Calculus 2 Final Exam version 2

Version 1: For who has the last digit of your student ID being odd.

Lecturer: Ryosuke Takahashi; Date: 2021/6/23

Let ab be the final two digits of your student's ID. Define A and B as the following:

$$A = \begin{cases} \frac{1}{2} & \text{when } a \text{ is even} \\ 2 & \text{when } a \text{ is odd} \end{cases}.$$

and

$$B = \begin{cases} -2 & \text{when } b \in \{0, 3, 6, 9\} \\ 1 & \text{when } b \in \{1, 4, 7\} \\ 3 & \text{when } b \in \{2, 5, 8\} \end{cases}$$

1. (12 points) Compute the following double integral:

$$\iint_\Omega x^2 dxdy$$

with $\Omega := \{(x, y) \in \mathbb{R}^2 | \frac{x^2}{A^2} + \frac{y^2}{B^2} \leq 1\}$.
Hint. Change the variables to make Ω a disc of radius 1.

2. (12 points) Let $F = (9 + 6xy, 3x^2 - 9y^3)$ be a vector field defined on $\mathbb{R}^2$. Show that F is conservative and find the function f such that $F = \nabla f$.

3. (12 points) Let $R = \{(x, y) \in \mathbb{R}^2 | 1 \leq x^2 + y^2 \leq (B - 2)^2 + 1, x \leq 0\}$. Find

$$\iint_R (Ax + 2By^2) dxdy.$$

4. (16 points) Let $S = \{(x, y, z) | x^2 + y^2 + z^2 = 4, |z| \leq 1\}$ be a surface.
a. Find the unit normal vector $\mathbf{n}$ of this surface.
b. Find the area of this surface.

5. (12 points) Let $\Omega = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq |B|\}$. Find

$$\iiint_\Omega (3x^2 + y^2)\sqrt{x^2 + y^2 + z^2} dxdydz.$$

6. (12 points) Let C be the boundary of the region enclosed by $y = x^2$, $x = y^2$. Find

$$\int_C (2y + \cos x^2)dx + (x + e^{\sqrt{y}})dy.$$

7. (12 points) Let $\Omega = \{(x, y, z) \in \mathbb{R}^3 | z^2 \geq x^2 + y^2, -2 \leq z \leq 0\}$ and $V = (3x^3, 3y^3, 3z^3)$. Find

$$\iint_{\partial\Omega} V \cdot \vec{n} dS.$$

where $\vec{n}$ is the outer unit normal of $\partial\Omega$.

8. (12 points) Let $S = \{(x, y, z) \in \mathbb{R}^3 | x^2 + \frac{y^2}{B^2} + \frac{z^2}{A^2} = 1\}$. Explain why

$$\iint_S \text{curl}(V) \cdot \vec{n} dS = 0$$

for any vector field V defined on $\mathbb{R}^3$.