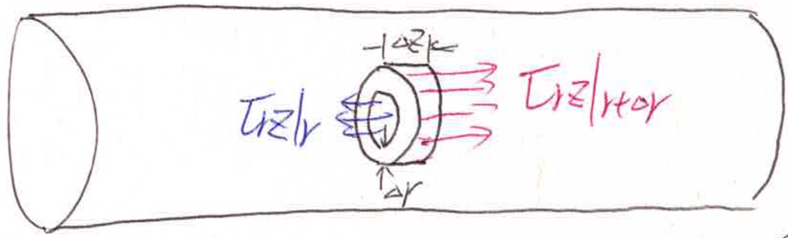
Chap 8.

Assumption:

1. The fluid is Newtonian, incompressible
   behaves as continuum
2. The flow laminar, fully developed
   steady-state

   fully developed ⇓ the velocity profile doesn't vary along the axis of flow (z direction)
3. No entrance effects
4. No-slip condition



$$\vec{v} = (\cancel{v_r}^{\,0}, \cancel{v_\theta}^{\,0}, v_z)$$

laminar flow
only flows in z direction

$$v_z(r, \cancel{\theta}, \cancel{z})$$

symmatric fully-developed.

$$\overleftrightarrow{\tau} = \begin{pmatrix} \sigma_{rr}^{\,-P} & \cancel{\tau_{r\theta}}^{\,0} & \tau_{rz} \\ \cancel{\tau_{\theta r}}^{\,0} & \sigma_{\theta\theta}^{\,-P} & \cancel{\tau_{\theta z}}^{\,0} \\ \tau_{zr} & \cancel{\tau_{z\theta}}^{\,0} & \sigma_{zz}^{\,-P} \end{pmatrix}$$

$$\sigma_{ii} = \underbrace{f\left(\frac{\partial v_i}{\partial i}\right)}_{=0} - P$$

Mass B.

$$0 = \iint_{C.S.} \rho \vec{v} \cdot \vec{n}\, dA + \cancel{\frac{\partial}{\partial t} \iiint_{C.V.} \rho\, dV}^{\,0 \text{ steady-state}}$$

$$A = \pi\left[(r+\Delta r)^2 - r^2\right] = \pi$$

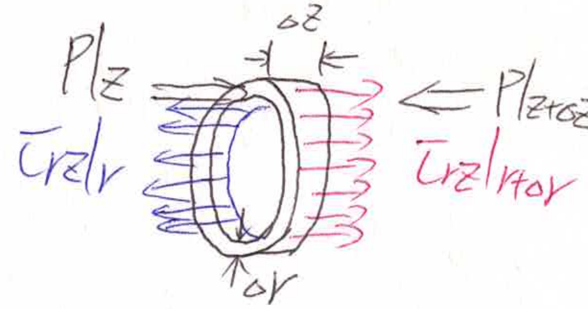
$$0 = \left(\rho v_z|_{z+\Delta z} - \rho v_z|_z\right) \cdot A$$

$$\therefore v_z|_{z+\Delta z} = v_z|_z$$

Momentum B.

$$\sum \vec{F} = \iint_{C.S.} \rho \vec{v}(\vec{v}\cdot\vec{n})\,dA + \overset{0 \text{ steady-state}}{\cancel{\frac{\partial}{\partial t}\iiint_{C.V.} \rho \vec{v}\,dV}}$$

z direction



$$\tau_{rz}(2\pi (r+\Delta r)\Delta z)|_{r+\Delta r} - \tau_{rz}(2\pi r\Delta z)|_r$$

$$-\left[P|_{z+\Delta z} - P|_z\right]\pi\left[\underbrace{(r+\Delta r)^2 - r^2}_{(2r\Delta r+\cancel{\Delta r^2})}\right]$$

$$= \left[\underbrace{(\rho v_z)v_z|_{z+\Delta z} - (\rho v_z)v_z|_z}_{0}\right]\pi\left[\underbrace{(r+\Delta r)^2 - r^2}_{(2r\Delta r+\cancel{\Delta r^2})}\right]$$

$$(\tau_{rz}\cdot r\Delta z)|_{r+\Delta r} - (\tau_{rz}\cdot r\Delta z)|_r - (P|_{z+\Delta z} - P|_z)r\Delta r = 0$$

$$\frac{(r\tau_{rz})|_{r+\Delta r} - (r\tau_{rz})|_r}{\Delta r} - \frac{(P|_{z+\Delta z} - P|_z)r}{\Delta z} = 0$$

$$\frac{d(r\tau_{rz})}{dr} - r\frac{dP}{dz} = 0$$

The force of the fluid acting on the wetted surface of the pipe

$$F_z = \left(\tau_{rz}\big|_{r=R}\right)\cdot 2\pi RL$$

$$= \left(\frac{dp}{dz}\right)\cdot\frac{R}{2}\cdot 2\pi RL$$

$$= \pi R^2 L\left(\frac{dp}{dz}\right)$$

Mass rate of flow

$$Q = \int_0^{2\pi}\int_0^R \rho v_z\, r\,dr\,d\theta = \left(-\frac{dp}{dz}\right)\frac{\rho\pi R^4}{8\mu} = \rho\pi R^2 V_{avg}$$

$$\left(-\frac{dp}{dz}\right) = \frac{8\mu Q}{\rho\pi R^4} \quad \text{or} \quad \left(-\frac{dp}{dz}\right) = \frac{8\mu V_{avg}}{R^2} = \frac{32\mu V_{avg}}{D^2}$$

Exp. Observations in a pipe

Reynolds number $Re = \dfrac{\rho V_{avg} D}{\mu}$ $\dfrac{\text{Inertial force}}{\text{Viscous force}}$

(I) $Re < 2100$ laminar flow

(II) $Re > 2100$ turbulent flow

The above analysis is valid only for $Re < 2100$

$\rho\vec{v}$: mass flux

$\rho\vec{v}\vec{v}$: convective momentum flux

$\vec{\tau}$ = ① momentum flux by viscous transfer

② stress

For a vertical tube (or pipe)

No pressure gradient

$$\tau_{rz}(2\pi(r+\cancel{\Delta r})\Delta z)|_{r+\Delta r} - \tau_{rz}(2\pi r\Delta z)|_r$$

$$+\rho g\cdot\pi[\underbrace{(r+\Delta r)^2 - r^2}_{(2r\Delta r+\cancel{\Delta r^2})}]\Delta z = [\underbrace{(\rho v_z)v_z|_{z+\Delta z} - (\rho v_z)v_z|_z}_{=0}]\pi((r+\Delta r)^2 - r$$

$$(\tau_{rz}\cdot r\Delta z)|_{r+\Delta r} - (\tau_{rz}\cdot r\Delta z)|_r + \rho g r\Delta z = 0$$

$$\frac{(r\tau_{rz})|_{r+\Delta r} - (r\tau_{rz})|_r}{\Delta r} + \rho g r = 0$$

$$\frac{d(r\tau_{rz})}{dr} + \rho g r = 0$$

$$V_{avg} = \rho g\frac{R^2}{8\mu}$$

$$\mu = \frac{\rho g R^2}{8V_{avg}} \qquad \mu \propto \frac{\rho}{V_{avg}} \propto \frac{\rho t}{L}$$

Fluid flow down an inclined-plane surface

$$V_x = \frac{\rho g L^2 \sin\theta}{\mu}\left[\frac{y}{L} - \frac{1}{2}\left(\frac{y}{L}\right)^2\right]$$

$$V_{avg} = \frac{\int_0^w \int_0^L V_x \, dy\, dz}{\int_0^w \int_0^L dy\, dz}$$

$$V_{avg} = \frac{\iint V\, dA}{\iint dA}$$

$$= \frac{\rho g L^2 \sin\theta}{3\mu} \quad \left(= \frac{2}{3} V_{max}\right)$$

$$V_{max} = \frac{\rho g L^2 \sin\theta}{2\mu} \quad \text{at } y = L$$

The mass rate of flow

$$Q = \int_0^w \int_0^L \rho V_x \, dy\, dz$$

$$= \frac{\rho^2 g w L^3 \sin\theta}{3\mu} = \rho w L V_{avg}$$

Film thickness

$$L = \sqrt{\frac{3\mu V_{avg}}{\rho g \sin\theta}} = \sqrt[3]{\frac{3\mu Q}{\rho^2 g w \sin\theta}}$$

The force exerted by the fluid on the wall

$$F_x = \int_0^w \int_0^D \left(\tau_{yx}|_{x=0}\right) dx\, dz$$

$$= \rho g L D W \sin\theta$$

Exp. observations

There are actually three "flow regimes":

I) $Re < 20$, laminar flow with negligible rippling

II) $20 < Re < 1500$, laminar flow with pronounced rippling

III) $Re > 1500$ turbulent flow

The above analysis is valid only for regime I.

$$Re = \frac{\rho V_{avg} (4L)}{\mu}$$