Linear Algebra: Final Exam-B

This is a 120-minutes closed-book exam.

Calculator is allowed.

4 pages in total

1. (54 pts) Determine whether the following statements are true (T) or false (F)? (Reasoning is required.)

(1) If U, V, and W are subspaces of R^3 and if $U \perp V$ and $V \perp W$, then $U \perp W$.

(2) If **x** and **y** are unit vectors in R^n and $|\mathbf{x}^T\mathbf{y}| = 1$, then **x** and **y** are linearly independent.

(3) It is possible to find a nonzero vector **y** in the column space of A such that $A^T\mathbf{y} = \mathbf{0}$.

(4) If the characteristic polynomial of a matrix A is $det(A - \lambda I) = \lambda^2 - 2\lambda$, then A is invertible.

(5) The characteristic equation of a 2×2 matrix A can be expressed as $det(\lambda I - A) = \lambda^2 - tr(A)\lambda + det(A) = 0$, where $tr(A)$ is the trace of A.

(6) Let A be an $n \times n$ matrix and $det(A) \neq 0$. If the matrix A has an eigenvalue $\boldsymbol{\lambda}$ and its corresponding eigenvector **v**, then $\boldsymbol{A^{-1}}$ has a eigenvector **v** with eigenvalue $\frac{\mathbf{1}}{\boldsymbol{\lambda}}$.

(7) If A and B are similar matrices, then they have the same eigenvalues.

(8) Let $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ be an orthonormal basis for an inner product space V and let $\mathbf{u} = 2\,\mathbf{u_1} - \mathbf{u_2} + \mathbf{u_3}$ and $\mathbf{v} = \mathbf{u_1} + \mathbf{u_2} - \mathbf{u_3}$. We have $\mathbf{u} \perp \mathbf{v}$.

(9) If A is a 4×3 matrix of rank 2, then the dimensions of $N(A^T)$ is 2.

(10) If the inner product on P_3 is defined by $\langle f,\ g\rangle = \int_{-1}^{1} f(x)g(x)dx$, then $\langle 1 - x,\ 1 + x - x^2\rangle = \frac{2}{3}$.

(11) If the inner product on P_3 is defined by $\langle f,\ g\rangle = \int_{-1}^{1} f(x)g(x)dx$, then

$p_1(x) = 1, p_2(x) = x$ and $p_3(x) = x^2 - \frac{1}{2}$ are orthogonal.

(12) Let $\langle \mathbf{u}, \mathbf{v} \rangle$ be the Euclidean inner product on R^2, and if $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, then $\langle \mathbf{u}, \mathbf{2v} \rangle = 12$.

(13) If a matrix A has the vector $\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$ in row space of A, then the $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ is in the null space of A.

(14) The orthogonal complement（正交補餘）of the subspace of R^3 spanned by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$ is spanned by $\left\{ \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \right\}$.

(15) Let **u** and **v** be vectors in an inner product space *V*. If $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal, then $||\mathbf{u}|| = ||\mathbf{v}||$.

(16) The set $\boldsymbol{S} = \{\mathbf{1}, \mathbf{cos}(\boldsymbol{x}), \boldsymbol{sin}(\boldsymbol{x})\}$ is a linearly independent subset of $C\ [-\pi, \pi]$ with respect to the inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) dx$

(17) Let A be an $n \times n$ matrix and let $\boldsymbol{B} = \boldsymbol{A}^{\mathbf{2}} - \boldsymbol{A} + \mathbf{2}\,\boldsymbol{I}$. If the eigenvalues of A are $\boldsymbol{\lambda}_{\boldsymbol{i}}, i = 1,2,\cdots,n$, then the eigenvalues of B are $\boldsymbol{\lambda}_{\boldsymbol{i}}{}^{\mathbf{2}} - \boldsymbol{\lambda}_{\boldsymbol{i}} + \mathbf{2},\ i = 1,2,\cdots,n$.

(18) Let Q be an orthogonal matrix, then $det(Q) = \pm 1$.

(19) If A is a square matrix and $||A\mathbf{u}|| = ||\mathbf{u}||$ for all vectors $\mathbf{u} \neq 0$, then A is orthogonal.

(20) The matrix $A = \begin{bmatrix} 1 & 1+1i & 2i \\ 1-i & 2 & 1+2i \\ -2i & 1+2i & 3 \end{bmatrix}$ is Hermitian.

(21) If A is Hermitian and c is a complex scalar, then cA is Hermitian.

(22) If three points $(x_1, y_1) = (0, 1)$, $(x_2, y_2) = (1, 2)$ and $(x_3, y_3) = (2,\ 4)$ are given, then the straight line $y = 1 + x$ to minimize $\sum_{i=1}^{3} [y_i - (1 +$

$x_i)]^2$.

(23) Let M_{33} denote the vector space consisting of all 3×3 matrices. If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix}$, we define $\langle A, B \rangle = \sum_{i=1}^{9} a_i b_i$ Now, if we have $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -2 \\ -1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 4 & 1 \\ -2 & 1 & 1 \end{bmatrix}$, then A and B are orthogonal.

(24) Let $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$ be the standard basis for R^3 and if $L : R^3 \rightarrow R^3$ be a linear transformation with the properties $L(\mathbf{e_1}) = \mathbf{e_2}, L(\mathbf{e_2}) = 2\mathbf{e_1} + \mathbf{e_2}$, $L(\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}) = \mathbf{e_3}$, then $L\left(\begin{bmatrix} \mathbf{2} \\ \mathbf{2} \\ \mathbf{1} \end{bmatrix}\right) = \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \\ \mathbf{1} \end{bmatrix}$.

(25) If a matrix $A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$ is given, then the $GM_{\lambda_i} = AM_{\lambda_i}, i = 1,2$.

(26) If a matrix A has singular values $\sigma_1 = 4 > \sigma_2 = 2$, then the eigenvalues of $A^T A$ are 2 and $\sqrt{2}$.

(27) If a matrix A has singular value decomposition as
$A = \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{bmatrix}$, then its rank is 2.

2. (16 pts) Consider a matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 3 & 6 & 3 \end{bmatrix}$, determine

(1) The basis set for the column space of A.

(2) The basis set for the row space of A.

(3) The range space of A.

(4) The basis set for the null space of A.

3. (12 pts) Let $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

(1) Use the Gram–Schmidt process to find an orthonormal basis for the column space of A.

(2) Factor A into a product QR, where Q has an orthonormal set of column vectors and R is upper triangular.

(3) Solve the least squares problem $A\mathbf{x} = \mathbf{b}$.

4. (18 pts) Suppose that A is a 3×3 matrix with eigenvalues $\boldsymbol{\lambda_1} = -1, \boldsymbol{\lambda_2} = 0$ and $\boldsymbol{\lambda_3} = 1$, and corresponding eigenvectors

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} (\lambda_1 = -1), \mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} (\lambda_2 = 0), \mathbf{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (\lambda_3 = 1).$$

(1) Find the matrix A.

(2) Find A^{20}.

(3) Find the unique solution of the differential equation $\frac{d\,Y(t)}{dt} = AY(t), t \geq 0$ with the initial condition $Y(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.