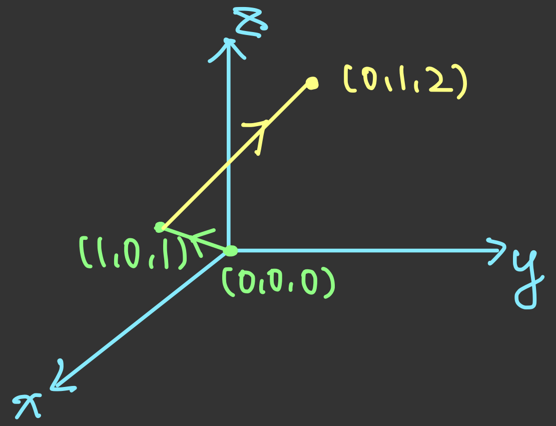
2021.6.11 助教課內容

Evaluate $\int_C (y+z)dx + (x+z)dy + (x+y)dz$

where C consists of line segments from $(0,0,0)$ to $(1,0,1)$ and from $(1,0,1)$ to $(0,1,2)$



$C = C_1 \cup C_2$

$C_1 : r_1(t) = \langle t, 0, t \rangle,\ t \in [0,1]$

$C_2 : r_2(t) = \langle 1-t, t, 1+t \rangle,\ t \in [0,1]$

$$\int_{C_1} (y+z)dx + (x+z)dy + (x+y)dz = \int_0^1 (t \cdot 1 + 2t \cdot 0 + t \cdot 1)\,dt = t^2\Big|_0^1 = 1$$

$x=t,\ y=0,\ z=t,\ dx=dt,\ dy=0,\ dz=dt$

$$\int_{C_2} (y+z)dx + (x+z)dy + (x+y)dz = \int_0^1 \big((1+2t)\cdot(-1) + 2 \cdot 1 + 1 \cdot 1\big)dt = (2t - t^2)\Big|_0^1 = 1$$

$x=1-t,\ y=t,\ z=1+t,\ dx=-dt,\ dy=dt,\ dz=dt$

$$\int_C (y+z)dx + (x+z)dy + (x+y)dz = \int_{C_1} + \int_{C_2} = 1 + 1 = 2$$

Let $\vec{F}(x,y) = \langle \sin y + e^x,\ x\cos y \rangle$, $C: x = t,\ y = t(3-t),\ 0 \le t \le 3$

(a) Show that $\vec{F}$ is conservative and find a potential function f

$P(x,y) = \sin y + e^x$, $Q(x,y) = x\cos y$, $\frac{\partial P}{\partial y} = \cos y = \frac{\partial Q}{\partial x}$, $\vec{F}$ is conservative.

To find a function f such that $\vec{F} = \nabla f$,

Solve $\begin{cases} f_x = \sin y + e^x \\ f_y = x\cos y \end{cases}$

contant for x

$f = \int f_x \, dx = x\sin y + e^x + g(y)$

$x\cos y = f_y = \frac{\partial}{\partial y}(x\cos y + e^x + g(y)) = x\cos y + g'(y)$

$g'(y) = 0$ imply $g(y)$ is a constant function.

Therefore, $f(x,y) = x\sin y + e^x + C$ are potential functions.

(b) Evaluate $\int \vec{F} \cdot d\vec{r}$ by using potential function

$\int_C \vec{F} \cdot d\vec{r} = \int \nabla f \cdot d\vec{r} = f(\vec{r}(3)) - f(\vec{r}(0)) = f(3,0) - f(0,0) = e^3 - 1$

$\vec{F}(x,y,z) = \langle y^2z + 2xz^2,\ 2xyz,\ xy^2 + 2x^2z \rangle$, where $C: x=\sqrt{t},\ y=t+1,\ z=t^2,\ 0\le t\le 1$

(a) Find a function f such that $\vec{F} = \nabla f$

Solve $\begin{cases} f_x = y^2z + 2xz^2 \\ f_y = 2xyz \\ f_z = xy^2 + 2x^2z \end{cases}$

constant for x

From $f_x = y^2z + 2xz^2$, $f(x,y,z) = xy^2z + x^2z^2 + h(y,z)$

$2xyz = \frac{\partial}{\partial y}\left(xy^2z + x^2z^2 + h(y,z)\right)$

$= 2xyz + h_y(y,z)$ constant for y

$h_y(y,z) = 0 \Rightarrow h(y,z) = g(z)$

$xy^2 + 2x^2z = \frac{\partial}{\partial z}\left(xy^2z + x^2z^2 + g(z)\right)$

$= xy^2 + 2x^2z + g'(z)$

$g'(z) = 0 \Rightarrow g(z) = C$, C is a constant.

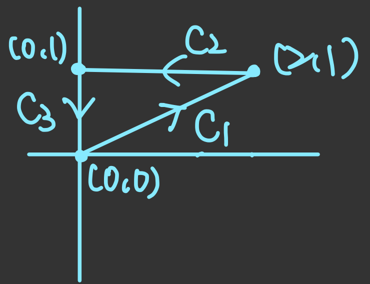
Therefore, $f(x,y,z) = xy^2z + x^2z^2 + C$ are functions such that $\vec{F} = \nabla f$

(b) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by using the function in (a).

$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) = f(1,2,1) - f(0,1,0) = 5$

Evaluate $\int_C (x^2+y^2)dx+(x^2-y^2)dy$, C is the triangle with vertices $(0,0),(2,1),(0,1)$. counterclockwisely.

Method I. Directly compute the line integral



$C_1: r_1(t)=\langle 2t,t\rangle,\ t\in[0,1]$

$C_2: r_2(t)=\langle 2-t,1\rangle,\ t\in[0,2]$

$C_3: r_3(t)=\langle 0,1-t\rangle,\ t\in[0,1]$

$$\int_{C_1}(x^2+y^2)dx+(x^2-y^2)dy=\int_0^1((4t^2+t^2)\cdot 2+(4t^2-t^2)\cdot 1)dt=\int_0^1 13t^2dt=\frac{13}{3}$$

$$\int_{C_2}(x^2+y^2)dx+(x^2-y^2)dy=\int_0^2((t^2-4t+5)\cdot(-1)+(t^2-4t+3)\cdot 0)dt$$

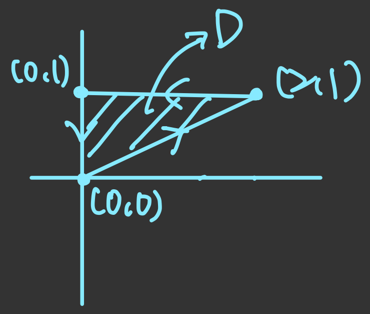
$$=\int_0^2(-t^2+4t-5)dt=(-\frac{1}{3}t^3+2t^2-5t)\Big|_0^2=-\frac{14}{3}$$

$$\int_{C_3}(x^2+y^2)dx+(x^2-y^2)dy=\int_0^1((1-t)^2\cdot 0+(-(1-t)^2)\cdot(-1))dt$$

$$=\int_0^1(t^2-2t+1)dt=(\frac{1}{3}t^3-t^2+t)\Big|_0^1=\frac{1}{3}$$

$$\int_C(x^2+y^2)dx+(x^2-y^2)dy=\frac{13}{3}-\frac{14}{3}+\frac{1}{3}=0$$

Method 2, Use Green's theorem



$P(x,y)=x^2+y^2,\ Q(x,y)=x^2-y^2$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=\frac{\partial}{\partial x}(x^2-y^2)-\frac{\partial}{\partial y}(x^2+y^2)=2x-2y$$

$D=\{(x,y)\mid 0\le x\le 2y,\ 0\le y\le 1\}$

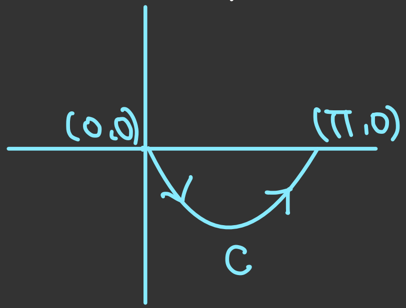
$$\int_C P\,dx+Q\,dy \overset{\text{Green's}}{=} \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$

$$=\int_0^1\int_0^{2y}(2x-2y)\,dx\,dy$$

$$=\int_0^1 (x^2-2xy)\Big|_{x=0}^{x=2y}\,dy$$

$$=\int_0^1 0\,dy=0$$

Let $\vec{F}(x,y)=\langle e^{-x}+y^2, e^{-y^3}+x^2\rangle$, C is the arc of the curve $y=-\sin x$ from $(0,0)$ to $(0,\pi)$



$C=\vec{r}(\theta)=\langle\theta,-\sin\theta\rangle,\ \theta\in[0,\pi]$

$\int_C \vec{F}\cdot d\vec{r}=\int_0^\pi \vec{F}(\vec{r}(\theta))\cdot\vec{r}'(\theta)\,d\theta$

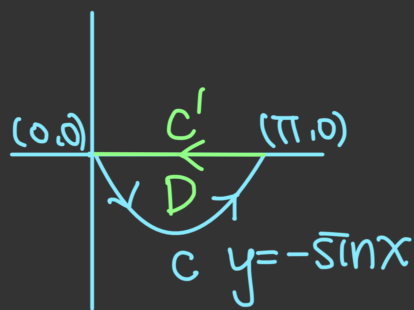
$=\int_0^\pi\langle e^{-\theta}+\sin^2\theta,\ e^{-\sin^3\theta}+\theta^2\rangle\cdot\langle 1,-\cos\theta\rangle\,d\theta$

$=\int_0^\pi\left(e^{-\theta}+\sin^2\theta\underbrace{-\cos\theta\, e^{-\sin^3\theta}}_{\text{不能計算}}-\theta\cos\theta\right)d\theta$

因為 $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=2x-2y$，或許使用 Green's theorem 可解

但 Green's theorem 要求封閉曲線. 我們自己增加一段曲線使原曲線封閉

Let $C'=r_1(t)=\langle\pi-t,0\rangle,\ t\in[0,\pi]$



$$\int_{C\cup C'} \vec{F}\cdot d\vec{r} \underset{\uparrow}{=} \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA = \iint_D (2x-2y)\,dA$$

Greens theorem

$$\int_{C'} \vec{F}\cdot d\vec{r} = \int_0^{\pi} \vec{F}(\vec{r}_1(t))\cdot \vec{r}_1'(t)\,dt = \int_0^{\pi} \langle e^{-(\pi-t)},\ 1+(\pi-t)^2\rangle\cdot\langle -1,0\rangle\,dt$$

$$= \int_0^{\pi} -e^{t-\pi}\,dt = -e^{-\pi}\int_0^{\pi} e^{t}\,dt = -e^{-\pi}(e^{\pi}-1) = 1+e^{-\pi}$$

$$\iint_D (2x-2y)\,dA = \int_0^{\pi}\int_{-\sin x}^{0} (2x-2y)\,dy\,dx$$

$$= \int_0^{\pi} (2xy-y^2)\Big|_{-\sin x}^{0}\,dx$$

$$= \int_0^{\pi} (2x\sin x+\sin^2 x)\,dx$$

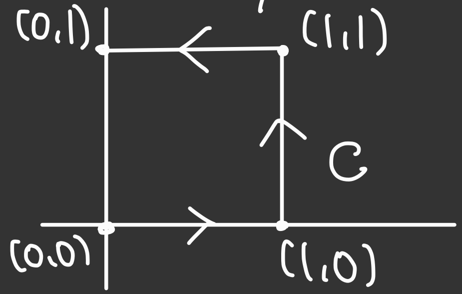
$$\int_0^{\pi} \underset{u}{\underline{2x}}\,\underset{dv}{\underline{\sin x\,dx}} = 2x(-\cos x)\Big|_0^{\pi} - \int_0^{\pi}(-\cos x)\cdot 2\,dx = 2\pi + 2\sin x\Big|_0^{\pi} = 2\pi$$

$$\int_0^{\pi} \sin^2 x\,dx = \int_0^{\pi} \frac{1-\cos 2x}{2}\,dx = \left(\frac{x}{2}-\frac{\sin 2x}{4}\right)\Big|_0^{\pi} = \frac{\pi}{2}$$

Therefore $\int_C \vec{F}\cdot d\vec{r} = \iint_D (2x-2y)\,dA - \int_{C'} \vec{F}\cdot d\vec{r}$

$$= \frac{5}{2}\pi - (1+e^{-\pi})$$

Let $\vec{F}(x,y) = \langle \sin y\, e^{x\sin y}, x\cos y\, e^{x\sin y} + x^2 \rangle$



Evaluate $\int_C \vec{F}\cdot d\vec{r}$

$$\frac{\partial P}{\partial y} = \cos y\, e^{x\sin y} + x\sin y\cos y\, e^{x\sin y}$$

$$\frac{\partial Q}{\partial x} = \cos y\, e^{x\sin y} + x\sin y\cos y\, e^{x\sin y} + 2x$$

$\vec{F}$ is not conservative.

But $\vec{F}_1(x,y) = \langle \sin y\, e^{x\sin y}, x\cos y\, e^{x\sin y} \rangle$ is conservative.

Let $F_2(x,y) = \langle 0, x^2 \rangle$, $\vec{F} = \vec{F}_1 + \vec{F}_2$, $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$

Since $\vec{F}_1$ is conservative and $f(x,y) = e^{x\sin y}$ is potential for $\vec{F}_1(x,y)$,

$\int_C \vec{F}_1\cdot d\vec{r} = f(0,1) - f(0,0) = 0$

$C = C_1 \cup C_2 \cup C_3$ where $C_1 : r_1(t) = \langle t, 0 \rangle,\ t \in [0,1]$

$C_2 : r_2(t) = \langle 1, t \rangle,\ t \in [0,1]$

$C_3 : r_3(t) = \langle 1-t, 1 \rangle,\ t \in [0,1]$

$$\int_C \vec{F_2} \cdot d\vec{r} = \int_{C_1} \vec{F_2} \cdot d\vec{r} + \int_{C_2} \vec{F_2} \cdot d\vec{r} + \int_{C_3} \vec{F_2} \cdot d\vec{r}$$

$$= \int_0^1 t^2 \cdot 0\, dt + \int_0^1 1 \cdot dt + \int_0^1 (1-t)^2 \cdot 0\, dt$$

$$= 1$$

Therefore, $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F_1} \cdot d\vec{r} + \int_C \vec{F_2} \cdot d\vec{r} = 0 + 1 = 1$

(a) If C is a line segment from (x_1,y_1) to (x_2,y_2), evaluate $\int_C x\,dy-y\,dx$

$C=r(t)=\langle x_1+t(x_2-x_1),\, y_1+t(y_2-y_1)\rangle,\ t\in[0,1]$

$$\int_C x\,dy-y\,dx=\int_0^1[(x_1+t(x_2-x_1))\cdot(y_2-y_1)-(y_1+t(y_2-y_1))\cdot(x_2-x_1)]dt$$

$$=\int_0^1[x_1(y_2-y_1)+\cancel{t(x_2-x_1)(y_2-y_1)}-y_1(x_2-x_1)-\cancel{t(x_2-x_1)(y_2-y_1)}]dt$$

$$=x_1y_2-\cancel{x_1y_1}-y_1x_2+\cancel{x_1y_1}$$

$$=x_1y_2-y_1x_2$$

(b) If the vertices of polygon, in counterclockwise order, are $(x_1,y_1),\dots,(x_n,y_n)$. Evaluate the area of the polygon.

$$\iint_D 1\,dA \underset{\text{Green's}}{=} \frac{1}{2}\int_C x\,dy-y\,dx$$

Set $C=C_1\cup\cdots\cup C_n$ where C_i is line segment from (x_i,y_i) to (x_{i+1},y_{i+1}) for $i=1,\dots,n-1$ and C_n is the line segment form (x_n,y_n) to (x_1,y_1)

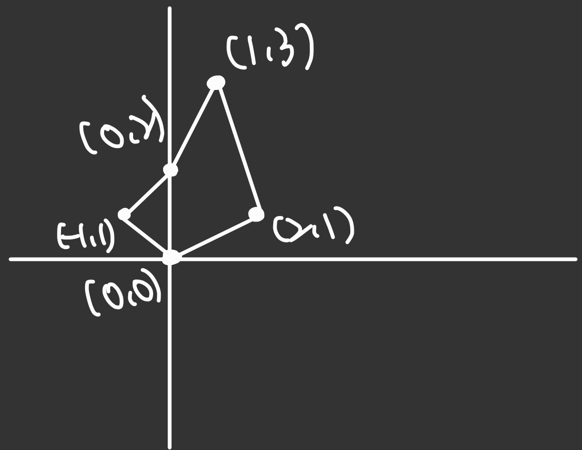
$$\int_C x\,dy - y\,dx = \sum_{i=1}^{n} \int_{C_i} x\,dy - y\,dx$$

$$= \sum_{i=1}^{n-1} (x_i y_{i+1} - x_{i+1} y_i) + (x_n y_1 - x_1 y_n)$$

Therefore the area of the polygon is

$$\frac{1}{2}[(x_1y_2 - x_2y_1) + \dots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 + x_1y_n)]$$

(c) Evaluate the area of



Let $(x_1,y_1)=(0,0)$, $(x_2,y_2)=(2,1)$, $(x_3,y_3)=(1,3)$
$(x_4,y_4)=(0,2)$, $(x_5,y_5)=(-1,1)$

$$\text{Area} = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - y_3x_4)$$
$$+ (x_4y_5 - x_5y_4) + (x_5y_1 - y_5x_1)]$$
$$= \frac{1}{2}[0 + 5 + 2 + 2] = \frac{9}{2}$$

$$\vec{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

(a) Show that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$$\frac{\partial P}{\partial y} = \frac{(-1)(x^2+y^2)-(-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}, \quad \frac{\partial Q}{\partial x} = \frac{1\cdot(x^2+y^2)-x\cdot(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

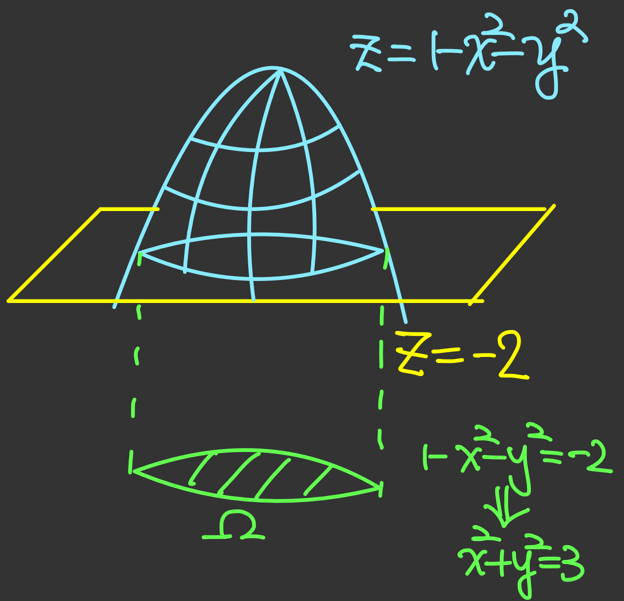
(b) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $C: r(\theta) = \langle \cos\theta, \sin\theta \rangle, \theta \in [0, 2\pi]$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle -\sin\theta, \cos\theta \rangle \cdot \langle -\sin\theta, \cos\theta \rangle \, d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi$$

Note that the result does not contradict to Green's theorem,

domain of $\vec{F}(x,y)$ is not whole $\mathbb{R}^2$.

Find the area of the surface of the part of the paraboloid $z=1-x^2-y^2$ that lies above the plane $z=-2$.



Consider $r(u,v)=\langle u,v,1-u^2-v^2\rangle$ is a parametrization for the surface

$$\frac{\partial r}{\partial u}=\langle 1,0,-2u\rangle,\quad \frac{\partial r}{\partial v}=\langle 0,1,-2v\rangle$$

$$\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix}=\langle 2u,2v,1\rangle$$

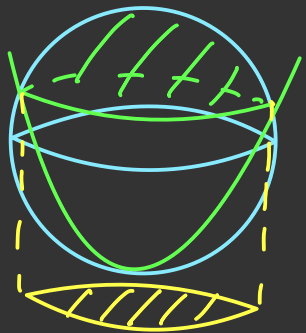
$$\left|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right|=\sqrt{4u^2+4v^2+1}$$

$$\text{Area}=\iint_\Omega \left|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right| du\,dv$$

$$=\int_0^{2\pi}\int_0^{\sqrt{3}}\sqrt{4r^2+1}\cdot r\,dr\,d\theta$$

$$=2\pi\cdot\frac{1}{12}(4r^2+1)^{\frac{3}{2}}\Big|_0^{\sqrt{3}}=\frac{\pi}{6}\left(13^{\frac{3}{2}}-1\right)$$

Find the area of the surface of the part of the sphere $x^2+y^2+z^2=4z$ that lies inside the paraboloid $z=x^2+y^2$



$\Omega: x^2+y^2\le 3$

$$x^2+y^2+z^2=4z \Rightarrow x^2+y^2+(x^2+y^2)^2=4(x^2+y^2) \Rightarrow (x^2+y^2)^2-3(x^2+y^2)=0$$

$$\Rightarrow (x^2+y^2)(x^2+y^2-3)=0$$

$$x^2+y^2+(z-2)^2=4$$
$$\Downarrow$$
$$z=2+\sqrt{4-x^2-y^2}$$

Consider $r(u,v)=\langle u, v, 2+\sqrt{4-u^2-v^2}\rangle$

$$r_u=\langle 1, 0, (4-u^2-v^2)^{-\frac{1}{2}}\cdot\tfrac{1}{2}\cdot(-2u)\rangle$$

$$r_v=\langle 0, 1, (4-u^2-v^2)^{-\frac{1}{2}}\cdot\tfrac{1}{2}\cdot(-2v)\rangle$$

$$r_u\times r_v=\begin{vmatrix} i & j & k \\ 1 & 0 & -u(4-u^2-v^2)^{-\frac{1}{2}} \\ 0 & 1 & -v(4-u^2-v^2)^{-\frac{1}{2}} \end{vmatrix}=\left\langle \frac{u}{\sqrt{4-u^2-v^2}}, \frac{v}{\sqrt{4-u^2-v^2}}, 1\right\rangle$$

$$|r_u\times r_v|=\sqrt{\frac{u^2}{4-u^2-v^2}+\frac{v^2}{4-u^2-v^2}+1}=\frac{2}{\sqrt{4-u^2-v^2}}$$

$$\text{Area}=\iint_\Omega \frac{2}{\sqrt{4-u^2-v^2}}\cdot du\,dv$$

$$=\int_0^{2\pi}\int_0^{\sqrt{3}}\frac{2r}{\sqrt{4-r^2}}\,dr\,d\theta$$

$$=\int_0^{2\pi}-2(4-r^2)^{\frac{1}{2}}\Big|_{r=0}^{r=\sqrt{3}}\,d\theta$$

$$=\int_0^{2\pi}2\,d\theta=2\theta\Big|_0^{2\pi}=4\pi$$