Linear Algebra: Final Exam-A

This is a 120-minutes closed-book exam.

Calculator is allowed.

4 pages in total

1. (54 pts) Determine whether the following statements are true (T) or false (F)? (Reasoning is required.)

(1) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space *V*. If $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal, then $||\mathbf{u}|| = ||\mathbf{v}||$.

(2) If $\mathbf{x}$ and $\mathbf{y}$ are unit vectors in R^n and $|\mathbf{x}^T\mathbf{y}| = 1$, then $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

(3) Let A be an $n \times n$ matrix and $det(A) \neq 0$. If the matrix A has an eigenvalue $\boldsymbol{\lambda}$ and its corresponding eigenvector $\mathbf{v}$, then $\boldsymbol{A^{-1}}$ has a eigenvector $\mathbf{v}$ with eigenvalue $\frac{\mathbf{1}}{\boldsymbol{\lambda}}$.

(4) If the characteristic polynomial of a matrix A is $det(A-\lambda I) = \lambda^2 - 2\lambda$, then *A* is invertible.

(5) The characteristic equation of a 2×2 matrix A can be expressed as $det(\lambda I - A) = \lambda^2 - tr(A)\lambda + det(A) = 0$, where $tr(A)$ is the trace of *A*.

(6) If A and B are similar matrices, then they have the same eigenvalues.

(7) If U, V, and W are subspaces of R^3 and if $U \perp V$ and $V \perp W$, then $U \perp W$.

(8) If *A* is a 4×3 matrix of rank 1, then the dimensions of $N(A^T)$ is 3.

(9) Let $\langle \mathbf{u}, \mathbf{v} \rangle$ be the Euclidean inner product on R^2, and if $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, then $\langle \mathbf{u}, \mathbf{2v} \rangle = 14$.

(10) It is possible to find a nonzero vector $\mathbf{y}$ in the column space of *A* such that $A^T\mathbf{y} = \mathbf{0}$.

(11) Let $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ be an orthonormal basis for an inner product space *V* and let

$\mathbf{u} = 2\,\mathbf{u_1} + \mathbf{u_2} + \mathbf{u_3}$ and $\mathbf{v} = \mathbf{u_1} - \mathbf{u_2} - \mathbf{u_3}$. We have $\mathbf{u} \perp \mathbf{v}$.

(12) If the inner product on P_3 is defined by $\langle f,\ g\rangle = \int_{-1}^{1} f(x)g(x)dx$, then $\langle 1+x,\ 1-x-x^2\rangle = \frac{2}{3}$.

(13) If the inner product on P_3 is defined by $\langle f,\ g\rangle = \int_{-1}^{1} f(x)g(x)dx$, then $p_1(x) = 1, p_2(x) = x$ and $p_3(x) = x^2 - \frac{1}{3}$ are orthogonal.

(14) If a matrix A has the vector $\begin{bmatrix}1\\1\\2\end{bmatrix}$ in row space of A, then the $\begin{bmatrix}2\\-4\\1\end{bmatrix}$ is in the null space of A.

(15) The orthogonal complement（正交補餘）of the subspace of R^3 spanned by $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\2\end{bmatrix}\right\}$ is spanned by $\left\{\begin{bmatrix}2\\2\\0\end{bmatrix}\right\}$.

(16) The set $\boldsymbol{S} = \{\mathbf{1}, \mathbf{cos}(\boldsymbol{x}), \boldsymbol{sin}(\boldsymbol{x})\}$ is a linearly independent subset of $C\ [-\pi,\ \pi]$ with respect to the inner product $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$

(17) Let A be an $n \times n$ matrix and let $\boldsymbol{B} = -\boldsymbol{A}^{\mathbf{2}} + \boldsymbol{A} + \mathbf{2}\,\boldsymbol{I}$. If the eigenvalues of A are $\boldsymbol{\lambda_i}, i = 1,2,\cdots,n$, then the eigenvalues of B are $-\boldsymbol{\lambda_i}^{\mathbf{2}} + \boldsymbol{\lambda_i} + \mathbf{2}$, $i = 1,2,\cdots,n$.

(18) Let M_{33} denote the vector space consisting of all 3×3 matrices. If $A = \begin{bmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\\ a_7 & a_8 & a_9\end{bmatrix}$ and $B = \begin{bmatrix}b_1 & b_2 & b_3\\ b_4 & b_5 & b_6\\ b_7 & b_8 & b_9\end{bmatrix}$, we define $\langle A,\ B\rangle = \sum_{i=1}^{9} a_i b_i$ Now, if we have $A = \begin{bmatrix}1 & 2 & 3\\ -1 & 1 & -2\\ -1 & 1 & 2\end{bmatrix}$ and $B = \begin{bmatrix}-2 & 1 & 0\\ -2 & 4 & 1\\ -2 & 1 & -1\end{bmatrix}$, then A and B are orthogonal.

(19) If three points $(x_1, y_1) = (0, 1)$, $(x_2, y_2) = (1, 3)$ and $(x_3, y_3) = (2,\ 4)$ are given, then the straight line $y = 1 + x$ to minimize $\sum_{i=1}^{3}[y_i - (1 +$

$x_i)]^2$.

(20) If A is a square matrix and $\|A\mathbf{u}\| = \|\mathbf{u}\|$ for all vectors $\mathbf{u} \neq 0$, then A is orthogonal.

(21) Let Q be an orthogonal matrix, then $det(Q) = \pm 1$.

(22) If A is Hermitian and c is a complex scalar, then cA is Hermitian.

(23) The matrix $A = \begin{bmatrix} 1 & 1+1i & 2i \\ 1+i & 2 & 1+2i \\ -2i & 1-2i & 3 \end{bmatrix}$ is Hermitian.

(24) Let $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$ be the standard basis for R^3 and if $L : R^3 \to R^3$ be a linear transformation with the properties $L(\mathbf{e_1}) = \mathbf{e_2}, L(\mathbf{e_2}) = 2\mathbf{e_1} + \mathbf{e_2}$, $L(\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}) = \mathbf{e_3}$, then $L\left(\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}\right) = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}$.

(25) If a matrix $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$ is given, then the $GM_{\lambda_i} = AM_{\lambda_i}, i = 1,2$.

(26) If a matrix A has singular values $\sigma_1 = 9 > \sigma_2 = 4$, then the eigenvalues of A^TA are 3 and 2.

(27) If a matrix A has singular value decomposition as $A = \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{bmatrix}$, then its rank is 2.

2. (16 pts) Consider a matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 7 & 3 \\ 2 & 6 & 2 \end{bmatrix}$, determine

(1) The basis set for the column space of A.

(2) The basis set for the row space of A.

(3) The range space of A.

(4) The basis set for the null space of A.

3. (12 pts) Let $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

(1) Use the Gram–Schmidt process to find an orthonormal basis for the column space of *A*.

(2) Factor *A* into a product *QR*, where *Q* has an orthonormal set of column vectors and *R* is upper triangular.

(3) Solve the least squares problem $A\mathbf{x} = \mathbf{b}$.

4. (18 pts) Suppose that *A* is a 3×3 matrix with eigenvalues $\boldsymbol{\lambda_1} = 0$, $\boldsymbol{\lambda_2} = -1$ and $\boldsymbol{\lambda_3} = 1$, and corresponding eigenvectors

$$\mathbf{v_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} (\lambda_1 = 0), \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} (\lambda_1 = -1), \mathbf{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (\lambda_3 = 1).$$

(1) Find the matrix *A*.

(2) Find A^{20}.

(3) Find the unique solution of the differential equation $\frac{d\,Y(t)}{dt} = AY(t), t \geq 0$

with the initial condition $Y(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.