CALCULUS I LECTURE 9: SEQUENCES, SERIES AND THEIR LIMITS I

1. Sequences

A sequence is a set A equipped with a bijective correspondence between A and $\mathbb{N}$ (or $\mathbb{N} \cup \{0\}$). We usually denote it by

$$A = \{a_i\}_{i\in\mathbb{N}}. \tag{1.1}$$

Any two sequence $\{a_i\}_{i\in\mathbb{N}}$, $\{b_i\}_{i\in\mathbb{N}}$ and a constant $c \in \mathbb{R}$, we define the sum of these two sequences to be

$$\{a_i\}_{i\in\mathbb{N}} + \{b_i\}_{i\in\mathbb{N}} = \{a_i + b_i\}_{i\in\mathbb{N}}; \tag{1.2}$$

and the scalar multiplication of $\{a_i\}_{i\in\mathbb{N}}$ to be

$$c\{a_i\}_{i\in\mathbb{N}} = \{ca_i\}_{i\in\mathbb{N}}. \tag{1.3}$$

The limit for sequences can be defined as the following.

Definition 1.1. We call $\lim_{n\to\infty} a_n = L$ if and only if there exists a function $N : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|a_n - L| < \varepsilon \tag{1.4}$$

when $n > N(\varepsilon)$.

Like the limits of functions, it is not always the case that a function has a limit at a point. Some sequences have no limit. For example, the sequence $\{(-1)^n\}_{n\in\mathbb{N}}$ has no limit. However, some sequences, if they satisfy some conditions, must have limit for sure.

Definition 1.2. We call a sequence $\{a_n\}$ is bounded from above if and only if there exists $M \in \mathbb{R}$ such that $a_n \leq M$; We call a sequence is bounded from below if and only if there exists $M \in \mathbb{R}$ such that $a_n \geq M$. If a sequence is bounded from both sides, we simply call that sequence bounded.

Definition 1.3. A sequence $\{a_n\}$ is non-decreasing if and only if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$; A sequence $\{a_n\}$ is non-increasing if and only if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$.

Proposition 1.4. If a sequence $\{a_n\}$ is monotonic increasing and bounded from above, then

$$\lim_{n\to\infty} a_n \tag{1.5}$$

exists.

Proof. (Omit)
The proof involves the property of completeness of $\mathbb{R}$, which will be explained in an advanced (math) course. So we omit the proof □

An equivalent way to define a sequence **convergent**, i.e. the limit exists, is the concept of Cauchy sequences.

Definition 1.5. We call a sequence $\{a_n\}$ Cauchy if and only if there exists $N^0 : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$|a_n - a_m| < \varepsilon \tag{1.6}$$

when $m, n > N^0(\varepsilon)$.

Proposition 1.6. A sequence is convergent if and only if it is a Cauchy sequence.

Proof. Suppose $\{a_n\}$ convergent. Then there exists $L \in \mathbb{R}$ such that

$$\lim_{n\to\infty} a_n = L. \tag{1.7}$$

By Definition, we have $N : \mathbb{R}^+ \to \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \tag{1.8}$$

when $n > N(\varepsilon)$. So, by taking $N^0(\varepsilon) := N(\frac{\varepsilon}{2})$, we will have

$$|a_m - a_n| \leq |a_m - L| + |L - a_n| < \varepsilon \tag{1.9}$$

when $m, n > N^0(\varepsilon)$.

Conversely, suppose $\{a_n\}$ is a Cauchy sequence, then we have

$$\cup_{n=k}^{\infty}(-\infty, a_n) = (-\infty, b_k) \text{ for some } b_k. \tag{1.10}$$

$\{b_k\}$ is a decreasing function. Meanwhile, since $\{a_n\}$ is Cauchy, we have

$$a_n > a_{N^0(1)+1} - 1 \tag{1.11}$$

for all $n > N^0(1)$. So $b_n > a_{N^0(1)+1} - 1$, which means this sequence has a lower bound. Therefore, b_k converges to some number L. We claim that $\{a_n\}$ also converges to this L.

To prove this claim, we notice that, since $\lim_{k\to\infty} b_k = L$, we have

$$|b_k - L| < \varepsilon \tag{1.12}$$

when $k > N(\varepsilon)$ with some function N defined. By the definition of b_k in (1.10), for any k, there is also an $a_{n(k)}$ such that $|a_{n(k)} - b_k| < \frac{\varepsilon}{3}$. Notice that $n(k) > k$ by (1.10). Since $\{a_n\}$ is Cauchy, we can choose $N^{new}(\varepsilon) = \max\{N^0(\frac{\varepsilon}{3}), N(\frac{\varepsilon}{3})\}$, so

$$\begin{aligned} |a_k - L| &\leq |a_k - a_{n(k)}| + |a_{n(k)} - b_k| + |b_k - L| \\ &< 3 \cdot \frac{\varepsilon}{3} = \varepsilon \end{aligned} \tag{1.13}$$

for all $k > N^{new}(\varepsilon)$. This proves the proposition. $\square$

2. Series and their limits

Suppose we have a sequence $\{a_n\}_{n\in\mathbb{N}}$. We can construct a new sequence form this one by

$$s_m := \sum_{i=1}^{m} a_i. \tag{2.1}$$

We usually call s_m the **partial sum** of $\{a_n\}$ and $\lim_{m\to\infty} s_m$ be the **series** of $\{a_n\}$. When $\{s_m\}$ is convergent, we call the series of $\{a_n\}$ convergent. Denote by

$$\sum_{i=1}^{\infty} a_i \tag{2.2}$$

or

$$a_1 + a_2 + a_3 + \cdots \tag{2.3}$$

the limit of the partial sums, $\lim_{m\to\infty} s_m$.

According to Proposition 1.6, we have a equivalent statement as the following.

$$\{s_m\} \text{ convergent if and only if } \{s_m\} \text{ Cauchy.} \tag{2.4}$$

Namely, there exists a function $N^0 : \mathbb{R}^+ \to \mathbb{N}$ with the following significance. For any $\varepsilon > 0$, we have

$$|s_k - s_l| < \varepsilon \tag{2.5}$$

when $k, l > N^0(\varepsilon)$. We can assume $k > l$, so (2.5) tells us that

$$|\sum_{i=l+1}^{k} a_i| < \varepsilon \tag{2.6}$$

when $k > l > N^0(\varepsilon)$. Once (2.6) holds for some function N^0, we have the series $\{s_m\}$ convergent.

Example 2.1. Show that $\sum_{i=1}^{\infty} \frac{1}{2^i}$ exists.

Notice that

$$\sum_{i=l+1}^{k} \frac{1}{2^i} = \frac{2^{-(l+1)} - 2^{-(k+1)}}{1 - \frac{1}{2}} = 2^{-l} - 2^{-k} < 2^{-l}. \tag{2.7}$$

So we can check that, by taking $N^0(\varepsilon) := \log_2(\frac{1}{\varepsilon})$, (2.6) holds. So $\sum_{i=1}^{\infty} \frac{1}{2^i}$ exists.

One can prove that, by using Proposition 1.6, a series is convergent only if $a_n \to 0$.

Proposition 2.2. Suppose that $\sum_{i=1}^{\infty} a_i$ exists. Then $\lim_{n\to\infty} a_i = 0$.

Proof. Exercise. $\square$