CALCULUS I LECTURE 3: CONTINUOUS FUNCTIONS AND DERIVATIVES

1. Continuous functions

When we say a function f is continuous at a point $a \in X$, it actually means that the limit of f and the value of f at a are the same. Namely, we have

Definition 1.1. Let f is define at a and near a. We call f continuous at a if and only if there exists a function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(x) - f(a)| < \varepsilon \text{ when } |x - a| < \delta. \tag{1.1}$$

We call f is a continuous function if f is continuous at every point in its domain X.

Let f, g be a function defined on X, continuous at $a \in X$. Then by the same argument in Proposition 2.1, Proposition 2.2 in Lecture 2, we have

Proposition 1.2.
a. For any $c \in \mathbb{R}$, $f + cg$ is also continuous at a.
b. If $f(a) > 0$, then for any $q \in \mathbb{Q}^+$, f^q is also continuous at a.

By this proposition, we can see that all algebraic functions are continuous.

In the following paragraphs, we will define exponential function in a rigorous way. Recall that we haven't define the meaning for a^x when $a > 0$ and $x \in \mathbb{R} - \mathbb{Q}$.

Proposition 1.3. Suppose that we have two continuous functions $f(x)$, $g(x)$ defined on $\mathbb{R}$ and $f(x) = g(x)$ for all $x \in \mathbb{Q}$, then $f = g$.

Proof. Suppose that $f \neq g$, i.e., $f(a) \neq g(a)$ for some irrational number a. Because both f and g are continuous, by taking $\varepsilon = \frac{1}{4}|f(a) - g(a)|$, there is a rational number$x$ near a such that $|f(a) - f(x)| < \varepsilon$ and $|g(a) - g(x)| < \varepsilon$. Therefore,

$$\begin{aligned} |f(a) - g(a)| \le |f(a) - f(x)| + |f(x) - g(x)| + |g(x) - g(a)| &\le 2\varepsilon \\ &= \frac{1}{2}|f(a) - g(a)| \end{aligned}$$

which is a contradiction. ∎

Now, the exponential function $y = a^x$ can be defined as

$$a^x = \min\{L \in \mathbb{R} \mid L > a^q \text{ for any } q < x\}.$$

Exercise 1.4.
1. Prove that the extended exponential function defined in this way is continuous.
2. Prove that $y = a^x$ is a monotone increasing function.

Now, we can use Proposition 2.3 in Lecture 2. Let t be an irrational number. For any $p < t < q$ with $p, q \in \mathbb{Q}$, we have

$$\lim_{x\to a} f^p(x) = L^p \leq \lim_{x\to a} f^t(x) \leq \lim_{x\to a} f^q(x) = L^q$$

when $\lim_{x\to a} f(x) = L$ Therefore, we have

$$\lim_{x\to a} f^t(x) = L^t$$

So we have

Proposition 1.5. For any rational number $q \in \mathbb{R}$ and $L > 0$, we have

$$\lim_{x\to a} f^q(x) = L^q \tag{1.2}$$

when $\lim_{x\to a} f(x) = L$.

Proposition 1.6. Let $f : X \to Y$ and $g : Y \to Z$. Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{y\to L} g(y)$ exists. Then we have

$$\lim_{x\to a} g(f(x)) = \lim_{y\to L} g(y). \tag{1.3}$$

Proof. By Definitions, there exists $\delta_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(x) - L| < \varepsilon \tag{1.4}$$

when $0 < |x - a| < \delta_1(\varepsilon)$; there exists $\delta_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|g(y) - g(L)| < \varepsilon \tag{1.5}$$

when $0 < |y - f(a)| < \delta_2(\varepsilon)$.

One can check that, if we take $\delta^{new} = \delta_1 \circ \delta_2$, then

$$|g(f(x)) - g(L)| < \varepsilon \tag{1.6}$$

when $0 < |x - a| < \delta^{new}(\varepsilon)$. $\square$

2. Derivatives

Suppose we have a continuous function $f : X \to Y$ with f defines near $a \in X$.

Definition 2.1. We define the derivative of f at a to be

$$\frac{df}{dx}(a) := \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \tag{2.1}$$

if the limit exists.

If we write $x = a + h$, then (2.1) can be written as

$$\lim_{h\to 0} \frac{f(a + h) - f(a)}{h} \tag{2.2}$$

Sometime we use the notation $f'(a)$ to denote $\frac{df}{dx}(a)$.

One can regard the derivative as the instantaneous rate of change of f. We call f is differentiable at $a \in X$ if and only if $\frac{df}{dx}(a)$ exists; we call f a differentiable function if and only if the derivative of f exists at every point in its domain. Obviously, not every function is differentiable, even it is continuous. However, a differentiable function must be continuous.

Since the derivative is defined by limit, we will have the following proposition.

Proposition 2.2.
1. Suppose f, g are differentiable at a, $c \in \mathbb{R}$, then $f + cg$ is differentiable at a.
2. Suppose $f(a) > 0$ and f is differentiable at a, then f^r is differentiable at a for any $r \in \mathbb{R}$.

The first property can be obtained directly from the definition. The second one, however, is not obvious. We leave the proof of this part as an exercise.

Exercise 2.3. All algebraic functions are differentiable.

Exercise 2.4. $\frac{dx^n}{dx}(a) = na^{n-1}$

3. Product rule and chain rule

Suppose we have two functions f, g mapping from X to Y, where $X, Y \subset \mathbb{R}$. Then we have

Proposition 3.1. (**Product Rule**). Suppose f, g are differentiable at a. Then

$$\frac{d(fg)}{dx}(a) = \frac{df}{dx}(a)g(a) + f(a)\frac{dg}{dx}(a). \tag{3.1}$$

Proof. By Definition 2.1, we have the right hand side of (3.1) equals

$$\lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a}. \tag{3.2}$$

Notice that

$$\begin{aligned} \frac{f(x)g(x) - f(a)g(a)}{x - a} &= \frac{f(x)g(x) - f(a)g(x)}{x - a} + \frac{f(a)g(x) - f(a)g(a)}{x - a} \\ &= \frac{f(x) - f(a)}{x - a} g(x) + f(a)\frac{g(x) - g(a)}{x - a}. \end{aligned}$$

for any $x \in X - \{a\}$. So

$$\lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} g(x) + f(a) \lim_{x \to a} \frac{g(x) - g(a)}{x - a}.$$

Then one can obtain (3.1) immediately from this equality. □

Now let us consider another scenario: Suppose $f : X \to Y$, $g : Y \to Z$ are two functions; X, Y and Z are subsets of $\mathbb{R}$. If f is differentiable at $a \in X$ and g is differentiable at $f(a) \in Y$, then we have the following rule for the composition function $g \circ f$.

Proposition 3.2. (**Chain Rule**). If f is differentiable at $a \in X$ and g is differentiable at $f(a) \in Y$, then $g \circ f$ is differentiable at a and

$$\frac{d(g \circ f)}{dx}(a) = \frac{dg}{dx}(f(a))\frac{df}{dx}(a). \tag{3.3}$$

Proof. By Definition 2.1, we have the right hand side of (3.3) can be written as

$$\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.4}$$

The first term can be written as $h \circ f$ with

$$h(y) = \frac{g(y) - g(f(a))}{y - f(a)}.$$

So by using Proposition 1.6, we have

$$\lim_{x \to a} h \circ f(x) = \lim_{y \to f(a)} h(y) = \frac{dg}{dx}(f(a)).$$

Combine this equality and (3.4), one can obtain (3.3) now. $\square$

4. Geometric meaning of derivatives

Starting form Definition 2.1, (2.1) can be rewritten as

$$\lim_{x \to} \frac{f(x) - f(a) - f'(a)(x-a)}{x-a} = 0 \tag{4.1}$$

Let us write $T(x) = f(a) + f'(a)(x-a)$, then

$$\lim_{x \to a} \frac{f(x) - T(x)}{x - a} = 0.$$

Notice that $(f - T)(a) = 0$, so this equality tells us that the derivative $\frac{d(f-T)}{dx}(a) = 0$. Namely, there is no instantaneous rate of change for $f - T$ at a. Meanwhile, for any other linear function $T^*(x) = c + m(x-a)$, $\frac{d(f-T^*)}{dx}(a) = 0$ holds if and only if $T^* = T$. We usually call T the **tangent line** of f at the point $(a, f(a))$ (or briefly, at a).

Under this setting, we have the slope of f at a is $f'(a)$.