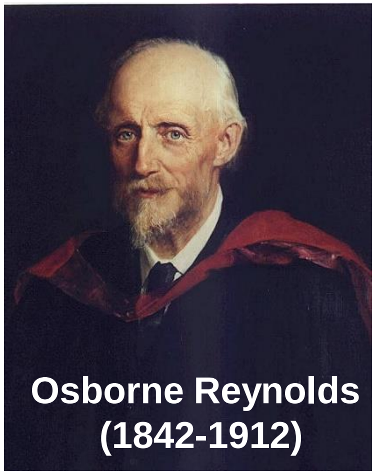
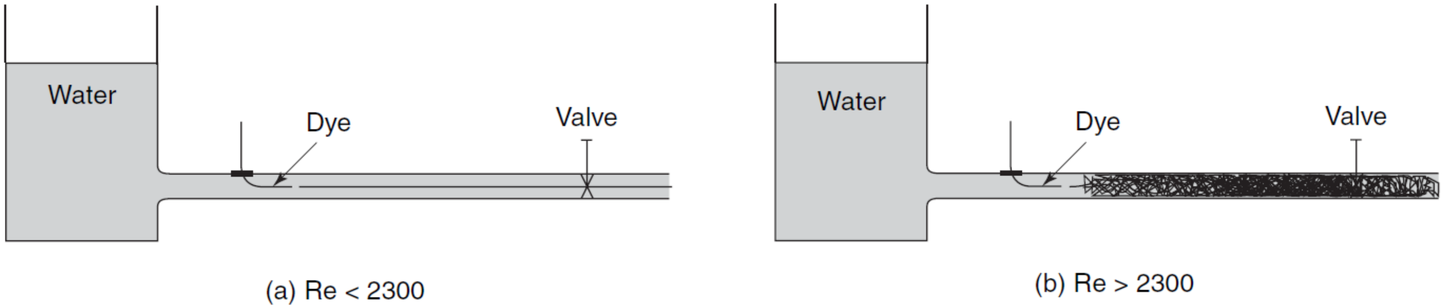
Viscous flow

- Although under certain conditions, the flow may be considered ideal or inviscid, in reality all fluids are viscous. Thus, the effect of viscosity and the interactions between the solid surface and the fluid flowing on it must be considered.





Reynolds' experiment in 1883

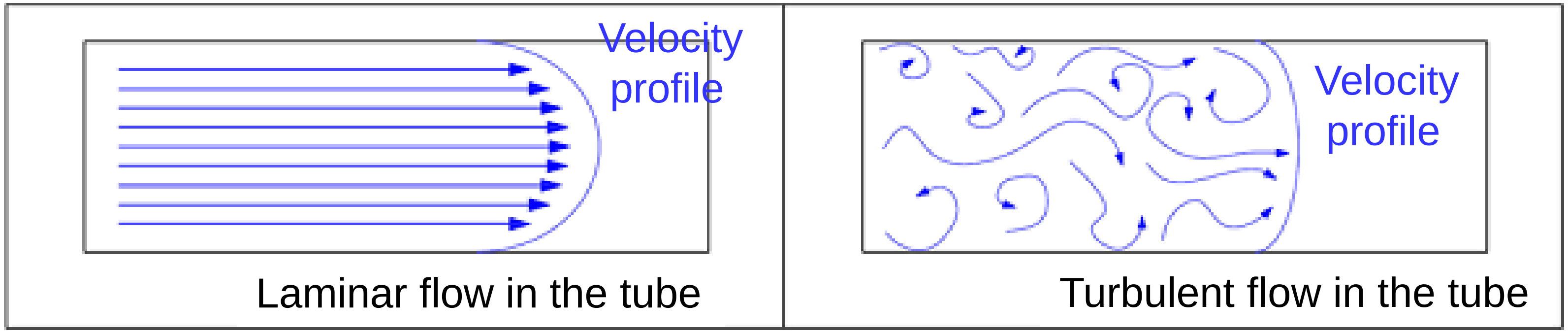
- **Laminar flow**

- **Turbulent flow**

Laminar flow *vs.* turbulent flow

- Laminar flow:

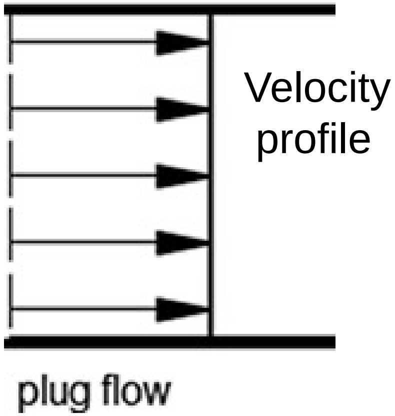
- Adjacent fluid layers slide smoothly over one another with mixing between layers or lamina occurring only on a molecular level.
- It was for this type of flow that Newton's viscosity relation was derived, and in order for us to measure the viscosity (μ), this laminar flow must exist.



- Plug flow:

An ideal flow assuming that no interactions between the fluid and the solid surface.

→ It usually appears at the region very close to the entrance of a tube.



Laminar flow *vs.* turbulent flow

- For flow in circular pipes, the flow is laminar when the Reynolds number is below 2,300.

- Above Re = 2,300, the flow may be laminar as well, and indeed, laminar flow has been observed for Reynolds numbers as high as 40,000 in experiments wherein external disturbances were minimized.

- Above a Reynolds number of 2300, small disturbances will cause a transition to turbulent flow.

$$\boxed{\mathrm{Re} \equiv L v_{\infty} \rho / \mu} \qquad \left(\frac{Inertial\ Force}{Viscous\ Force}\right)$$

Drag

- The drag force due to friction is caused by the shear stresses at the surface of a solid object moving through a viscous fluid:

$$\boxed{\frac{F}{A} \equiv C_f \frac{\rho v_\infty^2}{2}}$$

F: drag force due to *friction*
A: area of contact between the solid body and the fluid
C_f: coefficient of skin friction
V_∞: free-stream fluid velocity

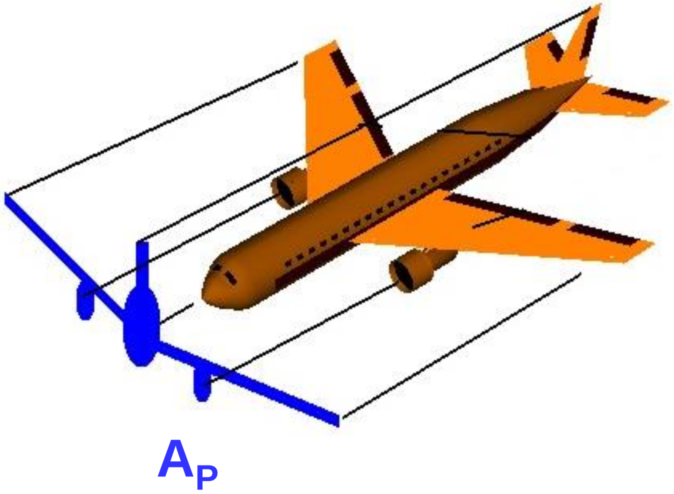
$$C_f = \frac{F/A}{\rho v^2/2} = \frac{\text{Pressure Force}}{\text{Inertial force}} \Rightarrow \frac{\rho v_\infty^2}{2} \equiv Dynamic\ pressure$$

$$\boxed{\underbrace{\frac{F}{A_P}}_{\text{Drag force (pressure + friction)}} \equiv \underbrace{C_D}_{\text{Drag coefficient (Find it from chart)}} \frac{\rho v_\infty^2}{2}}$$

Drag force
(pressure + friction)

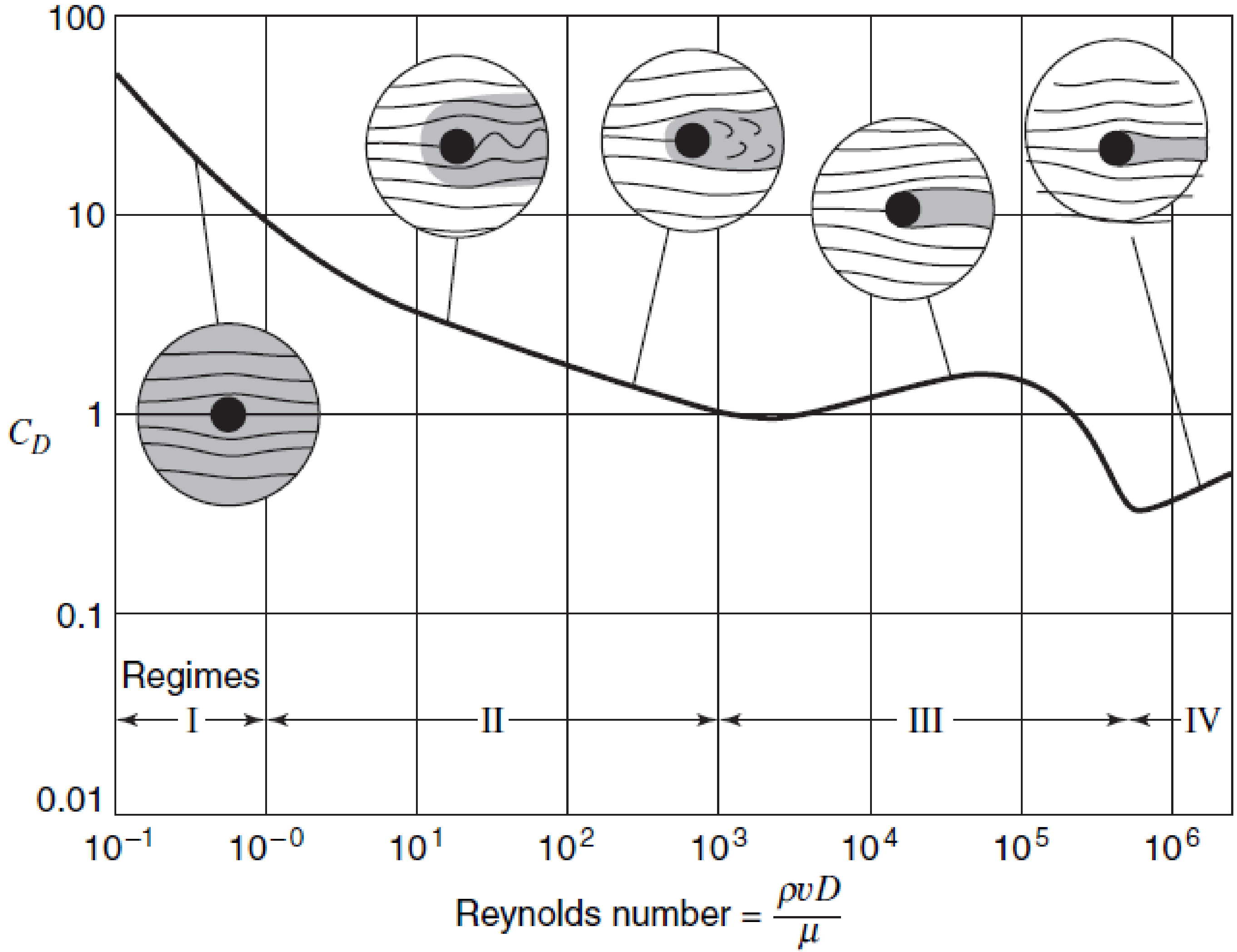
Projected area

Drag coefficient
(Find it from chart)

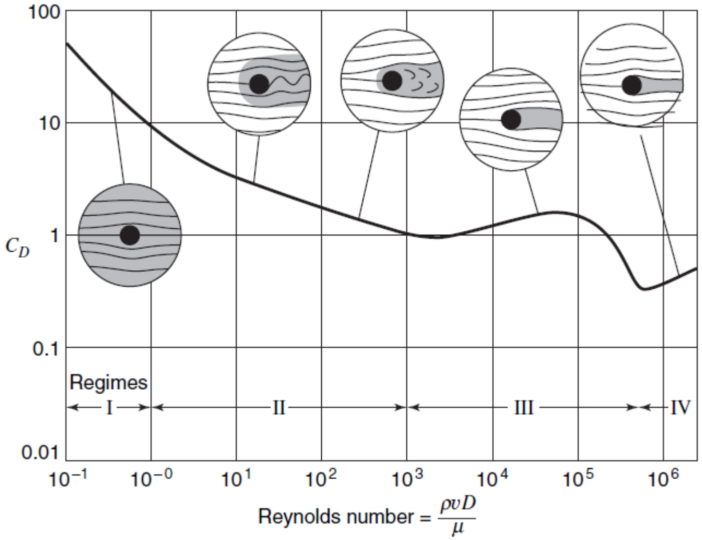


Drag for a flowing fluid

- For a viscous fluid flowing through a smooth circular cylinder:



Low Re region



Re<1: viscous force >> inertial force

- The flow adheres to the body without any oscillation.
- Also call "**creeping flow**"

Note:

$$\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = -\nabla P + \mu \nabla^2 v$$

(Navier-Stokes Equation)

- At a very low Re (<<1):

1. It is almost steady-state.
2. Inertial term is negligible.

$$\boxed{0 = -\nabla P + \mu \nabla^2 v}$$

Eq. of motion for ***creeping flow***

- At a very high Re:

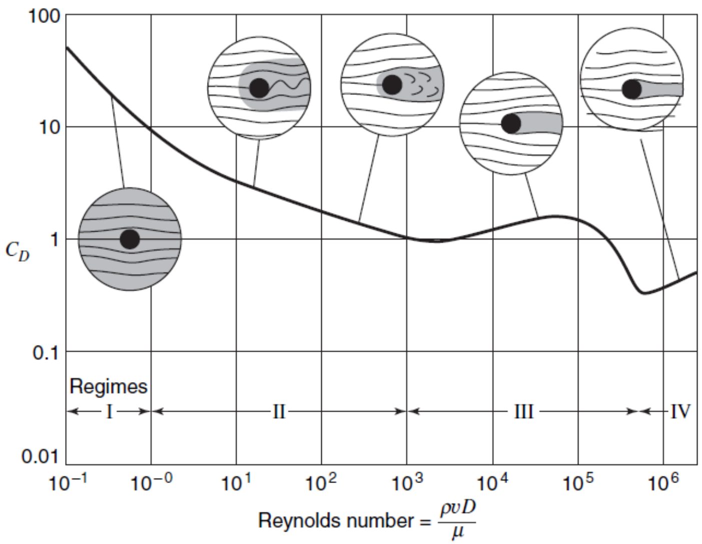
1. Viscous term is negligible.

$$\boxed{\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = -\nabla P}$$

Eq. of motion for ***inviscid flow***

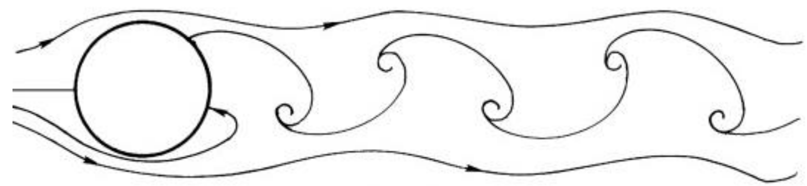
High Re regions

$$C_D = \frac{\text{Pressure Force}}{\text{Inertial force}}$$

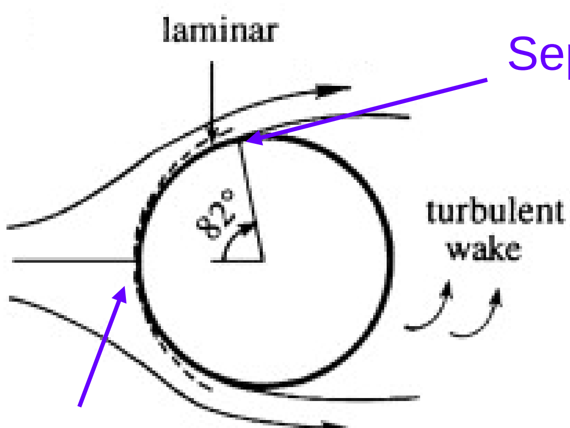


Region II: <u>1 < Re < 2300</u>

- Formation of unsteady eddies in the wake
- <u>Flow separation</u> from the body



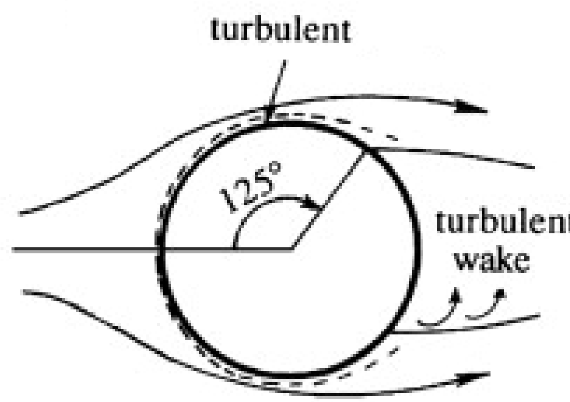
Region III: <u>2300 < Re < ~2 x 10^5</u>



Separation point

Stagnation point

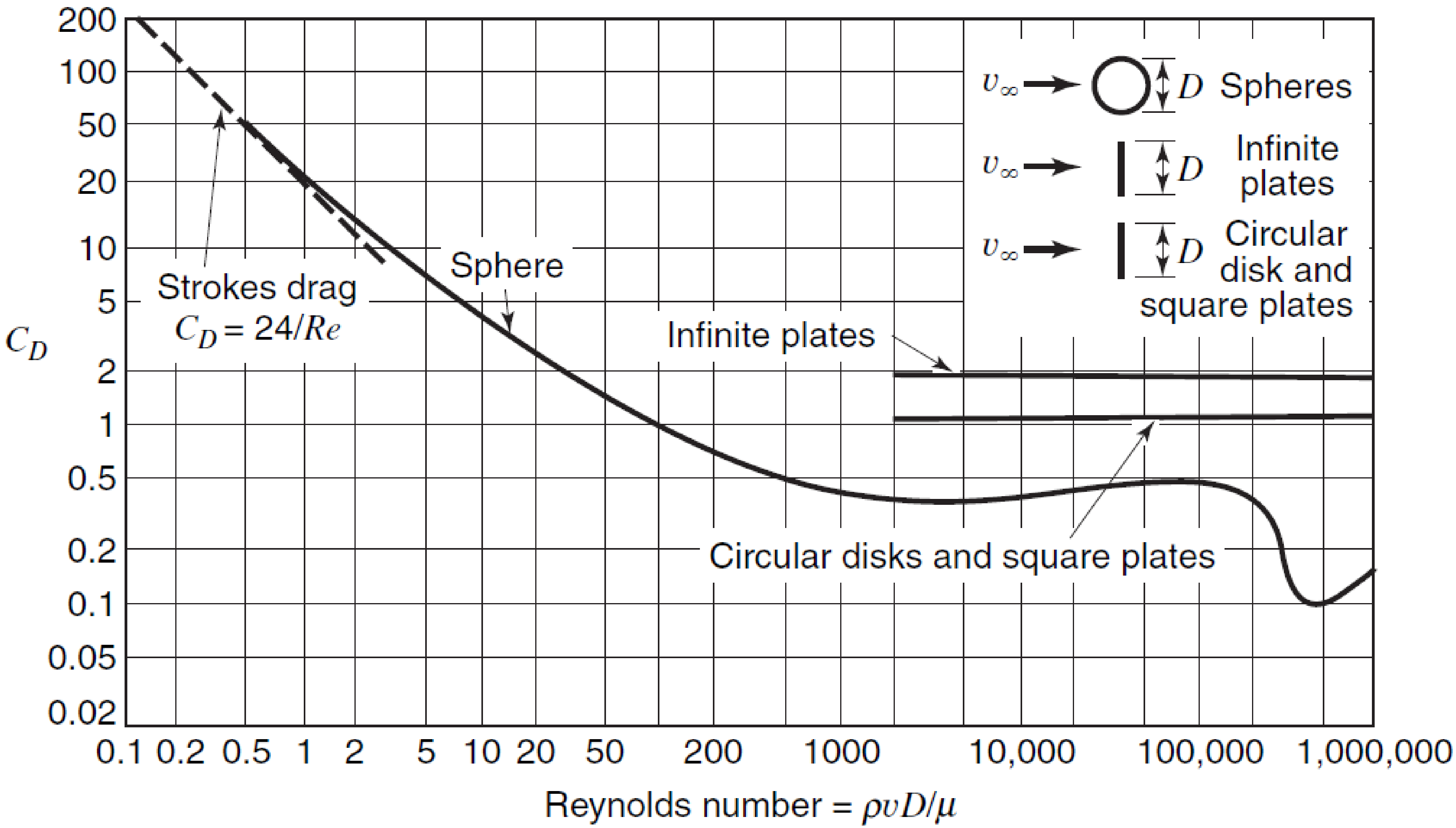
Region IV: <u>Re > 5 x 10^5</u>



"Turbulent flow resists flow separation better"

Flow patterns before separation agree well with the <u>inviscid flow</u> theory!

Chart: drag coefficient *vs.* Re

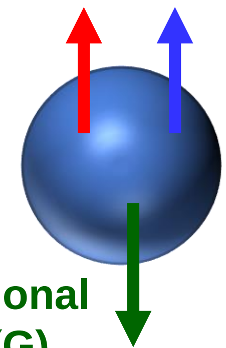


$$Drag\ force = A_P C_D \frac{\rho v_\infty^2}{2}$$

Example

Evaluate the terminal velocity of a 7.5-mm-diameter glass sphere falling freely through (a) air at 300 K and (b) glycerin at 300 K. The density of glass is 2250 kg/m³.

Solution:



$$Volume\ (V) = \frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3$$

$$A_P = \frac{1}{4}\pi d^2$$

ρ_s = density of glass
ρ_f = density of fluid

$$Drag\ force = \frac{1}{4}\pi d^2 C_D \frac{\rho_f v_\infty^2}{2} = G - B = \rho_s V g - \rho_f V g$$

$$C_D v_\infty^2 = \frac{4gd}{3}\left(\frac{\rho_s}{\rho_f} - 1\right)$$

d = 7.5 x 10^{-3} m
ρ_s = 2250 kg/m³
ρ_f = 1.177 kg/m³ (Air; from Appendix I)
ρ_f = 1260 kg/m³ (Glycerin; from Appendix I)

$$\begin{cases} Air:\ C_D v_\infty^2 = 187.2\ m^2/s^2 \\ Glycerin:\ C_D v_\infty^2 = 0.077\ m^2/s^2 \end{cases}$$

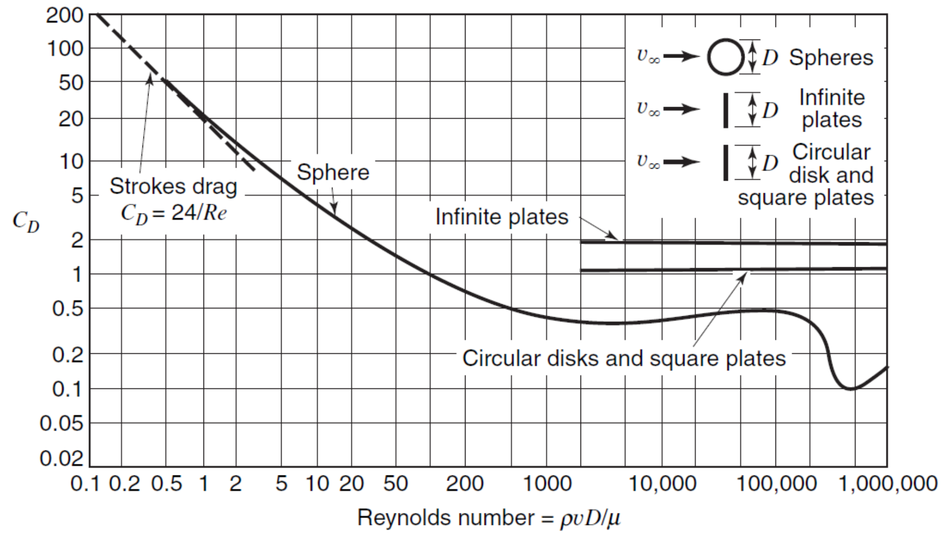
μ_f = 1.85 x 10^{-5} kg/m-s (Air; from Appendix I)
μ_f = 0.892 kg/m-s (Glycerin; from Appendix I)

Example

Trial & Error!

$$Air: Re = \frac{d\rho_f v_\infty}{\mu} = 477 v_\infty$$

$$Glycerin: Re = \frac{d\rho_f v_\infty}{\mu} = 10.6 v_\infty$$



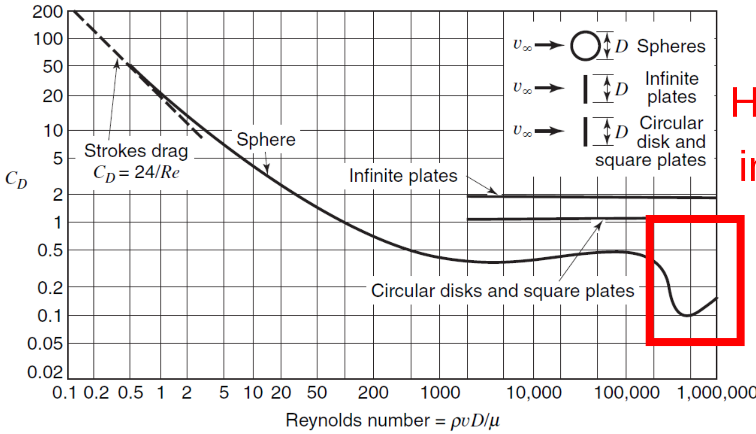
$$Air: \ C_D v_\infty^2 = 187.2\ m^2/s^2$$

$$Glycerin: \ C_D v_\infty^2 = 0.077\ m^2/s^2$$

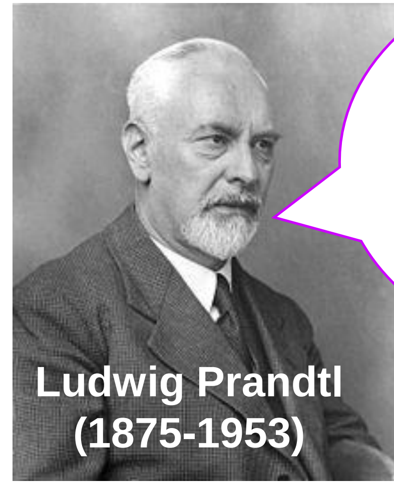
- Tip (1): In air, the velocity should be very fast, and there is a region of ***C_D*~0.4** for a wide Re values.
  → Let's assume C_D = 0.4 → **v_∞ = 21.63 m/s** → Re ~10000 (✔)

- Tip (2): In glycerin, the velocity should be very slow, so let's assume that C_D = 24/Re:
  → **v_∞ = 0.034 m/s** → Re = 0.36 (✔)

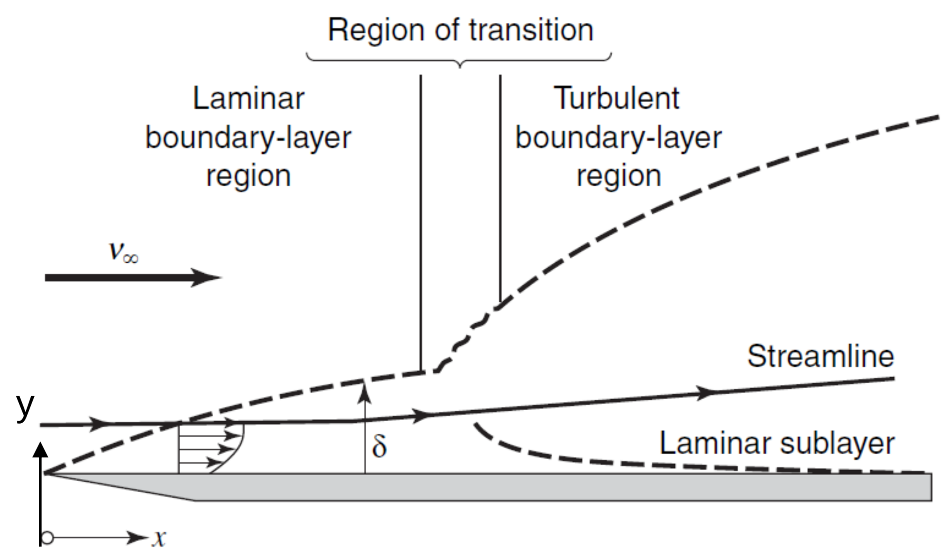
The boundary-layer concept



Huge decrease in drag at high Re



At high Re, the effect of <u>fluid friction</u> is limited to <u>*a thin layer*</u> near the surface!



(leading edge, x=0)

- No significant pressure change across the boundary layer
- Boundary layer thickness (δ):

$$v|_{y=\delta} \equiv 0.99 v_\infty$$

- Local Re (Re_x):

$$Re_x \equiv \frac{x\rho v}{\mu} = \frac{xv}{\nu}$$

Boundary-layer equations for a 2D flow

- Navier-Stokes Equations:

Normal stress

$$\sigma_{xx}$$

Affecting on the surface vertical to x axis

x direction

x direction $$\rho\left\{\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right\}=\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}$$

y direction $$\rho\left\{\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}\right\}=\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}$$

……and: $\tau_{yx}=\tau_{xy}=\mu(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x})$

- Assumptions:

(1) A thin layer → v_y is not changing so much → $\frac{\partial v_x}{\partial y} \gg \frac{\partial v_y}{\partial x}$

(2) Large Re → Both normal stress are approximately the same as ***-P***

(3) $\partial P/\partial y \sim 0$

From Bernoulli's eq:

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)=-\frac{\partial P}{\partial x}+\mu\frac{\partial^2 v_x}{\partial y^2}$$

and: $-\frac{dP}{dx}=\rho v_\infty\frac{dv_\infty}{dx}$

Boundary-layer equations for a 2D flow

$$\boxed{\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = v_\infty \frac{d v_\infty}{dx} + \nu \frac{\partial^2 v_x}{\partial y^2}}$$

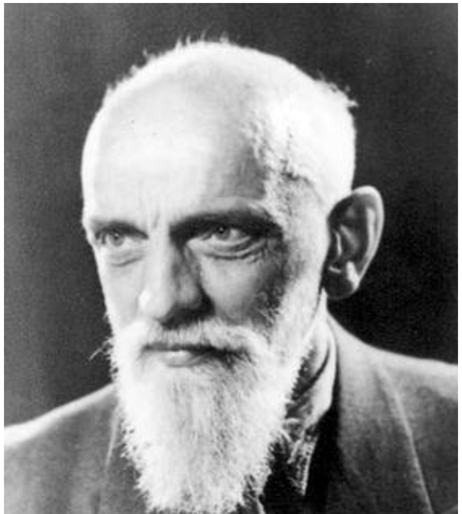
Eq. of motion

$$\boxed{\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0}$$

Eq. of continuity

Boundary-layer equations

"With boundary-layer simplifications, the analytical treatments of viscous flow become possible."



Paul Richard Heinrich Blasius (1883-1970)

- **The Blasius's solution:**

Assumptions: Laminar boundary layer on a **flat plate** and steady flow (*no pressure gradient*)

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Step 1: Use stream function (Ψ) to automatically satisfice the eq. of continuity.

Step 2: Transform the first PDE into ODE to solve it!

The Blasius's solution for *FLAT plate*

Step 1: Use stream function (Ψ) to automatically satisfice the eq. of continuity.

$$d\Psi \equiv v_x dy - v_y dx \rightarrow \frac{\partial \Psi(x,y)}{\partial x} = -v_y \qquad \frac{\partial \Psi(x,y)}{\partial y} = v_x$$

Step 2: Transform the PDE (eq. of motion) into ODE to solve it:

Set:

New variable $$\eta(x,y) = \frac{y}{2}\left(\frac{v_\infty}{\nu x}\right)^{1/2}$$

$$f(\eta) = \frac{\Psi(x,y)}{(\nu x v_\infty)^{1/2}}$$ New function

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2} \xrightarrow{\cdots\cdots} \boxed{f''' + ff'' = 0}$$

1 2 3 4 5

B.C.: (1) $y = 0$, $v_x = v_y = 0$ → $f = f' = 0$ at $\eta = 0$

(2) $y = \infty$, $v_x = v_\infty$ → $f' = 2$ at $\eta = \infty$

Ex: 1 $$v_x = \frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial f}\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} = (\nu x v_\infty)^{\frac{1}{2}}\left(\frac{\partial f}{\partial \eta}\right)\frac{1}{2}\left(\frac{v_\infty}{\nu x}\right)^{\frac{1}{2}} = \frac{v_\infty}{2} f'(\eta)$$

The Blasius's solution for *FLAT plate*

$$\boxed{f''' + ff'' = 0}$$

A nonlinear ODE

$$f = f' = 0 \qquad \text{at } \eta = 0$$
$$f' = 2 \qquad \text{at } \eta = \infty$$

Blasius solved this ODE by power law expansion……

$\eta = \frac{y}{2}\sqrt{\frac{v_\infty}{\nu x}}$	f	f'	f''	$\frac{v_x}{v_\infty}$
0	0	0	1.32824	0
0.2	0.0266	0.2655	1.3260	0.1328
0.4	0.1061	0.5294	1.3096	0.2647
0.6	0.2380	0.7876	1.2664	0.3938
0.8	0.4203	1.0336	1.1867	0.5168
1.0	0.6500	1.2596	1.0670	0.6298
1.2	0.9223	1.4580	0.9124	0.7290
1.4	1.2310	1.6230	0.7360	0.8115
1.6	1.5691	1.7522	0.5565	0.8761
1.8	1.9295	1.8466	0.3924	0.9233
2.0	2.3058	1.9110	0.2570	0.9555
2.2	2.6924	1.9518	0.1558	0.9759
2.4	3.0853	1.9756	0.0875	0.9878
2.6	3.4819	1.9885	0.0454	0.9943

Key findings:

(1) $v_x = 0.99 v_\infty$ at $\eta = 2.5$

(2) $f'' = 1.328$ at $\eta = 0$

The Blasius's solution for *FLAT plate*

Key findings:

(1) $v_x = 0.99 v_\infty$ at $\eta = 2.5$

(2) $f'' = 1.328$ at $\eta = 0$

(1) $\eta = 2.5 = \frac{\delta}{2}\left(\frac{v_\infty}{\nu x}\right)^{1/2}; \qquad \delta = 5\sqrt{\frac{\nu x}{v_\infty}} = \frac{5x}{\sqrt{Re_x}}$

(2) $\frac{\partial v_x}{\partial y} = \frac{\partial v_x}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{v_\infty}{2} f''(\eta)\frac{1}{2}\left(\frac{v_\infty}{\nu x}\right)^{\frac{1}{2}} \rightarrow \left.\frac{\partial v_x}{\partial y}\right|_{y=0} = \frac{v_\infty}{4}(1.328)\left(\frac{v_\infty}{\nu x}\right)^{\frac{1}{2}} = 0.332 v_\infty \frac{\sqrt{Re_x}}{x}$

$\rightarrow \tau = \mu \left.\frac{\partial v_x}{\partial y}\right|_{y=0} = 0.332 \mu v_\infty \frac{\sqrt{Re_x}}{x}$

(3) $C_{f,x} = \frac{\tau}{(\rho v_\infty^2/2)} = \frac{0.664}{\sqrt{Re_x}}$

"Local" coefficient of skin friction

(4)

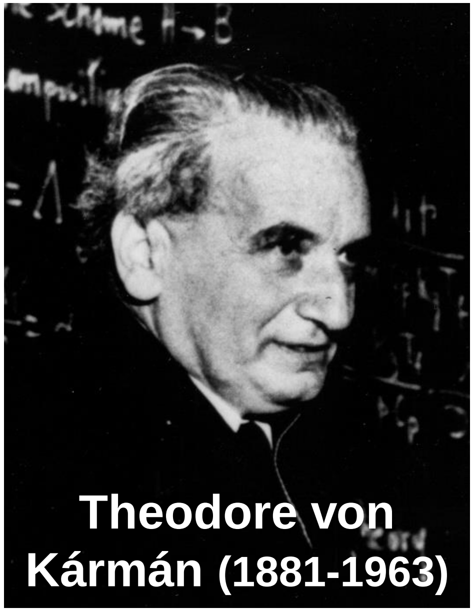
$$C_{fL} = \frac{1}{A}\int_A C_{fx}\, dA = \frac{1}{L}\int_0^L C_{fx}\, dx = \frac{1}{L}\int_0^L 0.664\sqrt{\frac{\nu}{v_\infty}}x^{-1/2}dx = 1.328\sqrt{\frac{\nu}{L v_\infty}}$$

(W = 1 here)

$$C_{fL} = \frac{1.328}{\sqrt{\mathrm{Re}_L}}$$

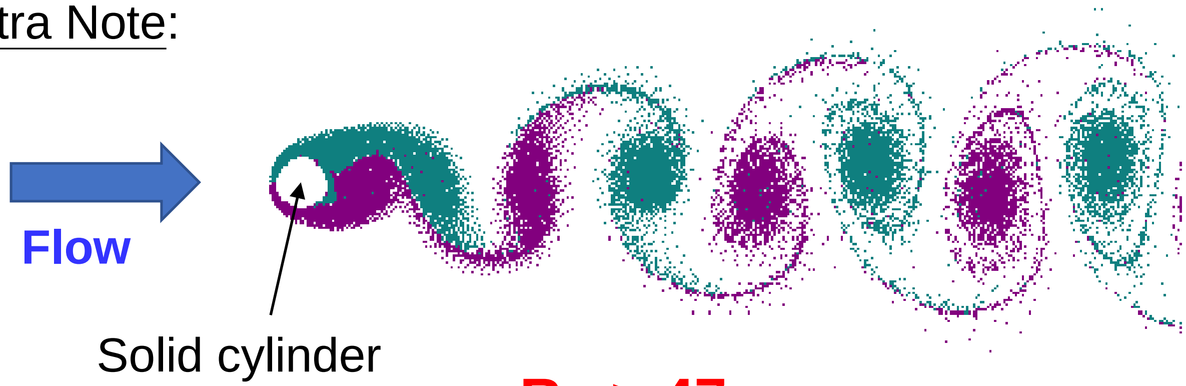
The von Kármán approximation

- The Blasius solution is obviously quite restrictive in application, applying only to the case of a laminar boundary layer over a **“flat plate”**.

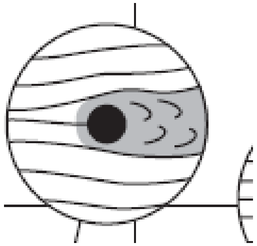


Theodore von Kármán (1881-1963)

Extra Note:



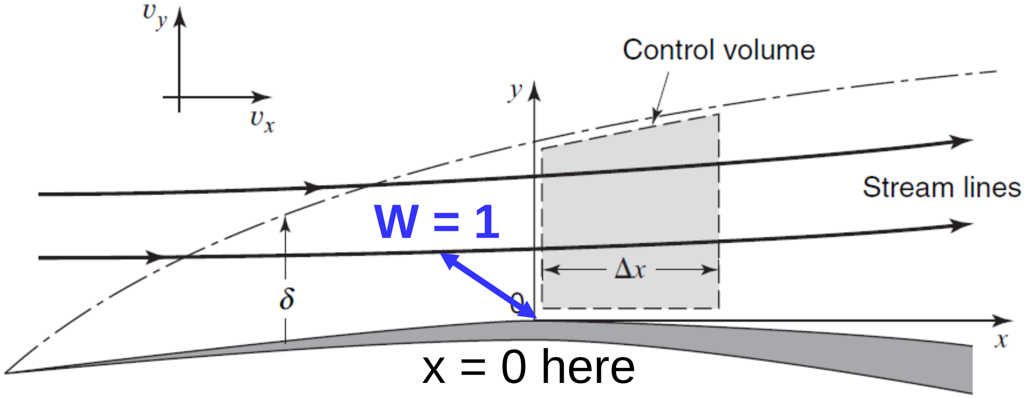
Re > 47



“von Kármán vortex”

- Another important contribution of von Kármán is the von Kármán’s approximation to boundary layer for any geometry.

The von Kármán approximation



(Pressure force) (Pressure force) (Friction force)

$$\sum F_x = P\delta|_x - P\delta|_{x+\Delta x} + \left(P|_x + \frac{P|_{x+\Delta x} - P|_x}{2}\right)(\delta|_{x+\Delta x} - \delta|_x) - \tau_0 \Delta x$$

And:

$$\sum F_x = \iint_{\text{c.s.}} \underbrace{v_x \rho (\mathbf{v} \cdot \mathbf{n})}_{\text{Convective momentum flux (output)}} dA + \cancel{\frac{\partial}{\partial t} \iiint_{\text{c.v.}} v_x \rho \, dV}$$

Steady-state

The von Kármán approximation

$$\iint_{\text{c.s.}} v_x \rho(\mathbf{v}\cdot\mathbf{n})dA = \int_0^\delta \rho v_x^2 dy\bigg|_{x+\Delta x} - \int_0^\delta \rho v_x^2 dy\bigg|_x - v_\infty \underline{\dot{m}_{\text{top}}}$$

mass-flow rate into the top of the control volume

$$\dot{m}_{\text{top}} = \int_0^\delta \rho v_x dy\bigg|_{x+\Delta x} - \int_0^\delta \rho v_x dy\bigg|_x$$

$$-(P\delta|_{x+\Delta x} - P\delta|_x) + \left(\frac{P|_{x+\Delta x} - P|_x}{2} + P|_x\right)(\delta|_{x+\Delta x} - \delta|_x) - \tau_0\,\Delta x$$

$$= \int_0^\delta \rho v_x^2\,dy\bigg|_{x+\Delta x} - \int_0^\delta \rho v_x^2\,dy\bigg|_x - v_\infty\left(\int_0^\delta \rho v_x\,dy\bigg|_{x+\Delta x} - \int_0^\delta \rho v_x\,dy\bigg|_x\right)$$

Rearrange & Divided by Δx

$$-\left(\frac{P|_{x+\Delta x} - P|_x}{\Delta x}\right)\delta|_{x+\Delta x} + \left(\frac{P|_{x+\Delta x} - P|_x}{2}\right)\left(\frac{\delta|_{x+\Delta x} - \delta|_x}{\Delta x}\right) + \left(\frac{P\delta|_x - P\delta|_x}{\Delta x}\right)$$

$$= \left(\frac{\int_0^\delta \rho v_x^2 dy|_{x+\Delta x} - \int_0^\delta \rho v_x^2\,dy|_x}{\Delta x}\right) - v_\infty\left(\frac{\int_0^\delta \rho v_x dy|_{x+\Delta x} - \int_0^\delta \rho v_x\,dy|_x}{\Delta x}\right) + \tau_0$$

Δx ~ 0

$$-\delta\frac{dP}{dx} = \tau_0 + \frac{d}{dx}\int_0^\delta \rho v_x^2\,dy - v_\infty\frac{d}{dx}\int_0^\delta \rho v_x\,dy$$

The von Kármán approximation

- The boundary-layer concept assumes inviscid flow outside the boundary layer (Bernoulli's equation):

$$\frac{dP}{dx} + \rho v_\infty \frac{dv_\infty}{dx} = 0$$

$$\left\{\begin{aligned} \frac{\delta}{\rho}\frac{dP}{dx} &= \frac{d}{dx}(\delta v_\infty^2) - v_\infty \frac{d}{dx}(\delta v_\infty) \\ -\delta \frac{dP}{dx} &= \tau_0 + \frac{d}{dx}\int_0^\delta \rho v_x^2\, dy - v_\infty \frac{d}{dx}\int_0^\delta \rho v_x\, dy \end{aligned}\right.$$

$$\boxed{\frac{\tau_0}{\rho} = \underline{\left(\frac{d}{dx} v_\infty\right) \int_0^\delta (v_\infty - v_x)\, dy} + \frac{d}{dx}\int_0^\delta v_x(v_\infty - v_x)\, dy}$$

von Kármán momentum integral equation

This term becomes zero when v_∞ is constant (ex: Flat plate)

The von Kármán approximation

$$\boxed{\frac{\tau_0}{\rho} = \left(\frac{d}{dx} v_\infty\right) \int_0^\delta (v_\infty - v_x)\, dy + \frac{d}{dx} \int_0^\delta v_x (v_\infty - v_x)\, dy}$$

$V_x(y)$ = ???

Example:

- Flat plate (Blasius's case)

$$\rightarrow \quad \frac{\mu}{\rho} \frac{dv_x}{dy}\bigg|_{y=0} = \frac{d}{dx} \int_0^\delta v_x (v_\infty - v_x) dy$$

- Guess: $\boxed{v_x = a + by + cy^2 + dy^3}$ $\rightarrow$ 4 B.C. are needed!

$$(1) \quad v_x = 0 \quad \text{at } y = 0$$

$$(2) \quad v_x = v_\infty \quad \text{at } y = \delta$$

$$(3) \quad \frac{\partial v_x}{\partial y} = 0 \quad \text{at } y = \delta$$

$$\cancel{\frac{\partial v_x}{\partial t}} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = v_\infty \cancel{\frac{dv_\infty}{dx}} + \nu \frac{\partial^2 v_x}{\partial y^2}$$

S.S At y = 0, $v_x = v_y = 0$

$$\rightarrow \quad (4) \quad \frac{\partial^2 v_x}{\partial y^2} = 0 \quad \text{at } y = 0$$

The von Kármán approximation

$$a = 0 \quad b = \frac{3}{2\delta} v_\infty \quad c = 0 \quad d = -\frac{v_\infty}{2\delta^3} \quad \rightarrow \quad \frac{v_x}{v_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

$$\frac{\mu}{\rho}\frac{dv_x}{dy}\bigg|_{y=0} = \frac{d}{dx}\int_0^\delta v_x(v_\infty - v_x)dy$$

$$\rightarrow \quad \frac{3\nu}{2}\frac{v_\infty}{\delta} = \frac{d}{dx}\int_0^\delta v_\infty^2 \left(\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right)\left(1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right) dy$$

$$\rightarrow \quad \frac{3}{2}\nu\frac{v_\infty}{\delta} = \frac{39}{280}\frac{d}{dx}(v_\infty^2\delta) \qquad \delta\, d\delta = \frac{140}{13}\frac{\nu\, dx}{v_\infty} \quad \rightarrow \quad \frac{\delta}{x} = \frac{4.64}{\sqrt{\mathrm{Re}_x}}$$

$$C_{fx} \equiv \frac{\tau_0}{\frac{1}{2}\rho v_\infty^2} = \frac{2\nu}{v_\infty^2}\frac{3}{2}\frac{v_\infty}{\delta} = \frac{0.646}{\sqrt{\mathrm{Re}_x}} \qquad C_{fL} = \frac{1.292}{\sqrt{\mathrm{Re}_L}}$$

The von Kármán approximation

For the case of a flat plate:

- Blasius's exact solution:

$$C_{fL} = \frac{1.328}{\sqrt{\mathrm{Re}_L}}$$

- von Kármán's approximation with the assmption of: $v_x = a + by + cy^2 + dy^3$

$$C_{fL} = \frac{1.292}{\sqrt{\mathrm{Re}_L}}$$

Error ~ 3%

v/v_∞	$\delta x^{-1}(\mathrm{Re})^{0.5}$	Error in C_f
Blasius's exact solution	5	-
y/δ	3.46	13%
$2y/\delta - y^2/\delta^2$	5.48	10%
$1.5y/\delta - 0.5y^3/\delta^3$	4.64	3%
$\sin(0.5\pi y/\delta)$	4.79	1.4%

- von Kármán's approximation can be used to estimate drag force for any geometry with a known $v_\infty(x)$.

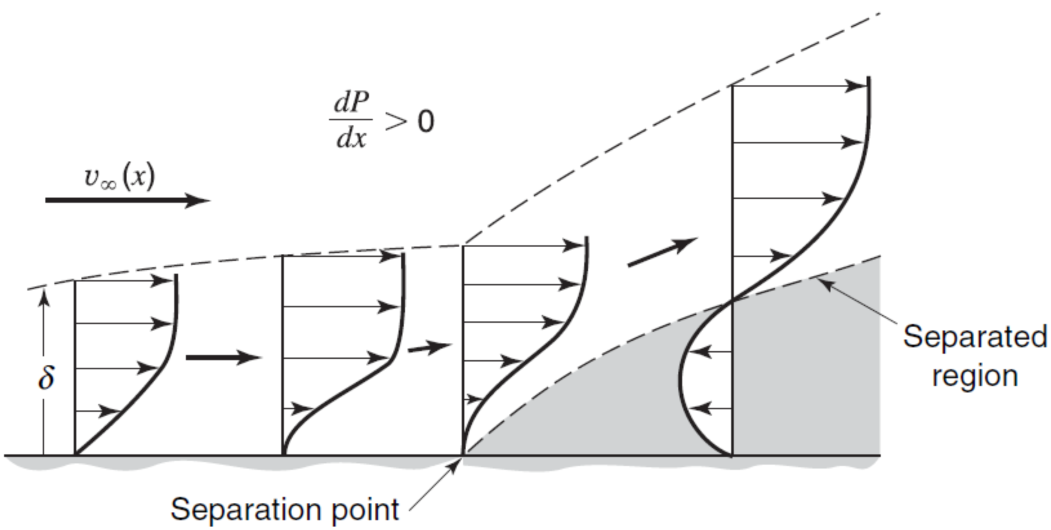
Reason for flow separation

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 v_x}{\partial y^2}$$

- Steady-state flow and **at y = 0** ($v_x = v_y = 0$):

$$\frac{dP}{dx} = \mu\frac{\partial}{\partial y}\left(\underline{\frac{\partial v_x}{\partial y}}\right)\Bigg|_{y=0}$$

Velocity gradient



"Adverse pressure gradient":

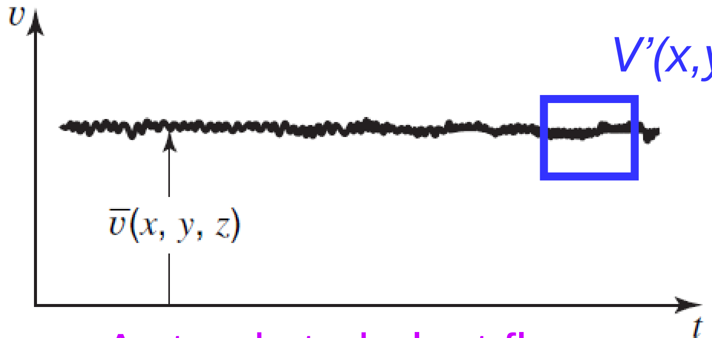
- Definition: **dP/dx > 0**
- Not always cause flow separation

Turbulent flow

- Turbulent flow is the most frequently encountered type of viscous flow, yet the analytical treatment of turbulent flow is not nearly well developed as that of laminar flow.

Several small random fluctuations:

V'(x,y,z,t)



A steady turbulent flow

$$v_x = \underline{\overline{v}_x(x, y, z)} + v_x'(x, y, z, t)$$

Time-averaged velocity

at the point (x, y, z)

- Defining by setting a long-enough time (t_1):

$$\overline{v}_x = \frac{1}{t_1}\int_0^{t_1} v_x(x, y, z, t)\,dt$$

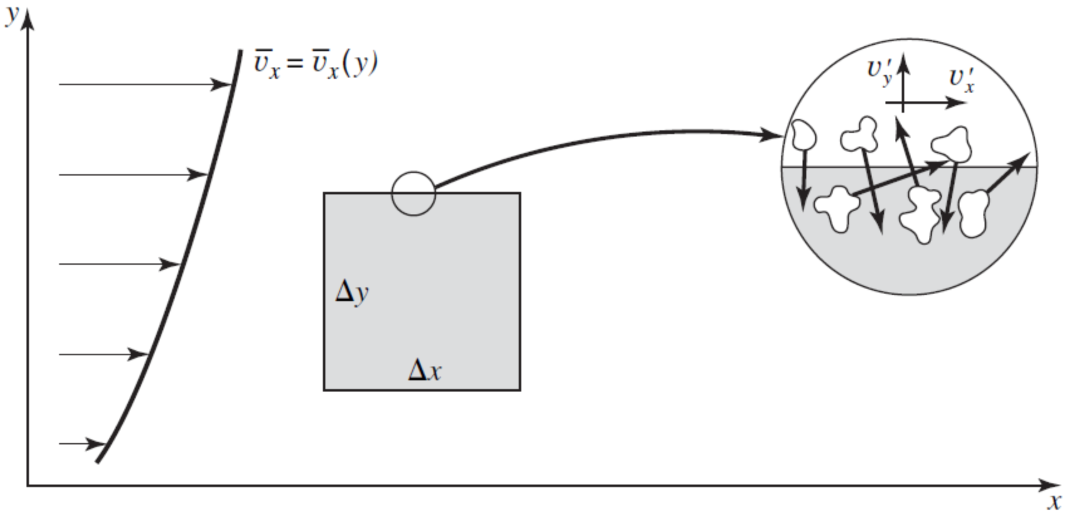
$$\overline{v}_x' = \frac{1}{t_1}\int_0^{t_1} v_x'(x, y, z, t)\,dt = 0$$

- **Intensity of turbulence (*I*)**:

$$\boxed{I \equiv \frac{\sqrt{\left(\overline{v_x'^2} + \overline{v_y'^2} + \overline{v_z'^2}\right)/3}}{v_\infty}}$$

v_∞: mean velocity of flow

Shear stress in turbulent flow



No analytical solution even for the simplest case!

$$\tau_{yx} = \underbrace{\mu \frac{d\bar{v}_x}{dy}}_{} - \underbrace{\overline{\rho v_x' v_y'}}_{}$$

Molecular (laminar) contribution

Turbulent contribution

"Reynolds stress"

- In turbulent flow, Reynolds stress is usually much larger than the molecular contribution except near the walls.
- The turbulent contribution is only related to the fluctuating properties of the flow (instead of the viscosity).

Experiments are required to measure them!

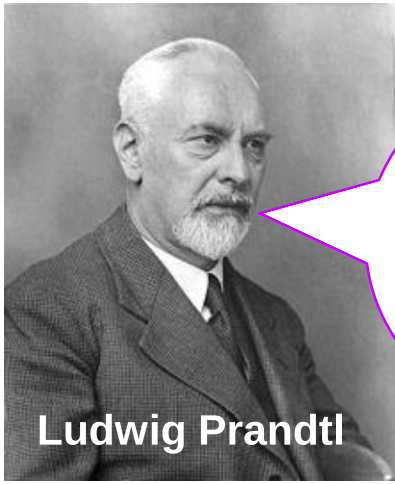
$$(\tau_{yx})_{\text{turb}} = A_t \frac{d\bar{v}_x}{dy}$$

$$\epsilon_M \equiv A_t/\rho.$$

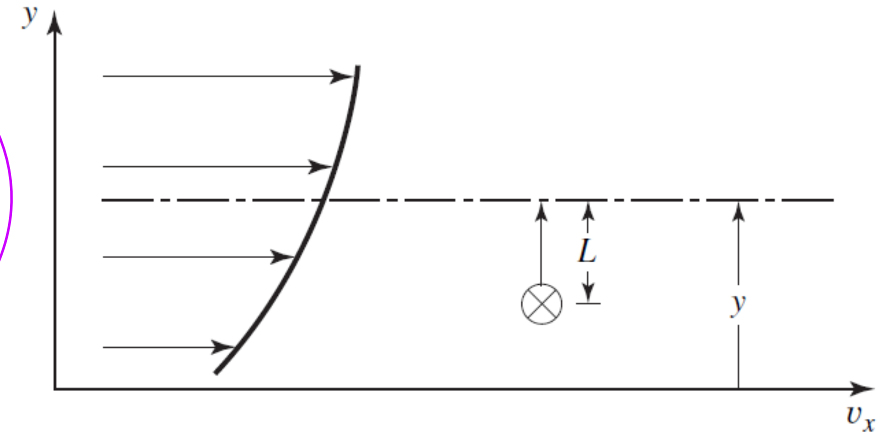
$(\tau_{yx})_{turb}$: Reynolds stress
A_t: Eddy viscosity

ϵ_M: Eddy diffusivity of momentum

The mixing length hypothesis



Let's consider the turbulent contribution as something like the gas molecules with mean free path.



$$\bar{v}_x\big|_{y\pm L} - \bar{v}_x\big|_y = \pm L\frac{d\bar{v}_x}{dy} \quad \rightarrow \quad v'_x = \pm L\frac{d\bar{v}_x}{dy}$$

- Prandtl assumed that v_x' must be proportional to v_y':

$$\overline{v'_x v'_y} = -(\text{constant})L^2\left|\frac{d\,\bar{v}_x}{dy}\right|\frac{d\,\bar{v}_x}{dy} \quad \longrightarrow \quad \overline{v'_x v'_y} = -L^2\left|\frac{d\,\bar{v}_x}{dy}\right|\frac{d\,\bar{v}_x}{dy}$$

- The mixing length is assumed to vary directly with y, and thus **L = Ky**:

$$(\tau_{yx})_{turb} = -\overline{\rho v_x{}' v_y{}'} = \rho K^2 y^2 \left(\frac{d\overline{v_x}}{dy}\right)^2$$ = a constant

The mixing length hypothesis

$$\frac{d\,\overline{v}_x}{dy} = \frac{\sqrt{\tau_0/\rho}}{Ky}$$

$$\rightarrow \quad \overline{v}_x = \frac{\sqrt{\tau_0/\rho}}{K}\ \text{In}\ y + C$$

<u>Assumptions</u>:

(1) Shear stress is only from the turbulent term.

(2) Shear stress is a constant (τ_0).

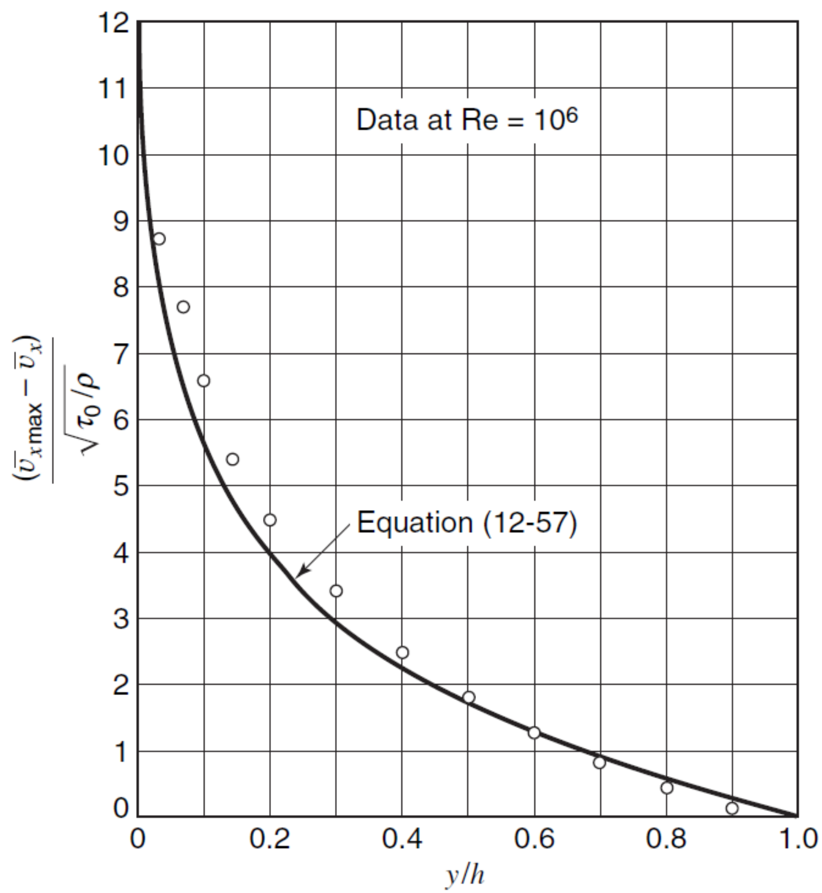
- By using the following B.C.:

$$\overline{v_x} = \overline{v_x}_{max}\ at\ y = h$$

$$\frac{\overline{v}_{x\,\max} - \overline{v}_x}{\sqrt{\tau_0/\rho}} = -\frac{1}{K}\left[\ \ln\frac{y}{h}\ \right]$$

(Eq. 12-57)

- Prandtl conducted experiments to find that K is around 0.4.



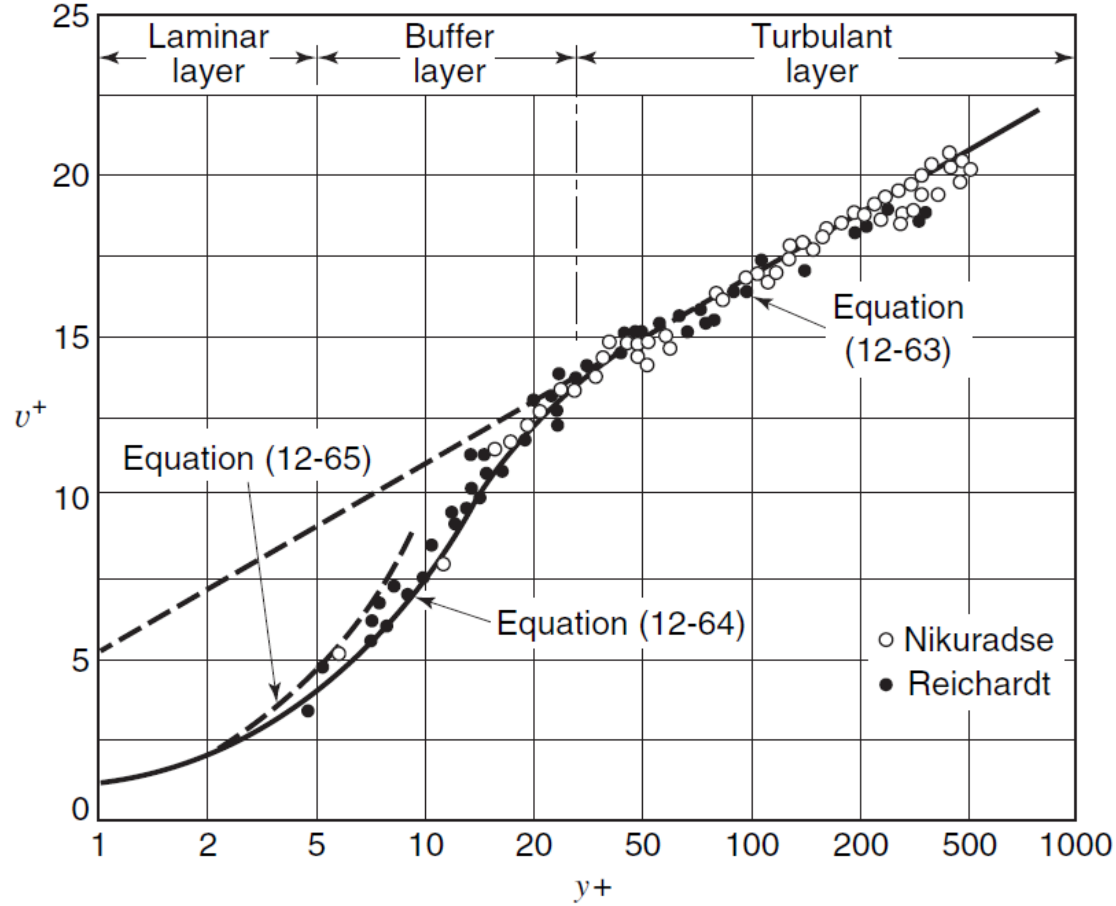
For turbulent flow in smooth tubes…

$$v^+ \equiv \frac{\bar{v}_x}{\sqrt{\tau_0/\rho}}$$

"Dimensionless velocity"

$$y^+ \equiv \frac{\sqrt{\tau_0/\rho}}{\nu} y$$

"pseudo-Reynolds number"



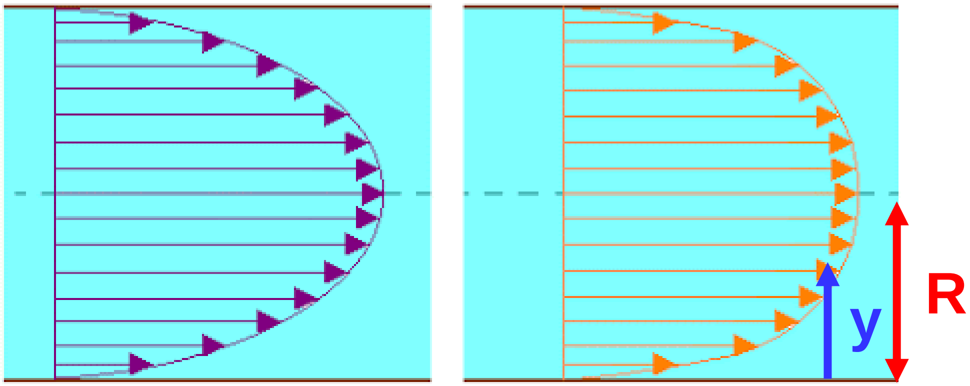
$$y^+ \geq 30 \qquad v^+ = 5.5 + 2.5 \ln y^+ \qquad (12\text{-}63)$$

$$30 \geq y^+ \geq 5 \qquad v^+ = -3.05 + 5 \ln y^+ \qquad (12\text{-}64)$$

$$5 > y^+ > 0 \qquad v^+ = y^+ \qquad (12\text{-}65)$$

The Prandtl's *one-seventh power law*

- For flow in *smooth circular tubes*:



Laminar flow **Turbulent flow**

Re~10^5

$$\boxed{\frac{\overline{v}_x}{\overline{v}_{x\,\max}} = \left(\frac{y}{R}\right)^{1/7}}$$

- For turbulent flow:

$$\frac{\overline{v}_x}{\overline{v}_{x\,\max}} = \left(\frac{y}{R}\right)^{1/n}$$

From experimental data:

Re=4,000, n~6

Re=3,200,000, n~10

- Useful for turbulent boundary layer:

$$\frac{\overline{v}_x}{\overline{v}_{x\,\max}} = \left(\frac{y}{\delta}\right)^{1/n}$$

The Prandtl's *one-seventh power law*

Example: For a turbulent flow in a smooth circular pipe at Re=100,000, find the relationship between the average velocity (v_{avg}) and v_{max}.

$$\boxed{\frac{\overline{v}_x}{\overline{v}_{x\,\max}} = \left(\frac{y}{R}\right)^{1/7}}$$

Solution:

$$v_{avg} = \frac{\int_0^R \int_0^{2\pi} vr\, d\theta dr}{\int_0^R \int_0^{2\pi} r\, d\theta dr} = \frac{\int_0^R 2\pi vr\, dr}{\pi R^2} = \frac{\int_0^R 2\pi v_{max}\left(\frac{y}{R}\right)^{\frac{1}{7}} r\, dr}{\pi R^2}$$

$$= \frac{2v_{max}}{R^2}\int_0^R \left(\frac{y}{R}\right)^{\frac{1}{7}} (R-y)\, d(R-y)$$

$$= \frac{2v_{max}}{R^2}\int_{y=R}^{y=0} R^{6/7}y^{1/7} - R^{-1/7}y^{8/7}(-dy) = \frac{2v_{max}}{R^2}\left[\frac{7}{8}R^2 - \frac{7}{15}R^2\right]$$

$$= \frac{49}{60}v_{max}$$

The Blasius's correlation for shear stress

- Another useful relation is Blasius's correlation for shear stress:

For turbulent flow in:

Pipe with Re < 10^5

Flat plate with Re < 10^7

$$\tau_0 = 0.0225 \rho \overline{v}_{x\max}^2 \left(\frac{\nu}{\overline{v}_{x\max}\, y_{\max}} \right)^{1/4}$$

Wall shear stress

Pipe: $y_{max} = R$

Flat plate: $y_{max} = \delta$

- In turbulent boundary layer (**Smooth plate** & **Re < 10^7**):

$$\tau_0 = 0.0225 \rho \overline{v}_{x\max}^2 \left(\frac{\nu}{\overline{v}_{x\max}\, \delta} \right)^{1/4}$$

$$\frac{\overline{v}_x}{\overline{v}_{x\max}} = \left(\frac{y}{\delta} \right)^{1/7}$$

Now we can utilize these two correlations in the von Kármán integral equation:

$$\frac{\tau_0}{\rho} = \left(\frac{d}{dx} v_\infty \right) \int_0^\delta (v_\infty - v_x)\, dy + \frac{d}{dx} \int_0^\delta v_x (v_\infty - v_x)\, dy$$

The turbulent boundary layer

$$\boxed{\frac{\tau_0}{\rho} = \left(\frac{d}{dx} v_\infty\right) \int_0^\delta (v_\infty - v_x)\, dy + \frac{d}{dx} \int_0^\delta v_x (v_\infty - v_x)\, dy}$$

$$\frac{0.0225 \rho v_\infty^2 (\frac{\nu}{v_\infty \delta})^{1/4}}{\rho} = \frac{d}{dx} \int_0^\delta \left(v_\infty (\frac{y}{\delta})^{1/7}\right)\left(v_\infty - (v_\infty (\frac{y}{\delta})^{1/7})\right) dy$$

$$0.0225 (\frac{\nu}{v_\infty \delta})^{1/4} = \frac{d}{dx} \int_0^\delta \left((\frac{y}{\delta})^{1/7} - (\frac{y}{\delta})^{2/7}\right) dy = \frac{d}{dx}\left(\frac{7\delta}{8} - \frac{7\delta}{9}\right) = \frac{7}{72}\frac{d\delta}{dx}$$

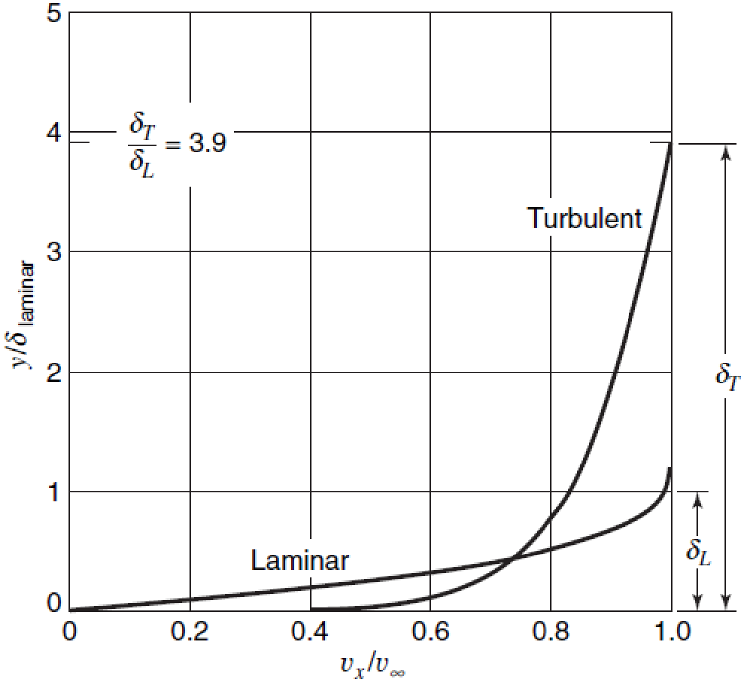
$$(\frac{\nu}{v_\infty})^{1/4} dx = 4.32 \delta^{1/4} d\delta \quad \rightarrow \quad (\frac{\nu}{v_\infty})^{1/4} x = 3.45 \delta^{5/4} + C$$

- If the boundary layer is assumed to be turbulent from x=0 (a poor assumption):

$$(\frac{\nu}{v_\infty})^{1/4} x = 3.45 \delta^{5/4} \qquad (\frac{\nu}{v_\infty})^{1/4} x^{-1/4} = 3.45 (\frac{\delta}{x})^{5/4} \qquad 0.376\, (Re_x)^{-1/5} = \frac{\delta}{x}$$

$$\rightarrow \quad C_{fx} = \frac{\tau_0}{0.5 \rho v_\infty^2} = 0.045 (\frac{\nu {Re_x}^{1/5}}{v_\infty (0.376 x)})^{1/4} = \frac{0.0576}{{Re_x}^{1/5}}$$

The turbulent boundary layer



- Turbulent boundary layer has a larger mean velocity, which <u>resists flow separation better</u> than the laminar boundary layer at an adverse (*unfavorable*) pressure gradient.

(Good for most engineering interests)

Table 12.2 Factors affecting the Reynolds number of transition from laminar to turbulent flow

Factor	Influence
Pressure gradient	Favorable pressure gradient retards transition; unfavorable pressure gradient hastens it
Free-stream turbulence	Free-stream turbulence decreases transition Reynolds number
Roughness	No effect in pipes; decreases transition in external flow
Suction	Suction greatly increases transition Re
Wall curvatures	Convex curvature increases transition Re. Concave curvature decreases it
Wall temperature	Cool walls increase transition Re. Hot walls decrease it