1. (28 points) Compute the following limit with **details**. If the limit doesn't exist, explain why.

(a) $\lim\limits_{h\to 0} \dfrac{(3+h)^3-27}{4h}$.

**Solution:**

$$\lim_{h\to 0} \frac{(3+h)^3-27}{4h} = \lim_{h\to 0} \frac{27+27h+9h^2+h^3-27}{4h} = \lim_{h\to 0} \frac{27+9h+h^2}{4} = \frac{27}{4}$$

(b) $\lim\limits_{x\to \frac{\pi}{4}} \sqrt{\sin^4 x + \frac{3}{4}}$.

**Solution:** Since $\sqrt{x}, \sin x, x^4$ are continuous on its domain respectively, we have

$$\lim_{x\to \frac{\pi}{4}} \sqrt{\sin^4 x + \frac{3}{4}} = \sqrt{\lim_{x\to \frac{\pi}{4}} (\sin^4 x + \frac{3}{4})} = \sqrt{(\lim_{x\to \frac{\pi}{4}} \sin x)^4 + \frac{3}{4}} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

(c) $\lim\limits_{x\to 2} \dfrac{2-x}{\sqrt{x+2}-2}$.

**Solution:**

$$\lim_{x\to 2} \frac{2-x}{\sqrt{x+2}-2} = \lim_{x\to 2} \frac{(2-x)(\sqrt{x+2}+2)}{(\sqrt{x+2}-2)(\sqrt{x+2}+2)} = \lim_{x\to 2} -(\sqrt{x+2}+2) = -4$$

(d) $\lim\limits_{x\to 0^+} (x+1)\cos\left(\dfrac{1}{x}\right)$.

**Solution:** If $\lim\limits_{x\to 0^+} (x+1)\cos\left(\dfrac{1}{x}\right)$ exists and we say $\lim\limits_{x\to 0^+} (x+1)\cos\left(\dfrac{1}{x}\right) = L$, then

$$\lim_{x\to 0^+} \cos\left(\frac{1}{x}\right) = \lim_{x\to 0^+} \left((x+1)\cos\left(\frac{1}{x}\right)\cdot \frac{1}{x+1}\right) = \lim_{x\to 0^+} (x+1)\cos\left(\frac{1}{x}\right) \cdot \lim_{x\to 0^+} \frac{1}{x+1} = L$$

But since $\lim\limits_{x\to 0^+} \cos\left(\dfrac{1}{x}\right)$ does not exist, there is a contradiction and $\lim\limits_{x\to 0^+} (x+1)\cos\left(\dfrac{1}{x}\right)$ does not exist.

2. (28 points) Differentiate the following functions.

(a) $f(x) = \dfrac{x^2+4x+3}{\sqrt{x}}$.

**Solution:** Write $f(x) = x^{3/2} + 4x^{1/2} + 3x^{-1/2}$ and $f'(x) = \dfrac{3}{2}x^{1/2} + 2x^{-1/2} - \dfrac{3}{2}x^{-3/2}$.

(b) $f(x) = x^2 + \cot x$.

**Solution:** $f'(x) = 2x - \csc^2 x$

(c) $f(x) = e^{\theta}(\tan\theta - \theta)$.

**Solution:** $f'(x) = e^\theta(\tan\theta - \theta) + e^\theta(\sec^2\theta - 1)$

(d) $f(x) = x5^x$.

**Solution:** $f'(x) = 5^x + x5^x \ln 5$

3. (20 points) (a) Suppose that $f(3) = 7$ $f'(3) = 4$. Find the value of

$$\frac{d}{dx}\sqrt{3+4f^2(x)}\bigg|_{x=3}.$$

**Solution:** Note that

$$\frac{d}{dx}\sqrt{3+4f^2(x)} = \frac{1}{2}(3+4f^2(x))^{-1/2}\cdot 8f(x)\cdot f'(x)$$

and

$$\frac{d}{dx}\sqrt{3+4f^2(x)}\bigg|_{x=3} = \frac{1}{2}(3+4f^2(3))^{-1/2}\cdot 8f(3)\cdot f'(3) = \frac{1}{2\sqrt{199}}\cdot 56\cdot 4 = \frac{112}{\sqrt{199}}.$$

(b) Let $f(0) = 0$, $f'(0) = 2$. Find the value of

$$\frac{d}{dx}[f(3f(4f(x)))]\bigg|_{x=0}.$$

**Solution:** Note that

$$\frac{d}{dx}[f(3f(4f(x)))] = f'(3f(4f(x)))\cdot 3f'(4f(x))\cdot 4f'(x)$$

and

$$\frac{d}{dx}[f(3f(4f(x)))]\bigg|_{x=0} = f'(3f(4f(0)))\cdot 3f'(4f(0))\cdot 4f'(0) = f'(0)\cdot 3f'(0)\cdot 4f'(0) = 96.$$

4. (14 points) Let $f(x) = \sqrt{x+4}$ be a function defined on $\{x\in\mathbb{R}|x\geq -4\}$.

(a) Find $\delta > 0$ such that

$$|f(x) - f(5)| < \frac{1}{2}$$

when $|x-5| < \delta$.

**Solution:**

$$|\sqrt{x+4} - 3| < \frac{1}{2} \Leftrightarrow \frac{5}{2} < \sqrt{x+4} < \frac{7}{2} \Leftrightarrow \frac{25}{4} < x+4 < \frac{49}{4} \Leftrightarrow \frac{9}{4} < x < \frac{33}{4} \Leftrightarrow -\frac{11}{4} < x-5 < \frac{13}{4}$$

Therefore, if $|x-5| < \frac{11}{4}$, then $|f(x) - f(5)| < \frac{1}{2}$. For any $\delta$ with $0 < \delta \leq \frac{11}{4}$, the statement is true.

(b) Find a function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $\varepsilon$,

$$|f(x) - f(5)| < \varepsilon$$

when $|x - 5| < \delta(\varepsilon)$.

**Solution:** Consider

$$|f(x) - f(5)| = \frac{|x-5|}{\sqrt{x+4}+3}$$

Since $\sqrt{x+4} \geq 0$, $\dfrac{1}{\sqrt{x+4}+3} \leq \dfrac{1}{3}$ and

$$|f(x) - f(5)| = \frac{|x-5|}{\sqrt{x+4}+3} \leq \frac{|x-5|}{3}$$

Moreover, since domain of $f(x)$ is $\{x \in \mathbb{R} | x \geq -4\}$, we can not choose $\delta > 9$. That is, if we choose $\delta = \min(3\varepsilon, 9)$, then

$$|f(x) - f(5)| = \frac{|x-5|}{\sqrt{x+4}+3} \leq \frac{|x-5|}{3} < \varepsilon \text{ when } |x-5| < \delta.$$

Therefore, define $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ by $\delta(x) = \min(3x, 9)$.

5. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} x^2 \left[\sin\left(\dfrac{1}{x^2}\right) + 7\right] & \text{when } x \neq 0, \\ 0 & \text{when } x = 0. \end{cases}$$

(a) Is $f$ continuous at 0? Explain your reason.

**Solution:** Since $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$, we have $6 \leq \sin\left(\dfrac{1}{x^2}\right) + 7 \leq 8$ for all $x \in \mathbb{R}$ and

$$6x^2 \leq x^2\left(\sin\left(\frac{1}{x^2}\right) + 7\right) \leq 8x^2$$

for all $x \in \mathbb{R}$. Since $\lim_{x \to 0} 6x^2 = \lim_{x \to 0} 8x^2 = 0$, by squeeze theorem, $\lim_{x \to 0} f(x) = 0 = f(0)$. Therefore, $f$ is continuous at 0.

(b) Is $f$ differentiable at 0? Explain your reason.

**Solution:** Since

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2\left(\sin\left(\frac{1}{h^2}\right) + 7\right)}{h} = \lim_{h \to 0} h\left(\sin\left(\frac{1}{h^2}\right) + 7\right)$$

For $h \to 0^+$, consider $6h \leq h\left(\sin\left(\dfrac{1}{h^2}\right) + 7\right) \leq 8h$ and $\lim_{h \to 0^+} 6h = \lim_{h \to 0^+} 8h = 0$. By squeeze theorem, $\lim_{h \to 0^+} h\left(\sin\left(\dfrac{1}{h^2}\right) + 7\right) = 0$.

For $h \to 0^-$, consider $6h \geq h\left(\sin\left(\frac{1}{h^2}\right)+7\right) \geq 8h$ and $\lim_{h\to 0^-} 6h = \lim_{h\to 0^-} 8h = 0$. By squeeze theorem, $\lim_{h\to 0^-} h\left(\sin\left(\frac{1}{h^2}\right)+7\right) = 0$. Therefore,

$$f'(0) = \lim_{h\to 0} h\left(\sin\left(\frac{1}{h^2}\right)+7\right) = 0$$

and $f$ is differetiable at 0.