# Chapter 2: Fundamentals of Logic

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# Outline

- ***Basic Connectives and Truth Tables***
- Logical Equivalence: The Laws of Logic
- Logical Implication: Rules of Inference
- The Use of Quantifiers
- Quantifiers, Definitions, and the Proofs of Theorems

# Statements (1/5)

- Statements (or propositions): declarative sentences that are either true or false but not both.
- Examples:
  - $p$: Combinatorics is a required course for sophomores.
  - $q$: Margaret Mitchell wrote *Gone with the Wind*.
  - $r$: 2 + 3 = 5.
- Non-statements: (do not have truth values)
  - "What a beautiful evening!" (exclamation)
  - "Get up and do your exercises." (command)

# Statements (2/5)

- Primitive statements: there is really no way to break them down into anything simpler.
- New statements can be obtained from existing ones in two ways.

1. Negation: We do not consider the negation of a primitive statement to be a primitive statement.

   $\neg p$ : Combinatorics is **not** a required course for sophomores.

# Statements (3/5)

2. Compound statement, using the following logical connectives:
   a) Conjunction $\wedge$

      $p \wedge q$: "Combinatorics is a required course for sophomores, **and** Margaret Mitchell wrote *Gone with the Wind*."

   b) Disjunction $\vee$

      $p \vee q$ (inclusive): true if one or the other of $p$, $q$ is true or if both of the statements $p$, $q$ are true.

      $p \veebar q$ (exclusive): true if one or the other of $p$, $q$ is true but not both of the statements $p$, $q$ are true.

# Statements (4/5)

c) Implication $\rightarrow$

$p \rightarrow q$: "If combinatorics is a required course for sophomores, **then** Margaret Mitchell wrote Gone with the Wind."

Alternatively, we can say

1. "If $p$, then $q$."
2. "$p$ is sufficient for $q$."
3. "$p$ is a sufficient condition for $q$."
4. "$q$ is necessary for $p$."
5. "$q$ is a necessary condition for $p$."
6. "$q$ only if $p$."

# Statements (5/5)

- The statement $p$ is called the hypothesis of the implication; $q$ is called the conclusion.

d) Biconditional $\leftrightarrow$

$p \leftrightarrow q$: "Combinatorics is a required course for sophomores, **if and only if** Margaret Mitchell wrote *Gone with the Wind*."

- "$p$ if and only if $q$" or "$p$ is necessary and sufficient for $q$."
- Abbreviate "$p$ if and only if $q$" as "$p$ iff $q$."

- **Note:** A sentence such as "The number $x$ is an integer." is not a statement because its truth value cannot be determined until a numerical value is assigned for $x$.

# Truth Table

"0" for false and "1" for true

| $p$ | $\neg p$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

| $p$ | $q$ | $p \wedge q$ | $p \vee q$ | $p \veebar q$ | $p \rightarrow q$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 0 | 1 | 1 |

# Example of implication (1/2)

- **Example 2.2:** It is almost the week before Christmas and Penny will be attending several parties that week. Ever conscious of her weight, she plans not to weigh herself until the day after Christmas. Considering what those parties may do to her waistline by then, she makes the following resolution for the December 26 outcome: “If I weigh more than 120 pounds, then I shall enroll in an exercise class.”

# Example of implication (2/2)

- **Example 2.2 (cont.):**
  Let $p$ and $q$ denote the (primitive) statements

  $p$: I weigh more than 120 pounds.
  $q$: I shall enroll in an exercise class.

  Then Penny's statement (implication) is given by $p \rightarrow q$.

  ➡ We shall consider the truth values of this particular example of $p \rightarrow q$

# Example of Compound Statement (1/2)

- **Example 2.4:** Let us examine the truth table for the compound statement "Margaret Mitchell wrote *Gone with the Wind*, and if $2 + 3 \neq 5$, then combinatorics is a required course for sophomores."
  - $p$: Combinatorics is a required course for sophomores.
  - $q$: Margaret Mitchell wrote "*Gone with the Wind*".
  - $r$: $2 + 3 = 5$.

➡ $q \wedge (\neg r \rightarrow p)$

# Example of Compound Statement (2/2)

- **Example 2.4 (cont.):**

| $p$ | $q$ | $r$ | $\neg r$ | $\neg r \rightarrow p$ | $q \wedge (\neg r \rightarrow p)$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | | | |
| 0 | 0 | 1 | | | |
| 0 | 1 | 0 | | | |
| 0 | 1 | 1 | | | |
| 1 | 0 | 0 | | | |
| 1 | 0 | 1 | | | |
| 1 | 1 | 0 | | | |
| 1 | 1 | 1 | | | |

# Tautology and Contradiction

- **Definition 2.1:** A compound statement is called a tautology if it is true for all truth value assignments for its component statement. If a compound statement is false for all such assignments, then it is called a contradiction.

| $p$ | $q$ | $p \vee q$ | $p \rightarrow (p \vee q)$ | $\neg p$ | $\neg p \wedge q$ | $p \wedge (\neg p \wedge q)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| [illegible] | [illegible] | [illegible] | 1 | [illegible] | [illegible] | 0 |
| [illegible] | [illegible] | [illegible] | 1 | [illegible] | [illegible] | 0 |
| [illegible] | [illegible] | [illegible] | 1 | [illegible] | [illegible] | 0 |

Use the symbol $T_0$ to denote any tautology.

Use the symbol $F_0$ to denote any contradiction.

- **EXERCISES 2.1:** 4, 6

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- Basic Connectives and Truth Tables
- ***Logical Equivalence: The Laws of Logic***
- Logical Implication: Rules of Inference
- The Use of Quantifiers
- Quantifiers, Definitions, and the Proofs of Theorems

# Logical Equivalence

- **Definition 2.2:** Two statements $s_1$, $s_2$ are said to be logically equivalent, and we write $s_1 \Leftrightarrow s_2$, when the statement $s_1$ is true (respectively, false) if and only if the statement $s_2$ is true (respectively, false).

| $p$ | $q$ | $\neg p$ | $\neg p \vee q$ | $p \rightarrow q$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 |
| [illegible] | [illegible] | 0 | 0 | 0 |
| [illegible] | [illegible] | 0 | 1 | 1 |

We can eliminate the connective $\rightarrow$ from compound statements.

$$\therefore \quad \neg p \vee q \Leftrightarrow p \rightarrow q$$

# Eliminate Connectives from Compound Statements

| $p$ | $q$ | $p \to q$ | $q \to p$ | $(p \to q) \land (q \to p)$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 |

$$(p \leftrightarrow q) \Leftrightarrow (\neg p \lor q) \land (\neg q \lor p)$$

| $p$ | $q$ | $p \veebar q$ | $p \lor q$ | $p \land q$ | $\neg(p \land q)$ | $(p \lor q) \land \neg(p \land q)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 0 |

# Important Properties (1/2)

- **Example 2.8:** (DeMorgan's Laws)

| $p$ | $q$ | $p \wedge q$ | $\neg(p \wedge q)$ | $\neg p$ | $\neg q$ | $\neg p \vee \neg q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |

$$\therefore \quad \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

# Important Properties (2/2)

- **Example 2.9:** (Distributive Law)

| $p$ | $q$ | $r$ | $p \wedge (q \vee r)$ | $(p \wedge q) \vee (p \wedge r)$ | $p \vee (q \wedge r)$ | $(p \vee q) \wedge (p \vee r)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

# The Laws of Logic (1/2)

- For any primitive statements $p$, $q$, $r$, any tautology $T_0$, and any contradiction $F_0$,

**1)** $\neg\neg p \Leftrightarrow p$ — Law of *Double Negation*

**2)** $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ — *DeMorgan's* Laws

$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

**3)** $p \vee q \Leftrightarrow q \vee p$ — *Commutative* Laws

$p \wedge q \Leftrightarrow q \wedge p$

**4)** $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ — *Associative* Laws

$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

# The Laws of Logic (2/2)

**5)** $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ *Distributive* Laws

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

**6)** $p \vee p \Leftrightarrow p$ *Idempotent* Laws

$p \wedge p \Leftrightarrow p$

**7)** $p \vee F_0 \Leftrightarrow p$ *Identity* Laws

$p \wedge T_0 \Leftrightarrow p$

**8)** $p \vee \neg p \Leftrightarrow T_0$ *Inverse* Laws

$p \wedge \neg p \Leftrightarrow F_0$

**9)** $p \vee T_0 \Leftrightarrow T_0$ *Domination* Laws

$p \wedge F_0 \Leftrightarrow F_0$

**10)** $p \vee (p \wedge q) \Leftrightarrow p$ *Absorption* Laws

$p \wedge (p \vee q) \Leftrightarrow p$

# Dual of Statement (1/3)

- **Definition 2.3:** Let $s$ be a statement. If $s$ contains no logical connectives other than $\wedge$ and $\vee$, then the dual of $s$, denoted $s^d$, is the statement obtained from $s$ by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$, respectively, and each occurrence of $T_0$ and $F_0$ by $F_0$ and $T_0$, respectively.

# Dual of Statement (2/3)

- If $p$ is any primitive statement, then
  - $p^d$ is the same as $p$
  - $(\neg p)^d$ is the same as $\neg p$
  - $p \vee \neg p$ and $p \wedge \neg p$ are duals of each other
  - $p \vee T_0$ and $p \wedge F_0$ are duals of each other

# Dual of Statement (3/3)

- **Theorem 2.1** (The Principle of Duality): Let $s$ and $t$ be statements that contain no logical connectives other than $\wedge$ and $\vee$. If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

➡ Laws 2 through 10 in our list can be established by proving one of the laws in each pair and then invoking this principle.

# Substitution Rules

1. Suppose that the compound statement $P$ is a tautology. If $p$ is a primitive statement that appears in $P$ and we replace each occurrence of $p$ by the same statement $q$, then the resulting compound statement $P_1$ is also a tautology.
2. Let $P$ be a compound statement where $p$ is an arbitrary statement that appears in $P$, and let $q$ be a statement such that $q \Leftrightarrow p$. Suppose that in $P$ we replace one or more occurrences of $p$ by $q$. Then this replacement yields the compound statement $P_1$. Under these circumstances $P_1 \Leftrightarrow P$.

# Examples of Substitution Rules (1/3)

- **Example 2.10: (**Substitution Rule 1)
  a) (From DeMorgan's Laws) $P$: $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology.

     $P_1$: $\neg[(r \wedge s) \vee q] \leftrightarrow [\neg(r \wedge s) \wedge \neg q]$ is a tautology.

     $P_2$: $\neg[(r \wedge s) \vee (t \rightarrow u)] \leftrightarrow [\neg(r \wedge s) \wedge \neg\,(t \rightarrow u)]$ is also a tautology.

# Examples of Substitution Rules (2/3)

- **Example 2.10 (cont.): (Substitution Rule 1)**

b)

| $p$ | $q$ | $p \to q$ | $p \wedge (p \to q)$ | $[p \wedge (p \to q)] \to q$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 |

$\therefore\ [p \wedge (p \to q)] \to q$ is a tautology.

➡ $(r \to s) \wedge [(r \to s) \to (\neg t \vee u)] \to (\neg t \vee u)$ is also a tautology.

# Examples of Substitution Rules (3/3)

- **Example 2.11: (**Substitution Rule 2)

  $P$: $p \to (p \lor q)$ is a tautology.

  $P_1$: $p \to (\neg\neg p \lor q)$ is also a tautology.
  (Since $\neg\neg p \Leftrightarrow p$.)

# Examples (1/6)

- **Example 2.12:** Negate and simplify the compound statement $(p \vee q) \rightarrow r$.

1) $(p \vee q) \rightarrow r \Leftrightarrow \neg(p \vee q) \vee r$
2) $\neg[(p \vee q) \rightarrow r] \Leftrightarrow \neg[\neg(p \vee q) \vee r]$
3) $\neg[\neg(p \vee q) \vee r] \Leftrightarrow \neg\neg(p \vee q) \wedge \neg r$
4) $\neg\neg(p \vee q) \wedge \neg r \Leftrightarrow (p \vee q) \wedge \neg r$

➡ $\neg[(p \vee q) \rightarrow r] \Leftrightarrow (p \vee q) \wedge \neg r$

# Examples (2/6)

- **Example 2.13:** Let $p$, $q$ denote the primitive statements

  $p$: Joan goes to Lake George.

  $q$: Mary pays for Joan's shopping spree.

  and consider the implication

  $p \rightarrow q$: If Joan goes to Lake George, then Mary pays for Joan's shopping spree.

  How to write the negation of $p \rightarrow q$ in a way other than simply $\neg(p \rightarrow q)$?

# Examples (3/6)

- **Example 2.13 (cont.):**

  $p$: Joan goes to Lake George.

  $q$: Mary pays for Joan's shopping spree.

$$\neg(p \to q) \Leftrightarrow \neg(\neg p \vee q) \Leftrightarrow \neg\neg p \wedge \neg q \Leftrightarrow p \wedge \neg q$$

$\neg(p \to q)$: Joan goes to Lake George, but Mary does not pay for Joan's shopping spree.

**Note:** The negation of an if-then statement does not begin with the word if. It is not another implication.

# Examples (4/6)

- **Example 2.15:**

| $p$ | $q$ | $p \to q$ | $\neg q \to \neg p$ | $q \to p$ | $\neg p \to \neg q$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

  - The statement $\neg q \to \neg p$ is called the contrapositive of the implication $p \to q$.
  - The statement $q \to p$ is called the converse of $p \to q$.
  - The statement $\neg p \to \neg q$ is called the inverse of $p \to q$.

# Examples (5/6)

- **Example 2.15 (cont.):**
  Let us consider a specific example where $p$, $q$ represent the statements
  $p$: Jeff is concerned about his cholesterol levels.
  $q$: Jeff walks at least 2 miles 3 times a week.

  (The implication: $p \to q$). If Jeff is concerned about his cholesterol levels, then he will walks at least 2 miles 3 times a week.

  (The contrapositive: $\neg q \to \neg p$). If Jeff does not walk at least 2 miles 3 times a week, then he is not concerned about his cholesterol levels.

# Examples (6/6)

- **Example 2.15 (cont.):**

  (The converse: $q \rightarrow p$). If Jeff walks at least 2 miles 3 times a week, then he is concerned about his cholesterol levels.

  (The inverse: $\neg p \rightarrow \neg q$). If Jeff is not concerned about his cholesterol levels, then he will not walks at least 2 miles 3 times a week.

# Simplification of Compound Statements (1/2)

- **Example 2.16:** For primitive statements $p$, $q$, is there any simpler way to express the compound statement $(p \vee q) \wedge \neg(\neg p \wedge q)$ — that is, can we find a simpler statement that is logically equivalent to the one given?

| $(p \vee q) \wedge \neg(\neg p \wedge q)$ | **Reasons** |
| --- | --- |
| $\Leftrightarrow (p \vee q) \wedge (\neg\neg p \vee \neg q)$ | DeMorgan's Law |
| $\Leftrightarrow (p \vee q) \wedge (p \vee \neg q)$ | Law of Double Negation |
| $\Leftrightarrow p \vee (q \wedge \neg q)$ | Distributive Law of $\vee$ over $\wedge$ |
| $\Leftrightarrow p \vee F_0$ | Inverse Law |
| $\Leftrightarrow p$ | Identify Law |

Consequently, we see that $(p \vee q) \wedge \neg(\neg p \wedge q) \Leftrightarrow p$

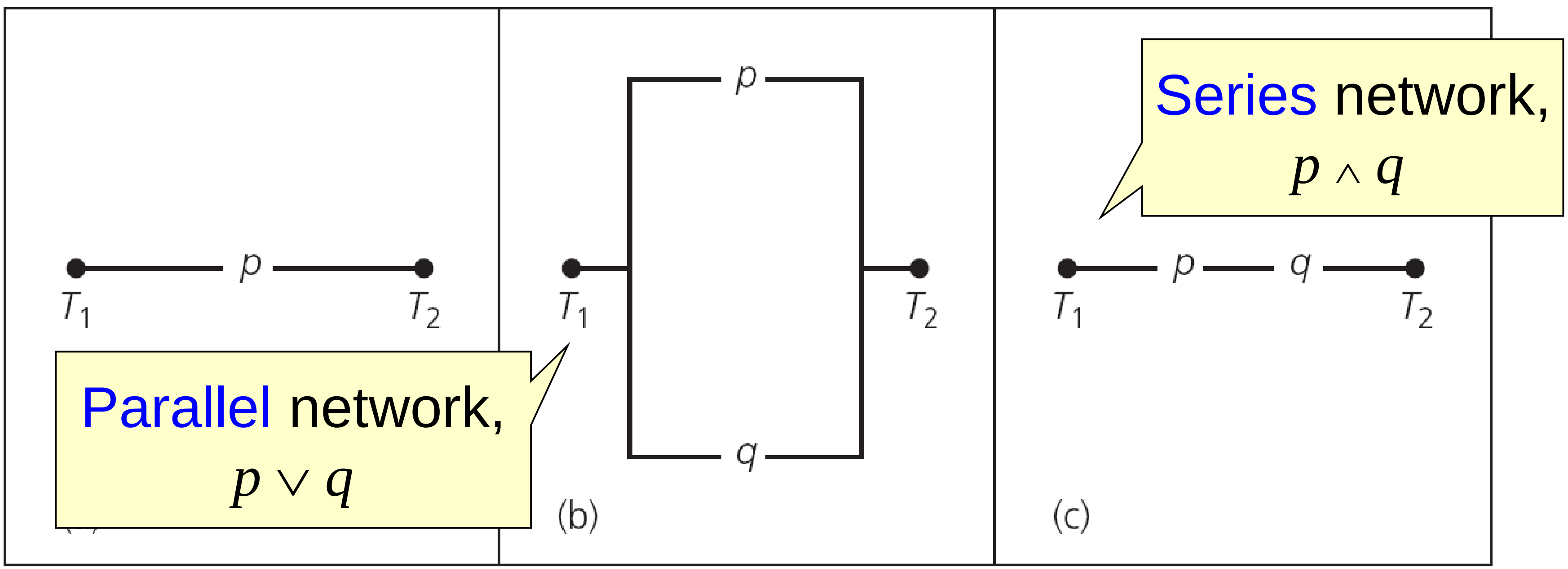
# Simplification of Compound Statements (2/2)

- **Example 2.17:** Consider the compound statement $\neg[\neg[(p \vee q) \wedge r] \vee \neg q]$, where $p$, $q$, $r$ are primitive statements.

| $\neg[\neg[(p \vee q) \wedge r] \vee \neg q]$ | **Reasons** |
|---|---|
| $\Leftrightarrow \neg\neg[(p \vee q) \wedge r] \wedge \neg\neg q]$ | DeMorgan's Law |
| $\Leftrightarrow [(p \vee q) \wedge r] \wedge q$ | Law of Double Negation |
| $\Leftrightarrow (p \vee q) \wedge (r \wedge q)$ | Associative Law of $\wedge$ |
| $\Leftrightarrow (p \vee q) \wedge (q \wedge r)$ | Commutative Law of $\wedge$ |
| $\Leftrightarrow [(p \vee q) \wedge q] \wedge r$ | Associative Law of $\wedge$ |
| $\Leftrightarrow q \wedge r$ | Absorption Law (as well as the Commutative Laws for $\wedge$ and $\vee$) |

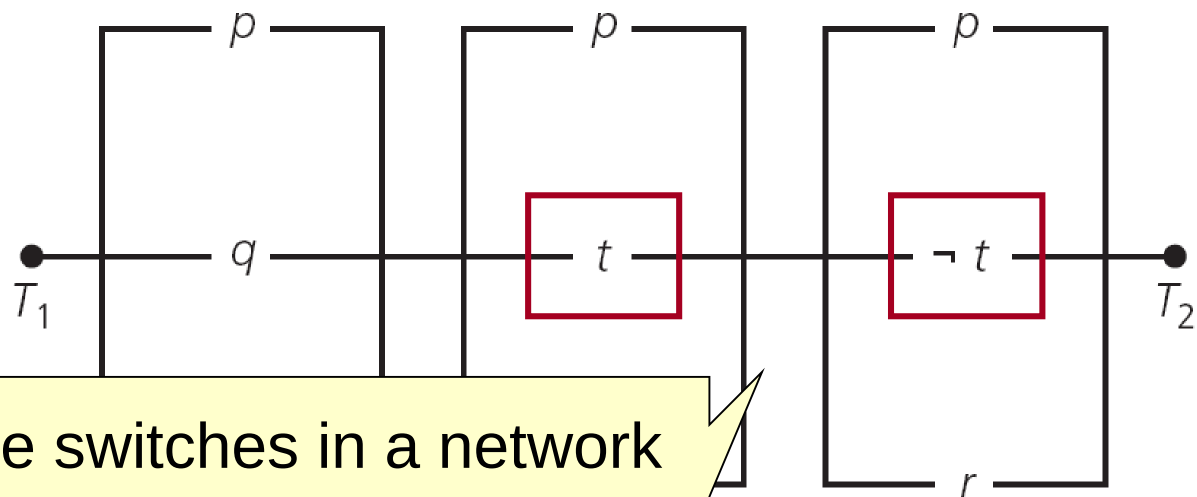
# Simplification of Switching Networks (1/4)

- A switching network is made up of wires and switches connecting two terminals $T_1$ and $T_2$.
  - Open (0): No current flows through it.
  - Closed (1): Current does flow through it.



# Simplification of Switching Networks (2/4)

- **Example 2.18:**




The switches in a network need not act independently of each other.

This network is represented by the statement $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$.

# Simplification of Switching Networks (3/4)

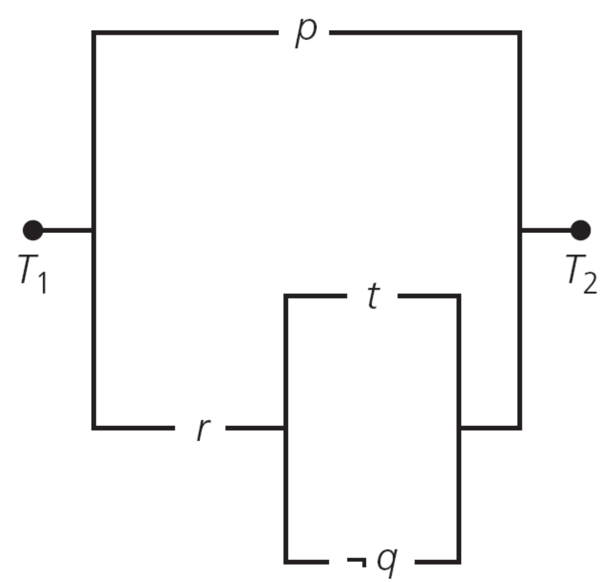
- **Example 2.18 (cont.):**

| | **Reasons** |
|---|---|
| $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$ | |
| $\Leftrightarrow p \vee [(q \vee r) \wedge (t \vee \neg q) \wedge (\neg t \vee r)]$ | Distributive Law of $\vee$ over $\wedge$ |
| $\Leftrightarrow p \vee [(q \vee r) \wedge (\neg t \vee r) \wedge (t \vee \neg q)]$ | Commutative Law of $\wedge$ |
| $\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (t \vee \neg q)]$ | Distributive Law of $\vee$ over $\wedge$ |
| $\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (\neg\neg t \vee \neg q)]$ | Law of Double Negation |
| $\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge \neg(\neg t \wedge q)]$ | DeMorgan's Law |
| $\Leftrightarrow p \vee [\neg(\neg t \wedge q) \wedge ((\neg t \wedge q) \vee r)]$ | Commutative Law of $\wedge$ (twice) |
| $\Leftrightarrow p \vee [(\neg(\neg t \wedge q) \wedge (\neg t \wedge q)) \vee (\neg(\neg t \wedge q) \wedge r)]$ | Distributive Law of $\wedge$ over $\vee$ |

# Simplification of Switching Networks (4/4)

- **Example 2.18 (cont.):**

| | |
|---|---|
| $\Leftrightarrow p \vee [F_0 \vee (\neg(\neg t \wedge q) \wedge r)]$ | $\neg s \wedge s \Leftrightarrow F_0$, for any statement $s$ |
| $\Leftrightarrow p \vee [(\neg(\neg t \wedge q)) \wedge r]$ | $F_0$ is the identity for $\vee$ |
| $\Leftrightarrow p \vee [r \wedge \neg(\neg t \wedge q)]$ | Commutative Law of $\wedge$ |
| $\Leftrightarrow p \vee [r \wedge (t \vee \neg q)]$ | DeMorgan's Law and the Law of Double Negation |




- **EXERCISES 2.2:** 4, 8, 10

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- Basic Connectives and Truth Tables
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# Valid Argument (1/2)

- Let us consider the implication

$$(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \rightarrow q.$$

  Here $n$ is a positive integer.
- The statements $p_1, p_2, p_3, \ldots, p_n$ are called the premises of the argument.
- The statement $q$ is the conclusion of the argument.
- The argument is called valid if whenever each of the premises $p_1, p_2, p_3, \ldots, p_n$ is true, then the conclusion $q$ is likewise true.

# Valid Argument (2/2)

- If any one of $p_1, p_2, p_3, \ldots, p_n$ is false, then the hypothesis $(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n)$ is false and the implication $(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \rightarrow q$ is automatically true, regardless of the truth value of $q$.
- One way to establish the validity of a given argument is to show that the statement $(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology.

# Example (1/3)

- **Example 2.19:** Let $p$, $q$, $r$ denote the primitive statements given as

  $p$: Roger studies.
  $q$: Roger plays tennis.
  $r$: Roger passes discrete mathematics.

  Now let $p_1$, $p_2$, $p_3$ denote the premises

  $p_1$: If Roger studies, then he will pass discrete mathematics.

  $p_2$: If Roger doesn't play tennis, then he'll study.

  $p_3$: Roger failed discrete mathematics.

# Example (2/3)

- **Example 2.19 (cont.):**
  We want to determine whether the argument

  $$(p_1 \wedge p_2 \wedge p_3) \rightarrow q$$

  is valid.

  To do so, we rewrite $p_1$, $p_2$, $p_3$ as

  $$p_1 \colon p \rightarrow r \qquad p_2 \colon \neg q \rightarrow p \qquad p_3 \colon \neg r$$

  and examine the truth table for the implication

  $$[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$$

# Example (3/3)

- **Example 2.19 (cont.):**

| | | | $p_1$ | $p_2$ | $p_3$ | $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ |
|---|---|---|---|---|---|---|
| $p$ | $q$ | $r$ | $p \rightarrow r$ | $\neg q \rightarrow p$ | $\neg r$ | $[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$ |
| 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 | 1 |

➡ Since the implication is a tautology, we can say that $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ is a valid argument.

➡ The truth of the conclusion $q$ is deduced or inferred from the truth of the premises $p_1$, $p_2$, and $p_3$.

# Logical Implication

- **Definition 2.4:** If $p$, $q$ are arbitrary statements such that $p \to q$ is a tautology, then we say that $p$ logically implies $q$ and we write $p \Rightarrow q$ to denote this situation. (We refer to $p \to q$ as a logical implication.)

# Rule of Inference: Modus Ponens

- **Example 2.22** (Modus Ponens or Rule of Detachment)**:** In symbolic form this rule is expressed by the logical implication

If (1) $p$ is true, and (2) $p \to q$ is true (or $p \Rightarrow q$), then the conclusion $q$ must also be true.

| $p$ | $q$ | | | |
|---|---|---|---|---|
| 0 | 0 | | | |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 |

$$\begin{array}{r} p \\ p \to q \\ \hline \therefore q \end{array}$$

# Rule of Inference: Law of the Syllogism (1/2)

- **Example 2.23** (Law of the Syllogism)**:** This rule is given by the logical implication

$$[(p \to q) \wedge (q \to r)] \to (p \to r),$$

where $p$, $q$, and $r$ are any statements.

In tabular form it is written

$$\begin{array}{r} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

# Rule of Inference: Law of the Syllogism (2/2)

- **Example 2.23 (cont.):**
  We may use it as follows:

1) If the integer 35244 is divisible by 396, then the integer 35244 is divisible by 66. $\quad p \to q$
2) If the integer 35244 is divisible by 66, then the integer 35244 is divisible by 3. $\quad q \to r$
3) Therefore, if the integer 35244 is divisible by 396, then the integer 35244 is divisible by 3. $\quad \therefore\ p \to r$

# Example (1/3)

- **Example 2.24:** Consider the following argument.
    1) Rita is baking a cake.
    2) If Rita is baking a cake, then she is not practicing her flute.
    3) If Rita is not practicing her flute, then her father will not buy her a car.
    4) Therefore Rita's father will not buy her a car.

  We may write the argument as

$$
\begin{array}{l}
\quad p \\
\quad p \rightarrow \neg q \\
\quad \neg q \rightarrow \neg r \\
\hline
\therefore \neg r
\end{array}
$$

# Example (2/3)

- **Example 2.24 (cont.):**

  We establish the validity of the argument as follows:

| **Steps** | **Reasons** |
| --- | --- |
| 1) $p \rightarrow \neg q$ | Premise |
| 2) $\neg q \rightarrow \neg r$ | Premise |
| 3) $p \rightarrow \neg r$ | This follows from steps (1) and (2) and the Law of the Syllogism |
| 4) $p$ | Premise |
| 5) $\therefore \neg r$ | This follows from steps (4) and (3) and the Rule of Detachment |

# Example (3/3)

- **Example 2.24 (cont.):**

  A second way to validate the argument as follows:

| **Steps** | **Reasons** |
| --- | --- |
| 1) $p$ | Premise |
| 2) $p \to \neg q$ | Premise |
| 3) $\neg q$ | Steps (1) and (2) and the Rule of the Detachment |
| 4) $\neg q \to \neg r$ | Premise |
| 5) $\therefore \neg r$ | Steps (3) and (4) and the Rule of Detachment |

# Rule of Inference: Modus Tollens (1/4)

- **Example 2.25** (Modus Tollens)**:** The rule of inference called Modus Tollens ("method of denying") is given by

$$\begin{array}{r} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

This follows from the logical implication $[(p \to q) \land \neg q] \to \neg p$.

# Rule of Inference: Modus Tollens (2/4)

- **Example 2.25 (cont.):**

  The following exemplifies the use of Modus Tollens in making a valid inference:

  1) If Connie is elected president of Phi Delta sorority, then Helen will pledge that sorority. $p \rightarrow q$
  2) Helen did not pledge Phi Delta sorority. $\neg q$
  3) Therefore Connie was not elected president of Phi Delta sorority. $\therefore \neg p$

# Rule of Inference: Modus Tollens (3/4)

- **Example 2.25 (cont.):**

  Use Modus Tollens to show that the following argument is valid (for primitive statements $p$, $r$, $s$, $t$, and $u$).

$$
\begin{array}{l}
p \rightarrow r \\
r \rightarrow s \\
t \vee \neg s \\
\neg t \vee u \\
\neg u \\
\hline
\therefore \neg p
\end{array}
$$

# Rule of Inference: Modus Tollens (4/4)

- **Example 2.25 (cont.):**

| **Steps** | **Reasons** |
|---|---|
| 1) $p \rightarrow r, r \rightarrow s$ | Premises |
| 2) $p \rightarrow s$ | Step (1) and the Law of the Syllogism |
| 3) $t \vee \neg s$ | Premise |
| 4) $\neg s \vee t$ | Step (3) and the Commutative Law of $\vee$ |
| 5) $s \rightarrow t$ | Step (4) and the fact that $\neg s \vee t \Leftrightarrow s \rightarrow t$ |
| 6) $p \rightarrow t$ | Steps (2) and (5) and the Law of the Syllogism |
| 7) $\neg t \vee u$ | Premise |
| 8) $t \rightarrow u$ | Step (7) and the fact that $\neg t \vee u \Leftrightarrow t \rightarrow u$ |
| 9) $p \rightarrow u$ | Steps (6) and (8) and the Law of the Syllogism |
| 10) $\neg u$ | Premise |
| 11) $\therefore \neg p$ | Steps (9) and (10) and Modus Tollens |

# Rule of Inference: Rule of Conjunction

- **Example 2.26** (Rule of Conjunction)**:** If $p$, $q$ are true statements, then $p \wedge q$ is a true statement. We call this rule the Rule of Conjunction and write it in tabular form as

$$\begin{array}{l} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$$

# Rule of Inference:
# Rule of Disjunctive Syllogism (1/2)

- **Example 2.27** (Rule of Disjunctive Syllogism)**:** The Rule of Disjunctive Syllogism comes about from the logical implication

$$[(p \vee q) \wedge \neg p] \rightarrow q,$$

which we can derive from Modus Ponens by observing that $p \vee q \Leftrightarrow \neg p \rightarrow q$.

In tabular form we write

$$\begin{array}{l} \quad p \vee q \\ \quad \neg p \\ \hline \therefore q \end{array}$$

# Rule of Inference: Rule of Disjunctive Syllogism (2/2)

- **Example 2.27 (cont.):**

1) Bart's wallet is in his back pocket or it is on his desk. $\quad p \vee q$
2) Bart's wallet is not in his back pocket. $\quad \neg p$
3) Therefore Bart's wallet in on his desk. $\quad \therefore q$

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

# Rule of Inference: Rule of Contradiction (1/2)

- **Example 2.28** (Rule of Contradiction)**:** Let $p$ denote an arbitrary statement, and $F_0$ a contradiction. Then $(\neg p \rightarrow F_0) \rightarrow p$ is a tautology.

$$\begin{array}{l} \neg p \rightarrow F_0 \\ \hline \therefore p \end{array}$$

| $p$ | $\neg p$ | $F_0$ | $\neg p \rightarrow F_0$ | $(\neg p \rightarrow F_0) \rightarrow p$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 |

If $p$ is a statement and $\neg p \rightarrow F_0$ is true, then $\neg p$ must be false because $F_0$ is false. So then we have $p$ true.

# Rule of Inference: Rule of Contradiction (2/2)

- **Example 2.28 (cont.):** The Rule of Contradiction is the basis of a method for establishing the validity of an argument – namely, the method of Proof by Contradiction.

  In general, when we want to establish the validity of the argument

  $$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q\,,$$

  We can establish the validity of the logically equivalent argument

  $$(p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge \neg q) \rightarrow F_0$$

# Summary of Inference Rules (1/2)

| Rule of Inference | Related Logical Implication | Name of Rule |
|---|---|---|
| **1)** $p$<br>$p \to q$<br>$\therefore q$ | $[p \land (p \to q)] \to q$ | Rule of Detachment (Modus Ponens) |
| **2)** $p \to q$<br>$q \to r$<br>$\therefore p \to r$ | $[(p \to q) \land (q \to r)] \to (p \to r)$ | Law of the Syllogism |
| **3)** $p \to q$<br>$\neg q$<br>$\therefore \neg p$ | $[(p \to q) \land \neg q] \to \neg p$ | Modus Tollens |
| **4)** $p$<br>$q$<br>$\therefore p \land q$ | | Rule of Conjunction |
| **5)** $p \lor q$<br>$\neg p$<br>$\therefore q$ | $[(p \lor q) \land \neg p] \to q$ | Rule of Disjunctive Syllogism |

# Summary of Inference Rules (2/2)

| | Rule | Tautology | Name |
|---|---|---|---|
| 6) | $\dfrac{\neg p \to F_0}{\therefore p}$ | $(\neg p \to F_0) \to p$ | Rule of Contradiction |
| 7) | $\dfrac{p \wedge q}{\therefore p}$ | $(p \wedge q) \to p$ | Rule of Conjunctive Simplification |
| 8) | $\dfrac{p}{\therefore p \vee q}$ | $p \to p \vee q$ | Rule of Disjunctive Amplification |
| 9) | $\dfrac{\begin{array}{l} p \wedge q \\ p \to (q \to r) \end{array}}{\therefore r}$ | $[(p \wedge q) \wedge [p \to (q \to r)]] \to r$ | Rule of Conditional Proof |
| 10) | $\dfrac{\begin{array}{l} p \to r \\ q \to r \end{array}}{\therefore (p \vee q) \to r}$ | $[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$ | Rule for Proof by Cases |
| 11) | $\dfrac{\begin{array}{l} p \to q \\ r \to s \\ p \vee r \end{array}}{\therefore q \vee s}$ | $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$ | Rule of the Constructive Dilemma |
| 12) | $\dfrac{\begin{array}{l} p \to q \\ r \to s \\ \neg q \vee \neg s \end{array}}{\therefore \neg p \vee \neg r}$ | $[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)] \to (\neg p \vee \neg r)$ | Rule of the Destructive Dilemma |

# Apply Rules (1/3)

- **Example 2.29:** Our first example demonstrates the validity of the argument

$$\begin{array}{l} p \to r \\ \neg p \to q \\ \underline{q \to s} \\ \therefore \neg r \to s \end{array}$$

| **Steps** | **Reasons** |
|---|---|
| 1) $p \to r$ | Premise |
| 2) $\neg r \to \neg p$ | Step (1) and $p \to r \Leftrightarrow \neg r \to \neg p$ |
| 3) $\neg p \to q$ | Premise |
| 4) $\neg r \to q$ | Steps (2) and (3) and the Law of the Syllogism |
| 5) $q \to s$ | Premise |
| 6) $\therefore\ \neg r \to s$ | Steps (4) and (5) and the Law of the Syllogism |

# Apply Rules (2/3)

- **Example 2.30:** Establish the validity of the argument

$$\begin{array}{l} p \to q \\ q \to (r \wedge s) \\ \neg r \vee (\neg t \vee u) \\ p \wedge t \\ \hline \therefore u \end{array}$$

| **Steps** | **Reasons** |
|---|---|
| 1) $p \to q$ | Premise |
| 2) $q \to (r \wedge s)$ | Premise |
| 3) $p \to (r \wedge s)$ | Steps (1) and (2) and the Law of the Syllogism |
| 4) $p \wedge t$ | Premise |
| 5) $p$ | Step (4) and the Rule of Conjunctive Simplification |
| 6) $r \wedge s$ | Step (5) and (3) and the Rule of Detachment |
| 7) $r$ | Step (6) and the Rule of Conjunctive Simplification |

# Apply Rules (3/3)

- **Example 2.30 (cont.):**

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow (r \wedge s) \\ \neg r \vee (\neg t \vee u) \\ p \wedge t \\ \hline \therefore u \end{array}$$

| | |
|---|---|
| 8) $\neg r \vee (\neg t \vee u)$ | Premise |
| 9) $\neg(r \wedge t) \vee u$ | Step (8), the Associative Law of $\vee$, and DeMorgan's Laws |
| 10) $t$ | Step (4) and the Rule of Conjunctive Simplification |
| 11) $r \wedge t$ | Steps (7) and (10) and the Rule of Conjunction |
| 12) $\therefore u$ | Steps (9) and (11) and the Rule of Disjunctive Syllogism |

# Proof by Contradiction (1/3)

- **Example 2.32:** Consider the argument

$$\begin{array}{l} \neg p \leftrightarrow q \\ q \rightarrow r \\ \neg r \\ \hline \therefore p \end{array}$$

- To establish the validity for this argument, we assume the negation $\neg p$ of the conclusion $p$ as another premise. The objective now is to use these four premises to derive a contradiction of $F_0$. Our derivation follows.

# Proof by Contradiction (2/3)

- **Example 2.32 (cont.):**

$$\begin{array}{l} \neg p \leftrightarrow q \\ q \rightarrow r \\ \neg r \\ \hline \therefore p \end{array}$$

| **Steps** | **Reasons** |
|---|---|
| 1) $\neg p \leftrightarrow q$ | Premise |
| 2) $(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)$ | Step (1) and $(\neg p \leftrightarrow q) \Leftrightarrow [(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)]$ |
| 3) $\neg p \rightarrow q$ | Step (2) and the Rule of Conjunctive Simplification |
| 4) $q \rightarrow r$ | Premise |
| 5) $\neg p \rightarrow r$ | Steps (3) and (4) and the Law of the Syllogism |

# Proof by Contradiction (3/3)

- **Example 2.32 (cont.):**

$$\begin{array}{l} \neg p \leftrightarrow q \\ q \rightarrow r \\ \neg r \\ \hline \therefore p \end{array}$$

| | |
|---|---|
| 6) $\neg p$ | Premise (the one assumed) |
| 7) $r$ | Steps (5) and (6) and the Rule of Detachment |
| 8) $\neg r$ | Premise |
| 9) $r \wedge \neg r$ ($\Leftrightarrow F_0$) | Steps (7) and (8) and the Rule of Conjunction |
| 10) $\therefore p$ | Steps (6) and (9) and the method of Proof by Contradiction |

# Principle

- $[p \rightarrow (q \rightarrow r)] \Leftrightarrow [(p \wedge q) \rightarrow r]$

| $p$ | $q$ | $r$ | $p \wedge q$ | $(p \wedge q) \rightarrow r$ | $q \rightarrow r$ | $p \rightarrow (q \rightarrow r)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

$$[(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow (q \rightarrow r)] \Longleftrightarrow [(p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge q)^{\dagger} \rightarrow r]$$

# Example (1/3)

- **Example 2.33:** In order to establish the validity of the argument

$$\begin{array}{l} u \rightarrow r \\ (r \wedge s) \rightarrow (p \vee t) \\ q \rightarrow (u \wedge s) \\ \neg t \\ \hline \therefore q \rightarrow p \end{array}$$

We consider the corresponding argument

$$\begin{array}{l} u \rightarrow r \\ (r \wedge s) \rightarrow (p \vee t) \\ q \rightarrow (u \wedge s) \\ \neg t \\ q \\ \hline \therefore p \end{array}$$

# Example (2/3)

$$\begin{array}{l} u \to r \\ (r \wedge s) \to (p \vee t) \\ q \to (u \wedge s) \\ \neg t \\ q \\ \hline \therefore p \end{array}$$

- **Example 2.33 (cont.):**

| **Steps** | **Reasons** |
|---|---|
| 1) $q$ | Premise |
| 2) $q \to (u \wedge s)$ | Premise |
| 3) $u \wedge s$ | Steps (1) and (2) and the Rule of Detachment |
| 4) $u$ | Step (3) and the Rule of Conjunctive Simplification |
| 5) $u \to r$ | Premise |
| 6) $r$ | Steps (4) and (5) and the Rule of Detachment |
| 7) $s$ | Step (3) and the Rule of Conjunctive Simplification |

# Example (3/3)

- **Example 2.33 (cont.):**

$$\begin{array}{l} u \to r \\ (r \wedge s) \to (p \vee t) \\ q \to (u \wedge s) \\ \neg t \\ q \\ \hline \therefore p \end{array}$$

| | |
|---|---|
| 8) $r \wedge s$ | Steps (6) and (7) and the Rule of Conjunction |
| 9) $(r \wedge s) \to (p \vee t)$ | Premise |
| 10) $p \vee t$ | Steps (8) and (9) and the Rule of Detachment |
| 11) $\neg t$ | Premise |
| 12) $\therefore p$ | Steps (10) and (11) and the Rule of Disjunctive Syllogism |

# Counterexample of the Argument (1/3)

- Given an argument $p_1 \wedge p_2 \wedge p_3 \wedge \ldots p_n \rightarrow q$
  we say that the argument is invalid if it is possible for each of the premises $p_1, p_2, p_3, \ldots, p_n$ to be true (with truth value 1), while the conclusion $q$ is false (with truth value 0).

# Counterexample of the Argument (2/3)

- **Example 2.34:** Consider the primitive statements $p$, $q$, $r$, $s$, and $t$ and the argument

$$\begin{array}{l} p \\ p \vee q \\ q \rightarrow (r \rightarrow s) \\ t \rightarrow r \\ \hline \therefore \neg s \rightarrow \neg t \end{array}$$

Show that this is an invalid argument.

# Counterexample of the Argument (3/3)

- **Example 2.35:** What can we say about the validity or invalidity of the following argument? Here $p$, $q$, $r$, and $s$ denote primitive statements.

$$\begin{array}{l} \quad p \rightarrow q \\ \quad q \rightarrow s \\ \quad r \rightarrow \neg s \\ \quad \neg p \veebar r \\ \hline \therefore \neg p \end{array}$$

- **EXERCISES 2.3:** 4, 6, 8

# Outline

- Basic Connectives and Truth Tables
- Logical Equivalence: The Laws of Logic
- Logical Implication: Rules of Inference
- ***<u>The Use of Quantifiers</u>***
- Quantifiers, Definitions, and the Proofs of Theorems

# Open Statement

- **Definition 2.5:** A declarative sentence is an open statement if
  1. it contains one or more variables, and
  2. it is not a statement, but
  3. it becomes a statement when the variables in it are replaced by certain allowable choices.

These allowable choices constitute what is called the universe or universe of discourse for the open statement.

# Notation of Open Statement (1/2)

- The open statement “The number $x + 2$ is an even integer” is denoted by $p(x)$ [or $q(x)$, etc.].
- Then $\neg p(x)$ may be read “The number $x + 2$ is not an even integer.”
- We use $q(x, y)$ to represent an open statement that contains two variables.

# Notation of Open Statement (2/2)

- Two types of quantifiers
  - The existential quantifier $\exists$
  - The universal quantifier $\forall$
- **Example:** $r(x)$: "$2x$ is an even integer" with the universe of all integers.

$\Rightarrow$ $\forall x\ r(x)$ and $\exists x\ r(x)$ both are true statements.

- The variable $x$ in each of open statements $p(x)$ is called a free variable (of the open statement).
- For $\exists x\ p(x, y)$, $\forall x\ p(x, y)$
  - $x$ is a bounded variable
  - $y$ is a free variable

# Example (1/3)

- **Example 2.36:** Here the universe comprises all real numbers.

$p(x)$: $x \geq 0$ $\qquad$ $r(x)$: $x^2 - 3x - 4 = 0$

$q(x)$: $x^2 \geq 0$ $\qquad$ $s(x)$: $x^2 - 3 > 0$

1) $\exists x\, [p(x) \wedge r(x)]$ TRUE
2) $\forall x\, [p(x) \rightarrow q(x)]$ TRUE
3) $\forall x\, [q(x) \rightarrow s(x)]$ FALSE
4) $\forall x\, [r(x) \vee s(x)]$ FALSE
5) $\forall x\, [r(x) \rightarrow p(x)]$ FALSE

# Example (2/3)

- **Example 2.39:** In the following program segment, $n$ is an integer variable and the variable $A$ is an array $A[1], A[2], \ldots, A[20]$ of 20 integer values.

```
for n := 1 to 20 do
    A[n] := n * n − n
```

Represent the following statements about the array $A$ in quantified form, where the universe consists of all integers from 1 to 20, inclusive.

# Example (3/3)

- **Example 2.39 (cont.):**

1) Every entry in the array is nonnegative:

$$\forall n\ (A[n] \geq 0).$$

2) There exist two consecutive entries in *A* where the larger entry is twice the smaller:

$$\exists n\ (A[n+1] = 2A[n]).$$

3) The entries in the array are sorted in (strictly) ascending order:

$$\forall n\ [(1 \leq n \leq 19) \rightarrow (A[n] < A[n+1])].$$

4) The entries in the array are distinct:

$$\forall m\ \forall n\ [(m \neq n) \rightarrow (A[m] \neq A[n])], \qquad \text{or}$$

$$\forall m, n\ [(m < n) \rightarrow (A[m] \neq A[n])].$$

# Definitions (1/2)

- **Definition 2.6:** Let $p(x)$, $q(x)$ be open statements defined for a given universe. The open statements $p(x)$ and $q(x)$ are called (logically) equivalent, and we write $\forall x\ [p(x) \Leftrightarrow q(x)]$ when the biconditional $p(a) \leftrightarrow q(a)$ is true for each replacement $a$ from the universe (that is, $p(a) \Leftrightarrow q(a)$ for each $a$ in the universe). If the implication $p(a) \rightarrow q(a)$ is true for each $a$ in the universe (that is, $p(a) \Rightarrow q(a)$ for each $a$ in the universe), then we write $\forall x\ [p(x) \Rightarrow q(x)]$ and say that $p(x)$ logically implies $q(x)$.

# Definitions (2/2)

- **Definition 2.7:** For open statements $p(x)$, $q(x)$ — defined for a prescribed universe — and the universally quantified statement $\forall x\,[p(x) \rightarrow q(x)]$ we define:
  1) The contrapositive of $\forall x\,[p(x) \rightarrow q(x)]$ to be $\forall x\,[\neg q(x) \rightarrow \neg p(x)]$.
  2) The converse of $\forall x\,[p(x) \rightarrow q(x)]$ to be $\forall x\,[q(x) \rightarrow p(x)]$.
  3) The inverse of $\forall x\,[p(x) \rightarrow q(x)]$ to be $\forall x\,[\neg p(x) \rightarrow \neg q(x)]$.

# Summarization of Quantifiers

| Statement | When Is It True? | When Is It False? |
|---|---|---|
| $\exists x\ p(x)$ | For some (at least one) $a$ in the universe, $p(a)$ is true. | For every $a$ in the universe, $p(a)$ is false. |
| $\forall x\ p(x)$ | For every replacement $a$ from the universe, $p(a)$ is true. | There is at least one replacement $a$ from the universe for which $p(a)$ is false. |
| $\exists x\ \neg p(x)$ | For at least one choice $a$ in the universe, $p(a)$ is false, so its negation $\neg p(a)$ is true. | For every replacement $a$ in the universe, $p(a)$ is true. |
| $\forall x\ \neg p(x)$ | For every replacement $a$ from the universe, $p(a)$ is false and its negation $\neg p(a)$ is true. | There is at least one replacement $a$ from the universe for which $\neg p(a)$ is false and $p(a)$ is true. |

# Example

- **Example 2.42:** Here the universe consists of all the integers, and the open statements $r(x)$, $s(x)$ are given by

$$r(x):\quad 2x + 1 = 5 \qquad\qquad s(x):\quad x^2 = 9.$$

$\exists x\, r(x) \wedge \exists x\, s(x)$ is true.

$\exists x\, [r(x) \wedge s(x)]$ is false.

$\Rightarrow$ $\exists x\, [r(x) \wedge s(x)] \not\Leftrightarrow [\exists x\, r(x) \wedge \exists x\, s(x)]$

(is not logically equivalent to)

$[\exists x\, r(x) \wedge \exists x\, s(x)] \not\Rightarrow \exists x\, [r(x) \wedge s(x)]$

(does not logically imply)

# Logical Equivalences and Logical Implications for Quantified Statements

- For a prescribed universe and any statements $p(x)$, $q(x)$ in the variable $x$:

$$\exists x\,[p(x) \wedge q(x)] \Rightarrow [\exists x\, p(x) \wedge \exists x\, q(x)]$$

$$\exists x\,[p(x) \vee q(x)] \Leftrightarrow [\exists x\, p(x) \vee \exists x\, q(x)]$$

$$\forall x\,[p(x) \wedge q(x)] \Leftrightarrow [\forall x\, p(x) \wedge \forall x\, q(x)]$$

$$[\forall x\, p(x) \vee \forall x\, q(x)] \boxed{\Rightarrow} \forall x[\, p(x) \vee q(x)]$$

How about the other direction?

# Rules for Negating Statements

Rules for Negating Statements with One Quantifier

- $\neg[\forall x\ p(x)] \Leftrightarrow \exists x\ \neg p(x)$
- $\neg[\exists x\ p(x)] \Leftrightarrow \forall x\ \neg p(x)$
- $\neg[\forall x\ \neg p(x)] \Leftrightarrow \exists x\ \neg\neg p(x) \Leftrightarrow \exists x\ p(x)$
- $\neg[\exists x\ \neg p(x)] \Leftrightarrow \forall x\ \neg\neg p(x) \Leftrightarrow \forall x\ p(x)$

# Order of Quantifiers

- **Example 2.48:** We restrict ourselves here to the universe of all integers and let $p(x, y)$ denote the open statement "$x + y = 17$."

The statement $\forall x \, \exists y \; p(x, y)$ is true.

The statement $\exists y \, \forall x \; p(x, y)$ is false.

# Negate the Definition of Limit

- **Example 2.50:** Negate the definition of limit.

$$\lim_{x \to a} f(x) = L \Leftrightarrow \forall \epsilon > 0 \;\; \exists \delta > 0$$

$$\forall x \left[ (0 < |x - a| < \delta) \to (|f(x) - L| < \epsilon) \right]$$

$$\lim_{x \to a} f(x) \neq L$$

$$\Leftrightarrow \neg[\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \left[ (0 < |x - a| < \delta) \to (|f(x) - L| < \epsilon) \right]]$$

$$\Leftrightarrow \exists \epsilon > 0 \;\; \forall \delta > 0 \;\; \exists x \; \neg[(0 < |x - a| < \delta) \to (|f(x) - L| < \epsilon)]$$

$$\Leftrightarrow \exists \epsilon > 0 \;\; \forall \delta > 0 \;\; \exists x \; \neg[\neg(0 < |x - a| < \delta) \vee (|f(x) - L| < \epsilon)]$$

$$\Leftrightarrow \exists \epsilon > 0 \;\; \forall \delta > 0 \;\; \exists x \; [\neg\neg(0 < |x - a| < \delta) \wedge \neg(|f(x) - L| < \epsilon)]$$

$$\Leftrightarrow \exists \epsilon > 0 \;\; \forall \delta > 0 \;\; \exists x \; [(0 < |x - a| < \delta) \wedge (|f(x) - L| \geq \epsilon)]$$

- **EXERCISE 2.4:** 18

# Outline

- Basic Connectives and Truth Tables
- Logical Equivalence: The Laws of Logic
- Logical Implication: Rules of Inference
- The Use of Quantifiers
- ***Quantifiers, Definitions, and the Proofs of Theorems***

# The Rule of Universal Specification

- If an open statements becomes true for all replacements by the members in a given universe, then that open statement is true for each specific individual member in that universe. (A bit more symbolically — if $p(x)$ is an open statement for a given universe, and if $\forall x\, p(x)$ is true, then $p(a)$ is true for each $a$ in the universe.)

# Example (1/9)

- **Example 2.53:** a) For the universe of all people, consider the open statements

  $m(x)$: $x$ is a mathematics professor

  $c(x)$ : $x$ has studied calculus.

  Now consider the following argument.

  – All mathematics professors have studied calculus.

  – Leona is a mathematics professor.

  – Therefore Leona has studied calculus.

# Example (2/9)

- **Example 2.53 (cont.):**
  Let $l$ represent this particular woman (in our universe) named Leona.

$$\begin{array}{l} \forall x\ [m(x) \rightarrow c(x)] \\ m(l) \\ \hline \therefore c(l) \end{array}$$

| **Steps** | **Reasons** |
|---|---|
| 1. $\forall x\ [m(x) \rightarrow c(x)]$ | Premise |
| 2. $m(l)$ | Premise |
| 3. $m(l) \rightarrow c(l)$ | Step (1) and the Rule of Universal Specification |
| 4. $\therefore c(l)$ | Steps (2) and (3) and the Rule of Detachment |

# Example (3/9)

- **Example 2.53 (cont.):** b) For an example of a more mathematical nature let us consider the universe of all triangles in the plane in conjunction with the open statements

  $p(t)$: $t$ has two sides of equal length.
  $q(t)$: $t$ is an isosceles triangle.
  $r(t)$: $t$ has two angles of equal measure.

  Let triangle $XYZ$ with no two angles of equal measure be designed by $c$.

# Example (4/9)

- **Example 2.53 (cont.):**

In triangle $XYZ$ there is no pair of angles of equal measure.
If a triangles has two sides of equal length, then it is isosceles.
If a triangle is isosceles, then is has two angles of equal measure.
Therefore triangle $XYZ$ has no two sides of equal length.

$$\begin{array}{l} \neg r(c) \\ \forall t\ [p(t) \rightarrow q(t)] \\ \forall t\ [q(t) \rightarrow r(t)] \\ \hline \therefore \neg p(c) \end{array}$$

is a valid one — as evidenced by the following.

# Example (5/9)

- **Example 2.53 (cont.):**

| **Steps** | **Reasons** |
|---|---|
| 1. $\forall t\ [p(t) \rightarrow q(t)]$ | Premise |
| 2. $p(c) \rightarrow q(c)$ | Step (1) and the Rule of Universal Specification |
| 3. $\forall t\ [q(t) \rightarrow r(t)]$ | Premise |
| 4. $q(c) \rightarrow r(c)$ | Step (3) and the Rule of Universal Specification |
| 5. $p(c) \rightarrow r(c)$ | Steps (2) and (4) and the Law of the Syllogism |
| 6. $\neg r(c)$ | Premise |
| 7. $\therefore\ \neg p(c)$ | Steps (5) and (6) and Modus Tollens |

# Example (6/9)

- **Example 2.53 (cont.):** c) Here we'll consider the universe to be made up of the entire student body at a particular college. One specific student, Mary Gusberti, will be designated by $m$.
  For this universe and the open statements.

  $j(x)$: $x$ is a junior     $s(x)$: $x$ is a senior
  $p(x)$: $x$ is enrolled in a physical education class.

# Example (7/9)

- **Example 2.53 (cont.):**

  We consider the argument :
  No junior or senior is enrolled in a physical education class.
  Mary Gusberti is enrolled in a physical education class.
  Therefore Mary Gusberti is not a senior.

$$\begin{array}{l} \quad \forall x\ [(j(x) \vee s(x)) \rightarrow \neg p(x)] \\ \quad p(m) \\ \hline \therefore \neg s(m) \end{array}$$

# Example (8/9)

- **Example 2.53 (cont.):**

| **Steps** | **Reasons** |
|---|---|
| 1. $\forall x\,[(j(x) \vee s(x)) \rightarrow \neg p(x)]$ | Premise |
| 2. $p(m)$ | Premise |
| 3. $(j(m) \vee s(m)) \rightarrow \neg p(m)$ | Step(1) and the Rule of Universal Specification |
| 4. $p(m) \rightarrow \neg(j(m) \vee s(m))$ | Step(3), $(q \rightarrow t) \Leftrightarrow (\neg t \rightarrow \neg q)$, and the Law of Double Negation |
| 5. $p(m) \rightarrow (\neg j(m) \wedge \neg s(m))$ | Step (4) and DeMorgan's Law |

# Example (9/9)

- **Example 2.53 (cont.):**

| | |
|---|---|
| 6. $\neg j(m) \wedge \neg s(m)$ | Steps (2) and (5) and the Rule of Detachment (or Modus Ponens) |
| 7. $\therefore\ \neg s(m)$ | Step (6) and the Rule of Conjunctive Simplification |

# Some Possible Errors (1/3)

- Within the universe of all polygons in the plane, let $c$ denote one specific polygon – the quadrilateral *EFGH*, where the measure of angle *E* is 91°. For the open statements
  $p(x)$: $x$ is a square $\quad$ $q(x)$: $x$ has four sides,
  the following argument is invalid.

  All squares have four sides.

  Quadrilateral EFGH has four sides.

  Therefore quadrilateral *EFGH* is a square.

# Some Possible Errors (2/3)

- Within the universe of all polygons in the plane, let $c$ denote one specific polygon – the quadrilateral *EFGH*, where the measure of angle *E* is 91$^\circ$. For the open statements
  $p(x)$: $x$ is a square $q(x)$: $x$ has four sides,
  the following argument is invalid.

  All squares have four sides.

  Quadrilateral EFGH has four sides.

  Therefore quadrilateral *EFGH* is a square.

$$\begin{array}{l} \forall x\ [p(x) \rightarrow q(x)] \\ q(c) \\ \hline \therefore p(c) \end{array}$$

# Some Possible Errors (3/3)

- $p(x)$: $x$ is a square $\quad$ $q(x)$: $x$ has four sides,
  The following argument is invalid.

  All squares have four sides.

  Quadrilateral EFGH is not a square.
  Therefore quadrilateral *EFGH* does not have four sides.

$$\begin{array}{l} \forall x\ [p(x) \rightarrow q(x)] \\ \neg p(c) \\ \hline \therefore \neg q(c) \end{array}$$

# The Rule of Universal Generalization

- If an open statement $p(x)$ is proved to be true when $x$ is replaced by any arbitrarily chosen element $c$ from our universe, then the universally quantified statement $\forall x\, p(x)$ is true. Furthermore, the rule extends beyond a single variable. So if, for example, we have an open statement $q(x, y)$ that is proved to be true when $x$ and $y$ are replaced by arbitrarily chosen elements from the same universe, or their own respective universes, then the universally quantified statement $\forall x \forall y\, q(x, y)$[or, $\forall x, y\, q(x, y)$] is true. Similar results hold for the cases of three or more variables.

# Example (1/5)

- **Example 2.54:** Let $p(x)$, $q(x)$, and $r(x)$ be open statements that are defined for a given universe. We show that the argument

$$\begin{array}{r} \forall x\ [p(x) \rightarrow q(x)] \\ \forall x\ [q(x) \rightarrow r(x)] \\ \hline \therefore \forall x\ [p(x) \rightarrow r(x)] \end{array}$$

  is valid by considering the following.

# Example (2/5)

- **Example 2.54 (cont.):**

| **Steps** | **Reasons** |
|---|---|
| 1) $\forall x\,[p(x) \to q(x)]$ | Premise |
| 2) $p(c) \to q(c)$ | Step (1) and the Rule of Universal Specification |
| 3) $\forall x\,[q(x) \to r(x)]$ | Premise |
| 4) $q(c) \to r(c)$ | Step (3) and the Rule of Universal Specification |
| 5) $p(c) \to r(c)$ | Steps (2) and (4) and the Law of the Syllogism |
| 6) $\therefore \forall x\,[p(x) \to r(x)]$ | Step (5) and the Rule of Universal Generalization |

# Example (3/5)

- **Example 2.56:** The steps and reasons needed to establish the validity of the argument

$$\begin{array}{l} \quad \forall x\,[p(x) \vee q(x)] \\ \quad \forall x\,[(\neg p(x) \wedge q(x)) \rightarrow r(x)] \\ \hline \therefore \forall x\,[\neg r(x) \rightarrow p(x)] \end{array}$$

are given as follows.

| **Steps** | **Reasons** |
|---|---|
| 1) $\forall x\,[p(x) \vee q(x)]$ | Premise |
| 2) $p(c) \vee q(c)$ | Step (1) and the Rule of Universal Specification |

# Example (4/5)

- **Example 2.56 (cont.):**

| | |
|---|---|
| 3) $\forall x\,[(\neg p(x) \wedge q(x)) \rightarrow r(x)]$ | Premise |
| 4) $[\neg p(c) \wedge q(c)] \rightarrow r(c)$ | Step (3) and the Rule of Universal Specification |
| 5) $\neg r(c) \rightarrow \neg[\neg p(c) \wedge q(c)]$ | Step (4) and $s \rightarrow t \Leftrightarrow \neg t \rightarrow \neg\, s$ |
| 6) $\neg r(c) \rightarrow [p(c) \vee \neg q(c)]$ | Step (5), DeMorgan's Law, and the Law of Double Negation |
| 7) $\neg r(c)$ | Premise (assumed) |
| 8) $p(c) \vee \neg q(c)$ | Step (7) and (6) and Modus Ponens |

# Example (5/5)

- **Example 2.56 (cont.):**

| | |
|---|---|
| 9) $[p(c) \vee q(c)] \wedge [p(c) \vee \neg q(c)]$ | Steps (2) and (8) and the Rule of Conjunction |
| 10) $p(c) \vee [q(c) \wedge \neg q(c)]$ | Step (9) and the Distributive Law of $\vee$ over $\wedge$ |
| 11) $p(c)$ | Step (10), $q(c) \wedge \neg q(c) \Leftrightarrow F_0$, and $p(c) \vee F_0 \Leftrightarrow p(c)$ |
| 12) $\therefore \forall x\, [\neg r(x) \rightarrow p(x)]$ | Steps (7) and (11) and the Rule of Universal Generalization |

# Read by Yourself

- **Definition 2.8**
- **Theorem 2.2**
- **Theorem 2.3**
- **Theorem 2.4**
- **Theorem 2.5**
- Suppose we want to prove $\forall m\ [p(m) \rightarrow q(m)]$, we could prove it by the contrapositive method or by contradiction.

| | Assumption | Result Derived |
|---|---|---|
| Contraposition | $\neg q(m)$ | $\neg p(m)$ |
| Contradiction | $p(m)$ and $\neg q(m)$ | $F_0$ |

# Homework Assignment #2

- **EXERCISES 2.1**

  4, 6
- **EXERCISES 2.2**

  4, 8, 10
- **EXERCISES 2.3**

  4, 6, 8
- **EXERCISE 2.4**

  18
- **EXERCISE 2.5**

  10