v}}{\Delta t} = \dfrac{d\vec{v}}{dt}$

$\therefore \vec{a} = \dfrac{d\vec{v}}{dt} = \dot{\vec{v}}$

Note : (i) direction of $\vec{a}$ is neither tangent to the path nor normal to the path

(ii) normal component of $\vec{a}$ does point toward the center of curvature.

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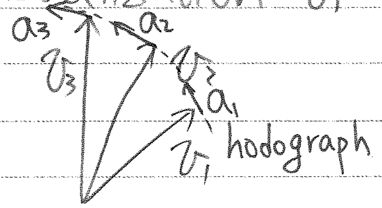
Visualization of Motion (hodograph) 速端曲線



Derivatives of Vector

$\underset{\sim}{p}(t) = P_x(t)\vec{i} + P_y(t)\vec{j} + P_z(t)\vec{k}$, i, j, k unit vector of fixed rectangular coordinate

$\frac{d\underset{\sim}{P}(t)}{dt} = \dot{p}(t)$

$= \dot{P}_x(t)\vec{i} + \dot{P}_y(t)\vec{j} + \dot{P}_z(t)\vec{k}$

$u \sim$ real-valued time function

$\frac{d(u\underset{\sim}{P})}{dt} = \dot{u}\underset{\sim}{P} + u\dot{\underset{\sim}{P}}(t)$

$\frac{d(\underset{\sim}{P}\cdot\underset{\sim}{Q})}{dt} = \dot{\underset{\sim}{P}}\underset{\sim}{Q} + \underset{\sim}{P}\cdot\dot{\underset{\sim}{Q}}$

Three different coordinate systems are used for discribing Vector relationships for planer curvilinear motion of a particle

- ☐ rectangular coordinates (x, y)
- ☐ normal and tangent coordinates (motion along a fixed curve)
- ☐ polar coordinates (motion on a circle)

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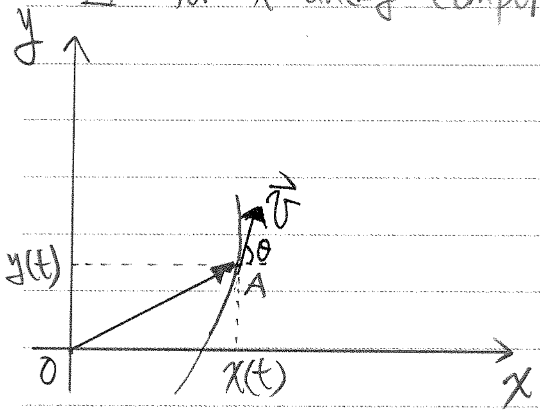
Rectangular coordinates (x-y)

□ for x- and y components are independently generated.



position vector

$$\vec{r} = x\vec{i} + y\vec{j}$$

Velocity

$$\vec{v} = \dot{\vec{r}} = \dot{x}\vec{i} + \dot{y}\vec{j} = v_x\vec{i} + v_y\vec{j}$$

Accelation

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{x}\vec{i} + \ddot{y}\vec{j} = a_x\vec{i} + a_y\vec{j}$$

$\Rightarrow v_x = \dot{x}, \ v_y = \dot{y}$

$\Rightarrow a_x = \ddot{x}, \ a_y = \ddot{y}$

$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$

$a = \sqrt{a_x^2 + a_y^2}$

$$\frac{v_y}{v_x} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \tan\theta$$

note: If coord. x and y are known indepently is function of time $x = f_1(t)$

$y = f_2(t)$

$$\vec{r}(t) = f_1(t)\vec{i} + f_2(t)\vec{j}$$

$$v(t) = \frac{d\vec{r}(t)}{dt} = \dot{f}_1(t)\vec{i} + \dot{f}_2(t)\vec{j}$$

$\boxed{\dot{\vec{i}} = \vec{0}, \ \dot{\vec{j}} = \vec{0}}$ ← x, y, fixed coord.

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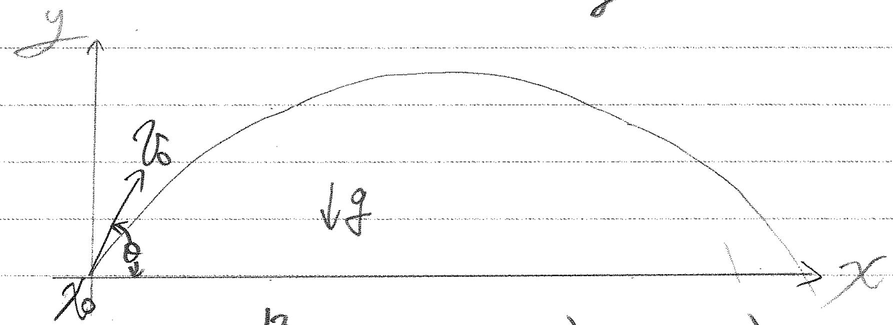
Special case: Projectile Motion
拋射運動

Assume that aneodynamic drg and curvature and rotation of earth are negelected.



$a_x = 0 = \frac{dv_x}{dt}$ $\quad v_0 = (v_x)_0 \vec{i} + (v_y)_0 \vec{j}$

$a_y = -g = \frac{dv_y}{dt}$ $\quad = v_0\cos\theta \vec{i} + v_0 \sin\theta \vec{j}$

$\Rightarrow dv_x = 0\,dt = 0 \Rightarrow \boxed{v_x = \text{constant} = (v_x)_0}$

$\int_{(v_y)_0}^{v_y} dv_y = -\int_0^t g\,dt = -g\int_0^t dt = -g(t-0) = -gt$

$\Rightarrow v_y - (v_y)_0 = -gt \Rightarrow \boxed{v_y = (v_y)_0 - gt}$

$v_x = \frac{dx}{dt} = (v_x)_0 \Rightarrow \int_{x_0}^{x} dx = \int_0^t (v_x)_0\, dt$

$\Rightarrow (x - x_0) = (v_x)_0 \cdot (t-0)$

$\Rightarrow \boxed{x(t) = x = x_0 + (v_x)_0\, t}$

$\int_{y_0}^{y} dy = \int_0^t v_y\, dt = \int_0^t ((v_y)_0 - gt)\,dt$

$\Rightarrow (y - y_0) = v_{y_0}(t-0) - \frac{1}{2}gt^2$

$\Rightarrow \boxed{y = y(t) = y_0 + v_{y_0} t - \frac{1}{2}gt^2}$

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$$\because \begin{cases} a_y = \dfrac{dv_y}{dt} \\ v_y = \dfrac{dy}{dt} \end{cases} \Rightarrow \frac{a_y}{v_y} = \frac{dv_y}{dy}$$

$$\Rightarrow \int_{(v_y)_0}^{v_y} v_y \cdot dv_y = \int_{y_0}^{y} a_y \cdot dy = \int_{y_0}^{y} -g\, dy$$

$$\Rightarrow \frac{1}{2} v_y^2 \Big]_{(v_y)_0}^{v_y} = -g(y - y_0)$$

$$\Rightarrow \frac{1}{2} v_y^2 - \frac{1}{2} (v_y)_0^2 = -g(y - y_0)$$

$$\Rightarrow v_y^2 = (v_y)_0^2 - 2g(y - y_0)$$

Double A

$de_t = \square$

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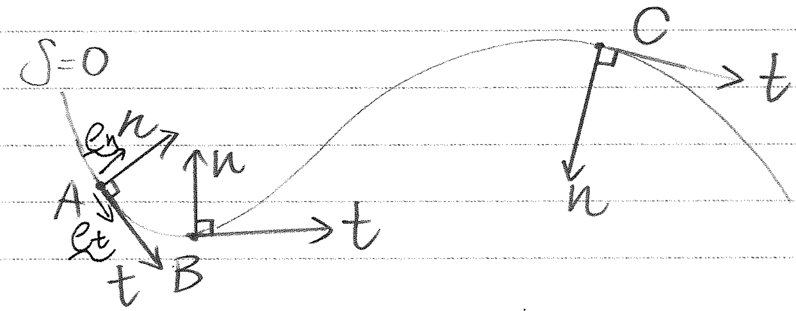
2/5 Normal and Tangential Coordinates

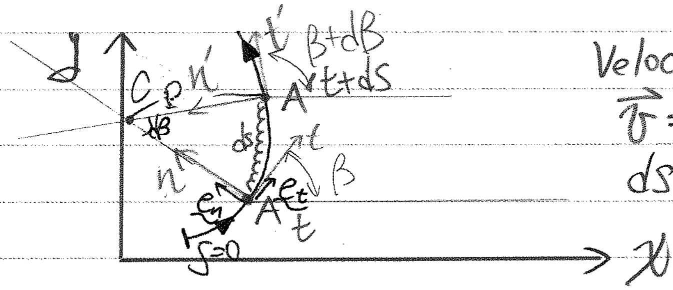
- measurement along the tangent and normal to path of particle
- nature description for curvilinear motion



$\dot{e}_t \neq 0$ $\quad$ $\dot{e}_n \neq 0$

$|e_t| = 1$ $\quad$ $|e_n| = 1$

note: n-t coordinates are moving!!



Velocity of particle at time t

$\vec{v} = \dot{s}e_t = |\vec{v}|e_t = \rho\dot{\beta}e_t$

$ds = \rho \, d\beta$

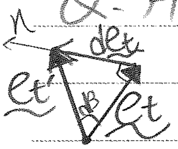
$\dot{s} = \frac{ds}{dt} = \rho \cdot \frac{d\beta}{dt} = \rho\dot{\beta}$

$\Rightarrow |\vec{v}| = \rho\dot{\beta}$

Accelation

$$a = \frac{dv}{dt} = \frac{d(ve_t)}{dt} = \dot{v}e_t + v \cdot \dot{e}_t$$

Q: How to find $\dot{e}_t$?



$$\dot{e}_t = \lim_{\Delta t \to 0} \frac{e_t(t+\Delta t) - e_t(t)}{\Delta t} = \frac{de_t}{dt}$$

$|de_t| = 1 \cdot d\beta = d\beta$ $\quad$ $\therefore de_t = d\beta \cdot e_n$

direction of $de_t \Rightarrow e_n$ $\quad$ $\boxed{\dot{e}_t} = \frac{d\beta}{dt} \cdot e_n = \boxed{\dot{\beta}e_n}$ $\quad$ 2/9

$\because v = \rho\dot{\beta} \Rightarrow \dot{\beta} = \frac{v}{\rho}$ $\quad$ $\therefore \boxed{\dot{e}_t = \frac{v}{\rho}e_n}$

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sub. into (2/8) yield:

$\underset{\sim}{a} = v\dot{\underset{\sim}{e}}_t + \dot{v}\underset{\sim}{e}_t$

$= v(\frac{v}{\rho})\underset{\sim}{e}_n + \dot{v}e_t = \frac{v^2}{\rho}\underset{\sim}{e}_n + \dot{v}e_t$

$\Rightarrow \boxed{\underset{\sim}{a} = \frac{v^2}{\rho}\underset{\sim}{e}_n + \dot{v}e_t}$ 2/10

given $s \cdot v \cdot \dot{v} \cdot \cdot \rho$

we can caculate accelation of the particle

a_n: normal accelation

a_t: tangential accelation

$a_n = \frac{v^2}{\rho} = \rho \cdot \dot{\beta}^2 = v\dot{\beta}$

$a_t = \dot{v} = \ddot{s}$

magnitude of $\underset{\sim}{a} = \sqrt{a_n^2 + a_t^2}$

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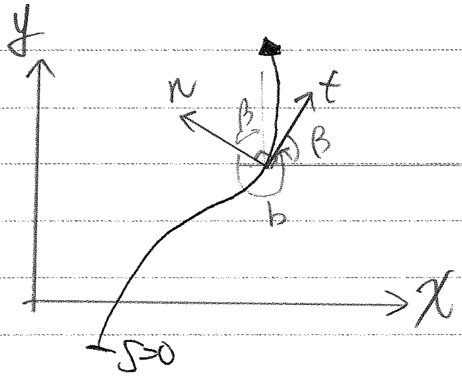
Another derivation of $\dot{e}_t$



define $e_b = e_t \times e_n$
and a new axis b
to n– and t–axes

FACT: coordinate (t–n–b)
magnitude of $a = \sqrt{a_n^2 + a_t^2}$
is rotating at angular velocity

$\omega = \dot{\beta}\, e_b$

then $\boxed{\dot{e}_t} = \omega \times e_t = \dot{\beta}\, e_b \times e_t = \boxed{\dot{\beta} e_n}$

$\boxed{\dot{e}_n} = \omega \times e_n = \dot{\beta}\, e_b \times e_n = \boxed{-\dot{\beta} e_t}$

Third method

$$e_t = \cos\beta\, \vec{i} + \sin\beta\, \vec{j}$$

$$e_n = -\sin\beta\, \vec{i} + \cos\beta\, \vec{j}$$

$$\because \beta = \beta(t)$$

$$\dot{e}_t = \frac{de_t}{dt} = -\sin\beta \cdot \dot{\beta}\, \vec{i} + \cos\beta \cdot \dot{\beta}\, \vec{j}$$

$$= \dot{\beta}(-\sin\beta\, \vec{i} + \cos\beta\, \vec{j}) = \dot{\beta}\, e_n$$

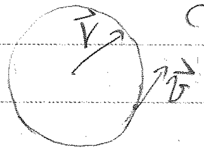
$$\dot{e}_n = \frac{de_n}{dt} = -\cos\beta \cdot \dot{\beta}\, \vec{i} - \sin\beta \cdot \dot{\beta}\, \vec{j}$$

$$= -\dot{\beta}(\cos\beta\, \vec{i} + \sin\beta\, \vec{j})$$

$$= -\dot{\beta} \cdot e_t$$

Special case: circular motion

constant speed

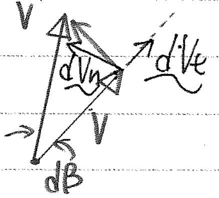


$$\rho = r = \text{constant}$$

$$\dot{v} = 0$$

$$a_n = \frac{v^2}{r}, \quad a_t = 0$$

Geometric Interpretation



Vector Change

$$d\mathbf{v} = d\mathbf{v}_t + d\mathbf{v}_n$$

$$|d\mathbf{v}_n| = v\, d\beta$$

$$\therefore a_n = \frac{|d\mathbf{v}_n|}{dt} = v\frac{d\beta}{dt} = v\dot{\beta} = \frac{v^2}{\rho}$$

Remarks:

- The normal compenent of accelation a_n is always directed toward the center of curvature C.
- The tangential compenent a_t directed in the postive t-direction if the speed is increasing and in negative t

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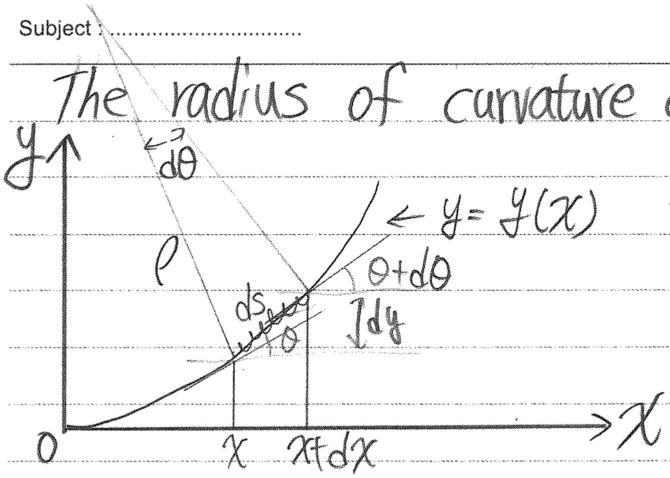
The radius of curvature of a plane curve



$\tan\theta = \frac{dy}{dx} = y'(x)$, $y'(x) = \frac{dy}{dx}$

$ds = \rho \cdot d\theta$

Curvature $C \triangleq \frac{d\theta}{ds}$

radius of curvature $\rho = \left|\frac{1}{C}\right| = \left|\frac{ds}{d\theta}\right|$

$ds^2 = dx^2 + dy^2 \Rightarrow \boxed{ds = \sqrt{dx^2+dy^2} = \sqrt{1+y'^2}\,dx}$

$\because \theta = \arctan y'$

$\therefore \boxed{\frac{d\theta}{dx}} = \frac{d\arctan y'}{dx} = \frac{d\arctan y'}{dy'} \cdot \frac{dy'}{dx}$

$= \boxed{\frac{1}{1+y'^2} \cdot y''}$

$\therefore C = \frac{\frac{1}{1+y'^2} \cdot y''\,dx}{\sqrt{1+y'^2}\,dx} = \frac{y''}{(1+y'^2)^{3/2}}$

$\rho = \frac{1}{C} = \frac{(1+y'^2)^{3/2}}{y''}$

整數　　浮點數

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整數	byte	浮點數	byte
char	1	float	4
short	2	double	8
int	4	long double	8
long	8		

int i = 3.5 // implicitane
= int(3.5) // explicitant

$0 \leq rand() \leq 32767$ 　 0~32767

static_cast< >( )

rand()%(max - min+1) + min

-3.75 　 10.7

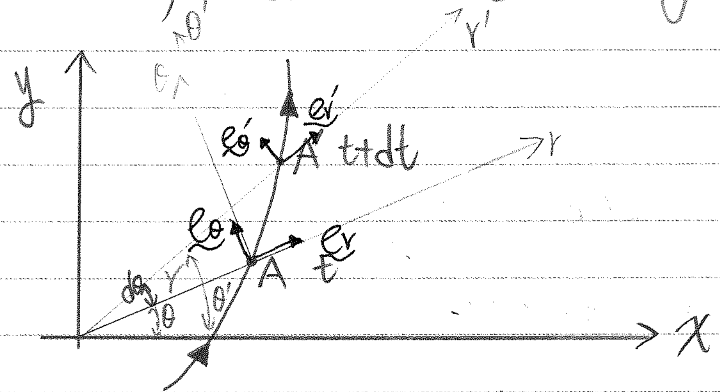
rand() x (max-min) / RAND_MAX + min

0X_ _ _ _ _ _ _ _

alpha R G B

2/6 polar coordinates (r-θ)

useful when a motion is constrained through control of a radial distance and an angular position or unconstrained motion observed by radial distance & angular position

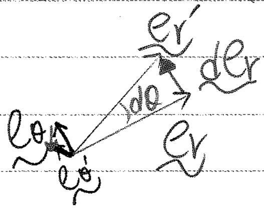


Note: x-y coordinates are used for reference to measure θ

position $\mathbf{r} = r\,\mathbf{e}_r$

Velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r$

Q: How to find $\dot{\mathbf{e}}_r$ · $\dot{\mathbf{e}}_\theta$



$\dot{\mathbf{e}}_r = \lim_{\Delta t \to 0} \frac{\mathbf{e}_r' - \mathbf{e}_r}{\Delta t} = \frac{d\mathbf{e}_r}{dt}$

$\boxed{\frac{d\mathbf{e}_r}{dt} = \frac{d\theta}{dt} \times \mathbf{e}_\theta = \dot{\theta}\,\mathbf{e}_\theta}$

$|d\mathbf{e}_r| = 1\, d\theta$

direction of $d\mathbf{e}_r = \mathbf{e}_\theta$

$d\mathbf{e}_\theta = -d\theta \times \mathbf{e}_r$

$\therefore \boxed{\frac{d\mathbf{e}_\theta}{dt} = -\frac{d\theta}{dt} \times \mathbf{e}_r = -\dot{\theta}\,\mathbf{e}_r}$

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$\therefore \underset{\sim}{v} = \dot{r}\,\underset{\sim}{e_r} + r\dot{\theta}\,\underset{\sim}{e_\theta}$

$= v_r\,\underset{\sim}{e_r} + v_\theta e_\theta$

$v_r = \dot{r}$, $v_\theta = r\dot{\theta}$

$v = \sqrt{v_r^2 + v_\theta^2}$, magnitude of $\underset{\sim}{v}$

$\underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt} = \dot{\underset{\sim}{v}} = \frac{d}{dt}(\dot{r}\,\underset{\sim}{e_r} + r\dot{\theta}\,\underset{\sim}{e_\theta})$

$= \ddot{r}\,\underset{\sim}{e_r} + \dot{r}\,\dot{\underset{\sim}{e_r}} + \dot{r}\dot{\theta}\,\underset{\sim}{e_\theta} + r\ddot{\theta}\,\underset{\sim}{e_\theta} + r\dot{\theta}\,\dot{\underset{\sim}{e_\theta}}$

$= \ddot{r}\,\underset{\sim}{e_r} + \dot{r}\dot{\theta}\,\underset{\sim}{e_\theta} + \dot{r}\dot{\theta}\,\underset{\sim}{e_\theta} + r\ddot{\theta}\,\underset{\sim}{e_\theta} - r\dot{\theta}^2\,\underset{\sim}{e_r}$

$= (\ddot{r} - r\dot{\theta}^2)\,\underset{\sim}{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underset{\sim}{e_\theta}$

Accelation

$\underset{\sim}{a} = (\ddot{r} - r\dot{\theta}^2)\,\underset{\sim}{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underset{\sim}{e_\theta}$

$= a_r\,\underset{\sim}{e_r} + a_\theta\,\underset{\sim}{e_\theta}$

$a_r = \ddot{r} - r\dot{\theta}^2$

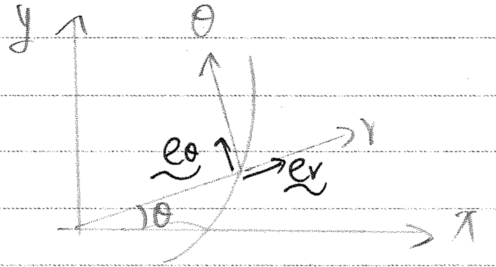
$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

$a = |\underset{\sim}{a}| = \sqrt{a_r^2 + a_\theta^2}$

Note: $a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

$=$

Geomatric Interpretation

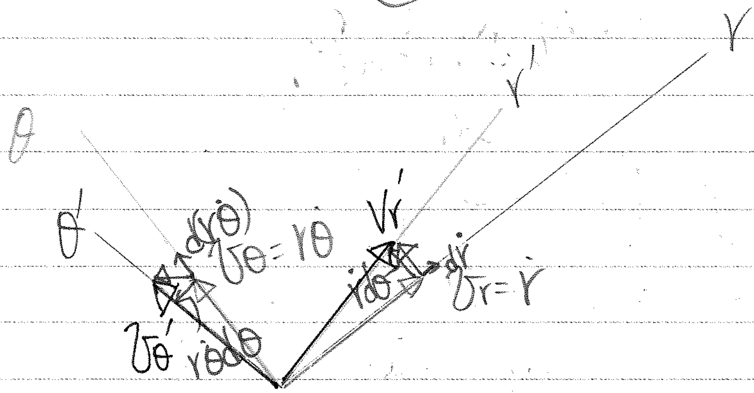


$$\underset{\sim}{e_r} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$$\underset{\sim}{e_\theta} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$$

$$\vec{i} = ?\,e_r + ?\,e_\theta$$

$$j = ?\,\underset{\sim}{e_r} + ?\,e_\theta$$



(a) magnitude change of v_r

$dv_r = d\dot{r}$ $\quad \underset{\sim}{+r}$ $\quad \Rightarrow \dfrac{dv_r}{dt} = \underset{\sim}{\dfrac{d\dot{r}}{dt} = \ddot{r}}$

(b) direction change of v_r

$v_r\,d\theta = \dot{r}\,d\theta$ $\quad \underset{\sim}{+\theta}$ $\quad \Rightarrow \dfrac{v_r\,d\theta}{dt} = \dot{r}\dfrac{d\theta}{dt} = \underset{\sim}{\dot{r}\dot{\theta}}$

(c) magnitude change of v_θ

$d(r\dot{\theta})$ $\quad \underset{\sim}{+\theta}$ $\quad \Rightarrow \dfrac{d(r\dot{\theta})}{dt} = \underset{\sim}{r\ddot{\theta} + \dot{r}\dot{\theta}}$

(d) direction change of v_θ

$r\dot{\theta}\,d\theta$ $\quad \underset{\sim}{-r}$ $\quad \Rightarrow r\dot{\theta}\dfrac{d\theta}{dt} = \underset{\sim}{r\dot{\theta}^2}$

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$\therefore a_r = \ddot{r} - r\dot{\theta}^2$

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

$\ddot{r}$ ~ acceleration along the radius without changing θ

$-r\dot{\theta}^2$ ~ normal component if r is constant

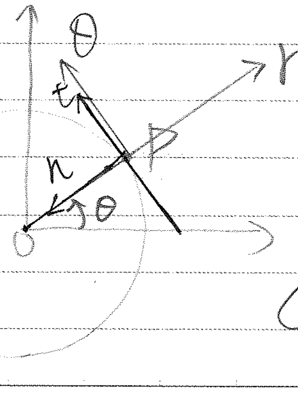
$r\ddot{\theta}^2$ ~ tangential acceleration if r were constant

$2\dot{r}\dot{\theta}$ ~ part of magnitude change of v_θ ⊕ chang of direction

→ Coriolis acceleration

Special Case: Circular motion

For motion in a circular path



$r = \text{constant}$

$\Rightarrow \dot{r} = 0 \; , \; \ddot{r} = 0$

$v_r = 0 \; , \; v_\theta = r\dot{\theta}$

$a_r = -r\dot{\theta}^2 \; , \; a_\theta = r\ddot{\theta}$

Compared with n-t coordinates

$\underline{e_r = -e_n}$, $a_t = r\ddot{\theta}$

$\underline{e_\theta = e_t}$, $a_n = r\dot{\theta}^2$

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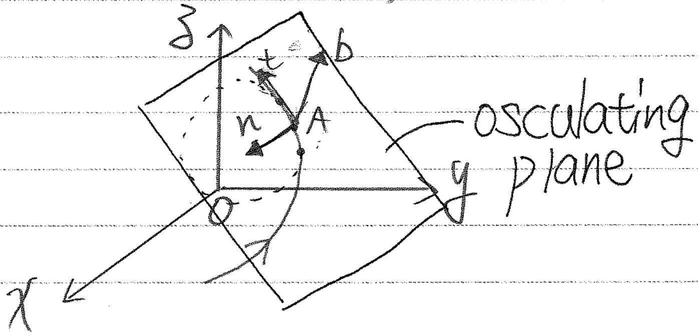
2/17 Space curvilinear Motion
空間

- General case of 3 dimensionel motion of a particle along a space curve.

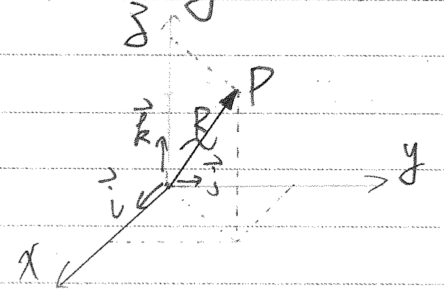
- Three coordinates systems:
  rectangular ($x-y-z$)
  Cylindrical ($r-\theta-z$), and
  spherical ($R-\theta-\phi$)

- normal and tangential coordinates
  $\Rightarrow$ ($t-n-b$)



$e_b = e_t \times e_n$

□ Rectangular coordinates $(x\text{-}y\text{-}z)$



$$\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

$$\vec{v}(t) = \frac{d\vec{R}(t)}{dt}$$

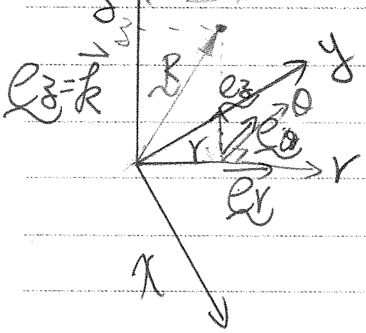
$$= \dot{x}(t)\vec{i} + \dot{y}(t)\vec{j} + \dot{z}(t)\vec{k}$$

$$\dot{\vec{i}} = \frac{d\vec{i}}{dt} = \vec{0}\ ,\quad \dot{\vec{j}} = \frac{d\vec{j}}{dt} = \vec{0}\ ,\quad \dot{\vec{k}} = \frac{d\vec{k}}{dt} = \vec{0}$$

$$a(t) = \frac{dv(t)}{dt} = \ddot{x}(t)\vec{i} + \ddot{y}(t)\vec{j} + \ddot{z}(t)\vec{k}$$

□ Cylindrical coordinate

圓柱



$$R(t) = r(t)e_r + z(t)e_z\ ,\quad e_z = \vec{k}$$

$$v(t) = \frac{dR(t)}{dt} = \dot{r}(t)e_r + r(t)\dot{e}_r + \dot{z}(t)e_z$$

$$= \dot{r}(t)e_r + r(t)\dot{\theta}(t)e_\theta + \dot{z}(t)e_z$$

$$a(t) = \frac{dv(t)}{dt} = (\ddot{r} - r\dot{\theta}^2)e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta + \ddot{z}\,e_z$$

$$\therefore a_r = \ddot{r} - r\dot{\theta}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$a_z = \ddot{z}$$

$$a = \sqrt{a_r^2 + a_\theta^2 + a_z^2}$$

□ spherical coordinates (R-θ-ϕ)

difine $\underset{\sim}{e_R}$, $\underset{\sim}{e_\theta}$, $\underset{\sim}{e_\phi}$, θ方向角, ϕ:仰角

position Vector

$$\underset{\sim}{R} = R\underset{\sim}{e_R}$$

Velocity

$$\underset{\sim}{v} = \frac{d\underset{\sim}{R}}{dt} = \dot{R}\underset{\sim}{e_R} + R\cdot\dot{\underset{\sim}{e_R}}$$

Q: How to determine $\dot{\underset{\sim}{e_R}}$ · $\dot{\underset{\sim}{e_\theta}}$ · $\dot{\underset{\sim}{e_\phi}}$?

Angular Velocity of R-θ-ϕ coordinate

$\underset{\sim}{\omega}$ = 繞 z axis rotate $\dot{\theta}$
繞 θ axis rotate $-\dot{\phi}$

$$\underset{\sim}{\omega} = -\dot{\phi}\underset{\sim}{e_\theta} + \dot{\theta}\cos\phi\,\underset{\sim}{e_\phi} + \dot{\theta}\sin\phi\,\underset{\sim}{e_R}$$

$$\dot{\underset{\sim}{e_R}} = \underset{\sim}{\omega} \times \underset{\sim}{e_R} = -\dot{\phi}\underset{\sim}{e_\theta} \times \underset{\sim}{e_R} + \dot{\theta}\cos\phi\,\underset{\sim}{e_\phi} \times \underset{\sim}{e_R}$$

$$\dot{\underset{\sim}{e_\theta}} = \underset{\sim}{\omega} \times \underset{\sim}{e_\theta} = \dot{\phi}\underset{\sim}{e_\phi} + \dot{\theta}\cos\phi\,\underset{\sim}{e_\theta}$$

$$\dot{\underset{\sim}{e_\phi}} = \underset{\sim}{\omega} \times \underset{\sim}{e_\phi}$$

$$\therefore \underset{\sim}{v} = \dot{R}\underset{\sim}{e_R} + R\dot{\phi}\underset{\sim}{e_\phi} + R\dot{\theta}\cos\phi\,\underset{\sim}{e_\theta}$$

$$= v_R\underset{\sim}{e_R} + v_\phi\underset{\sim}{e_\phi} + v_\theta\underset{\sim}{e_\theta}$$

$v_R = \dot{R}$

$v_\theta = R\dot{\theta}\cos\phi$ $\quad v = \sqrt{v_R^2 + v_\theta^2 + v_\phi^2}$ $\quad \underset{\sim}{e_R} = ?\,\vec{i} + ?\,\vec{j}$

$v_\phi = R\dot{\phi}$

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acceleration

θ: 方向角

$$a = \frac{dv}{dt} = \ddot{R}\, e_R + \dot{R}\, \dot{e}_R$$

$$+ \dot{R}\dot{\phi}\, e_\phi + R\ddot{\phi}\, e_\phi + R\dot{\phi}\, \dot{e}_\phi$$

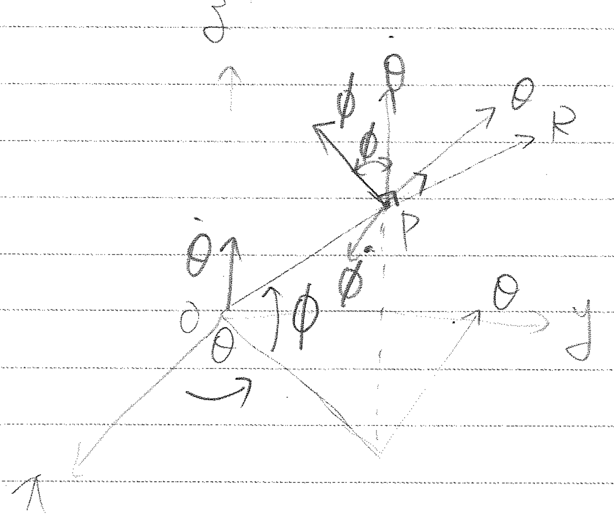
$$+ \dot{R}\dot{\theta}\cos\phi\, e_\theta + R\ddot{\theta}\cos\phi\, e_\theta - R\dot{\theta}\dot{\phi}\sin\phi\, e_\theta$$

$$+ R\cos\phi\, \dot{e}_\theta$$

$$\therefore a = (\ddot{R} - R\dot{\phi}^2 - R\dot{\theta}^2\cos^2\phi)\, e_R + (R\ddot{\theta} - 2R\dot{\theta}\dot{\phi}\sin\phi + 2\dot{R}\dot{\theta}\cos\phi)\, e_\theta$$

$$+ (R\ddot{\phi} + 2\dot{R}\dot{\phi} + R\dot{\theta}^2\sin^2\phi)\, e_\phi$$

$$\omega \times \dot{R} = \begin{vmatrix} e_R & e_\theta & e_\phi \\ \dot{\theta}\sin\phi & -\dot{\phi} & \dot{\theta}\cos\phi \\ \dot{R} & R\dot{\theta}\cos\phi & R\dot{\phi} \end{vmatrix}$$



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$\boxed{2/8}$ Relative Motion (translating axes)

- ☐ coordinates reffered to fixed coordinates
  absolute displacement, velocity & acceleration

- ☐ Analysis can be simplified using measurement
  with respect to a moving reference system
  (移動參考坐標)

- ☐ relative motion Analysis
  Choice of coordinates sys. Newtonian mechanics:

  fixed coordinate system
  ~ primary inertial frame

Engineering problems:

- Earth-bound ~ set of axes
  attached of Earth.
- Satellite ~ non-rotating coordinates
  fixed to axis of rotation
  of Earth.

Interplanetary travel ~ non-rotating axes
fixed to the sun

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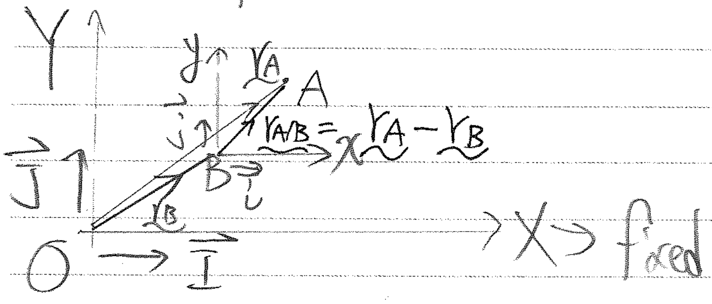
Vector representation of relative motion



$$\underset{\sim}{r_A} = \underset{\sim}{r_B} + \underset{\sim}{r_{A/B}}$$

$$\vec{i} = \vec{I}$$

$$\vec{j} = \vec{J}$$

$$\underset{\sim}{v_A} = \underset{\sim}{v_B} + \underset{\sim}{v_{A/B}}$$

$$\underset{\sim}{a_A} = \underset{\sim}{a_B} + \underset{\sim}{a_{A/B}}$$

$$\underset{\sim}{r_A} = \underset{\sim}{r_B} + \underset{\sim}{r_{A/B}}$$

$$\underset{\sim}{r_{A/B}} = \underset{\sim}{r_A} - \underset{\sim}{r_B}$$

$$\underset{\sim}{r_{A/B}} = x\vec{i} + y\vec{j}$$

$$\dot{\underset{\sim}{r_A}} = \dot{\underset{\sim}{r_B}} + \dot{\underset{\sim}{r_{A/B}}}$$

$$= \dot{\underset{\sim}{r_B}} + \dot{x}\vec{i} + \dot{y}\vec{j} + \cancel{x\dot{\vec{i}}} + \cancel{y\dot{\vec{j}}} \qquad \dot{\vec{i}} = \vec{0} \quad \dot{\vec{j}} = \vec{0}$$

$$\underset{\sim}{v_A} = \underset{\sim}{v_B} + \underset{\sim}{v_{A/B}}$$

$$\ddot{\underset{\sim}{r_A}} = \ddot{\underset{\sim}{r_B}} + \ddot{\underset{\sim}{r_{A/B}}} = \ddot{\underset{\sim}{r_B}} + \ddot{x}\vec{i} + \ddot{y}\vec{j}$$

$$\underset{\sim}{a_A} = \underset{\sim}{a_B} + \underset{\sim}{a_{A/B}}$$

$$\underset{\sim}{r_B} = \underset{\sim}{r_A} + \underset{\sim}{r_{B/A}}$$

$$\underset{\sim}{r_{B/A}} = \underset{\sim}{r_B} - \underset{\sim}{r_A} = -\underset{\sim}{r_{A/B}}$$

$$\underset{\sim}{v_{B/A}} = \underset{\sim}{v_B} - \underset{\sim}{v_A} = -\underset{\sim}{v_{A/B}}$$

$$\underset{\sim}{a_{B/A}} = \underset{\sim}{a_B} - \underset{\sim}{a_A} = -\underset{\sim}{a_{A/B}}$$

※ In relative motion analysis if the translating sys. x-y has a constant velocity the accelation of a particle observed in x-y is equal to that observed in fixed sys X-Y.

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$\underset{\sim}{a_A} = \underset{\sim}{a_B} + \underset{\sim}{a_{A/B}}$

If B has a constant velocity

$\underset{\sim}{a_B} = \underset{\sim}{0}$

$\therefore \underset{\sim}{a_A} = \underset{\sim}{a_{A/B}}$

2/9 Constrained Motion of Conneted particle

- Motion of particles are interrelated due constraint imposed by interconnecting members

- Constraints have to be considerd (Pulley Sys.)

DOF (Degree of fredom)

= # of variable decribing particle motion − # constraints

P99, Figure 2/19

$(x, y) = 2$ — x: position of A

$-$ Constraint $= 1$ — y: position of B

1 — Constraint: Cable total length is constraint

$L = x + \frac{\pi}{2} \times \boxed{r_2} + y + \pi \boxed{r_1} + y + \boxed{b} =$ constant

$\because L \cdot r_2 \cdot r_1 \cdot b$ are constraints

$\therefore \dot{L} = \dot{x} + \frac{\pi}{2} \times \cancel{\dot{r}_2} + 2\dot{y} + \pi \cancel{\dot{r}_1} + \cancel{\dot{b}} = 0$

$= \dot{x} + 2\dot{y} = 0$ or $v_A + 2v_B = 0$ or $v_A = -2v_B$

$\ddot{L} = 0 \rightarrow \ddot{x} + 2\ddot{y} = 0$ or $a_A + 2a_B = 0$ or $a_A = -2a_B$

Double A

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Figure 2/20

particles 4
− constraints 2
DOF = 2

position of lower pulley C & Cylinder depeneds y_A and y_B
Length of cables attached to particles A & B

$$L_A = 2y_D + y_A + \text{constant}$$
$$L_B = y_B + y_C + y_C - y_D + \text{constant}$$

$$\therefore \dot{L}_A = 0, \ \ddot{L}_A = 0$$
$$\dot{L}_B = 0, \ \ddot{L}_B = 0$$

$$\Rightarrow 2\dot{y}_D + \dot{y}_A = 0, \quad \dot{y}_D = \frac{-1}{2}\dot{y}_A$$
$$2\ddot{y}_D + \ddot{y}_A = 0, \quad \ddot{y}_D = \frac{-1}{2}\ddot{y}_A$$
$$\dot{y}_B + 2\dot{y}_C - \dot{y}_D = 0$$
$$\Rightarrow \ddot{y}_B + 2\ddot{y}_C - \ddot{y}_D = 0$$

or
$$\dot{y}_A + 2\dot{y}_B + 4\dot{y}_C = 0$$
$$\ddot{y}_A + 2\ddot{y}_B + 4\ddot{y}_C = 0$$

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2/10 Chapter review

categories of Motion

- ☐ Rectangular motion (1 coordinate)
- ☐ Plane curvilinear motion (2 coordinates)
- ☐ Space curvilinear motion (3 coordinates)

Use of fixed Axes

- ☐ fixed reference axes (absolute motion)
- ☐ moving axes (relative motion)

Choice of coordinates

- ☐ Rectangular (Cartesian) cord. $(x\text{-}y)$ $(x\text{-}y\text{-}z)$
- ☐ Normal and Tangentical cord. $(n\text{-}t)$ $(t\text{-}n\text{-}b)$
- ☐ polar cord. $(r\text{-}\theta)$
- ☐ Cylindrical cor. $(r\text{-}\theta\text{-}z)$
- ☐ Spherical cord. $(R\text{-}\theta\text{-}\phi)$

※ P108 Figure 2/21

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CH3 Kinetics of Particles

§3/1 Introduction

- ☐ Study relations between unbalanced force and resulting changes in motion.
- ☐ combine knowledge of properties force and kinematics of particle motion.
- ☐ Apply Newton's second law to solve engineering problems involving force, mass, and motion.

Three approaches:

(A) Force – mass – acceleration Method
(Direct application of Newton's law
Vector Eqn.)

(B) Work and energy principle
(Scalar Eqn.)

(C) Impulse and Momentum Methods.
(Vector Eqn.)

§3/2 Newton's second law

$\underline{F} = m\underline{a}$ <Inertial system>

SI units: force N | mass kg | acceleration m/s^2

$1\,N = kg \cdot m/s^2$

$g = 9.81\ m/s^2$

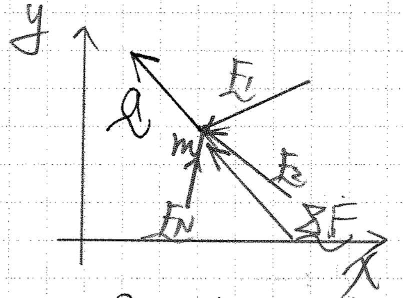
US units: force lb
mass slug
acceleration ft/sec^2

$\boxed{1 \text{ slug} = 1 \text{ lb} \cdot sec^2/ft}$

$g = 32.2 \, ft/sec^2$, $1 ft = 12 in$

↳ 3/3 Equations of Motion and Solutions of Problems.

When a particle mass m is subjected to action of concurrent forces $F_1 \cdot F_2 \cdots F_N$



$$\sum_{i=1}^{N} F_i = m \cdot a$$

for 2-D motion

$\Sigma F_x = m \cdot a_x$

$\Sigma F_y = m \cdot a_y$

Two types of dynamics problems:

- Inverse dynamics

Two types of dynamics problems

- inverse dynamics:

  acceleration of particle is <u>specified</u> or <u>determined</u> from <u>kinematics condition</u> and then determined <u>corresponding forces</u>

- Forward dynamics:

  forces acting on particles are specified and determine resulting motion

$$\mathbf{a} = \frac{1}{m} \cdot \Sigma \mathbf{F}$$

$$\int_{v_0}^{v} d\mathbf{v} = \int_0^t \frac{1}{m} \Sigma \mathbf{F} \, dt$$

$$\int_{r_0}^{r} dr = \int_0^t v \, dt$$

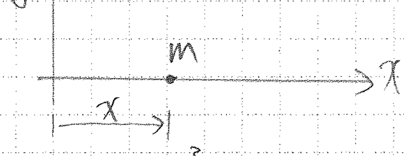
Key concepts

free-body diagram — P.P 123-124

§3/4 Rectilinear Motion

直線

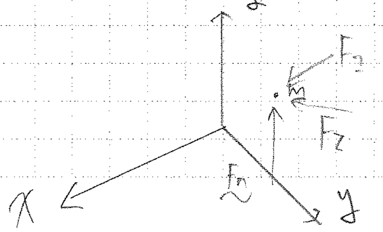
Let x-axis be path of rectilinear motion



$$\Sigma F_x = m a_x$$

$$\Sigma F_y = 0$$

$$\Sigma F_z = 0$$



$$\sum_{i=1}^{n} F_i = m\mathbf{a}$$

$$\mathbf{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$$

$$\mathbf{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$$

$$\sum_{i=1}^{n} F_{ix} + \sum_{i=1}^{n} F_{iy} + \sum_{i=1}^{n} F_{iz} = m a_x \vec{i} + m a_y \vec{j} + m a_z \vec{k}$$

$$|\Sigma F| = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$

$$|a| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

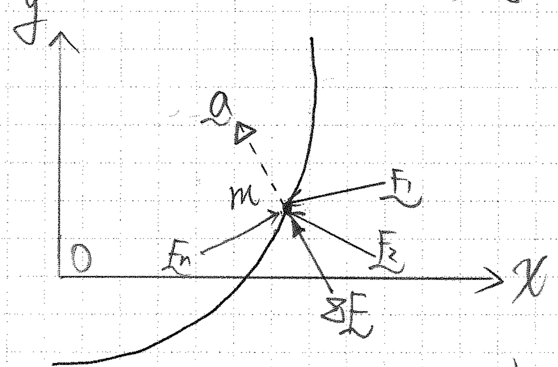
§3/5 Curvilinear Motion

□ Choice of appropriate coordinate system depends on condition of problem

Rectangular coord.

Given $x(t) \cdot y(t)$

$\rightarrow \dot{x}(t) \cdot \dot{y}(t) \cdot \ddot{x}(t) \cdot \ddot{y}(t)$



$$\sum_{i=1}^{n} F_i = ma$$

$$F_1 = F_{1x}\vec{i} + F_{1y}\vec{j}$$

$$\vdots$$

$$F_n = F_{nx}\vec{i} + F_{ny}\vec{j}$$

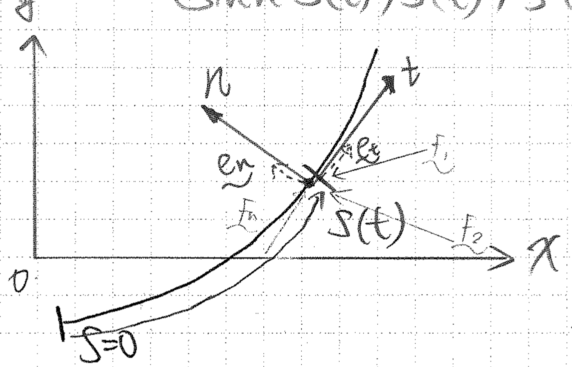
$$\therefore \sum_{i=1}^{n} F_{ix}\vec{i} + \sum_{i=1}^{n} F_{ix}\vec{j} = ma_x\vec{i} + ma_y\vec{j}$$

$$\Rightarrow \sum_{i=1}^{n} F_{ix} = ma_x \quad a_x = \ddot{x}$$

$$\sum_{i=1}^{n} F_{iy} = ma_y \quad a_y = \ddot{y}$$

Normal and Tangential coordinates

Given $S(t), \dot{s}(t), \ddot{s}(t), \rho$



$\underset{\sim}{F_1} = F_{1t}\underset{\sim}{e_t} + F_{1n}\underset{\sim}{e_n}$

$\underset{\sim}{F_2} = F_{2t}\underset{\sim}{e_t} + F_{2n}\underset{\sim}{e_n}$

$\vdots$

$\underset{\sim}{F_N} = F_{Nt}\underset{\sim}{e_t} + F_{Nn}\underset{\sim}{e_n}$

$\underset{\sim}{a} = a_t\underset{\sim}{e_t} + a_n\underset{\sim}{e_n}$

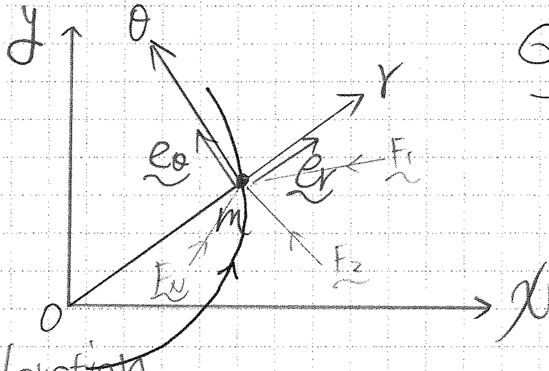
$$\sum_{i=1}^{N}\underset{\sim}{F_i} = m\underset{\sim}{a}$$

$$\sum_{i=1}^{N}F_{it}\underset{\sim}{e_t} + \sum_{i=1}^{N}F_{in}\underset{\sim}{e_n} = ma_t\underset{\sim}{e_t} + ma_n\underset{\sim}{e_n}$$

$$\Rightarrow \sum_{i=1}^{N}F_{it} = ma_t \qquad a_t = \ddot{s}$$

$$\sum_{i=1}^{N}F_{in} = ma_n \qquad a_n = \frac{\dot{s}^2}{\rho} = \frac{v^2}{\rho}$$

polar coordinates



Given $r(t), \theta(t)$

$\rightarrow \dot{r}, \dot{\theta}, \ddot{r}, \ddot{\theta}$

$\underset{\sim}{F_1} = F_{1r}\underset{\sim}{e_r} + F_{1\theta}\underset{\sim}{e_\theta}$

$\underset{\sim}{F_2} = F_{2r}\underset{\sim}{e_r} + F_{2\theta}\underset{\sim}{e_\theta}$

$\vdots$

$\underset{\sim}{F_N} = F_{Nr}\underset{\sim}{e_r} + F_{N\theta}\underset{\sim}{e_\theta}$

$a_r = \ddot{r} - r\dot{\theta}$

$a_\theta = r\ddot{\theta} + 2\dot{r}$

acceleration

$\underset{\sim}{a} = a_r\underset{\sim}{e_r} + a_\theta\underset{\sim}{e_\theta}$

$$\sum_{i=1}^{N}\underset{\sim}{F_i} = m\underset{\sim}{a}$$

$$\Rightarrow \sum F_{ir}\underset{\sim}{e_r} + \sum F_{i\theta}\underset{\sim}{e_\theta} = m(a_r\underset{\sim}{e_r} + a_\theta\underset{\sim}{e_\theta})$$

$$\Rightarrow \sum_{i=1}^{N}F_{ir} = ma_r$$

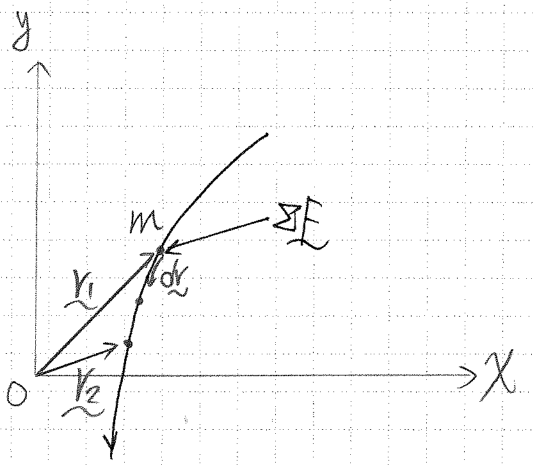
$$\sum_{i=1}^{N}F_{i\theta} = ma_\theta$$

3/6 Work and Kinetic Energy (只有力會作功)

previous: $\sum \underset{\sim}{F} = m\underset{\sim}{a}$ law.

another approaches: (i) integration of force with respect to displacement

(ii) integration of force with respect to time



①

$$\sum \underset{\sim}{F} = m\underset{\sim}{a}$$

$$\int \sum \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int m\underset{\sim}{a} \cdot d\underset{\sim}{r}$$

$$= \int m\ddot{\underset{\sim}{r}} \cdot d\underset{\sim}{r} = \int m\ddot{\underset{\sim}{r}} \cdot \frac{d\underset{\sim}{r}}{dt} dt$$

$$= \int m\ddot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}}\, dt$$

Note: $(\ddot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}}) = \frac{1}{2}\frac{d}{dt}(\dot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}})$

$$\therefore \int_{\underset{\sim}{r}_1}^{\underset{\sim}{r}_2} \sum \underset{\sim}{F} d\underset{\sim}{r} = \frac{1}{2}\int m\, d(\dot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}})$$

$$= \frac{1}{2}\int_{v_1}^{v^2} m\, d(v^2)$$

$$\Rightarrow \int_{\underset{\sim}{r}_1}^{\underset{\sim}{r}_2} \sum \underset{\sim}{F} \cdot d\underset{\sim}{r} = \frac{1}{2} m (v_2^2 - v_1^2)$$

$$= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

$$\boxed{\int_{r_1}^{r_2} \overset{①}{\Sigma F \cdot dr} = \overset{②}{\frac{1}{2}m(v_2^2 - v_1^2)}} \Rightarrow \text{功能原理}$$

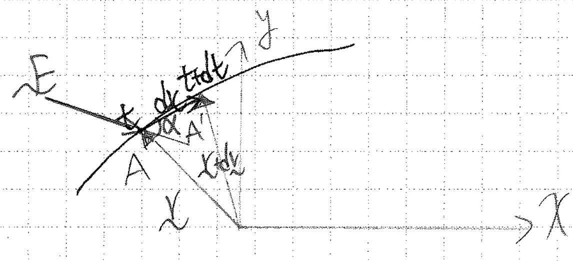
① work done by ΣF on particle
② ΔKE change of kinetic energy

② $\Sigma F = ma$

$$\Sigma F\,dt = ma\,dt$$
$$= m\frac{dv}{\cancel{dt}}\cancel{dt}$$

$$\underbrace{\int_{t_1}^{t_2} \Sigma F\,dt}_{\text{impulse}} = \int_{v_1}^{v_2} m\,dv$$
$$= mv(t_2) - mv(t_1)$$
$$= \underbrace{mv_2 - mv_1}_{\Delta\,\text{momentum}}$$

Definition of Work



Force F acting on particle at A.

r: position vector at time t

$r+dr$: position vector at time $t+d$

dr: differential displacement from A to A'

Work done by F during displacement dr

$$dU = F \cdot dr = F\,ds\cos\alpha$$
$$|dr| = ds$$

Note: (i) another inteprelation is displacement multiplea by force compenent in the direction of displacement.

(ii) $\because F_n \perp dr \therefore F_n \cdot dr = 0$

$\therefore F_n$ does No WORK.

(iii) sign convertion of Work:
position if F_t is in the direction of dr
[illegible] if F_t opposite to direction of [illegible]

(iv) Force which do work are called active force.
Constraint force ~ reactive force does not WORK

Unit of Work:

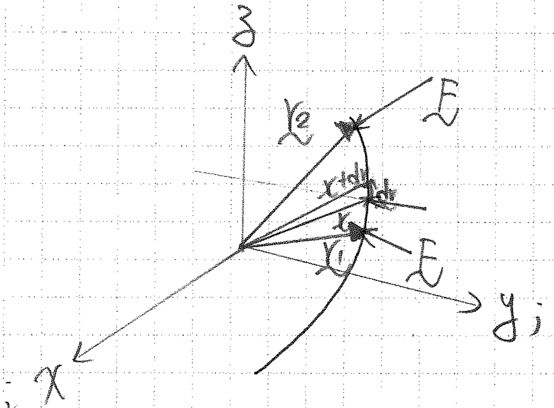
SI: joule (J), 1 J = 1 N·m

US: ft-lb, note: unit of $\boxed{\text{moment}}$ lb-ft (vector not scalar)

Caculation of Work

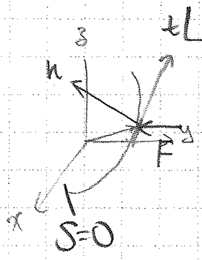
$$\underset{\sim}{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$$

$$d\underset{\sim}{r} = dx\vec{i} + dy\vec{j} + dz\vec{k} \quad (\vec{r} = x\vec{i} + y\vec{j} + z\vec{k})$$

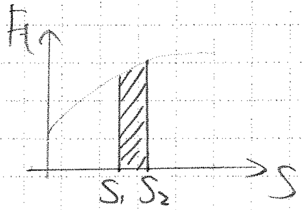


$$dU = \underset{\sim}{F} \cdot d\underset{\sim}{r}$$
$$= (F_x\vec{i} + F_y\vec{j} + F_z\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$
$$= F_x dx + F_y dy + F_z dz$$

$$U_{1-2} = \int_1^2 dU = \int_{\underset{\sim}{r_1}}^{r_2} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_1^2 F_x dx + F_y dy + F_z dz$$



$$\text{or} \quad U_{1\sim 2} = \int_{S_1}^{S_2} \underset{\sim}{F_t}(S) \cdot dS$$

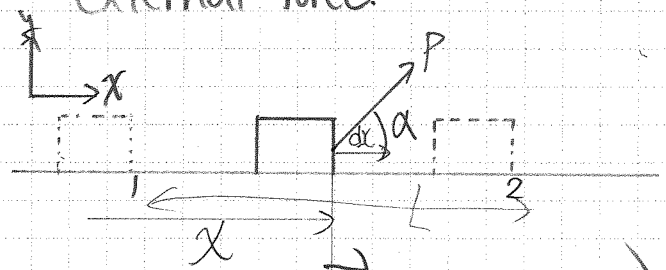


$\boxed{(r\text{-}\theta)\text{導 ?}}$

Examples of work

Three forces: constant force
spring force
weight

(1) Work associated with a constant external force.



$$\underset{\sim}{P} = P\cos\alpha\,\vec{i} + P\sin\alpha\,\vec{j}$$

$$d\underset{\sim}{r} = dx\,\vec{i}$$

$$dU = \underset{\sim}{P}\cdot d\underset{\sim}{r}$$

$$= \left(P\cos\alpha\,\vec{i} + P\sin\alpha\,\vec{j}\right)\left(dx\,\vec{i}\right)$$

$$= P\cos\alpha\,dx$$

$$U_{1-2} = \int_1^2 dU = \int_{x_1}^{x_2} P\cos\alpha\,dx$$

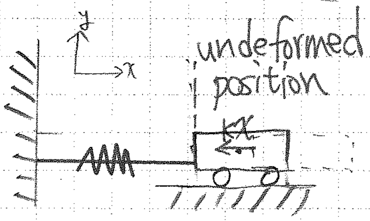
$$= P\cos\alpha\,(x_2 - x_1)$$

$$= PL\cos\alpha$$ work done by $\underset{\sim}{P}$ from position x_1 to x_2

Note: if $\alpha = 90^\circ$, then $U_{1-2} = 0$
if $90^\circ < \alpha < 270^\circ$, then $U_{1-2} < 0$

(2) Work associated with a spring force

Consider a linear spring with stiffness k ($F = k x$) (spring constant)



Determine work done on the body by the spring force on the body displaces from initial position x_1 to final position x_2

$$\underset{\sim}{F} = -kx\vec{i}, \quad d\underset{\sim}{r} = dx\vec{i}$$

$$U_{1-2} = \int_1^2 dU = \int_1^2 \underset{\sim}{F}\, d\underset{\sim}{r}$$

$$= \int_1^2 -kx\vec{i}\cdot dx\vec{i}$$

$$= \int_1^2 -kx\,dx = -k\int_{x_1}^{x_2} x\,dx = -k\cdot\frac{x^2}{2}\Big]_{x_1}^{x_2}$$

$$\Rightarrow \boxed{U_{1-2} = -\frac{k}{2}(x_2^2 - x_1^2)} \text{ — 3/10}$$

Note: (1) If $x_1 = 0$ (neutral position)

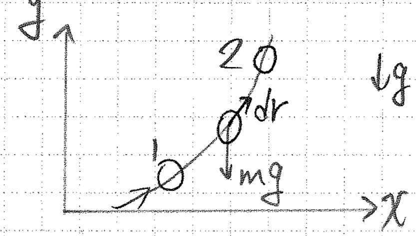
$$U_{1-2} = \frac{-k}{2}x_2^2 < 0 \text{ — negative work}$$

(2) If $x_1 \neq 0$, $x_2 = 0$

$$U_{1-2} = \frac{-k}{2}(0 - x_1^2) = \frac{k}{2}x_1^2 > 0$$

(3) Work associated with Weight
(gravitational force)

case(a), g = constant



$$d\underset{\sim}{r} = dx\vec{i} + dy\vec{j}$$

$$\underset{\sim}{F} = -mg\vec{j}$$

$$U_{1-2} = \int_1^2 dU = \int_1^2 \underset{\sim}{F} \cdot d\underset{\sim}{r}$$

$$= \int_1^2 -mg\vec{j} \cdot (dx\vec{i} + dy\vec{j})$$

$$= \int_{y_1}^{y_2} -mg\, dy$$

$$= -mgy\Big]_{y_1}^{y_2}$$

$$= -mg(y_2 - y_1)$$

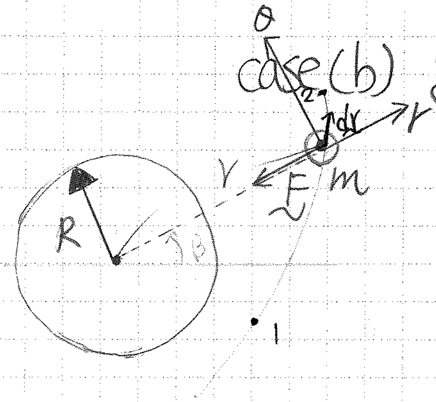
$$\therefore \boxed{U_{1-2} = -mg(y_2 - y_1)}$$

discussion:

$U_{1-2} > 0$ if $y_2 < y_1$, failing

$U_{1-2} < 0$ if $y_2 > y_1$, rising

case (b) $g \neq$ constant



$$d\underset{\sim}{r} = dr\,\underset{\sim}{e_r} + r\,d\theta\,\underset{\sim}{e_\theta}$$

$$F = \frac{G m_e m}{r^2}$$

$$\underset{\sim}{F} = \frac{-G m_e m}{r^2}\underset{\sim}{e_r}$$

$$dU = \underset{\sim}{F} \cdot d\underset{\sim}{r}$$

$$= \frac{-G m_e m}{r^2}\underset{\sim}{e_r} \cdot (dr\,\underset{\sim}{e_r} + r\,d\theta\,\underset{\sim}{e_\theta})$$

$$= \frac{-G m_e m}{r^2} dr$$

$$U_{1-2} = \int_1^2 dU = \int_{r_1}^{r_2} \frac{-G m_e m}{r^2} dr$$

$$= -G m_e m \int_{r_1}^{r_2} r^{-2} dr$$

$$= G m_e m \frac{1}{r}\Big]_{r_1}^{r_2}$$

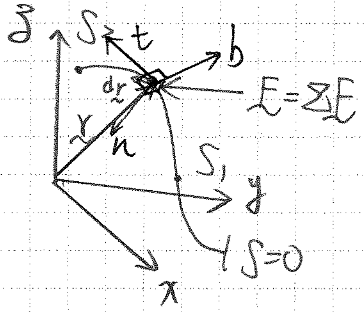
$$= G m_e m \left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$\because G m_e = g R^2$$

$$\therefore \boxed{U_{1-2} = m g R^2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$$

Work and Curvilinear motion

Fig 3/17



Consider work done on a particle of mass m moving along a curved path under action of resultant of all forces.

$\underset{\sim}{r}$ = position vector

using n-t-b coordinates

$\underset{\sim}{e_t}$ · $\underset{\sim}{e_n}$ · $\underset{\sim}{e_b} = \underset{\sim}{e_n} \times \underset{\sim}{e_t}$, $\underset{\sim}{e_t} \perp \underset{\sim}{e_n}$

$$d\underset{\sim}{r} = ds\, \underset{\sim}{e_t}$$

$$\underset{\sim}{F} = \Sigma \underset{\sim}{F} = \Sigma F_t \underset{\sim}{e_t} + \Sigma F_t \underset{\sim}{e_n} + \Sigma F_t \underset{\sim}{e_b}$$

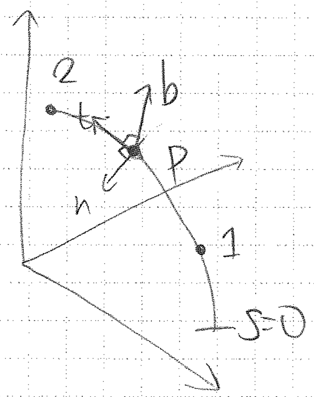
$$U_{1-2} = \int_1^2 dU = \int_{S_1}^{S_2} \Sigma F_t\, ds$$

From Newton's law

$$\Sigma F_t = m a_t$$

$$U_{1-2} = \int_{S_1}^{S_2} m a_t \cdot ds \quad \boxed{a_t ds = v dv} \rightarrow \text{近行於線}$$

$$= \int_1^2 m v dv = \frac{1}{2} m (v_2^2 - v_1^2)$$



$$W_{1-2} = \int_{v_1}^{v_2} m v \, dv$$

$$= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

$$= T_2 - T_1$$

※ Principle of Work and Kinetic Energy

define K.E. of the particle as

$T = \frac{1}{2} m v^2 \sim$ scalar

Eq 3/13 $\Rightarrow \underbrace{W_{1-2}}_{\text{total Work done by all force acting on a particle}} = \underbrace{T_2 - T_1}_{\text{change in kinetic energy from 1 to 2}} = \Delta T$ (3/15) Work energy eqn for a particle

$\Rightarrow T_2 = T_1 + W_{1-2}$

Remarks:

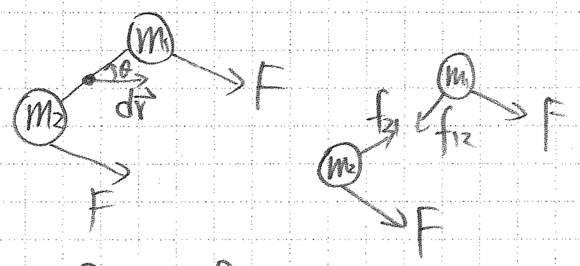
(1) Work and kinetic energy principle have been proved in retangular coordinates & normal-tangential coordinates.

(2) It can be also proved in polar coordinates or Cylindrical

Advantage of Work-and Energy Work

(1) avoid computing acceleration and leads directly to Velocity changes as function of the forces which do work.

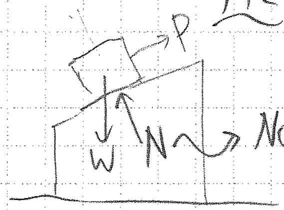
(2) Extendable to system of <u>two</u> or more particle connecting by connection which is frictionless and non-deformable.



$\because \ \mathbf{f}_{21} = -\mathbf{f}_{12}$

$\therefore$ the net work done by interior forces $\mathbf{f}_{21} = \mathbf{f}_{12}$ is zero

$$\mathbf{f}_{12} \cdot d\mathbf{r} + \mathbf{f}_{21} \cdot d\mathbf{r} = 0$$



Not neccessory to considor the constraint force which does work

4 power (功率)

Capacity of a machine is rated by its power (馬力)

power is defined as time rate of doing work

$$P = \frac{dU}{dt} = \frac{\mathbf{F} \cdot d\mathbf{r}}{dt} = \mathbf{F} \cdot \frac{dx}{dt} = \mathbf{F} \cdot \mathbf{v}$$

$\Rightarrow P = \mathbf{F} \cdot \mathbf{V}$ (3/16)

(single particle)

$P \sim$ scalar

Unit: $N\,m/s$ or J/s 　　　1度 = 1 km·hr

$\therefore 1\,W = 1\,J/s$

US unit hp (horse power)

$1\,hp = 550\ ft\cdot lb/sec$

$= 33000\ ft\cdot lb/min$

$1\,hp = 746\,W$

$= 0.746\,kW \quad 1\,kW = 10^3\,W$

4 Efficiency 效率

Def. e_m (mechanical efficiency) ratio of Work done by a machine to the work done on the machine during same time interval.

$$\longrightarrow \boxed{\text{machine}} \longrightarrow$$

$$e_m = \frac{P_0}{P_I} \quad (3/17)$$

Note: $e_m < 1$

$\because$ every machine operates with same loss of energy due to negative work of Kinetic friction force.

other energy loss

e_e electrical, e_t thermal

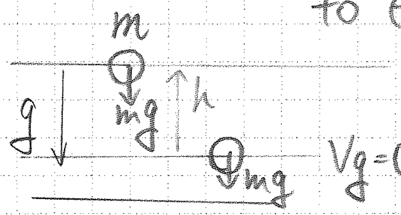
$$e = e_m e_e e_t$$

63/7 potential energy (位能、勢能)

Gravitational P.E.

(1) particle in close procimity to earth surface

Fig 3/8



gravitational P.E. ≡ work done against the gravitational field to elevate particle a distance of above a datum. ($V_g=0$)

particle from $h=h_1$ to $h=h_2$, $h_2>h_1$

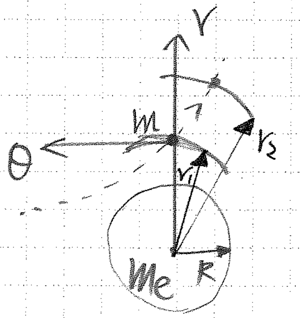
$$\Delta V_g = mg(h_2-h_1)$$

The corresponding work done by the gravitational force on the particle

$$\underbrace{U_{1-2}}_{\text{work done by gravity}} = -mg\,(h_2-h_1) = -mg\,\Delta h = \underbrace{-\Delta V_g}_{-\text{ change in P.E.}}$$

$$\therefore \boxed{\begin{aligned} U_{1-2} &= -\Delta V_g \\ \Delta V_g &= -U_{1-2} \end{aligned}}$$

(2) Large change in attitude



$$F = \frac{G M_e m}{r^2} = \frac{mgR^2}{r^2} \propto r^{-2}$$

$$\underset{\sim}{F} = -\frac{mgR^2}{r^2}\underset{\sim}{e_r}, \quad \underset{\sim}{dr} = dr \times \underset{\sim}{e_r}$$

$$\Delta V_g = -W_{1\text{-}2}$$

$$= -\int_{r_1}^{r_2} dW = -\int_{r_1}^{r_2} \underset{\sim}{F} \cdot \underset{\sim}{dr}$$

$$= +\int_{r_1}^{r_2} mgR^2 r^{-2} dr$$

$$= mgR^2 \frac{r^{-2+1}}{-2+1}\Bigg|_{r_1}^{r_2}$$

$$= -mgR^2\left(r_2^{-1} - r_1^{-1}\right)$$

$$\cancel{= mgR^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

$$= V_{g_2} - V_{g_1}$$

Take $(V_{g_2}) = 0$ when $r = \infty$ as the datum

$$-V_{g_1} = mgR^2 \times \frac{1}{r}$$

$$\Rightarrow \boxed{V_g = \frac{-mgR^2}{r}} \quad (3/19)$$

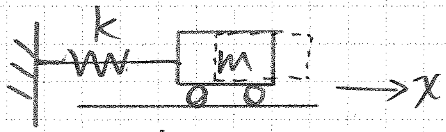
In going from r_1 to r_2, $\Delta V_g = -mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$

Elastic P.E.

due to deformation of an elastic body
e.g. spring

definition: the work done on the spring
to deform it.

V_e = elastic P.E.



$$\int_0^x kx\,dx = \frac{1}{2}kx^2$$

datum $x=0$ neutral position

$$\therefore \boxed{V_e = \frac{1}{2}kx^2}$$

deformation x_1 to x_2

$$\Delta V_e = \frac{1}{2}k(x_2^2 - x_1^2)$$

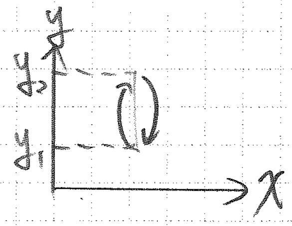
Similarily

$$-U_{1-2} = \Delta V_e \quad x_1 \rightarrow x_2$$

$$-\int_{x_1}^{x_2} -kx\,dx = \int_{x_1}^{x_2} kx\,dx = \frac{1}{2}k(x_2^2 - x_1^2)$$

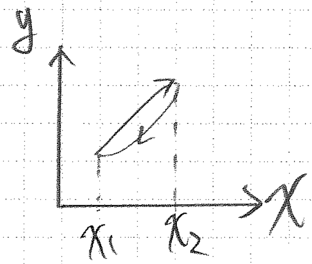
$\because$ gravitational and elastic force are conservative fac

$$\oint \underset{\sim}{F} \cdot \underset{\sim}{dr} = 0$$



$$\oint \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_{y_1}^{y_2} -mg\,dy + \int_{y_2}^{y_1} -mg\,dy$$

$$= 0$$



$$\oint \underset{\sim}{F} \cdot d\underset{\sim}{r}$$

$$= \int_{x_1}^{x_2} kx\,dx + \int_{x_2}^{x_1} -kx\,dx = 0$$

Work done by conservative force
depend on net charge of position not path

Work - Energy Equation

From work-energy Eqn.

$$U_{1-2} = \Delta T$$ → 動能

work done by external force ①

$$\text{Let: } U_{1-2} = U'_{1-2} + U^{g}_{1-2} + U^{e}_{1-2}$$

g gravitational force, e spring force

$$-U^{e}_{1-2} = \Delta V_e \Rightarrow \Delta V_e = -U^{e}_{1-2}$$

$$-U^{g}_{1-2} = \Delta V_g \Rightarrow \Delta V_g = -U^{g}_{1-2}$$

$$\therefore U'_{1-2} - \Delta V_g - \Delta V_e = \Delta T$$

$$\Rightarrow U'_{1-2} = \Delta V_g + \Delta V_e + \Delta T$$

$$= \Delta V + \Delta T$$

$\Delta V = \Delta V_g + \Delta V_e$: Chang in total P.E.

Let $\Delta T = T_2 - T_1$

$\Delta V = V_2 - V_1$

$U'_{1-2} = V_2 - V_1 + T_2 - T_1$

$= \underbrace{(V_2 + T_2)}_{\text{機械能}} - (V_1 + T_1)$

$T_1 + V_1 + U'_{1-2} = T_2 + V_2$ (3/21 a)

$E = T + V$ = mechanical energy of particle

$\Rightarrow U'_{1-2} = \Delta T + \Delta V$

$= \Delta E$

→ work done by non-constraint force

※ Constraint force does No work

※ Special Case

For problems when the only forces are gravitational and elastic force and non-working constraint forces $\Rightarrow U'_{1-2} = 0$

The energy eqn. becomes $\boxed{T_1 + V_1 = T_2 + V_2}$ law of conservation of dynamical (mechnical) Energy

or $E_1 = E_2$

$E = T + V$

§C Impulse and Momentum

衝量 向量 動量

3/8 Introduction

- Intergrate equation of motion with respect to time ⇒ relation between impulse and change of momentum.

- Impact problems

3/9 Linear Impulse and Linear Momentum

$$\Sigma \underset{\sim}{F} = m\dot{\underset{\sim}{v}} = \frac{d}{dt}(m\underset{\sim}{v}) \quad (\text{if } m \text{ is constant})$$

$$= \frac{d}{dt}\underset{\sim}{G} = \dot{\underset{\sim}{G}}$$

$\underset{\sim}{G} \triangleq m\underset{\sim}{v}$: linear momentum

$$\Sigma \underset{\sim}{F} = \Sigma F_x \vec{i} + \Sigma F_y \vec{j} + \Sigma F_z \vec{k}$$

$$\underset{\sim}{G} = G_x \vec{i} + G_y \vec{j} + G_z \vec{k}$$

$$\underset{\sim}{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$$

$$\dot{\underset{\sim}{G}} = \dot{G}_x \vec{i} + \dot{G}_y \vec{j} + \dot{G}_z \vec{k}$$

$$\Sigma F_x = \dot{G}_x = \frac{d}{dt}(mv_x)$$

$$\Sigma F_y = \dot{G}_y = \frac{d}{dt}(mv_y)$$

$$\Sigma F_z = \dot{G}_z = \frac{d}{dt}(mv_z)$$

Linear Impulse-Momentum Principle

$\because \Sigma \underset{\sim}{F} = \frac{\Delta \underset{\sim}{G}}{\Delta t}$

$\Rightarrow \int_{t_1}^{t_2} \Sigma \underset{\sim}{F}\, dt = \int_{t_1}^{t_2} d\underset{\sim}{G}$

$= \underset{\sim}{G}(t_2) - \underset{\sim}{G}(t_1) = \Delta \underset{\sim}{G} \quad (3/27)$

$= m\underset{\sim}{v}(t_2) - m\underset{\sim}{v}(t_1)$

$\Rightarrow \int_{t_1}^{t_2} \Sigma \underset{\sim}{F}\, dt = \Delta \underset{\sim}{G} \quad (3/27)$

$= \underset{\sim}{G}_2 - \underset{\sim}{G}_1$

$$\therefore \boxed{\underset{\sim}{G}_1 + \underbrace{\int_{t_1}^{t_2} \Sigma \underset{\sim}{F}\, dt}_{\text{linear Impulse on mass 'm'}} = \underset{\sim}{G}_2} \quad (3/27a)$$

$\underset{\sim}{G}_1$: linear momentum at time t_1

$\underset{\sim}{G}_2$: linear momentum at time t_2

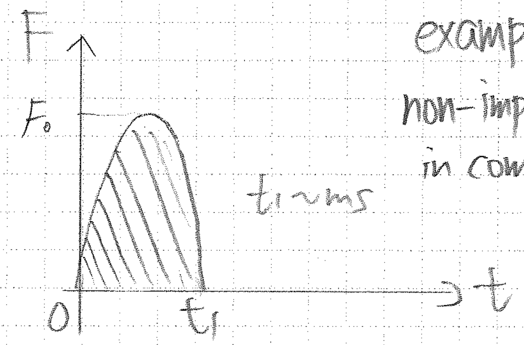
$$m(v_1)_x \boxed{\vec{i}} + m(v_1)_y \boxed{\vec{j}} + m(v_1)_z \boxed{\vec{k}}$$
$$+ \int_{t_1}^{t_2} \Sigma F_x\, dt \boxed{\vec{i}} + \int_{t_1}^{t_2} \Sigma F_y\, dt \boxed{\vec{j}} + \int_{t_1}^{t_2} \Sigma F_z\, dt \boxed{\vec{k}}$$
$$= m(v_2)_x \boxed{\vec{i}} + m(v_2)_y \boxed{\vec{j}} + m(v_2)_z \boxed{\vec{k}}$$

$$\therefore m(v_1)_x + \int_{t_1}^{t_2} \Sigma F_x\, dt = m(v_2)_x$$
$$m(v_1)_y + \int_{t_1}^{t_2} \Sigma F_y\, dt = m(v_2)_y \quad (3/27b)$$
$$m(v_1)_z + \int_{t_1}^{t_2} \Sigma F_z\, dt = m(v_2)_z$$

Impulsive Force: large force last for short duration

example: force of sharp impact

non-impulsive force can be neglected in comparison with impulsive force



Example of Inpulsive force

$$F = F_0 \sin\left(\frac{\pi t}{t_1}\right)$$

$t_1 \sim 10^{-3}\,s$

$F_0 = 10^4\,N$

Show that impulse by F during $[0, t_1]$ is finite

$$I = \int_0^{t_1} F(t)\,dt = \int_0^{t_1} F_0 \sin\frac{\pi t}{t_1}\,dt$$

$$= \frac{2}{\pi} F_0 t_1 = \frac{2}{\pi} \times 10^4 \times 10^{-3} = \frac{20}{\pi}\ N\cdot S \rightarrow \text{finite}$$

compared with gravitational force $m = 500g$

$$I_g = \int_0^{t_1} mg\,dt$$

$$= \int_0^{t_1} 0.5 \times 9.81\,dt = 4.9 \times 10^{-3}\ N\cdot S \rightarrow 0$$

Conservation of Linear Momentum

If resultant force on particle is zero during $[t_1, t_2]$, then the linear momentum $\mathbf{G}$ remains constant

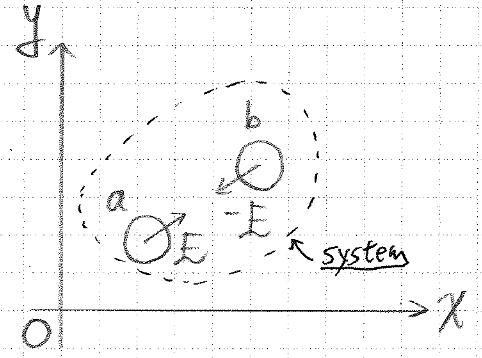
From 3/27

$$\int_{t_1}^{t_2} \Sigma \mathbf{F}\, dt = \int_{t_1}^{t_2} \mathbf{0}\, dt = \mathbf{G}_2 - \mathbf{G}_1 = 0$$

$$\therefore \mathbf{G}_2 = \mathbf{G}_1$$

$$\int_{t_1}^{t_2} \Sigma \mathbf{F}\, dt = \mathbf{0} \Rightarrow \mathbf{G}_2 = \mathbf{G}_1$$

Consider motion of two interacting particles a & b



during $t \in [t_1, t_2]$

$$\Delta \mathbf{G}_a = \int_{t_1}^{t_2} \mathbf{F}_t\, dt \quad \text{for particle a}$$

$$\Delta \mathbf{G}_b = \int_{t_1}^{t_2} -\mathbf{F}_t\, dt$$

$$\Rightarrow \Delta \mathbf{G}_a = -\Delta \mathbf{G}_b$$

$$\Rightarrow \Delta \mathbf{G}_a + \Delta \mathbf{G}_b = \mathbf{0}$$

principle of conservation of linear momentum for two-mass sys.

$$\text{or } \Delta \mathbf{G} = 0 \Rightarrow \boxed{\mathbf{G}(t_2) = \mathbf{G}(t_1)}$$

$$\mathbf{G}_a(t_2) + \mathbf{G}_b(t_2) = \mathbf{G}_a(t_1) + \mathbf{G}_b(t_1)$$

$$\mathbf{G}(t) = \mathbf{G}_a(t) + \mathbf{G}_b(t)$$

total linear momentum of the system

3/10 Angular Impulse and Angular Momentum
角衝量　　　　角動量

$$H_o \triangleq \underset{\sim}{r} \times m\underset{\sim}{v} \quad (3/29)$$

Angular momentum of P about O.

compenent of H_o

$$\underset{\sim}{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\underset{\sim}{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

$$H_o = (x\vec{i} + y\vec{j} + z\vec{k}) \times m(v_x\vec{i} + v_y\vec{j} + v_z\vec{k})$$

$$\left(\begin{array}{ll} \vec{i}\times\vec{j} = \vec{k} & \vec{i}\times\vec{i} = 0 \\ \vec{k}\times\vec{i} = \vec{j} & \vec{j}\times\vec{j} = 0 \\ \vec{j}\times\vec{k} = \vec{i} & \vec{k}\times\vec{k} = 0 \end{array}\right)$$

$$= m(yv_z - zv_y)\vec{i} + m(zv_x - xv_z)\vec{j} + m(xv_y - yv_x)\vec{k}$$

another method:

$$H_o = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

Unit of angular momentum

$$SI: m \cdot kg \cdot \frac{m}{s} = m\,kg \cdot \frac{ms}{s^2}$$
$$= kg \cdot \frac{m}{s^2} \cdot ms$$
$$= N \cdot m \cdot s$$

$$US: [{}^{lb}/_{ft/sec^2}][{}^{ft}/_{sec}][ft] = lb\ sec\ ft$$

Rate of change of Angular momentum

$$\dot{\underset{\sim}{H}}_o = ?$$

$\Sigma \underset{\sim}{F}$ = resultant force acting on P

$$\underset{\sim}{M}_o = \underset{\sim}{r} \times \Sigma \underset{\sim}{F}$$
$$= \underset{\sim}{r} \times m \dot{\underset{\sim}{v}} \text{ (from Newton's law)}$$

$$\because \underset{\sim}{H}_o = \underset{\sim}{r} \times m \underset{\sim}{v}$$
$$\left(\underset{\sim}{H}_o(t) = \underset{\sim}{r}(t) \times m \underset{\sim}{v}(t) \right)$$

$$\therefore \frac{d \underset{\sim}{H}_o(t)}{dt} = \dot{\underset{\sim}{H}}_o = \dot{\underset{\sim}{r}}(t) \times m \underset{\sim}{v}(t) + \underset{\sim}{r}(t) \times m \dot{\underset{\sim}{v}}(t)$$
$$= \underbrace{\underset{\sim}{v}(t) \times m \underset{\sim}{v}(t)}_{0} + \underset{\sim}{r}(t) \times m \dot{\underset{\sim}{v}}(t)$$

$$\therefore \boxed{\Sigma \underset{\sim}{M}_o = \dot{\underset{\sim}{H}}_o \quad (3/31)}$$

time rate of change of angular momentum about fixed point O.

moment about fixed point of all force

$$\Sigma M_{Ox}\vec{i} + \Sigma M_{Oy}\vec{j} + \Sigma M_{Oz}\vec{k}$$
$$= \dot{H}_{Ox}\vec{i} + \dot{H}_{Oy}\vec{j} + \dot{H}_{Oz}\vec{k}$$

$$\text{or} \begin{cases} \Sigma M_{Ox} = \dot{H}_{Ox} \\ \Sigma M_{Oy} = \dot{H}_{Oy} \\ \Sigma M_{Oz} = \dot{H}_{Oz} \end{cases} \quad (3/32)$$

Angular Impulse = Momentum Principle

from (3/31)

$$\Sigma \underset{\sim}{M_O} = \frac{d H_O}{dt}$$

$$\Rightarrow \int_{t_1}^{t_2} \Sigma \underset{\sim}{M_O}\, dt = \int_{t_1}^{t_2} d(H_O) = (\underset{\sim}{H_O})_2 - (\underset{\sim}{H_O})_1$$
$$= \underset{\sim}{H_O}(t_2) - \underset{\sim}{H_O}(t_1)$$
$$= \Delta \underset{\sim}{H_O} \quad (3/33)$$

$$(\underset{\sim}{H_O})_2 = \underset{\sim}{r_2} \times m \underset{\sim}{v_2}, \quad \underset{\sim}{H_O}(t_2) = \underset{\sim}{r}(t_2) \times m \underset{\sim}{v}(t_2)$$

$$(\underset{\sim}{H_O})_1 = \underset{\sim}{r_1} \times m \underset{\sim}{v_1}, \quad \underset{\sim}{H_O}(t_1) = \underset{\sim}{r}(t_1) \times m \underset{\sim}{v}(t_1)$$

Alternative

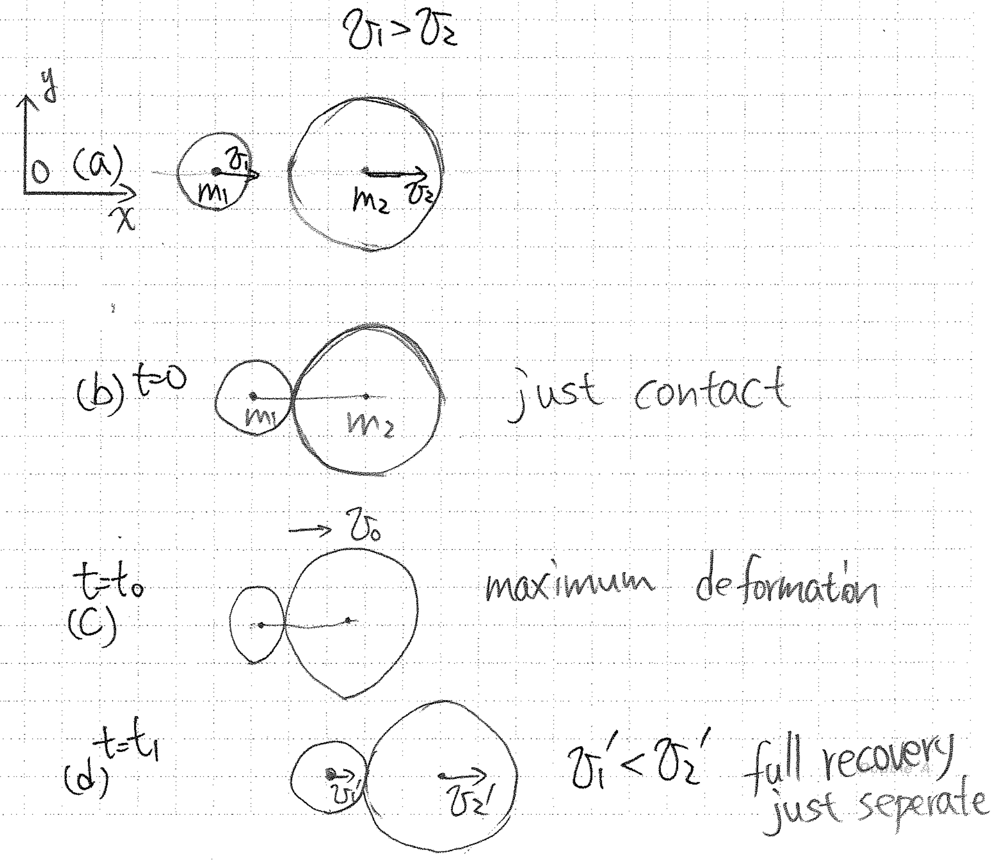
$$\boxed{(\underset{\sim}{H_O})_1 + \int_{t_1}^{t_2} \Sigma \underset{\sim}{M_O}\, dt = (\underset{\sim}{H_O})_2} \quad 3/33a$$

Compared with

$$\boxed{\underset{\sim}{G}_1 + \int_{t_1}^{t_2} \Sigma F \, dt = \underset{\sim}{G}_2}$$

3/12 Impact 撞擊

- Collision between two bodies
- generate relative large contact force act over a very short interval of time.
- Complex event involving material deformation and recovery and generation of heat and sound.



(b) → (c) increasing deformation period

(c) → (d) restoration motion period

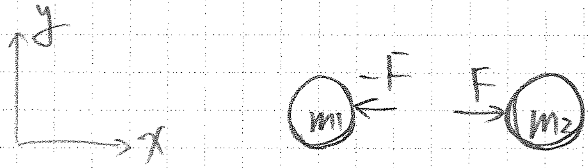
Assumption:

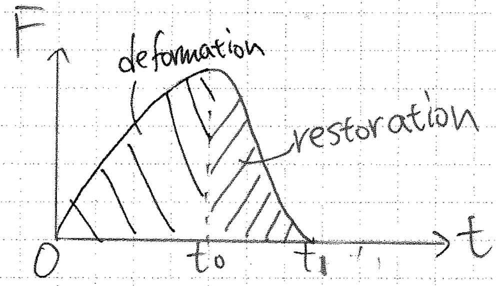
(1) impact is not overly severe

(2) spheres are highly elastic

(3) internal large contact forces are dominant.

Free Body Diagram





Contact force have equal magnitude & opposite direction

From law of conservation of linear momentum

$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1{}' + m_2\vec{v}_2{}'$ (3/35)

$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$ (v_1' and v_2' are unknown)

need one more condition for determining V_1' & V_2'

Coefficient of Restitution, e, 恢復係數

$$e \triangleq \frac{\text{mag. of restoration impulse}}{\text{mag. of deformation impulse}}$$

for particle ※1

$$e = \frac{\int_{t_0}^{t_1} -F\,dt}{\int_0^{t_0} -F\,dt} = \frac{m_1(v_1' - v_0)}{m_1(v_0 - v_1)} = \frac{v_1' - v_0}{v_0 - v_1} \quad *$$

for particle ※2

$$e = \frac{\int_{t_0}^{t_1} F\,dt}{\int_0^{t_0} F\,dt} = \frac{m_2(v_2' - v_0)}{m_2(v_0 - v_2)} = \frac{v_2' - v_0}{v_0 - v_2} \quad **$$

$$e = \frac{a}{b} = \frac{d}{c} = \frac{a-d}{b-c}$$

From $*$, $**$

$$\Rightarrow \boxed{e = \frac{v_2' - v_1'}{v_1 - v_2}} \quad (3/36)$$

$v_2' - v_1'$ → relative vel. of departion

$v_1 - v_2$ → relative vel. of approaching

Remarks: using (3/13 3/36)

i.e. given v_1, v_2, e

we can solve for v_1' & v_2'

Energy loss during impact

- Impact phenomena ~ energy loss $\Delta T < 0$
- Energy loss (heat, dissipetion of elastic stress wave)

$e=1$, elastic impact
(without loss of energy)

$e=0$, inelastic (plastic) impact
塑性
particles cling together after impact,
max energy loss.

$0 \le e \le 1$ for real materials
depending on a pair of contacting bodies (materials)

Oblique central Impact
斜向

using conservation of linear momentum

$$m_1 \underset{\sim}{v_1} + m_2 \underset{\sim}{v_2} = m_1 \underset{\sim}{v_1}' + m_2 \underset{\sim}{v_2}' \quad (*)$$

$$\underset{\sim}{v_1} = v_1 \cos\theta_1 \underset{\sim}{e_t} - v_1 \sin\theta_1 \underset{\sim}{e_n}$$
$$= (v_1)_t \underset{\sim}{e_t} + (v_1)_n \underset{\sim}{e_n}$$

$$\underset{\sim}{v_2} = v_2 \cos\theta_2 \underset{\sim}{e_t} + v_2 \sin\theta_2 \underset{\sim}{e_n}$$
$$= (v_2)_t \underset{\sim}{e_t} + (v_2)_n \underset{\sim}{e_n}$$

$$\underset{\sim}{v_1}' = v_1' \cos\theta_1' \underset{\sim}{e_t} + v_1' \sin\theta_1' \underset{\sim}{e_n}$$
$$= (v_1')_t \underset{\sim}{e_t} + (v_1')_n \underset{\sim}{e_n}$$

$$\underset{\sim}{v_2}' = v_2' \cos\theta_2' \underset{\sim}{e_t} - v_2' \sin\theta_2' \underset{\sim}{e_n}$$
$$= (v_2')_t \underset{\sim}{e_t} + (v_2')_n \underset{\sim}{e_n}$$

sub into Eq(*)

$$\Rightarrow m_1(v_1)_t \underset{\sim}{e}_t + m_1(v_1)_n \underset{\sim}{e}_n + m_2(v_2)_t \underset{\sim}{e}_t + m_2(v_2)_n \underset{\sim}{e}_n$$

$$= m(v_1')_t \underset{\sim}{e}_t + m_1(v_1')_n \underset{\sim}{e}_n + m_2(v_2')_t \underset{\sim}{e}_t + m_2(v_2')_n \underset{\sim}{e}_n$$

$\underset{\sim}{e}_t$: $m_1(v_1)_t + m_2(v_2)_t = m_1(v_1')_t + m_2(v_2')_t$ (1)

$\underset{\sim}{e}_n$: $m(v_1)_n + m_2(v_2)_n = m_1(v_1')_n + m_2(v_2')_n$ (2)

$\because$ impulse contact force $\underset{\sim}{F} = F\underset{\sim}{e}_n + 0\underset{\sim}{e}_t$

has zero in t-component

$\therefore$ For m_1 & m_2 have

$(v_1)_t = (v_1')_t$ (conservation of linear momentum

$(v_2)_t = (v_2')_t$ along t-axis)

If the coefficient restitution is e

$$e = \frac{(v_2')_n - (v_1')_n}{(v_1)_n - (v_2)_n} \quad (3)$$

From (1), (2), (3), we can solve for

$(v_1')_n$ & $(v_2')_n$

$$\theta_1' = \arctan\frac{(v_1')_n}{(v_1')_t}$$

$$\theta_2' = \arctan\frac{(v_2')_n}{(v_2')_t}$$

§ 3/14 Relative Motion 相對運動

Relative-Motion eqn.

(1) XYZ ~ fixed coordinates system (inertial frames)

(2) xyz ~ translates with respect to XYZ

i.e. $xyz \parallel XYZ$

$\boxed{\text{particle}}$ A with mass m

position $\underset{\sim}{r}_A = \underset{\sim}{r}_B + \underset{\sim}{r}_{A/B}$

$\underset{\sim}{r}_{A/B} = \underset{\sim}{r}_A - \underset{\sim}{r}_B$

$\Rightarrow \ddot{\underset{\sim}{r}}_A = \ddot{\underset{\sim}{r}}_B + \ddot{\underset{\sim}{r}}_{A/B}$

or $\underset{\sim}{a}_A = \underset{\sim}{a}_B + \underset{\sim}{a}_{A/B}$

From Newton's law:

$\Sigma \underset{\sim}{F} = m\underset{\sim}{a}_A = m(\underset{\sim}{a}_B + \underset{\sim}{a}_{A/B})$

If $\underset{\sim}{a}_B = \underset{\sim}{0}$ or $\underset{\sim}{v}_B$ = constant vector

then

$$\boxed{\Sigma \underset{\sim}{F} = m\,\underset{\sim}{a}_{A/B}} \quad (*)$$

resultant force on particle A; mass of A; relative acceleration

Work Energy principle

$$\Sigma \underset{\sim}{F} = m\underset{\sim}{a}_A = m\underset{\sim}{a}_{A/B} + m\cancel{\underset{\sim}{a}_B}^{\,0}$$

$$\underset{\sim}{a}_{A/B} \cdot d\underset{\sim}{r}_{A/B} = \frac{d\underset{\sim}{v}_{A/B}}{dt} \cdot d\underset{\sim}{r}_{A/B}$$

$$= d\underset{\sim}{v}_{A/B} \cdot \frac{d\underset{\sim}{r}_{A/B}}{dt}$$

$$= \underset{\sim}{v}_{A/B} \cdot d\underset{\sim}{v}_{A/B}$$

$$= d\left(\frac{1}{2} v_{A/B}^2\right)$$

$$\therefore dU_{A/B} = \Sigma \underset{\sim}{F} \cdot d\underset{\sim}{r}_{A/B} = m\underset{\sim}{a}_{A/B} \cdot d\underset{\sim}{r}_{A/B}$$

$$= m v_{A/B} \cdot d v_{A/B}$$

$$\Rightarrow \int_1^2 dU_{A/B} = \int_1^2 d\left(\frac{1}{2} m v_{AB}^2\right) = \frac{1}{2} m (v_{A/B})_2^2 - \frac{1}{2} m (v_{A/B})_1^2$$

$$\Rightarrow U_{A/B_1}{}^2 = \Delta T_{A/B} \quad (3/52$$

Linear Impulse – Momentum principle

$$\Sigma \underset{\sim}{F}\, dt = m\underset{\sim}{a}_A\, dt = m\cancel{\underset{\sim}{a}_B}^{\,0}\, dt + m\underset{\sim}{a}_{A/B}\, dt$$

$$= m\underset{\sim}{a}_{A/B}\, dt$$

$$= m \frac{d\underset{\sim}{v}_{A/B}}{\cancel{dt}} \cancel{dt} = m\, d\underset{\sim}{v}_{A/B}$$

$$\underset{\sim}{G}_{A/B} \triangleq m\underset{\sim}{v}_{A/B}$$

$$\therefore \Sigma \underset{\sim}{F} = m \frac{d v_{AB}}{dt} = m\underset{\sim}{a}_{A/B} = \dot{\underset{\sim}{G}}_{A/B}$$

$$\boxed{\int_{t_1}^{t_2} \Sigma \underset{\sim}{F}\, dt = \int_{t_1}^{t_2} d\, \underset{\sim}{G}_{A/B} = \Delta \underset{\sim}{G}_{AB}}$$

Angular Impulse – Momentum

$$(\underset{\sim}{H}_{\underset{參考點}{B}})_{A/B} = \underset{\sim}{r}_{A/B} \times m \underset{\sim}{v}_{A/B}$$

$$= \underset{\sim}{r}_{A/B} \times \underset{\sim}{G}_{A/B}$$

$$(\dot{\underset{\sim}{H}}_B)_{A/B} = \dot{\underset{\sim}{r}}_{A/B} \times m \underset{\sim}{v}_{A/B} + \underset{\sim}{r}_{A/B} \times m \dot{\underset{\sim}{v}}_{A/B}$$

$$= \cancelto{0}{\underset{\sim}{v}_{A/B} \times m \underset{\sim}{v}_{A/B}} + \underset{\sim}{r}_{A/B} \times m \underset{\sim}{a}_{A/B}$$

$$= \underset{\sim}{r}_{A/B} \times \Sigma \underset{\sim}{F} = \Sigma \underset{\sim}{M}_{\underset{參考點}{B}}$$

$$\Rightarrow \boxed{\Sigma \underset{\sim}{M}_B = (\dot{\underset{\sim}{H}}_B)_{A/B}}$$

Summary:

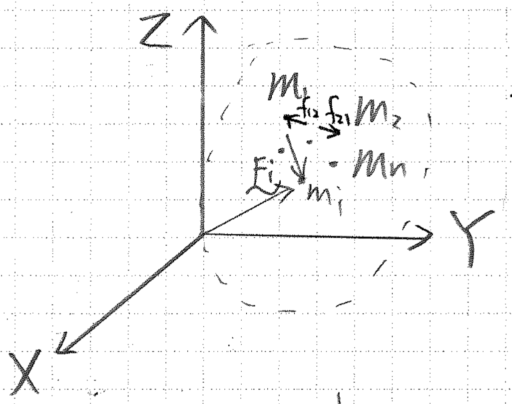
- Kinetics of particles
- Force–mass–acceleration $\left\{\begin{matrix} \text{rectangular} \\ n\text{-}t \\ r\text{-}\theta\text{-}z \end{matrix}\right\}$ coord.
- Work–Energy method
- Impulse–momentum method
- Impact problems
- Relative Motion

Ch4 Kinetics of systems of particles (質點系)

4/1 Introduction

- ch3 deals with kinetics of a particles
- The principles will be extended for a system of particles.

4/2 Generalized Newton's law



$\underset{\sim}{f_{12}} = -\underset{\sim}{f_{21}}$

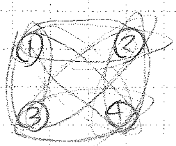
Consider n mass particles that consistutes a system. For example, the solar system in consisted of Sun, venus, earth, Mars...

m_i, $i = 1 \cdot 2 \cdots n$

Let the force extend on particle i by particle j be $\underset{\sim}{f_{ij}}$

we have $\underset{\sim}{f_{ji}} = \underset{\sim}{f_{ij}}$

Let the external force acted on particle i be $\underset{\sim}{F_i}$



From Newton's law for i^{th} particle

$$\underbrace{F_i}_{\text{External}} + \underbrace{\sum_{j=1}^{n} f_{ij}}_{\text{internal force}} = m_i \ddot{\underset{\sim}{r_i}}, \quad i = 1 \cdot 2 \cdots n \quad (1)$$

相消 ∵ $\underset{\sim}{f_{ij}} = -\underset{\sim}{f_{ji}}$

$$\sum_{i=1}^{n} \underset{\sim}{F_i} + \cancelto{0}{\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \underset{\sim}{f_{ij}}} = \sum_{i=1}^{n} m_i \underset{\sim}{\ddot{r}_i} \quad (2)$$

$$\Rightarrow \sum_{i=1}^{n} \underset{\sim}{F_i} = \sum_{i=1}^{n} m_i \underset{\sim}{\ddot{r}_i} \quad (3)$$

Define Center of mass position of the particle system as $\underset{\sim}{\bar{r}}$

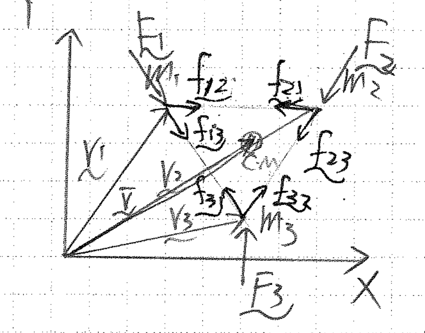
$$\underset{\sim}{\bar{r}}(t) = \frac{\sum_{i=1}^{n} m_i \underset{\sim}{r_i}(t)}{\sum_{i=1}^{n} m_i} \quad (4)$$

$$\underset{\sim}{\dot{\bar{r}}} = \frac{\sum_{i=1}^{n} m_i \underset{\sim}{\dot{r}_i}}{\sum_{i=1}^{n} m_i}$$

$$\boxed{\underset{\sim}{\ddot{\bar{r}}} = \frac{\sum_{i=1}^{n} m_i \underset{\sim}{\ddot{r}_i}}{\sum_{i=1}^{n} m_i}} \quad (5) \Rightarrow \left(\sum_{i=1}^{n} m_i\right) \underset{\sim}{\ddot{\bar{r}}} = \sum_{i=1}^{n} m_i \underset{\sim}{\ddot{r}_i}$$

$$\boxed{\begin{aligned} \Rightarrow \sum_{i=1}^{n} \underset{\sim}{F_i} &= \left(\sum_{i=1}^{n} m_i\right) \underset{\sim}{\ddot{\bar{r}}} \\ &= m \underset{\sim}{\ddot{\bar{r}}} \qquad \underbrace{m = \sum_{i=1}^{n} m_i}_{\text{total mass of sys.}} \\ &= m \underset{\sim}{\bar{a}} \end{aligned}}$$

EXAMPLE 3-mass system



mass 1 : $m_1 \ddot{r}_1 = \underset{\sim}{F_1} + \underset{\sim}{f_{12}} + \underset{\sim}{f_{13}}$

mass 2 : $m_2 \ddot{r}_2 = \underset{\sim}{F_2} + \underset{\sim}{f_{21}} + \underset{\sim}{f_{23}}$

mass 3 : $m_3 \ddot{r}_3 = \underset{\sim}{F_3} + \underset{\sim}{f_{31}} + \underset{\sim}{f_{32}}$

+)

∵ $f_{ij} = -f_{ji}$ 消

$(m_1 \ddot{r}_1 + m_2 \ddot{r}_2 + m_3 \ddot{r}_3) = \underset{\sim}{F_1} + \underset{\sim}{F_2} + \underset{\sim}{F_3} + \underline{0}$

$\Rightarrow \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} m_i \ddot{r}_i \quad (3)$

$$\underset{\sim}{f_{ij}} = f_{ij}(\underset{\sim}{r_i} - \underset{\sim}{r_j})$$

4/3 Work-Energy principle

Consider a system of n particles with mass: $m_1, m_2 \cdots m_n$

$\underset{\sim}{F_i}$: external force on mass m_i

$\underset{\sim}{f_i}$: $\sum_{\substack{j=1 \\ j \neq i}}^{n} \underset{\sim}{f_{ij}}$: resultant of internal force from other particles

Newton's law for m_i

$$\underset{\sim}{F_i} + \underset{\sim}{f_i} = m_i \ddot{\underset{\sim}{r_i}} \quad (1)$$

From work-energy principle

$$\int_1^2 (\underset{\sim}{F_i} + \underset{\sim}{f_i}) \cdot d\underset{\sim}{r} = \int_1^2 m_i \ddot{\underset{\sim}{r_i}} \cdot d\underset{\sim}{r_i}$$

$$\Rightarrow \int_1^2 (\underset{\sim}{F_i} + \underset{\sim}{f_i}) \cdot d\underset{\sim}{r} = \frac{1}{2} m_i (v_i)_2^2 - \frac{1}{2} m_i (v_i)_1^2 \quad (2)$$

$(v_i)_2$: velocity magnitude of m_i at particle 2

$(v_i)_1$: velocity magnitude of m_i at particle 1

Summation of Eq(2) for all particles:

$$\sum_{i=1}^{n}\int_1^2 (\underset{\sim}{F_i}+\underset{\sim}{f_i})\cdot d\underset{\sim}{r_i} = \sum_{i=1}^{n}\frac{1}{2}m_i(v_i)_2^2 - \sum_{i=1}^{n}\frac{1}{2}m_i(v_i)_1^2$$

$$= T_2 - T_1 \quad (4/2)$$

or $U_{1-2} = T_2 - T_1$

T_2 ← total kinetic energy at postition 2

U_{1-2} ← work done by all forces external and internal

Note: $U_{1-2} = \sum_{i=1}^{n}(U_{1-2})_i$ include work done by internal force

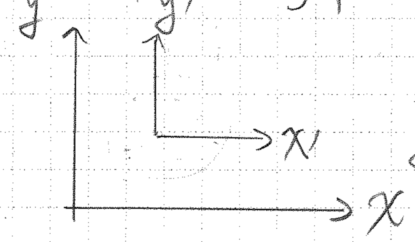
special case:

If the particle system are belong to a translating rigid body

平移

$d\underset{\sim}{r_1} = d\underset{\sim}{r_2} = \cdots = d\underset{\sim}{r_n}$

$$\Rightarrow \int_1^2 \sum_{i=1}^{n}\left(\underset{\sim}{F_i}+\sum_{\substack{j=1\\ j\neq i}}^{n}\underset{\sim}{f_{ij}}\right)\cdot d\underset{\sim}{r_i} = \int_1^2 \sum_{i=1}^{n}\underset{\sim}{F_i}\cdot d\underset{\sim}{r_i} + \cancel{\int_1^2 \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\underset{\sim}{f_{ij}}\cdot d\underset{\sim}{r_i}}$$



$x' /\!/ x$

$y' /\!/ y$

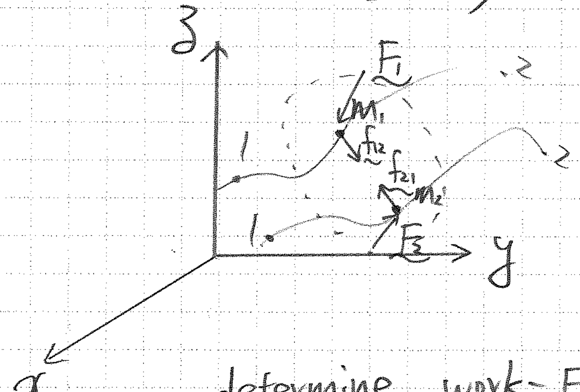
$\because$ dr_i 同

$f_{ij} = -f_{ji}$

$$\therefore U_{1-2} = \int_1^2 \sum_{i=1}^{n} F_i \cdot dr_i$$

The total work is contribution by external forces only!

EXAMPLE : System of two particles



determine work-Energy eqn. of the sys.

※1: $\underset{\sim}{F_1} + \underset{\sim}{f_{12}} = m_1\ddot{\underset{\sim}{r_1}} \Rightarrow \int_1^2 (\underset{\sim}{F_1}+\underset{\sim}{f_{12}})\cdot d\underset{\sim}{r_1} = \frac{1}{2}m_1(v_1)_2^2 - \frac{1}{2}m_1(v_1)_1^2$

※2: $\underset{\sim}{F_2} + \underset{\sim}{f_{21}} = m_2\ddot{\underset{\sim}{r_2}} \Rightarrow \int_1^2 (\underset{\sim}{F_2}+\underset{\sim}{f_{21}})\cdot dr_2 = \frac{1}{2}m_2(v_2)_2^2 - \frac{1}{2}m_2(v_2)_1^2$

※1+※2

$$\Rightarrow \int_1^2 (\underset{\sim}{F_1}\cdot d\underset{\sim}{r_1} + F_2\cdot d\underset{\sim}{r_2}) + \int_1^2 (\underset{\sim}{f_{12}}\cdot d\underset{\sim}{r_1} + \underset{\sim}{f_{21}}\cdot d\underset{\sim}{r_2})$$

$$= \frac{1}{2}m_1(v_1)_2^2 + \frac{1}{2}m_2(v_2)_2^2 - \frac{1}{2}m_1(v_1)_1^2 - \frac{1}{2}m_2(v_2)_1^2$$

$$= T_2 - T_1$$

When $d\underset{\sim}{r_1} = d\underset{\sim}{r_2}$

$$W_{1-2} = \int_1^2 (\underset{\sim}{F_1}\cdot d\underset{\sim}{r_1} + F_2\cdot d\underset{\sim}{r_2}) + \int_1^2 \cancel{(\underset{\sim}{f_{12}}+\underset{\sim}{f_{21}})}^{0}\cdot d\underset{\sim}{r_1}$$

$$= \int_1^2 (\underset{\sim}{F_1}\cdot d\underset{\sim}{r_1} + F_2\cdot d\underset{\sim}{r_2})$$

For a particle system which include elastic members and acted by gravitational force.

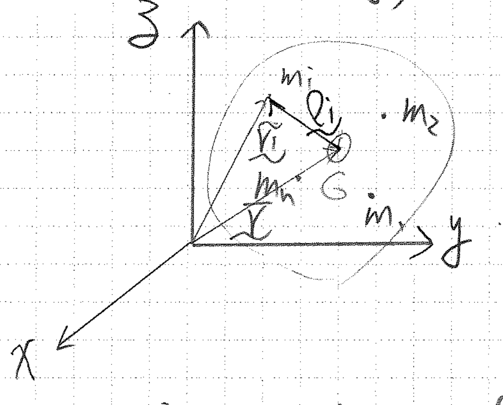
$$U_{1-2} = U'_{1-2} - (V_2 - V_1) = (T_2 - T_1)$$

$$\left(V_2 = Ve_2 + Vg_2 \ , \ V_1 = Ve_1 + Vg_1 \right)$$

$$U'_{1-2} = (T_2 - T_1) + (V_2 - V_1) \quad (4/3)$$

$$\Rightarrow \underbrace{T_1 + V_1}_{\substack{\text{mechanical} \\ \text{energy at} \\ \text{position 1}}} + U'_{1-2} = T_2 + V_2$$

Kinetic Energy at particle system



G: center of mass

for i^{th} particle

$$r_i = \bar{r} + \rho_i$$

$$v_i = \dot{\bar{r}} + \dot{\rho}_i$$

Kinetic Energy of system

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i v_i \cdot v_i = \sum_{i=1}^{n} \frac{1}{2} m_i (\dot{\bar{r}} + \dot{\rho}_i) \cdot (\dot{\bar{r}} + \dot{\rho}_i)$$

$$= \sum_{i=1}^{n} \frac{1}{2} m_i (\dot{\bar{r}} \cdot \dot{\bar{r}} + \dot{\rho}_i \cdot \dot{\rho}_i + 2 \dot{\bar{r}} \cdot \dot{\rho}_i)$$

$$= \sum_{i=1}^{n} \frac{1}{2} m_i |\dot{\bar{r}}|^2 + \sum_{i=1}^{n} \frac{1}{2} m_i |\dot{\rho}_i|^2 + \sum_{i=1}^{n} m_i \dot{\bar{r}} \cdot \dot{\rho}_i$$

$$= \sum_{i=1}^{n} \frac{1}{2} m_i \overline{v}^2 + \sum_{i=1}^{n} \frac{1}{2} m_i |\dot{\underset{\sim}{\rho}}_i|^2 + \dot{\underset{\sim}{\overline{r}}} \cdot \underbrace{\sum_{i=1}^{n} m_i \dot{\underset{\sim}{\rho}}_i}_{\to 0}$$

Note: $\sum_{i=1}^{n} m_i \underset{\sim}{\rho_i} = \underset{\sim}{0} \Rightarrow \sum_{i=1}^{n} m_i \dot{\underset{\sim}{\rho}}_i = \underset{\sim}{0}$

$$\Rightarrow T = \frac{1}{2} m \overline{v}^2 + \sum_{i=1}^{n} \frac{1}{2} m_i |\dot{\rho}_i|^2$$

$$m = \sum_{i=1}^{n} m_i$$

proof: $\because \underset{\sim}{r_i} = \underset{\sim}{\overline{r}} + \rho_i$

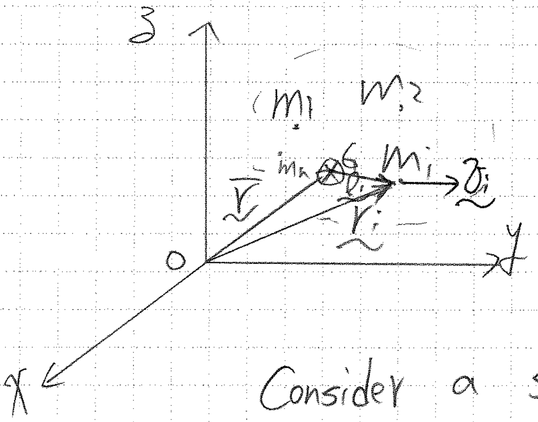
$$\sum_{i=1}^{n} m_i \underset{\sim}{r_i} = \sum_{i=1}^{n} m_i (\underset{\sim}{\overline{r}} + \rho_i)$$

$$= \sum_{i=1}^{n} m_i \underset{\sim}{\overline{r}} + \underbrace{\sum_{i=1}^{n} m_i \rho_i}_{\to 0}$$

$$\therefore \overline{r} = \frac{\sum_{i=0}^{n} m_i \underset{\sim}{r_i}}{\sum_{i=0}^{n} m_i}$$

4/4 Impulse-Momentum for particle system

Linear Momentum



Consider a system of n particles

m_i : $i = 1, 2, \cdots, n$

Linear momentum of the sys.

$$\underset{\sim}{G} = \sum_{i=1}^{n} m_i \underset{\sim}{v_i} \quad (1)$$

$$\underset{\sim}{r_i} = \underset{\sim}{\bar{r}} + \underset{\sim}{\rho_i} \quad (2)$$

$\underset{\sim}{\bar{r}}$: position of center of mass

$$\underset{\sim}{v_i} = \underset{\sim}{\dot{r}_i} = \underset{\sim}{\dot{\bar{r}}} + \underset{\sim}{\dot{\rho}_i} \quad (3)$$

sub into (1)

$$\underset{\sim}{G} = \sum_{i=1}^{n} m_i (\underset{\sim}{\dot{\bar{r}}} + \underset{\sim}{\dot{\rho}_i}) = \sum_{i=1}^{n} m_i \underset{\sim}{\dot{\bar{r}}} + \cancel{\sum_{i=1}^{n} m_i \underset{\sim}{\dot{\rho}_i}}^{\,0}$$

$$\therefore \boxed{\underset{\sim}{G} = m \underset{\sim}{\bar{v}}} \quad (4/5) \quad m = \sum_{i=1}^{n} m_i$$

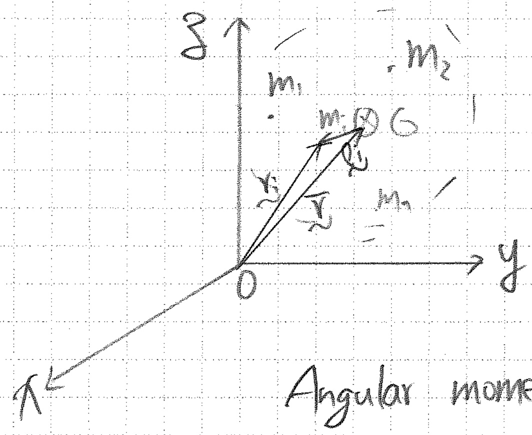
$$\because \sum_{i=1}^{n} m_i \underset{\sim}{\rho_i} = \underset{\sim}{0} \quad \therefore \frac{d}{dt} \sum_{i=1}^{n} (m_i \underset{\sim}{\rho_i}) = \underset{\sim}{0}$$

From § 4/2

$$\sum_{i=1}^{n} \underset{\sim}{F_i} = m\ddot{\underset{\sim}{r_i}} = \frac{d\underset{\sim}{G}}{dt} = \dot{G}$$

$$\boxed{\Rightarrow \underbrace{\int_{t_1}^{t_2} \sum_{i=1}^{n} \underset{\sim}{F_i}\, dt}_{\text{net impulse by external forces}} = \int_{t_1}^{t_2} d\underset{\sim}{G} = \underbrace{\underset{\sim}{G}(t_2) - \underset{\sim}{G}(t_1)}_{\text{change of total linear momentum of sys.}}}$$

Angular momentum of particle system



$$m_i \ddot{\underset{\sim}{r_i}} = \underset{\sim}{F_i} + \sum_{\substack{j=1\\ j\neq i}}^{n} \underset{\sim}{f_{ij}}$$

Angular momentum about point O.

$$\underset{\sim}{H_o} = \sum_{i=1}^{n} \underset{\sim}{r_i} \times m_i \underset{\sim}{v_i}$$

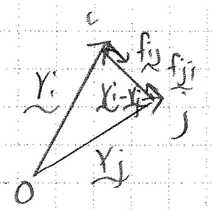
$$\dot{\underset{\sim}{H_o}} = \sum_{i=1}^{n} \left(\dot{\underset{\sim}{r_i}} \times m_i \underset{\sim}{v_i} + \underset{\sim}{r_i} \times m_i \dot{\underset{\sim}{v_i}} \right)$$

$$= \underbrace{\cancel{\sum_{i=1}^{n} \dot{\underset{\sim}{r_i}} \times m_i \underset{\sim}{v_i}}}_{0} + \sum_{i=1}^{n} \underset{\sim}{r_i} \times m_i \ddot{\underset{\sim}{r_i}}$$

$$= \sum_{i=1}^{n} \underset{\sim}{r_i} \times \left(\underset{\sim}{F_i} + \sum_{\substack{j=1\\ j\neq i}}^{n} \underset{\sim}{f_{ij}} \right)$$

$$= \sum_{i=1}^{n} \underset{\sim}{r_i} \times \underset{\sim}{F_i} + \underbrace{\cancel{\sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \underset{\sim}{r_i} \times \underset{\sim}{f_{ij}}}}_{0}$$

pf:

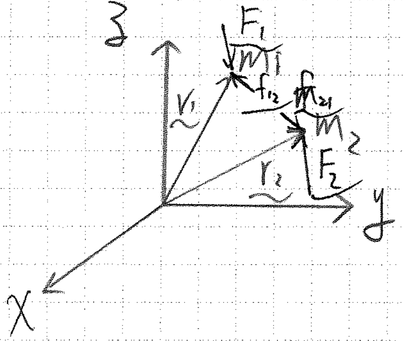


$$r_i \times f_{ij} + r_j \times f_{ji}$$

$$= r_i \times f_{ij} - r_j \times f_{ij} \quad (\because f_{ij} = -f_{ji})$$

$$= (r_i - r_j) \times f_{ij} = 0$$

Example : 2 particle sys.



$$M_o = r_1 \times F_1 + r_1 \times f_{12} + r_2 \times F_2 + r_2 \times f_{21}$$

$$(\because f_{21} = -f_{12})$$

$$= r_1 \times F_1 + r_1 \times F_2 + \overset{0}{\cancel{(r_1 - r_2) \times f_{12}}}$$

$$\therefore \boxed{\sum_{i=1}^{n} M_{oi} = \dot{H}_o}$$

net moment by external force only

time rate of change of total angular momentum

$$\Rightarrow \sum_{i=1}^{n} M_{oi} = \frac{dH_o}{dt}$$

$$\Rightarrow \boxed{\begin{aligned} \int_{t_1}^{t_2} \sum_{i=1}^{n} M_{oi}\,dt &= \int_{t_1}^{t_2} dH_o \\ &= H_o(t_2) - H_o(t_1) \end{aligned}}$$

net impulse by external forces

change of total angular momentum

About center of mass G

$$\underset{\sim}{H_G} \triangleq \sum_{i=1}^{n} \underset{\sim}{\rho_i} \times m_i \underset{\sim}{v_i}$$

$$\dot{\underset{\sim}{H_G}} = \sum_{i=1}^{n}\left(\dot{\rho}_i \times m_i v_i + \rho_i \times m_i \dot{v}_i\right)$$

$$\left(\because\ \underset{\sim}{r_i} = \underset{\sim}{\bar{r}} + \underset{\sim}{\rho_i} \quad \Rightarrow v_i = \bar{v} + \dot{\rho}_i = \dot{\bar{r}} + \dot{\rho}_i \right)$$

$$\therefore\ \dot{\underset{\sim}{H_G}} = \sum_{i=1}^{n}\left(\dot{\rho}_i \times m_i(\dot{\bar{r}} + \dot{\rho}_i) + \rho_i \times m_i \dot{v}_i\right)$$

$$= \sum_{i=1}^{n}\left(\dot{\rho}_i \times m_i \dot{\bar{r}} + \cancelto{0}{\dot{\rho}_i \times m_i \dot{\rho}_i} + \rho_i \times m_i \dot{v}_i\right)$$

$$= -\sum_{i=1}^{n}\left(\dot{\bar{r}} \times m_i \dot{\rho}_i + \rho_i \times \left(F_i + \sum_{\substack{j=1\\ j\neq i}}^{n} f_{ij}\right)\right)$$

$$= -\dot{\bar{r}} \times \cancelto{0}{\frac{d}{dt}\sum_{i=1}^{n} m_i \rho_i} + \sum_{i=1}^{n} \rho_i \times F_i + \cancelto{0}{\sum_{i=1}^{n} \rho_i \times \sum_{\substack{j=1\\ j\neq i}}^{n} f_{ij}}$$

$$= \sum_{i=1}^{n} \underset{\sim}{M_{Gi}}$$

$$\boxed{\underset{\sim}{M_{Gi}} = \rho_i \times F_i}$$

$$\boxed{\sum_{i=1}^{n} \underset{\sim}{M_{Gi}} = \dot{\underset{\sim}{H_G}} = \frac{dH_G}{dt}}$$

$$\boxed{\int_{t_1}^{t_2} \sum_{i=1}^{n} \underset{\sim}{M_{Gi}}\, dt = \int_{t_1}^{t_2} d\underset{\sim}{H_G} = H_G(t_2) - H_G(t_1)}$$

net moment about center of mass for all external force

change of total angular momen

4/5 守恒 Conservation of Energy and Conservation of Momentum

A particle sys. is conservative
(1) no internal friction forces (do negtive work)
(2) no inelastic member (dissipate energy cyclic motion)
(3) only extended by gravitational forces or fixed forces

$U'_{1-2} = 0$

Then

$$\Delta T + \Delta V = 0 \quad (4/4)$$

or

$$\boxed{T_1 + V_1 = T_2 + V_2 \quad (4/14a)}$$

conservation of energy

Conservation of Linear Momentum

during $[t_1, t_2]$

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \underset{\sim}{F_i} \, dt = \underset{\sim}{0} \quad \text{for a particle}$$

$$\because \sum_{i=1}^{n} \underset{\sim}{F_i} = \frac{dG}{dt}$$

system

$$\Rightarrow \int_{t_1}^{t_2} \sum_{i=1}^{n} \underset{\sim}{F_i} \, dt = G(t_2) - G(t_1)$$

Conservation of Angular Momentum

$$\sum_{i=1}^{n} \underset{\sim}{M_{oi}} = \frac{d\underset{\sim}{H_o}}{dt}$$, where O ~ origin of fixed frame

during $t \in [t_1, t_2]$, $\int_{t_1}^{t_2} \sum_{i=1}^{n} \underset{\sim}{M_{oi}}\, dt = \underset{\sim}{0}$

then $\int_{t_1}^{t_2} d\underset{\sim}{H_o} = \underset{\sim}{0} = \underset{\sim}{H_o}(t_2) - \underset{\sim}{H_o}(t_1) = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{H_o}(t_2) - \underset{\sim}{H_o}(t_1) = \underset{\sim}{0}$

Similiarly

$$\sum_{i=1}^{n} \underset{\sim}{M_{Gi}} = \frac{d\underset{\sim}{H_G}}{dt}$$, G: center of mass of particle sys.

during $t \in [t_1, t_2]$

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \underset{\sim}{M_{Gi}}\, dt = \underset{\sim}{0} = \int_{t_1}^{t_2} d\underset{\sim}{H_G}$$

$\Rightarrow \underset{\sim}{H_G}(t_2) = \underset{\sim}{H_G}(t_1)$ (4/17)

Ch5 Plane Kinematics of Rigid Bodies

剛體平面運動學

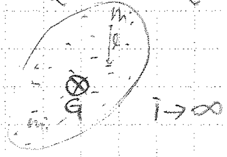
5/1 Introduction

- □ particle Kinematics — displacement, Velocity, acceler
- □ Rigid body Kinematics: linear and angular displacement, Velocity, acceleration.

Rigid Body assumption

A system of particles for which the distance between particles remained unchanged.

l: constant



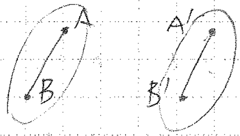
Plane Motion

When all parts of the body move in parallel planes. The plane of motion contains the center of mass of the rigid body.

The plane-motion of rigid-body

Fig. 5/1

(a) rectilinear translation

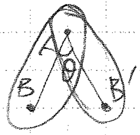


$\overline{AB} \parallel \overline{A'B'}$

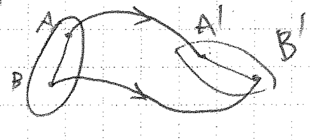
(b) curvilinear translation

$\overline{AB} \parallel \overline{A'B'}$

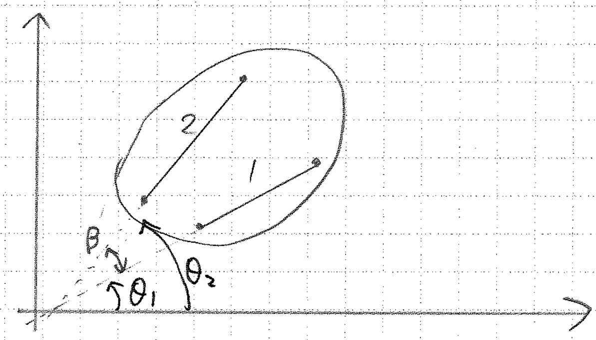
(c) fixed axis rotation



(d) general plane motion



5/2 Rotation (Angular displacement)



$\theta_2 = \theta_1 + \beta$

line 1 specified by θ_1

line 2 specified by θ_2

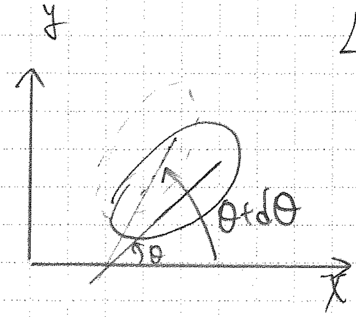
$\theta_2(t) = \theta_1(t) + \beta$

β is a constant

$\Rightarrow \dot{\theta}_2(t) = \dot{\theta}_1(t)$

$\boxed{\ddot{\theta}_2(t) = \ddot{\theta}_1(t)}$ during Δt $\boxed{\Delta\theta_2 = \Delta\theta_1}$

$\therefore$ All lines on a rigid body have same angular displacement, angular velocity and angular acceleration.



Let θ be angular position of any line of the body.

differential angular displacement

$+d\theta \cdot \underset{\sim}{k}$

Angular velocity

$\underset{\sim}{\omega} = \frac{d\theta}{dt} \underset{\sim}{k}$

angular velocity ω

$\omega = \frac{d\theta}{dt}$

angular acceleration α

$$\boxed{\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta}} \quad (5/1)$$

$$\omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt}$$

$$\Rightarrow \omega\, d\omega = \alpha\, d\theta$$
$$= \ddot{\theta}\, d\theta \quad \text{(繞定軸)}$$

Unit:

angular displacement $d\underset{\sim}{\theta} = d\theta\, \underset{\sim}{k}$

$[\text{rad}] \xrightarrow[\times\frac{180}{\pi}]{} [\text{deg}]$

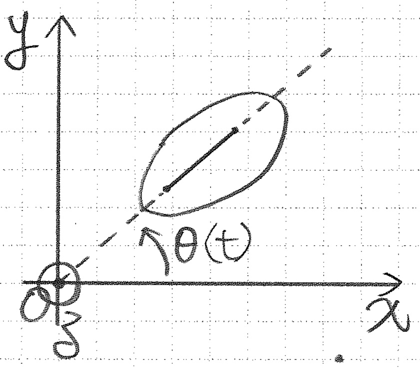
angular velocity $\underset{\sim}{\omega} = \frac{d\underset{\sim}{\theta}}{dt} \left[\text{rad}/s\right] = \frac{d\theta}{dt}\underset{\sim}{k} = \dot{\theta}(t)\, \underset{\sim}{k}$

rpm: rounds per minute

1 rpm = ? deg/s = ? rad/s

angular acceleration $\underset{\sim}{\alpha} = \frac{d\omega}{dt} \left[\text{rad}/s^2\right] = \frac{d\frac{d\theta(t)}{dt}}{dt}\underset{\sim}{k} = \ddot{\theta}(t)\, \underset{\sim}{k}$

Note: the sign convention of α & ω is the same as θ



$$d\underset{\sim}{\theta} = d\theta\, \underset{\sim}{k}$$

$$\underset{\sim}{\omega} = \frac{d\underset{\sim}{\theta}}{dt} = \frac{d\theta(t)}{dt}\underset{\sim}{k} = \dot{\theta}(t)\underset{\sim}{k}$$

$$\underset{\sim}{\alpha} = \frac{d\underset{\sim}{\omega}}{dt} = \frac{d\frac{d\theta(t)}{dt}}{dt} = \ddot{\theta}(t)\,\underset{\sim}{k}$$

$(\underset{\sim}{i} \cdot \underset{\sim}{j} \cdot \underset{\sim}{k})$ $\dot{\underset{\sim}{i}} = 0$

$\dot{\underset{\sim}{j}} = 0$

$\dot{\underset{\sim}{k}} = 0$

Special Case:

rotation with constant angular acceleration

$$\alpha = \frac{d\omega}{dt}$$

$$\int_0^t \alpha\, dt = \int_{\omega_0}^{\omega} d\omega$$

$$\Rightarrow \alpha(t-0) = \omega(t) - \omega_0$$

$$\Rightarrow \boxed{\omega(t) = \omega_0 + \alpha t}$$

$$\int_{\omega_0}^{\omega} \omega\, d\omega = \int_{\theta_0}^{\theta} \alpha\, d\theta \quad \left(\begin{array}{l}\omega = \frac{d\theta}{dt}\\ \alpha = \frac{d\omega}{dt}\end{array}\right)$$

$$\Rightarrow \frac{1}{2}\omega^2 \Big|_{\omega_0}^{\omega} = \alpha(\theta - \theta_0)$$

$$\Rightarrow \frac{1}{2}\omega^2(t) - \frac{1}{2}\omega_0^2 = \alpha(\theta - \theta_0)$$

$$\Rightarrow \boxed{\omega^2(t) = \omega_0^2 + 2\alpha(\theta - \theta_0)}$$

$$\because \frac{d\theta}{dt} = \omega(t) = \omega_0 + \alpha t$$

$$\Rightarrow \int_{\theta_0}^{\theta} d\theta = \int_0^t (\omega_0 + \alpha t)\, dt$$

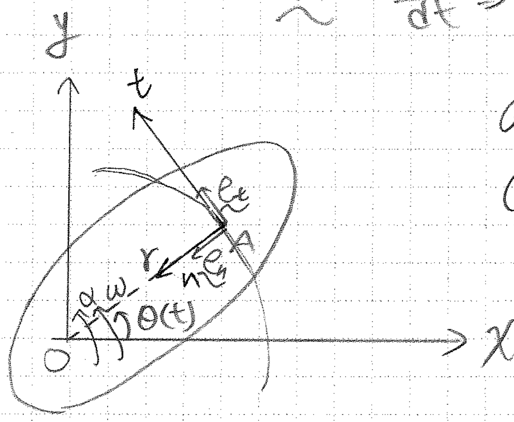
$$\Rightarrow \theta(t) - \theta_0 = \omega_0 (t-0) + \frac{1}{2}\alpha t^2 \Big|_0^t$$

$$\Rightarrow \boxed{\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2}$$

Note: the relationships are similiar to rectlinear motion of partide with constanst acceleration

$$\underset{\sim}{\omega} = \frac{d\underset{\sim}{\theta}}{dt} \Rightarrow \omega \underset{\sim}{k} = \frac{d\theta}{dt}\underset{\sim}{k}$$

$$\underset{\sim}{\alpha} = \frac{d\underset{\sim}{\omega}}{dt} \Rightarrow \alpha \underset{\sim}{k} = \frac{d\omega}{dt}\underset{\sim}{k}$$



Rotation about a fixed axis

Consider a Rigid Body rotates about a fixed axis (z-axis) normal to plane of motion.

Point A on the body moves in a circle of radius r define: $\underset{\sim}{e_b} = \underset{\sim}{e_t} \times \underset{\sim}{e_n}$

$$\underset{\sim}{v_A} = r\omega \underset{\sim}{e_t} = \underset{\sim}{\omega} \times \underset{\sim}{r}$$

$$\underset{\sim}{a_A} = r\omega^2 \underset{\sim}{e_n} + r\alpha \underset{\sim}{e_t} = \underset{\sim}{\alpha} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times$$

$$\underset{\sim}{\omega} = \dot{\theta} \underset{\sim}{e_b} = \omega \underset{\sim}{e_b}$$

$$\underset{\sim}{\alpha} = \alpha \underset{\sim}{e_b} = \dot{\omega} \underset{\sim}{e_b} = \ddot{\theta} \underset{\sim}{e_b}$$

$$\underset{\sim}{r} = -r \underset{\sim}{e_n}$$

$$\begin{aligned}
\underset{\sim}{v_A} = \dot{\underset{\sim}{r}} &= -r \dot{\underset{\sim}{e_n}} \\
&= -r(\underset{\sim}{\omega} \times \underset{\sim}{e_n}) \\
&= \underset{\sim}{\omega} \times (-r \underset{\sim}{e_n}) \\
&= \boxed{\underset{\sim}{\omega} \times \underset{\sim}{r}} = \omega \underset{\sim}{e_b} \times (-r \underset{\sim}{e_n}) \\
&= \omega r \, \underset{\sim}{e_b} \times (-\underset{\sim}{e_n}) \\
&= r\omega \underset{\sim}{e_t}
\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{a_A} = \frac{d\underset{\sim}{v_A}}{dt} &= \frac{d\underset{\sim}{\omega}}{dt} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times \frac{d\underset{\sim}{r}}{dt} \\
&= \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times \dot{\underset{\sim}{r}} \\
&= \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}) \\
&= \underset{\sim}{a_t} + \underset{\sim}{a_n}
\end{aligned}$$

$$\therefore \underset{\sim}{a_t} = \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}$$

$$\underset{\sim}{a_n} = \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})$$

$$\dot{\boldsymbol{\omega}} = \frac{d}{dt}(\omega \boldsymbol{e}_b) = \dot{\omega}\boldsymbol{e}_b + \underbrace{\omega \dot{\boldsymbol{e}}_b}_{=\,\boldsymbol{0}\ (\because \text{平面運動})}$$

$$= \alpha \boldsymbol{e}_b$$

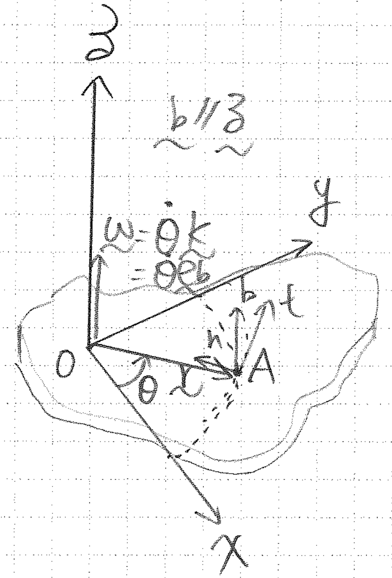
$$\dot{\boldsymbol{\omega}} \times \boldsymbol{r} = \alpha \boldsymbol{e}_b \times (-r\boldsymbol{e}_n)$$

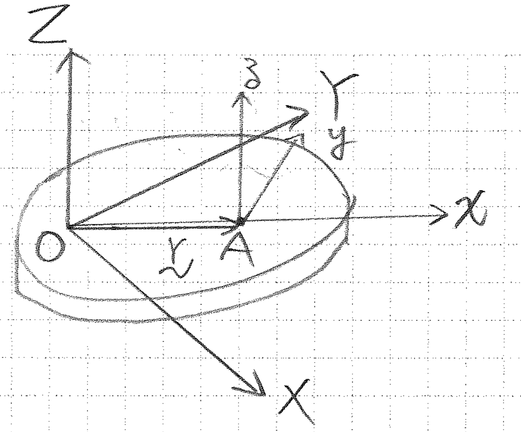
$$= r\alpha \boldsymbol{e}_t$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = \omega \boldsymbol{e}_b \times (\omega \boldsymbol{e}_b \times (-r\boldsymbol{e}_n))$$

$$= \omega \boldsymbol{e}_b \times (r\omega \boldsymbol{e}_t)$$

$$= r\omega^2 \boldsymbol{e}_n$$





$OXYZ$ fixed coordinates
$(\underset{\sim}{I} \cdot \underset{\sim}{J} \cdot \underset{\sim}{K})$ ~ unit vectors
$Axyz$ moving coordi.
fixed to the rigid body
$(\underset{\sim}{i} \cdot \underset{\sim}{j} \cdot \underset{\sim}{k})$ unit vectors

$$\boxed{\dot{\underset{\sim}{i}} = \underset{\sim}{\omega} \times \underset{\sim}{i}, \ \dot{\underset{\sim}{j}} = \underset{\sim}{\omega} \times \underset{\sim}{j}, \ \dot{\underset{\sim}{k}} = \underset{\sim}{\omega} \times \underset{\sim}{k}}$$

position of A: $\underset{\sim}{r} = r\underset{\sim}{i}$

velocity of A: $\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt} = r\dot{\underset{\sim}{i}} = r(\underset{\sim}{\omega} \times \underset{\sim}{i})$

angular velocity of $Axyz$

$$\underset{\sim}{\omega} = \dot{\theta}\underset{\sim}{k} = \omega\underset{\sim}{k}$$

$$\Rightarrow \underset{\sim}{v} = \dot{\underset{\sim}{r}} = r\,\underset{\sim}{\omega} \times \underset{\sim}{i} = r \cdot (\omega\underset{\sim}{k} \times \underset{\sim}{i})$$

$$= r\omega\underset{\sim}{j} = \underset{\sim}{\omega} \times r\underset{\sim}{i} = \underset{\sim}{\omega} \times \underset{\sim}{r}$$

Acceleration of A

$$\underset{\sim}{a} = \dot{\underset{\sim}{v}} = \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times \dot{\underset{\sim}{r}}$$

$$= \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})$$

$$= \dot{\omega}\underset{\sim}{k} \times r\underset{\sim}{i} + \omega\underset{\sim}{k} \times (\omega\underset{\sim}{k} \times r\underset{\sim}{i})$$

$$= \dot{\omega} r\underset{\sim}{j} + \omega\underset{\sim}{k} \times (\omega r\underset{\sim}{j})$$

$$= \alpha r\underset{\sim}{j} - r\omega^2\underset{\sim}{i} \qquad \underset{\sim}{a}_n = -r\omega^2\underset{\sim}{i}$$

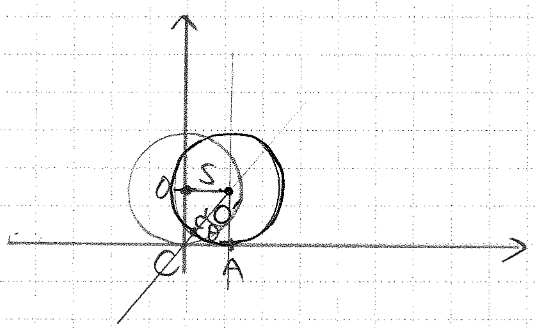
$$= \underbrace{\underset{\sim}{a}_t}_{\text{tangential acceleration}} + \underbrace{\underset{\sim}{a}_n}_{\text{Centripetal acceleration}} \qquad \underset{\sim}{a}_t = \alpha r\underset{\sim}{j}$$

5/3 Absolute Motion

- ☐ using geometric relations which define configuration of R.B. (Rigid Body) (DOF=3)
- ☐ Take time derivatives of geometric relations to obtain acceleration.

Example:

A wheel of radius r rolls on a flat surface without slipping.



from no slipping condition

$\overline{CA} = s$

$\overset{\frown}{C'A} = s$

(a) Given ω, α find $\dot{s}$, $\ddot{s}$

(b) determine acceleration of a point on rim of the wheel as the point come into contact with surface.

displacement relationship

$$s = r\theta$$

$$\Rightarrow v_O = \frac{ds}{dt} = r\dot{\theta} = r\omega$$

$$a_O = \frac{d^2 s}{dt^2} = r\dot{\omega} = r\alpha$$

position of $C'(x, y)$

$$x = s - r\sin\theta = r\theta - r\sin\theta$$
$$y = r - r\cos\theta$$

$$\underset{\sim}{r}_{c'} = r(\theta - \sin\theta)\underset{\sim}{i} + r(1-\cos\theta)\underset{\sim}{j}$$

$$\underset{\sim}{v}_{c'} = \dot{\underset{\sim}{r}}_{c'} = r(\dot{\theta} - \dot{\theta}\cos\theta)\underset{\sim}{i} + r(\dot{\theta}\sin\theta)\underset{\sim}{j}$$
$$= v_0(1-\cos\theta)\underset{\sim}{i} + v_0\sin\theta\,\underset{\sim}{j}$$

$$\underset{\sim}{a}_{c'} = \ddot{\underset{\sim}{r}}_{c'} = r(\ddot{\theta} - \ddot{\theta}\cos\theta + \dot{\theta}^2\sin\theta)\underset{\sim}{i}$$
$$+ (r\ddot{\theta}\sin\theta + r\dot{\theta}^2\cos\theta)\underset{\sim}{j}$$

at contact point C

$$\theta = 0^\circ \Rightarrow a_c = 0\underset{\sim}{i} + r\omega^2\underset{\sim}{j}$$

check $\theta = \frac{\pi}{2}$
$\theta = \pi$

show this result by using relative motion analysis.

5/4 Relative Velocity

Relative Velocity due to rotation

Idea: Choose two points on some rigid body, the motion of one point as seen by observer translating with the other point must be circular.

Fig 5/5
Rigid body moves from position AB to position A'B' during Δt

1°: The body translates to parallel position A''B' with displacement $\Delta \underset{\sim}{r}_B$

2°: The body rotates about B' through angle $\Delta\theta$ give rise to displacement $\Delta \underset{\sim}{r}_{A/B}$

Total displacement of A

$$\Delta \underset{\sim}{r}_A = \Delta \underset{\sim}{r}_{A/B} + \Delta \underset{\sim}{r}_B$$

$$|\Delta r_{A/B}| = r\Delta\theta, \quad \text{when } \begin{matrix}\Delta t \to 0 \\ \Delta\theta \to 0\end{matrix}$$

$$\Rightarrow \frac{\Delta \underset{\sim}{r}_A}{\Delta t} = \frac{\Delta \underset{\sim}{r}_B}{\Delta t} + \frac{\Delta \underset{\sim}{r}_{A/B}}{\Delta t}$$

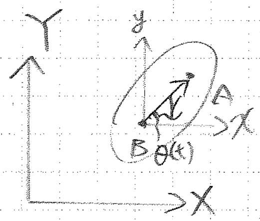
$$\lim_{\Delta t\to 0} \frac{\Delta \underset{\sim}{r}_A}{\Delta t} = \lim_{\Delta t\to 0} \frac{\Delta \underset{\sim}{r}_B}{\Delta t} + \lim_{\Delta t\to 0} \frac{\Delta \underset{\sim}{r}_{A/B}}{\Delta t}$$

$$\Rightarrow \underset{\sim}{v_A} = \underset{\sim}{v_B} + \underset{\sim}{v_{A/B}}$$

$$v_{A/B} = \lim_{\Delta t \to 0} \frac{|\Delta r_{A/B}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{r \Delta \theta}{\Delta t} = r|\dot{\theta}|$$

$$= r|\omega|$$

$$\underset{\sim}{v_{A/B}} = \underset{\sim}{\omega} \times \underset{\sim}{r} \quad (\underset{\sim}{\omega} = \dot{\theta} \underset{\sim}{k})$$



$$\underset{\sim}{r} = x \underset{\sim}{i} + y \underset{\sim}{j}$$

$$\underset{\sim}{\omega} \times \underset{\sim}{r} = \dot{\theta} \underset{\sim}{k} \times (x \underset{\sim}{i} + y \underset{\sim}{j})$$

$$= \dot{\theta} x \underset{\sim}{j} - \dot{\theta} y \underset{\sim}{i}$$

$$= -\dot{\theta} y \underset{\sim}{i} + \dot{\theta} x \underset{\sim}{j}$$

$$|\underset{\sim}{\omega} \times \underset{\sim}{r}| = \sqrt{(-\dot{\theta} y)^2 + (\dot{\theta} x)^2}$$

$$= |\dot{\theta}| \sqrt{x^2 + y^2}$$

$$= |\dot{\theta}| r$$

angular velocity of R.B.

$$\therefore \boxed{\underset{\sim}{v_A} = \underset{\sim}{v_B} + \underset{\sim}{\omega} \times \underset{\sim}{r}}$$

relative position of point A to point B

translation velocity of ref. point

5/5 Instaneous center of zero velocity (瞬心)

A unique reference point which momentarily has zero velocity can be found.

- □ The RB can be considered in pure rotation about an axis passing through the reference point.

- □ The axis is called instaneous axis of zero Velocity.

- □ The point is called instaneous center of zero velocity.

Fig 5/7

(a) $\underset{\sim}{v_A} = \underset{\sim}{\omega} \times \underset{\sim}{r_A}$

$\underset{\sim}{v_B} = \underset{\sim}{\omega} \times \underset{\sim}{r_B}$

$\because$ velocity of $C = \underset{\sim}{0}$

$$\omega = \frac{v_A}{r_A}$$

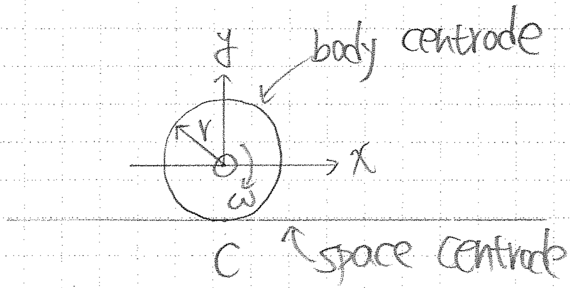
$$v_B = \omega r_B = r_B \frac{v_A}{r_A} = \left(\frac{r_B}{r_A}\right) v_A$$

when $v_A = v_B$, $\overline{AC} = \infty$

(b) $v_A \parallel v_B$ how to determine C?

$$|\omega| = \frac{v_A}{\overline{AC}} = \frac{v_B}{\overline{BC}} = \frac{v_A - v_B}{\overline{AC} - \overline{BC}} = \frac{v_A - v_B}{\overline{AB}}$$

$$\therefore \overline{AC} = \frac{v_A}{v_A - v_B} \overline{AB}$$



$$\underset{\sim}{v}_C = \underset{\sim}{v}_0 + \underset{\sim}{\omega} \times \underset{\sim}{r}$$

$$\underset{\sim}{v}_0 = r\omega \underset{\sim}{i}$$

$$\underset{\sim}{\omega} = -\omega \underset{\sim}{k}$$

$$\therefore \underset{\sim}{v}_C = r\omega \underset{\sim}{i} + (-\omega)\underset{\sim}{k} \times (-r)\underset{\sim}{j}$$

$$= r\omega \underset{\sim}{i} - r\omega \underset{\sim}{i}$$

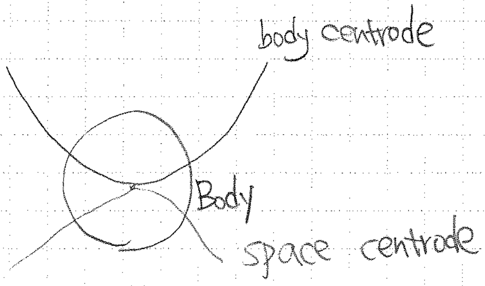
$$= \underset{\sim}{0}$$

$\therefore$ C is the instaneous center of vel.

$$\underset{\sim}{v}_A = \underset{\sim}{\omega} \times 2r\underset{\sim}{j} = -2\omega r \underset{\sim}{i}$$

$$\underset{\sim}{v}_B = ? = \underset{\sim}{\omega} \times (-r\underset{\sim}{i} + r\underset{\sim}{j})$$

$$= -\omega r \underset{\sim}{i} - \omega r \underset{\sim}{j}$$

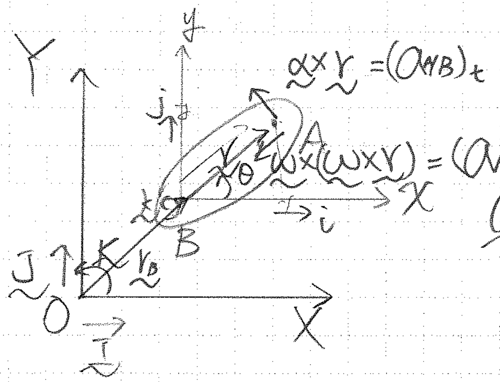


NOTE: The concept of body centrode and space centrode can be used in 3-dimensional rigid body motion

$\Rightarrow$ space cone / body cone

5-6 Relative Acceleration

Consider relative velocity of two particles A & B in terms of non-rotating reference axes



$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{a}_A = \dot{\mathbf{v}}_A = \dot{\mathbf{v}}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}}$$

$$= \dot{\mathbf{v}}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$\mathbf{i} = \mathbf{I}$, $\mathbf{j} = \mathbf{J}$ ($\because$ non-rotating reference)

$\mathbf{a}_B = \ddot{\mathbf{r}}_B \qquad \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$

$\boldsymbol{\omega} = \dot{\theta}\,\mathbf{k}$

$\boldsymbol{\alpha} = \ddot{\theta}\,\mathbf{k}$

proof: $\underset{\sim}{r_A} = \underset{\sim}{r_B} + \underset{\sim}{r} = x_B \underset{\sim}{i} + y_B \underset{\sim}{j} + r\cos\theta \underset{\sim}{i} + r\sin\theta \underset{\sim}{j}$

$$\underset{\sim}{v_A} = \dot{\underset{\sim}{r_A}} = \underbrace{\dot{x}_B \underset{\sim}{i} + \dot{y}_B \underset{\sim}{j}} - r\dot{\theta}\sin\theta \underset{\sim}{i} + r\dot{\theta}\cos\theta \underset{\sim}{j}$$

$$^{*} = \boxed{\underset{\sim}{v_B} + \dot{\theta}\,(r\cos\theta \underset{\sim}{j} - r\sin\theta \underset{\sim}{i})}$$

$$= \boxed{\underset{\sim}{v_B} + \dot{\theta}\,(r\cos\theta(\underset{\sim}{k} \times \underset{\sim}{i}) + r\sin\theta(\underset{\sim}{k} \times \underset{\sim}{j}))}$$

$$= \underset{\sim}{v_B} + \dot{\theta}\underset{\sim}{k} \times (r\cos\theta \underset{\sim}{i} + r\sin\theta \underset{\sim}{j})$$

$$= \underset{\sim}{v_B} + \underset{\sim}{\omega} \times \underset{\sim}{r}$$

$$\underset{\sim}{a_A} = \dot{\underset{\sim}{v_A}} = \dot{\underset{\sim}{v_B}} + \ddot{\theta}(r\cos\theta \underset{\sim}{j} - r\sin\theta \underset{\sim}{i})$$

$$+ \dot{\theta}(-r\dot{\theta}\sin\theta \underset{\sim}{j} - r\dot{\theta}\cos\theta \underset{\sim}{i})$$

$$\boxed{\begin{array}{l} \underset{\sim}{\omega} = \dot{\theta}\underset{\sim}{k} \\ \underset{\sim}{\alpha} = \ddot{\theta}\underset{\sim}{k} \end{array}}$$

$$= \underset{\sim}{a_B} + \alpha\,[r\cos\theta(\underset{\sim}{k} \times \underset{\sim}{i}) + r\sin\theta(\underset{\sim}{k} \times \underset{\sim}{j})]$$

$$+ \dot{\theta}\,[r\cos\theta \underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{i}) + r\sin\theta \underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{j})]$$

$$= \underset{\sim}{a_B} + \alpha \underset{\sim}{k} \times \underset{\sim}{r} + \dot{\theta}\underset{\sim}{k} \times (\dot{\theta}\underset{\sim}{k} \times r\cos\theta \underset{\sim}{i})$$

$$+ \dot{\theta}\underset{\sim}{k} \times (\dot{\theta}\underset{\sim}{k} \times r\sin\theta \underset{\sim}{j})$$

$$= \underset{\sim}{a_B} + \underset{\sim}{\alpha} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})$$

$$(a_{A/B})_n = \frac{(v_{A/B})^2}{r} = \frac{(\omega r)^2}{r} = r\omega^2$$

$$(a_{A/B})_t = r\alpha = \dot{v}_{A/B}$$

In vector rotation

$$\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= \mathbf{a}_B + (\mathbf{a}_{A/B})_t + (\mathbf{a}_{A/B})_n$$

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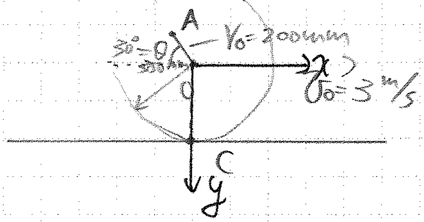
EXAMPLE 5/7



$$\mathbf{v}_0 = 3\mathbf{i} \text{ m/s}$$

$$r\omega = 3, \quad r = 0.3\text{m}$$

$$\omega = \frac{3}{0.3} = 10 \text{ rad/s}$$

$$\boldsymbol{\omega} = 10\mathbf{k} \text{ rad/s}$$

$$\mathbf{v}_C = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}$$

$$= 3\mathbf{i} + 10\mathbf{k} \times 0.3\mathbf{j}$$

$$= 3\mathbf{i} - 3\mathbf{i} = \mathbf{0} \text{ (m/s)}$$

$$\mathbf{v}_A = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{OA}$$

$$= 3\mathbf{i} + 10\mathbf{k} \times (-0.2\cos 30°\mathbf{i} - 0.2\sin 30°\mathbf{j})$$

$$= (3+0.1)\mathbf{i} - 1.732\mathbf{j} \text{ (m/s)}$$

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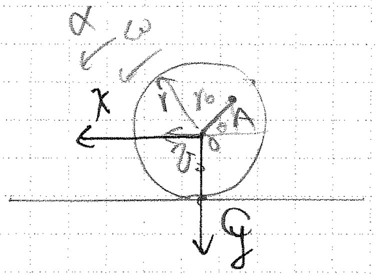
EXAMPLE 5/13



Center O.
Velocity V_0 to the left
acceleration a_0 to the left
Determin acceleration of A & C

$$\underset{\sim}{v_0} = v_0 \underset{\sim}{i} = r\omega \underset{\sim}{i} \quad \therefore \omega = \frac{v_0}{r}$$

$$\underset{\sim}{\omega} = \frac{v_0}{r} \underset{\sim}{k}$$

$$\underset{\sim}{a_0} = a_0 \underset{\sim}{i} = r\alpha \underset{\sim}{i} \quad \therefore \alpha = \frac{a_0}{r}$$

$$\underset{\sim}{\alpha} = \frac{a_0}{r} \underset{\sim}{k}$$

$$\underset{\sim}{a_A} = \underset{\sim}{a_0} + \underset{\sim}{\alpha} \times \underset{\sim}{OA} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{OA})$$

$$\left(\underset{\sim}{OA} = -r_0 \cos\theta \underset{\sim}{i} - r_0 \sin\theta \underset{\sim}{j} \right)$$

$$\underset{\sim}{a_A} = r\alpha \underset{\sim}{i} + \underset{\sim}{\alpha} \times (-r_0\cos\theta \underset{\sim}{i} - r_0 \sin\theta \underset{\sim}{j})$$
$$+ \frac{v_0}{r} \underset{\sim}{k} \times \left(\frac{v_0}{r} \underset{\sim}{k} \times (-r_0\cos\theta \underset{\sim}{i} - r_0\sin\theta \underset{\sim}{j}) \right)$$
$$= \left(r\alpha + \frac{a_0 r_0}{r}\sin\theta + \frac{r_0 v_0^2}{r^2}\cos\theta \right) \underset{\sim}{i}$$
$$+ \left(-\frac{a_0 r_0}{r}\cos\theta + \frac{r_0 v_0^2}{r^2}\sin\theta \right) \underset{\sim}{j}$$

$$\underset{\sim}{a_C} = \underset{\sim}{a_0} + \underset{\sim}{\alpha} \times \underset{\sim}{OC} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{OC})$$

$$\underset{\sim}{\rho}_C = 0\underset{\sim}{i} + r\underset{\sim}{j}$$

$$a_C = a_0\underset{\sim}{i} + \frac{a_0}{r}\underset{\sim}{k} \times r\underset{\sim}{j} + \frac{v_0}{r}\underset{\sim}{k} \times \left(\frac{v_0}{r}\underset{\sim}{k} \times r\underset{\sim}{j}\right)$$

$$= \left(a_0 - \frac{a_0}{r}r\right)\underset{\sim}{i} + \frac{v_0^2}{r}\underset{\sim}{k} \times (-\underset{\sim}{i})$$

$$= \frac{-v_0^2}{r}\underset{\sim}{j}$$

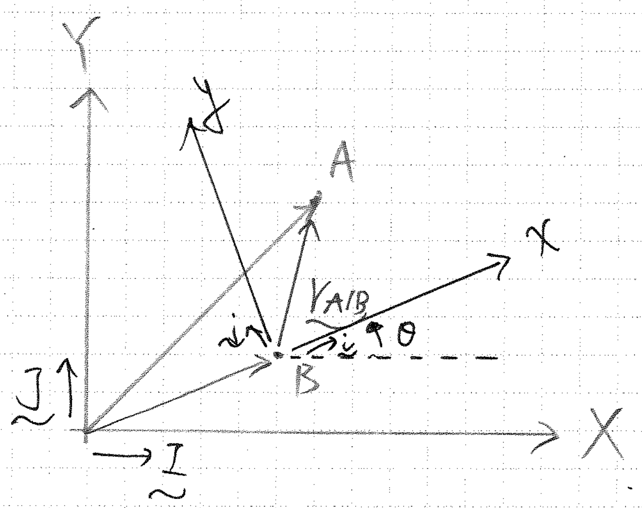
如換座標軸
需小心正負號

5/17 Motion relative to Rotating Axes

using <u>rotating coordinate system</u> we can analyze complicated movement problem in kinematics.

For example movement of fluid particle along curve vane of centrifugal pump

Consider plane motion of two particles A & B in fixed plane X–Y. We observe the motion of A from a moving reference axes xy which has origin attached at B and rotates with angular velocity $\underset{\sim}{\omega} = \dot{\theta}\underset{\sim}{k}$



$$\boldsymbol{\omega} = \dot{\theta}\,\mathbf{k}$$

$$\mathbf{K} = \mathbf{k}$$

$$\mathbf{i} = \cos\theta\,\mathbf{I} + \sin\theta\,\mathbf{J}$$

$$\mathbf{j} = -\sin\theta\,\mathbf{I} + \cos\theta\,\mathbf{J}$$

$$\dot{\mathbf{i}} = \frac{d\mathbf{i}}{dt} = -\sin\theta\,\dot{\theta}\,\mathbf{I} + \cos\theta\,\dot{\theta}\,\mathbf{J}$$

$$= \dot{\theta}\,(\cos\theta\,\mathbf{J} - \sin\theta\,\mathbf{I})$$

$$= \dot{\theta}\,(\cos\theta\,\mathbf{K}\times\mathbf{I} + \sin\theta\,\mathbf{K}\times\mathbf{J})$$

$$= \dot{\theta}\,\mathbf{K}\times(\cos\theta\,\mathbf{I} + \sin\theta\,\mathbf{J})$$

$$= \dot{\theta}\,\mathbf{k}\times(\cos\theta\,\mathbf{I} + \sin\theta\,\mathbf{J})$$

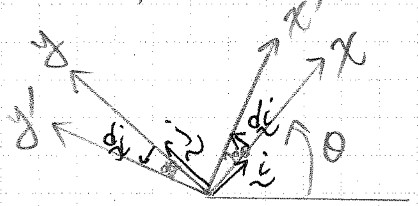
$$= \boldsymbol{\omega}\times\mathbf{i}$$

$$\dot{\underset{\sim}{j}} = \frac{d\underset{\sim}{j}}{dt} = \dot{\theta}(-\sin\theta\, \underset{\sim}{K} \times \underset{\sim}{I} + \cos\theta\, \underset{\sim}{K} \times \underset{\sim}{J})$$

$$= \dot{\theta}\, \underset{\sim}{k} \times (-\sin\theta\, \underset{\sim}{I} + \cos\theta\, \underset{\sim}{J})$$

$$= \dot{\theta}\underset{\sim}{k} \times \underset{\sim}{j} = \dot{\theta}\underset{\sim}{k} \times \underset{\sim}{j} = \underset{\sim}{\omega} \times \underset{\sim}{j}$$

Graphical method



during dt

$d\underset{\sim}{i} = d\theta\, \underset{\sim}{j}$

$d\underset{\sim}{j} = -d\theta\, \underset{\sim}{i}$

$$\frac{d\underset{\sim}{i}}{dt} = \frac{d\theta}{dt}\underset{\sim}{i} = \dot{\theta}\,\underset{\sim}{i} = \dot{\theta}\underset{\sim}{k} \times \underset{\sim}{i} = \underset{\sim}{\omega} \times \underset{\sim}{i}$$

Similiarly

$$\frac{d\underset{\sim}{j}}{dt} = \frac{-d\theta}{dt}\underset{\sim}{j} = -\dot{\theta}\,\underset{\sim}{j} = -\dot{\theta}\underset{\sim}{k} \times \underset{\sim}{j} = \underset{\sim}{\omega} \times \underset{\sim}{j}$$

$$\therefore \boxed{\begin{aligned} \dot{\underset{\sim}{i}} &= \underset{\sim}{\omega} \times \underset{\sim}{i} \\ \dot{\underset{\sim}{j}} &= \underset{\sim}{\omega} \times \underset{\sim}{j} \end{aligned}} \quad \underset{\sim}{\omega} = \dot{\theta}\underset{\sim}{k}$$

∵ A & B are particles moving on XY plane

∵ $\underset{\sim}{r}_{A/B}$ is not a constant vector
& it is expressed in terms of the rotating references axes!

$$\underset{\sim}{r}_A = \underset{\sim}{r}_B + \underset{\sim}{r}_{A/B}$$

$$= \underset{\sim}{r}_B + (x\underset{\sim}{i} + y\underset{\sim}{j})$$

∵ $\dot{\underset{\sim}{i}} \neq 0$ & $\dot{\underset{\sim}{j}} \neq 0$

$$\underset{\sim}{v}_A = \dot{\underset{\sim}{r}}_A = \dot{\underset{\sim}{r}}_B + (\dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j} + x\dot{\underset{\sim}{i}} + y\dot{\underset{\sim}{j}})$$

$$= \underset{\sim}{v}_B + x\underset{\sim}{\omega} \times \underset{\sim}{i} + y\underset{\sim}{\omega} \times \underset{\sim}{j} + \underbrace{\dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}}_{V_{rel}}$$

$$= \underset{\sim}{v}_B + \underset{\sim}{\omega} \times (x\underset{\sim}{i} + y\underset{\sim}{j}) + \underset{\sim}{V}_{rel}$$ relative

$$= \boxed{\underset{\sim}{v}_B + \underset{\sim}{\omega} \times \underset{\sim}{r}_{A/B} + \underset{\sim}{V}_{rel}} \quad (1)$$

Relative acceleration

$$\underset{\sim}{a}_A = \dot{\underset{\sim}{v}}_A = \dot{\underset{\sim}{v}}_B + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}_{A/B} + \underset{\sim}{\omega} \times \dot{\underset{\sim}{r}}_{A/B} + \dot{\underset{\sim}{V}}_{rel} \quad (2)$$

∵ $\underset{\sim}{r}_{A/B} = x\underset{\sim}{i} + y\underset{\sim}{j}$

∴ $\dot{\underset{\sim}{r}}_{A/B} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j} + x\underset{\sim}{\omega} \times \underset{\sim}{i} + y\underset{\sim}{\omega} \times \underset{\sim}{j} = \underset{\sim}{V}_{rel} + \underset{\sim}{\omega} \times \underset{\sim}{r}_{A/B}$

∵ $\underset{\sim}{v}_{rel} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j} \quad (3)$

∴ $\dot{\underset{\sim}{v}}_{rel} = \ddot{x}\underset{\sim}{i} + \ddot{y}\underset{\sim}{j} + \dot{x}\underset{\sim}{\omega} \times \underset{\sim}{i} + \dot{y}\underset{\sim}{\omega} \times \underset{\sim}{j}$

$= \underset{\sim}{a}_{rel} + \underset{\sim}{\omega} \times \underset{\sim}{v}_{rel} \quad (4)$

sub(3)(4) into (2) yeild

$$\Rightarrow \underset{\sim}{a_A} = \underset{\sim}{a_B} + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r_{A/B}} + \underset{\sim}{\omega} \times \left(\underset{\sim}{v_{rel}} + \underset{\sim}{\omega} \times \underset{\sim}{r_{A/B}}\right)$$

$$+ \underset{\sim}{a_{rel}} + \underset{\sim}{\omega} \times \underset{\sim}{v_{rel}}$$

$$= \underset{\sim}{a_B} + \underbrace{\dot{\underset{\sim}{\omega}} \times \underset{\sim}{r_{A/B}}}_{\text{tangential relative accel}} + \underbrace{\underset{\sim}{\omega} \times \left(\underset{\sim}{\omega} \times \underset{\sim}{r_{A/B}}\right)}_{\text{Centripetal relative accel.}}$$

$$+ \underbrace{2\, \underset{\sim}{\omega} \times \underset{\sim}{v_{rel}}}_{\text{Coriolis accel.}} + \underbrace{\underset{\sim}{a_{rel}}}_{\text{relative accel } A \text{ to } B}$$

$$\left(\frac{d\underset{\sim}{v_{rel}}}{dt}\right)_{XY} = \left(\frac{d\underset{\sim}{v_{rel}}}{dt}\right)_{xy} + \underset{\sim}{\omega} \times \underset{\sim}{v_{rel}}$$

向量在固定座標之全微分 相對於動座標之微分 轉動座標角速度 × 該向量

Fig 5/14 (a)

$$\underset{\sim}{v}_A = \underset{\sim}{\omega} \times \underset{\sim}{r} + \underset{\sim}{v}_{rel}$$

$$= \omega \underset{\sim}{k} \times x \underset{\sim}{i} + \dot{x} \underset{\sim}{i}$$

$$= \omega x \underset{\sim}{j} + \dot{x} \underset{\sim}{i}$$

$$\underset{\sim}{a}_A = \underset{\sim}{\alpha} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}) + 2 \underset{\sim}{\omega} \times \underset{\sim}{v}_{rel} + \underset{\sim}{a}_{rel}$$

$$= \underset{\sim}{0} \times \underset{\sim}{r} + \omega \underset{\sim}{k} \times (\omega \underset{\sim}{k} \times x \underset{\sim}{i}) + 2 \omega \underset{\sim}{k} \times \dot{x} \underset{\sim}{i} + \underset{\sim}{0}$$

$$= -x\omega^2 \underset{\sim}{i} + 2\omega \dot{x} \underset{\sim}{j}$$

Ch6 plane kinetics of Rigid Bodies

6/1 Introduction

□ Kinetics of RB ~ relationships between external forces and translational and relational motion of body.

□ Assumptions plane kinetics

(1) plane of motion contains mass center

(2) all force be projected on to plane of motion

Background:

(1) Kinetics of system of particles & Kinetics of R.B.

(2) free-body diagram

(3) mass moment of Inertia: property of resistence to rotational acceleration of R.B.

(A) Force, Mass & Acceleration general eqn. of motion $\begin{cases} \text{Newton's law} \\ \text{Euler's equation : fixed axis rotation} \end{cases}$

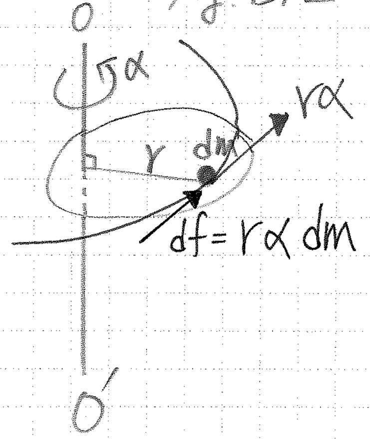
(B) Work and Energy

(C) Impulse and Momentum

§ Mass Moment of Inertia (P.627) about the axi

The rotation of a R.B. depends on the distribution of mass with respect to rotating axis

Fig. B/1



Consider a body of mass m rotates about o-o' axis with angular acceleration α differential element with mass dm has acceleration rα, resultant force on dm $df = r\alpha\, dm$

For RB, α is the same for all differential mass

moment of resultant force

$$d\tau = r\,df = r^2\alpha\, dm$$

Sum of moment of there force

扭矩 $\tau = \int_B d\tau = \int_B r^2\alpha\, dm = \left[\int_B r^2 dm\right]\alpha = I\alpha$

$\boxed{I = \int r^2 dm}$ ~ mass moment of inertia of the RB about o-o' axis.

$$= \frac{\tau}{\alpha}$$

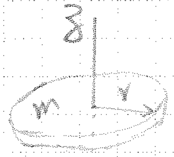
unit: $kg\cdot m^2$ $\quad \dfrac{N\cdot m}{rad/s^2} = \dfrac{kg\cdot m/s^2\cdot m}{rad/s^2} = kg\cdot m^2$

If the density of R.B. is constant
homogeneous 均質

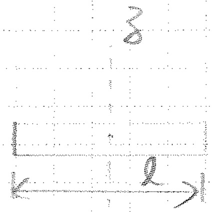
$$dm = \rho dV$$

$$I = \int_B r^2 \rho dV = \rho \underbrace{\int_B r^2 dV}_{\text{pure geometrical property}}$$

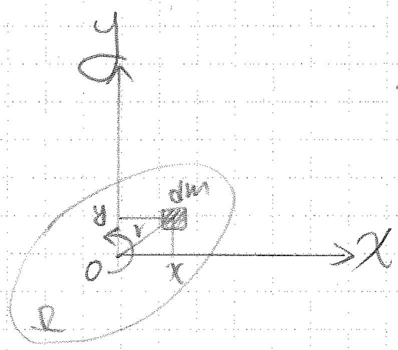
Example :



$$I_{zz} = \frac{1}{2} m r^2$$



$$I_{zz} = \frac{1}{12} m l^2$$



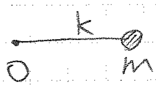
Moment of Inertia about z-axis

$$I_{zz} = \int_\Omega r^2 dm$$

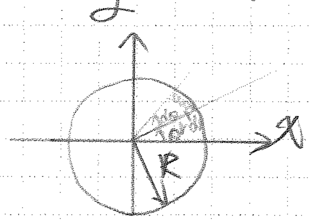
$$= \int_\Omega (x^2 + y^2) dm$$

Def. radius of gyration K (迴轉半徑)

$$I_{zz} = m k^2 \Rightarrow K = \sqrt{\frac{I_{zz}}{m}}$$

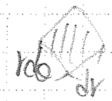


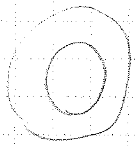
Example: disk of radius R & mass m



Assume disk is homogeneous
with thickness h and density ρ

$$I_{zz} = \int_0^{2\pi}\int_0^R r^2\,dm$$





$$= \int_0^{2\pi}\int_0^R r^2 \rho\, r\,d\theta\,dr\,h$$ 厚度

$$= \rho h \int_0^{2\pi}\int_0^R r^3\,d\theta\,dr$$

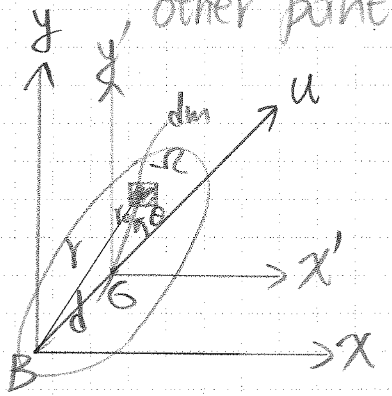
$$= \rho h\, 2\pi \times \frac{r^4}{4}\Big|_0^R = \frac{1}{2}\rho\pi R^4 h = \frac{1}{2}(\rho\pi R^2 h)R^2$$

$$= \frac{1}{2} mR^2 \;,\; k = \sqrt{\frac{I_{zz}}{m}} = \sqrt{\frac{\frac{1}{2}mR^2}{m}} = \frac{R}{\sqrt{2}}$$

other shape: cylinder, sphere, rod, ref
to Sample problems. B/1
B/2
B/3

Note: the moment of inertia depends on the
selection of base point

The moment of inertia about center and other point is related by parallel axis theorem



$x' \parallel x$
$y' \parallel y$

G: mass center

$$I_G = \int_\Omega r_0^2 \, dm$$

$$I_B = \int_\Omega r^2 \, dm$$

$$r^2 = r_0^2 + d^2 - 2 r_0 d \cos(\pi - \theta)$$
$$= r_0^2 + d^2 + 2 r_0 d \cos\theta$$

$$\therefore I_B = \int_\Omega (r_0^2 + d^2 + 2 r_0 d \cos\theta) \, dm$$
$$= \int_\Omega r_0^2 \, dm + \int_\Omega d^2 \, dm + 2d \cancel{\int_\Omega r_0 \cos\theta \, dm}$$

0

u

($\because$ G is the center of mass)

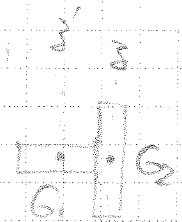
$$\boxed{I_B = I_G + m d^2}$$

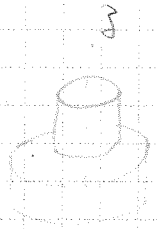
$$I = \bar{I} + m d^2$$

I: about base point
$\bar{I}$: about M.C.
d: distance between G & B

$\sum_{i=1}^{n} m_i \ell_i = 0$

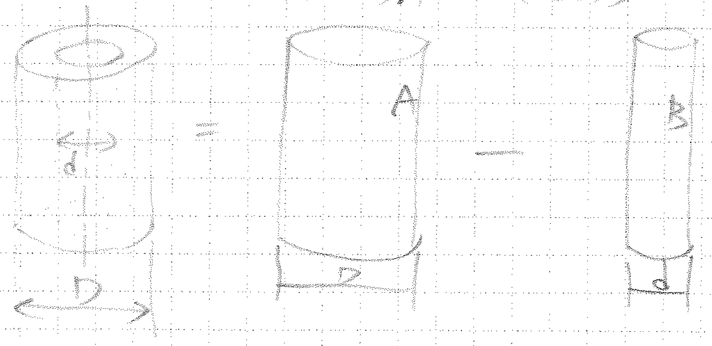
Moment of Inertia of composite Bodies

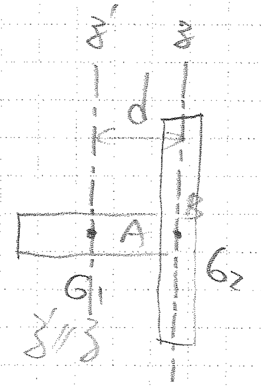




For a body consisted of simple shapes (disk, cylinder, sphere, rod) the moment of inertia of the body can be determined by algebraic sum of component's the moment of inertia and parallel axis theorem.

$$I_{zz} = (I_{zz})_A - (I_{zz})_B$$

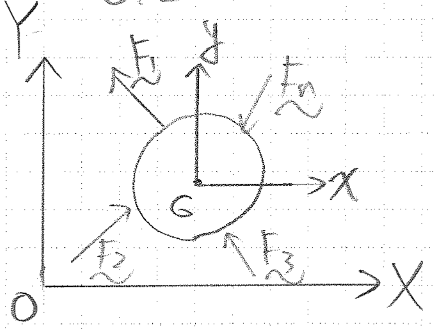




$$I_{zz} = \left((\bar{I}_{G_1})_A + m_A d^2\right) + (\bar{I}_{G_2})_B$$

A. Force, Mass and Acceleration

6/2 General Equations of Motion



In 4/2, we have derived the equations of motion for a system of particles.

The rigid body can be treated as system with infinite # (number) of particles in which the distance of any two particles remains constants.

Newton's law

$$\Sigma \underset{\sim}{F} = m \bar{\underset{\sim}{a}}$$

$\Sigma \underset{\sim}{F}$: resultant of external force

$\bar{\underset{\sim}{a}}$: acceleration of mass center

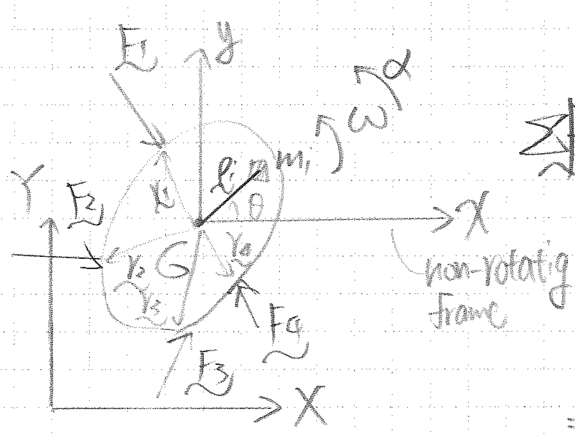
Euler's Eqn.

$$\Sigma \underset{\sim}{M_G} = \dot{\underset{\sim}{H_G}}$$

$\Sigma \underset{\sim}{M_G}$: resultant moment about mass center

$\dot{\underset{\sim}{H_G}}$: time derivative of angular momentum about M.C.

$$\underset{\sim}{H_G} = I \omega \underset{\sim}{k}$$



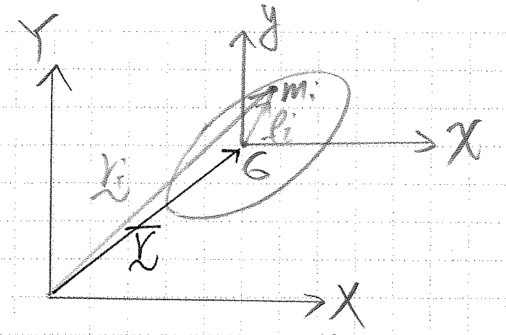
Generalize Newton's law

$$\Sigma \underset{\sim}{F} = \underset{\sim}{F_1} + \underset{\sim}{F_2} + \underset{\sim}{F_3} + \underset{\sim}{F_4} = m \bar{\underset{\sim}{a}}$$

For Euler's Eqn.

$$\underset{\sim}{r_1} \times \underset{\sim}{F_1} + \underset{\sim}{r_2} \times \underset{\sim}{F_2} + \underset{\sim}{r_3} \times \underset{\sim}{F_3} + \underset{\sim}{r_4} \times \underset{\sim}{F_4} = \bar{I} \dot{\omega} \underset{\sim}{k}$$

$\bar{I}$: moment of inertia about G.



$\mathbf{r}_i = \bar{\mathbf{r}} + \boldsymbol{\rho}_i$

$\dot{\mathbf{r}}_i = \dot{\bar{\mathbf{r}}} + \dot{\boldsymbol{\rho}}_i$

$\boldsymbol{\rho}_i = x_i \mathbf{i} + y_i \mathbf{j}$

$\boldsymbol{\omega} = \omega \mathbf{k}$

$\rightarrow \boldsymbol{\rho}_i \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) = ?$

$$\mathbf{H}_G = \sum_{i=1}^{n} \boldsymbol{\rho}_i \times m_i \dot{\mathbf{r}}_i$$

$$= \sum_{i=1}^{n} \boldsymbol{\rho}_i \times m_i (\dot{\bar{\mathbf{r}}} + \dot{\boldsymbol{\rho}}_i)$$

$$= \sum_{i=1}^{n} \boldsymbol{\rho}_i \times m_i \dot{\bar{\mathbf{r}}} + \sum_{i=1}^{n} \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i$$

$$= -\dot{\bar{\mathbf{r}}} \underbrace{\sum_{i=1}^{n} m_i \times \boldsymbol{\rho}_i}_{=0} + \sum_{i=1}^{n} \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i$$

$$= \sum_{i=1}^{n} \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i = \sum_{i=1}^{n} \boldsymbol{\rho}_i \times m_i (\boldsymbol{\omega} \times \boldsymbol{\rho}_i)$$

$$= \sum_{i=1}^{n} m_i \rho_i^2 \omega \mathbf{k} = \bar{I} \omega \mathbf{k}$$

$$\therefore \boxed{\mathbf{H}_G = \bar{I} \omega \mathbf{k}}$$

$$\dot{\mathbf{H}}_G = \bar{I} \alpha \mathbf{k}$$

$m_i \rightarrow dm \rightarrow \overset{\text{density}}{\rho}\, dx\, dy$

$\rho_i \rightarrow \rho(x, y)$

$\sum_i = \int_{xy}$

$$\underset{\sim}{H_G} = \sum_{i=1} \underset{\sim}{\rho_i} \times m_i \underset{\sim}{\dot{r}_i}$$

Euler's Eqn.
resultant force on differential mass m.

$$\underset{\sim}{\dot{H}_G} = \sum_{i=1} \underset{\sim}{\dot{\rho}_i} \times m_i \underset{\sim}{\dot{r}_i} + \sum_{i=1} \underset{\sim}{\rho_i} \times m_i \underset{\sim}{\ddot{r}_i} = \underset{\sim}{F_i}$$

$$= \sum_{i=1} \underset{\sim}{\dot{\rho}_i} \times m_i (\underset{\sim}{\dot{\bar{r}}} + \underset{\sim}{\dot{\rho}_i}) + \sum_{i=1} \underset{\sim}{\rho_i} \times \underset{\sim}{F_i}$$

$$= -\underset{\sim}{\dot{\bar{r}}} \times \cancelto{0}{\sum_{i=1} m_i \underset{\sim}{\dot{\rho}_i}} + \sum_{i=1} \cancelto{0}{\underset{\sim}{\dot{\rho}_i} \times m_i \underset{\sim}{\dot{\rho}_i}} + \sum_{i=1} \underset{\sim}{\rho_i} \times \underset{\sim}{F_i}$$

$$= \sum \underset{\sim}{M_{G_i}}$$

$$\boxed{\therefore \sum \underset{\sim}{M_G} = \underset{\sim}{\dot{H}_G} = \bar{I} \dot{\omega} \underset{\sim}{k} = \bar{I} \alpha \underset{\sim}{k}} \quad (6/1)$$

For the example:

$$\underset{\sim}{r_1} \times \underset{\sim}{F_1} + \underset{\sim}{r_2} \times \underset{\sim}{F_2} + \underset{\sim}{r_3} \times \underset{\sim}{F_3} + \underset{\sim}{r_4} \times \underset{\sim}{F_4} = \bar{I} \dot{\omega} \underset{\sim}{k}$$

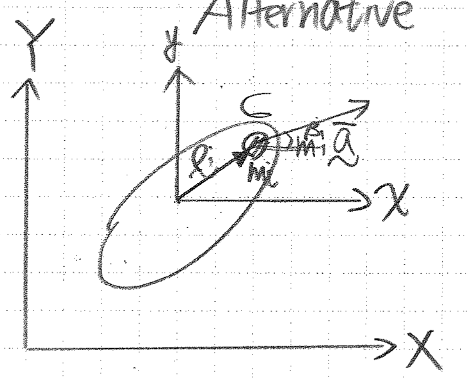
$$\Rightarrow \underset{\sim}{M_G} = \bar{I} \underset{\sim}{\dot{\omega}}$$

General Equations of motion for a R.B.

$$\left.\begin{aligned} \sum \underset{\sim}{F} &= m \underset{\sim}{\bar{a}} \\ \sum M_G &= I \alpha \end{aligned}\right)$$

3 scalar eqs.
3 DOF (Degree of Freedom)

Alternative derivation



acceleration of m_i

$$\bar{a} + \omega\times(\omega\times\rho_i) + \dot{\omega}\times\rho_i$$

$$= \bar{a} + \omega k\times(\omega k\times(x_i i + y_i j)) + \alpha k\times(x_i i + y_i j)$$

$$= \bar{a} - \omega^2(x_i i + y_i j) + \alpha x_i i - \alpha y_i j$$

$$(\bar{a}\cos\beta\, i + \bar{a}\sin\beta\, j)$$

resultant force on m_i

$$= m_i[\bar{a}\cos\beta\, i + \bar{a}\sin\beta\, j - \omega^2 x_i i - \omega^2 y_i j + \alpha x_i i - \alpha y_i j]$$

Moment of resultant force on m

$$M_{Gi} = \rho_i\times[m_i\bar{a}\cos\beta\, i + m_i\bar{a}\sin\beta\, j - m\omega^2 x_i i - m\omega^2 y_i j + m\alpha x_i i - m\alpha y_i j]$$

$$= (x_i i + y_i j)\times[\;\text{"}\quad\text{"}\;]$$

$$= [m_i\rho_i^2\alpha + (m_i\bar{a}\sin\beta)x_i - (m_i\bar{a}\cos\beta)y_i]$$

$$\Sigma M_{Gi} = \Sigma m_i\rho_i^2\alpha k + \bar{a}\sin\beta\,\overset{0}{\Sigma m_i x_i}\, k - \bar{a}\cos\beta\,\overset{0}{\Sigma m_i y_i}\, k$$

($\because$ G is mass center)

$$= \Sigma m_i\rho_i^2\alpha k$$

$$= \bar{I}\alpha k = \bar{I}\alpha$$

$$\boxed{\sum M_G = \sum m_i \rho_i^2 \alpha = \bar{I}\alpha}$$

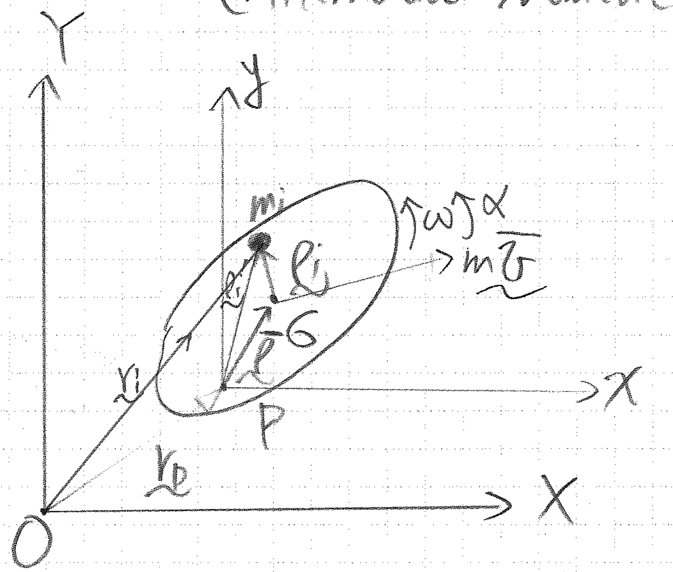
Remarks:

(1) internal forces of R.B. do not contribute to $\sum M_G$ become they occur in pair of equal and opposite direction.

$\ddot{\bar{r}} = \frac{\sum m_i \ddot{r}_i}{\sum m_i}$

(2) Force component $m_i \rho_i \omega^2$ has no moment about G $\therefore$ Right hand side of the eqn. is free of ω.

※ When the base is not Mass Center (Alternative Moment Eqn.)



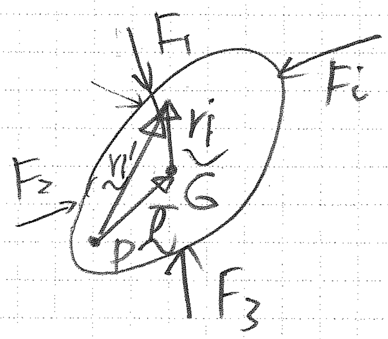
$$\underset{\sim}{H_P} = \sum_i \underset{\sim}{\rho_i'} \times m_i \underset{\sim}{\dot{r}_i}$$

$$= \sum_i (\underset{\sim}{\bar{\rho}} + \underset{\sim}{\rho_i}) \times m_i \underset{\sim}{\dot{r}_i}$$

$$= \sum_i \underset{\sim}{\bar{\rho}} \times m_i \underset{\sim}{\dot{r}_i} + \sum_i \underset{\sim}{\rho_i} \times m_i \underset{\sim}{\dot{r}_i}$$

$$= \underset{\sim}{\bar{\rho}} \times \sum_i m_i \underset{\sim}{\dot{r}_i} + H_G$$

$$\therefore \underset{\sim}{H_P} = \underset{\sim}{\bar{\rho}} \times m \underset{\sim}{\bar{v}} + \underset{\sim}{H_G}$$



$$\underset{\sim}{r}_i' = \bar{\underset{\sim}{\ell}} + \underset{\sim}{r}_i$$

$$\underset{\sim}{r}_1' \times F_1 = (\bar{\underset{\sim}{\ell}} + \underset{\sim}{r}_i) \times F_1$$

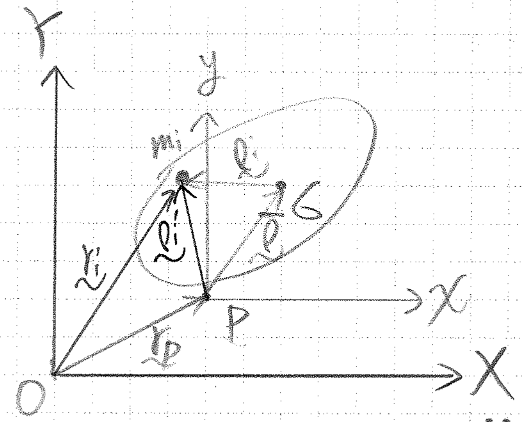
$$= \bar{\ell} \times \underset{\sim}{F}_1 + \underset{\sim}{r}_i \times \underset{\sim}{F}_1$$

$$\therefore \Sigma \underset{\sim}{M}_P = \sum_i \underset{\sim}{r}_i' \times \underset{\sim}{F}_i = \bar{\underset{\sim}{\ell}} \times \Sigma \underset{\sim}{F}_i + \Sigma \underset{\sim}{r}_i \times \underset{\sim}{F}_i$$

$$= \bar{\ell} \times \sum_i m_i \ddot{\underset{\sim}{r}}_i + \underset{\sim}{H}_G = \boxed{\bar{\underset{\sim}{\ell}} \times \Sigma \underset{\sim}{F} + \Sigma \underset{\sim}{M}}$$

$$\underset{\sim}{H}_P = \bar{\ell} \times m \bar{\underset{\sim}{v}} + \underset{\sim}{H}_G$$

□ Another alternative moment eqn. about P



Define :

$$(H_P)_{rel} \triangleq \sum_i \underset{\sim}{\ell}_i' \times m_i \dot{\underset{\sim}{\ell}}_i'$$

$$(\dot{H}_P)_{rel} = \sum_i \cancel{\dot{\underset{\sim}{\ell}}_i' \times m_i \dot{\underset{\sim}{\ell}}_i'}^{\,0} + \sum_i \underset{\sim}{\ell}_i' \times m_i \ddot{\underset{\sim}{\ell}}_i'$$

$$\underset{\sim}{\ell}_i' = \underset{\sim}{r}_i - \underset{\sim}{r}_P \Rightarrow \ddot{\underset{\sim}{\ell}}_i' = \ddot{\underset{\sim}{r}}_i - \ddot{r}_P$$

result force

$$\therefore (\dot{H}_P)_{rel} = \sum_i \underset{\sim}{\ell}_i' \times m_i (\ddot{\underset{\sim}{r}}_i - \ddot{\underset{\sim}{r}}_P) = \sum_i \underset{\sim}{\ell}_i' \times \boxed{m_i \ddot{\underset{\sim}{r}}_i} - \sum_i \underset{\sim}{\ell}_i' \times m_i$$

$$= \Sigma \underset{\sim}{M}_P - m \bar{\ell} \times \ddot{\underset{\sim}{r}}_P$$

$$\therefore (\dot{H}_P)_{rel} = \Sigma \underset{\sim}{M}_P - \bar{\underset{\sim}{\ell}} \times m\underset{\sim}{a}_P$$

$$\therefore \boxed{\Sigma \underset{\sim}{M}_P = (\dot{\underset{\sim}{H}}_P)_{rel} + \bar{\underset{\sim}{\ell}} \times m\underset{\sim}{a}_P} \quad (4/13)$$

※ When P is fixed to RB

$$(\underset{\sim}{H}_P)_{rel} = \sum_i \underset{\sim}{\ell_i}' \times m_i \dot{\underset{\sim}{\ell_i}}' = \sum_i \underset{\sim}{\ell_i}' \times m_i (\underset{\sim}{\omega} \times \underset{\sim}{\ell_i}')$$

$$= \underline{\sum_i m_i (\ell_i')^2} \omega \underset{\sim}{k}$$

$$= I_P \omega \underset{\sim}{k}$$

$$\therefore (\dot{\underset{\sim}{H}}_P)_{rel} = I_P \alpha \underset{\sim}{k}$$

$$\Rightarrow \Sigma \underset{\sim}{M}_P = I_P \alpha \underset{\sim}{k} + \bar{\underset{\sim}{\ell}} \times m\underset{\sim}{a}_P$$

Special Case: (1) when $\bar{\underset{\sim}{\ell}} = \underset{\sim}{0}$, P is mass center

$$\Rightarrow \Sigma \underset{\sim}{M}_G = \bar{I} \alpha \underset{\sim}{k}$$

(2) when P ~ fixed point O in inertal frame

$$\therefore \underset{\sim}{a}_P = \underset{\sim}{0}$$

$$\Rightarrow \Sigma \underset{\sim}{M}_O = I_O \alpha \underset{\sim}{k}$$

(3) when $\bar{\underset{\sim}{\ell}}$ parrell to $\underset{\sim}{a}_P$

$$\Rightarrow \Sigma \underset{\sim}{M}_P = I_P \alpha \underset{\sim}{k}$$

unstrained and constrained Motion

RB in 2D motion has 3 DOF

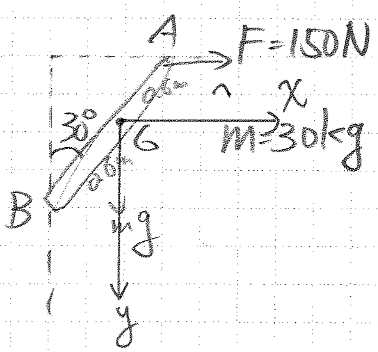
2 translation

1 rotation

fig 6-6 (b)

$\left.\begin{array}{l} y_A = 0 \\ x_A = 0 \end{array}\right\rangle$ 2 constraints DOF = 3 - 2 = 1

Sample 6/7



determine α, reactie force at A&B

$\boxed{\omega(0) = 0}$

$\underset{\sim}{a_A} = (a_A)_x \underset{\sim}{i} + (a_A)_y \underset{\sim}{j}$

$\boxed{(a_A)_y = 0}$ --- (1) constraint eqn.

$\underset{\sim}{a_B} = (a_B)_x \underset{\sim}{i} + (a_B)_y \underset{\sim}{j}$

$\boxed{(a_B)_y = 0}$ --- (2) constraint eqn.

$\alpha = \alpha \underset{\sim}{k}$, at t=0 the link is extend horizonlly by force $\vec{F}$=150

at t=0, $\underset{\sim}{\omega} = \underset{\sim}{0}$, $\underset{\sim}{\alpha} = \alpha \underset{\sim}{k}$

$$\underset{\sim}{a}_A = \bar{\underset{\sim}{a}} + \underset{\sim}{\alpha} \times \underset{\sim}{GA} + \underset{\sim}{\omega} \times (\cancelto{0}{\underset{\sim}{\omega}} \times \underset{\sim}{GA})$$

$$\underset{\sim}{a}_B = \bar{\underset{\sim}{a}} + \underset{\sim}{\alpha} \times \underset{\sim}{GB} + \underset{\sim}{\omega} \times (\cancelto{0}{\underset{\sim}{\omega}} \times \underset{\sim}{GB})$$

$$\underset{\sim}{a}_A = (\bar{a}_x \underset{\sim}{i} + \bar{a}_y \underset{\sim}{j}) + \alpha \underset{\sim}{k} \times (0.6\cos 60^\circ \underset{\sim}{i} - 0.6\sin 60^\circ \underset{\sim}{j})$$

$$= (\bar{a}_x + 0.3\sqrt{3}\alpha)\underset{\sim}{i} + \underbrace{(\bar{a}_y + 0.3\alpha)}_{=0}\underset{\sim}{j}$$

$$\Rightarrow \bar{a}_y = -0.3\alpha$$

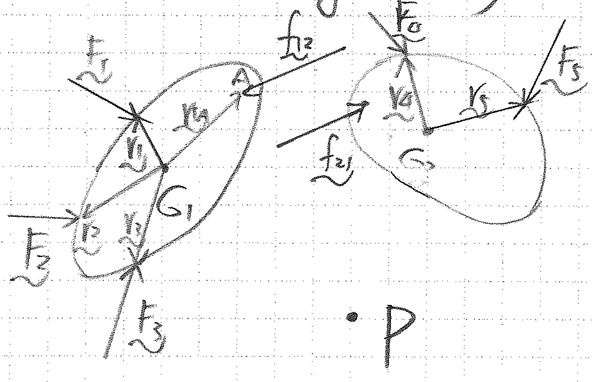
$$\underset{\sim}{a}_B = (\bar{a}_x \underset{\sim}{i} + \bar{a}_y \underset{\sim}{j}) + \alpha \underset{\sim}{k} \times (-0.6\cos 60^\circ \underset{\sim}{i} + 0.6\sin 60^\circ \underset{\sim}{j})$$

$$= \cancelto{0}{(\bar{a}_x - 0.3\sqrt{3}\alpha)}\underset{\sim}{i} + (\bar{a}_y - 0.3\alpha)\underset{\sim}{j}$$

$$\bar{a}_x = 0.3\sqrt{3}\,\alpha$$

sys. of interconnected R.B.

multi-rigid body



Body G1

$$F_1 + F_2 + F_3 + f_{12} = m_1 \bar{a}_1 \cdots (1)$$

$$\bar{I}_1 \alpha_1 = r_1 \times F_1 + r_2 \times F_2 + r_3 \times F_3 + r_{1A} \times f_{12} \quad (3)$$

Body G2

$$f_{21} + F_4 + F_5 = m_2 \bar{a}_2 \cdots (2)$$

$$\bar{I}_2 \alpha_2 = r_{2A} \times f_{21} + r_4 \times F_4 + r_5 \times F_5 \cdots (4)$$

$$(1)+(2) \Rightarrow F_1 + F_2 + F_3 + F_4 + F_5 = m_1 \bar{a}_1 + m_2 \bar{a}_2$$

$$\boxed{\Sigma F = m_1 \bar{a}_1 + m_2 \bar{a}_2}$$

$$(3)+(4) \Rightarrow \bar{I}_1 \alpha_1 + \bar{I}_2 \alpha_2 = r_1 \times F_1 + r_2 \times F_2 + r_3 \times F_3 + r_4 \times F_4 + r_5 \times F_5 + r_{1A} \times f_{12} + r_{2A} \times f_{21}$$

※ $\Sigma M_P = \underset{\sim}{r}_{1/P} \times \underset{\sim}{F}_1 + \underset{\sim}{r}_{2/P} \times \underset{\sim}{F}_2 + \underset{\sim}{r}_{3/P} + \underset{\sim}{F}_3 + \underset{\sim}{r}_{A/P} \times \underset{\sim}{f}_{12}$

$+ \underset{\sim}{r}_{A/P} \times \underset{\sim}{f}_{21} + \underset{\sim}{r}_{4/P} \times \underset{\sim}{F}_4 + \underset{\sim}{r}_{5/P} \times \underset{\sim}{F}_5$

$\therefore \Sigma M_P = \bar{I}_1 \alpha_1 + \bar{I}_2 \alpha_2 + m_1 \bar{a}_1 d_1 + m_2 \bar{a}_2 d_2$ (6/5)

6/3 Translation

When a R.B. undergoes a translation (rectilinear, curvilinear)

$\alpha = 0 \Rightarrow \Sigma M_G = 0$

Equation of Motion fig 6-8

$\Sigma \underset{\sim}{F} = m\underset{\sim}{\bar{a}} \rightarrow \begin{cases} \Sigma F_x = m\bar{a}_x \\ \Sigma F_y = 0 \end{cases}$

(a) $\Sigma \underset{\sim}{M}_G = \underset{\sim}{0}$

or $\Sigma M_P = \bar{I} \cdot 0 + m\bar{a}d = m\bar{a}d$

$\Sigma M_A = 0 \; (\because d = 0)$

(b) Note t-n coordinates

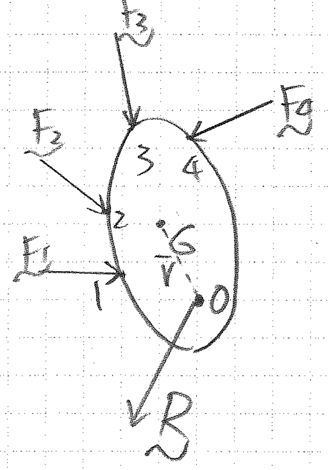
$\Sigma F_n = m\bar{a}_n \qquad \Sigma M_A = \bar{I} \cdot 0 + m\bar{a}_n d_A$

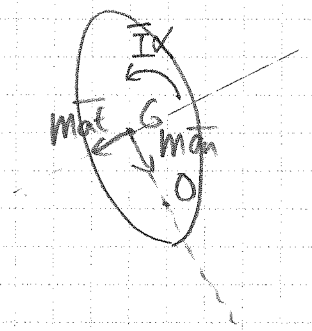
$\Sigma F_t = m\bar{a}_t \qquad \Sigma M_B = \bar{I} \cdot 0 - m\bar{a}_t d_B$

$\Sigma M_G = 0$

6/4 Fixed-Axis Rotation

$$\bar{a}_n = \bar{r}\omega^2$$
$$\bar{a}_t = \bar{r}\alpha$$





$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \rightarrow \begin{cases} \Sigma F_t = m\bar{a}_t \\ \Sigma F_n = m\bar{a}_n \end{cases}$$

$$\Sigma \mathbf{M}_G = \bar{I}\alpha$$

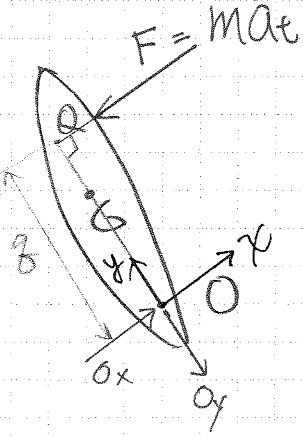
or

$$\Sigma \mathbf{M}_O = \bar{I}\alpha + \bar{r}\, m\bar{a}_t$$
$$= \bar{I}\alpha + \bar{r} m \bar{r}\alpha$$
$$= (\bar{I} + m\bar{r}^2)\alpha$$

$$\Rightarrow \Sigma M_O = I_O \alpha$$

Note: $\mathbf{R}$ does not contribute to ΣM_O

Center of percussion (敲打中心)



Q: Find location for $\vec{F}$ to yield reactive force component along x-axis.

$$\Sigma M_O = \bar{I}\alpha + m\bar{r}^2\alpha = I_O\alpha$$

$$= Fq = m\bar{a}_t q$$

$$= m\bar{r}\alpha q$$

$\because \bar{I} = m\bar{k}^2$, $I_O = \bar{I} + m\bar{r}^2 = mk_O^2$

$\because \bar{I} = m\bar{k}^2$ ($\bar{k}$: radius of gyration)

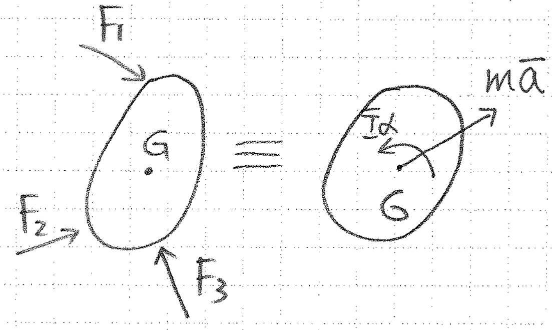
$$I_O = \bar{I} + m\bar{r}^2$$

$$= m(\bar{k}^2 + \bar{r}^2) = mk_O^2$$

$$\bar{k}^2 + \bar{r}^2 = \bar{r}q$$

$$\therefore q = \frac{\bar{k}^2 + \bar{r}^2}{\bar{r}}$$

6/5 General plane Motion



$\Sigma \underset{\sim}{F} = m\underset{\sim}{\bar{a}}$

$\Sigma M_G = \bar{I}\alpha$

6/6 Work Energy Relations

$dU = \underset{\sim}{F} \cdot d\underset{\sim}{r}$

$\therefore U = \int \underset{\sim}{F} \cdot d\underset{\sim}{r}$

$M = F \cdot b$

during dt : translation from AB to $A'B''$

and rotation from $A'B''$ to $A'B'$

work done by translation

$\underset{\sim}{F} \cdot \underset{\sim}{BB''} + (-\underset{\sim}{F}) \cdot AA' = 0$

work done by rotation

$\underbrace{F \cdot b \cdot d\theta + 0}_{B'' \to B'}$ $\therefore \underbrace{dU}_{\text{differential work done by couple}} = F b \, d\theta = M d\theta$

During a finite rotation

$\boxed{U = \int M \, d\theta}$

vector form

$$\vec{M} = -Fb\,\vec{k}$$

$$d\vec{\theta} = -d\theta\,\vec{k}$$

$$dU = \vec{M}\cdot d\vec{\theta} = -Fb\,\vec{k}\cdot(-d\theta)\vec{k}$$

$$= Fb\,d\theta$$

Kinetic Energy of R.B.

(a) translation

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i v^2 = \frac{1}{2}\left(\sum_i m_i\right) v^2$$

$(\because v_i = v)$ $= \frac{1}{2} m v^2$

v : speed of M.C

(b) Fixed-Axis Rotation

$$T = \sum_i \frac{1}{2} m_i v_i^2$$

$$\because v_i = r_i \omega$$

$$\therefore T = \frac{1}{2}\sum_i m_i (r_i \omega)^2$$

$$= \frac{1}{2}\sum_i m_i r_i^2 \omega^2$$

$$= \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

$$= \frac{1}{2} I_o \omega^2$$

(C) plane Motion

velocity of M.C. $\bar{v}$

angular velocity ω

vel. of differential mass m_i:

$$v_i = \bar{v} + \omega \times \rho_i$$

$$v_i^2 = \bar{v}^2 + (\rho_i \omega)^2 - 2\bar{v}\rho_i\omega \cdot \cos(\pi - \theta)$$

$$\therefore T = \sum_i \frac{1}{2} m_i v_i^2$$

$$= \sum_i \frac{1}{2} m_i (\bar{v}^2 + (\rho_i \omega)^2 - 2\bar{v}\rho_i\omega\cos\theta)$$

$$= \frac{1}{2}\sum_i m_i \bar{v}^2 + \frac{1}{2}\sum_i m_i \rho_i \omega^2 + \sum_i \bar{v}\omega m_i \rho_i c$$

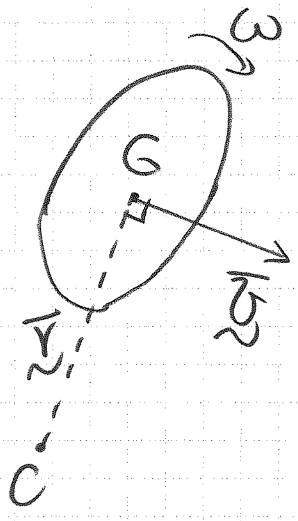
$$= \frac{1}{2} m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 + \omega\bar{v}\cancel{\sum m_i y_i}^{\,0}$$

($\because$ G is mass center)

$$= \frac{1}{2} m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 \quad (6/9)$$

Remarks: translational energy and rotational energy are decoupled.

If we express T in terms of rotation about intaneous center of zero velocity C



$$\bar{v} = \omega \times \bar{r}$$

$$I_C = \bar{I} + m\bar{r}^2$$

$$T = \frac{1}{2} m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$

$$= \frac{1}{2} m(\bar{r}\omega^2) + \frac{1}{2}\bar{I}\omega^2$$

$$= \frac{1}{2}(m\bar{r}^2 + \bar{I})\omega^2$$

$$\therefore T = \frac{1}{2} I_C \omega^2$$

potential Energy & work-Energy Eqn.

Recalled: gravitational energy V_g

elastic energy V_e

Work - Energy relation of R.B

force kinetic energy

$$U_{1-2} = T_2 - T_1$$

or $T_1 + U_{1-2} = T_2$ (4/2)

For R.B., T_1, T_2 include rotational energy and translational energy

$$T_1 = \frac{1}{2} m (\bar{v})_1^2 + \frac{1}{2} \bar{I} (\omega)_1^2$$

$$T_2 = \frac{1}{2} m (\bar{v})_2^2 + \frac{1}{2} \bar{I} (\omega)_2^2$$

Express the effects of weight & springs by potential energy

$$U_{1-2} = U'_{1-2} + (-\Delta V_g) + (-\Delta V_e)$$

$$\Rightarrow T_1 + U'_{1-2} + (-\Delta V_g) + (-\Delta V_e) = T_2$$

$$\Rightarrow U'_{1-2} = (T_2 - T_1) + \Delta V_g + \Delta V_e$$

$$= T_2 - T_1 + V_2 - V_1$$

$$\therefore T_1 + V_1 + U'_{1-2} = T_2 + V_2 \quad (4/3\,a)$$

$\hookrightarrow$ work done by all force other than weight and springs.

work done by conservative forces

Note: (4/3 a) can be applied to interconnected R.B. system

V: include stored elastic energy in connections, and gravitational energy of all members.

T: include kinetic energy of all moving parts.

<u>Power</u>

for a force acting on a R.B.

$$P = \frac{dU}{dt} = \frac{\underset{\sim}{F} \cdot d\underset{\sim}{r}}{dt} = \underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt}$$

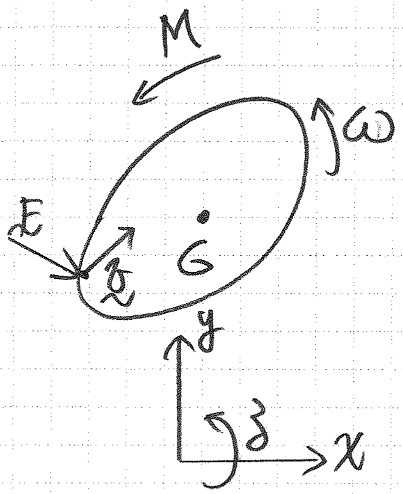
$$= \underset{\sim}{F} \cdot \underset{\sim}{v}$$

For a couple M acting on a R.B

$$P = \frac{dU}{dt} = \frac{M d\theta}{dt} = M\omega$$

vector form

$$P = \frac{dU}{dt} = \frac{\underset{\sim}{M} \cdot d\underset{\sim}{\theta}}{dt} = \underset{\sim}{M} \cdot \frac{d\underset{\sim}{\theta}}{dt} = \underset{\sim}{M} \cdot \underset{\sim}{\omega}$$



Total power

$$\boxed{P = \underset{\sim}{F} \cdot \underset{\sim}{v} + \underset{\sim}{M} \cdot \underset{\sim}{\omega}}$$

Work–Energy relation for an infinitesimal displacement

$$dU' = dT + dV$$

$U' \sim$ work of active force & couple

exclude gravitiational & spring forces.

$$P = \frac{dU'}{dt} = \frac{dT}{dt} + \frac{dV}{dt}$$

↳ power by active force & couple

$$T = \frac{1}{2} m \underset{\sim}{\bar{v}} \cdot \underset{\sim}{\bar{v}} + \frac{1}{2} \bar{I} \omega^2$$

$$\frac{dT}{dt} = \frac{d}{dt}\left(\frac{1}{2} m \underset{\sim}{\bar{v}} \cdot \underset{\sim}{\bar{v}} + \frac{1}{2} \bar{I} \omega^2\right)$$

$$= m \underset{\sim}{\bar{a}} \cdot \underset{\sim}{\bar{v}} + \bar{I} \omega \dot{\omega}$$

Resultant moment about M.C

$$= m \underset{\sim}{\bar{a}} \cdot \underset{\sim}{\bar{v}} + \bar{I} \alpha \omega = \underset{\sim}{R} \cdot \underset{\sim}{v} + \bar{M} \omega$$

Resultant of all force

C: Impulse and Momentum

Linear Momentum

$$\dot{\underset{\sim}{r}}_i = \overline{\underset{\sim}{v}} + \underset{\sim}{\omega} \times \underset{\sim}{\rho}_i$$

$$\underset{\sim}{G} = \sum_i m_i \dot{\underset{\sim}{r}}_i$$

$$= \sum_i m_i (\overline{\underset{\sim}{v}} + \underset{\sim}{\omega} \times \underset{\sim}{\rho}_i)$$

$$= \sum m_i \overline{\underset{\sim}{v}} + \sum_i m_i \underset{\sim}{\omega} \times \underset{\sim}{\rho}_i$$

$$= m \overline{\underset{\sim}{v}} + \underset{\sim}{\omega} \cancelto{0}{\sum_i m_i \underset{\sim}{\rho}_i}$$

$$= m \overline{\underset{\sim}{v}}$$

$$\therefore \underset{\sim}{G} = m \overline{\underset{\sim}{v}}$$

$$\dot{\underset{\sim}{G}} = m\dot{\underset{\sim}{\bar{v}}} = \Sigma \underset{\sim}{F}$$

$$\frac{d\underset{\sim}{G}}{dt} = \Sigma \underset{\sim}{F}$$

linear impulse

$$\Rightarrow \int_1^2 d\underset{\sim}{G} = \boxed{\int_{t_1}^{t_2} \Sigma \underset{\sim}{F}\, dt}$$

$$\Rightarrow \underset{\sim}{G_2} - \underset{\sim}{G_1} = \int_{t_1}^{t_2} \Sigma \underset{\sim}{F}\, dt$$

$$\boxed{\underset{\sim}{G_1} + \int_{t_1}^{t_2} \Sigma \underset{\sim}{F} \cdot dt = \underset{\sim}{G_2}}$$

scalar component

$$\Sigma F_x = \dot{G}_x \quad , \quad (G_x)_1 + \int_{t_1}^{t_2} \Sigma F_x \cdot dt = (G_x)_2$$

$$\Sigma F_y = \dot{G}_y \quad , \quad (G_y)_1 + \int_{t_1}^{t_2} \Sigma F_y \cdot dt = (G_y)_2$$

(6/12a)

Angular momentum

$$\underset{\sim}{H_G} = \bar{I}\omega \underset{\sim}{k} \qquad \frac{d\underset{\sim}{H_G}}{dt} = \Sigma \underset{\sim}{M_G}$$

$$\Sigma \underset{\sim}{M_G} = \dot{\underset{\sim}{H_G}} \quad \rightarrow \quad \frac{dH_G}{dt} = \Sigma M_G$$

$$\Rightarrow \int_1^2 dH_G = \int_{t_1}^{t_2} \Sigma M_G\, dt$$

$$\dot{\underset{\sim}{G}} = m\dot{\underset{\sim}{v}} = \Sigma \underset{\sim}{F}$$

$$\frac{d\underset{\sim}{G}}{dt} = \Sigma \underset{\sim}{F}$$

$$\Rightarrow \int_1^2 d\underset{\sim}{G} = \int_{t_1}^{t_2} \Sigma \underset{\sim}{F}\, dt$$

$$\underset{\sim}{G}_1 + \int_{t_1}^{t_2} \Sigma \underset{\sim}{F}\, dt = \underset{\sim}{G}_2$$

$$\Rightarrow (H_G)_2 - (H_G)_1 = \int_{t_1}^{t_2} \Sigma M_G \cdot dt$$

angular impuse during t_1 to t_2

$$\therefore (H_G)_1 + \int_{t_1}^{t_2} \Sigma M_G \cdot dt = (H_G)_2 \quad (6/14)$$

Double A

Angular momentum about any point O.

$$H_o = \bar{I}\omega + m\bar{v}d \quad (6/15)$$

When a body rotates about a fixed point O.

$$\bar{v} = \bar{r}\omega$$

$$d = \bar{r}$$

$$H_o = \bar{I}\omega + m\bar{r}\omega\bar{r}$$

$$= (\bar{I} + m\bar{r}^2)\omega$$

$$= I_o\omega$$

$$\Sigma M_o = \dot{H}_o$$

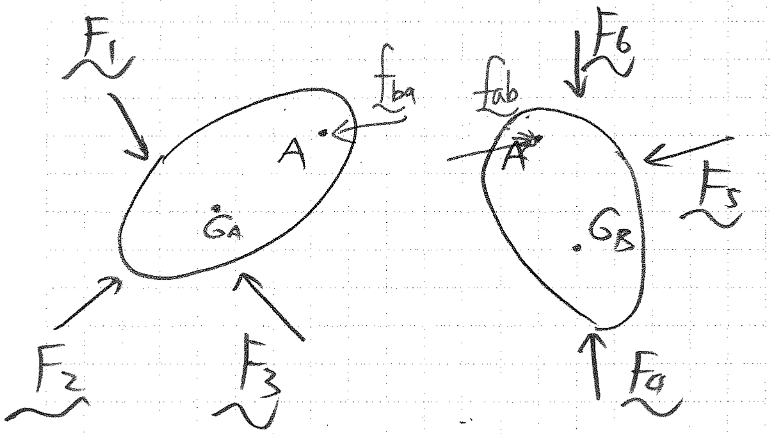
$$\frac{dH_o}{dt} = \Sigma M_o$$

$$\int_1^2 dH_o = \int_{t_1}^{t_2} \Sigma M_o \, dt$$

$$\Rightarrow (H_o)_2 - (H_o)_1 = \int_{t_1}^{t_2} \Sigma M_o \, dt$$

$$\Rightarrow (H_o)_1 + \int_{t_1}^{t_2} \Sigma M_o \, dt = (H_o)_2$$

Interconnected Rigid Bodies



$$f_{ab} = -f_{ba}$$

$$F_1 + F_2 + F_3 + f_{ba} = m_a \dot{\bar{v}}_A \quad (1)$$

$$F_4 + F_5 + F_6 + f_{ab} = m_b \dot{\bar{v}}_B \quad (2)$$

$$(1)+(2) \Rightarrow F_1 + F_2 + \cdots + F_6 = m_a \dot{\bar{v}}_a + m_B \dot{\bar{v}}_B$$

$$\Sigma F = \dot{G}_a + \dot{G}_b$$

$$r_1 \times F_1 + r_2 \times F_2 + r_3 \times F_3 + r_a \times f_{ba} = (\dot{H}_o)_a \quad (3)$$

$$r_4 \times F_4 + r_5 \times F_5 + r_6 \times F_6 + r_a \times f_{ab} = (\dot{H}_o)_b \quad (4)$$

$$(3)+(4): \quad r_1 \times F_1 + \cdots + \cancelto{0}{r_a \times f_{ba} + r_a \times f_{ab}} = (\dot{H}_o)_a + (\dot{H}_o)_b$$

$$\therefore \Sigma M_o = (\dot{H}_o)_a + (\dot{H}_o)_b$$

General system:

$$\sum \underset{\sim}{F} = \dot{\underset{\sim}{G}}_a + \dot{\underset{\sim}{G}}_b + \cdots$$

$$\sum \underset{\sim}{M}_o = (\dot{\underset{\sim}{H}}_o)_a + (\dot{\underset{\sim}{H}}_o)_b + \cdots$$

$$\Rightarrow \int_{t_1}^{t_2} \sum \underset{\sim}{F} \, dt = (\Delta \underset{\sim}{G})_{sys.}$$

$$\Rightarrow \int_{t_1}^{t_2} \sum \underset{\sim}{M}_o \, dt = (\Delta \underset{\sim}{H}_o)_{sys.} \quad (6/19)$$

Conservation of Momentum

If $\sum \underset{\sim}{F} = \underset{\sim}{0} \rightarrow \underset{\sim}{G}_1 = \underset{\sim}{G}_2$ → conservation of linear momentum

$t \in [t_1, t_2]$

If $\sum \underset{\sim}{M}_o = \underset{\sim}{0} \rightarrow (\underset{\sim}{H}_o)_1 = (\underset{\sim}{H}_o)_2$ → conservation of Angular momentum

$\sum \underset{\sim}{M}_G = \underset{\sim}{0} \rightarrow (H_G)_1 = (H_G)_2$

Summary

Kinetics of Plane Rigid Bodies

- Force-mass-acceleration
- $\sum M_G = \bar{I}\alpha$, $\sum M_o = I_o\alpha$
- work-Energy relation
  - ↳ rotational + translation
- -Impulse-and Momentum

$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$

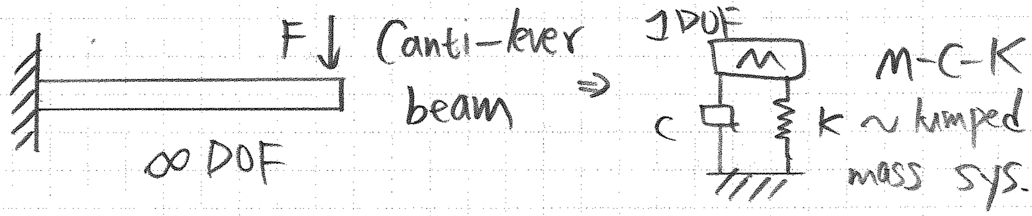
CH8 Vibration and Time Response

spring & shock absorber

8/1 Introduction

Examples of <u>vibration promblems</u> in engineering

distributed parametor v.s. lumped mass system



8/2 Free vibrations of particles

- ☐ When a spring-mounted body disturbed from equilibrium position, its motion is absence of any imposed external forces.
- ☐ Damping force tends to diminish vibration due to mechanical & fluid friction

※ undamped free friction

$\Sigma F = m\ddot{x}$

$-kx = m\ddot{x}$

$\therefore m\ddot{x} + kx = 0$ (8/1), ordinary differential equation

$x(0) = x_0$, $x'(0) = \dot{x}_0$, initial conditions

$\frac{(8/1)}{m} \Rightarrow \ddot{x} + \frac{k}{m}x = 0$ (8/2)

Let $\frac{k}{m} = \omega_n^2$, $\omega_n = \sqrt{\frac{k}{m}}$, natual circular frequency

unit: rad/s

$\ddot{x} + \omega_n^2 x = 0$ char. eqn: $\lambda^2 + \omega_n^2 = 0$, $\lambda = \pm\omega_n i$

observe: $x = \cos\omega_n t$, $\dot{x} = \omega_n \sin\omega_n t$, $\ddot{x} = -\omega_n^2 \cos$

$x = \sin\omega_n t$, $\dot{x} = \omega_n \cos\omega_n t$, $\ddot{x} = -\omega_n^2 \sin\omega_n t$

solution to eq.(8/2)

Let $x(t) = A\cos\omega_n t + B\sin\omega_n t$ (8/4)

or

$x(t) = C\sin(\omega_n t + \psi)$

(I) $x(t) = A\cos\omega_n t + B\sin\omega_n t$

$\ddot{x}(t) = -\omega_n^2 A\cos\omega_n t - \omega_n^2 B\sin\omega_n t$

$\ddot{x}(t) + \omega_n^2 x = -\omega_n^2(\cancel{A\cos\omega_n t - B\sin\omega_n t}) + \omega_n^2 (A\cancel{\cos\omega t - B\sin\omega t}) = 0$

From initial conditions:

$$x(0) = \boxed{A = x_0}$$

$$\dot{x}(t) = -A\omega_n \sin\omega_n t + B\omega_n \cos(\omega t)$$

$$\dot{x}(0) = B\omega_n = \dot{x}_0 \Rightarrow \boxed{B = \frac{\dot{x}_0}{\omega_n}}$$

$$\therefore x(t) = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n} \sin\omega_n t$$

$$= \underbrace{\sqrt{A^2+B^2}}_{C}\left(\underbrace{\frac{A}{\sqrt{A^2+B^2}}}_{\sin\psi} \cos\omega_n t + \underbrace{\frac{B}{\sqrt{A^2+B^2}}}_{\cos\psi} \sin\omega_n t\right)$$

$$= C \sin(\omega_n t + \psi)$$

$$x(t) = C \sin(\omega_n t + \psi)$$

$$\dot{x}(t) = C\omega_n \cos(\omega_n t + \psi)$$

$$\ddot{x}(t) = -C\omega_n^2 \sin(\omega_n t + \psi)$$

$$\ddot{x}(t) + \omega_n^2 x(t) = -C\omega_n^2 \sin(\omega_n t + \psi) + \omega_n^2 C \sin(\omega_n t + \psi) = 0$$

$$x(0) = C \sin\psi = x_0$$

$$\dot{x}(0) = C\omega_n \cos\psi = \dot{x}_0 \Rightarrow C\cos\psi = \frac{\dot{x}_0}{\omega_n}$$

$$C^2 = x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2 \therefore C = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}$$

$$\tan\psi = \frac{x_0\omega_n}{\dot{x}_0} \therefore \psi = \arctan\left(\frac{x_0\omega_n}{\dot{x}_0}\right)$$

(II) $$x(t) = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} \sin\left(\omega_n t + \arctan\left(\frac{x_0\omega_n}{\dot{x}_0}\right)\right)$$

using superposition principle:

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

$$= A_1 e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + A_2 e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}$$

(I) $\zeta > 1$ (overdamped)

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

(II) $\zeta = 1$ (critical damped)

$$x(t) = (A_1 + A_2 t) e^{-\omega_n t} \quad (\because \zeta = 1)$$

(III) $\zeta < 1$ (under damped) 共軛

$$x(t) = A_1 e^{i\sqrt{1-\zeta^2}\,\omega_n t} + A_2 e^{-i\sqrt{1-\zeta^2}\,\omega_n t}\,] e^{-\zeta\omega_n t}$$

$i = \sqrt{-1}$, A_1 & A_2 are complex conjugates

$$\boxed{\omega_d = \sqrt{1-\zeta^2}\,\omega_n}$$

Let $A_1 = \frac{1}{2}(A_3 - iA_4)$

$A_2 = \frac{1}{2}(A_3 + iA_4)$

$$\therefore x(t) = (A_3 \cos\omega_d t + A_4 \cos\omega_d t) e^{-\zeta\omega_n t}$$

(8/11)

Similiar to undamped system

$$x(t) = C\sin(\omega_d t + \psi)\, e^{-\zeta\omega_n t}$$

$$= C e^{-\zeta\omega_n t} \sin(\omega_d t + \psi)$$

$$\tau_d = \frac{2\pi}{\omega_d}$$

$$\delta = \ln\frac{x_1}{x_2}$$

$$= \zeta\omega_n\tau_d$$

$$= \zeta\omega_n \times \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$$

$$\Rightarrow \zeta = \frac{\delta}{\sqrt{(2\pi)^2+\delta^2}}$$

$$\zeta \doteq \frac{\delta}{2\pi}$$

8/13

Force Vibration of particles

$$-c\dot{x}-kx+F_0\sin\omega t=m\ddot{x}$$

$$\Rightarrow m\ddot{x}+c\dot{x}+kx=F_0\sin\omega t$$

$$\omega_d=\sqrt{1-\zeta^2}\,\omega_n$$

$\Rightarrow$ divided by m:

$$\boxed{\begin{aligned}&\ddot{x}+\frac{c}{m}\dot{x}+\frac{k}{m}x=\frac{F_0}{m}\sin\omega t \quad (8/13)\\&x(0)=x_0\ ,\ \dot{x}(0)=v_0\end{aligned}}$$

$$-k(x-x_B)-c\dot{x}=m\ddot{x}$$

$$\Rightarrow m\ddot{x}+c\dot{x}+kx=kx_B$$

$$=kb\sin\omega t$$

divide by m:

$$\Rightarrow \boxed{\begin{aligned}&\ddot{x}+\frac{c}{m}\dot{x}+\frac{k}{m}x=\frac{kb}{m}\sin\omega t\\&x(0)=x_0\ ,\ \dot{x}(0)=v_0\end{aligned}}$$

undamped Force vibration ($c=0$)

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega t \quad (8/15)$$

$$x(0) = x_0, \quad \dot{x}(0) = v_0$$

Complete solution, $x(t) = \underbrace{x_c(t)}_{\text{complemendary}} + \underbrace{x_p(t)}_{\text{partial}}$

$x_c(t)$: solution of $\ddot{x} + \omega_n^2 x = 0$, $x(0) = x_0$, $\dot{x}(0) = v_0$

$x_p(t)$: solution of $\ddot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega t$

steady-state solution

Let $x_p(t) = X\sin\omega t$

X: amplitude of particular solution.

$$\dot{x}_p(t) = X\omega\cos\omega t$$

$$\ddot{x}_p(t) = -X\omega^2\sin\omega t$$

sub into (8/15)

$$\Rightarrow -X\omega^2\cancel{\sin\omega t} + \omega_n^2 X\cancel{\sin\omega t} = \frac{F_0}{m}\cancel{\sin\omega t}$$

$$\Rightarrow (\omega_n^2 - \omega^2)X = \frac{F_0}{m}$$

$$\Rightarrow X = \frac{F_0}{m(\omega_n^2 - \omega^2)}, \quad X(\omega) = \frac{F_0}{m(\omega_n^2 - \omega^2)}$$

$$X(\omega) = \frac{\frac{F_0}{k}}{\frac{m}{k}(\omega_n^2 - \omega^2)} = \frac{\frac{F_0}{k}}{\frac{1}{\omega_n^2}(\omega_n^2 - \omega^2)} = \boxed{\frac{F_0/k}{1 - \left(\frac{\omega}{\omega_n}\right)^2}} \quad \frac{F_0}{k} = \delta_{st}$$

$\therefore$ The particular solution

$$x_p = \frac{\frac{F_0}{k}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \sin\omega t \quad (8/18)$$

Note:

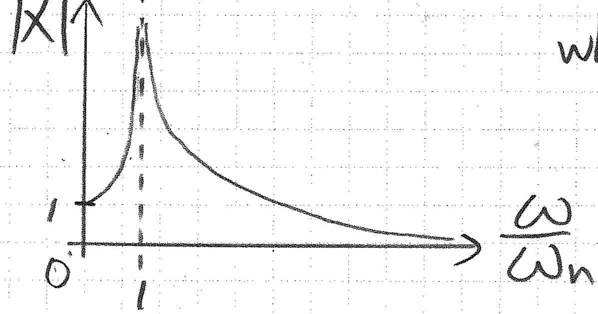
(1) x_p describes the continuing motion

steady-state solution

(2) period: $\frac{2\pi}{\omega}$, Not $\frac{2\pi}{\omega_n}$

(3) Amplitude ratio

$$\boxed{M = \frac{X}{\delta_{st}} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}} \quad (8/19)$$



when $\omega \to \omega_n$, $|X| \to \infty$

$\omega < \omega_n$, $M > 0$

$\omega > \omega_n$, $\underline{M < 0}$

out of phase 180°

Damped Force vibration

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega t \quad (8/13)$$

(i) let $x_p(t) = X\sin(\omega t - \phi)$

sub into (8/13) and solve for X and ϕ

(ii) using harmonic analysis

Let the force be $f(t) = \frac{F_0}{m}e^{i\omega t}$

$$= Fe^{i\omega t}$$

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$$f_0(t) = Im(Fe^{i\omega t}) = F\sin\omega t$$

$$= \frac{F_0}{m}\sin\omega t$$

$$x_p(t) = Im(Xe^{i\omega t})$$

$$\dot{x}(t) = i\omega Xe^{i\omega t} \quad -(1)$$

$$\ddot{x}(t) = -\omega^2 Xe^{i\omega t} \quad -(2)$$

sub (1)·(2) into 8/13

$$\Rightarrow [(\omega_n^2 - \omega^2)X + i2\zeta\omega\omega_n X]\cancel{e^{i\omega t}} = F\cancel{e^{i\omega t}}$$

$$\Rightarrow (\omega_n^2 - \omega^2 + i2\zeta\omega\omega_n)X = F$$

$$X = \frac{F}{\omega_n^2 - \omega^2 + i2\zeta\omega\omega_n}$$

$$= \frac{F_0/k}{\frac{m}{k}[\omega_n^2 - \omega^2 + i2\zeta\omega\omega_n]}$$

$$= \frac{F_0/k}{\frac{1}{\omega_n^2}[\omega_n^2 - \omega^2 + i2\zeta\omega\omega_n]}$$

$$= \boxed{\frac{F_0/k}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\left(\frac{\omega}{\omega_n}\right)}}$$ 8/20

(Recall: $X + iY = Re^{i\phi}$

$R = \sqrt{X^2 + Y^2}$, $\phi = \arctan\left(\frac{Y}{X}\right)$)

$$= \frac{F_0}{k} \times \frac{1}{Re^{i\phi}} = \frac{F_0}{k}\frac{1}{R} \times e^{-i\phi}$$

$$\therefore R = \sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}, \ \phi = \arctan\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

8/21

$$M = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$e^{i(\omega t-\phi)}$$

$$\therefore x_p(t) = I_m\left(\frac{F_0}{k} M e^{-i\phi} e^{i\omega t}\right)$$

$$= \frac{F_0}{k} M \sin(\omega_d t - \phi)$$ phase lag

$$\therefore x(t) = C e^{-\zeta\omega_n t} \sin(\omega t - \phi)$$

$$+ \frac{F_0}{k} \times \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \times \sin(\omega t - \phi)$$

$$\phi = \arctan\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$$