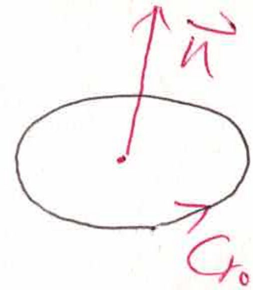
Chap 10

Vorticity

Stokes's theorem



$$\oint_{C_{r_0}} \vec{F}\cdot\vec{r}\,ds = \iint_{S_{r_0}} (\nabla\times\vec{F})\cdot\vec{n}\,ds$$

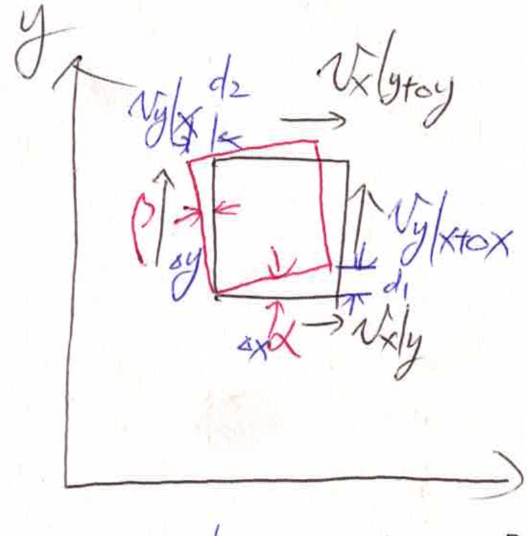
$$= (\nabla\times\vec{F})\cdot\vec{n}(p^*)A_{r_0}$$

$$\Rightarrow \nabla\times\vec{F}\cdot\vec{n}(p^*) = \frac{1}{A_{r_0}}\oint_{C_{r_0}} \vec{F}\cdot\vec{r}\,ds$$

Specific circulation(circulation per unit area) of the flow in the surfac

$$\frac{1}{2}(\nabla\times\vec{v}) = \vec{w}$$ vorticity $(\nabla\times\vec{v})$

$\nabla\times\vec{v} = 0$ irrotational.



$$\tan\alpha = \frac{d_1}{\Delta x}$$

$$\tan\beta = \frac{d_2}{\Delta y}$$

$$d_1 = (v_y|_{x+\Delta x} - v_y|_x)\Delta t$$

$$d_2 = (v_x|_y - v_x|_{y+\Delta y})\Delta t$$

$$w_z = \frac{d}{dt}\left(\frac{\alpha+\beta}{2}\right)$$

$$= \lim_{\Delta t,\Delta x,\Delta y\to 0}\left(\frac{\alpha+\beta}{2\Delta t}\right)$$

$$= \frac{1}{2}\lim_{\Delta t,\Delta x,\Delta y\to 0}\left[\frac{(v_y|_{x+\Delta x} - v_y|_x)}{\Delta x} - \frac{(v_x|_{y+\Delta y} - v_x|_y)}{\Delta y}\right]$$

$$= \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)$$

$$\Rightarrow \vec{\omega} = \frac{1}{2}(\nabla \times \vec{v})$$

From Navier-Stokes eq. (9-19)

$$\rho \frac{D\vec{v}}{Dt} = \rho\vec{g} - \nabla p + \mu \nabla^2 \vec{v}$$

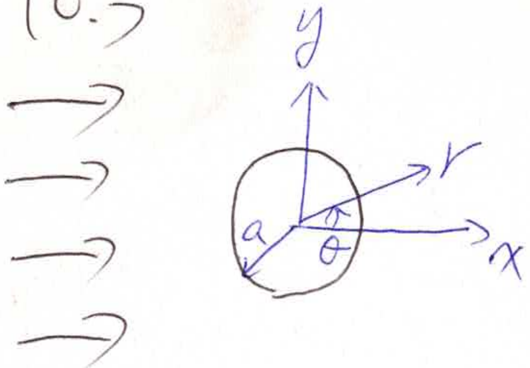
ρ = const (incompr

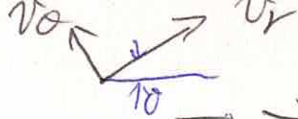
μ = const (Newtonia

$$\Rightarrow \rho \frac{D\vec{v}}{Dt} = \rho\vec{g} - \nabla p + \mu\left[\nabla\underbrace{(\nabla\cdot\vec{v})}_{=0} - \nabla\times(\nabla\times\vec{v})\right]$$

Frome the above eq.,
If viscous forces act on a fluid, the flow must be rotational.

10.3





$\nabla \cdot \vec{v} = 0$

$\nabla \times \vec{v} = 0$

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} = 0$$ 除r^2

$$v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \quad v_\theta = -\frac{\partial \Psi}{\partial r}$$

B.C. ×10

1. $v_r|_{r=a} = 0$ or $\frac{\partial \Psi}{\partial \theta}\Big|_{r=a} = 0$

   The velocity normal to a streamline is ~~zero~~ zero.

2. $\boxed{v_\theta|_{\theta=0} = 0, \text{ or } \frac{\partial \Psi}{\partial r}\Big|_{\theta=0} = 0}$

3. $v_x|_{r\to\infty}$ is finite (v_∞)

$v_x = v_r \cos\theta - v_\theta \sin\theta$

Apply separation of variable

$\Psi(r,\theta) = f(r)\,g(\theta)$ PDE.

$$f''(r)g(\theta) + \frac{1}{r}f'(r)g(\theta) + \frac{1}{r^2}f(r)g''(\theta) = 0$$

Trial solution

$g(\ )$ 找 sin or cos function

$\Psi(r,\theta) = c r^n g(\theta)$

$c\,n(n-1)\cancel{r^{n-2}}\,g(\theta) + c\,n\cdot\cancel{\frac{1}{r}\cdot r^{n-1}}\cdot g(\theta) + c\,\cancel{\frac{1}{r^2}}\cdot r^n g''(\theta)$

If $\Psi(r,\theta) = c r^n g(\theta)$ is the solution,

$g(\theta) = g''(\theta)$ or $-g''(\theta)$

From B.C. 2

变数 $\neq 0$

$f'(r)\,g(0) = 0 \Rightarrow g(0) = 0$

then $g(\theta) = \sin\theta$

$\Rightarrow \Psi(r,\theta) = c r^n \sin\theta$

$\Rightarrow \cancel{c}\,n(n-1)\cancel{r^{n-2}\sin\theta} + \cancel{c}\,n\cdot\cancel{r^{n-2}\sin\theta} - \cancel{c\,r^{n-2}\sin\theta}$

$\Rightarrow n(n-1) + n - 1 = 0$

$n^2 - 1 = 0$

$n = 1$ or -1

$\therefore \Psi(r,\theta) = \left(Ar + \frac{B}{r}\right)\sin\theta$

代其它 B.C. ①.③

Apply B.C. 1

$$\left.\frac{\partial \Psi}{\partial \theta}\right|_{r=a} = 0$$

$$\left.\left(\underbrace{Ar + \frac{B}{r}}_{=0}\right)\right|_{r=a} \cos\theta = 0$$

实现

$$Aa + \frac{B}{a} = 0$$

$$B = -Aa^2$$

$$\Psi = A\left(r - \frac{a^2}{r}\right)\sin$$

Apply B.C. 3

$$\Rightarrow \quad A = V_\infty, \quad B = -a^2 V_\infty$$

$$\psi = A \quad r - \frac{a^2}{r})\sin\theta$$

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = v_r = \frac{1}{r}\left[A\left(r - \frac{a^2}{r}\right)\right]\cos\theta$$

$$\frac{\partial\psi}{\partial r} = v_\theta = -A\left[1 + \frac{a^2}{r^2}\right]\sin\theta$$

$$v_x = v_r\cos\theta - v_\theta\sin\theta$$

$$= A\left[1 - \frac{a^2}{r^2}\right]\cos^2\theta + A\left[1 + \frac{a^2}{r^2}\right]\sin^2\theta$$

$$= A\,\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + A\frac{a^2}{r^2}(\cancel{\sin^2\theta} - \cos^2\theta)$$

$$V_\infty|_{r\to\infty} \to V_\infty$$

$$A = V_\infty$$