05/28 助教課

1. $\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dx\,dy = \int_0^1 \underbrace{\int_0^1 xy\sqrt{x^2+y^2}\,dx}_{\text{在固定 } y \text{ 下計算}}\,dy$

$$= \int_0^1 \left(\frac{1}{3} y (x^2+y^2)^{\frac{3}{2}} \Big|_{x=0}^{x=1} \right) dy$$

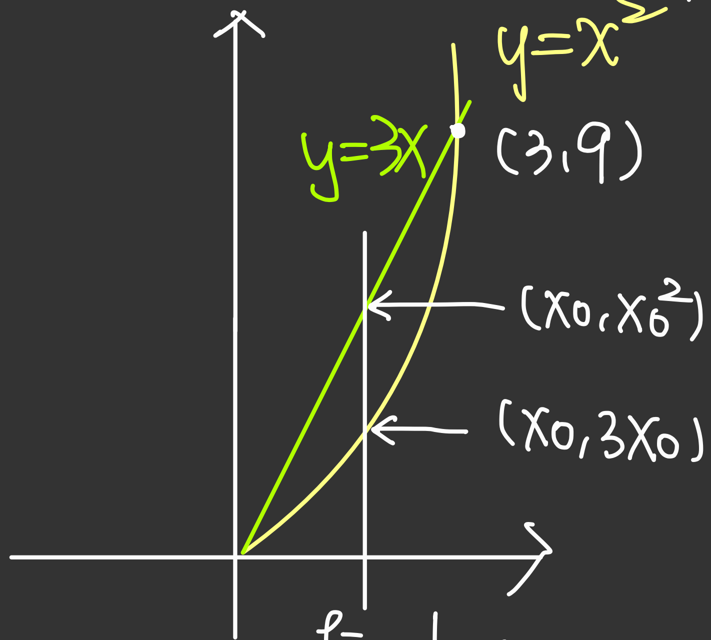
$$= \int_0^1 \left(\frac{1}{3} y (y^2+1)^{\frac{3}{2}} - \frac{1}{3} y^4 \right) dy$$

$$= \left(\frac{1}{15} (y^2+1)^{\frac{5}{2}} - \frac{1}{15} y^5 \right) \Big|_0^1$$

$$= \left(\frac{1}{15} \cdot 2^{\frac{5}{2}} - \frac{1}{15} \right) - \left(\frac{1}{15} \right)$$

$$= \frac{1}{15} \cdot 2^{\frac{5}{2}} - \frac{2}{15}$$

2. Evaluate $\iint_D xy\,dA$ where R is enclosed by $y=x^2$, $y=3x$



y between $3x_0$ and x_0^2

fixed x_0, $x_0 \in [0,3]$

$$D=\{(x,y) \mid 0 \le x \le 3,\ x^2 \le y \le 3x\}$$

$$\iint_D xy\,dA$$

$$= \int_0^3 \int_{x^2}^{3x} xy\,dy\,dx$$

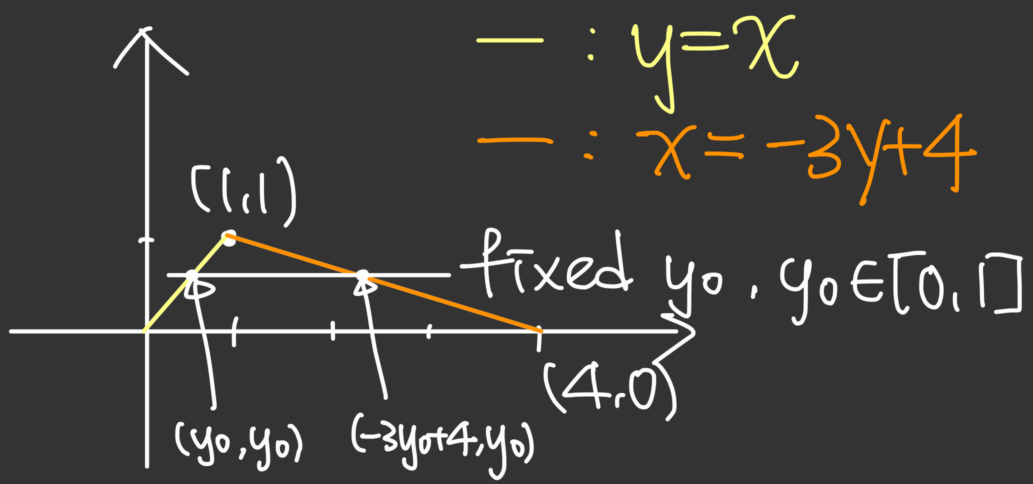
$$= \int_0^3 \left(\frac{1}{2}xy^2 \Big|_{x^2}^{3x}\right) dx$$

$$= \int_0^3 \left(\frac{9}{2}x^3 - \frac{1}{2}x^5\right) dx$$

$$= \left(\frac{9}{8}x^4 - \frac{1}{12}x^6\right)\Big|_0^3$$

$$= \frac{9}{8}\cdot 3^4 - \frac{1}{12}\cdot 3^6$$

3. Evaluate $\iint_D y\,dA$, D is enclosed by (0,0), (1,1), (4,0)



$$D=\{(x,y)\mid y\le x\le -3y+4,\ 0\le y\le 1\}$$

$$\iint_D y\,dA$$

$$=\int_0^1\int_y^{-3y+4} y\,dx\,dy$$

$$=\int_0^1\left(xy\ \Big|_{x=y}^{x=-3y+4}\right)dy$$

$$=\int_0^1(-3y^2+4y-y^2)\,dy$$

$$=\left(-\frac{4}{3}y^3+2y^2\right)\Big|_0^1$$

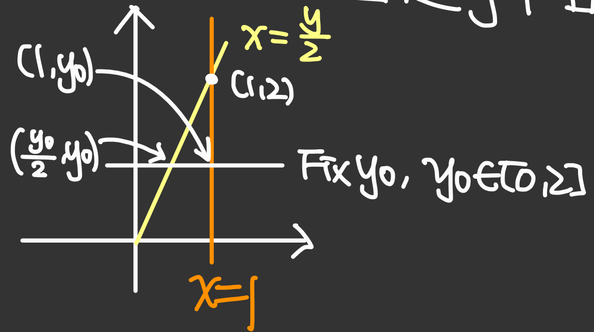
$$=\frac{2}{3}$$

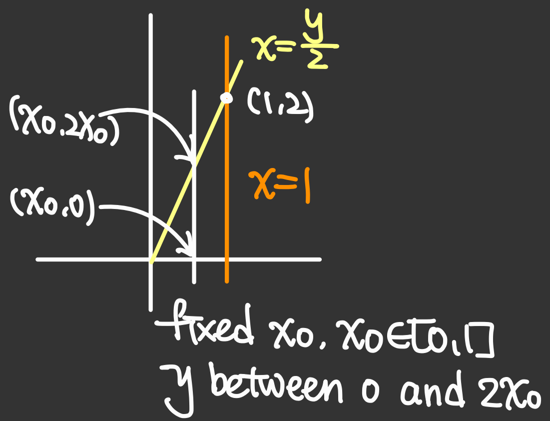
4. $\int_0^2 \int_{y/2}^1 y\cos(x^3-1)\,dx\,dy$

因為 $\int_{y/2}^1 y\cos(x^3-1)\,dx$ 我們不會算，所以我們先交換積分順序

$D=\{(x,y) \mid \frac{y}{2} \le x \le 1,\ 0 \le y \le 2\}$

在固定y下的範圍





$D=\{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 2x\}$

$$\int_0^2 \int_{y/2}^1 y\cos(x^3-1)\,dx\,dy$$

$$= \int_0^1 \int_0^{2x} y\cos(x^3-1)\,dy\,dx$$

$$= \int_0^1 \left(\frac{y^2}{2}\cos(x^3-1) \Big|_{y=0}^{y=2x} \right) dx$$

$$= \int_0^1 2x^2\cos(x^3-1)\,dx$$

$$= \frac{2}{3}\sin(x^3-1)\Big|_0^1$$

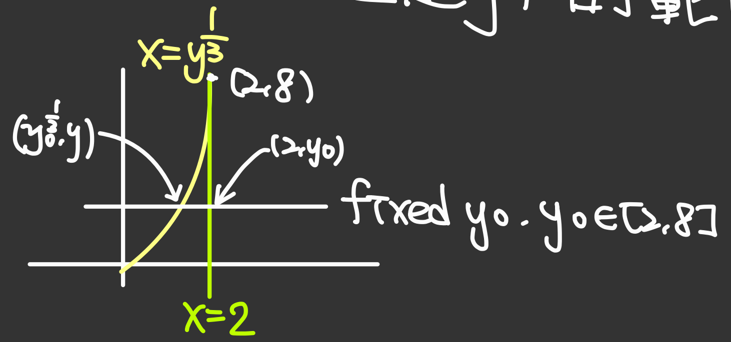
$$= -\frac{2}{3}\sin 1$$

5. $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy$

同理, $\int_{\sqrt[3]{y}}^2 e^{x^4}\,dx$ 不會計算, 所以先交換積分順序

$D=\{(x,y) \mid \sqrt[3]{y} \le x \le 2,\ 0 \le y \le 8\}$

在固定y下的範圍



$D=\{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le x^3\}$

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy$$

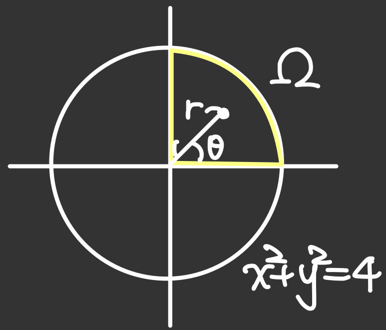
$$= \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$

$$= \int_0^2 x^3 e^{x^4}\,dx$$

$$= \frac{1}{4} e^{x^4} \Big|_0^2$$

$$= \frac{1}{4}e^{16} - \frac{1}{4}$$

6. $\iint_{\Omega} (2x-y)\,dA$. Ω in first quadrant enclosed by $\begin{array}{l} x^2+y^2=4 \\ x=0 \\ y=0 \end{array}$



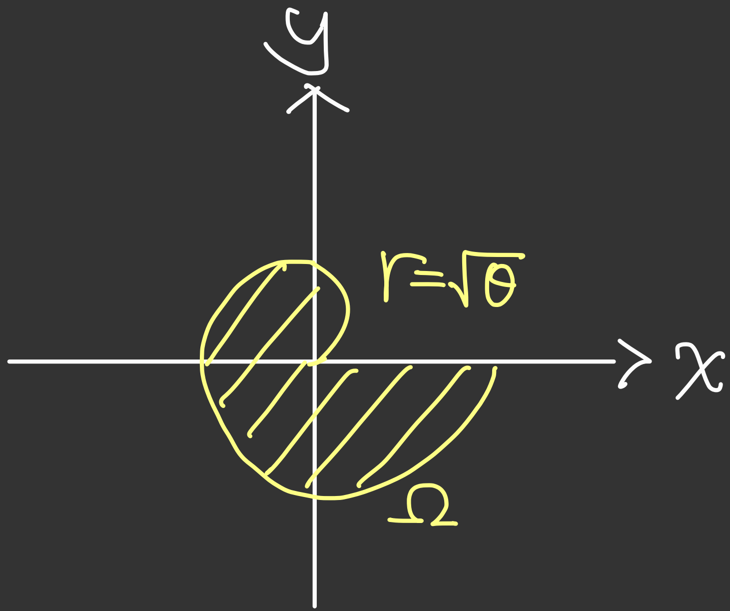
$$\tilde{\Omega} = \{(r.\theta) \mid (r\cos\theta, r\sin\theta) \in \Omega\}$$
$$= \{(r.\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le \tfrac{\pi}{2}\}$$

$$f(x,y) = 2x-y$$

$$g(r.\theta) = g(r\cos\theta, r\sin\theta)$$
$$= 2r\cos\theta - r\sin\theta$$

$$\iint_{\Omega} f(x,y)\,dx\,dy$$
$$= \iint_{\tilde{\Omega}} g(r.\theta)\cdot r\,dr\,d\theta$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{2} (2r\cos\theta - r\sin\theta)\cdot r\,dr\,d\theta$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{2} r^2(2\cos\theta - \sin\theta)\,dr\,d\theta$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{r^3}{3}(2\cos\theta - \sin\theta)\Big|_{r=0}^{r=2}\right) d\theta$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{16}{3}\cos\theta - \frac{8}{3}\sin\theta\right) d\theta$$
$$= \left(\frac{16}{3}\sin\theta + \frac{8}{3}\cos\theta\right)\Big|_0^{\frac{\pi}{2}}$$
$$= \frac{16}{3} - \frac{8}{3} = \frac{8}{3}$$

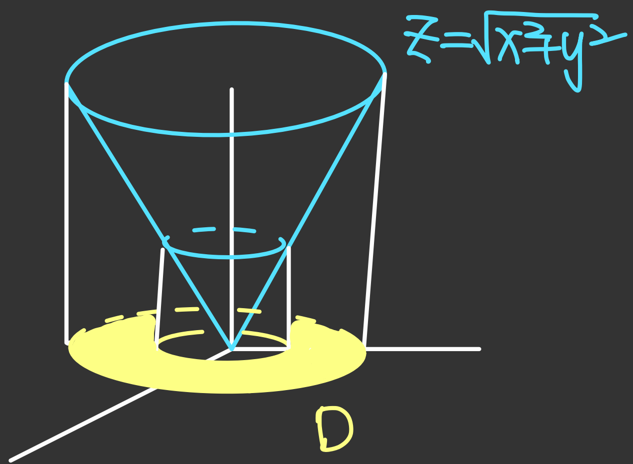
7. Find the area of D



$$\tilde{\Omega} = \{(r,\theta) \mid (r\cos\theta, r\sin\theta) \in \Omega\}$$
$$= \{(r,\theta) \mid 0 \le r \le \sqrt{\theta},\ 0 \le \theta \le 2\pi\}$$

$$\text{Area}(D) = \iint_D 1\, dA$$
$$= \int_0^{2\pi}\int_0^{\sqrt{\theta}} 1 \cdot r\,dr\,d\theta$$
$$= \int_0^{2\pi} \left(\frac{r^2}{2} \Big|_{r=0}^{r=\sqrt{\theta}} \right) d\theta$$
$$= \int_0^{2\pi} \frac{\theta}{2}\, d\theta$$
$$= \pi$$

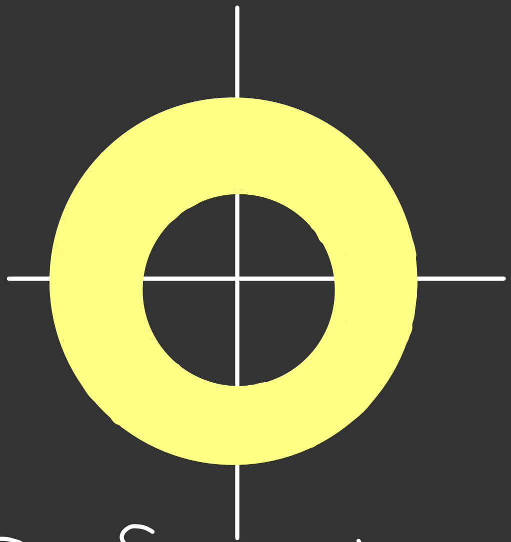
8. Find the volume of the solid below $z=x^2+y^2$ and above $1\le x^2+y^2\le 4$



$$z=f(x,y)=\sqrt{x^2+y^2}$$

$$g(r.\theta)=f(r\cos\theta, r\sin\theta)=r$$

$$\text{Volume}=\iint_{\underset{\text{底面}}{D}} \underset{\text{高}}{z}\, dA$$



$$=\int_0^{2\pi}\int_1^2 r\cdot r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left(\frac{1}{3}r^3\Big|_1^2\right)d\theta$$

$$=\int_0^{2\pi}\frac{7}{3}\,d\theta$$

$$=\frac{14}{3}\pi$$

$$\Omega=\{(x,y)\mid 1\le x^2+y^2\le 4\}$$

$$\tilde{\Omega}=\{(r.\theta)\mid 1\le r\le 2,\ 0\le\theta\le 2\pi\}$$

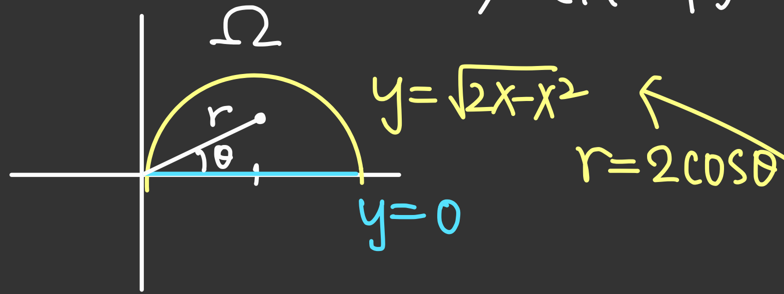
9. $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,dy\,dx$

$f(x,y)=\sqrt{x^2+y^2}$

$g(r,\theta)=\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}=r$

$y=\sqrt{2x-x^2}\Rightarrow y^2=2x-x^2$

$\Rightarrow (x-1)^2+y^2=1$



$(r\sin\theta)^2=2r\cos\theta-r^2\cos^2\theta$

$\Rightarrow r\sin^2\theta=2\cos\theta-r\cos^2\theta$

$\Rightarrow r=r(\sin^2\theta+\cos^2\theta)=2\cos\theta$

$\Omega=\{(r,\theta)\mid 0\le r\le 2\cos\theta,\ 0\le\theta\le\frac{\pi}{2}\}$

$\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,dy\,dx$

$\displaystyle=\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} r\cdot r\,dr\,d\theta$

$\displaystyle=\int_0^{\frac{\pi}{2}}\left(\frac{r^3}{3}\Big|_{r=0}^{r=2\cos\theta}\right)d\theta$

$\displaystyle=\int_0^{\frac{\pi}{2}}\frac{8}{3}\underbrace{\cos^3\theta}_{\cos\theta(1-\sin^2\theta)}\,d\theta$

$u=\sin\theta$

$du=\cos\theta\,d\theta$

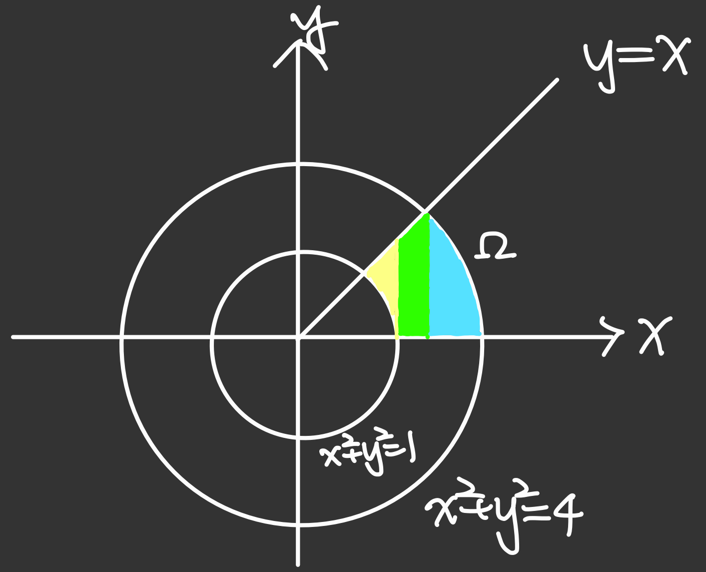
$\displaystyle\int_0^1\frac{8}{3}(1-u^2)\,du$

$\displaystyle=\frac{8}{3}\left(u-\frac{u^3}{3}\right)\Big|_0^1$

$\displaystyle=\frac{8}{3}\cdot\frac{2}{3}=\frac{16}{9}$

10. $\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx$

D_1 D_2 D_3



$\tilde{\Omega} = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{4}\}$

$f(x,y) = xy$

$g(r,\theta) = r^2\cos\theta\sin\theta$

$$原積分 = \int_0^{\frac{\pi}{4}}\int_1^2 (r^2\cos\theta\sin\theta)\cdot r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{r^4}{4}\cos\theta\sin\theta\right)\Big|_{r=1}^{r=2} d\theta$$

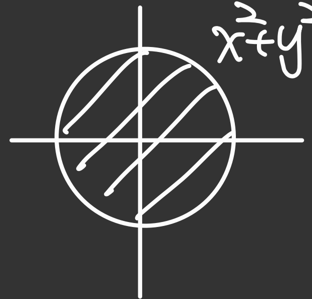
$$= \int_0^{\frac{\pi}{4}} \frac{15}{4}\cos\theta\sin\theta\, d\theta$$

$$= \frac{15}{8}\sin^2\theta \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{15}{16}$$

11. Let $I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx$

$$:= \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\,dA$$

$D_a =$ 

Show that $I = \pi$

$\widetilde{D_a} = \{(r,\theta) \mid 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$

$f(x,y) = e^{-(x^2+y^2)}$

$g(r,\theta) = e^{-r^2}$

$$\iint_{D_a} e^{-(x^2+y^2)}\,dA$$

$$= \int_0^{2\pi}\int_0^a e^{-r^2}\cdot r\,dr\,d\theta$$

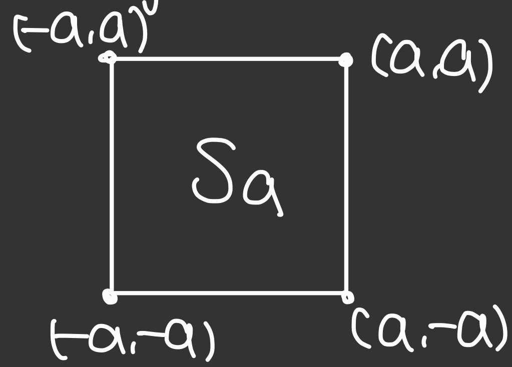
$$= \int_0^{2\pi}\left(-\frac{1}{2}e^{-r^2}\Big|_0^a\right)d\theta = \pi(1-e^{-a^2})$$

$$I = \lim_{a\to\infty} \pi(1-e^{-a^2})$$

$$= \pi$$

12. An equivalent definition of improper integral in 11.

is $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA := \lim_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}\,dA$



Show that $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$

$$\iint_{S_a} e^{-(x^2+y^2)}\,dA = \int_{-a}^{a}\int_{-a}^{a} e^{-x^2}\cdot \underbrace{e^{-y^2}}_{\text{independent of } x}\,dx\,dy = \int_{-a}^{a} e^{-y^2}\cdot \underbrace{\int_{-a}^{a} e^{-x^2}\,dx}_{\substack{\text{definite integral} \\ \text{the value} < +\infty}}\,dy$$

$$= \int_{-a}^{a} e^{-x^2}\,dx \cdot \int_{-a}^{a} e^{-y^2}\,dy = \left(\int_{-a}^{a} e^{-x^2}\,dx\right)^2$$

$$\pi = I = \lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)^2$$

First, we show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges

$$\text{and } \lim_{a\to\infty}\int_{-a}^{a} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

Note that $e^{-x^2} \le e^{-x}$ for $x \ge 1$

$$\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$$

$$\le \underbrace{\int_0^1 e^{-x^2}dx}_{\text{Definite Integral}} + \lim_{a\to\infty} \underbrace{\int_1^a e^{-x}dx}_{(1-e^{-a})} < +\infty$$

Note that $e^{-x^2} \le e^{x}$ for $x \le -1$

$$\int_{-\infty}^0 e^{-x^2}dx = \int_{-1}^0 e^{-x^2}dx + \int_{-\infty}^{-1} e^{-x^2}dx < +\infty.$$

Therefore, $\int_0^\infty e^{-x^2}dx$. $\int_{-\infty}^0 e^{-x^2}dx$. $\int_{-\infty}^\infty e^{-x^2}dx$ are convergent.

$$\lim_{a\to\infty} \int_{-a}^a e^{-x^2}dx \underset{e^{-x^2}\text{ is even}}{=} \lim_{a\to\infty} \left(2\int_0^a e^{-x^2}dx\right) = 2\lim_{a\to\infty}\int_0^a e^{-x^2}dx$$

$$\int_{-\infty}^\infty e^{-x^2}dx = \lim_{a\to\infty}\int_0^a e^{-x^2}dx + \underline{\lim_{b\to-\infty}\int_b^0 e^{-x^2}dx}$$

Let $x=-t$

$$= \lim_{a\to\infty}\int_0^a e^{-x^2}dx + \lim_{b\to-\infty}\int_0^{-b} e^{-t^2}dt = 2\lim_{a\to\infty}\int_0^a e^{-x^2}dx$$

Therefore, $\int_{-\infty}^{\infty} e^{-x^2} dx = \lim_{a\to\infty} \int_{-a}^{a} e^{-x^2} dx$.

$$\pi = \lim_{a\to\infty} \left(\int_{-a}^{a} e^{-x^2} dx\right)^2 = \left(\lim_{a\to\infty} \int_{-a}^{a} e^{-x^2} dx\right)^2$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \lim_{a\to\infty} \int_{-a}^{a} e^{-x^2} dx = \sqrt{\pi}$$

Let $t = \sqrt{2}x$. $dt = \sqrt{2}dt$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \lim_{a\to\infty} \int_{-a}^{a} e^{-\frac{x^2}{2}} dx$$

$$= \lim_{a\to\infty} \int_{-\sqrt{2}a}^{\sqrt{2}a} e^{-t^2} \cdot \sqrt{2} dt$$

$$= \sqrt{2} \cdot \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \sqrt{2} \cdot \sqrt{\pi} = \sqrt{2\pi}$$