# Chapter 3: Set Theory

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# Outline

- ***Sets and Subsets***
- Set Operations and the Laws of Set Theory
- Counting and Venn Diagrams
- A First Word on Probability

# Sets and Subsets (1/3)

- Set: A well-defined collection of objects. These objects are called elements and are said to be members of the set.
- We use capital letters, such as *A*, *B*, *C*, …, to represent sets and lowercase letters to represent elements.
- For a set *A* we write $x \in A$ if $x$ is an element of *A*; $y \notin A$ indicates that $y$ is not a member of *A*.

# Sets and Subsets (2/3)

- **Example 3.1:** *A* set can be designated by listing its elements within set braces. For example, if *A* is the set consisting of the first five positive integers, then we write $A = \{1, 2, 3, 4, 5\}$. Here $2 \in A$ but $6 \notin A$.
- Another standard notation: $A = \{x \mid x$ is an integer and $1 \leq x \leq 5\}$.

# Sets and Subsets (3/3)

- **Example 3.2:** For $U = \{1, 2, 3, \ldots\}$, the set of positive integers, Let

  a) $A = \{1, 4, 9, \ldots, 64, 81\} = \{x^2 \mid x \in U, x^2 < 100\}$ $= \{x^2 \mid x \in U \wedge x^2 < 100\}$.

  b) $B = \{2, 4, 6, 8, \ldots\} = \{2k \mid k \in U\}$.
- $U$: universe, or universe of discourse
- $A$: finite set, $B$: infinite set
- $|A|$: the number of elements in $A$ and is referred to as the cardinality, or size. $|A| = 9$.

# Subset and Proper Subset

- **Definition 3.1:** If $C$, $D$ are sets from a universe $U$, we say that $C$ is a subset of $D$ and write $C \subseteq D$, or $D \supseteq C$, if every element of $C$ is an element of $D$. If, in addition, $D$ contains an element that is not in $C$, then $C$ is called a proper subset of $D$, and this is denoted by $C \subset D$ or $D \supset C$.
- For all subsets $C$, $D$ of $U$, $C \subset D \Rightarrow C \subseteq D$
- When $C$, $D$ are finite, $C \subseteq D \Rightarrow |C| \leq |D|$, and $C \subset D \Rightarrow |C| < |D|$

# Set Equality

- **Definition 3.2:** For a given universe $U$, the sets $C$ and $D$ (taken from $U$) are said to be equal, and we write $C = D$, when $C \subseteq D$ and $D \subseteq C$.
- Neither order nor repetition is relevant for a general set, i.e., {1, 2, 3} = {3, 1, 2} = {2, 2, 1, 3} = {1, 2, 1, 3, 1}.

# Example (1/2)

- **Example 3.5:** Let $U$ = {1, 2, 3, 4, 5, 6, $x$, $y$, {1, 2}, {1, 2, 3}, {1, 2, 3, 4}} (where $x$, $y$ are the 24th, 25th lowercase letters of the alphabet and do not represent anything else, such as 3, 5, or {1, 2}). Then $|U|$ = 11.
- If $A$={1, 2, 3, 4}, then $|A|$=4 and here we have

i) $A \subseteq U$; ii) $A \subset U$; iii) $A \in U$;

iv) $\{A\} \subseteq U$; v) $\{A\} \subset U$; vi) $\{A\} \notin U$.

# Example (2/2)

- **Example 3.5 (cont.):** $A = \{1, 2, 3, 4\}$
- Now let $B = \{5, 6, x, y, A\} = \{5, 6, x, y, \{1, 2, 3, 4\}\}$. Then $|B| = 5$, *not* 8. And now we find that

i) $A \in B$; ii) $\{A\} \subseteq B$; iii) $\{A\} \subset B$.

But

iv) $\{A\} \notin B$;

v) $A \not\subseteq B$ (that is, $A$ is not a subset of $B$); and

vi) $A \not\subset B$ (that is, $A$ is not a proper subset of $B$).

# Subset Relations

- **Theorem 3.1:** Let $A$, $B$, $C \subseteq U$.

  a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

  b) If $A \subset B$ and $B \subseteq C$, then $A \subset C$.

  c) If $A \subseteq B$ and $B \subset C$, then $A \subset C$.

  d) If $A \subset B$ and $B \subset C$, then $A \subset C$.

  **Proof:** Read by yourself.

# Example

- **Example 3.6:** Let $U = \{1, 2, 3, 4, 5\}$ with $A = \{1, 2, 3\}$, $B = \{3, 4\}$, and $C = \{1, 2, 3, 4\}$. Then the following subset relations hold:

  a) $A \subseteq C$
  b) $A \subset C$
  c) $B \subset C$
  d) $A \subseteq A$
  e) $B \nsubseteq A$
  f) $A \not\subset A$ (that is, $A$ is not proper subset of A)

# Null Set

- **Definition 3.3:** The null set, or empty set, is the (unique) set containing no elements. It is denoted by $\varnothing$ or $\{\ \}$.
- $|\varnothing| = 0$, but $\{0\} \neq \varnothing$.
- $\varnothing \neq \{\varnothing\}$ because $\{\varnothing\}$ is a set with one element, namely, the null set.
- **Theorem 3.2:** For any universe $U$, let $A \subseteq U$. Then $\varnothing \subseteq A$, and if $A \neq \varnothing$, then $\varnothing \subset A$.
  **Proof.** Read by yourself.

# Power Set (1/2)

- **Definition 3.4:** If $A$ is a set from universe $U$, the power set of $A$, denoted $P(A)$, is the collection (or set) of all subsets of $A$.

- **Example 3.8:** For the set $C = \{1, 2, 3, 4\}$, $P(C) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\} \{1, 3, 4\} \{2, 3, 4\}, C\}$.

# Power Set (2/2)

For any finite set $A$ with $|A| = n \geq 0$, we find that $A$ has $2^n$ subsets and that $|P(A)| = 2^n$. For any $0 \leq k \leq n$, there are $\binom{n}{k}$ subsets of size $k$. Counting the subsets of $A$ according to the number, $k$, of elements in a subset, we have the combinatorial identity

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n, \text{ for } n \geq 0.$$

# Gray Code

- A systematic way to represent the subsets of a given nonempty set can be accomplished by using a coding scheme known as a Gray code.

| | |
|---|---|
| 0 | ∅ |
| 1 | $\{x\}$ |

(a)

| | | |
|---|---|---|
| 0 | 0 | ∅ |
| 1 | 0 | $\{x\}$ |
| 1 | 1 | $\{x, y\}$ |
| 0 | 1 | $\{y\}$ |

(b)

| | | |
|---|---|---|
| 0 0 | 0 | ∅ |
| 1 0 | 0 | $\{x\}$ |
| 1 1 | 0 | $\{x, y\}$ |
| 0 1 | 0 | $\{y\}$ |
| 0 1 | 1 | $\{y, z\}$ |
| 1 1 | 1 | $\{x, y, z\}$ |
| 1 0 | 1 | $\{x, z\}$ |
| 0 0 | 1 | $\{z\}$ |

(c)

| (d) | (e) | (f) |
|---|---|---|
| 000 | 000 | 000 |
| 100 | 010 | 001 |
| 110 | 011 | 101 |
| 010 | 001 | 100 |
| 011 | 101 | 110 |
| 111 | 111 | 010 |
| 101 | 110 | 011 |
| 001 | 100 | 111 |

# Example (1/3)

- **Example 3.11:** Consider the composition of 7 as a sum of one or more positive integers, where the order of the summands is relevant.

  For the set {1, 2, 3, 4, 5, 6} there are $2^6$ subsets. What does this have to do with the composition of 7?

  Consider a subset {1, 4, 6}. This results in the composition 2 + 1 + 2 + 2.

$$(1 \underset{\substack{\big\downarrow \\ \text{1st plus} \\ \text{sign}}}{+} 1) + 1 + (1 \underset{\substack{\big\downarrow \\ \text{4th plus} \\ \text{sign}}}{+} 1) + (1 \underset{\substack{\big\downarrow \\ \text{6th plus} \\ \text{sign}}}{+} 1)$$

# Example (2/3)

- **Example 3.11 (cont.):**

  The subset {1, 2, 5, 6} gives us the composition $3 + 1 + 3$.

$$(1 \underset{\substack{\downarrow \\ \text{1st plus} \\ \text{sign}}}{+} 1 \underset{\substack{\downarrow \\ \text{2nd plus} \\ \text{sign}}}{+} 1) + 1 + (1 \underset{\substack{\downarrow \\ \text{5th plus} \\ \text{sign}}}{+} 1 \underset{\substack{\downarrow \\ \text{6th plus} \\ \text{sign}}}{+} 1)$$

  For each positive integer $m$, there are $2^{m-1}$ compositions of $m$.

# Example (3/3)

- **Example 3.12:** For integers $n$, $r$ with $n \geq r \geq 1$,

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}.$$

**Hint:** Let $A = \{x, a_1, a_2, \ldots, a_n\}$ and consider all subsets of *A* that contain $r$ elements.

# Pascal’s Triangle

$$
\begin{array}{lccccccccccc}
(n=0) & & & & & & \binom{0}{0} & & & & & \\
(n=1) & & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
(n=2) & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
(n=3) & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
(n=4) & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
(n=5) & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

$$
\begin{array}{lccccccccccc}
(n=0) & & & & & & 1 & & & & & \\
(n=1) & & & & & 1 & & 1 & & & & \\
(n=2) & & & & 1 & & 2 & & 1 & & & \\
(n=3) & & & 1 & & 3 & & 3 & & 1 & & \\
(n=4) & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(n=5) & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

# Set of Numbers

a) $\mathbf{Z}$ = the set of *integers* = $\{0, 1, -1, 2, -2, 3, -3, \ldots\}$

b) $\mathbf{N}$ = the set of *nonnegative integers* or *natural numbers* = $\{0, 1, 2, 3, \ldots\}$

c) $\mathbf{Z}^+$ = the set of *positive integers* = $\{1, 2, 3, \ldots\} = \{x \in \mathbf{Z} \mid x > 0\}$

d) $\mathbf{Q}$ = the set of *rational numbers* = $\{a/b \mid a, b \in \mathbf{Z}, b \neq 0\}$

e) $\mathbf{Q}^+$ = the set of *positive rational numbers* = $\{r \in \mathbf{Q} \mid r > 0\}$

f) $\mathbf{Q}^*$ = the set of *nonzero rational numbers*

g) $\mathbf{R}$ = the set of *real numbers*

h) $\mathbf{R}^+$ = the set of *positive real numbers*

i) $\mathbf{R}^*$ = the set of *nonzero real numbers*

j) $\mathbf{C}$ = the set of *complex numbers* = $\{x + yi \mid x, y \in \mathbf{R}, i^2 = -1\}$

k) $\mathbf{C}^*$ = the set of *nonzero complex numbers*

l) For each $n \in \mathbf{Z}^+$, $\mathbf{Z}_n = \{0, 1, 2, \ldots, n - 1\}$

# Set of Numbers

**m)** For real numbers $a$, $b$ with $a < b$, $[a, b] = \{x \in \mathbf{R} \mid a \le x \le b\}$, $(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$, $[a, b) = \{x \in \mathbf{R} \mid a \le x < b\}$, $(a, b] = \{x \in \mathbf{R} \mid a < x \le b\}$. The first set is called a *closed interval*, the second set an *open interval*, and the other two sets *half-open intervals*.

- **EXERCISES 3.1:** 4, 12

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- Sets and Subsets
- ***Set Operations and the Laws of Set Theory***
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- A First Word on Probability

# Closed Binary Operations

- The addition and multiplication of positive integers are said to be closed binary operations on $\mathbf{Z}^+$.
- Two operands, hence the operation is called binary.
- Since $a + b \in \mathbf{Z}^+$ when $a, b \in \mathbf{Z}^+$, we say that the binary operation of addition (on $\mathbf{Z}^+$) is closed.

# Binary Operations for Sets

- **Definition 3.5:** For $A$, $B \subseteq U$ we define the following:
  a) $A \cup B$ (the union of $A$ and $B$)
  $= \{x \mid x \in A \vee x \in B\}$.
  b) $A \cap B$ (the intersection of $A$ and $B$)
  $= \{x \mid x \in A \wedge x \in B\}$.
  c) $A \triangle B$ (the symmetric difference of $A$ and $B$)
  $= \{x \mid (x \in A \vee x \in B) \wedge x \notin A \cap B\}$
  $= \{x \mid x \in A \cup B \wedge x \notin A \cap B\}$.

# Example

- **Example 3.15:** With $U = \{1, 2, 3, \ldots, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$, and $C = \{7, 8, 9\}$, we have:

  a) $A \cap B = \{3, 4, 5\}$

  b) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

  c) $B \cap C = \{7\}$

  d) $A \cap C = \emptyset$

  e) $A \triangle B = \{1, 2, 6, 7\}$

  f) $A \cup C = \{1, 2, 3, 4, 5, 7, 8, 9\}$

  g) $A \triangle C = \{1, 2, 3, 4, 5, 7, 8, 9\}$

# Disjoint (1/2)

- **Definition 3.6:** Let $S$, $T \subseteq U$. The sets $S$ and $T$ are called disjoint, or mutually disjoint, when $S \cap T = \varnothing$.
- **Theorem 3.3:** If $S$, $T \subseteq U$, then $S$, $T \subseteq U$ are disjoint if and only if $S \cup T = S \triangle T$.

**Proof:**

(Only if part) Consider each $x \in U$.
If $x \in S \cup T$, then $x \in S$ or $x \in T$.
$\because S, T$ disjoint $\therefore x \notin S \cap T$, so $x \in S \triangle T$
$\therefore S \cup T \subseteq S \triangle T$ (1)

# Disjoint (2/2)

- **Theorem 3.3 (cont.):**

  If $y \in S \triangle T$, then $y \in S$ or $y \in T$. So $y \in S \cup T$.

  $\therefore S \triangle T \subseteq S \cup T$ (2)

  It follows from **Definition 3.2** that $S \cup T = S \triangle T$.

  (If part) (By contradiction) Assume $S \cup T = S \triangle T$ and $S \cap T \neq \varnothing$. Then, let $x \in S \cap T$.

  Since $x \in S$ and $x \in T$, $x \in S \cup T$ $(= S \triangle T)$

  But when $x \in S \cup T$ and $x \in S \cap T$, then $x \notin S \triangle T$.

  From this contradiction, we have $S$ and $T$ disjoint.

# Complement

- **Definition 3.7:** For a set $A \subseteq U$, the complement of $A$, denoted $U - A$, or $\overline{A}$, is given by $\{x \mid x \in U \wedge x \notin A\}$.
- **Definition 3.8:** For $A, B \subseteq U$, the (relative) complement of $A$ in $B$, denoted $B - A$, is given by $\{x \mid x \in B \wedge x \notin A\}$.
- **Example 3.17:** With $U = \{1, 2, 3, \ldots, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$, and $C = \{7, 8, 9\}$,

a) $B - A = \{6, 7\}$ b) $A - B = \{1, 2\}$ c) $A - C = A$

d) $C - A = C$ e) $A - A = \emptyset$ f) $U - A = \overline{A}$

# Subset, Union, Intersection, and Complement (1/4)

- **Theorem 3.4:** For any universe $U$ and any sets $A$, $B \subseteq U$, the following statements are equivalent:

  a) $A \subseteq B$ b) $A \cup B = B$

  c) $A \cap B = A$ d) $\overline{B} \subseteq \overline{A}$

  **Proof:** In order to prove the theorem, we prove that (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d), and (d) $\Rightarrow$ (a).

  (a) $\Rightarrow$ (b) If $A$, $B$ are any sets, then $\boldsymbol{B \subseteq A \cup B}$. For the opposite inclusion, if $x \in A \cup B$, then $x \in A$ or $x \in B$, but since $A \subseteq B$, in either case we have $x \in B$. So $\boldsymbol{A \cup B \subseteq B}$ and, since we now have both inclusions, it follows that $A \cup B = B$.

# Subset, Union, Intersection, and Complement (2/4)

- **Theorem 3.4 (cont.) :**

  a) $A \subseteq B$ b) $A \cup B = B$

  c) $A \cap B = A$ d) $\overline{B} \subseteq \overline{A}$

  **Proof:**

  (b) $\Rightarrow$ (c) Given Sets $A$, $B$, we always have $\boldsymbol{A \supseteq A \cap B}$. For the opposite inclusion, let $y \in A$. With $A \cup B = B$, $y \in A \Rightarrow y \in A \cup B \Rightarrow y \in B$ (since $A \cup B = B$) $\Rightarrow y \in A \cap B$, so $\boldsymbol{A \subseteq A \cap B}$ and we conclude that $A = A \cap B$.

# Subset, Union, Intersection, and Complement (3/4)

- **Theorem 3.4 (cont.) :**

a) $A \subseteq B$

b) $A \cup B = B$

c) $A \cap B = A$

d) $\overline{B} \subseteq \overline{A}$

**Proof:**

(c) $\Rightarrow$ (d) We know that $\boldsymbol{z \in \overline{B}} \Rightarrow z \notin B$. Now if $z \in A \cap B$, then $z \in B$, since $A \cap B \subseteq B$. The contradiction — namely, $z \notin B \wedge z \in B$ — tells us that $z \notin A \cap B$. Therefore, $z \notin A$ because $A \cap B = A$. But $z \notin A \Rightarrow \boldsymbol{z \in \overline{A}}$, so $\overline{B} \subseteq \overline{A}$.

# Subset, Union, Intersection, and Complement (4/4)

- **Theorem 3.4 (cont.) :**

  a) $A \subseteq B$ b) $A \cup B = B$

  c) $A \cap B = A$ d) $\overline{B} \subseteq \overline{A}$

  **Proof:**

  (d) $\Rightarrow$ (a) Last, $\boldsymbol{w \in A}$ $\Rightarrow w \notin \overline{A}$. If $w \notin B$, then $w \in \overline{B}$. With $\overline{B} \subseteq \overline{A}$ it then follows that $w \in \overline{A}$. This time we get the contradiction $w \notin \overline{A} \wedge w \in \overline{A}$, and this tells us that $\boldsymbol{w \in B}$. Hence $A \subseteq B$.

# The Laws of Set Theory (1/2)

- For any sets $A$, $B$, and $C$ taken from a universe $U$

| | |
|---|---|
| **1)** $\overline{\overline{A}} = A$ | Law of *Double Complement* |
| **2)** $\overline{A \cup B} = \overline{A} \cap \overline{B}$ | *DeMorgan's* Laws |
| $\overline{A \cap B} = \overline{A} \cup \overline{B}$ | |
| **3)** $A \cup B = B \cup A$ | *Commutative* Laws |
| $A \cap B = B \cap A$ | |
| **4)** $A \cup (B \cup C) = (A \cup B) \cup C$ | *Associative* Laws |
| $A \cap (B \cap C) = (A \cap B) \cap C$ | |
| **5)** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ | *Distributive* Laws |
| $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | |
| **6)** $A \cup A = A$ | *Idempotent* Laws |
| $A \cap A = A$ | |

# The Laws of Set Theory (2/2)

| | | |
|---|---|---|
| **7)** | $A \cup \emptyset = A$ | *Identity* Laws |
| | $A \cap \mathcal{U} = A$ | |
| **8)** | $A \cup \overline{A} = \mathcal{U}$ | *Inverse* Laws |
| | $A \cap \overline{A} = \emptyset$ | |
| **9)** | $A \cup \mathcal{U} = \mathcal{U}$ | *Domination* Laws |
| | $A \cap \emptyset = \emptyset$ | |
| **10)** | $A \cup (A \cap B) = A$ | *Absorption* Laws |
| | $A \cap (A \cup B) = A$ | |

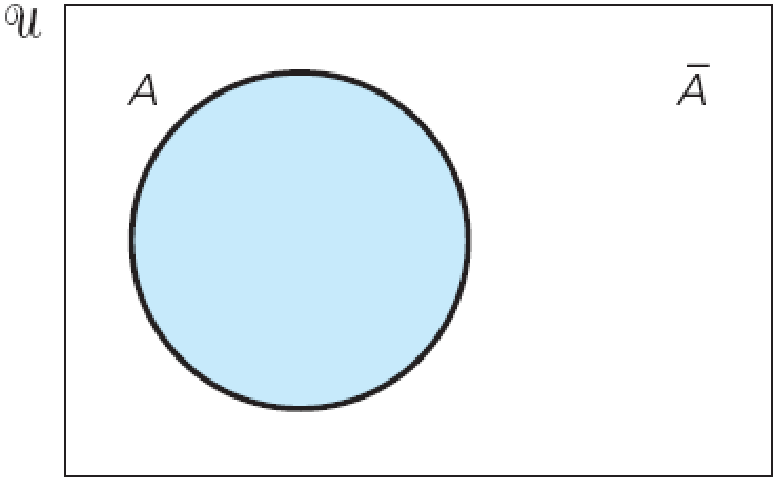
# Dual

- **Definition 3.9:** Let $s$ be a (general) statement dealing with the equality of two set expressions. Each such expression may involve one or more occurrences of sets (such as $A$, $\overline{A}$, $B$, $\overline{B}$, etc.), one or more occurrences of Ø and $U$, and only the set operation symbols $\cap$ and $\cup$. The dual of $s$, denoted $s^d$, is obtained from $s$ by replacing (1) each occurrence of Ø and $U$ (in $s$) by $U$ and Ø, respectively; and (2) each occurrence of $\cap$ and $\cup$ (in $s$) by $\cup$ and $\cap$, respectively.

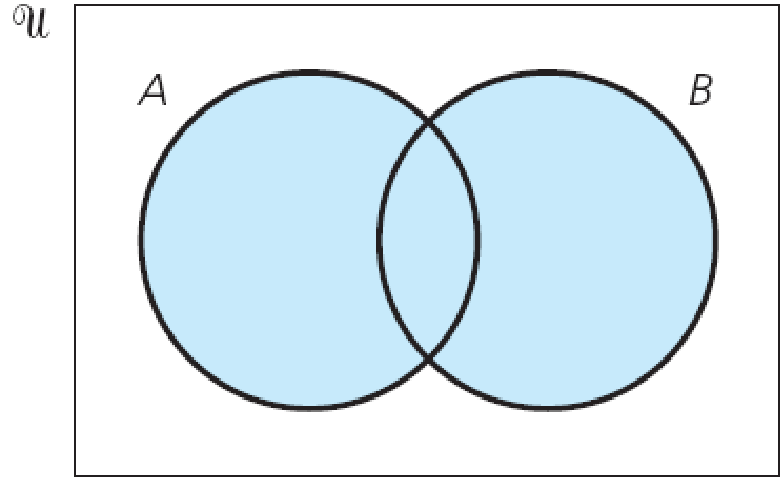
# The Principle of Duality

- **Theorem 3.5:** Let $s$ denote a theorem dealing with the equality of two set expressions (involving only the set operations $\cap$ and $\cup$ as described in **Definition 3.9**). Then $s^d$, the dual of $s$, is also a theorem.
- This result cannot be applied to particular situations but only to results (theorems) about sets in general.
- **Example:** $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 5\}$, $C = \{1, 2\}$, and $D = \{1, 3,\}$
$A \cap B = C \cup D$, but $A \cup B \neq C \cap D$

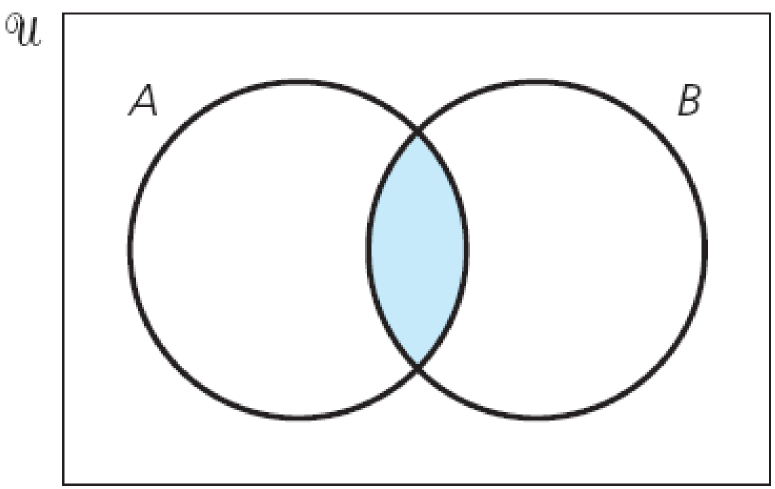
# Venn Diagram (1/3)



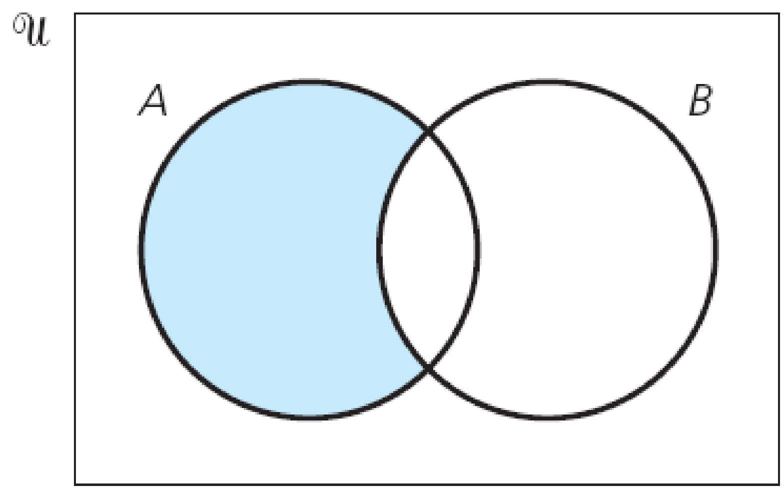

(a)




(b) $A \cup B$

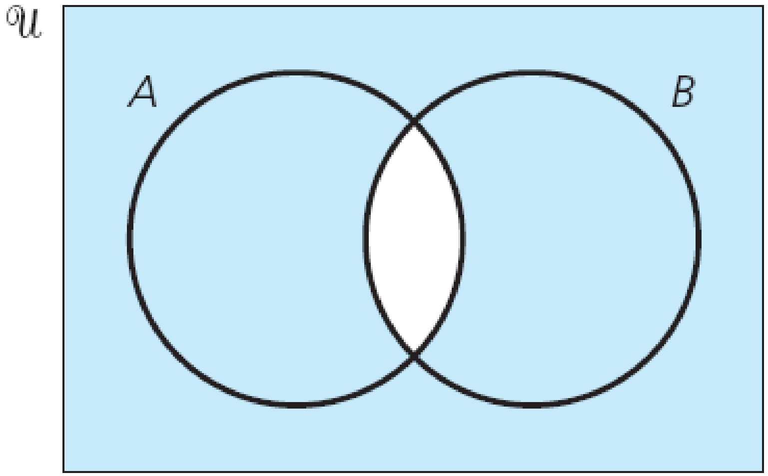



(c) $A \cap B$

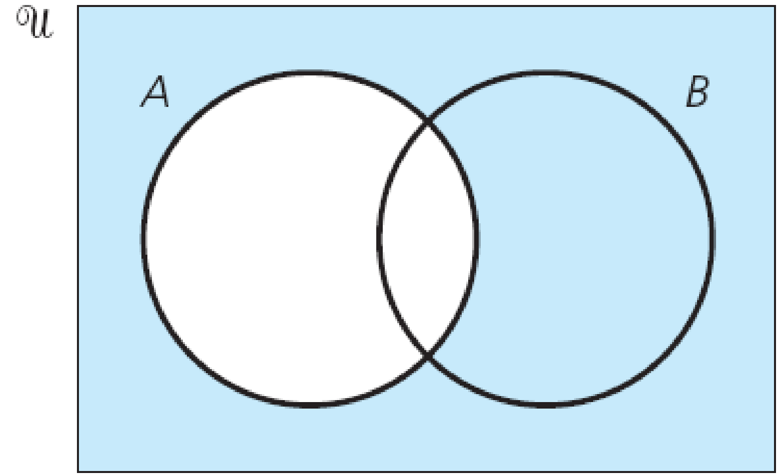



(d) $A - B$

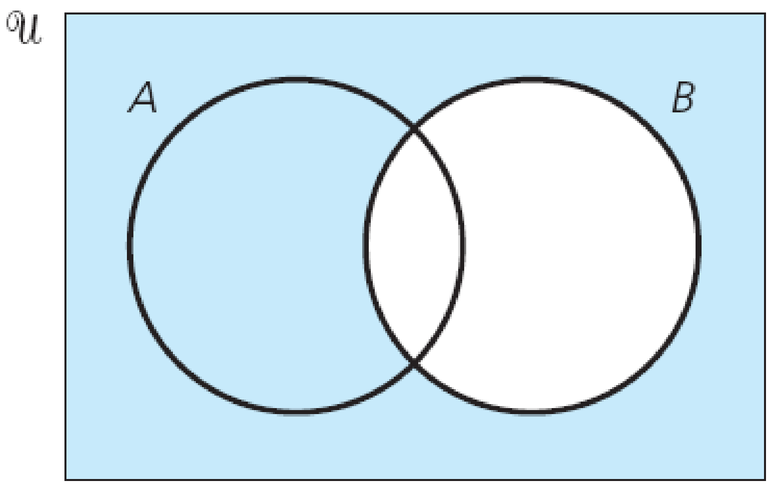
# Venn Diagram (2/3)



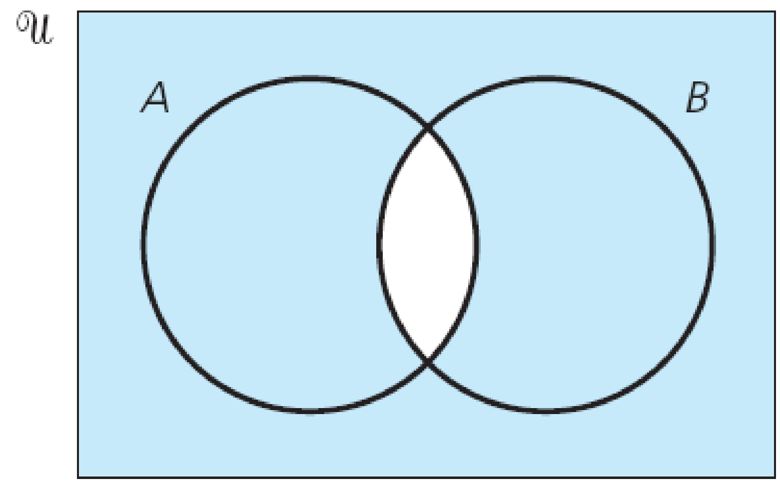

(a) $\overline{A \cap B}$




(b) $\overline{A}$

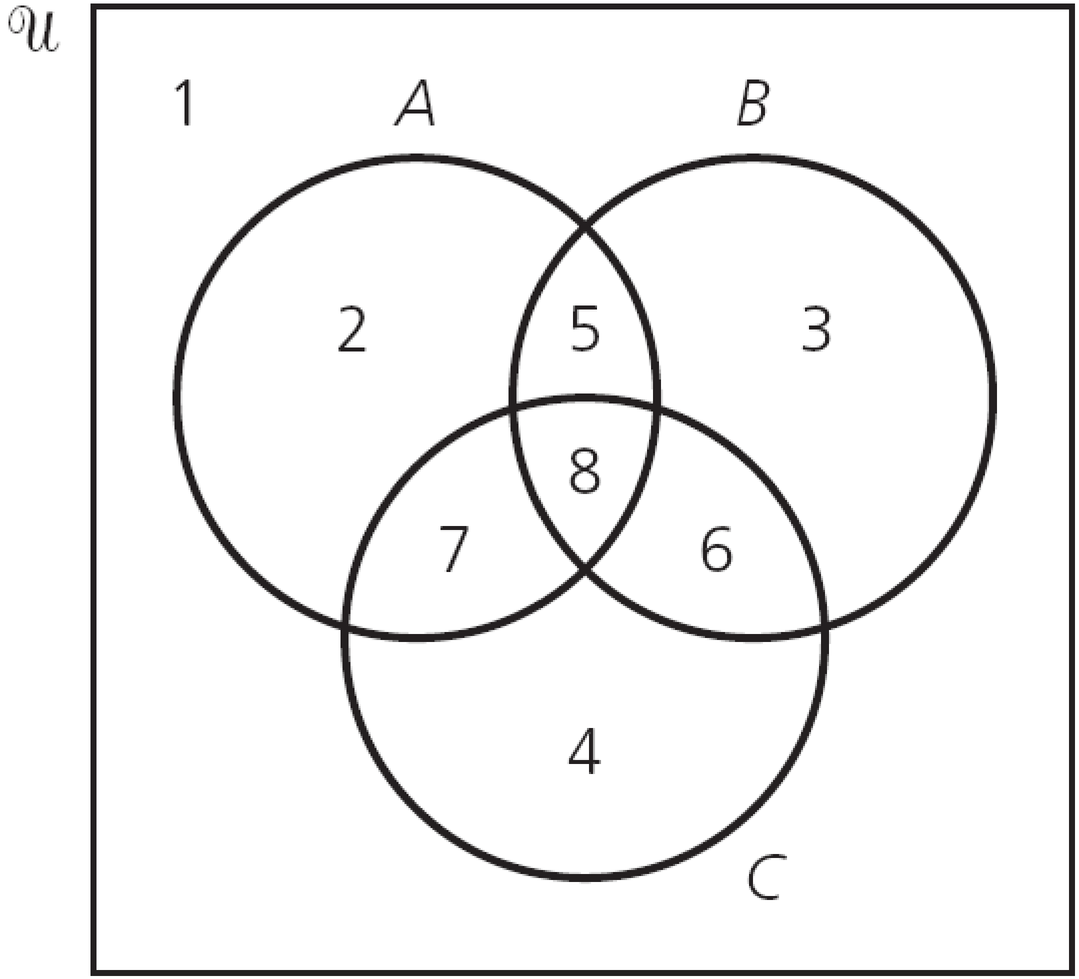



(c) $\overline{B}$




(d) $\overline{A} \cup \overline{B}$

# Venn Diagram (3/3)




$$\overline{(A \cup B) \cap C} = (\overline{A} \cap \overline{B}) \cup \overline{C}$$

# Membership Table (1/3)

- We observe that for sets $A, B \subseteq U$, an element $x \in U$ satisfies exactly one of the following four situations:

  a) $x \notin A, x \notin B$ b) $x \notin A, x \in B$

  c) $x \in A, x \notin B$ d) $x \in A, x \in B$

- When $x$ is an element of a given set, we write a 1 in the column representing that set in the membership table; when $x$ is not in the set, we enter a 0.

# Membership Table (2/3)

| $A$ | $B$ | $A \cap B$ | $A \cup B$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 |

(a)

| $A$ | $\overline{A}$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

(b)

- Using membership tables, we can establish the equality of two sets by comparing their respective columns in the table.

# Membership Table (3/3)

Correspond with region 1: $\overline{A} \cap \overline{B} \cap \overline{C}$

| $A$ | $B$ | $C$ | $B \cap C$ | $A \cup (B \cap C)$ | $A \cup B$ | $A \cup C$ | $(A \cup B) \cap (A \cup C)$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Since these columns are identical, we conclude that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

# Simplify the Expression

- **Example 3.20:** Simplify the expression
  $\overline{\overline{(A \cup B) \cap C} \cup \overline{B}}$.

$\overline{\overline{(A \cup B) \cap C} \cup \overline{B}}$ **Reasons**

# Example (1/3)

- **Example 3.21:** Express $\overline{A - B}$ in terms of $\cup$ and $\overline{\phantom{A}}$.

  From the definition of relative complement, $A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$. Therefore,

| | |
|---|---|
| $\overline{A - B} = \overline{A \cap \overline{B}}$ | **Reasons** |
| $= \overline{A} \cup \overline{\overline{B}}$ | DeMorgan's Law |
| $= \overline{A} \cup B$ | Law of Double Complement |

# Example (2/3)

- **Example 3.22:** We have $A \triangle B = \{x \mid x \in A \cup B \wedge x \notin A \cap B\} = (A \cup B) - (A \cap B) = (A \cup B) \cap \overline{(A \cap B)}$, so

$\overline{A \triangle B} = \overline{(A \cup B) \cap \overline{(A \cap B)}}$ **Reasons**

# Example (3/3)

- **Example 3.22 (cont.):**

# Index Set

- **Definition 3.10:** Let $I$ be a nonempty set and $U$ a universe. For each $i \in I$ let $A_i \subseteq U$. Then $I$ is called an index set (or set of indices), and each $i \in I$ is called an index.

  Under these conditions,

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \quad \text{for at least one } i \in I\},$$

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \quad \text{for every } i \in I\}.$$

# Example

- **Example 3.23:** Let $I = \{3, 4, 5, 6, 7\}$, and for each $i \in I$ let $A_i = \{1, 2, 3, \ldots, i\} \subseteq U = \mathbf{Z}^+$. Then $\cup_{i \in I} A_i = \cup_{i=3}^{7} A_i = \{1, 2, 3, \ldots, 7\} = A_7$, whereas $\cap_{i \in I} A_i = \{1, 2, 3\} = A_3$.

- **Example 3.24:** Let $U = \mathbf{R}$ and $I = \mathbf{R}^+$. If for each $r \in \mathbf{R}^+$, $A_r = [-r, r]$, then $\cup_{r \in I} A_r = \mathbf{R}$ and $\cap_{r \in I} A_r = \{0\}$.

# Generalized DeMorgan's Laws

- **Theorem 3.6:** Generalized DeMorgan's Laws*.*
  Let $I$ be an index set where for each $i \in I$, $A_i \subseteq U$.
  Then
  a) $\overline{\bigcup_{i \in I} A_i} = \bigcap_{i \in I} \overline{A_i}$ b) $\overline{\bigcap_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$

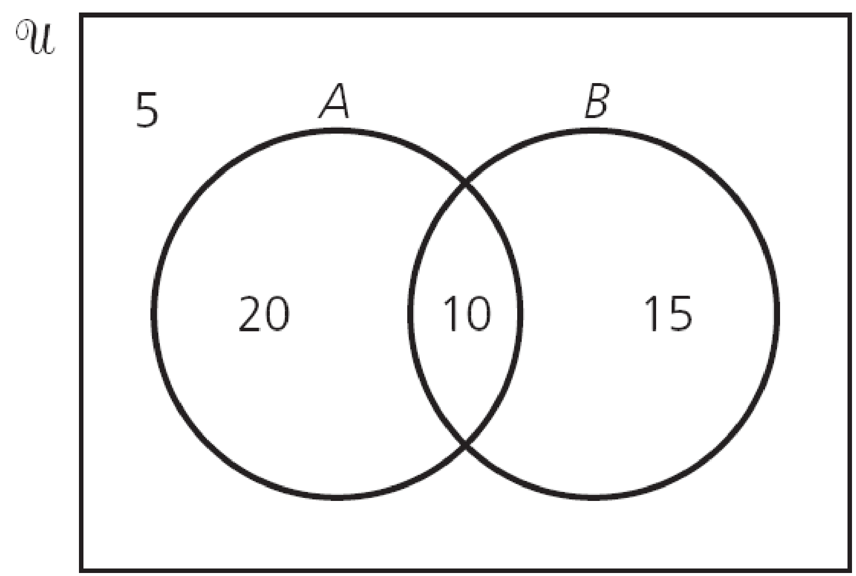
- **EXERCISES 3.2:** 2, 8

# Outline

- Sets and Subsets
- Set Operations and the Laws of Set Theory
- ***<u>Counting and Venn Diagrams</u>***
- A First Word on Probability

# Counting Formulas (1/5)

- **Example 3.25:** In a class of 50 college freshman, 30 are studying C++, 25 are studying Java, and 10 are studying both languages. How many freshman are studying either computer language?



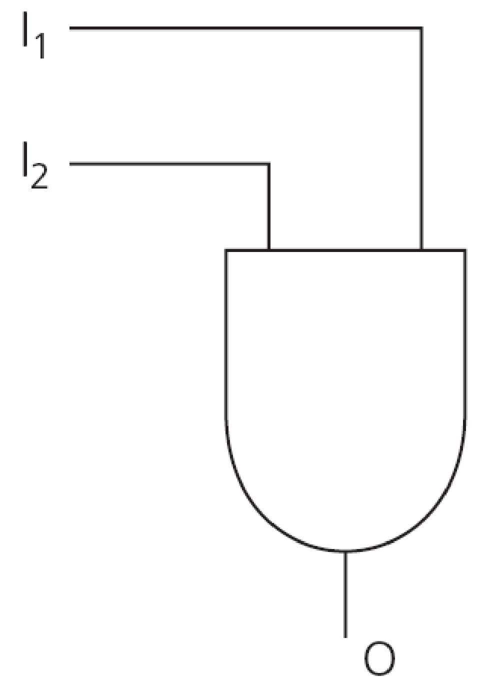
# Counting Formulas (2/5)

- If $A$ and $B$ are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$. Consequently, finite sets $A$ and $B$ are (mutually) disjoint if and only if $|A \cup B| = |A| + |B|$. In addition, when $U$ is finite, from DeMorgan's Law we have $|\overline{A} \cap \overline{B}| = |\overline{A \cup B}| = |U| - |A \cup B| = |U| - |A| - |B| + |A \cap B|$.

# Counting Formulas (3/5)

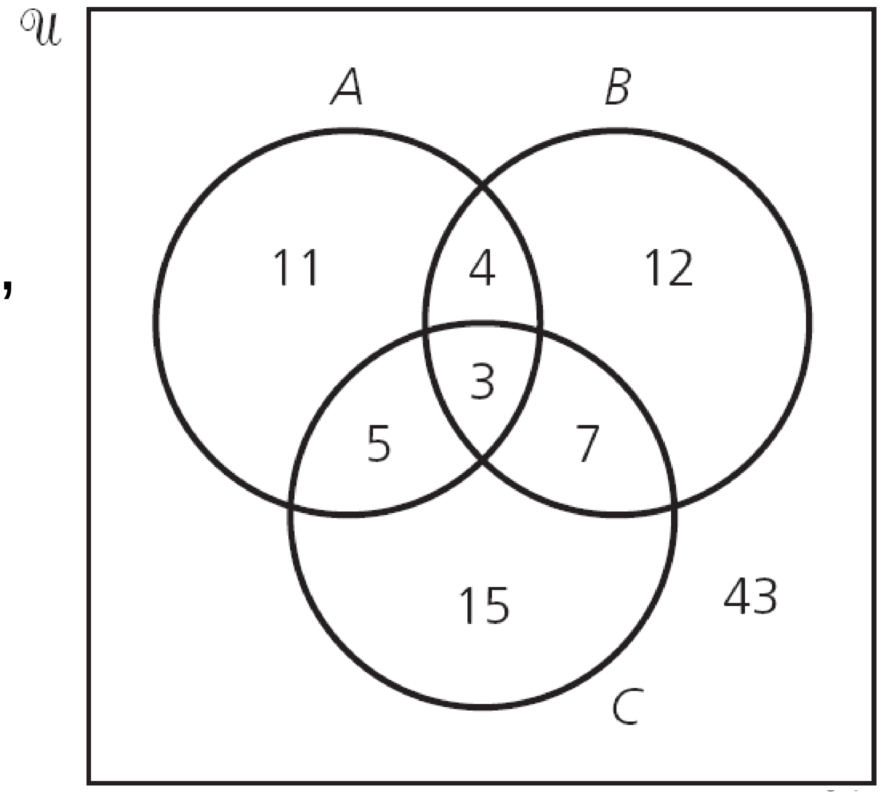
- **Example 3.26:** An AND gate in an ASIC (Application Specific Integrated Circuit) has two inputs: $I_1$, $I_2$, and one output: O. Such an AND gate can have any or all of the following defects:

  $D_1$: The input $I_1$ is stuck at 0.
  $D_2$: The input $I_2$ is stuck at 0.
  $D_3$: The input O is stuck at 1.



# Counting Formulas (4/5)

- **Example 3.26 (cont.):** For a sample of 100 such gates we let $A$, $B$, and $C$ be the subsets (of these 100 gates) having defects $D_1$, $D_2$, and $D_3$, respectively. With $|A| = 23$, $|B| = 26$, $|C| = 30$, $|A \cap B| = 7$, $|A \cap C| = 8$, $|B \cap C| = 10$, and $|A \cap B \cap C| = 3$, how many gates in the sample have at least one of the defects $D_1$, $D_2$, $D_3$?



# Counting Formulas (5/5)

- If $A$, $B$, $C$ are finite sets, then $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. From the formula for $|A \cup B \cup C|$ and DeMorgan's Law, we find that if the universe $U$ is finite, then $|\overline{A} \cap \overline{B} \cap \overline{C}| = |\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C| = |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$.

- **EXERCISES 3.3:** 10

# Outline

- Sets and Subsets
- Set Operations and the Laws of Set Theory
- Counting and Venn Diagrams
- ***A First Word on Probability***

# Definition for Probability

Under the assumption of equal likelihood, let $I$ be the sample space for an experiment $E$. Any subset $A$ of $I$, including the empty subset, is called an event. Each element of $I$ determines an outcome, so if $|I| = n$ and $a \in I$, $A \subseteq I$, then

$Pr(\{a\})$ = The probability that $\{a\}$ (or, $a$) occurs $= |\{a\}|/I = 1/n$, and

$Pr(A)$ = The probability that $A$ occurs $= |A|/|I| = |A|/n$.

[**Note:** We often write $Pr(a)$ for $Pr(\{a\})$.]

# Example (1/4)

- **Example 3.29:** If Dillon rolls a fair die, what is the probability he gets (a) a 5 or a 6, (b) an even number?

  The sample space $I = \{1, 2, 3, 4, 5, 6\}$.
  In part (a) we have event $A = \{5, 6\}$ and $Pr(A) = 1/3$.
  For part (b) we consider event $B = \{2, 4, 6\}$ and find that $Pr(B) = 1/2$.

  Furthermore we also notice here that $Pr(\overline{A}) = Pr(\{1, 2, 3, 4\}) = 4/6 = 1 - 1/3 = 1 - Pr(A)$.

# Example (2/4)

- **Example 3.30:** There are 20 students enrolled in Mrs. Arnold's class. Hence, if she wants to select two of her students, at random, to take care of the class rabbit, she may make her selection in C(20, 2) = 190 ways, so $|I| = 190$. Now suppose that Kyle and Kody are two of the 20 students in the class and we let $A$ be the event that Kyle is one of the students selected and $B$ be the event that the selection includes Kody. Consequently, upon choosing the students, at random, the probability that Mrs. Arnold selects

# Example (3/4)

- **Example 3.30 (cont.):**

  a) both Kyle and Kody is

  $$Pr(A \cap B) = \binom{2}{2} / \binom{20}{2} = 1/190;$$

  b) neither Kyle nor Kody is

  $$Pr(\overline{A} \cap \overline{B}) = \binom{18}{2} / \binom{20}{2} = 153/190;$$

  c) Kyle but not Kody is

  $$Pr(A \cap \overline{B}) = \binom{1}{1} \binom{18}{1} / \binom{20}{2} = 18/190 = 9/95.$$

# Example (4/4)

- **Example 3.31:** Consider drawing five cards from a standard deck of 52 cards. This can be done in C(52, 5) = 2,598,960 ways. Now suppose that Tanya draws five cards, at random, from a standard deck. What is the probability she gets (a) three aces and two jacks; (b) three aces and a pair; (c) a full house (that is, three of one kind and a pair)?

  (a) C(4, 3)C(4, 2)/C(52, 5)

  (b) C(4, 3)C(12, 1)C(4, 2)/C(52, 5)

  (c) C(13, 1)C(4, 3)C(12, 1)C(4, 2)/C(52, 5)

# Cross Product (1/2)

- **Definition 3.11:** For sets $A$, $B$, the Cartesian product, or cross product, of $A$ and $B$ is denoted by $A \times B$ and equals $\{(a, b) \mid a \in A, b \in B\}$.

- **Example 3.32:** If $A = \{1, 2, 3\}$ and $B = \{x, y\}$, then $A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$ while $B \times A = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$. Here $(1, x) \in A \times B$ but $(1, x) \notin B \times A$, although $(x, 1) \in B \times A$. So $A \times B \neq B \times A$, but $|A \times B| = 6 = 2 \cdot 3 = |A||B| = |B||A| = |B \times A|$.

# Cross Product (2/2)

- **Example 3.35:** If Charles tosses a fair coin four times, what is the probability that he gets two heads and two tails?

  For this experiment of tossing a fair coin four times, we have the sample space $I = I_1 \times I_2 \times I_3 \times I_4$, where $I_1 = I_2 = I_3 = I_4 = \{\mathrm{H}, \mathrm{T}\}$. $|I| = 2^4 = 16$.

  The event $A$ we are concerned about contains all arrangement of H, H, T, T, so $|A| = 4!/(2!2!) = 6$.

  $Pr(A) = 6/16 = 3/8$.

# Concept of a Venn Diagram (1/2)

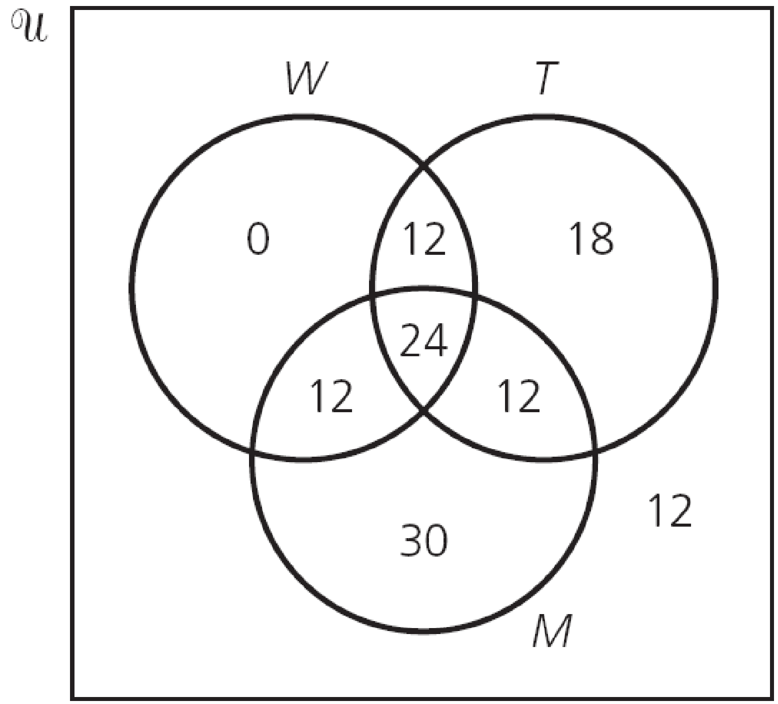
- **Example 3.37:** In a survey of 120 passengers, an airline found that 48 enjoyed wine with their meals, 78 enjoyed mixed drinks, and 66 enjoyed iced tea. In addition, 36 enjoyed any given pair of these beverages and 24 passengers enjoyed them all. If two passengers are selected at random from the survey sample of 120, what is the probability that

  a) (Event $A$) they both want only iced tea with their meals?

  b) (Event $B$) they both enjoy exactly two of the three beverage offerings?

# Concept of a Venn Diagram (2/2)

- **Example 3.37 (cont.):**




$Pr(A)$ = C(18, 2)/C(120, 2) = 51/2380

$Pr(B)$ = 3/34

# Homework Assignment #3

- **EXERCISES 3.1**

  4, 12
- **EXERCISES 3.2**

  2, 8
- **EXERCISES 3.3**

  10
- **EXERCISE 3.4**

  5, 8