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Fundamentals of Momentum, Heat, and Mass Transfer

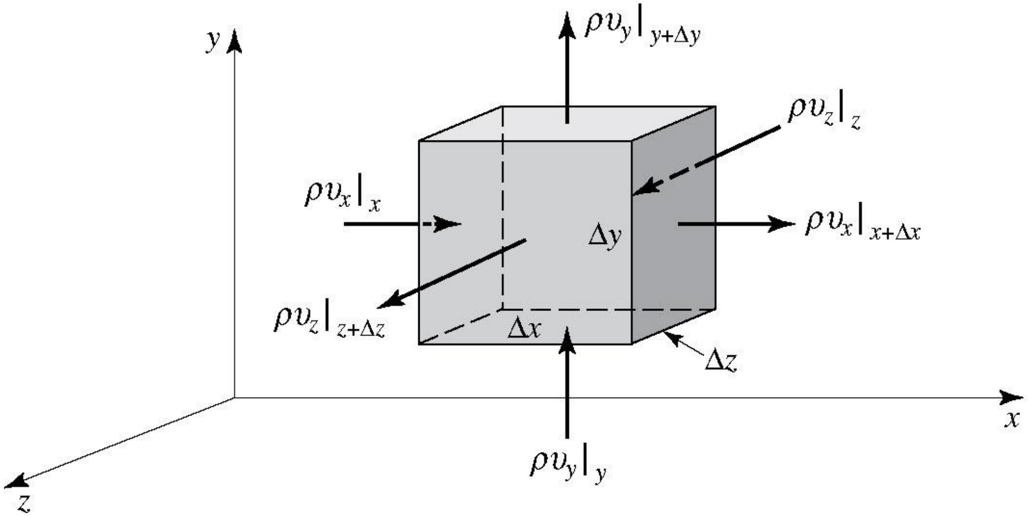
Sixth Edition

Welty • Rorrer • Foster

Chapter 9

Differential Equations of Fluid Flow

Conservation of Mass



$$\iint \rho(\mathbf{v} \cdot \mathbf{n})dA + \frac{\partial}{\partial t}\iiint \rho \, dv = 0 \qquad (4\text{-}1)$$

$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) + \frac{\partial \rho}{\partial t} = 0 \qquad (9\text{-}1)$$

$$\boxed{\nabla \cdot \rho\mathbf{v} + \frac{\partial \rho}{\partial t} = 0} \qquad (9\text{-}2)$$

Continuity equation

No assumption, only the fluid has to be continuous in the microscale.

Fig_9-1

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For an **incompressible** fluid (ρ = const),

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0 \tag{9-3}$$

Substantial derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + vx\,\frac{\partial}{\partial x} + vy\,\frac{\partial}{\partial y} + vz\,\frac{\partial}{\partial z} \tag{9-4}$$

Continuity equation

$$\frac{D\rho}{Dt} + \rho \boldsymbol{\nabla} \cdot \mathbf{v} = 0 \tag{9-5}$$

(2) $$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \frac{dx}{dt}\frac{\partial P}{\partial x} + \frac{dy}{dt}\frac{\partial P}{\partial y} + \frac{dz}{dt}\frac{\partial P}{\partial z} \qquad (9\text{-}6)$$

r' = v

$$\frac{dP}{dt} = \frac{DP}{Dt} = \underbrace{\frac{\partial P}{\partial t}}_{\substack{\text{local}\\ \text{rate of}\\ \text{change of}\\ \text{pressure}}} + \underbrace{v_x \frac{\partial P}{\partial t} + v_y \frac{\partial P}{\partial y} + v_z \frac{\partial P}{\partial z}}_{\substack{\text{rate of change}\\ \text{of pressure}\\ \text{due to motion}}} \qquad (9\text{-}7)$$

(3) **(1)**

= v·▼p

(1): local rate of change of pressure in **a weather station**
(2): rate of change of pressure on **an aircraft**
(3): rate of change of pressure on **a balloon**

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Navier-Stokes Equation

$$\Sigma \mathbf{F} = \iint \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) dA + \frac{\partial}{\partial t} \iiint \rho \mathbf{v} dV \qquad (5\text{-}4)$$

How to apply **divergence theorem** to derive Navier-Stokes equation?

$$\underbrace{\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\Sigma \mathbf{F}}{\Delta x \Delta y \Delta z}}_{①} = \underbrace{\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\iint \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) dA}{\Delta x \Delta y \Delta z}}_{②} + \underbrace{\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\partial / \partial t \iiint \rho \mathbf{v}\, dV}{\Delta x \Delta y \Delta z}}_{③} \qquad (9\text{-}8)$$

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\Sigma F_x}{\Delta x \Delta y \Delta z} = \boxed{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}} + \rho g_x \qquad (9\text{-}9)$$

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\Sigma F_y}{\Delta x \Delta y \Delta z} = \boxed{\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}} + \rho g_y \qquad (9\text{-}10)$$

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\Sigma F_z}{\Delta x \Delta y \Delta z} = \boxed{\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}} + \rho g_z \qquad (9\text{-}11)$$

Div τ

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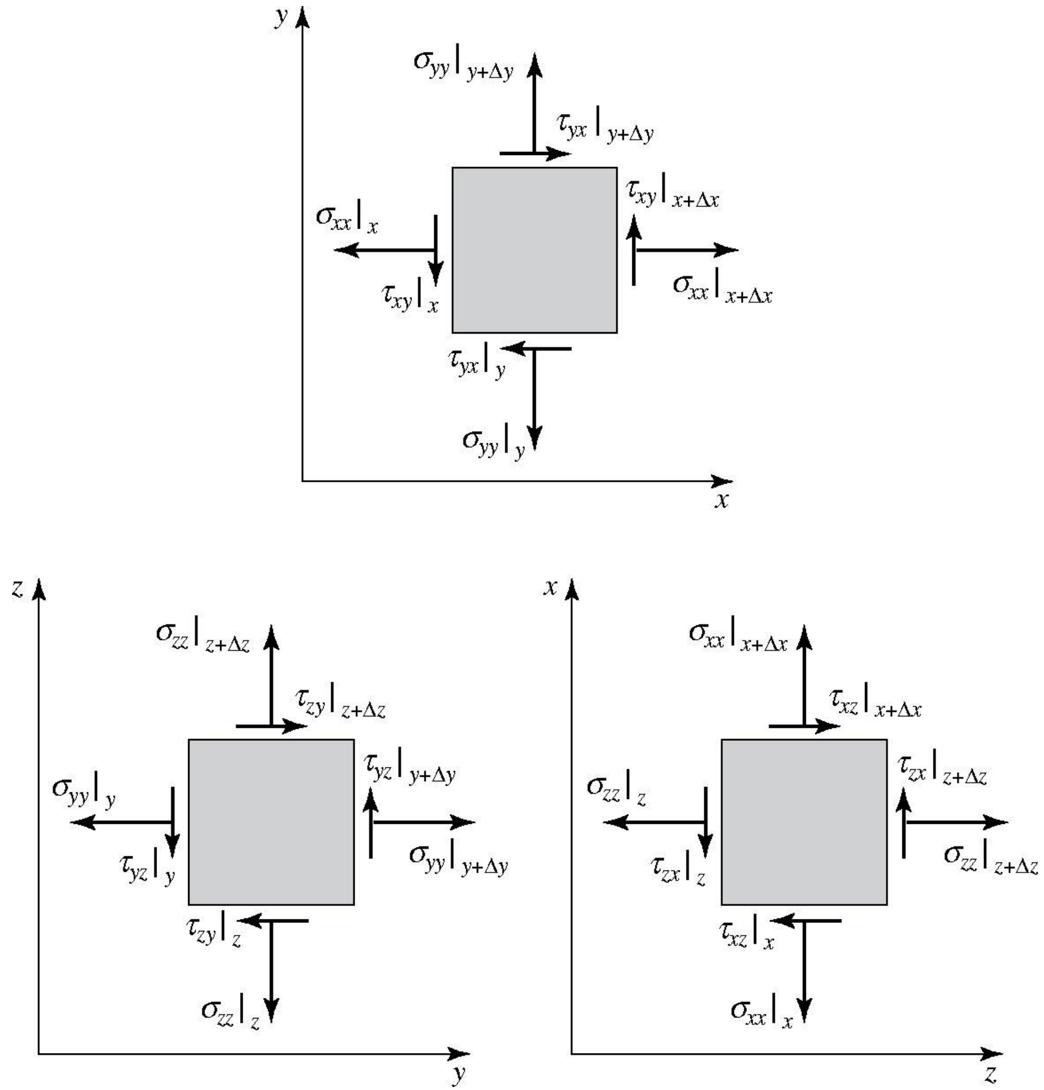
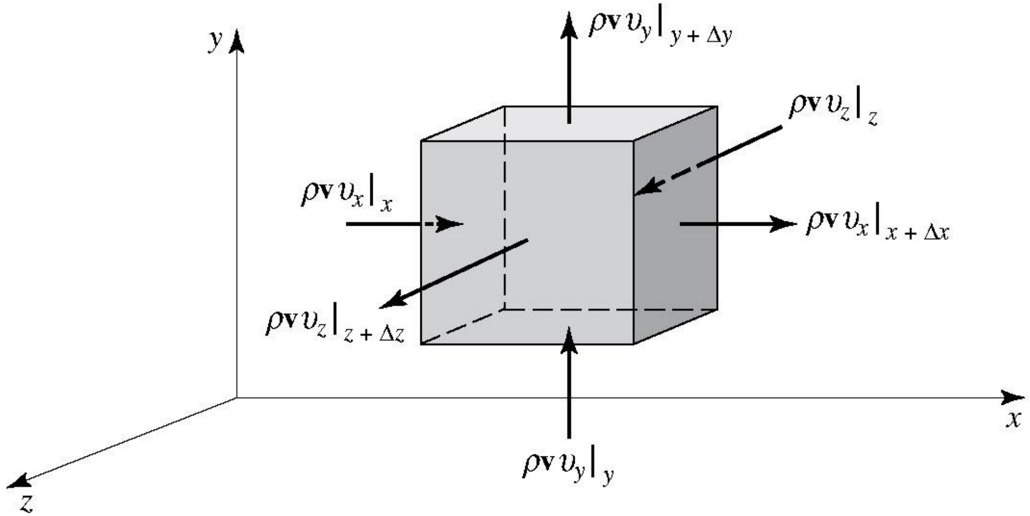

y
$\sigma_{yy}|_{y+\Delta y}$
$\tau_{yx}|_{y+\Delta y}$
$\tau_{xy}|_{x+\Delta x}$
$\sigma_{xx}|_{x}$
$\sigma_{xx}|_{x+\Delta x}$
$\tau_{xy}|_{x}$
$\tau_{yx}|_{y}$
$\sigma_{yy}|_{y}$
x
z
$\sigma_{zz}|_{z+\Delta z}$
$\tau_{zy}|_{z+\Delta z}$
$\tau_{yz}|_{y+\Delta y}$
$\sigma_{yy}|_{y}$
$\sigma_{yy}|_{y+\Delta y}$
$\tau_{yz}|_{y}$
$\tau_{zy}|_{z}$
$\sigma_{zz}|_{z}$
y
x
$\sigma_{xx}|_{x+\Delta x}$
$\tau_{xz}|_{x+\Delta x}$
$\tau_{zx}|_{z+\Delta z}$
$\sigma_{zz}|_{z}$
$\sigma_{zz}|_{z+\Delta z}$
$\tau_{zx}|_{z}$
$\tau_{xz}|_{x}$
$\sigma_{xx}|_{x}$
z

Fig_9-2



$$
\begin{aligned}
\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\iint \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) dA}{\Delta x \Delta y \Delta z} = \lim_{\Delta x, \Delta y, \Delta z \to 0} \Bigg[& \frac{(\rho \mathbf{v} v_x|_{x+\Delta x} - \rho \mathbf{v} v_x|_x) \Delta y \, \Delta z}{\Delta x \, \Delta y \, \Delta z} \\
& + \frac{(\rho \mathbf{v} v_y|_{y+\Delta y} - \rho \mathbf{v} v_y|_y) \Delta x \, \Delta z}{\Delta x \, \Delta y \, \Delta z} \\
& + \frac{(\rho \mathbf{v} v_z|_{z+\Delta z} - \rho \mathbf{v} v_z|_z) \Delta x \, \Delta y}{\Delta x \, \Delta y \, \Delta z} \\
= \quad & \frac{\partial}{\partial x}(\rho \mathbf{v} v_x) + \frac{\partial}{\partial y}(\rho \mathbf{v} v_y) + \frac{\partial}{\partial z}(\rho \mathbf{v} v_z)
\end{aligned} \tag{9-12}
$$

Fig_9-3

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$$\lim_{\Delta x,\Delta y,\Delta z \to 0} \frac{\iint \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) dA}{\Delta x \Delta y \Delta z} = -\mathbf{v}\frac{\partial \rho}{\partial t} + \rho\left[v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} + vz \frac{\partial \mathbf{v}}{\partial z}\right] \tag{9-13}$$

$$\lim_{\Delta x,\Delta y,\Delta z \to 0} \frac{\partial/\partial t \iiint \rho \mathbf{v} dV}{\Delta x \Delta y \Delta z} = \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial \rho}{\partial t} \tag{9-14}$$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{9-15a}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial x} + v_z \frac{\partial v_y}{\partial z}\right) = \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{9-15b}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial x} + v_z \frac{\partial v_z}{\partial z}\right) = \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \tag{9-15c}$$

$$\rho \frac{Dv_x}{Dt} = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{9-16a}$$

$$\rho \frac{Dv_y}{Dt} = \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{9-16b}$$

$$\rho \frac{Dv_z}{Dt} = \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \tag{9-16c}$$

$$\rho \frac{Dv_x}{Dt} = \rho g_x - \frac{\partial P}{\partial x} - \frac{\partial}{\partial x}\left(\frac{2}{3}\mu \boldsymbol{\nabla} \cdot \mathbf{v}\right) + \boldsymbol{\nabla} \cdot \left(\mu \frac{\partial \mathbf{v}}{\partial x}\right) + \boldsymbol{\nabla} \cdot (\mu \boldsymbol{\nabla} v_x) \tag{9-17a}$$

$$\rho \frac{Dv_y}{Dt} = \rho g_y - \frac{\partial P}{\partial x} - \frac{\partial}{\partial y}\left(\frac{2}{3}\mu \boldsymbol{\nabla} \cdot \mathbf{v}\right) + \boldsymbol{\nabla} \cdot \left(\mu \frac{\partial \mathbf{v}}{\partial y}\right) + \boldsymbol{\nabla} \cdot (\mu \boldsymbol{\nabla} v_y) \tag{9-17b}$$

$$\rho \frac{Dv_z}{Dt} = \rho g_z - \frac{\partial P}{\partial z} - \frac{\partial}{\partial z}\left(\frac{2}{3}\mu \boldsymbol{\nabla} \cdot \mathbf{v}\right) + \boldsymbol{\nabla} \cdot \left(\mu \frac{\partial \mathbf{v}}{\partial z}\right) + \boldsymbol{\nabla} \cdot (\mu \boldsymbol{\nabla} v_z) \tag{9-17c}$$

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For an **incompressible** fluid (ρ = const) with constant viscosity,

$$\rho \frac{Dv_x}{Dt} = \rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) \quad \text{(9-18a)}$$

$$\rho \frac{Dv_y}{Dt} = \rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) \quad \text{(9-18b)}$$

$$\rho \frac{Dv_z}{Dt} = \rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) \quad \text{(9-18c)}$$

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla P + \mu \nabla^2 \mathbf{v} \quad \text{(9-19)}$$

For an **inviscid** fluid ($\mu = 0$) or an **irrotational** flow (curl **v = 0**),

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla P \quad \text{(9-20)}$$

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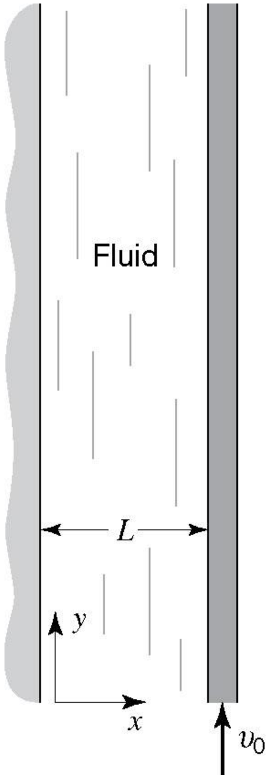

$$v_y = \underbrace{\frac{1}{2\mu}\left\{-\rho g - \frac{dP}{dy}\right\}\{Lx - x^2\}}_{①} + \underbrace{v_0 \frac{x}{L}}_{②} \tag{9-21}$$

Fig_9-4

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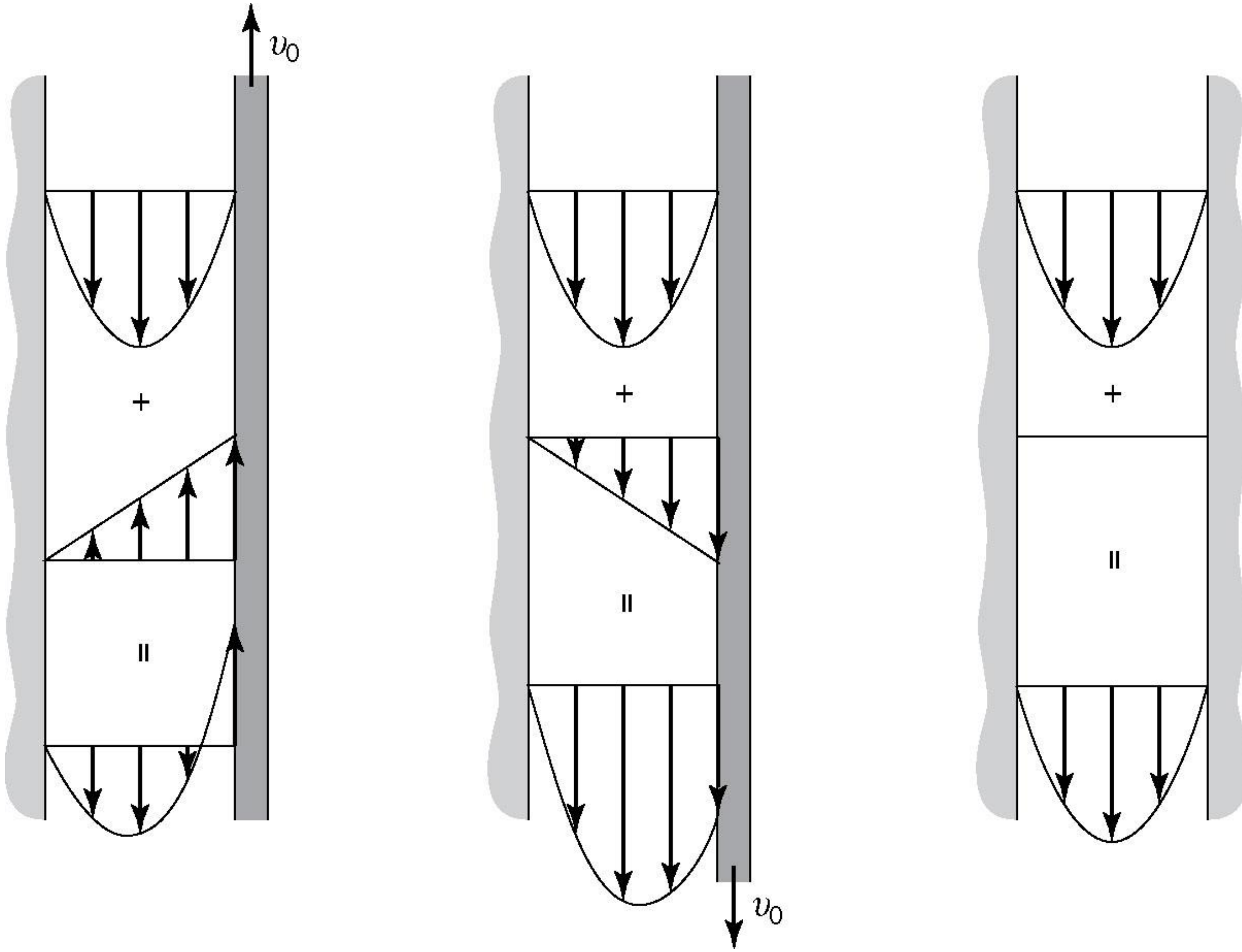

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Fig_9-5

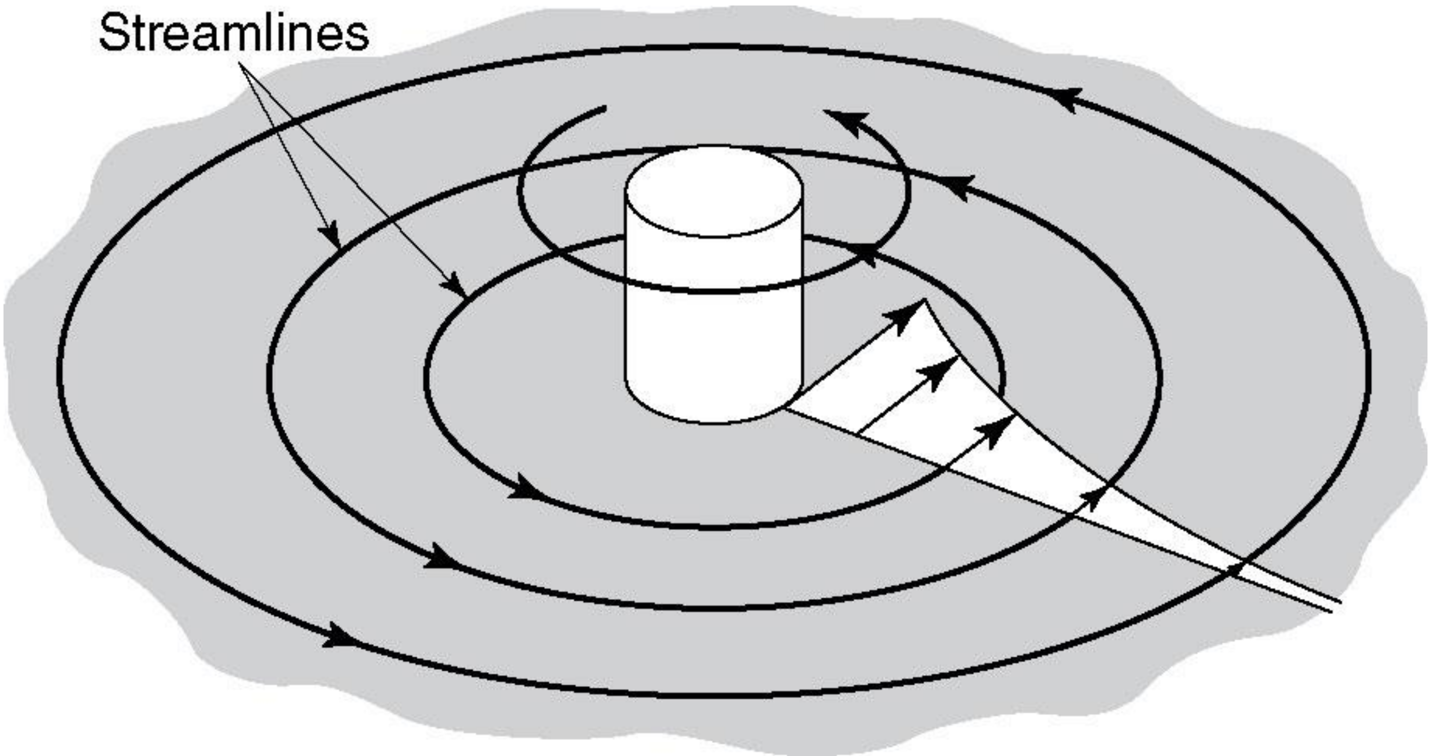

Fig_9-6

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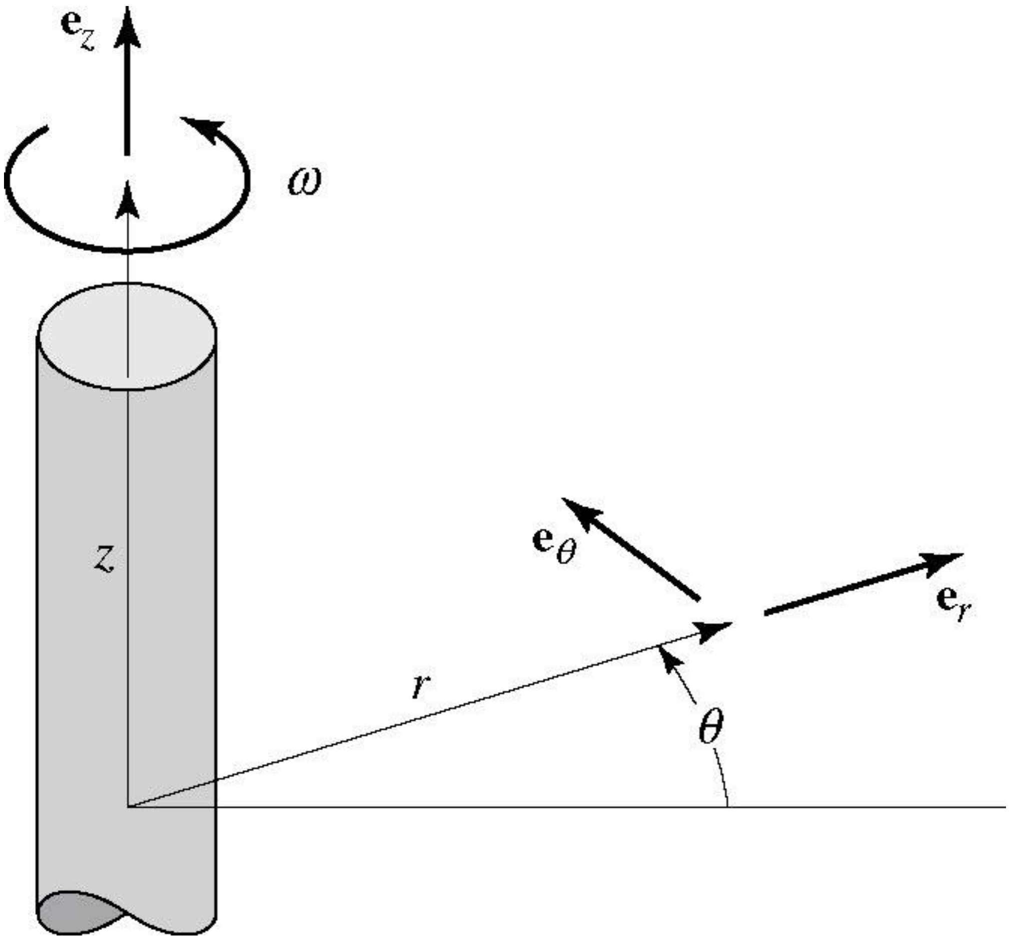

Fig_9-7

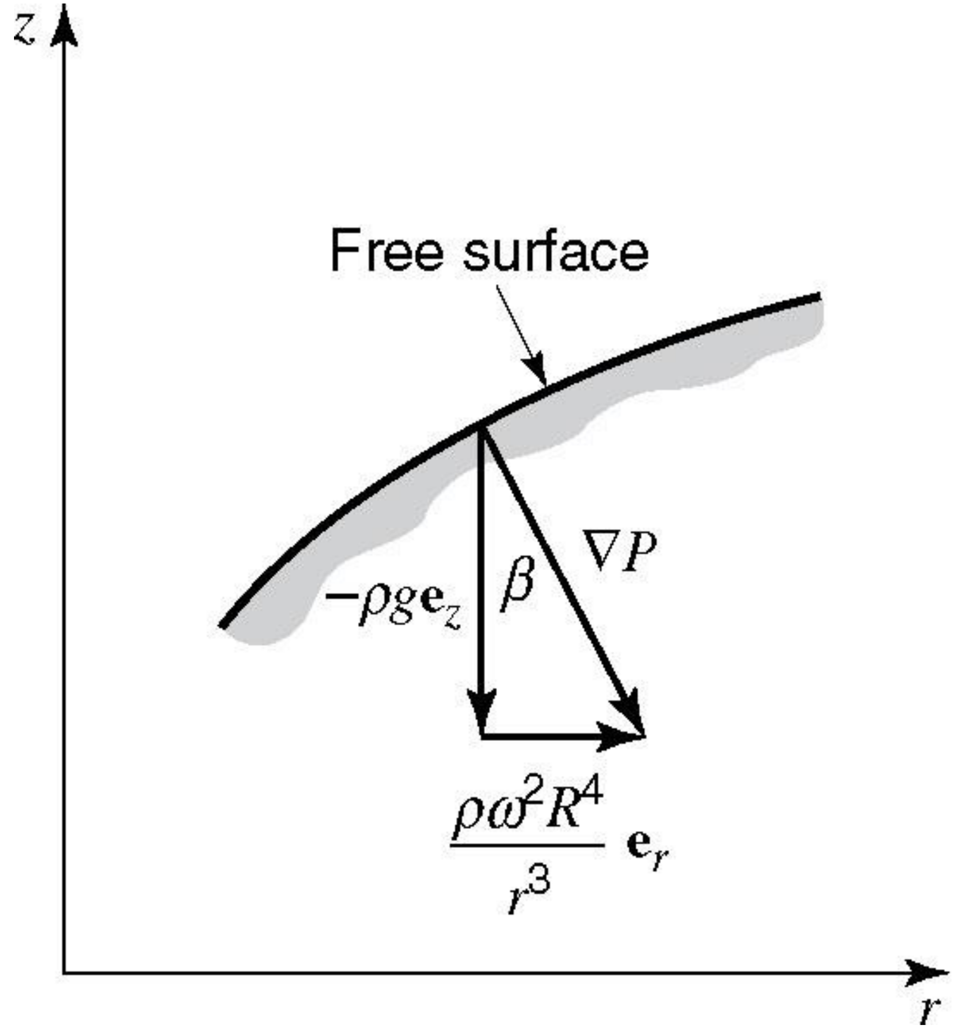
z
Free surface
∇P
β
$-\rho g \mathbf{e}_z$
$\dfrac{\rho\omega^2 R^4}{r^3}\,\mathbf{e}_r$
r

Fig_9-8

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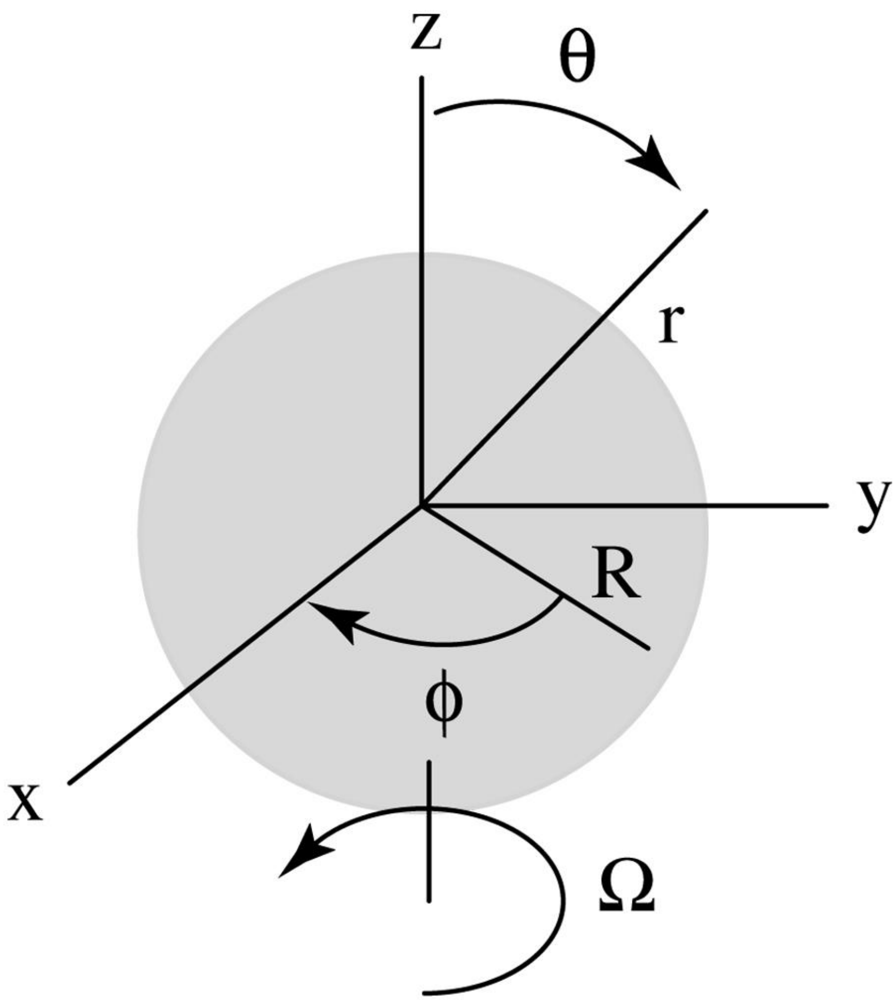
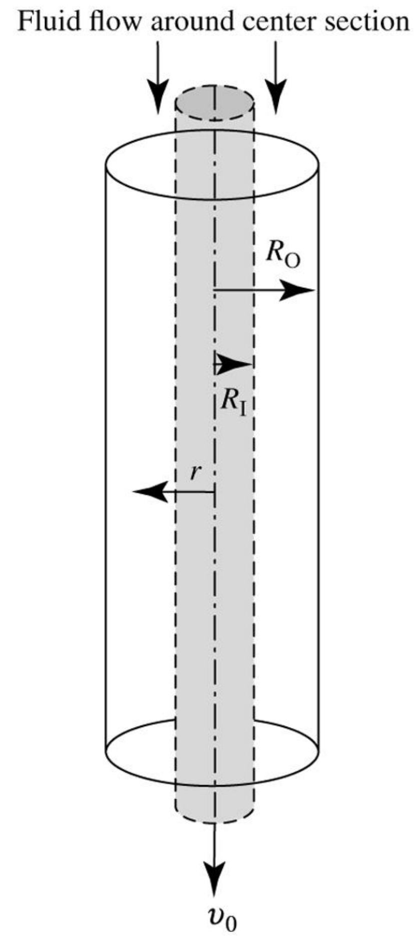
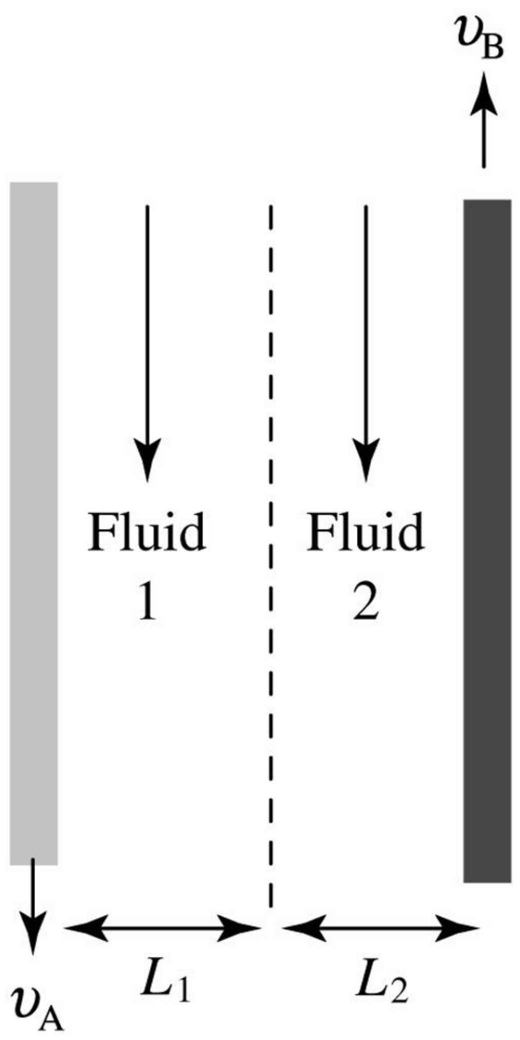

z
θ
r
y
R
x
ϕ
Ω


Fluid flow around center section
R_O
R_I
r
v_0



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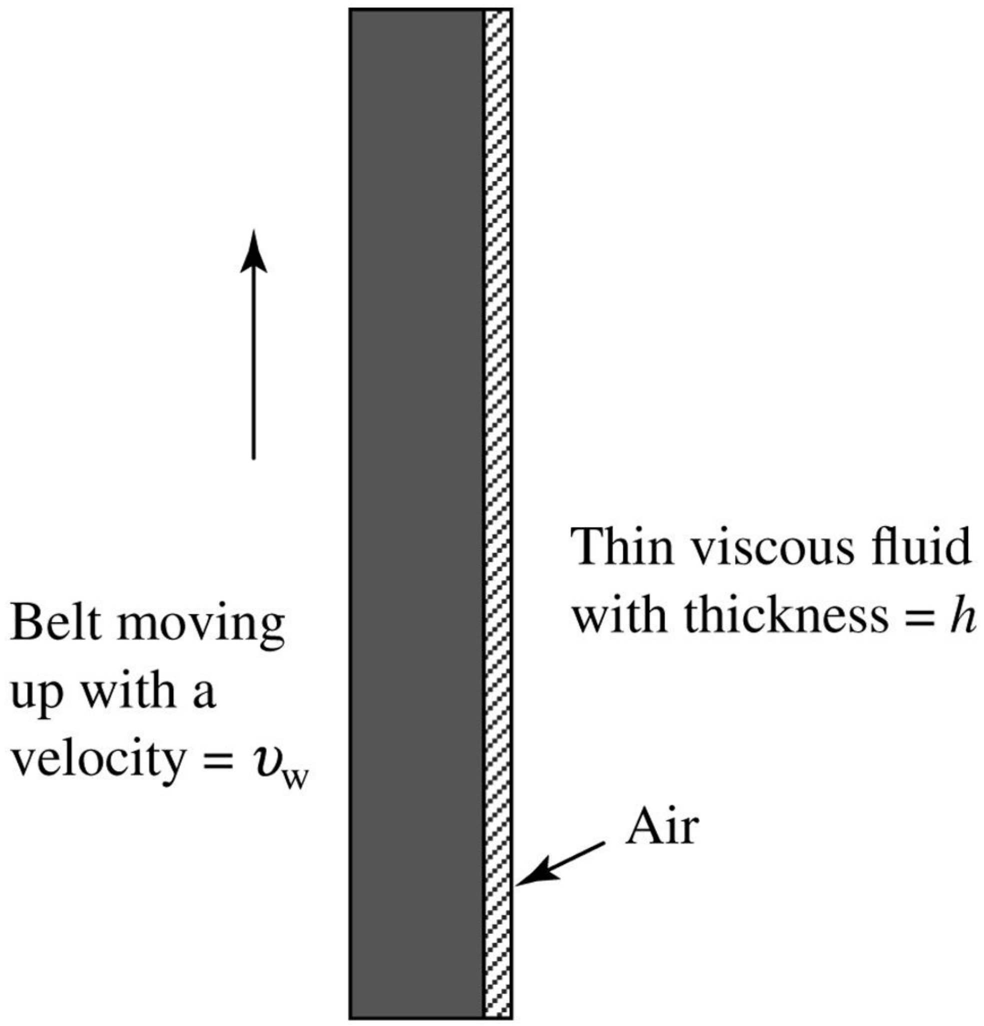
Belt moving
up with a
velocity = v_w
Thin viscous fluid
with thickness = h
Air