Linear Algebra: Midterm Exam 2A

This is a 120-minutes exam.

4 pages in total

1. (30 pts) Determine whether the following statements are true (T) or false (F)? (A reasoning is required.)

(1) If a 5×3 matrix A has a rank 3, then the dimension of $N(A)$ is 3.

(2) The set $\{x+1, x+3, x^2-1\}$ is spanning the vector space P_3.

(3) Let A be a $n \times n$ matrix. If $N(A) = \{0\}$, then then system $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ has infinite solution for a given vector $\mathbf{b} \in R^n$.

(4) The following vectors $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}$ are linearly independent in $\boldsymbol{M_{2\times 2}}$.

(5) Let $\boldsymbol{V}$ have two bases by $B = \{\sin(x) + \cos(x),\ 2\sin(x)\}$ and $B' = \{\sin(x),\ \cos(x)\}$, then the transition matrix from B to B' is $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$.

(6) Let A be a 4×5 matrix. If $\mathbf{a_1}, \mathbf{a_2}$, and $\mathbf{a_4}$ are linearly independent and $\mathbf{a_3} = \mathbf{a_1} + 2\mathbf{a_2}, \mathbf{a_5} = 2\mathbf{a_1} - \mathbf{a_2} + 3\mathbf{a_4}$, the reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$.

(7) The following transformation $L: R^3 \to R^3$ defined by $\boldsymbol{L}(\mathbf{x}) = \begin{bmatrix} 1 + x_1 \\ x_1 + x_2 \\ x_1 - 2x_3 \end{bmatrix}$ is a linear transformation.

(8) If A is a 4×5 matrix, then rank of A is at most 5.

(9) Let A be a 2×2 matrix, and let $\boldsymbol{L_A}$ be the linear operator defined by $\boldsymbol{L_A}(\mathbf{x}) = A\mathbf{x}$. We can conclude that $\boldsymbol{L_A}$ maps the vector space R^2 onto the row space of A.

(10) The matrix $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is a rotation matrix.

(11) Let $\boldsymbol{L_1}: R^2 \longrightarrow R^2$ be the orthogonal projection onto the x-axis, and $\boldsymbol{L_2}: R^2 \longrightarrow R^2$ be the rotation counterclockwise with the angle θ, then $\boldsymbol{L_2} \circ \boldsymbol{L_1} = \boldsymbol{L_1} \circ \boldsymbol{L_2}$.

(12) If A is a nonsingular matrix, and B is similar to A, then A^{-1} is also similar to B^{-1}.

(13) If the subspace S of R^3 is spanned by $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, then $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the orthogonal complement of S

(14) Is it possible for a matrix A to have the vector $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ in row space of A and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ in the null space of A?

(15) Let $\mathrm{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \mathrm{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then the angle between x and y is $\theta = 30^{\circ}$.

2. (8 pts) Determine the kernel and range of each of the following linear operators on R^3

(a) $L(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 \\ x_2 - 2x_3 \\ x_1 + x_3 \end{bmatrix}$ **(b)** $L(\mathbf{x}) = \begin{bmatrix} x_1 \\ 2x_1 \\ x_1 \end{bmatrix}$

3. (5 pts) Given the two bases $V = \left\{\mathrm{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathrm{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$, and $U = \left\{\mathrm{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathrm{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$, please find the transition matrix from V to U.

4. (12 pts) Consider a set of matrices with size 2×2: $\boldsymbol{S} = \left\{A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}\right\}$.

(1) Find the coordinate vector of the matrix $B_1 = \begin{bmatrix} 3 & -18 \\ -18 & 10 \end{bmatrix}$ with respect to the basis $\boldsymbol{S}$.

(2) Find the coordinate vector of the matrix $B_2 = \begin{bmatrix} 3 & 9 \\ -9 & 10 \end{bmatrix}$ with respect to the basis $\boldsymbol{S}$.

(3) Does the matrix $B_3 = \begin{bmatrix} 3 & -9 \\ -9 & 10 \end{bmatrix}$ be in the vector space $M = Span\{A_1, A_2, A_3\}$.

5. (9 pts) Let $\boldsymbol{L}: R^3 \longrightarrow R^4$ be a linear transformation such that

$$\boldsymbol{L}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\\2\\0\end{bmatrix},\ \boldsymbol{L}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\-3\\1\end{bmatrix}, \boldsymbol{L}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\2\\-9\\2\end{bmatrix}.$$

(1) Find the representing matrix A for the linear transformation $\boldsymbol{L}$.

(2) If $\boldsymbol{L}(\mathbf{x}) = \begin{bmatrix}3\\3\\-26\\5\end{bmatrix}$, then find $\mathbf{x}$.

(3) For the matrix A found in (1), if we are given a vector $\mathbf{b} \in R^4$, then please find the solution of the system $A\mathbf{x} = \mathbf{b}$.

6. (12 pts) Let $\boldsymbol{L_1}, \boldsymbol{L_2}, \boldsymbol{L_3}, \boldsymbol{L_4}: R^2 \longrightarrow R^2$ be the following linear transformations:

$\boldsymbol{L_1}$: Reflection with respect to the *x-axis;*

$\boldsymbol{L_2}$: Reflection with respect to the line l of angle $30°$;

$\boldsymbol{L_3}$: Rotation counterclockwise by the angle $60°$;

$\boldsymbol{L_4}$: Deformation: x_1 direction with factor $k = 2$.

(1) Find the representing matrices A_1, A_2, A_3 and A_4 of $\boldsymbol{L_1}(\mathrm{x}) = A_1\mathrm{x}, \boldsymbol{L_2}(\mathrm{x}) = A_2\mathrm{x}$, $\boldsymbol{L_3}(\mathrm{x}) = A_3\mathrm{x}$, and $\boldsymbol{L_4}(\mathrm{x}) = A_4\mathrm{x}$, respectively.

(2) Find the representing matrix C of the composition transformation $\boldsymbol{L_2} \circ \boldsymbol{L_1}$

(3) Find the vector of $\boldsymbol{L_1} \circ \boldsymbol{L_3}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$.

(4) Plot the composite transformation $\boldsymbol{L_4} \circ \boldsymbol{L_2}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$ in geometric diagram.

7. (12 pts) Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 3 \\ 3 & 0 & 3 & 4 & 5 \\ 0 & 4 & 0 & 4 & 4 \end{bmatrix}$.

(1) Find a basis for $N(A)$ (the null space of A) and the nullity of A.

(2) Find a basis for the row space of A and find a basis for the column space of A.

(3) Find the $N(A^T)$ and the nullity of A^T

(4) Find the range space of A^T and prove that $N(A) \perp R(A^T)$.

8. (12 pts) Let the operator $\boldsymbol{L}: P_3 \to P_3$ be defined by $\boldsymbol{L}\,(f(x)) = x^2 f(1) + f'(x)$.

(1) Find the matrix A representing $\boldsymbol{L}$ with respect to the basis $\{x^2, x, 1\}$.

(2) Find the matrix B representing $\boldsymbol{L}$ with respect to the basis $\{x^2+1, x+1, 1\}$.

(3) Find the matrix S such that $B = S^{-1}AS$.