Chap 9.

Application of divergence theorem to obtain differential forms of conservation of mass and conservation of momentum

$\iint \vec{F}\cdot\vec{n}\,dA = \iiint \nabla\cdot\vec{F}\,d$

piece-wise smooth surface, $\vec{F}$ different

(I) Conservation of mass

$$\underbrace{\iint_{c.s} \rho(\vec{v}\cdot\vec{n})\,dA}_{\iiint_{c.v.} \nabla\cdot(\rho\vec{v})\,dV} + \frac{\partial}{\partial t}\iiint_{c.v.} \rho\,dV = 0$$

$$\Rightarrow \iiint_{c.v.} \nabla\cdot(\rho\vec{v})\,dV + \iiint_{c.v.} \frac{\partial\rho}{\partial t}\,dV = 0$$

$$\Rightarrow \iiint_{c.v.} \left[\nabla\cdot(\rho\vec{v}) + \frac{\partial\rho}{\partial t}\right] dV = 0$$

$$\Rightarrow \underbrace{\nabla\cdot(\rho\vec{v})}_{\text{Net rate of mass efflux per unit volume}} + \underbrace{\frac{\partial\rho}{\partial t}}_{\text{rate of accumulation of mass per unit volume}} = 0 \quad (9\text{-}2)$$

$\rho\vec{v}$ = mass flu $\left(\frac{M}{L^2 t}\right)$

$$\Rightarrow \left[\vec{v}\cdot\nabla\rho + \rho(\nabla\cdot\vec{v})\right] + \frac{\partial\rho}{\partial t} = 0$$

$\underbrace{\frac{D\rho}{Dt}}_{\text{substantial derivative of }\rho} = \frac{\partial\rho}{\partial t} + \vec{v}\cdot\nabla$

$$\Rightarrow \frac{D\rho}{Dt} + \rho(\nabla\cdot\vec{v}) = 0 \quad (9\text{-}5)$$

Substantial derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$$

For example

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + \underline{v_x \frac{\partial P}{\partial x} + v_y \frac{\partial P}{\partial y} + v_z \frac{\partial P}{\partial z}}$$

$$= \frac{\partial P}{\partial t} + \underline{\vec{v} \cdot \nabla P}$$

local rate of change of pressure

rate of change of pressure due to fluid motion (conve

Total derivative

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \underbrace{\frac{dx}{dt}\frac{\partial P}{\partial x} + \frac{dy}{dt}\frac{\partial P}{\partial y} \frac{dz}{dt}\frac{\partial P}{\partial z}}_{\text{rate of change of pressure due to motion}}$$

To measure the rate change of pressure

(i) At a weather station $\frac{\partial P}{\partial t}$

(ii) ~~At~~ ~~a~~On an aircraft $\frac{dP}{dt}$

(iii) On a balloon $\frac{DP}{Dt}$

$$\nabla\cdot(\rho\vec{v})+\frac{\partial\rho}{\partial t}=0 \quad \text{or} \quad \frac{D\rho}{Dt}+\rho(\nabla\cdot\vec{v})=0$$

is called continuity eq.

(II) Conservation of momentum

$$\sum\vec{F}=\iint_{C.S}\rho\vec{v}(\vec{v}\cdot\vec{n})\,dA+\frac{\partial}{\partial t}\iiint_{C.V}\rho\vec{v}\,dV$$

$$\underbrace{\iint_{C.S.}\vec{\tau}\cdot\vec{n}\,dA+\iiint_{C.V}\rho\vec{g}\,dV}_{\text{sum of the external forces acting on the C.V}}=\underbrace{\iint_{C.S}\rho\vec{v}(\vec{v}\cdot\vec{n})\,dA}_{\text{net rate of linear momentum efflux from C.V.}}+\underbrace{\iiint_{C.V}\frac{\partial(\rho\vec{v})}{\partial t}\,dV}_{\text{time rate of change of linear momentum within the C.V.}}$$

$\vec{\tau}$: stresses (normal and shear)

momentum flux by viscous transfer

$\iiint_{C.V.}\nabla\cdot\vec{\tau}\,dV$ $\qquad$ $\iint_{C.S}\vec{\tau}\cdot\vec{n}\,dA$

Apply divergence theorem

$$\iiint_{C.V.}\left[\nabla\cdot\vec{\tau}\,\cancel{dV}+\rho\vec{g}\right]dV=\iiint_{C.V}\nabla(\rho\vec{v}\vec{v})\,dV+\iiint_{C.V}\frac{\partial(\rho\vec{v})}{\partial t}\,dV$$

$$\Rightarrow\ \nabla\cdot\vec{\tau}+\rho g=\nabla\cdot(\rho\vec{v}\vec{v})+\frac{\partial(\rho\vec{v})}{\partial t}$$

$$\nabla\cdot(\rho\vec{v}\vec{v}) = (\rho\vec{v})\cdot\nabla\vec{v}+(\nabla\cdot\rho\vec{v})\vec{v}$$

$$\Rightarrow \nabla\cdot\vec{\tau} + \rho\vec{g} = (\rho\vec{v})\cdot\nabla\vec{v} + (\nabla\cdot\rho\vec{v})\vec{v} + \underbrace{\frac{\partial(\rho\vec{v})}{\partial t}}_{\left(\frac{\partial\rho}{\partial t}\vec{v} + \rho\frac{\partial\vec{v}}{\partial t}\right.}$$

$$\Rightarrow \nabla\cdot\vec{\tau} + \rho\vec{g} = (\rho\vec{v})\cdot\nabla\vec{v} + (\nabla\cdot\rho\vec{v})\vec{v} + \frac{\partial\rho}{\partial t}\vec{v} + \rho\frac{\partial\vec{v}}{\partial t}$$

$$\Rightarrow \nabla\cdot\vec{\tau} + \rho\vec{g} = \rho\left(\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) + \underbrace{\left(\nabla\cdot\rho\vec{v} + \frac{\partial\rho}{\partial t}\right)}_{=0}\vec{v}$$

$$\Rightarrow \underbrace{\nabla\cdot\vec{\tau} + \rho\vec{g}}_{\frac{\sum\vec{F}}{V}} = \rho\frac{D\vec{v}}{Dt} \qquad (9\text{-}16)$$

$$= \frac{M}{V}\cdot\vec{a}$$ Newton's 2nd law of motio

where $$\frac{D\vec{v}}{Dt} = \underbrace{\frac{\partial\vec{v}}{\partial t}}_{\text{local change of } \vec{v}\text{, that is, local acceleration}} + \underbrace{\vec{v}\cdot\nabla\vec{v}}_{\text{rate of change in } \vec{v} \text{ due to fluid motion (convection), that is, convective acceleration}}$$

Strictly speaking, the only assumption for (9-16) is the continuity of fluid.

If the fluid applies Stokes's viscosity relati then gives eq. (9-17)

$$\rho\frac{Dv_x}{Dt} = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \qquad (9\text{-}16a)$$

$$\rho\frac{Dv_y}{Dt} = \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \qquad (9\text{-}16b)$$

$$\rho\frac{Dv_z}{Dt} = \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \qquad (9\text{-}16c)$$

For 9-16a, applying Stokes's viscosity relation

one

$$\rho\frac{Dv_x}{Dt} = \rho g_x + \cancel{\mu}\frac{\partial}{\partial x}\left[\left(2\boxed{\frac{\partial v_x}{\partial x}} - \frac{2}{3}(\nabla\cdot\vec{v})\right)\mu\right] - \frac{\partial P}{\partial x}$$

$$\cancel{\mu}\frac{\partial}{\partial y}\left[\mu\left(\boxed{\frac{\partial v_x}{\partial y}} + \frac{\partial v_y}{\partial x}\right)\right] + \cancel{\mu}\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_z}{\partial x} + \boxed{\frac{\partial v_x}{\partial z}}\right)\right]$$

$$\rho\frac{Dv_x}{Dt} = \rho g_x - \frac{\partial P}{\partial x} - \frac{\partial}{\partial x}\left(\frac{2}{3}\mu(\nabla\cdot\vec{v})\right)$$

$$+ \cancel{2\frac{\partial}{\partial x}\left(\mu\frac{\partial v_x}{\partial x}\right) +}$$

$$\nabla\cdot\left(\mu\frac{\partial \vec{v}}{\partial x}\right) + \nabla\cdot(\mu\nabla v_x) \qquad (9\text{-}17a)$$

same for (9-~~16~~17b) and (9-~~16~~17c)

Navier-Stokes eq.

(9-17)

If the fluid is incompressible (ρ = const), and const viscosity (μ = const)

then the continuity eq. becomes

$$\nabla \cdot \vec{v} = 0$$

and Navier-Stokes eq. becomes

$$\rho \frac{D\vec{v}}{Dt} = \rho \vec{g} - \nabla P + \mu \nabla^2 \vec{v} \qquad (9\text{-}18) \text{ or } (9\text{-}19)$$

Reynolds number

From
$$\rho \frac{\partial \vec{v}}{\partial t} + \underbrace{\rho \vec{v} \cdot \nabla \vec{v}}_{\rho \frac{v^2}{a}} = \rho \vec{g} - \nabla p + \underbrace{\mu \nabla^2 \vec{v}}_{\mu \frac{v}{a^2}}$$

$\rho \frac{v^2}{a}$: inertial force per unit volum; a — characteristic length

$\mu \frac{v}{a^2}$: viscous force per unit volume

$$Re = \frac{\text{inertial force}}{\text{viscous force}} = \frac{\rho \frac{v^2}{a} \cdot a^3}{\mu \frac{v}{a^2} \cdot a^3} = \frac{\rho v a}{\mu}$$

$$\underline{\nabla\cdot\vec{\tau}}+\rho\vec{g} = \underline{\nabla\cdot(\rho\vec{v}\vec{v})}+\frac{\partial(\rho\vec{v})}{\partial t}$$

rate of momentum gain by viscous transfer per unit volume

+

pressure force on element per unit volume

(or) ~~shear~~

Normal and shear forces acting on the C.V. per unit volume

+

pressure force on element per unit volume

$\cancel{\nabla\cdot(\rho\vec{v}\vec{v})}$

rate of momentum gain by convection per unit volume

rate of increase of momentum per unit volume

$$\underline{\nabla\cdot\vec{\tau}+\rho\vec{g}} = \rho\frac{D\vec{v}}{Dt} \quad (9\text{-}16)$$

$$\frac{\Sigma\vec{F}}{V} \qquad \frac{M}{V}\cdot\vec{a}$$

Newton's 2nd law of motion

$$(\underline{\mu\nabla^2\vec{v}}-\nabla P)+\rho\vec{g} = \rho\frac{D\vec{v}}{Dt} \quad \begin{pmatrix}9\text{-}18\\9\text{-}19\end{pmatrix}$$

9.4 A rotating solid sphere (p100-101)

Creeping flow

Assumption

1. The fluid : Newtonian and behaves as continuum
2. The flow : laminar, incompressible, steady-state, fully developed
3. No-slip condition applies.
4. Neglect gravity and pressure.

$$\vec{v} = (\overset{0}{\cancel{v_r}}, \overset{0}{\cancel{v_\theta}}, v_\phi)$$

laminar flow

$$\overline{\overline{\tau}} = \begin{bmatrix} \cancel{\sigma_{rr}} & \cancel{\tau_{r\theta}} & \tau_{r\phi} \\ \cancel{\tau_{\theta r}} & \cancel{\sigma_{\theta\theta}} & \tau_{\theta\phi} \\ \tau_{\phi r} & \tau_{\phi\theta} & \cancel{\sigma_{\phi\phi}} \end{bmatrix}$$

$$v_\phi(r, \theta, \overset{0}{\cancel{\phi}})$$

symmetric ~~symmetric~~

$$\tau_{\phi\theta} = \tau_{\theta\phi} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right)\right]$$

ϕ - direction $\quad 0 = \mu\nabla^2\vec{v}$

$$\tau_{\phi r} = \tau_{r\phi} = \cancel{\mu\left[\frac{1}{r\sin\theta}\right]}$$

$$\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$

$$0 = \mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_\phi}{\partial\theta}\right) - \frac{v_\phi}{r^2\sin^2\theta}\right)$$

Trial solution

$$v_\phi = f(r)\sin\theta$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 f'\sin\theta\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \cdot f\cos\theta\right) - \frac{f}{r^2\sin\theta}$$

$$0 = \frac{1}{r^2}\left[2rf'\sin\theta + r^2 f''\sin\theta\right] + \frac{f}{r^2\sin\theta}\left[\underbrace{\cos^2\theta - \sin^2\theta}_{1-2\sin^2\theta}\right] - \frac{f}{r^2\sin\theta}$$

$$0 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 f'\sin\theta\right) - \frac{2f\sin\theta}{r^2} \qquad \left(f' = \frac{df}{dr}\right)$$

$$0 = \frac{d}{dr}\left(r^2\frac{df}{dr}\right) - 2f$$

This is a Cauchy-typ eq.

the solution $f(r) = r^n$

$$\cancel{\frac{d}{dr}\left(r^{n+2}\right) - 2r^n}$$

$$\frac{d}{dr}\left(r^2 \cdot n r^{(n-1)}\right) - 2r^n = 0$$

$$n(n+1)\cdot r^n - 2r^n = 0$$

$$n^2 + n - 2 = 0$$

$$(n-1)(n+2) = 0$$

$$n = 1, -2$$

General solution $f(r) = C_1 r + \frac{C_2}{r^2}$

From B.C.s $v_\phi|_{r=R} = R\Omega\sin\theta$, $v_\phi|_{r\to\infty} = 0$

$$\Rightarrow v_\phi = \frac{R^3\Omega}{r^2}\sin\theta$$

shear stress

$$\tau_{r\phi} = \tau_{\phi r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right] = \mu(-3)\frac{R^3}{r}