CALCULUS I LECTURE 2: LIMITS OF FUNCTIONS

1. Definition of limits

In this lecture, we always assume $X, Y \subset \mathbb{R}$. We use the notations

$$(a,b) := \{x \in \mathbb{R} \mid a < x < b\}, \tag{1.1}$$

$$(a,b] := \{x \in \mathbb{R} \mid a < x \leq b\}, \tag{1.2}$$

$$[a,b) := \{x \in \mathbb{R} \mid a \leq x < b\}, \tag{1.3}$$

$$[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\} \tag{1.4}$$

to denote different types of intervals in $\mathbb{R}$. We call $f : X \to Y$ is defined everywhere near a point a if there exists $c > 0$ such that

$$(a-c, a+c) - \{a\} \subset X. \tag{1.5}$$

Definition 1.1. Let $f : X \to Y$ and f is defined everywhere near a. We call f converges to L at a, or

$$\lim_{x \to a} f(x) = L, \tag{1.6}$$

if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ when $0 < |x - a| < \delta$.

This logical statement tells us that when we consider those x which is sufficiently close to a (but not equal to), then the corresponding value $f(x)$ will be close to L.

This definition of limit is very cumbersome. We can actually simplified it by using the axiom of choice. Here we take $I = \mathbb{R}^+$. By Definition 1.1, if $\lim_{x \to a} f(x) = L$, then we can define the set S_ε for every $\varepsilon \in \mathbb{R}^+$ as the following:

$$S_\varepsilon = \{r > 0 \mid |f(x) - L| < \varepsilon \text{ for any } x \in (a-r, a+r) - \{a\}\}. \tag{1.7}$$

According to Definition 1.1, $S_\varepsilon \neq \emptyset$ (because there exists $\delta \in S_\varepsilon$). By the axiom of choice, we will have a function, which is still called $\delta : \mathbb{R}^+ \to \mathbb{R}^+$, such that

$$|f(x) - L| < \varepsilon \text{ when } 0 < |x - a| < \delta(\varepsilon). \tag{1.8}$$

Conversely, if we have the function δ satisfies (1.8), then the statement of Definition 1.1 holds. So we have $\lim_{x \to a} f(x) = L$.

Proposition 1.2. Let $f : X \to Y$ and f is defined everywhere near a. Then $\lim_{x \to a} f(x) = L$ if and only if there exist a function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that (1.8) holds for every $\varepsilon \in \mathbb{R}^+$.

Example 1.3. Show that $\lim_{x \to 0} a^x = 1$ for any $a > 0$.

Notice that the limit of a function may not exists: it can blow up to infinity, minus infinity or oscillating. For example, we have $\frac{1}{x^2}$ blow up to infinity at 0,

$\sin(\frac{1}{x})$ oscillating at 0. When we have a function blow up to infinity at a point a, we write

$$\lim_{x\to a} f(x) = \infty$$

Definition 1.4. We call

$$\lim_{x\to a} f(x) = \infty \tag{1.9}$$

if and only if for any $N > 0$, there exist $\delta > 0$ such that $f(x) > N$ when $0 < |x-a| < \delta$.

We can also write $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ as a function by the axiom of choice, such that $f(x) > N$ when $|x-a| < \delta(N)$.

In some cases, f only defines on $x > a$ or $x < a$. So we can only obtain the limit from one side.

Definition 1.5. If f is defined on $(a, a+c)$ for some $c > 0$, we call

$$\lim_{x\to a^+} f(x) = L \tag{1.10}$$

if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ when $0 < x - a < \delta$.

Similarly, one can define $\lim_{x\to a^-} f(x) = L$ when f is defined on $(a-c, a)$ for some $c > 0$.

One can also define the limit at infinity as the following.

Definition 1.6. Suppose that f is defined for $x > a$ with some $a \in \mathbb{R}$, then

$$\lim_{x\to\infty} f(x) = L \tag{1.11}$$

if and only if there exists $N : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(x) - L| < \varepsilon \text{ when } x > N(\varepsilon). \tag{1.12}$$

2. Properties for limit

We will prove here that all algebraic operators commute with limit.

Proposition 2.1. Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then we have

$$\lim_{x\to a} f(x) + cg(x) = L + cM \tag{2.1}$$

for any $c \in \mathbb{R}$.

Proof. By Proposition 1.2, there are two functions δ_1, δ_2 defining for the limit of f and g respectively. We define

$$\delta(\varepsilon) := \min\left\{\delta_1\Big(\frac{\varepsilon}{1+|c|}\Big), \delta_2\Big(\frac{\varepsilon}{1+|c|}\Big)\right\}.$$

Then if $|x-a| < \delta(\varepsilon)$, we have:
1. $|x-a| < \delta_1\Big(\frac{\varepsilon}{1+|c|}\Big)$, so $|f(x) - L| \le \frac{1}{1+|c|}\varepsilon$.

2. $|x-a| < \delta_2\left(\frac{\varepsilon}{1+|c|}\right)$, so $|g(x)-M| \leq \frac{1}{1+|c|}\varepsilon$.
Therefore, we have

$$|(f(x)+cg(x))-(L+cM)| \leq |f(x)-L|+|c||g(x)-M| < \varepsilon$$

by triangle inequality. □

Proposition 2.2. For any rational number $q \in \mathbb{Q}^+$ and $L > 0$, we have

$$\lim_{x\to a} f^q(x) = L^q \tag{2.2}$$

when $\lim_{x\to a} f(x) = L$.

Proof. Let $m, n \in \mathbb{N}$. We firstly prove that

$$\lim_{x\to a} f^{\frac{1}{n}}(x) = L^{\frac{1}{n}} \tag{2.3}$$

and then prove that

$$\lim_{x\to a} f^m(x) = L^m. \tag{2.4}$$

By these two equality, one can easily obtain the general case.

To prove (2.3), we first notice that $f^{\frac{1}{n}}$ is well-defined near a ($f(x) > 0$ near a). Since $\lim_{x\to a} f(x) = L$, there exits $\delta(\frac{L}{2})$ such that $|f(x)-L| < \frac{L}{2}$ when $|x-a| < \delta(\frac{L}{2})$. We denote $\delta(\frac{L}{2})$ by k. Then

$$f(x) > 0 \text{ when } |x-a| < k.$$

Here δ is the function defined by f. To obtain (2.3), we define the new delta as

$$\delta^{new}(\varepsilon) := \min\left\{k, \delta(L^{\frac{n-1}{n}}\varepsilon)\right\}. \tag{2.5}$$

So when $|x-a| < \delta^{new}(\varepsilon)$, we have
1. $|f(x)-L| < L^{\frac{n-1}{n}}\varepsilon$.
2. $f(x) > 0$.

One can obtain that

$$\begin{aligned} |f^{\frac{1}{n}(x)} - L^{\frac{1}{n}}| &= \frac{|f(x)-L|}{\left|f^{\frac{n-1}{n}}(x) + f^{\frac{n-2}{n}}(x)L^{\frac{1}{n}} + \cdots + L^{\frac{n-1}{n}}\right|} \\ &< \frac{L^{\frac{n-1}{n}}\varepsilon}{\left|f^{\frac{n-1}{n}}(x) + f^{\frac{n-2}{n}}(x)L^{\frac{1}{n}} + \cdots + L^{\frac{n-1}{n}}\right|} \end{aligned} \tag{2.6}$$

with $\left|f^{\frac{n-1}{n}}(x) + f^{\frac{n-2}{n}}(x)L^{\frac{1}{n}} + \cdots + L^{\frac{n-1}{n}}\right| > L^{\frac{n-1}{n}}$ (because $f(x) > 0$). So we get (2.3).

We leave the proof of (2.4) as the exercise. One candidate for the new delta is

$$\delta^{new}(\varepsilon) = \min\left\{k, \delta(\frac{\varepsilon}{m(2L)^{m-1}})\right\}. \tag{2.7}$$

□

Proposition 2.3. Let $\lim_{x\to a} f(x) = L \neq 0$. Then

$$\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{L}. \tag{2.8}$$

Proof. By definition, there exists $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(x) - L| < \frac{L}{2}$ when $|x-a| < \delta(\frac{L}{2})$. We denote $\delta(\frac{L}{2})$ by k. Define the new delta to be

$$\delta^{new}(\varepsilon) = \min\Big\{\delta\big(\frac{L^2\varepsilon}{2}\big), k\Big\}. \tag{2.9}$$

Then we will have

$$\Big|\frac{1}{f(x)} - \frac{1}{L}\Big| \leq \frac{|f(x)-L|}{|f(x)L|} \leq \frac{|f(x)-L|}{\frac{L^2}{2}} < \varepsilon$$

when $|x-a| < \delta^{new}(\varepsilon)$. □

Proposition 2.4. Let $f(x) \leq g(x)$ for all x near a, i.e., $f(x) \leq g(x)$ for all $x \in (a-c, a+c) - \{a\}$ with some $c > 0$. Then we have

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

if both limits exist.

Proof. Suppose $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$. For any $\varepsilon > 0$, there exist δ_1 and δ_2 such that $|f(x) - L| < \varepsilon$ and $|g(x) - M| < \varepsilon$ when $|x-a| < \min\{\delta_1, \delta_2\}$. So in particular, we have

$$L - \varepsilon < f(x) \leq g(x) < M + \varepsilon$$

when $|x-a| < \min\{\delta_1, \delta_2\}$. Namely,

$$L \leq M + 2\varepsilon$$

for any $\varepsilon > 0$. Therefore $L \leq M$. □

Exercise 2.5. Show that $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$ (Hint: use Proposition 2.1 and Proposition 2.2).