Unit Operation I Exam II **2010**

1. From the Navier-Stokes equation

$$\rho \frac{D\vec{\upsilon}}{Dt} = \rho\vec{g} + \nabla \bullet \vec{\tau}$$

 Where $\vec{\upsilon}$ is the velocity and $\vec{\tau}$ is the stress.

 (1) What is the physical law underlying the above equation? (6%)

 (2) What are the units of $\nabla \bullet \vec{\tau} = 0$ and $\frac{D\vec{\upsilon}}{Dt}$? (6%)

2. What is the physical meaning of $\nabla \times \vec{\upsilon} = 0$? (4%) What would the Navier-Stokes equation for an incompressible and Newtonian fluid become if $\nabla \times \vec{\upsilon} = 0$? (4%)

3. As you might know $\rho\vec{\upsilon}\vec{\upsilon}$ is the momentum flux per unit volume (it is a tensor.), please indicate the magnitude, direction, and orientation with respect to a plane for the two components $\rho\upsilon_y\upsilon_x\vec{e}_y\vec{e}_x$ and $\tau_{xy}\vec{e}_x\vec{e}_y$. (8%)
 Where $\vec{\upsilon}$ is the velocity of a fluid and

$$\rho\vec{\upsilon}\vec{\upsilon} = \left[\rho\vec{\upsilon}\upsilon_x \quad \rho\vec{\upsilon}\upsilon_y \quad \rho\vec{\upsilon}\upsilon_z\right] = \begin{bmatrix} \rho\upsilon_x\upsilon_x & \rho\upsilon_x\upsilon_y & \rho\upsilon_x\upsilon_z \\ \rho\upsilon_y\upsilon_x & \rho\upsilon_y\upsilon_y & \rho\upsilon_y\upsilon_z \\ \rho\upsilon_z\upsilon_x & \rho\upsilon_z\upsilon_y & \rho\upsilon_z\upsilon_z \end{bmatrix}$$

4. Which of the following assumptions were made in obtaining Hagen-Poiseuille equation (a) The flow is laminar, (b) The flow is fully developed, (c) The flow can be either steady or unsteady, (d) There is no slip at the wall, (e) The fluid is Newtonian, (f) The flow is incompressible, (g) The density doesn't have to be constant, (h) The fluid behaves as a continuum. (8%)

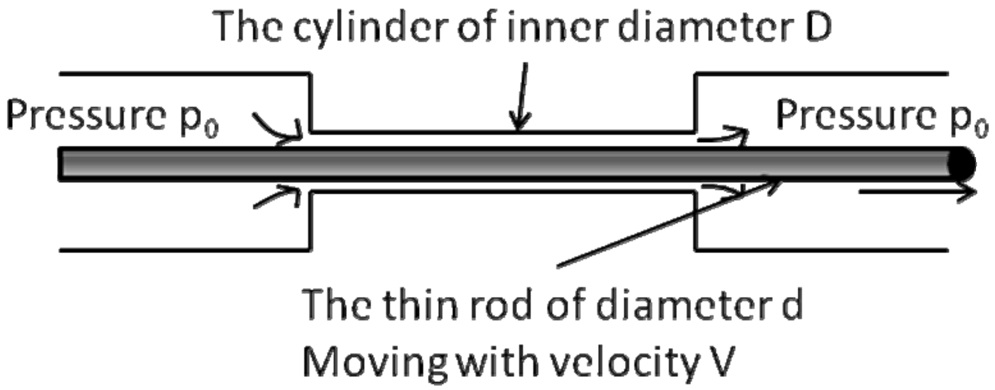
5. A realistic molecular model utilizing Lennard-Jones potential energy function yields a viscosity-temperature relationship consistant with the experimental data. The expression for the viscosity of a pure gas is

$$\mu = 2.6693 \times 10^{-6} \frac{\sqrt{MT}}{\sigma^2 \Omega_\mu},$$

 Please calculate the viscosity of carbon dioxide at 600 K. Comparing with the experimental viscosity shown in the table, please calculate the percentage of deviation. (8%)

6. An auto lift consists of 25.02-cm-diameter ram that slides in a 25.04-cm-diameter cylinder. The annular region is filled with oil having a kinematic viscosity of 0.00027 m^2/s and a specific gravity of 0.85. If the rate of travel of the ram is 0.12 m/s, estimate the frictional resistance when 3.0 m of the ram is engaged in the cylinder. (10%)

7. Consider the system (a wire-coating die) as shown below, in which the cylindrical rod is being moved with a velocity V. The rod is at the center of the cylinder. The fluid filling the space between the rod and the inner cylinder wall has density ρ and viscosity μ. Assume the flow is laminar and steady. No-slip condition is applied on the surface of the cylinder.
   (1) Derive the fluid velocity distribution. (8%)
   (2) Derive the shear force (drag) per unit length of rod. (8%)



where the shear stress components

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{\upsilon_\theta}{r}\right) + \frac{1}{r}\frac{\partial \upsilon_r}{\partial \theta}\right]$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left[\frac{\partial \upsilon_\theta}{\partial z} + \frac{1}{r}\frac{\partial \upsilon_z}{\partial \theta}\right]$$

$$\tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial \upsilon_z}{\partial r} + \frac{\partial \upsilon_r}{\partial z}\right]$$

8. An incompressible Newtonian fluid approaches a stationary cylinder with a uniform, steady velocity in the positive direction. Assume the flow is laminar and steady. No-slip condition is applied on the surface of the cylinder.
   (1) Please derive the governing equation and the boundary conditions. (8%) We know the velocity (υ_r and υ_θ) and pressure are the function of r and θ . You should know that υ_z is equal to zero. (You don't need to solve it!)
   Hint: the governing equation can be derived from the equation of continuity and Navier-Stokes equation.
   (2) The solution was derived and shown below. The pressure and velocity in the immediate vicinity of the cylinder are

$$p(r,\theta) = p_\infty - C\mu\frac{\upsilon_\infty \cos\theta}{r} - \rho g r \sin\theta,$$

$$\upsilon_r(r,\theta) = C\upsilon_\infty\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4} + \frac{1}{4}\left(\frac{R}{r}\right)^2\right]\cos\theta,$$

$$\upsilon_\theta(r,\theta) = -C\upsilon_\infty\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) + \frac{1}{4} - \frac{1}{4}\left(\frac{R}{r}\right)^2\right]\sin\theta,$$

Please derive the shear stress components, $\tau_{r\theta}$, $\tau_{\theta r}$, $\tau_{\theta z}$, $\tau_{z\theta}$, τ_{zr}, and τ_{rz}. Also, calculate the shear stress on the cylinder at θ=0 ° and 45°, respectively. (6%)

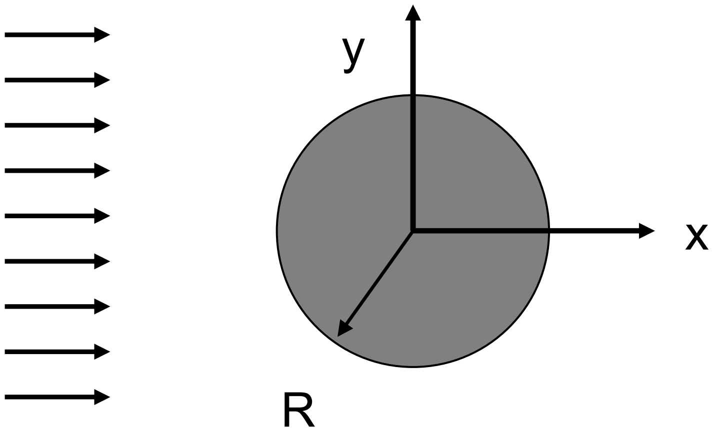
Where the shear stress components

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{\upsilon_\theta}{r}\right) + \frac{1}{r}\frac{\partial \upsilon_r}{\partial \theta}\right]$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left[\frac{\partial \upsilon_\theta}{\partial z} + \frac{1}{r}\frac{\partial \upsilon_z}{\partial \theta}\right]$$

$$\tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial \upsilon_z}{\partial r} + \frac{\partial \upsilon_r}{\partial z}\right]$$

Fluid approaches from x= -∞ with uniform velocity υ_∞



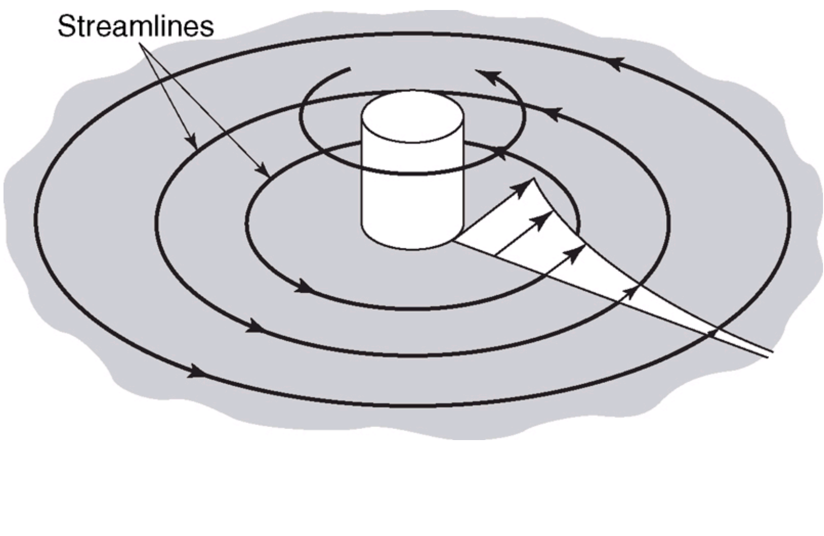
9. A rotating shaft, as shown in the figure, causes the incompressible fluid to move in circular streamlines with a velocity profile v_r=v_z=0, and $\upsilon_\theta = \omega R^2/r$, where R and ω are the radius and angular velocity of the shaft.

(1) Is it a rotational flow? Please give your reasons. (10%)

(2) Find the pressure distribution of the flow field. (6%)

Hint: The vorticity

$$\nabla\times\vec{\upsilon} = \left(\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\vec{e}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\vec{e}_z\right),$$



Unit Operation I Exam II 2011

Equations

Momentum balance

$$\sum \vec{F} = \oiint \rho \vec{\upsilon}(\vec{\upsilon} \bullet \vec{n}) dA + \frac{\partial}{\partial t} \iiint \rho \vec{\upsilon} dV,$$

Equation of continuity

$$\nabla \bullet (\rho \vec{\upsilon}) + \frac{\partial \rho}{\partial t} = 0,$$

Navier-Stokes equation

$$\rho \frac{D\vec{\upsilon}}{Dt} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{\upsilon},$$

Vorticity in cartesian coordinates

$$\nabla \times \vec{\upsilon} = \left(\left(\frac{\partial \upsilon_z}{\partial y} - \frac{\partial \upsilon_y}{\partial z} \right) \vec{e}_x + \left(\frac{\partial \upsilon_x}{\partial z} - \frac{\partial \upsilon_z}{\partial x} \right) \vec{e}_y + \left(\frac{\partial \upsilon_y}{\partial x} - \frac{\partial \upsilon_x}{\partial y} \right) \vec{e}_z \right),$$

The shear stress components in spherical coordinates

$$\tau_{r\theta} = \tau_{\theta r} = \mu \left[r \frac{\partial}{\partial r} \left(\frac{\upsilon_\theta}{r} \right) + \frac{1}{r} \frac{\partial \upsilon_r}{\partial \theta} \right]$$

$$\tau_{\phi\theta} = \tau_{\theta\phi} = \mu \left[\frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \left(\frac{\upsilon_\phi}{\sin\theta} \right) + \frac{1}{r\sin\theta} \frac{\partial \upsilon_\theta}{\partial \phi} \right]$$

$$\tau_{\phi r} = \tau_{r\phi} = \mu \left[\frac{1}{r\sin\theta} \frac{\partial \upsilon_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{\upsilon_\phi}{r} \right) \right]$$

The z-component of shear force acting on a sphere

$$F_z = \iint \left(\tau_{r\theta}\big|_{r=R} \sin\theta \right) R^2 \sin\theta d\theta d\phi$$

Unit Operation I Exam II **2011**

Problems

6. From the Navier-Stokes equation
   (a) What are the underlying assumptions for the above equation? (3%)
   (b) What is the contribution of viscous force shown in this equation? (3%)
   (c) What is the physical law underlying the above equation? (3%)

7. Please explain the physical meanings of
   (a) Reynolds number (3%)
   (b) $\nabla \times \vec{\upsilon} = 0$ (3%)
   (c) $\dfrac{D\vec{\upsilon}}{Dt}$ (3%)

8. What are the SI units of the following terms:
   (a) Viscosity (3%)
   (b) Momentum flux $\rho\vec{\upsilon}\vec{\upsilon}$ (3%)
   (c) Rate of shear strain (3%)
   (d) $\nabla \times \vec{\upsilon}$ (3%)

9. Which of the following assumptions were made in obtaining Bernoulli's equation?
   (a) The flow is incompressible, (b) The flow can be either steady or unsteady, (c) The flow can be either inviscid or irrotational, (d) It is developed from energy balance. (4%)

10. The velocity is shown below.
   (a) Please specify the condition so that the fluid satisfies the conservation of mass for an incompressible fluid. (5%)
   (b) Please specify the condition if the flow is irrotational. (5%)

$$\vec{\upsilon} = [xz + x]\vec{e}_x + [z]\vec{e}_y + [(-1/2)yz^2 + z]\vec{e}_z$$

11. An oil has a kinematic viscosity of 2.0×10^{-4} m^2/s and a density of 8.0×10^{-4} kg/m^3. If we want to have a falling film of thickness (L) of 2.5 mm on a declined wall (θ=45°). The wall is 5 m in length and 1m in width.

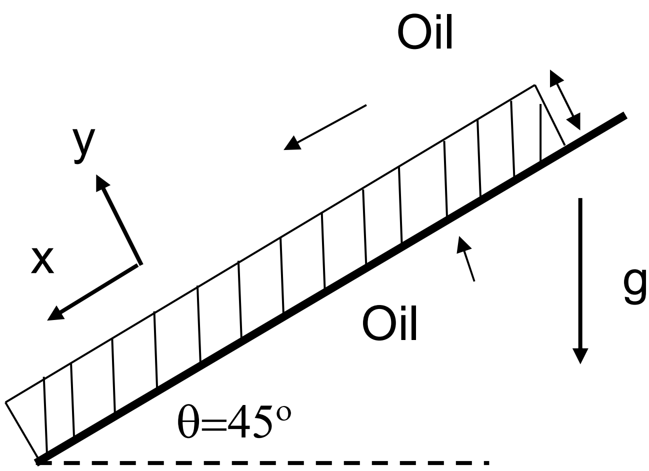
(a) What is the volumetric flow rate? (6%)

(b) Please check if the flow is laminar. (6%)

Hint: the Reynolds number for fluids flowing in tubes

$$\mathrm{Re} = \frac{4L\upsilon_{avg}\rho}{\mu}$$

If the Reynolds number is less than about 20, the flow is laminar.



7. An incompressible Newtonian fluid approaches a stationary sphere with a uniform, steady velocity in the positive direction. Assume the flow is laminar and steady. No-slip condition is applied on the surface of the sphere. (This approach enables one to further derive a force balance equation, which is the basic theory for **falling sphere viscometers**.)
The solution was derived and shown below. The pressure and velocity in the immediate vicinity of the sphere are

$$p(r,\theta) = p_\infty - \rho g z - \frac{3}{2}\frac{\mu\upsilon_\infty}{R}\left(\frac{R}{r}\right)^2 \cos\theta,$$

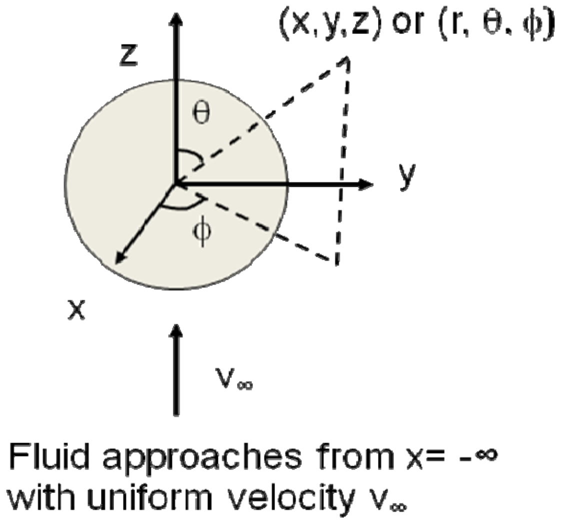
$$\upsilon_r(r,\theta) = \upsilon_\infty\left[1-\frac{3}{2}\left(\frac{R}{r}\right)+\frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta,$$

$$\upsilon_\theta(r,\theta) = -\upsilon_\infty\left[1-\frac{3}{4}\left(\frac{R}{r}\right)-\frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta,$$

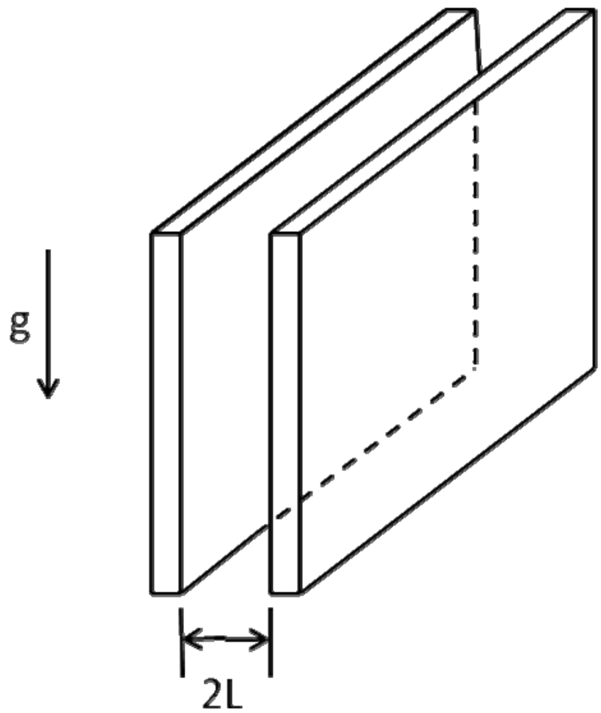
(a) Please derive the shear stress components. (5%)

(b) Calculate the shear stress acting on the sphere at θ=0 ° and 45°, respectively. (5%)

(c) The z-component of shear force exerting on the sphere. Please specify the direction. (5%)



8. Consider a vertical slit formed by two parallel walls as shown below, in which the distance is 2L. The fluid filling the space between the walls has density ρ and viscosity μ. Assume the flow is laminar and steady. No-slip condition is applied on the surface of the walls. Neglect the entrance and exit effects.
(a) Derive the fluid velocity distribution in the space between the walls. (7 %)
(b) Determine the ratio of average to maximum velocity in the slit. (5 %)



9. The flow of an incompressible, inviscid fluid in irrotational flow around a sphere with an uniform velocity. We know that the velocity potential (ϕ) must satisfy Laplace's equation.

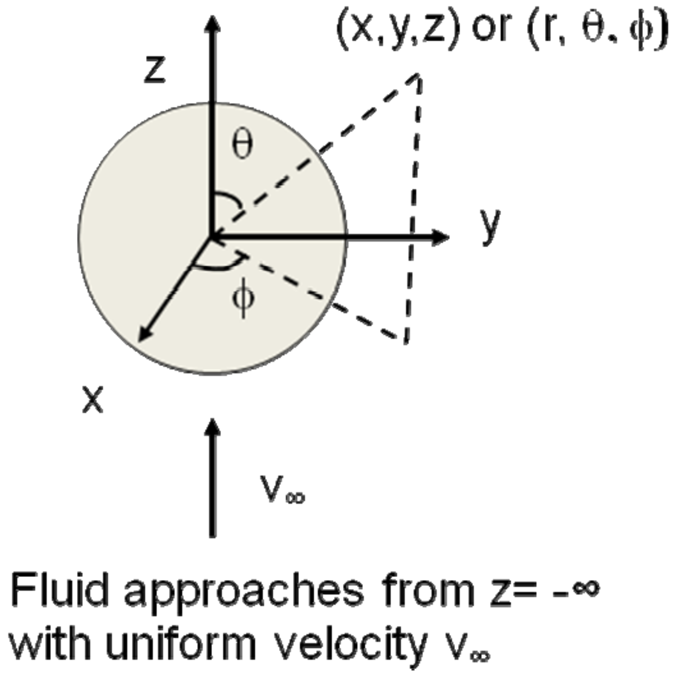
$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0$$

(a) We guess the solution to be $\phi(r,\theta) = \left[Ar + \frac{B}{r^2}\right]\cos\theta$.
Please prove the solution satisfies the Laplace's equation. (6%)

(b) Please write down the boundary conditions. Then solve the problem and derive the velocity. (6%)

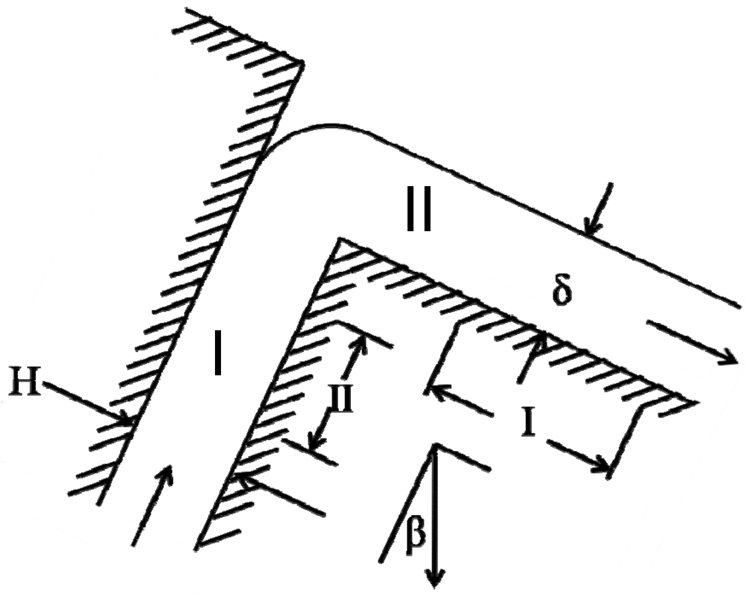
where $\upsilon_r = -\frac{\partial\phi}{\partial r}, \upsilon_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}$

(c) Show that the pressure distribution on the surface of the sphere. (5%)



Additional problems (optional):

10. A Newtonian liquid with viscosity μ and density ρ is flowing out of a slit and onto a slide as shown in the figure. The slit gap and the film thickness on the slide are H and δ, respectively. The angle between the slit wall and the direction of gravity is β. Assume that the flow in section I and II is laminar, steady, and fully developed. What is the minimum entrance pressure in section I in order for $H = \delta/2$?
 Hint: the mass rate of flow in section I is equal to the mass rate of flow in section II.



Unit Operation I Exam II **2012**

Equations

Momentum balance

$$\sum \vec{F} = \oiint \rho\vec{\upsilon}(\vec{\upsilon}\bullet\vec{n})dA + \frac{\partial}{\partial t}\oiiint \rho\vec{\upsilon}dV,$$

Equation of continuity

$$\nabla\bullet(\rho\vec{\upsilon}) + \frac{\partial\rho}{\partial t} = 0,$$

Navier-Stokes equation

$$\rho\frac{D\vec{\upsilon}}{Dt} = \rho\vec{g} + \nabla\bullet\vec{\tau}$$

Euler's equation

$$\rho\frac{D\vec{\upsilon}}{Dt} = \rho\vec{g} - \nabla p$$

Stream function Ψ and velocity potential ϕ in cylindrical coordinates

$$\upsilon_r = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = \frac{\partial\phi}{\partial r},$$

$$\upsilon_\theta = -\frac{\partial\Psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta},$$

The shear stress components in cylindrical coordinates

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{\upsilon_\theta}{r}\right) + \frac{1}{r}\frac{\partial\upsilon_r}{\partial\theta}\right]$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left[\frac{\partial\upsilon_\theta}{\partial z} + \frac{1}{r}\frac{\partial\upsilon_z}{\partial\theta}\right]$$

$$\tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial\upsilon_z}{\partial r} + \frac{\partial\upsilon_r}{\partial z}\right]$$

The z-component of shear force acting on a cylinder

$$F_z = \iint\left(\tau_{r\theta}\big|_{r=R}\right)Rd\theta dz$$

The vorticity and divergence of velocity in cylindrical coordinates

$$\nabla\times\vec{\upsilon} = \left(\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\vec{e}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\vec{e}_z\right),$$

$$\nabla\bullet\vec{\upsilon} = \frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) + \frac{1}{r}\frac{\partial\upsilon_\theta}{\partial\theta} + \frac{\partial\upsilon_z}{\partial z},$$

Unit Operation I Exam II 2012

Problems

12. The Navier-Stokes equation can be written in mainly three forms as shown in the following

(1) $\rho\frac{D\vec{\upsilon}}{Dt} = \rho\vec{g} + \nabla \bullet \vec{\tau}$, **(2)** $\rho\frac{D\vec{\upsilon}}{Dt} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{\upsilon}$, **(3)** $\rho\frac{D\vec{\upsilon}}{Dt} = \rho\vec{g} - \nabla p$

(d) What are the underlying assumptions for the three equations? (9%)
(e) What is the physical law underlying the Navier-Stokes equation? (4%)

13. Please explain the physical meanings and SI units of the following terms.
(d) $\rho\vec{\upsilon}\vec{\upsilon}$ (6%)
(e) $\nabla \bullet \vec{\tau}$ (6%)
(f) $\frac{D\vec{\upsilon}}{Dt}$ (6%)

14. The flow of a fluid can be described by $\nabla^2\Psi = 0$ or $\nabla^2\phi = 0$. Ψ and ϕ are stream function and velocity potential, respectively. What type of fluid flow is it? (6%)

15. Glycine at 25 °C is flowing through a horizontal tube 30 cm long and 0.25 cm inside diameter (D). The density of glycine at 25 °C is 1.261 g/cm^3. For a pressure drop of 2.75× 10^6 dyne/cm^2, the volumetric flow rate is 100 cm^3/min.
(c) Please calculate the viscosity of glycine. (8%)
(d) Please check if the flow is laminar. (6%)
Hint: The Hagen-Poiseuille equation

$$-\frac{dP}{dx} = \frac{8\mu\upsilon_{avg}}{R^2}$$

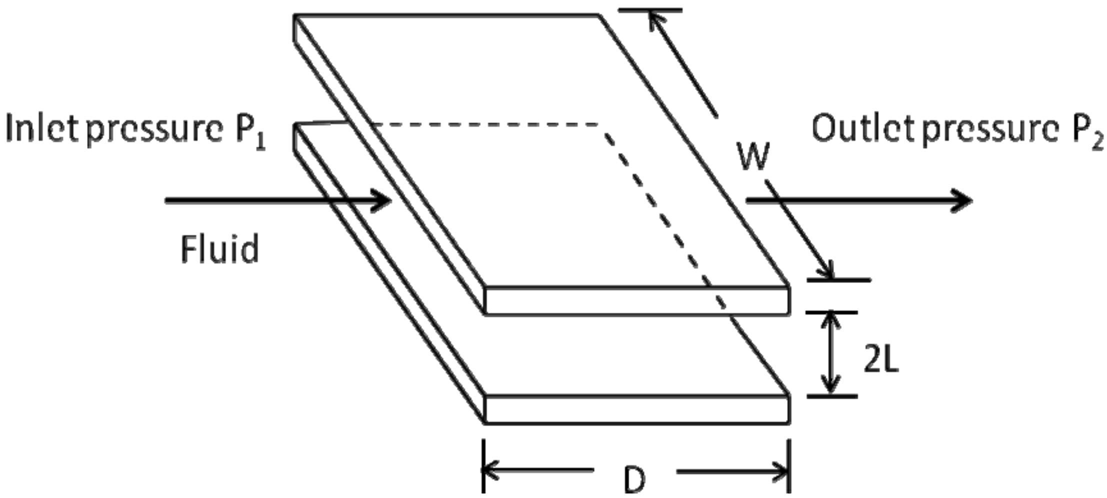
And the Reynolds number for fluids flowing in tubes

$$\mathrm{Re} = \frac{D\upsilon_{avg}\rho}{\mu}$$

If the Reynolds number is less than about 2100, the flow is laminar.

5. Consider a **film blowing process**, which is the polymer melt that is flowing through a die with pressure gradient (a horizontal slit formed by two parallel walls with the two sides being sealed) as shown below. The distance and width are 2L and W, respectively. The polymer melt, which is a Newtonian fluid, filling the space between the walls has density ρ and viscosity μ. Assume that the width is much larger than the space between the walls (that is, W>>L). The flow is laminar and steady. No-slip condition is applied on the surface of the walls. Neglect the entrance and exit effects.
(a) Derive the fluid velocity distribution in the space between the walls. (10 %)

(b) Determine the average velocity in the slit. (6 %)



6. A rotational viscometer, which is so-called **Couette-Hatschek viscometer**, consists essentially of two concentric cylinders, the inner of which rotates while the outer is held stationary. The instrument consists essentially of a rotating inner cylinder with R_1 in diameter and L in length surrounded by a concentric cylinder with R_2 in diameter and the same length. The flow is steady. No-slip condition is applied on the surface of the stationary cylinder.

(a) Derive the fluid velocity distribution in this kind of apparatus for the laminar flow of a Newtonian fluid. (10%)

(b) Derive the shear force acting on the outer cylinder. (5%)

7. A rotating shaft causes the incompressible fluid to move in a circular fashion with a velocity profile , $\vec{v} = (\omega R^2 / r)\vec{e}_\theta$, where R and ω are the radius and angular velocity of the shaft. Please note that **the fluid does not have to be inviscid**.

(1) The fluid flow can be described by the Euler's equation, why? Please give your reasons. (7%)

(2) The problem can be considered to be two dimensional. Please derive the stream function Ψ and velocity potential ϕ. (6%)

(3) Please draw the streamlines and constant velocity potential lines. (5%)

Unit Operation I Exam II **2013**

Equations

Momentum balance

$$\sum \vec{F} = \oiint \rho\vec{\upsilon}(\vec{\upsilon}\bullet\vec{n})dA + \frac{\partial}{\partial t}\oiiint \rho\vec{\upsilon}dV,$$

Equation of continuity

$$\nabla\bullet(\rho\vec{\upsilon}) + \frac{\partial\rho}{\partial t} = 0,$$

Navier-Stokes equation

$$\rho\frac{D\vec{\upsilon}}{Dt} = \rho\vec{g} + \nabla\bullet\vec{\tau} \qquad \rho\frac{D\vec{\upsilon}}{Dt} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{\upsilon}$$

Euler's equation

$$\rho\frac{D\vec{\upsilon}}{Dt} = \rho\vec{g} - \nabla p$$

Stream function Ψ and velocity potential ϕ in cylindrical coordinates

$$\upsilon_r = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = \frac{\partial\phi}{\partial r},$$

$$\upsilon_\theta = -\frac{\partial\Psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta},$$

The shear stress components in cylindrical coordinates

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{\upsilon_\theta}{r}\right) + \frac{1}{r}\frac{\partial\upsilon_r}{\partial\theta}\right]$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left[\frac{\partial\upsilon_\theta}{\partial z} + \frac{1}{r}\frac{\partial\upsilon_z}{\partial\theta}\right]$$

$$\tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial\upsilon_z}{\partial r} + \frac{\partial\upsilon_r}{\partial z}\right]$$

The z-component of shear force acting on a cylinder

$$F_z = \iint\left(\tau_{r\theta}\big|_{r=R}\right)Rd\theta dz$$

The vorticity and divergence of velocity in cylindrical coordinates

$$\nabla\times\vec{\upsilon} = \left(\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\vec{e}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\vec{e}_z\right),$$

$$\nabla\bullet\vec{\upsilon} = \frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) + \frac{1}{r}\frac{\partial\upsilon_\theta}{\partial\theta} + \frac{\partial\upsilon_z}{\partial z},$$

Unit Operation I Exam II 2013

Problems

16. Which of the following are the correct description(s) for the Continuity and Navier-Stokes equations (12%).
    (f) The physical law underlying the Navier-Stokes equation is Newton's first law of motion.
    (g) If a fluid is steady and compressible, the Continuity equation becomes $\nabla \bullet \rho\vec{\upsilon} = 0$.
    (h) If a Newtonian fluid is steady and incompressible, the Navier-Stokes equation becomes $0 = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{\upsilon}$.
    (i) If a Newtonian fluid is compressible, inviscid, and unsteady, the Navier-Stokes equation becomes $\rho\dfrac{D\vec{\upsilon}}{Dt} = \rho\vec{g} - \nabla p$.

17. Please explain the physical meanings and SI units of the following terms.
    (g) $\nabla \bullet \rho\vec{\upsilon}\vec{\upsilon}$ (4%)
    (h) $\vec{\upsilon} \bullet \nabla\vec{\upsilon}$ (4%)
    (i) $\mu\nabla^2\vec{\upsilon}$ (4%)

18. Based on two different models, the equations, Eq 1 and 2, for estimating viscosity were derived. It was found that the viscosity derived from Eq 2 was more consistant with the experimental data than that from Eq1. Please explain. (6%)

$$\mu = \frac{2}{3\pi^{3/2}}\frac{\sqrt{m\kappa T}}{d^2}, \qquad \text{(Eq 1)}$$

$$\mu = 2.6693\times10^{-6}\frac{\sqrt{MT}}{\sigma^2\Omega_\mu}, \qquad \text{(Eq 2)}$$

19. For a Newtonian and incompressible fluid flow, as long as there are shear forces acting on the fluid, the flow is rotational. Please explain why the statement is correct. (6%)

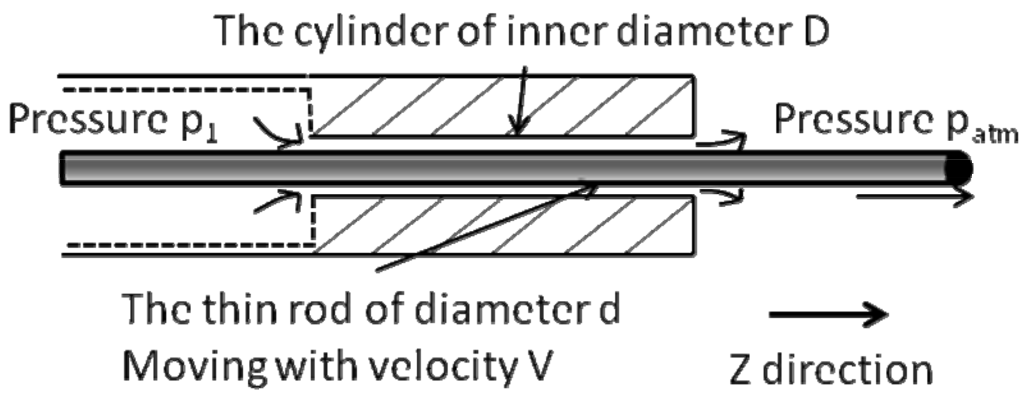
20. Sketch the streamlines of the flow due to a point source at (a, 0) plus an equivalent sink at (-a, 0). You don't have to solve it. (6%)

21. An incompressible Newtonian fluid flowing through a horizontal pipe with radius of R and length of L. The pressure gradient between the two ends is $-\left(\frac{dP}{dz}\right)$. The fluid filling the pipe has density ρ and viscosity μ. The flow is laminar and fully developed. Neglect the entrance and exit effects. On the tube wall, $r = R$. The derived velocity was found to be

$$\upsilon_z = -\left(\frac{dP}{dz}\right)\frac{R^2}{4\mu}\left[\frac{3}{2}-\left(\frac{r}{R}\right)^2\right]$$

(a) Is no-slip condition applied in this case? Explain. (4%)
(b) Please draw a representative differential control volume and determine the net rate of momentum due to convection for it. (6%)
(c) Derive the average velocity, υ_{avg}, in terms of the maximum velocity. (3%)
(d) Determine the shear stresses on the wall ($r = R$). (3%)

7. Consider a wire-coating process as shown below, in which the cylindrical rod is being moved with a velocity V. The rod is at the center of the cylindrical die. The fluid filling the space between the rod and the inner cylinder wall has density ρ and viscosity μ. Assume the flow is laminar and steady. No-slip condition is applied on the surface of the cylinder and rod. The pressure in the die is p_1, which is higher than the outside pressure, p_{atm}. Ignore the effect of gravity.
(a) Derive the differential momentum balance in the z direction. (6 %)
(b) Derive the fluid velocity profile in the space between the rod and the inner cylinder wall. (6 %)
(c) Derive the shear stress acting on the surface of the rod. (5 %)



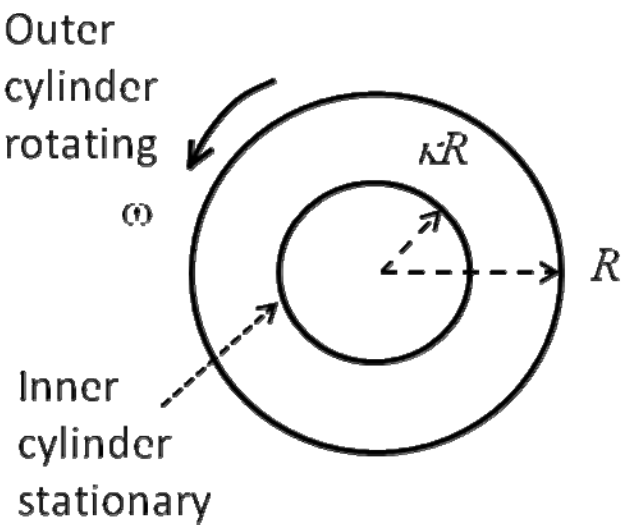
8. A continuous belt passes upward through a chemical bath at velocity V and picks up a film of liquid of thickness h, density ρ and viscosity μ. Gravity tends to make the liquid drain down, but the movement of the belt keeps the fluid from running off completely. Assume that the flow is a well-developed laminar flow with zero pressure gradient, and that the atmosphere produces no shear at the outer surface of the film.
(a) Derive the fluid velocity profile. (8%)
(b) Derive the film thickness h in terms of average velocity υ_{avg} , density ρ and viscosity μ. (5%)

9. A rotational viscometer consists essentially of two concentric cylinders, the outer of which rotates while the inner is held stationary. The instrument consists essentially of a rotating outer cylinder with R in radius and L in length surrounded on a concentric cylinder with κR in radius and the same length. The angular velocity of the outer cylinder is ω. The flow with fluid density ρ and viscosity μ is steady. No-slip condition is applied on the surface of the stationary cylinder. The derived velocity was found to be

$$\upsilon_\theta = \omega R\left[\frac{(\kappa R/r - r/\kappa R)}{\kappa - 1/\kappa}\right]$$

(a) Derive the torque T required to turn the outer cylinder. (6%)
(b) Derive the radial pressure distribution at fixed z. The pressure at the surface of the outer cylinder is p_0. (6%)

Top view



Unit Operation I Exam II 2014

Equations

Equation of continuity

$$\nabla \bullet (\rho \vec{\upsilon}) + \frac{\partial \rho}{\partial t} = 0,$$

In cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r \upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \upsilon_\theta)\frac{\partial}{\partial z}(\rho \upsilon_z) = 0$$

Navier-Stokes equation

$$\rho \frac{D\vec{\upsilon}}{Dt} = \rho \vec{g} + \nabla \bullet \vec{\tau} \qquad \rho \frac{\partial \vec{\upsilon}}{\partial t} + \rho \vec{\upsilon} \cdot \nabla \vec{\upsilon} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{\upsilon}$$

In cylindrical coordinates

r direction

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \rho g_r + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

The shear stress components in cylindrical coordinates

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{\upsilon_\theta}{r}\right) + \frac{1}{r}\frac{\partial \upsilon_r}{\partial \theta}\right]$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left[\frac{\partial \upsilon_\theta}{\partial z} + \frac{1}{r}\frac{\partial \upsilon_z}{\partial \theta}\right]$$

$$\tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial \upsilon_z}{\partial r} + \frac{\partial \upsilon_r}{\partial z}\right]$$

Unit Operation I Exam II 2014

Part I: Closed book

22. Which of the following are the correct description(s). (12%)
   (j) The physical law underlying the Navier-Stokes equation is Newton's second law of motion. .
   (k) The flow must be rotational if viscous forces act on a fluid.
   (l) Bernoulli's equation and Navier-Stokes equation are based on conservation of energy and momentum, respectively.
   (m) If a Newtonian fluid is compressible, irrotational, and unsteady, the Navier-Stokes equation becomes $\rho \frac{D\vec{\upsilon}}{Dt} = \rho\vec{g} - \nabla p$.

23. Which of the following assumptions were made in obtaining Hagen-Poiseuille equation. (12%)
   (a) The flow is laminar, (b) The fluid is Newtonian, (c) The flow can be either steady or unsteady, (d) No slip condition does not apply, (e), The flow is incompressible, and (f) The density doesn't have to be constant.

24. Please explain the physical meanings and SI units of the following terms.
   (j) $\mu\nabla^2\vec{\upsilon}$ (4%)
   (k) $\nabla\cdot\rho\vec{\upsilon}$ (4%)
   (l) $\nabla\times\vec{\upsilon}=0$ (4%)

25. Please explain the difference between $\frac{\partial\vec{\upsilon}}{\partial t}$ and $\frac{D\vec{\upsilon}}{Dt}$ in the terms of their physical meanings. (6%)

26. Please derive the differential equation for a radial flow in which $\upsilon_z = \upsilon_\theta = 0$ and $\upsilon_r = f(r)$ using the continuity and Navier-Stokes equations. Show that the solution does not involve viscosity. Assume that the fluid is Newtonian and incompressible. (8%)

27. An oil at 25 °C is flowing through a horizontal tube 40 cm long and 0.3 cm inside diameter (D). The density of glycine at 25 °C is 15.8 g/cm^3. For a pressure drop of 4.0× 10^6 dyne/cm^2, the volumetric flow rate is 150 cm^3/min.
   (e) Please calculate the viscosity of glycine. (6%)
   (f) Please check if the flow is laminar. (6%)
   Hint: The Hagen-Poiseuille equation

$$-\frac{dP}{dx} = \frac{8\mu\upsilon_{avg}}{R^2}$$

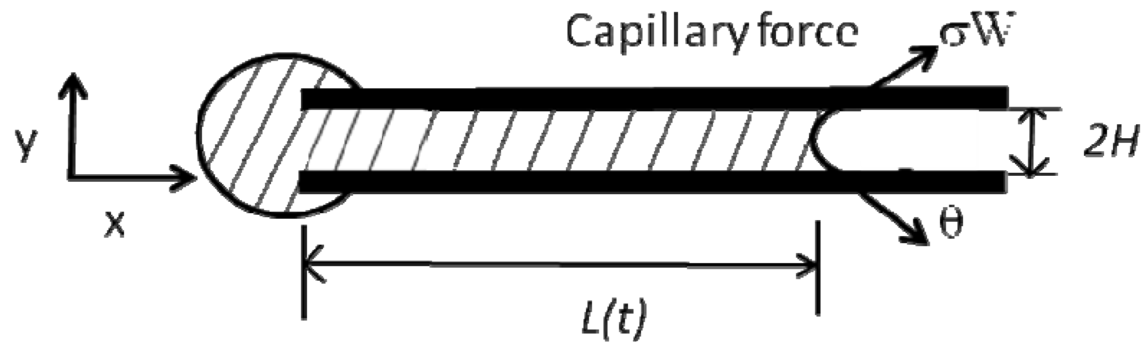
   And the Reynolds number for fluids flowing in tubes

$$\mathrm{Re} = \frac{D\upsilon_{avg}\rho}{\mu}$$

If the Reynolds number is less than about 2100, the flow is laminar.

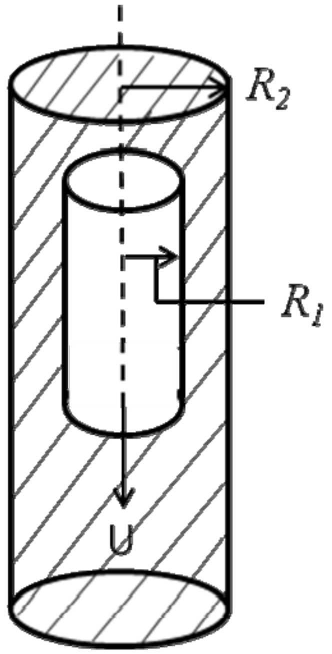
7. Consider the steady-state flow of an incompressible Newtonian liquid in a vertical tube. No-slip condition is applied on the surface of the tube.
   (a) Please derive the velocity distribution and average velocity. (8%)
   (b) Two fluids respectively flowed through this identical tube and their average velocities (υ_{x1} and υ_{x2}) were measured. The densities of the two fluids are ρ_1 and ρ_2, respectively. The viscosities of the two fluids are μ_1 and μ_2, respectively. Please derive an equation for the ratio of μ_1/μ_2. (6%)

8. In a typical flow device used in biomedical analysis as shown below, a drop of the sample to be tested is brought into contact with an opening in one end of the test cell. Capillary action is sufficient to draw the sample into the test chamber because the height of the opening is small. The test chamber is essentially a region bounded by a pair of parallel plates. W is the width of the planes. μ is the viscosity of the sample. Assume that the fluid is Newtonian and incompressible, and the flow is laminar and steady. No-slip condition is applied on the surface of the plates. Ignore the effect of gravity.
   (a) Derive the differential momentum balance in the x direction. (8 %)
   (b) Derive the fluid velocity profile in the space between the plates. (5 %)
   (c) Derive an equation analogue to the Hagen-Poiseuille equation for it. (5 %)



Part II: Open book 2014

9. A long solid cylinder of length L and radius R_1 falls coaxially under the attraction of gravity through a liquid held within a concentric cylindrical vessel of radius R_2, closed at the bottom. This setup can be designed as a viscometer (**falling rod viscometer**). Assume that the fluid is Newtonian and the flow is laminar and steady. No-slip condition is applied on the surface of the cylindrical vessel. The flow is fully developed in the region between R_1 and R_2, which means that the end effect can be ignored.
   (a) Derive the fluid velocity profile in the region between R_1 and R_2. (10%)
   (b) Derive the shear force acting on the solid cylinder. (6%)
   (c) Derive the terminal velocity for the solid circular cylinder. (6%)



Unit Operation I Exam II 2015

Equations

Equation of continuity

$$\nabla \bullet (\rho \vec{\upsilon}) + \frac{\partial \rho}{\partial t} = 0,$$

In cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r \upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \upsilon_\theta)\frac{\partial}{\partial z}(\rho \upsilon_z) = 0$$

Navier-Stokes equation

$$\rho \frac{D\vec{\upsilon}}{Dt} = \rho \vec{g} + \nabla \bullet \vec{\tau} \qquad \rho \frac{\partial \vec{\upsilon}}{\partial t} + \rho \vec{\upsilon} \cdot \nabla \vec{\upsilon} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{\upsilon}$$

In cylindrical coordinates

r direction

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z}\right)$$
$$= -\frac{\partial P}{\partial r} + \rho g_r + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta$$
$$+ \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right)$$
$$= -\frac{\partial P}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

The shear stress components in cylindrical coordinates

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{\upsilon_\theta}{r}\right) + \frac{1}{r}\frac{\partial \upsilon_r}{\partial \theta}\right]$$
$$\tau_{z\theta} = \tau_{\theta z} = \mu\left[\frac{\partial \upsilon_\theta}{\partial z} + \frac{1}{r}\frac{\partial \upsilon_z}{\partial \theta}\right]$$
$$\tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial \upsilon_z}{\partial r} + \frac{\partial \upsilon_r}{\partial z}\right]$$

Unit Operation I Exam II 2015

Part I: Closed book

28. Which of the following are the correct description(s). (15%)
    (n) If the flow follows $\nabla \bullet \vec{\upsilon} = 0$, it means the fluid is incompressible.
    (o) If the flow follows $\nabla \times \vec{\upsilon} = 0$, it means there is no viscous force acting on the fluid.
    (p) Continuity and Navier-Stokes equation are based on conservation of energy and momentum, respectively.
    (q) If a Newtonian fluid follows $\rho \frac{D\vec{\upsilon}}{Dt} = \rho\vec{g} - \nabla p$, the fluid is either inviscid or irrotational.
    (r) The physical meaning of $\frac{D\vec{\upsilon}}{Dt}$ is local acceleration.

29. The Navier-Stokes equation can be expressed in the following

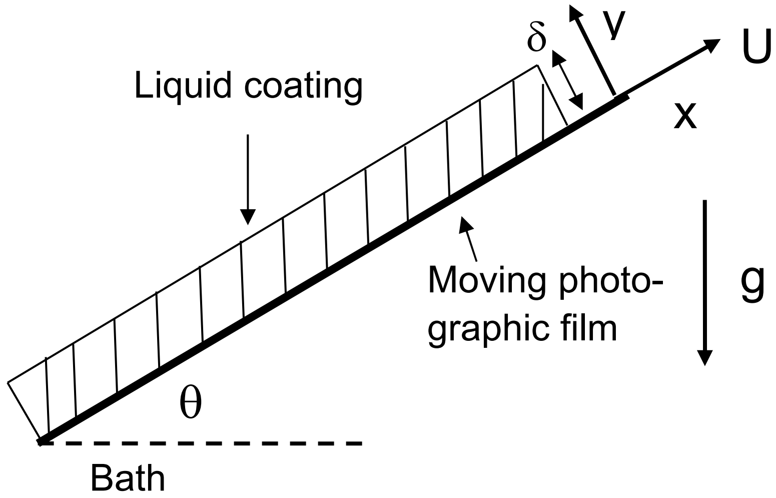
$$\rho \frac{\partial \vec{\upsilon}}{\partial t} + \rho\vec{\upsilon} \cdot \nabla \vec{\upsilon} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{\upsilon}$$

    (m) Please identify the assumption(s). (5%)
    (n) Please explain the physical meanings and SI units of $\rho\vec{\upsilon} \cdot \nabla \vec{\upsilon}$. (5%)
    (o) Please explain the physical meanings and SI units of $\mu\nabla^2\vec{\upsilon}$. (5%)
    (p) What is the physical law underlying the Navier-Stokes equation? (5%)

30. Please explain why the shear stress has to be symmetrical, that is $\tau_{ij} = \tau_{ji}$. (5%)

31. A Newtonian oil with a density of 0.9 kg/m^3 and viscosity of 0.6 kg/m s undergoes steady shear between two horizontal parallel plates. The lower plate is fixed and the upper plate moves with a constant velocity of 0.7 m/s. The distance between the plates is constant at 0.06 cm and the area of the upper plate in contact with the fluid is 0.48 m^2. What is the shear stress and force of the upper plate exerted on the fluid? (8%)

5. As shown below, a coating experiment involved a flat photographic film that is being pulled from a processing bath by rollers with a steady velocity U at angle θ to the horizontal. As the film leaves the bath, it entrains a liquid film with a constant thickness δ. The flow is steady and Newtonian, with constant density ρ and viscosity μ.
   (a) Derive the liquid velocity υ_x . (10%)
   (b) Derive the film thickness δ. (6%)
   (c) Descript how to tune the film thickness by varying certain parameters. (6%)

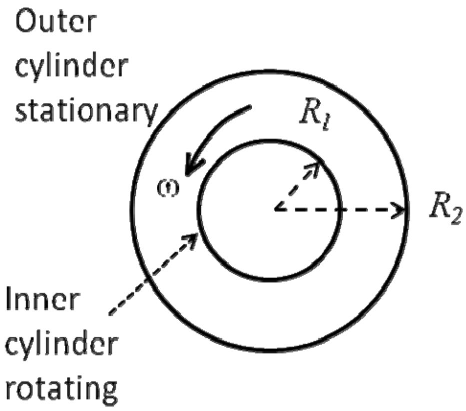


6. A rotational viscometer, which is so-called **Stormer viscometer**, consists essentially of two concentric cylinders, the inner of which rotates while the outer is held stationary. The instrument consists essentially of a rotating inner cylinder with R_1 in diameter and L in length surrounded by a concentric cylinder with R_2 in diameter and the same length. A torsion meter was installed on the inner cylinder in order to measure torque. An external torque is required to maintain the rotation of the inner cylinder at constant rotation speed ω. The flow is steady. No-slip condition is applied on the surface of the stationary cylinder.

(a) Derive the fluid velocity distribution in this kind of apparatus for the laminar flow of a Newtonian fluid. (10%)

(b) Derive the term, $\dfrac{D\vec{\upsilon}}{Dt}$. (4%)

(c) Descript how to calculate the viscosity of the fluid based on this setup. (6%)



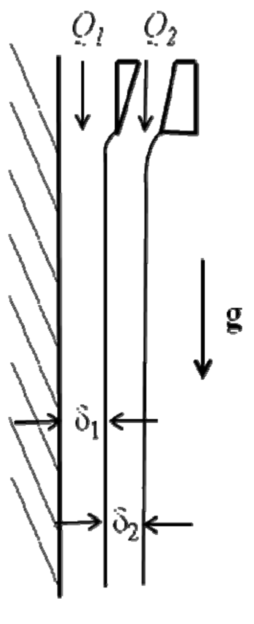
Part II: Open book 2015

7. A realistic molecular model utilizing Lennard-Jones potential energy function yields a viscosity-temperature relationship consistant with the experimental data. The expression for the viscosity of a pure gas is

$$\mu = 2.6693 \times 10^{-6} \frac{\sqrt{MT}}{\sigma^2 \Omega_\mu},$$

Please calculate the viscosity of nitrogen at 500 K. Calculate the percentage of deviation by comparing with the experimental viscosity. (10%)

8. Two immiscible viscous, Newtonian liquids are delivered to a vertical surface in such a way that the two liquids flow downward, under gravity, in the two adjacent planar parallel layers. The flow is steady. No-slip condition is applied on the surface of the plane.
(a) For the case of the volume flow rate ratio $Q_1/Q_2 = 1$, density ratio $\rho_1/\rho_2 = 1$, and viscosity ratio $\mu_1/\mu_2 = 1$, do you expect the film thickness ratio δ_1/δ_2 to be greater than, equal, or less than unity? Explain your reasoning without making any quantitative calculation. (6%)
(b) Derive the fluid velocity distribution in the two liquids. (8%)
(c) Assuming that the liquids have the same density but different viscosities, derive an expression for the film thickness ratio δ_1/δ_2 in terms of $Q_1/Q_2 = 1$. (6%)



Unit Operation I Exam II **2016**

Equations

Equation of continuity

$$\nabla \bullet (\rho \vec{\upsilon}) + \frac{\partial \rho}{\partial t} = 0,$$

Equation of continuity in cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r \upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \upsilon_\theta) + \frac{\partial}{\partial z}(\rho \upsilon_z) = 0$$

Navier-Stokes equation

$$\nabla \cdot \rho \vec{\upsilon}\vec{\upsilon} + \frac{\partial \rho \vec{\upsilon}}{\partial t} = \rho \vec{g} + \nabla \bullet \vec{\tau} \qquad \rho \frac{D\vec{\upsilon}}{Dt} = \rho \vec{g} + \nabla \bullet \vec{\tau} \qquad \rho \frac{\partial \vec{\upsilon}}{\partial t} + \rho \vec{\upsilon} \cdot \nabla \vec{\upsilon} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{\upsilon}$$

In cylindrical coordinates

r direction

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \rho g_r + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

The shear stress components in cylindrical coordinates

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{\upsilon_\theta}{r}\right) + \frac{1}{r}\frac{\partial \upsilon_r}{\partial \theta}\right]$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left[\frac{\partial \upsilon_\theta}{\partial z} + \frac{1}{r}\frac{\partial \upsilon_z}{\partial \theta}\right]$$

$$\tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial \upsilon_z}{\partial r} + \frac{\partial \upsilon_r}{\partial z}\right]$$

The vorticity and LaPlace's equation in cylindrical coordinates

$$\nabla \times \vec{\upsilon} = \left(\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\vec{e}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\vec{e}_z\right),$$

$$\nabla^2 \Psi = \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} = 0$$

Stream function Ψ and velocity potential ϕ in cylindrical coordinates

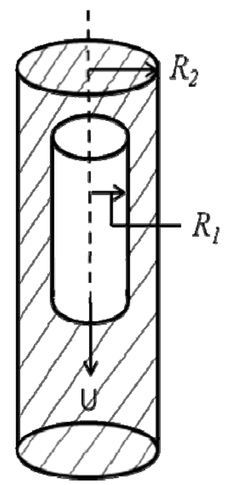
$$\upsilon_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \frac{\partial \phi}{\partial r}, \quad \upsilon_\theta = -\frac{\partial \Psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta},$$

Problems

32. Which of the following are the correct description(s). (15%)
   (s) If a fluid is steady and compressible, the Continuity equation becomes $\nabla \bullet \rho\vec{\upsilon} = 0$.
   (t) If the fluid is an irrotational flow, the fluid must be inviscid.
   (u) $\frac{\partial \vec{\upsilon}}{\partial t}$ and $\frac{D\vec{\upsilon}}{Dt}$ have the same physical meaning.
   (v) The physical meaning of $\nabla \bullet \rho\vec{\upsilon}\vec{\upsilon}$ is the rate of momentum due to convection per unit volume.
   (w) The physical meaning of $\vec{\upsilon}\cdot\nabla\vec{\upsilon}$ is convective acceleration.

33. Navier-Stokes equation is shown in the following, $\rho\frac{\partial \vec{\upsilon}}{\partial t} + \rho\vec{\upsilon}\cdot\nabla\vec{\upsilon} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{\upsilon}$.
   (a) What are the underlying assumptions for the above equation? (4%)
   (b) What is the physical law underlying the above equation? (4%)
   (c) What is the contribution of viscous force shown in this equation? (3%)
   (d) What is the contribution of convective acceleration shown in this equation? (3%)

34. What type of fluid can be described by the Euler's equation, $\rho\frac{D\vec{\upsilon}}{Dt} = \rho\vec{g} - \nabla p$? (8%)



Falling rod viscometer

35. This setup as shown on right is a **falling rod viscometer**. Please briefly describe how the viscosity can be derived based on this setup. (8%)

36. The stream function for an incompressible, two-dimensional flow field is $\Psi = 2r^3 \sin 3\theta$.
   (g) Please check if it satisfies Laplace's equation. (6%)
   (h) Please check if the flow is irrotational. (6%)

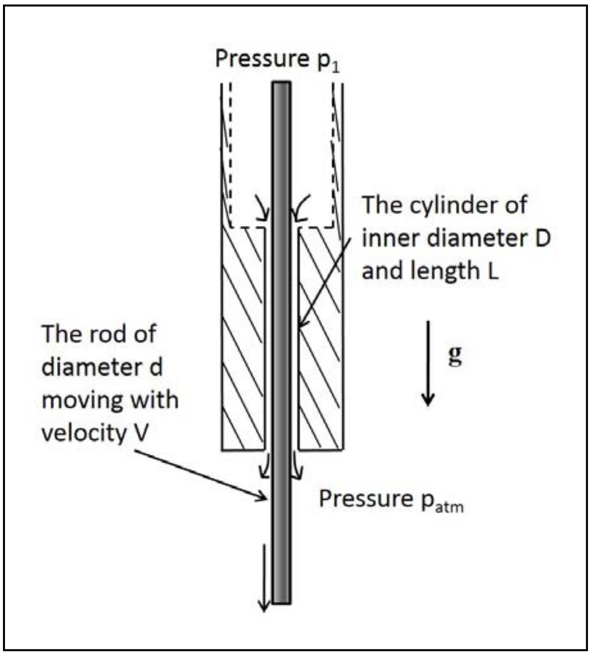
37. Based on two different models, the equations, Eq 1 and 2, for estimating viscosity were derived.
   (a) It was found that the viscosity derived from Eq 2 was much consistent with the experimental data as comparing with that from Eq1. Please explain. (6%)

$$\mu = \frac{2}{3\pi^{3/2}}\frac{\sqrt{m\kappa T}}{d^2}, \qquad \text{(Eq 1)}$$

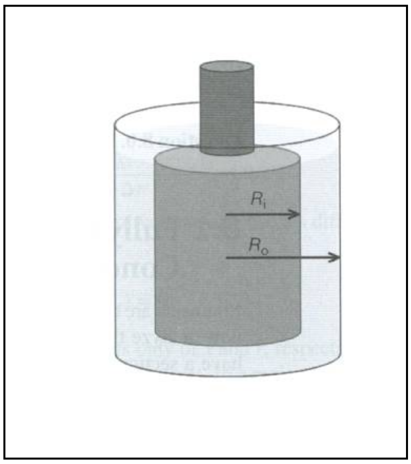
$$\mu = 2.6693\times10^{-6}\frac{\sqrt{MT}}{\sigma^2\Omega_\mu}, \qquad \text{(Eq 2)}$$

   (b) Please calculate the viscosity of oxygen at 300 K using Eq 2. (6%)

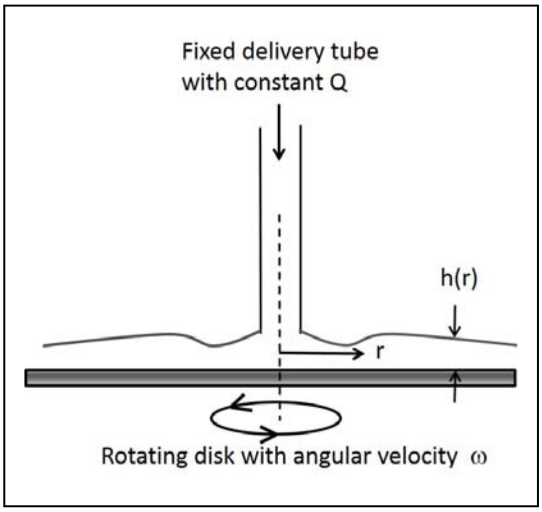
7. Consider a vertical **wire-coating process** as shown on right, in which the cylindrical rod is being moved with a velocity V. The rod is at the center of the cylindrical die. The fluid filling the space between the rod and the inner cylinder wall has density ρ and viscosity μ. Assume the flow is laminar and steady. No-slip condition is applied on the surface of the cylinder and rod. The pressure in the die is p_1, which is higher than the outside pressure, p_{atm}.
   (a) Derive the differential mass and momentum balances, as well as boundary conditions (10 %)
   (b) Derive the velocity profile in the space. (7 %)
   (c) Derive the shear stress acting on the surface of the inner cylinder. (5 %)



8. The right figure shows the geometry of a rheological experiment (**Stormer viscometer**). A fluid lies between an outer cylinder R_o (18.0 mm) and an inner cylinder R_i (17.0 mm). The inner cylinder is rotated at 5000 rpm. A torsion meter was installed on the outer cylinder and torque is measured to be 0.04 Nm. The length of the inner cylinder is 34 mm. Assume that that the fluid is Newtonian and that the viscous end effects are negligible. It is known that the velocity is $\vec{\upsilon} = (C_1/r + C_2 r)\vec{e}_\theta$, where C_1 and C_2 are constants.
   (a) Write down the boundary conditions and determine the constants, C_1 and C_2. (8%)
   (b) Explain how to determine the fluid viscosity and calculate it. (6%)



9. Consider a **spin coating process**, the coating fluid with a constant volumetric flowrate Q is fed to the axis of a rotating disk from a small tube centered at the axis of rotation. A thin film was deposited on the rotating disk through this process. The coating fluid with density ρ and viscosity μ is largely flowing in the radial direction. Assume the flow is laminar, steady, and axisymmetric with respect to θ direction. No-slip condition is applied on the surface of the rotating disk. Ignore the gravitational and entrance effects. Write down the Continuity and Navier-Stokes equations, as well as boundary conditions. (10%)



Equations

Equation of continuity

$$\nabla \bullet (\rho\vec{\upsilon}) + \frac{\partial \rho}{\partial t} = 0,$$

Equation of continuity in cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r \upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \upsilon_\theta) + \frac{\partial}{\partial z}(\rho \upsilon_z) = 0$$

Navier-Stokes equation

$$\nabla \cdot \rho\vec{\upsilon}\vec{\upsilon} + \frac{\partial \rho\vec{\upsilon}}{\partial t} = \rho\vec{g} + \nabla \bullet \vec{\tau} \qquad \rho\frac{D\vec{\upsilon}}{Dt} = \rho\vec{g} + \nabla \bullet \vec{\tau} \qquad \rho\frac{\partial \vec{\upsilon}}{\partial t} + \rho\vec{\upsilon}\cdot\nabla\vec{\upsilon} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{\upsilon}$$

In cylindrical coordinates

r direction

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right)$$
$$= -\frac{\partial P}{\partial r} + \rho g_r + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta$$
$$+ \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right)$$
$$= -\frac{\partial P}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

The shear stress components in cylindrical coordinates and spherical coordinates

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{\upsilon_\theta}{r}\right) + \frac{1}{r}\frac{\partial \upsilon_r}{\partial \theta}\right] \qquad \tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{\upsilon_\theta}{r}\right) + \frac{1}{r}\frac{\partial \upsilon_r}{\partial \theta}\right]$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left[\frac{\partial \upsilon_\theta}{\partial z} + \frac{1}{r}\frac{\partial \upsilon_z}{\partial \theta}\right] \qquad \tau_{\phi\theta} = \tau_{\theta\phi} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\upsilon_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial \upsilon_\theta}{\partial \phi}\right]$$

$$\tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial \upsilon_z}{\partial r} + \frac{\partial \upsilon_r}{\partial z}\right] \qquad \tau_{\phi r} = \tau_{r\phi} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial \upsilon_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{\upsilon_\phi}{r}\right)\right]$$

The vorticity and LaPlace's equation in cylindrical coordinates

$$\nabla \times \vec{\upsilon} = \left(\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\vec{e}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\vec{e}_z\right),$$

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial \theta^2} = 0$$

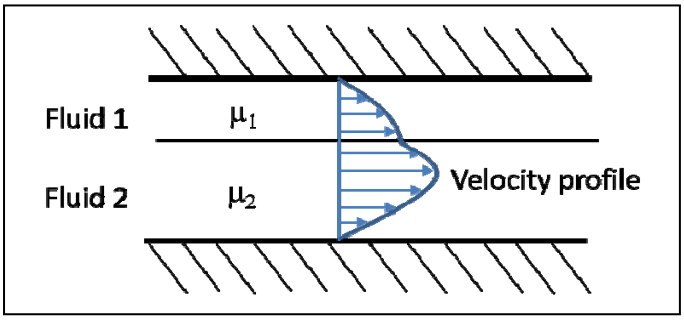
Stream function Ψ and velocity potential ϕ in cylindrical coordinates

$$\upsilon_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \frac{\partial \phi}{\partial r}, \qquad \upsilon_\theta = -\frac{\partial \Psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta},$$

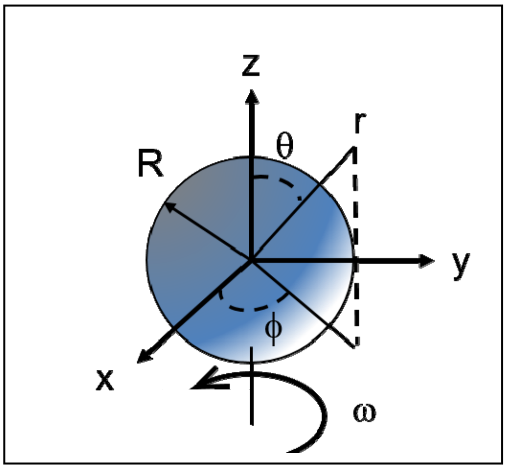
Part I: Closed book

38. For the Navier-Stokes equation $\nabla \cdot \rho \vec{\upsilon}\vec{\upsilon} + \dfrac{\partial \rho \vec{\upsilon}}{\partial t} = \rho \vec{g} + \nabla \bullet \vec{\tau}$,
   - (e) What is the physical law underlying the above equation? (4%)
   - (f) What is the underlying assumption for the above equation? (4%)
   - (g) What is the physical meaning of $\nabla \bullet \rho \vec{\upsilon}\vec{\upsilon}$ shown in this equation? (4%)
   - (h) What is the physical meaning of $\nabla \bullet \vec{\tau}$ shown in this equation? (4%)
   - (i) If the fluid is Newtonian and incompressible, what would the term, $\nabla \bullet \vec{\tau}$, become? (4%)

39. Two immiscible incompressible fluids are flowing in the z-direction in a horizontal thin slit of length L and width W under the influence of a pressure gradient as shown on the right. The velocity profiles for the two fluids were shown in the figure.
   - (a) Please determine whether $\mu_1 > \mu_2$ or $\mu_1 < \mu_2$, and give the reason. (6%)
   - (b) Please determine the position with the shear stress equals to zero, and give the reason. (6%)



40. Consider a solid sphere of radius R rotating in a large body of a stagnant, incompressible Newtonian fluid as shown on the right. Assume that that the fluid flow is laminar and steady.
   - (a) Determine the components of fluid velocity $(\upsilon_r, \upsilon_\theta, \upsilon_\phi)$ that can be eliminated (that is, zero). (4%)
   - (b) Determine the components of the stress that can be eliminated (that is, zero). (4%)
   - (c) Determine the position on the surface of the sphere exhibiting a maximum stress. (6%)



41. Please briefly describe how the viscosity can be derived based on a **falling sphere viscometer**. You have to draw the setup and write down the equation (as specific as possible) to explain how the viscosity can be derived. (8%)

42. A fluid is flowing through a horizontal tube with length L and radius R_i. The pressure drop ($-\Delta P/L$) follows the Hagen-Poiseuille equation and was measured to be 3.2×10^6 dyne/cm^2. The Reynolds number for the fluid flowing in the tube is 800. We would like to understand the effect of the tube radius on the fluid flow. The same fluid is flowing in another tube with the same length (L) and a half of the radius ($R_i/2$) with the volumetric flow rate kept the same. Please calculate the pressure drop and Reynolds number. (10%)

   Hint: The Hagen-Poiseuille equation and the Reynolds number for fluids flowing in tubes

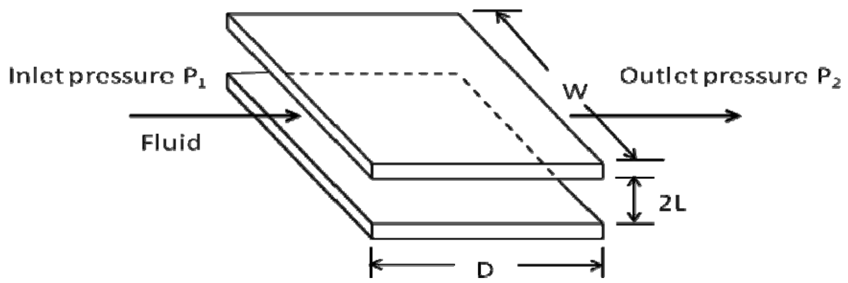
$$-\frac{\Delta P}{L} = \frac{8\mu\upsilon_{avg}}{R^2} \qquad \mathrm{Re} = \frac{2R\upsilon_{avg}\rho}{\mu}$$

43. Consider a **film blowing process**, which is the polymer melt that is flowing through a die with pressure gradient (a horizontal slit formed by two parallel walls with the two sides being sealed) as shown below. The distance and width are 2L and W, respectively. The polymer melt, which is a Newtonian fluid, filling the gap in the slit has density ρ and viscosity μ. Assume that the width is much larger than the gap (that is, W>>L). The flow is laminar and steady. No-slip condition is applied on the surface of the walls. Neglect the entrance and exit effects.

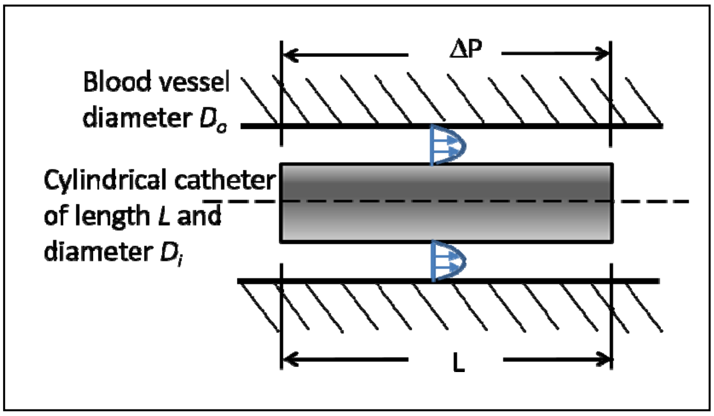
(a) Please draw a representative differential control volume and derive the differential momentum balance in the direction along the fluid flow. (6%)

(b) Derive the average velocity in the slit in terms of the maximum velocity. (8 %)

(c) Derive the shear stress for the fluid acting on the surface of the slit and specify the direction. (6 %)



44. In the process of performing the procedure called **coronary angioplasty**, a surgeon places a cylindrical catheter of length L and diameter D_i inside a coronary blood vessel of diameter D_o. It is important to understand the degree to which the presence of the catheter reduces blood flow within the vessel. Assume that the blood is Newtonian and the flow is laminar and steady. Assume that $\Delta P/L$ remains constant despite the presence or absence of the catheter. No-slip condition is applied on the surface of the catheter and blood vessel. The flow is fully developed in the region between the catheter and blood vessel, which means that the end effect can be ignored.



(a) Derive the blood velocity profile and the shear stress acting on the blood vessel without the catheter. (8%)

(b) Derive the blood velocity profile and shear stress acting on the blood vessel with the catheter. (12%)

(c) Determine the factor F without the catheter and with the presence of the catheter. F is a factor defined below. R and Q are the resistance factor and volumetric flow rate, respectively. (8%)

$$R = \frac{\Delta P}{LQ} = \frac{128\mu}{\pi D_o^4} F$$

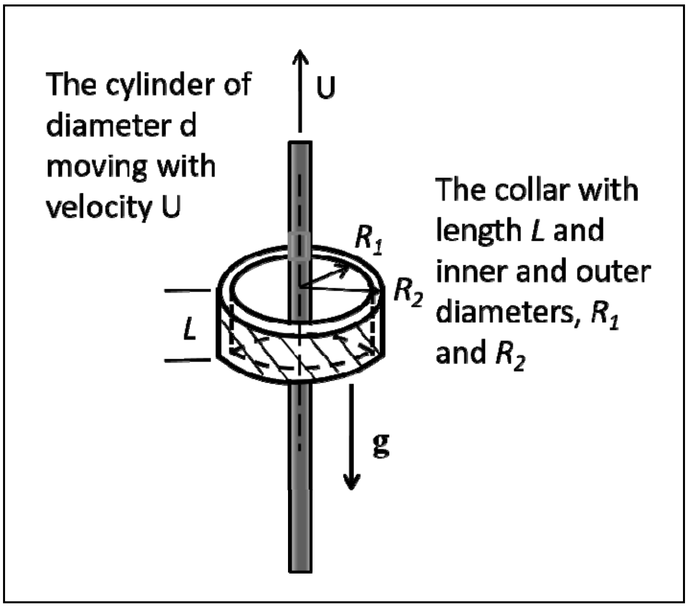
Part II: Open book

45. **A viscometer** can be designed based on the following setup as shown below. An infinitely long cylinder is continuously moving vertically upward through a large body of a stagnant, incompressible Newtonian fluid. This cylinder is surrounded by a short annular cylindrical collar, which is open at both ends. The collar is maintained coaxially aligning with the moving cylinder. The cylinder velocity U can be adjusted to maintain the collar at fixed position (does not fall). Assume that that the fluid flow is laminar.

(a) Write down the basic force balance (as specific as possible) for the collar in equilibrium. (6%)

(b) Derive the shear stress acting on the collar. (10%)

(c) Derive the viscosity of the fluid in terms of the geometry, the speed, and the densities of the fluid and collar. (8%)



Part I: Close book **2018**

46. For the continuity equation and Navier-Stokes equation

(j) What is the physical meaning of $\nabla \bullet (\rho \vec{\upsilon})$ in the continuity equation? (4%)

(k) What is the physical law underlying the Navier-Stokes equation? (4%)

(l) What is the physical meaning of $\nabla \bullet \rho \vec{\upsilon}\vec{\upsilon}$ shown in the following equation? (4%)

$$\nabla \cdot \rho \vec{\upsilon}\vec{\upsilon} + \frac{\partial \rho \vec{\upsilon}}{\partial t} = \rho \vec{g} + \nabla \bullet \vec{\tau}$$

(m) What is the contribution of viscous force shown in the following equation? (4%)

$$\rho \frac{\partial \vec{\upsilon}}{\partial t} + \rho \vec{\upsilon} \cdot \nabla \vec{\upsilon} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{\upsilon}$$

(n) What is the physical meaning of $\rho \vec{\upsilon} \cdot \nabla \vec{\upsilon}$ shown in the following equation? (4%)

$$\rho \frac{\partial \vec{\upsilon}}{\partial t} + \rho \vec{\upsilon} \cdot \nabla \vec{\upsilon} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{\upsilon}$$

(o) If the incompressible, Newtonian fluid is irrotational, please explain which term in the Navier-Stokes equation would be zero. (4%)

47. Based on two different models, the equations, Eq 1 and 2, for estimating viscosity were derived. It was found that the viscosity derived from Eq 2 was more consistent with the experimental data than that derived from Eq1. Please explain. (6%)

$$\mu = \frac{2}{3\pi^{3/2}} \frac{\sqrt{m\kappa T}}{d^2}, \qquad \text{(Eq 1)}$$

$$\mu = 2.6693 \times 10^{-6} \frac{\sqrt{MT}}{\sigma^2 \Omega_\mu}, \qquad \text{(Eq 2)}$$

48. Assume that the fluids in the following case are incompressible and Newtonian, with constant density ρ and viscosity μ. The fluid flow is laminar and steady. Please explain which of the following is(are) irrotational flow(s). (16%)

(d) A fluid flows between two plates.

(e) A fluid flows in z direction between two concentric tubes.

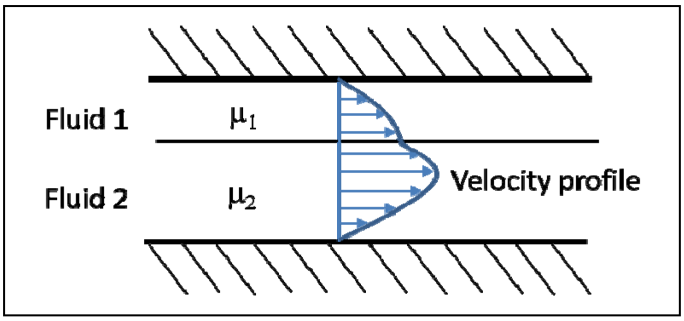
(f) A sphere rotating in a large body of stagnant fluid.

(g) A radial flow from a source.

49. Please explain the difference between $\frac{\partial \vec{\upsilon}}{\partial t}$ and $\frac{D\vec{\upsilon}}{Dt}$ in the terms of their physical meanings. (6%)

50. For a shaft rotating in a large body of stagnant fluid, please derive the stream function and velocity potential. Also, please draw the streamlines and velocity potential lines. (10%)

51. Two immiscible incompressible fluids are flowing in the z-direction in a horizontal thin slit of length L and width W under the influence of a pressure gradient as shown on the right. Assume

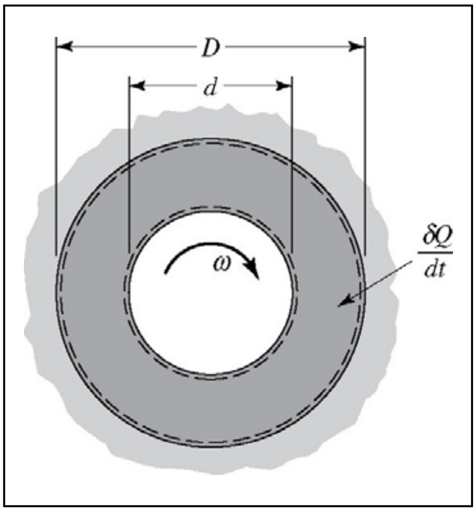


that the fluids in the following case are incompressible and Newtonian, with constant density ρ and viscosity μ. The fluid flow is laminar and steady. No-slip condition is applied on the interfaces. At the liquid-liquid interface, the shear stress and velocity are continuous.

(c) Please derive the velocity profiles for the two fluids. (8%)

(d) If the velocity profiles are shown in the figure, please determine whether $\mu_1 > \mu_2$ or $\mu_1 < \mu_2$, and give the reason. (5%)

52. Consider a **bearing** with a shaft of diameter *d* and length *L* rotating at a constant angular velocity as shown on the right. The shear stress acting on the shaft is τ. The lubricating oil between the rotating shaft and stationary bearing remain at constant temperature. The flow of lubricating oil is steady and Newtonian, with constant density ρ and viscosity μ. No-slip condition is applied on the surface of the bearing.



(a) Derive the velocity for the lubricating oil. (8%)

(b) Derive the torque τ required to turn the shaft. (6%)

(c) Derive the rate at which energy must be removed from the bearing in order for the lubricating oil to remain at constant temperature. (8%)

Part II: Open book

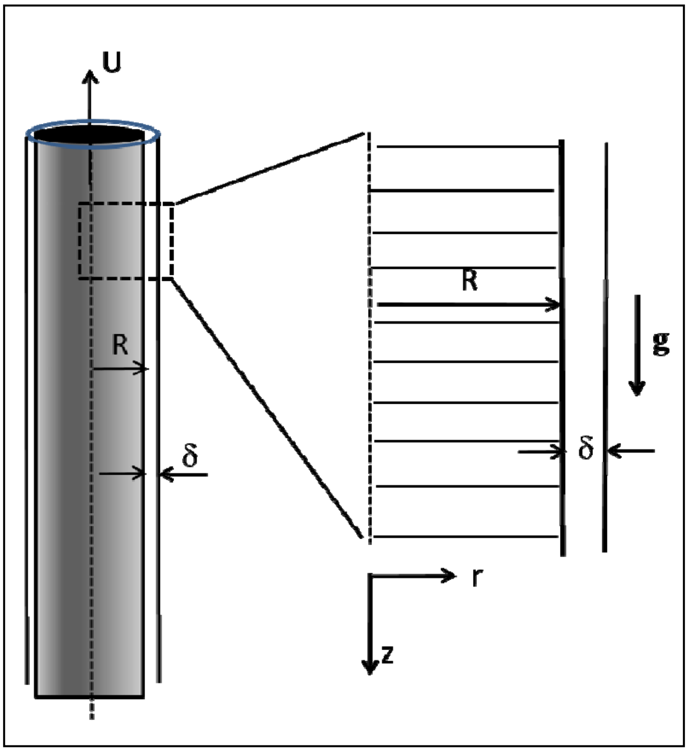
53. As shown below, a coating experiment involved a spherical wire with length ***L*** and radius ***R*** that is being vertically pulled from a processing bath by rollers with a steady velocity U. As the wire leaves the bath, it entrains a liquid film with a constant thickness δ. The fluid flow is steady and Newtonian, with constant density ρ_l and viscosity μ. The density of the wire is ρ_s. No-slip condition is applied on the surface of the wire.

    (a) Derive the liquid velocity υ_z . (10%)

    (b) Derive the film thickness δ if the volumetric flow rate is Q. (6%)

    (c) Descript how to tune the film thickness by varying certain parameters. (6%)

    (d) Derive the pulling force. (8%)



Unit Operation I Exam II May 22, 2019

Equations

Continuity equation

$$\nabla \bullet (\rho \vec{\upsilon}) + \frac{\partial \rho}{\partial t} = 0,$$

Continuity equation in cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r \upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \upsilon_\theta) + \frac{\partial}{\partial z}(\rho \upsilon_z) = 0$$

Navier-Stokes equation

$$\nabla \cdot \rho \vec{\upsilon}\vec{\upsilon} + \frac{\partial \rho \vec{\upsilon}}{\partial t} = \rho \vec{g} + \nabla \bullet \vec{\tau} \qquad \rho \frac{D\vec{\upsilon}}{Dt} = \rho \vec{g} + \nabla \bullet \vec{\tau} \qquad \rho \frac{\partial \vec{\upsilon}}{\partial t} + \rho \vec{\upsilon} \cdot \nabla \vec{\upsilon} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{\upsilon}$$

In cylindrical coordinates, the vorticity and LaPlace's equations, stream function Ψ and velocity potential ϕ

r direction

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \rho g_r + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

θ direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

$$\nabla \times \vec{\upsilon} = \left(\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\vec{e}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\vec{e}_z\right),$$

$$\nabla^2 \Psi = \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} + \frac{\partial^2 \Psi}{\partial z^2} = 0$$

$$\upsilon_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \frac{\partial \phi}{\partial r}, \quad \upsilon_\theta = -\frac{\partial \Psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta},$$

$$\nabla \bullet \vec{\upsilon} = \frac{1}{r}\frac{\partial}{\partial r}(r \upsilon_r) + \frac{1}{r}\frac{\partial \upsilon_\theta}{\partial \theta} + \frac{\partial \upsilon_z}{\partial z},$$

The shear stress components in cylindrical coordinates and spherical coordinates

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r \frac{\partial}{\partial r}\left(\frac{\upsilon_\theta}{r}\right) + \frac{1}{r}\frac{\partial \upsilon_r}{\partial \theta}\right] \qquad \tau_{r\theta} = \tau_{\theta r} = \mu\left[r \frac{\partial}{\partial r}\left(\frac{\upsilon_\theta}{r}\right) + \frac{1}{r}\frac{\partial \upsilon_r}{\partial \theta}\right]$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left[\frac{\partial \upsilon_\theta}{\partial z} + \frac{1}{r}\frac{\partial \upsilon_z}{\partial \theta}\right] \qquad \tau_{\phi\theta} = \tau_{\theta\phi} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\upsilon_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial \upsilon_\theta}{\partial \phi}\right]$$

$$\tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial \upsilon_z}{\partial r} + \frac{\partial \upsilon_r}{\partial z}\right] \qquad \tau_{\phi r} = \tau_{r\phi} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial \upsilon_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{\upsilon_\phi}{r}\right)\right]$$

Conservation of energy

$$\frac{\partial Q}{dt} - \frac{\partial W_s}{dt} = \iint_{c.s.}\left(e + \frac{P}{\rho}\right)\rho(\vec{\upsilon} \cdot \vec{n})\,dA + \frac{\partial}{\partial t}\iiint_{c.v} e\rho dV + \frac{\partial W_\mu}{dt}$$

Part I: Close book

54. For the conservation equations,

(p) In the cartesian coordinate system, one can derive the term, $\frac{\partial \upsilon_i}{\partial j}$, where *i* and *j* represent *x*, *y*, and *z*. Please explain the physical meaning of it. (4%)

(q) Please write down the continuity equation if the fluid flow is in steady state? (4%)

(r) Please write down the assumptions that can lead to the Navier-Stokes equation becoming the following form. (4%)

$$\rho\frac{\partial \vec{\upsilon}}{\partial t}+\rho\vec{\upsilon}\cdot\nabla\vec{\upsilon}=\rho\vec{g}-\nabla p+\mu\nabla^2\vec{\upsilon}$$

(s) What is the physical meaning of $\mu\nabla^2\vec{\upsilon}$ shown in the following equation? (4%)

$$\rho\frac{\partial \vec{\upsilon}}{\partial t}+\rho\vec{\upsilon}\cdot\nabla\vec{\upsilon}=\rho\vec{g}-\nabla p+\mu\nabla^2\vec{\upsilon}$$

(t) What is the physical meaning of $\iint_{c.s.}\left(e+\frac{P}{\rho}\right)\rho\left(\vec{\upsilon}\cdot\vec{n}\right)dA$ shown in the following equation? (4%)

$$\frac{\partial Q}{dt}-\frac{\partial W_s}{dt}=\iint_{c.s.}\left(e+\frac{P}{\rho}\right)\rho\left(\vec{\upsilon}\cdot\vec{n}\right)dA+\frac{\partial}{\partial t}\iiint_{c.v}e\rho dV+\frac{\partial W_\mu}{dt}$$

55. One graduate student and his advisor setup a home-made instrument that can measure the rate of shear strain of a polymer solution by applying a known shear force. Then the shear stress as a function the rate of shear strain for the polymer solution can be obtained. After several months of hard work, the student summarized his following findings.
(i) As the concentration of the polymer solution is much lower than a certain concentration (c*), the applied shear stress was found to be proportional to the rate of shear strain for the polymer solution.
(ii) As the concentration is higher than a certain concentration (c*), the applied shear stress was not proportional to the rate of shear strain. The more the shear stress is applied, the less the rate of shear strain is observed.
Please explain what type of the fluid for condition (i) and (ii) by plotting the shear stress vs. the rate of shear strain. (8%)

56. An incompressible Newtonian fluid with density (ρ = 0.8 g/cm^3) is flowing in a horizontal pipe (700 m) with the inside radius of 0.25 m. The pressure drop across the pipe is 1,000 Pascal and the average velocity is 0.5 m/s. The fluid flow is laminar and steady. No-slip condition is applied on the inner surface of the pipe.
(a) Please check if the fluid flow is laminar. (6%)
(b) Please calculate the fluid viscosity μ. (6%)
Hint: The Hagen-Poiseuille equation and Reynolds number are shown below. The fluid flow is laminar if the Reynolds number is less than 2,100.

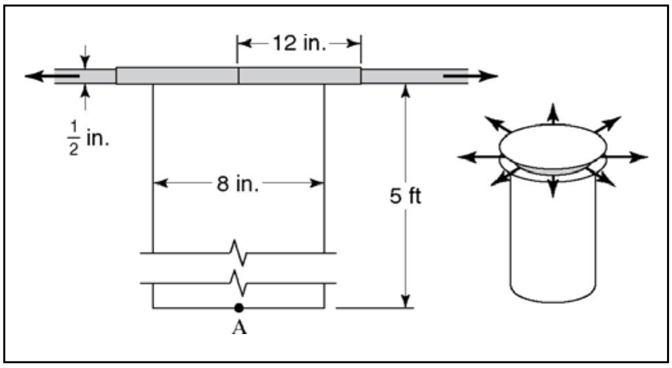
$$-\frac{\Delta P}{L}=\frac{8\mu\upsilon_{avg}}{R^2} \qquad \mathrm{Re}=\frac{D\upsilon_{avg}\rho}{\mu}$$

57. A solid sphere was placed in a container filled with an incompressible Newtonian fluid. The solid sphere was found to fall to the bottom of the container with a terminal velocity. The fluid flow is laminar and steady. The density of the solid sphere is higher than that of the fluid. The diameter of the container is much larger than that of the sphere.

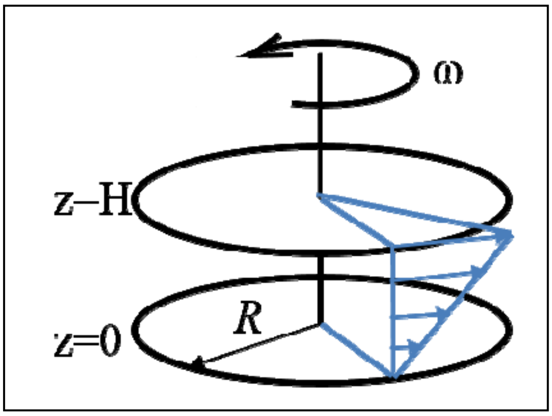
(a) The solid sphere would reach a terminal velocity. Why? (6%)

(b) Based on this setup, descript how to determine the viscosity of the fluid. (6%)

58. As shown below, a fluid of density ρ_I flows steadily up the vertical pipe and is then deflected to flow outward with a uniform radial velocity. If the friction is neglected, please write down the conservation of energy equation by eliminating the terms that are zero. (8%)



59. A rotational or torsional fluid flow between is shown below. The Newtonian fluid fills the space between the two disks. The upper disk is rotating about the z axis at a constant rotational speed ω. Consequently, the fluid between the two disks is in a rotational or torsional flow. The fluid flow is steady and laminar, with constant density ρ and viscosity μ. No-slip condition is applied on the surface of the disks and the radial flow is neglected.



(a) By eliminating the components of fluid velocity that are zero, only the θ component $\upsilon_\theta(r,z)$ exists and is a function of *r* and *z*. Please explain the reason. (8%)

(b) Write down the simplified Navier-Stokes equation and boundary conditions. (7%)

(c) The trial solution for the Navier-Stokes equation is $\upsilon_\theta(r,z) = Cr^n g(z)$ by using separation of variable method. It was found that $d^2 g(z)/dz^2 = 0$. Please determine the exponent *n* and the velocity profile of the fluid between the disks. (5%)

(d) Derive the torque required to maintain the rotation speed. (5%)

(e) Please check if the fluid flow is irrotational. (5%)

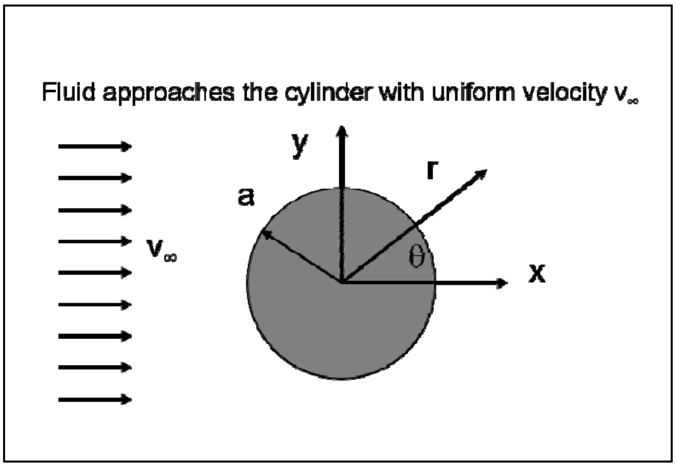
Part II: Open book

60. An incompressible, Newtonian fluid with constant density ρ and viscosity μ approaches a stationary cylinder with a uniform, steady velocity in the positive direction. Assume the flow is laminar and steady. No-slip condition is applied on the surface of the cylinder. The solution was derived and shown below. The pressure and velocity in the immediate vicinity of the cylinder are

$$p(r,\theta) = p_\infty - C\mu \frac{\upsilon_\infty \cos\theta}{r} - \rho g r \sin\theta,$$

$$\upsilon_r(r,\theta) = C\upsilon_\infty \left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4} + \frac{1}{4}\left(\frac{R}{r}\right)^2\right]\cos\theta,$$

$$\upsilon_\theta(r,\theta) = -C\upsilon_\infty \left[\frac{1}{2}\ln\left(\frac{r}{R}\right) + \frac{1}{4} - \frac{1}{4}\left(\frac{R}{r}\right)^2\right]\sin\theta,$$



(a) Please determine the place on the cylinder exhibiting maximum shear stress. Explain it! (6%)

(b) Please proof that $\nabla \bullet \bar{\upsilon} = 0$ and explain why it is the case. (7%)

(c) Please derive the shear force acting on the cylinder by the fluid at the x direction. (6%)

61. For an incompressible, Newtonian fluid, the flow is given by the velocity potential $\phi = 4xy - \frac{1}{2}x^2 + \frac{1}{2}y^2$.

(i) Please check if the velocity potential satisfies Laplace's equation. (7%)

(j) Please check if the flow satisfies Euler's equation. (6%)

(k) Please determine the stream function for the flow. (6%)

Part I: Close book **2020**

62. For the conservation equations,

(u) Please write down the physical law underlying the Navier-Stokes equation. (4%)

(v) What are the physical meanings of $\mu\nabla^2\vec{\upsilon}$ and $\vec{\upsilon} \bullet \nabla\vec{\upsilon}$ as shown in the following equation? (8%)

$$\rho\frac{\partial\vec{\upsilon}}{\partial t}+\rho\vec{\upsilon}\cdot\nabla\vec{\upsilon}=\rho\vec{g}-\nabla p+\mu\nabla^2\vec{\upsilon}$$

(w) What is the physical meaning of $\iint_{c.s.}\left(e+\frac{P}{\rho}\right)\rho\left(\vec{\upsilon}\cdot\vec{n}\right)dA$ shown in the following equation? (4%)

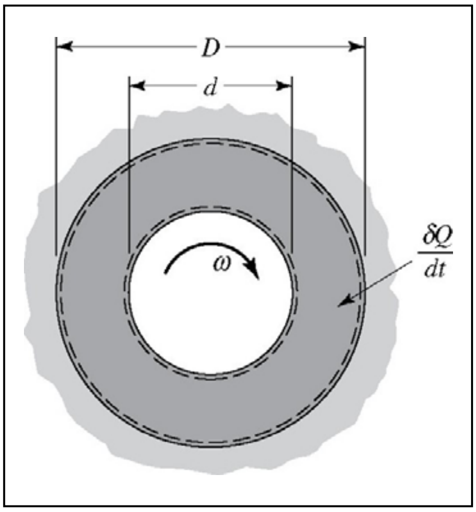
$$\frac{\partial Q}{dt}-\frac{\partial W_s}{dt}=\iint_{c.s.}\left(e+\frac{P}{\rho}\right)\rho\left(\vec{\upsilon}\cdot\vec{n}\right)dA+\frac{\partial}{\partial t}\iiint_{c.v}e\rho dV+\frac{\partial W_\mu}{dt}$$

(x) Bernoulli's equation is based on conservation of energy and please specify the type of energy involved in the equation. (4%)

63. For non-Newtonian fluids, please draw the shear stress vs. rate of shear strain curves for shear-thinning and shear-thickening fluids and explain the trend of fluid viscosity upon the increase of shear stress. (8%)

64. For an incompressible, Newtonian fluid with constant viscosity μ, please explain what type of fluid can be described by the Euler's equation, $\rho\frac{D\vec{\upsilon}}{Dt}=\rho\vec{g}-\nabla p$? (8%)

65. Consider a **bearing** with a shaft of diameter d and length L rotating at a constant angular velocity as shown on the right. The shear stress acting on the shaft is τ. The lubricating oil is incompressible and Newtonian with constant viscosity μ. No-slip condition is applied on the surface of the bearing. No shaft work is done by the system and a constant rate of heat ($\delta Q/dt$) is supplied to the system.



(b) Specify the components of velocity and stress that are not zero. (5%)

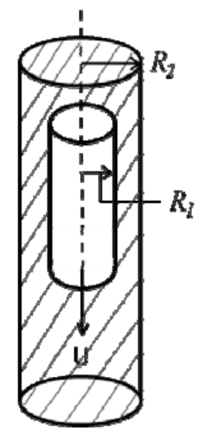
(c) Determine the orientation and direction of the shear stress (τ) acting on the shaft by the oil. (4%)

(d) Write down the conservation of momentum using Navier-Stokes equation. (5%)

(d) Write down the conservation of energy for the system and derive the rate of temperature increase in the system. (10%)

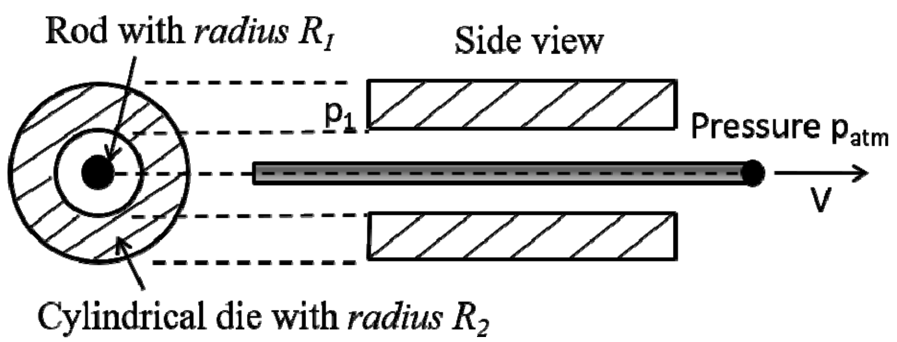
Hint: the internal energy $\Delta u = c\Delta T$

66. As shown on the right, this setup is a **falling rod viscometer**. The density of the rod (ρ_1) is higher than that of the fluid (ρ_2) so that the rod with R in radius and L in length would fall in the fluid due to the downward gravitational force. the shear stress acting on the rod is τ. Please briefly answer the following questions.
(a) The solid rod would reach a terminal velocity. Why? (4%)
(b) Based on this setup, please write down the force balance on the rod. (4%)
(c) Briefly descript how to determine the viscosity of the fluid based on this setup. You don't have to give detailed calculation. (4%)



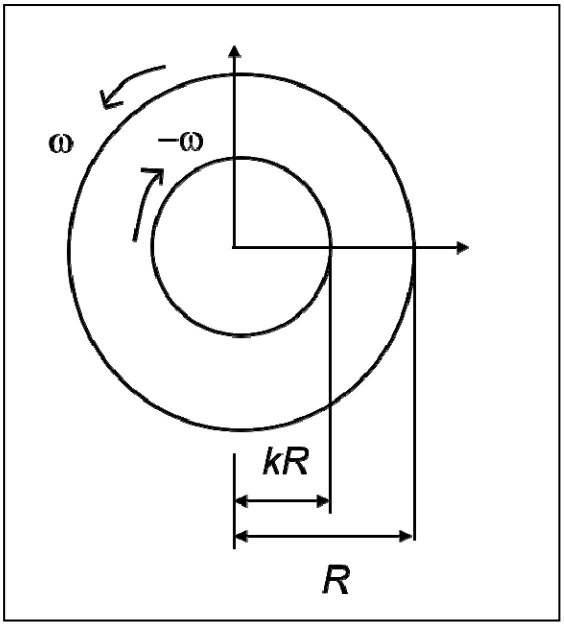
Falling rod viscometer

67. Consider a rod with R_1 radius moving with a velocity V at the center of the cylindrical die with R_2 radius as shown below. The incompressible, Newtonian fluid constant viscosity μ filling the space between the rod and the inner cylinder wall has. Assume the flow is laminar and steady. No-slip condition is applied on the die and rod. The innlet pressure is p_1, which is higher than the outlet pressure, p_{atm}. Ignore the effect of gravity.
(a) Draw the differential control volume (DCV) representing the system and specify the components of velocity and stress that are not zero. (5 %)
(b) Write down the equations based on conservation of mass and momentum as well as boundary conditions. (5%)
(c) Derive the fluid velocity profile. (6 %)
(d) Derive the pulling force which overcomes the shear force exerted on the rod by the fluid. (5 %)



Part II: Open book

68. A **milling process** consists essentially two concentric cylinders with a fluid in between as shown on the right. The outer of which rotates counterclockwisely while the inner one rotates clockwisely. The instrument consists essentially of a rotating outer cylinder with R in radius and L in length surrounded on a concentric cylinder with κR in radius and the same length. The angular velocities of the outer and inner cylinders are ω and $-\omega$, respectively. The flow with constant fluid density ρ and viscosity μ is steady. No-slip condition is applied on the surface.



(a) Derive the velocity profile in the fluid. (12%)

(b) Derive the place where the shear stress and velocity are zero. (6%)

(b) Derive the pressure gradient, ∇p. (6%)