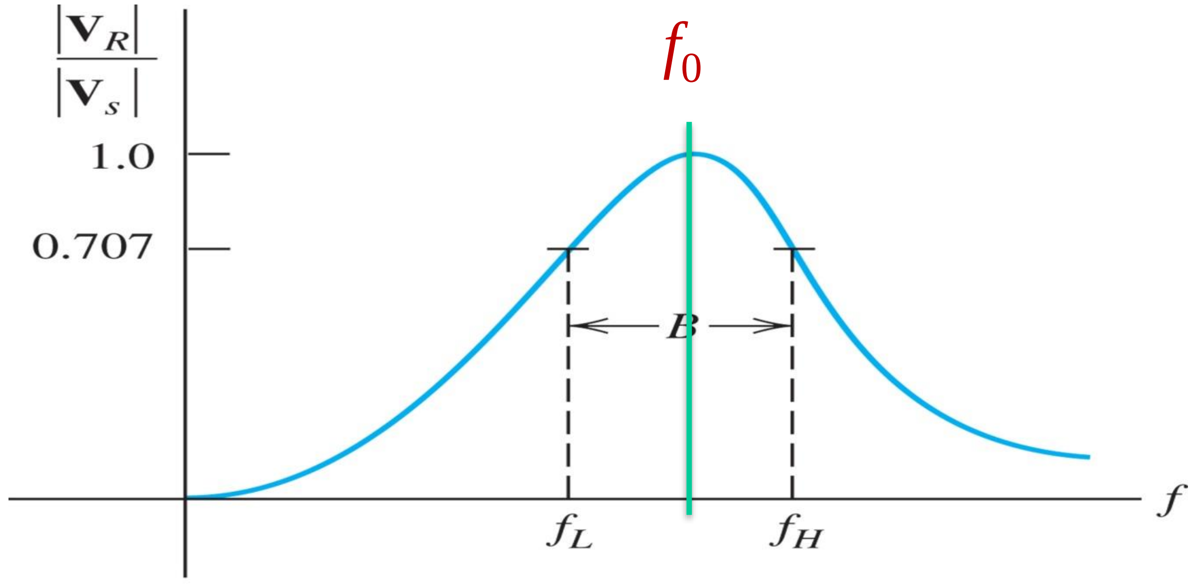
6.6 Series Resonance (串聯諧振)

- The resonant circuit behaves as a bandpass filter (帶通濾波器)。
- A band of components centered at the resonant frequency is passed (在共振頻率附近的頻帶會通過)。
- The components farther from the resonant frequency are rejected/reduced .



Bandwidth (頻寬): B

$$B = f_H - f_L$$

$$B = \frac{f_0}{Q_s}$$

f_0 :fundamental frequency

Q: quality factor (品質因子)

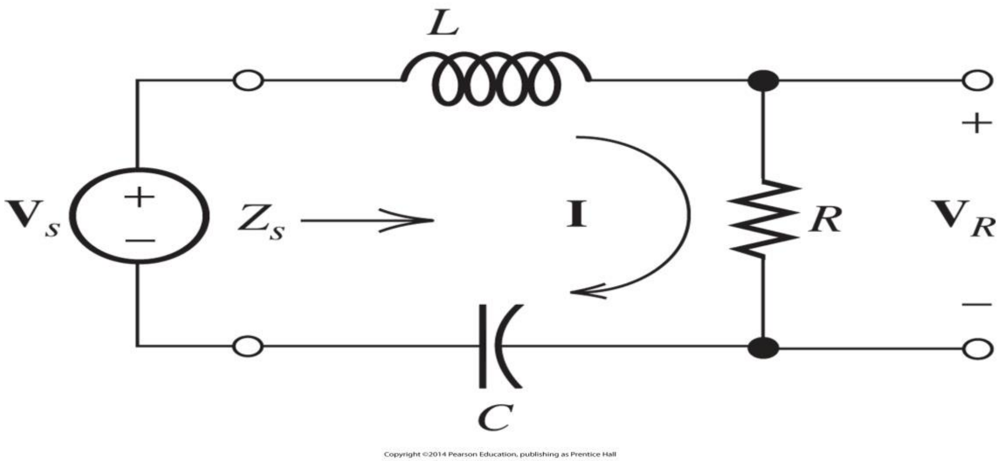
Q_s 越大通過的頻寬越窄(B越小)，越具選擇性，品質越好

$$f_H \cong f_0 + \frac{B}{2}$$

$$f_L \cong f_0 - \frac{B}{2}$$

f_H , f_L 代表高低兩個 half-power frequency

The series resonant circuit



整體阻抗 $Z_s(f) = j2\pi fL + R - j\dfrac{1}{2\pi fC}$

在共振頻率出現時，電容與電感的阻抗magnitude 相同，整體阻抗只剩電阻，電阻電壓最大(輸出最大)

$$2\pi f_0 L = \frac{1}{2\pi f_0 C}$$

$$f_0^2 = \frac{1}{4\pi^2 LC}$$

$$\Longrightarrow$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

For a series circuit, the quality factor (品質因子) Q_s is defined as

$$\frac{\text{reactance of the inductance}}{\text{resistance}}$$ (電感電抗與電阻的比值)

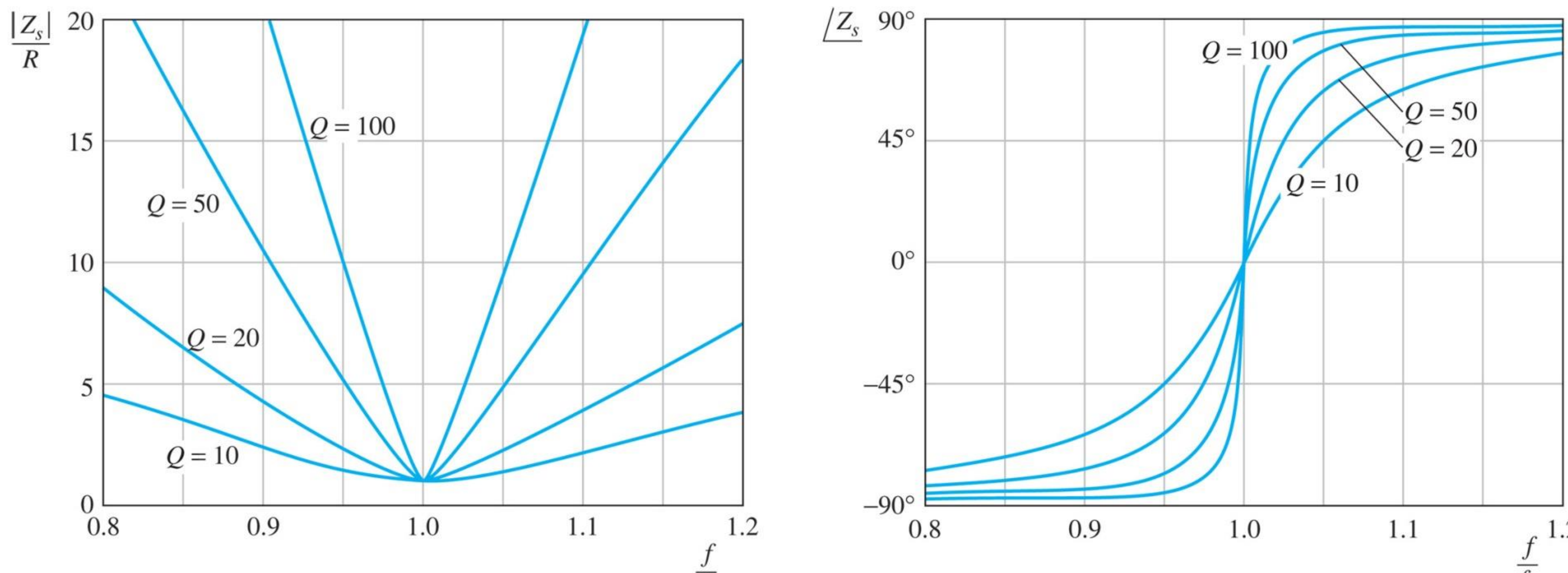
$$Q_s = \frac{2\pi f_0 L}{R} \quad \Longrightarrow \quad j2\pi fL = jQ_s R\frac{f}{f_0}$$

因為 $2\pi f_0 L = \dfrac{1}{2\pi f_0 C}$ 所以Q_s也可表示成

$$Q_s = \frac{1}{2\pi f_0 CR} \quad \Longrightarrow \quad -j\frac{1}{2\pi fC} = -jQ_s R\frac{f_0}{f}$$

整體阻抗 $Z_s(f) = j2\pi fL + R - j\dfrac{1}{2\pi fC}$ 可改寫為

$$Z_s(f) = R\left[1 + jQ_s\left(\frac{f}{f_0} - \frac{f_0}{f}\right)\right]$$



(a) Magnitude

(b) Phase

$$\frac{|Z_s|}{R} = \sqrt{1 + Q_s^2\left(\frac{f^2}{f_0^2} + \frac{f_0^2}{f^2} - 2\right)} \qquad \angle Z_s = \arctan Q_s\left(\frac{f}{f_0} - \frac{f_0}{f}\right)$$

1. For $f = f_0$

$$\frac{|Z_s|}{R} = 1 \qquad \angle Z_s = \arctan 0 = 0^\circ$$

2. For low frequency $f \to 0$

$$\frac{|Z_s|}{R} = \sqrt{1 + Q_s^2\left(\frac{f^2}{f_0^2} + \frac{f_0^2}{f^2} - 2\right)} \to \infty$$

$$\angle Z_s = \arctan Q_s\left(\frac{f}{f_0} - \frac{f_0}{f}\right) \to \arctan(-\infty) = -\frac{\pi}{2}$$

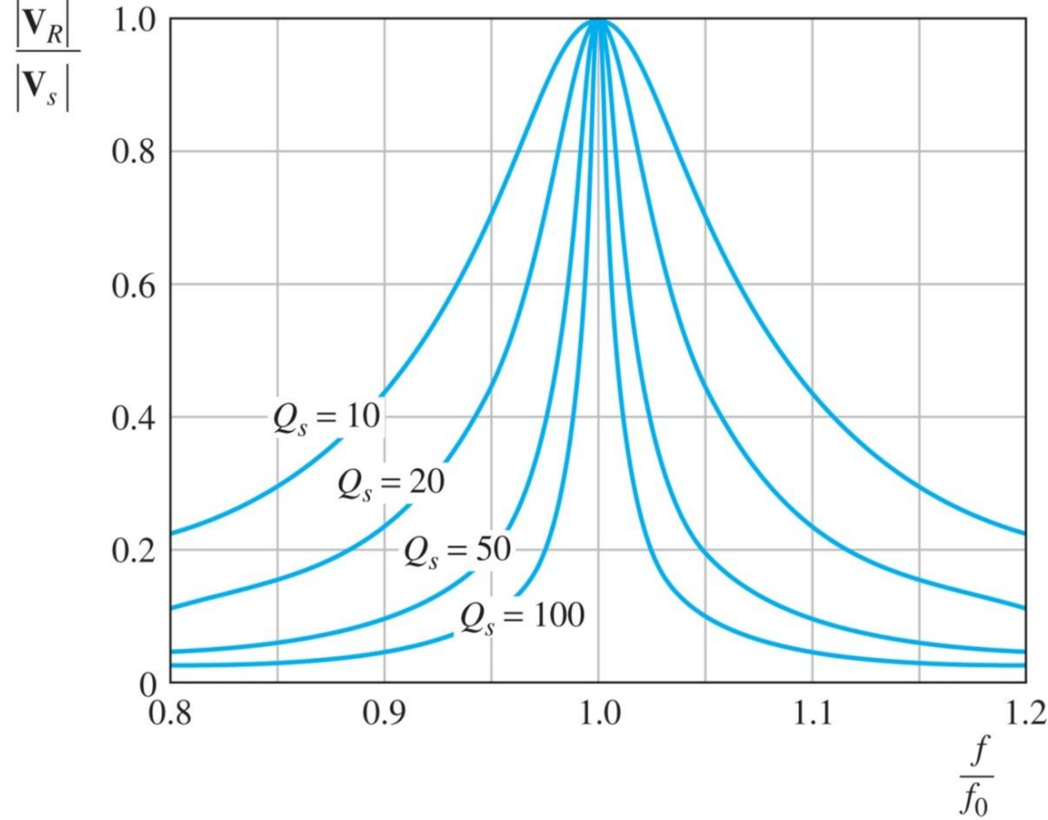
$$\frac{|Z_s|}{R} = \sqrt{1 + Q_s^2\left(\frac{f^2}{f_0^2} + \frac{f_0^2}{f^2} - 2\right)} \qquad \angle Z_s = \arctan Q_s\left(\frac{f}{f_0} - \frac{f_0}{f}\right)$$

3. For high frequency $f \to \infty$

$$\frac{|Z_s|}{R} = \sqrt{1 + Q_s^2\left(\frac{f^2}{f_0^2} + \frac{f_0^2}{f^2} - 2\right)} \to \infty$$

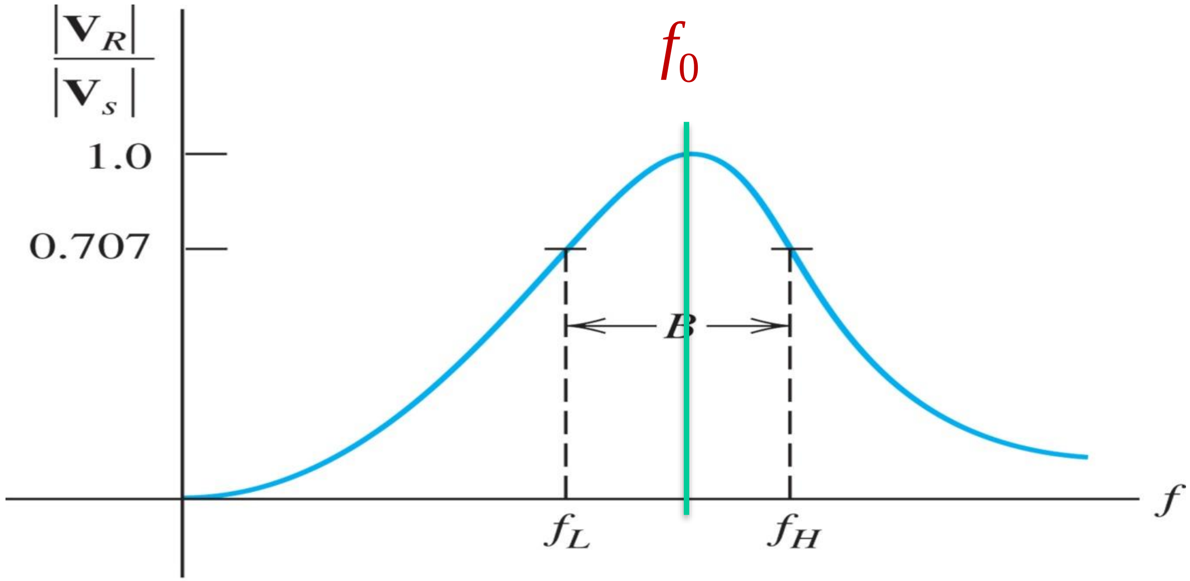
$$\angle Z_s = \arctan Q_s\left(\frac{f}{f_0} - \frac{f_0}{f}\right) \to \arctan(\infty) = \frac{\pi}{2}$$

Impedance magnitude 在 $f = f_0$ 最小，而 Q_s 越大斜率越大。



$$\frac{V_R}{Vs} = \frac{R}{Z_s} = \frac{1}{1 + jQ_s(\frac{f}{f_0} - \frac{f_0}{f})}$$

$$\frac{|V_R|}{|V_s|} \frac{R}{|Z_s|} = \frac{1}{\sqrt{1 + Q_s^2(\frac{f^2}{f_0^2} + \frac{f_0^2}{f^2} - 2)}}$$



$$\frac{|V_R|}{|V_s|} = \frac{1}{\sqrt{1+Q_s^2\left(\frac{f^2}{f_0^2}+\frac{f_0^2}{f^2}-2\right)}} = \frac{1}{\sqrt{2}}$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{\omega_0^2}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{\omega_0^2}}$$

Bandwidth (頻寬):

$$B = f_H - f_L = \frac{R}{2\pi L} = \frac{f_0}{Q_s}$$

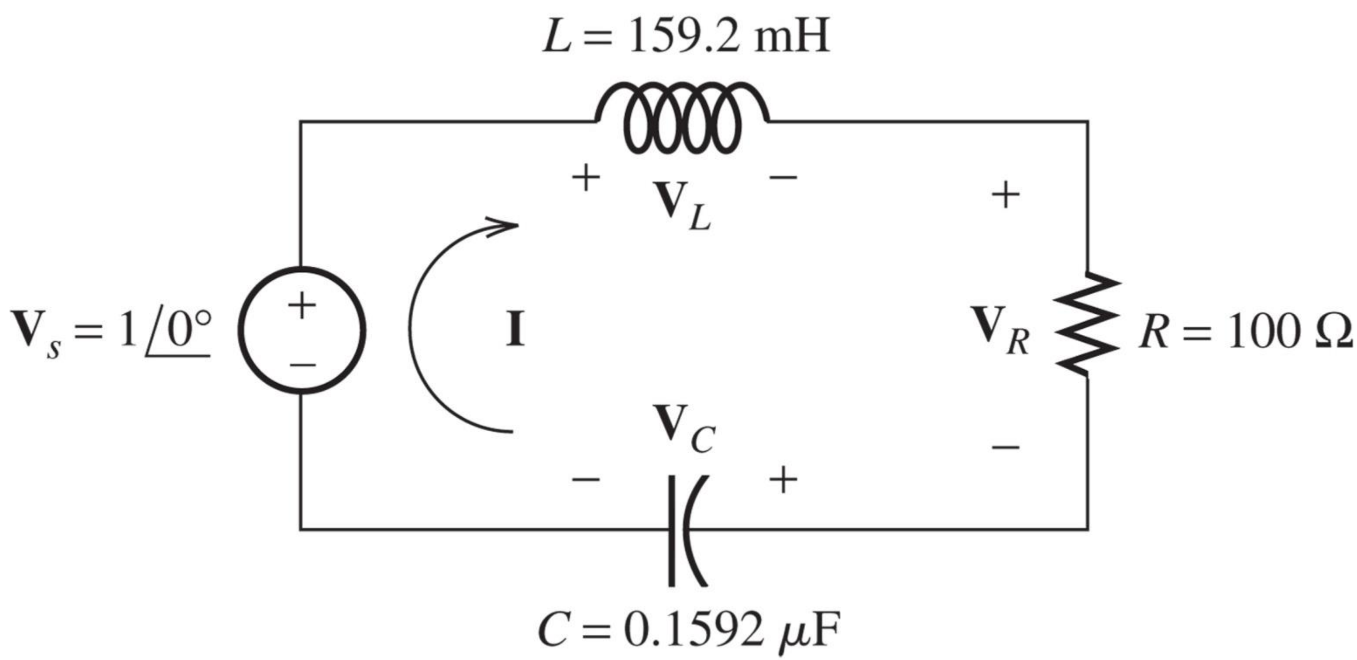
When Q_s >>1, f_H, f_L are given by the approximate expressions

$$f_H \cong f_0 + \frac{B}{2}$$

$$f_L \cong f_0 - \frac{B}{2}$$

Example 6.5 Series Resonant Circuit

Consider the series resonant circuit shown in Figure 6.27. Compute the resonant frequency, the bandwidth, and the half-power frequencies. Assuming that the frequency of the source is the same as the resonant frequency, find the phasor voltages across the elements and draw a phasor diagram.



Solution First, we use Equation 6.30 to compute the resonant frequency:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1592 \times 0.1592 \times 10^{-6}}} = 1000 \text{ Hz}$$

The quality factor is given by Equation 6.31:

$$Q_s = \frac{2\pi f_0 L}{R} = \frac{2\pi \times 1000 \times 0.1592}{100} = 10$$

The bandwidth is given by Equation 6.35:

$$B = \frac{f_0}{Q_s} = \frac{1000}{10} = 100 \text{ Hz}$$

Next, we use Equations 6.36 and 6.37 to find the approximate half-power frequencies:

$$f_H \cong f_0 + \frac{B}{2} = 1000 + \frac{100}{2} = 1050 \text{ Hz}$$

$$f_L \cong f_0 - \frac{B}{2} = 1000 - \frac{100}{2} = 950 \text{ Hz}$$

At resonance, the impedance of the inductance and capacitance are

$$Z_L = j2\pi f_0 L = j2\pi \times 1000 \times 0.1592 = j1000\ \Omega$$

$$Z_C = -j\frac{1}{2\pi f_0 C} = -j\frac{1}{2\pi \times 1000 \times 0.1592 \times 10^{-6}} = -j1000\ \Omega$$

As expected, the reactances are equal in magnitude at the resonant frequency. The total impedance of the circuit is

$$Z_s = R + Z_L + Z_C = 100 + j1000 - j1000 = 100\ \Omega$$

The phasor current is given by

$$\mathbf{I} = \frac{\mathbf{V}_s}{Z_s} = \frac{1\angle 0^\circ}{100} = 0.01\angle 0^\circ$$

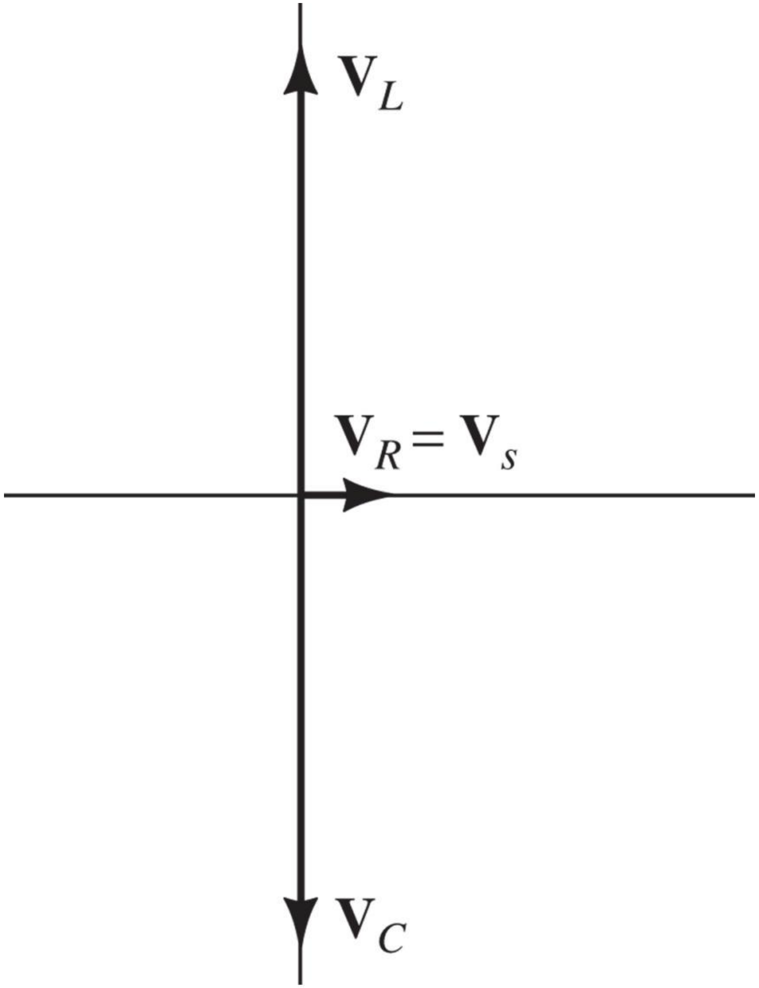
The voltages across the elements are

$$\mathbf{V}_R = R\mathbf{I} = 100 \times 0.01\angle 0^\circ = 1\angle 0^\circ$$
$$\mathbf{V}_L = Z_L\mathbf{I} = j1000 \times 0.01\angle 0^\circ = 10\angle 90^\circ$$
$$\mathbf{V}_C = Z_C\mathbf{I} = -j1000 \times 0.01\angle 0^\circ = 10\angle -90^\circ$$

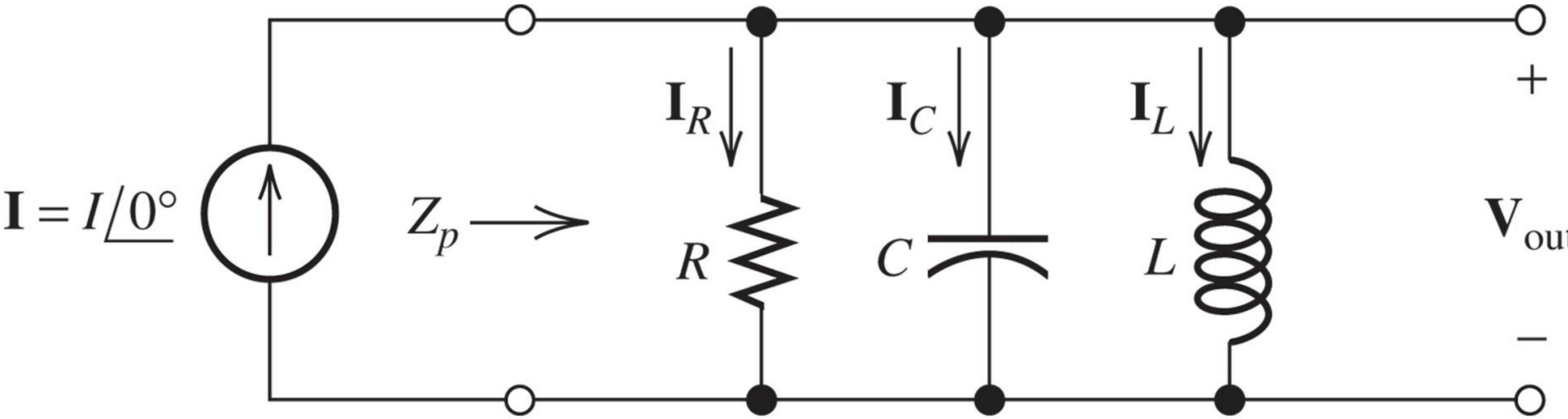
$$Q_s = 10$$

The phasor diagram is shown in Figure 6.28. Notice that the voltages across the inductance and capacitance are much larger than the source voltage in magnitude. Nevertheless, Kirchhoff's voltage law is satisfied because $\mathbf{V}_L$ and $\mathbf{V}_C$ are out of phase and cancel. ■

Phasor Diagram



6.7 Parallel Resonance



整體阻抗

$$Z_p = \frac{1}{1/R + j2\pi fC - j(1/2\pi fL)}$$

$$\frac{1}{j2\pi fL} = -j\frac{1}{2\pi fL}$$

在共振頻率出現時，電容與電感的阻抗magnitude 相同，整體阻抗只剩電阻

$$2\pi f_0 C = \frac{1}{2\pi f_0 L} \Rightarrow f_0 = \frac{1}{2\pi\sqrt{LC}}$$

For a parallel circuit, the quality factor Q_s is defined as

$$\frac{\text{resistance}}{\text{reactance of the inductance}}$$

(與series circuit 相反)

$$Q_p = \frac{R}{2\pi f_0 L}$$

亦可表示成

$$Q_p = 2\pi f_0 CR$$

整體阻抗

$$Z_p = \frac{1}{1/R + j2\pi fC - j(1/2\pi fL)}$$

可改寫為

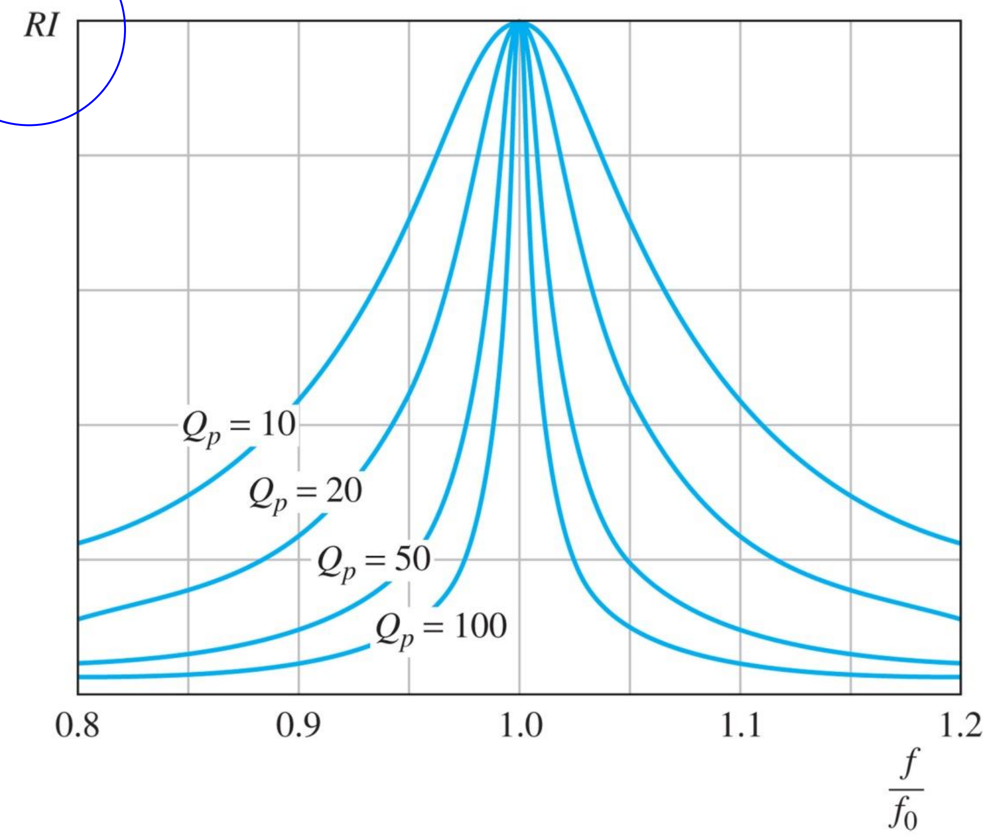
$$Z_p = \frac{R}{1 + jQ_p(f/f_0 - f_0/f)}$$

輸出 phasor voltage

$$\mathbf{V}_{\text{out}} = \frac{\mathbf{I}R}{1 + jQ_p(f/f_0 - f_0/f)}$$

Bandwidth (頻寬):

$$B = f_H - f_L = \frac{f_0}{Q_s}$$



Example 6.6 Parallel Resonant Circuit

Find the L and C values for a parallel resonant circuit that has $R = 10\,\text{k}\Omega$, $f_0 = 1\,\text{MHz}$, and $B = 100\,\text{kHz}$. If $\mathbf{I} = 10^{-3}\angle 0^\circ$ A, draw the phasor diagram showing the currents through each of the elements in the circuit at resonance.

$$B = \frac{f_0}{Q_s} \rightarrow Q_s = \frac{f_0}{B}$$

$$Q_p = \frac{R}{2\pi f_0 L} \quad \rightarrow L = \frac{R}{2\pi f_0 Q_p}$$

$$Q_p = 2\pi f_0 CR \quad \rightarrow C = \frac{Q_p}{2\pi f_0 R}$$

Example 6.6 Parallel Resonant Circuit

Find the L and C values for a parallel resonant circuit that has $R = 10\,\text{k}\Omega, f_0 = 1\,\text{MHz}$, and $B = 100\,\text{kHz}$. If $\mathbf{I} = 10^{-3}\underline{/0^\circ}$ A, draw the phasor diagram showing the currents through each of the elements in the circuit at resonance.

Solution First, we compute the quality factor of the circuit. Rearranging Equation 6.46 and substituting values, we have

$$Q_p = \frac{f_0}{B} = \frac{10^6}{10^5} = 10$$

Solving Equation 6.41 for the inductance and substituting values, we get

$$L = \frac{R}{2\pi f_0 Q_p} = \frac{10^4}{2\pi \times 10^6 \times 10} = 159.2\ \mu\text{H}$$

Similarly, using Equation 6.42, we find that

$$C = \frac{Q_p}{2\pi f_0 R} = \frac{10}{2\pi \times 10^6 \times 10^4} = 159.2\ \text{pF}$$

At resonance, the voltage is given by

$$\mathbf{V}_{\text{out}} = \mathbf{I}R = (10^{-3}\underline{/0^\circ}) \times 10^4 = 10\underline{/0^\circ}\ \text{V}$$

and the currents are given by

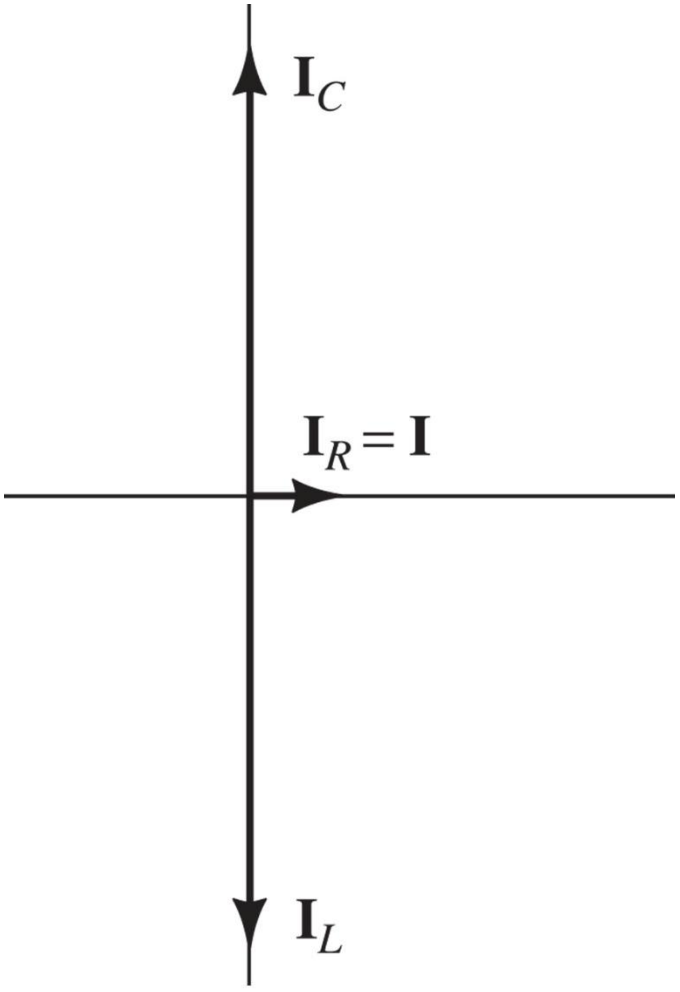
$$\mathbf{I}_R = \frac{\mathbf{V}_{\text{out}}}{R} = \frac{10\angle 0^\circ}{10^4} = 10^{-3}\angle 0^\circ \text{ A}$$

$$\mathbf{I}_L = \frac{\mathbf{V}_{\text{out}}}{j2\pi f_0 L} = \frac{10\angle 0^\circ}{j10^3} = 10^{-2}\angle -90^\circ \text{ A}$$

$$\mathbf{I}_C = \frac{\mathbf{V}_{\text{out}}}{-j/2\pi f_0 C} = \frac{10\angle 0^\circ}{-j10^3} = 10^{-2}\angle 90^\circ \text{ A}$$

The phasor diagram is shown in Figure 6.31. Notice that the currents through the inductance and capacitance are larger in magnitude than the applied source current. However, since $\mathbf{I}_C$ and $\mathbf{I}_L$ are out of phase, they cancel. ■

Phasor diagram



Resonance in a mechanical system

Tacoma Narrow Bridge

http://www.youtube.com/watch?v=j-zczJXSxnw