CALCULUS I LECTURE 13: INTEGRALS II

1. Basic properties of signed area

Suppose the integrals of f and g on $[a, b]$ exist (we call f, g **integrable**). Then we have the following properties.

Proposition 1.1. Suppose $f \leq g$ on $[a, b]$. Then we have

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx. \tag{1.1}$$

Proposition 1.2. For any $c \in \mathbb{R}$, we have

$$\int_a^b [f(x) + cg(x)]dx = \int_a^b f(x)dx + c\int_a^b g(x)dx. \tag{1.2}$$

Here we define the notation for **indefinite integrals**. Consider the integral for a function f on the interval $[a, s]$ for any $s \geq a$. It is equal to

$$F_a(s) := \int_a^s f(x)dx. \tag{1.3}$$

One can extend the definition of F to all s by taking

$$F_a(s) = \begin{cases} 0 & \text{if } s = a \\ -\int_s^a f(x)dx & \text{if } s < a. \end{cases} \tag{1.4}$$

Or we can simple define the rule

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \tag{1.5}$$

for any $a, b \in \mathbb{R}$. Now, for any $a, b \in \mathbb{R}$, we have

$$F_b(s) - F_a(s) = \int_a^b f(x)dx. \tag{1.6}$$

The right hand side of (1.6) is independent of s, which is a constant. Therefore, we can use the notation

$$\int f(x)dx = F_a(x) + C \tag{1.7}$$

for some constant C, to denote a function with an indeterminate constant term. We call this function the **indefinite integral** of f.

Remark 1.3. By (1.5), one can show that the following formula holds for any a, b and c (without any constraint of their ordering).

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \tag{1.8}$$

2. Fundamental theorem of calculus

So far we have introduced the concept of derivatives and integrals. These two different concepts are actually two side of a coin, which is called the fundamental theorem of calculus. It can be stated as a theorem in the following way.

Theorem 2.1. *Let $f(x)$ be a integrable, continuous function. Suppose we have the indefinite integral*

$$\int f(x)dx = F(x) + C \tag{2.1}$$

for some constant C. Then $\frac{dF}{dx} = f(x)$.

Proof. By definition, we have

$$\frac{dF}{dx}(a) = \lim_{h\to 0} \frac{F(a+h) - F(a)}{h} \tag{2.2}$$

$$= \lim_{h\to 0} \frac{1}{h} \int_a^{a+h} f(x)dx. \tag{2.3}$$

Since f is continuous, so $|f(x) - f(a)| \le \varepsilon$ when $|x - a| \le \delta(\varepsilon)$. By assuming $|h| \le \delta(\varepsilon)$, we have

$$\begin{aligned} \Big| \frac{1}{h} \int_a^{a+h} f(x)dx - f(a) \Big| &\le \frac{1}{h} \int_a^{a+h} |f(x) - f(a)| dx \\ &\le \varepsilon. \end{aligned} \tag{2.4}$$

So

$$\Big| \lim_{h\to 0} \frac{1}{h} \int_a^{a+h} f(x)dx - f(a) \Big| \le \varepsilon$$

for any $\varepsilon > 0$. That implies

$$\lim_{h\to 0} \frac{1}{h} \int_a^{a+h} f(x)dx = f(a). \tag{2.5}$$

□

According to the fundamental theorem of calculus (FTC), one can easily obtain the formula of integrals for many functions.

Example 2.2. Let $p \in \mathbb{R}$, $p \neq -1$.

$$\int x^p dx = \frac{1}{p+1} x^{p+1} + C. \tag{2.6}$$

When $p = -1$, we have

$$\int \frac{1}{x} dx = \log x + C. \tag{2.7}$$

Example 2.3.

$$\int \sin x dx = -\cos x + C; \tag{2.8}$$

$$\int \cos x dx = \sin x + C. \tag{2.9}$$

3. SUBSTITUTION RULE

Suppose there is a change of variable, $x = x(s)$ with $x = a$ when $s = c$, $x = b$ when $s = d$. Then we consider the indefinite integral

$$\int f(x)dx = F(x) + C \tag{3.1}$$

for some constant C. We substitute $x = x(s)$ into the right hand side of (3.1) and differentiate $F(x(s)) + C$ with respect to s, then the fundamental theorem of calculus and chain rule tells us

$$\begin{aligned} \frac{d}{ds}[F(x(s)) + C] &= \frac{dF}{dx}(x(s))\frac{dx}{ds} \\ &= f(x(s))\frac{dx}{ds}. \end{aligned} \tag{3.2}$$

Now, integrate (3.2) on both side, we have

$$\Big[\int f(x)dx\Big](x(s)) = \int [f(x(s))\frac{dx}{ds}]ds \tag{3.3}$$

One can also check that, if we consider definite integral on $x \in [a, b]$, (3.3) can be written as

$$\int_a^b f(x)dx = \int_c^d [f(x(s))\frac{dx}{ds}]ds. \tag{3.4}$$

We call (3.4) the **substitution rule** for integrals.

Example 3.1. Let $x = 2s$. Then we have

$$\int_a^b f(x)dx = \int_{\frac{a}{2}}^{\frac{b}{2}} 2f(2s)ds \tag{3.5}$$

by using substitution rule.

Example 3.2. By taking $x = \sin\theta$, we have

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta}\cos\theta d\theta = \int_0^{\frac{\pi}{2}} 1d\theta = \frac{\pi}{2}. \tag{3.6}$$

4. Generalized Riemann sums

Here we generalize the way to define the Riemann sum. Suppose f is defined on $[a, b]$. We generalize the definition of **partitions** as the following.

Definition 4.1. A partition of $[a, b]$ is a set $P = \{a = x_0, x_1, x_2, \cdots, x_n = b\}$ with $x_i < x_{i+1}$ for all i. We denote $(x_i - x_{i-1})$ by Δx_i and the maximum of them by Δx.

If we have two partitions P and P', we call P' is **finer** than P when $P \subset P'$. Or we call P' be a **refinement** of P. Under this setting, we can also define the upper Riemann sum and lower Riemann sum by

$$S_P = \sum_{k=1}^n M_k \Delta x_k \tag{4.1}$$

where $M_k = \max\{f(x) | x \in [x_{k-1}, x_k]\}$;

$$s_P = \sum_{k=1}^{n} m_k \Delta x_k \tag{4.2}$$

where $m_k = \min\{f(x) | x \in [x_{k-1}, x_k]\}$. According to these definitions, we have

$$S_{P'} \leq S_P; \tag{4.3}$$

$$s_{P'} \geq s_P \tag{4.4}$$

for any P' finer than P. We summarize this fact as the following proposition.

Proposition 4.2. Let P' be finer than P. Then we have

$$s_P \leq s_{P'} \leq \int_a^b f(x)dx \leq S_{P'} \leq S_P. \tag{4.5}$$

In fact, we have the following Proposition (we omit the proof since it is not a part of this course).

Proposition 4.3. In fact, we always have

$$\lim_{\Delta x \to 0} S_P = \int_a^b f(x)dx. \tag{4.6}$$