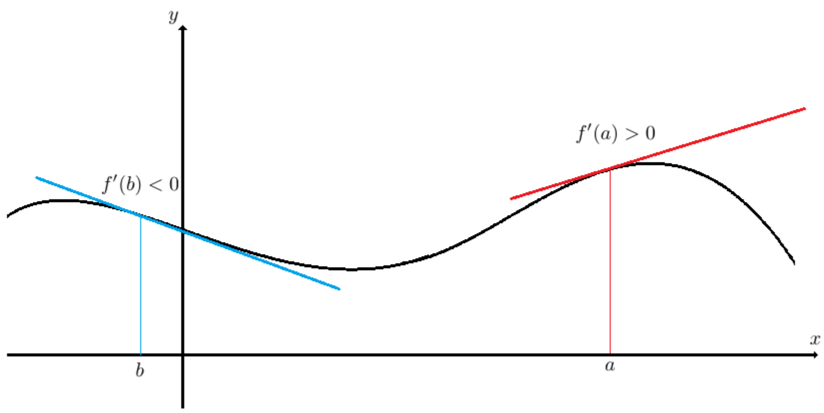
CALCULUS I LECTURE 6: MAXIMUM, MINIMUM AND MEAN VALUE THEOREM

1. Sign of derivatives and its geometric meaning

Let $f : X \to Y$ be a differentiable function, $X, Y \subseteq \mathbb{R}$. Recall that the derivative of f at $a \in X$ is the slope of the tangent line passing $(a, f(a))$. When $f'(a) > 0$, we can see that f is increasing near a; when $f'(a) < 0$, then f is decreasing near a. One can see the example in the following graph.



Pic.1

Definition 1.1. we call a a **critical point** of f if and only if $f'(a) = 0$.

We cannot make any conclusion about the increasing/decreasing of f near a critical point. For example, $f(x) = x^2$ has a critical point at 0, but it is decreasing from the negative side and increasing form the positive. $f(x) = \pm x^3$ are other examples, which show that at the critical point 0, the function may increasing or decreasing depending on some other information.

To obtain new information, one can consider the second derivative of f around a. Suppose $f''(a) > 0$, then we know that the $f'(x)$ is increasing near a, with $f'(a) = 0$ (because a is a critical point). Namely, we have $f'(x) < 0$ when $a - c < x < a$ for some $c > 0$ and $f'(x) > 0$ when $a < x < a + c$ for some $c > 0$. This means that f is decreasing on the negative side of a and increasing on the positive side of a. So we have $f(a)$ is a minimum in the small region near a. We will say f attends a **local minimum** at a. Similarly, one can obtain the same argument for a **local maximum** when $f''(a) < 0$.

Proposition 1.2. If $f''(a) > 0$ at a critical point a, then f attends a local minimum at a; if $f''(a) < 0$ at a critical point a, then f attends a local maximum at a.

Conversely, suppose that f has a local minimum at a (and f is differentiable), then we have

$$0 \leq \lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a) = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} \leq 0. \tag{1.1}$$

Therefore, we have $f'(a) = 0$. By this argument, we prove the following proposition.

Proposition 1.3. Suppose f has a local minimum/maximum at a and f is differentiable at a. Then $f'(a) = 0$.

We sometime call Proposition 1.2 and Proposition 1.3 the **second derivative test** of a critical point.

Suppose f is a continuous function defined on a closed interval $[c, d]$, $c < d \in \mathbb{R}$. Then one might notice that local minimums are not always an **absolute minimum** of the function, i.e. $f(x) \geq f(a)$ for all $x \in [c, d]$. The absolute minimum, however, must be the smallest value in the set

$$\{f(a) | f(a) \text{ is a local minimum of } f\} \cup \{f(c), f(d)\}. \tag{1.2}$$

Namely, either the absolute minimum is one of the local minimum, or the absolute minimum is the value at a boundary point. A similar version for the absolute maximum holds. I left the statement as an exercise for readers.

2. Mean value theorem

In any event, one can always find[1] an absolute maximum/minimum for a (differentiable) function defined on a closed interval $[c, d]$. By using this observation, one can prove the following theorem.

Theorem 2.1. *Let f be a differentiable function defined on $[c, d]$. Then there exists $a \in (c, d)$ such that*

$$f'(a) = \frac{f(d)-f(c)}{d-c}. \tag{2.1}$$

Proof. Let us consider a new function

$$g(x) = f(x) - \frac{f(d)-f(c)}{d-c}x. \tag{2.2}$$

One might notice that $g(c) = g(d) = \frac{f(d)c+f(c)d}{d-c}$. Also, g is differentiable on $[c, d]$. So there must be an absolute maximum and an absolute minimum attended.

Suppose both of them are attended by boundary points c, d. Then since $g(c) = g(d)$, we have $g(x) = g(c)$ for all $x \in [c, d]$. In this case, we can just take $a = \frac{c+d}{2}$ as desired.

Suppose one of the absolute minimum and absolute maximum is a local minimum or local maximum, then there exists $a \in (c, d)$ such that $g(a)$ is a local minimum/maximum. We take this a as desired.

[1] This is not actually rigorous, since we haven't prove the existence of local max/min for continuous functions. However, we will not prove this in this course.

In both of these cases, one has $a \in (c, d)$ such that $g'(a) = 0$. So

$$0 = g'(a) = f'(a) - \frac{f(d) - f(c)}{d - c}. \tag{2.3}$$

Therefore, we prove this theorem. □

Mean value theorem is the first powerful tool in this course. An immediate corollary is the following.

Corollary 2.2. *Suppose $f'(x) = g'(x)$ for all $x \in [c, d]$. Then we have $f(x) - g(x) = k$ for some constant k.*

Proof. Let $x \in (c, d]$. We apply the mean value theorem on the function $f - g$ defined on $[c, x]$. In this case, there exists $a \in (c, x)$ such that

$$f'(a) - g'(a) = \frac{[f(x) - g(x)] - [f(c) - g(c)])}{x - c}. \tag{2.4}$$

By the assumption, $f'(a) - g'(a) = 0$. So

$$f(x) - g(x) = [f(c) - g(c)] := k. \tag{2.5}$$

This result holds for any $x \in (c, d]$, so we prove this corollary. □

Exercise 2.3. Prove that if $f'(x) = 0$, then f is a constant function.

3. Applications of Mean value theorem

In Corollary 2.2, we have proved that any two functions are differ by just a constant provided their derivatives are the same. This is an important observation, which tells us when you have two functions satisfies same "condition" provided by their derivatives, then they are the same, up to at most a constant.

Exercise 3.1. Suppose that f is a positive function satisfying $f'(x) = af(x)$. Then $f(x) = Ce^{ax}$.

Before we start to solve this problem, one should notice that $g(x) = e^{ax}$ is a solution of this equation. Now, we take $F(x) = \ln(f(x))$ and $G(x) = ax$. Then we have

$$F'(x) = \frac{f'(x)}{f(x)} = a = G'(x) \tag{3.1}$$

for any $x \in \mathbb{R}$. By Corollary 2.2, we have

$$F(x) = G(x) + k \tag{3.2}$$

for some $k \in \mathbb{R}$. So

$$f(x) = e^{F(x)} = e^{G(x)+k} = e^k e^{ax} = Ce^{ax} \tag{3.3}$$

by taking $C = e^k$.