數值方法 Midterm exam_02

1. Please answer the following questions briefly.
   (a) Please describe the 3, 5, and 7 diagonal matrices and the numerical method that can be used to solve them? [4%]
   (b) Plot figures of the zero, first, second, and natural-degree spline. [3%]
   (c) Please explain the difference between the first-order-accuracy forward difference scheme and the second-order-accuracy central difference scheme. [3%]

2. A general Pade type boundary scheme (at i = 0) for the first derivative which does not alter the tridiagonal structure of the matrix can be written as

$$f_0' + \alpha f_1' = \frac{1}{h}(af_0 + bf_1 + cf_2 + df_3)$$

(a) [6%]Show that requiring this scheme to be at least third-order accurate would constrain the coefficients to

$a = -\frac{11+2\alpha}{2}$, $b = \frac{6-\alpha}{2}$, $c = \frac{2\alpha-3}{2}$, $d = \frac{2-\alpha}{6}$

Which value of α would you choose and why?

(b) [9%]Find all the coefficients such that the scheme would be fourth-order accurate.

3. [8%] Derive the following finite difference formula. $\sigma = \frac{x_{i+1}-x_i}{x_i-x_{i-1}}, \Delta x = x_i - x_{i-1}$

   (a) $\frac{\partial f_i}{\partial x} = \frac{f_{i+1}+(\sigma^2-1)f_i-\sigma^2 f_{i-1}}{\sigma(\sigma+1)\Delta x}$ + O(?)

   (b) $\frac{\partial^2 f_i}{\partial x^2} = 2\frac{f_{i+1}-(1+\sigma)f_i+\sigma f_{i-1}}{\sigma(\sigma+1)\Delta x^2}$ + O(?)

   what are the truncation errors of (a) and (b)?

4. (a)Derive the error term of Simpson's $\frac{1}{3}$ rule. [3%]

   (b)Derive the error term of Simpson's $\frac{3}{8}$ rule. [4%]

5. (a) Construct a natural cubic spline to approximate $f(x) = \cos\pi x$ by using the values given by $f(x)$ at x = 0, 0.5, and 1. [6%]

   (b) Integrate the spline over [0, 1], and compare the result to $\int_0^1 \cos\pi x \ dx$. [3%]

   (c) Use the derivatives of the spline to approximate $f'(0.5)$ and $f''(0.5)$, and compare these approximations to the actual values. [1%]

6. [12%] Evaluate $\int_1^{1.5} x^2 \ln x \, dx$ and $\int_0^6 \frac{dx}{1+x^2}$ by using (a) Trapezoidal rule, (b) Simpson's $\frac{1}{3}$ rule, (c) Simpson's $\frac{3}{8}$ rule and compare the error with the exact solution.

7. Use the Lagrange interpolating polynomial of degree 3 or less and four-digit chopping arithmetic to approximate cos0.750 using the following values. [3%]

cos0.698 = 0.7661 cos0.733 = 0.7432
cos0.768 = 0.7193 cos0.803 = 0.6946

8. [7%]Given the data

x	0	1	2	4	6
$f(x)$	1	9	23	93	259

(a) Construct the divided-difference table
(b) Using Newton's interpolation polynomial, find an approximation to $f(4.2)$

9. [10%] Use the TDMA method to solve the given equation: The length of the rod is 10 and number of grid points is 11.

$$\frac{\partial^2 T}{\partial x^2} = 3$$

i	1	2	3	4	5	6	7	8	9	10	11
T_i	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}

Boundary conditions: $T_1 = T_2, T_{11} = 100$ Find $T_i (i = 2 \sim 10)$

10. [18%] Use the fourth-order Pade' scheme and the fourth-order boundary schemes to differentiate

$$f(x) = \cos 3x + \cos 6x \quad 0 \le x \le 2\pi$$

use the TDMA method to plot the result. Comment on the minimum grid point number that is needed to solve the above equation. Please explain the aliasing error and the relation to this problem.