Midterm Exam II

November 28, 2016

Rules and Regulations: This is a take-home exam. You are all permitted and encouraged to discuss with your classmates. But, plagiarism is forbidden. You will get 0 point if you copy answers from others or lend your answers to someone else. Please hand in your solutions with the grading sheet before 3:10pm, December 5, 2016.

Problems for Solution:

1. (a) (5%) Please provide one signal $x(t)$ that has all of the properties listed below. Please also sketch the signal $x(t)$ and the spectrum $X(j\omega)$.
   - $x(t)$ is continuous;
   - $x(t)$ is periodic;
   - $x(t)$ is even;
   - $x(t)$ is real.

   (b) (5%) Please provide one signal $x(t)$ whose spectrum $X(j\omega)$ has all of the properties listed below. Please also sketch the signal $x(t)$ and the spectrum $X(j\omega)$.
   - $X(j\omega)$ is continuous;
   - $X(j\omega)$ is periodic;
   - $X(j\omega)$ is even;
   - $X(j\omega)$ is purely imaginary.

2. (a) (5%) For a continuous and periodic signal $x(t)$ with period T, we have the Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$

   where $\omega = 2\pi/T$. Please show the Parseval's relation:

$$\frac{1}{T}\int_T |x(t)|^2 d\,t = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

   (b) (5%) Please use the result of (a) to show that

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

   (Hint: Please consider the continuous-time periodic square wave which has been mentioned for many times in class.)

3. For a signal $x(t)$ with the Fourier transform $X(j\omega)$, we want to show that

$$\int_{-\infty}^{t} x(\tau)\, d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega).$$

(a) (5%) Please show that

$$\int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t)$$

where $u(t)$ is the unit step function.

(b) (5%) Please find the Fourier transform $U(j\omega)$ of $u(t)$.
(Hint: Please use the equation $u(t) = (1 + \text{sgn}(t))/2$.)

(c) (5%) Please use (a) and (b) to show that

$$\mathcal{F}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega).$$

4. (5%) Let $X(j\omega)$ be the spectrum of the signal $x(t)$. Please find the inverse Fourier transform of the following spectrum:

$$D(j\omega) = \frac{dX(j\omega)}{d\omega}.$$

That is, please find $d(t) = \mathcal{F}^{-1}\left[D(j\omega)\right] = ?$

5. (10%) Let

$$x(t) = \text{sinc}\left(\frac{t}{\pi}\right)$$

and

$$h(t) = \text{sinc}(t).$$

Please find $y(t) = x(t) * h(t)$.

6. Thank E34025010 for providing this problem which can be solved by using the Fourier transform of a rectangular function. Let

$$x(t) = \text{rect}\left(\frac{t}{2}\right) = \begin{cases} 1, & |t| \le 1 \\ 0, & |t| > 1. \end{cases}$$

(a) (5%) Please find the spectrum $X(j\omega)$ of $x(t)$.

(b) (5%) Please use the result of (a) to find the value of

$$\int_{0}^{\infty} \frac{\sin(\omega)}{\omega}\, d\omega.$$