CALCULUS I LECTURE 15: TECHNIQUES OF INTEGRATION I

1. Formula of integration by parts

Let f and g are two differentiable functions defined on $[a, b]$. Recall the product rule tells us that

$$\frac{d}{dx}(fg)(x) = f'(x)g(x) + f(x)g'(x). \tag{1.1}$$

If we integrate (1.1) from a to b, we will have

$$f(b)g(b) - f(a)g(a) = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx. \tag{1.2}$$

Namely, we have

$$\int_a^b f(x)g'(x)dx = [f(b)g(b) - f(a)g(a)] - \int_a^b f'(x)g(x)dx. \tag{1.3}$$

We call this formula the **integration by parts**. This formula is the most important formula when we work on integrations.

In many cases, one of the integrals $\int f'(x)g(x)$ and $\int f(x)g'(x)$ will be easier to compute. The integration by parts allows us to replace the difficult one by the easy one. Several examples will be shown in the following section.

An indefinite integral form of integration by parts can be written as:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx. \tag{1.4}$$

2. Examples

To use the formula of integration by parts, one should always remember that our goal is to obtain a simpler integral. To see which one is easier, it usually relies on our experience.

Example 2.1. Find $\int_0^1 xe^x dx$.

Let $f(x) = e^x$, $g(x) = x$. Then by (1.3), we have

$$\begin{aligned} \int_0^1 xe^x dx &= xe^x\Big|_0^1 - \int_0^1 e^x dx \\ &= e - e^x\Big|_0^1 = 1. \end{aligned}$$

Notice that if we take $f(x) = x$, $g(x) = e^x$ instead, the integral in the formula (1.3) on the right will be more difficult than the original one. So we shouldn't play in this way.

Example 2.2. Find $\int_0^\pi x \sin x dx$.

Let $f(x) = -\cos x$, $g(x) = x$. Then by (1.3),

$$\begin{aligned}\int_0^\pi x \sin x dx &= -x\cos x\Big|_0^\pi + \int_0^\pi \cos x dx \\ &= \pi + \sin x\Big|_0^\pi = \pi\end{aligned}$$

Example 2.3. Find $\int \ln x dx$.

Let $f(x) = x$, $g(x) = \ln x$. Then by (1.4), we have

$$\int \ln x dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - x + C.$$

for some constant C.

Example 2.4. Find $\int e^x \sin x dx$.

This example is trickier than others. One should apply the integration by parts twice and find the answer implicitly. To obtain the answer, we should start with $f(x) = -\cos x$, $g(x) = e^x$. So we have

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx. \tag{2.1}$$

Here we should apply the integration by parts again on the integral on the right hand side of (2.1), by taking $f(x) = \sin x$ and $g(x) = e^x$. We obtain

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx. \tag{2.2}$$

So it implies

$$\int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx. \tag{2.3}$$

Notice that we have the same integral on the both side of (2.3), but with different sign. Therefore, we can move them to one side and divide both side by two:

$$\int e^x \sin x dx = \frac{1}{2}(-e^x \cos x + e^x \sin x) + C \tag{2.4}$$

for some constant C. Notice that we need to put C on the right hand side because it is an indefinite integral.

3. Trigonometric integrals

Here we generalize the formula for the integrations of the form $f(x) = P(\sin x, \cos x)$ with $P(x, y)$ being a polynomial. To obtain the general formula, we firstly start with the following simple case.

$$\int \sin x \cos x dx = \sin^2 x + C \tag{3.1}$$

for some constant C. This formula can be achieved directly by using substitution rule. Notice that the right hand side of (3.1) can also be written as $\cos^2 x + C$.

By the same token, we have

$$\int \sin^n x \cos x dx = \frac{1}{n+1} \sin^{n+1} x + C \tag{3.2}$$

for some constant C.

Secondly, by using the integration by parts we have

$$\int \sin^2 x dx = -\sin x \cos x + \int \cos^2 x dx = -\sin x \cos x + \int (1-\sin^2 x) dx \tag{3.3}$$

So

$$\int \sin^2 x dx = -\frac{1}{2}\sin x \cos x + \frac{1}{2}x + C \tag{3.4}$$

for some constant C. Now, for any $n \in \mathbb{N}$, $n \geq 2$, we have

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx. \tag{3.5}$$

So we can write down the formula for the integrals of $\int \sin^n x dx$ (inductively) and $\int \sin^n x \cos x dx$ for any n. By using the equality $\cos^2 x = (1-\sin^2 x)$, we can write down the formula for the integral

$$\int P(\sin x, \cos x) dx$$

for any polynomial $P(x, y)$.

There is also a similar recipe for functions of the type $P(\tan x, \sec x)$. Firstly, we notice that

$$\int \sec x dx = \ln|\tan x + \sec x| + C; \tag{3.6}$$

$$\int \tan x \sec x dx = \sec x + C \tag{3.7}$$

for some constant C. By the integration by parts, we have

$$\int \tan^n x \sec x dx = \tan^{n-1} x \sec x - (n-1)\int \tan^{n-2} x \sec^3 x dx. \tag{3.8}$$

Here we use the identity $\sec^2 x = 1 + \tan^2 x$. (3.8) becomes

$$\int \tan^n x \sec x dx = \frac{1}{n}\tan^{n-1} x \sec x - \frac{n-1}{n}\int \tan^{n-2} x \sec x dx \tag{3.9}$$

when $n > 2$. So by (3.6), (3.7), we have general formula for

$$\int \tan^n x \sec x dx. \tag{3.10}$$

Secondly, we can prove that by using integration by parts,

$$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx \tag{3.11}$$

when $n > 2$ and

$$\int \tan x dx = -\ln|\sec x| + C. \tag{3.12}$$

This gives us the formula for $\int \tan^n x dx$. Therefore, by the using the formula of

$\int \tan^n x \sec x dx$, $\int \tan^n x dx$ and the identity $\sec^2 x = 1 + \tan^2 x$, we can obtain the formula of

$$\int P(\tan x, \sec x) dx \tag{3.13}$$

for any polynomial $P(x, y)$.