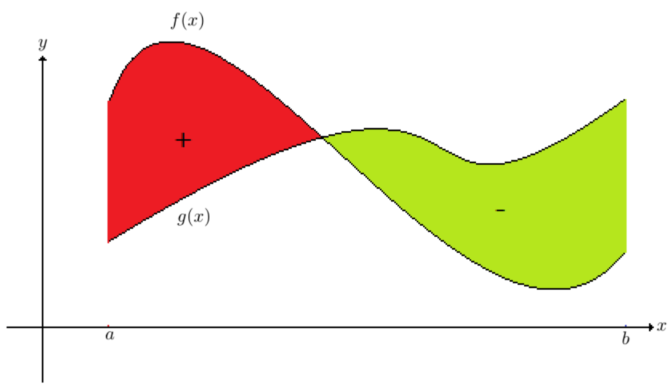
CALCULUS I LECTURE 14: AREAS AND VOLUMES

1. Area bounded by two curves

Let f and g are two integrable functions defined on $[a, b]$. Suppose $f(x) > g(x)$ for all $x \in [a, b]$. Then the area bounded by the graph of f, the graph of g, $x = a$ and $x = b$ can be written as

$$\int_a^b [f(x) - g(x)]dx. \tag{1.1}$$

In general, if we don't have the condition $f > g$, then the definite integral (1.1) represents the signed area which takes positive on those portions $f > g$ and takes negative on those portions $g > f$. See Pic 1. below.



Pic.1

2. Volume of solids of revolution

The concept of integral tells us that the area equals the limit of a sequence of Riemann sums. In fact, we can also compute the volume of solids of revolution by the same idea.

A region $\mathcal{R}$ in $\mathbb{R}^3$ is a **solid of revolution** if and only if there exist a function f defined on $[a, b]$ such that

$$\mathcal{R} = \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{y^2 + z^2} \leq f(x),\ x \in [a, b]\}. \tag{2.1}$$

One can imagine that $\mathcal{R}$ is the region bounded by rotating the graph of $f(x)$ about the x-axis.

To find the volume of $\mathcal{R}$, one should consider the "Riemann sum" for it. We cut the region $\mathcal{R}$ into many slices with each cut being perpendicular to the x-axis.

Again, we consider the width of these slices equal $\frac{b-a}{n} = \Delta_n$. Then we have the volume of n-th slice can be approximated by

$$\pi M_k^2 \Delta_n \tag{2.2}$$

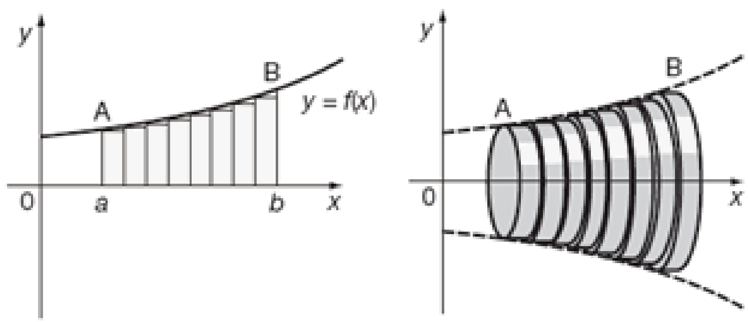
where $M_k = \max\{f(x)|x \in [a + (k - a)\Delta_n, a + k\Delta_n]\}$. So we have the (upper-)Riemann sum for $\mathcal{R}$ as follows

$$S_n = \sum_{k=1}^{n} \pi M_k^2 \Delta_n = \pi \sum_{k=1}^{n} M_k^2 \Delta_n. \tag{2.3}$$

Notice that $M_k^2 = \max\{f^2(x)|x \in [a + (k - a)\Delta_n, a + k\Delta_n]\}$, so the right hand side of (2.3) is the Riemann sum of $f^2(x)$ on $[a, b]$, multiplied by π. Take the limit $n \to \infty$ on (2.3), we obtain

$$\begin{aligned} \lim_{n\to\infty} S_n &= \pi \lim_{n\to\infty} \sum_{k=1}^{n} M_k^2 \Delta_n \\ &= \pi \int_a^b f^2(x)dx. \end{aligned} \tag{2.4}$$

This gives us the formula of the volume of solids of revolution. One can also use the lower Riemann sum, which is demonstrated in the picture below.



Pic.2 lower Riemann sum of a solid of revolution
(Cited from: https://www.emathhelp.net/notes/calculus-2/applications-of-integrals/volumes-solids-of-revolution-and-method-of-rings-disks/)

Proposition 2.1. Let $\mathcal{R}$ be the region bounded by rotating the function f about the x-axis, where f is defined on $[a, b]$. Then the volume of $\mathcal{R}$ equals

$$\pi \int_a^b f^2(x)dx. \tag{2.5}$$

One can also consider the case that $\mathcal{R}$ is the region rotating by the area bounded by $f(x)$ and $g(x)$ about x-axis. We suppose $f(x) > g(x)$ for all $x \in [a, b]$. Then the volume of $\mathcal{R}$ will be

$$\pi \int_a^b [f^2(x) - g^2(x)]dx. \tag{2.6}$$

There is another way to find the volume of solids of revolution. Consider this time that f is a non negative function defined on $[a, b]$ with $a, b > 0$. We rotate the region enclosed by the graph of f and x-axis about the y-axis. Again, we call this solid $\mathcal{R}$.

To find the volume of $\mathcal{R}$, we can first approximate this region by a family of **cylindrical shells**. Each of them has width $\Delta_n = \frac{b-a}{n}$. We use $f(a+(k-1)\Delta_n)$ as the height of the k-th shell. So the volume of the k-th shell will be

$$\begin{aligned}\pi\Big[(a+k\Delta_n)^2\Delta_n^2 - (a+(k-1)\Delta_n)^2\Big] f(a+(k-1)\Delta_n) \\ =2\pi(a+(k-1)\Delta_n)f(a+(k-1)\Delta_n)\Delta_n + \pi f(a+(k-1)\Delta_n)\Delta_n^2.\end{aligned} \tag{2.7}$$

Sum up all these volumes, we obtain

$$\sum_{k=1}^{n} 2\pi(a+(k-1)\Delta_n)f(a+(k-1)\Delta_n)\Delta_n + \sum_{k=1}^{n} \pi f(a+(k-1)\Delta_n)\Delta_n^2. \tag{2.8}$$

The second term in (2.8) will converge to zero because it is smaller than

$$\max\Big\{f(x)|x\in[a,b]\Big\}\frac{(b-a)^2}{n}. \tag{2.9}$$

The first one will converge to

$$\int_a^b 2\pi x f(x)dx. \tag{2.10}$$

This gives us the second formula for the solid of revolution.

Proposition 2.2. Let $\mathcal{R}$ be the region bounded by rotating the non negative function $f(x)$ about y-axis, where f is defined on $[a,b]$ with $a > 0$. Then the volume of $\mathcal{R}$ equals

$$\int_a^b 2\pi x f(x)dx. \tag{2.11}$$

One can also consider a solid $\mathcal{R}$ obtained by rotating the region bounded by $f(x)$ and $g(x)$, $f \geq g$, about the y-axis. Then the volume of $\mathcal{R}$ will be

$$\int_a^b 2\pi x[f(x)-g(x)]dx. \tag{2.12}$$

Example 2.3. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y=\sqrt{x}$ from 0 to 1.

By Proposition 2.1, we have the answer is

$$\pi\int_0^1 (\sqrt{x})^2 dx = \pi\frac{x^2}{2}\Big|_0^1 = \frac{\pi}{2}.$$

Example 2.4. Find the volume of the solid obtained by rotating the region bounded by $y=x^3$, $y=8$ and $x=0$ about the y-axis.

By Proposition 2.1, we have the answer

$$\pi\int_0^8 (y^{\frac{1}{3}})^2 dy = \pi\frac{3}{5}y^{\frac{5}{3}}\Big|_0^8 = \frac{96\pi}{5}.$$

Example 2.5. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y=x$ and $y=x^2$ from $x=0$ to $x=1$.

By the formula (2.12), the volume will be

$$\int_0^1 2\pi x[x - x^2]dx = \frac{\pi}{6}.$$

3. General cases

In general, we suppose $\mathcal{R}$ is a bounded region in $\mathbb{R}^3$. This means that there exists a ball with a (large) radius C contains the region. Because it is bounded, we can suppose that this region is contained in $\{(x, y, z) \in \mathbb{R}^3 \mid x \in [a, b]\}$.

To find the volume for region $\mathcal{R}$, we use the same strategy: Cut the region along the x-axis and write down the Riemann sum for the region. Then we take the limit to obtain the final result.

Let $\{a, a + \Delta_n, a + 2\Delta_n, \cdots, b\}$ be a partition of $[a, b]$. We denote by A_k the area of $\mathcal{R} \cap \{x = a + k\Delta_n\}$. Then a "right Riemann sum" can be obtained as the following:

$$S_n^r = \sum_{k=1}^{n} A_k \Delta_n. \tag{3.1}$$

In the case that $\lim_{n\to\infty} S_n^r$ exists, we obtain the volume of $\mathcal{R}$. In general, we shouldn't expect that the limit is a definite integral we can compute. Now, if we assume that $A(x)$ be the function which defines the area of cross-section of $\mathcal{R}$ and the plane $\{(x, y, z)|y, z \in \mathbb{R}\}$, then the limit of the Riemann sum will be

$$\int_a^b A(x)dx. \tag{3.2}$$