CALCULUS I LECTURE 8: L'HOSPITAL RULE

1. Indeterminate limits and l'Hospital Rule

Suppose f, g are two functions defined near a. Let us start with the limit of the form

$$\lim_{x\to a}\frac{f(x)}{g(x)} \tag{1.1}$$

with one of the following conditions holds:

$$\text{(a). } \lim_{x\to a} f(x) = 0;\ \ \lim_{x\to a} g(x) = 0. \tag{1.2}$$

$$\text{(b). } \lim_{x\to a} f(x) = \pm\infty;\ \ \lim_{x\to a} g(x) = \pm\infty. \tag{1.3}$$

When we face to this cases, we call the limit in (1.1) has an **indeterminate form**. Note that the limit of a indeterminate form may not exist. For example, the limit of

$$\lim_{x\to 0}\frac{x}{x^2} = \infty, \tag{1.4}$$

although both x and x^2 go to 0 as $x \to 0$. However, we have the following theorem

Theorem 1.1. (l'Hospital)
(a). *For any indeterminate form, we have*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{1.5}$$

when the limit on the right hand side exists.
(b). *Suppose the right hand side of (1.5) goes to $\pm\infty$, so does the left hand side of (1.5).*

Proof. First, we can assume (a) holds. Now, since $\lim_{x\to a} f(x) = 0;\ \lim_{x\to a} g(x) = 0$, we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{(f(x)-0)}{(g(x)-0)} = \lim_{x\to a}\frac{(f(x)-f(a))}{(g(x)-g(a))}. \tag{1.6}$$

Fix x for this moment, applying the mean value theorem on

$$F(y) = f(y) - \Big(\frac{f(x)-f(a)}{g(x)-g(a)}\Big)g(y), \tag{1.7}$$

one can obtain

$$\frac{F(x)-F(a)}{x-a} = f'(\xi) - \Big(\frac{f(x)-f(a)}{g(x)-g(a)}\Big)g'(\xi) = 0 \tag{1.8}$$

where $|\xi| < |x|$ (Note that $F(x) = F(a) = \frac{f(a)g(x)-f(x)g(a)}{g(x)-g(a)}$). So

$$\lim_{x\to a}\frac{(f(x)-f(a))}{(g(x)-g(a))} = \lim_{x\to a}\frac{f'(\xi)}{g'(\xi)} = \lim_{\xi\to a}\frac{f'(\xi)}{g'(\xi)} \tag{1.9}$$

Therefore, the left hand side of (1.6) equals the right hand side of (1.9), if the later exists (or goes to $\pm\infty$).

To prove the case with $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ (we can assume both of them go to ∞, by taking suitable sign in front), we choose b_x be a value with the following properties

$$b_x > x > a \text{ when } x > a; \tag{1.10}$$

$$b_x < x < a \text{ when } x < a; \tag{1.11}$$

$$g(b_x) = \sqrt{g(x)}. \tag{1.12}$$

One can check that such b_x exists when x is sufficient close to a and $\lim_{x\to a} b_x = \infty$. We claim that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - f(b_x)}{g(x) - g(b_x)}. \tag{1.13}$$

With this claim in our mind, together with the argument we used in (1.8) and (1.9), we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(\xi)}{g'(\xi)} \tag{1.14}$$

with $\xi \to a$ as $x \to a$. So we complete the proof.

To show the claim (1.13) is true. Here we suppose that $\frac{f(x)}{g(x)} \leq M$ for some $M > 0$ near a. Under this assumption, we have

$$\frac{f(x) - f(b_x)}{g(x) - g(b_x)} = \Big(\frac{f(x)}{g(x)} \cdot \frac{g(x)}{g(x) - \sqrt{g(x)}}\Big) + \frac{f(b_x)}{g(x) - \sqrt{g(x)}}. \tag{1.15}$$

Since $f(x) \leq Mg(x)$, the last term in (1.15) can be bounded by

$$\Big|\frac{f(b_x)}{g(x) - \sqrt{g(x)}}\Big| \leq \frac{M\sqrt{g(x)}}{g(x) - \sqrt{g(x)}} \tag{1.16}$$

which converges to 0. So by taking the limit, (1.15) implies

$$\lim_{x\to} \frac{f(x) - f(b_x)}{g(x) - g(b_x)} = \lim_{x\to a} \Big(\frac{f(x)}{g(x)} \cdot \frac{g(x)}{g(x) - \sqrt{g(x)}}\Big) = \lim_{x\to} \frac{f(x)}{g(x)}. \tag{1.17}$$

Suppose the condition $\frac{f(x)}{g(x)} \leq M$ fails for some $M > 0$. Then the claim will be more difficult to prove. We postpone the proof in the future.

□

Remark 1.2. This theorem holds when we replace the limit by the right or the left limit.

Example 1.3. Find the value $\lim_{x\to 1^+} \frac{\ln x}{x-1}$.

Firstly, we check that $\lim_{x\to 1^+} \ln x = 0 = \lim_{x\to 1^+}(x-1)$. So the limit **is** an indeterminate form. By l'Hospital rule and

$$\lim_{x\to 1^+} \frac{(\frac{1}{x})}{1} = 1, \tag{1.18}$$

we have

$$\lim_{x\to 1^+} \frac{\ln x}{x-1} = 1. \tag{1.19}$$

In some cases, we might face the limit of the form

$$\lim_{x\to a} f(x) - g(x) \tag{1.20}$$

with both $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. Here is an example that we can use l'Hospital rule to solve this type of problems.

Example 1.4. Compute $\lim_{x\to 1^+} \frac{1}{\ln x} - \frac{1}{x-1}$.

We can write

$$\frac{1}{\ln x} - \frac{1}{x-1} = \frac{x-1-\ln x}{(x-1)\ln x}. \tag{1.21}$$

So by l'Hospital

$$\lim_{x\to 1^+} \frac{1}{\ln x} - \frac{1}{x-1} = \lim_{x\to 1^+} \frac{1-\frac{1}{x}}{\ln x + 1 - \frac{1}{x}} = \lim_{x\to 1^+} \frac{x-1}{x\ln x + x - 1} \tag{1.22}$$

if the right hand side exists. Since both the denominator and numerator go to 0 as $x \to 1^+$, we can apply l'Hospital rule again:

$$\lim_{x\to 1^+} \frac{x-1}{x\ln x + x - 1} = \lim_{x\to 1^+} \frac{1}{\ln x + 2} = \frac{1}{2}. \tag{1.23}$$

Sometime we will not have a standard indeterminate form directly. We need to use the following Proposition to modify our problem first.

Proposition 1.5. Let h, k be two functions with h continuous, then

$$\lim_{x\to a} h \circ k(x) = h(\lim_{x\to a} k(x)). \tag{1.24}$$

if one of the limits exists.

Example 1.6. Calculate $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$.

Here we use Proposition 1.5. with $h(x) = \ln x$. So

$$\begin{aligned} \ln\left(\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}\right) &= \lim_{x\to 0^+} \ln(1+\sin 4x)\cot x \\ &= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x}. \end{aligned} \tag{1.25}$$

Both the denominator and numerator go to 0 as x goes to 0, by l'Hospital rule, we have

$$\lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} = \lim_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = 4. \tag{1.26}$$