Unit Operation I Exam II

May, 22, 2011

1. the Navier-Stokes equation

$$\rho \frac{D\vec{v}}{Dt} = \rho \vec{g} - \nabla P + \mu \nabla^2 \vec{v}$$

(a) incompressible, μ is const (or Newtonian fluid)

(b) $\mu \nabla^2 \vec{v}$

(c) Newtonian's Second Law of Motion

$F = ma$

2. (a) $\boxed{\dfrac{\text{inertial force}}{\text{viscous force}}}$ If the inertial force dominates, the flow is turbulent. If the inertial force can be neglected, the flow is laminar.

(b) irrotional flow

(c) acceleration

3. (a) kg/m·s , g/cm·s (M/Lt)

(b) kg/m·s², g/cm·s² (M/Lt²)

(c) 1/s

(d) 1/s

4、 a, c, d

5、 (a) $\nabla \cdot \vec{v} = 0$ (2分)

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$$z+1+0-\cancel{4}yz+1=0$$

$$-yz+z+2=0$$ ※

(b) $\nabla \times \vec{v} = 0$ (2分)

$$\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = 0 \Rightarrow -\frac{1}{2}z^2 - 1 = 0$$

$$\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} = 0 \Rightarrow x = 0$$

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0 \Rightarrow 0 = 0$$

6 (a) The volumetric flow rate

$$\boxed{Q = \int_0^{w=1}\int_0^{L=5} v_x \, dy \, dz}$$ (2分)

$$= wL \cdot V_{avg}$$

$$= \frac{\rho g w L^3 \sin\theta}{3\mu}$$

$$=$$

(b) $$Re = \frac{4L V_{avg} \rho}{\mu}$$

$$= \frac{4L\rho}{\mu}\left(\frac{Q}{wL}\right)$$ 算錯給2分

$$=$$

7. (a)

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

where

$$\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) = \frac{\partial}{\partial r}\left\{ -V_\infty\left[\frac{1}{r} - \frac{3}{4}\frac{R}{r^2} - \frac{1}{4}\frac{R^3}{r^4}\right]\sin\theta \right\}$$

$$= -V_\infty\left[-\frac{1}{r^2} + \frac{3}{2}\frac{R}{r^3} + \frac{R^3}{r^5}\right]\sin\theta$$

$$\Rightarrow \tau_{r\theta} = \tau_{\theta r} = \mu\left[-V_\infty\left(-\frac{1}{r} + \frac{3}{2}\frac{R}{r^2} + \frac{R^3}{r^4}\right)\sin\theta \right.$$

$$\left. - V_\infty\left(\frac{1}{r} - \frac{3}{2}\frac{R}{r^2} + \frac{1}{2}\frac{R^3}{r^4}\right)\sin\theta \right]$$

$$= -\frac{3}{2}\mu\frac{V_\infty R^3}{r^4}\sin\theta \quad (3分)$$

$$\tau_{\phi\theta} = \tau_{\theta\phi} = 0$$

(2分)

$$\tau_{\phi r} = \tau_{r\phi} = 0$$

(b) $$\tau_{r\theta}\big|_{r=R,\theta=45^\circ} = \tau_{\theta r}\big|_{r=R,\theta=45^\circ} = -\frac{3}{2}\mu\cdot\frac{1}{R}\sin 45^\circ$$

$$= -\frac{3}{2}\frac{\mu}{R}V_\infty \times\frac{\sqrt{2}}{2}$$

$$\tau_{r\theta}\big|_{r=R,\theta=0^\circ} = \tau_{\theta r}\big|_{r=R,\theta=45^\circ} = 0$$

7. (c)

$$F_z = \int_0^{2\pi}\int_0^{\pi} (+\tau_{r\theta}|_{r=R}\sin\theta) R^2 \sin\theta \, d\theta \, d\phi$$

$$= -2\pi \int_0^{\pi} \left[\frac{3}{2}\mu \frac{V_\infty}{R} \sin^3\theta \right] R^2 \, d\theta$$

$$= -3\pi\mu V_\infty R \int_0^{\pi} \sin^3\theta \, d\theta$$

$$= -3\pi\mu V_\infty R \cdot \left[-\cos\theta + \frac{1}{3}\cos^3\theta \right]_0^{\pi}$$ (3分)

$$= -3\pi\mu V_\infty R \cdot \left[+2 - \frac{2}{3} \right]$$

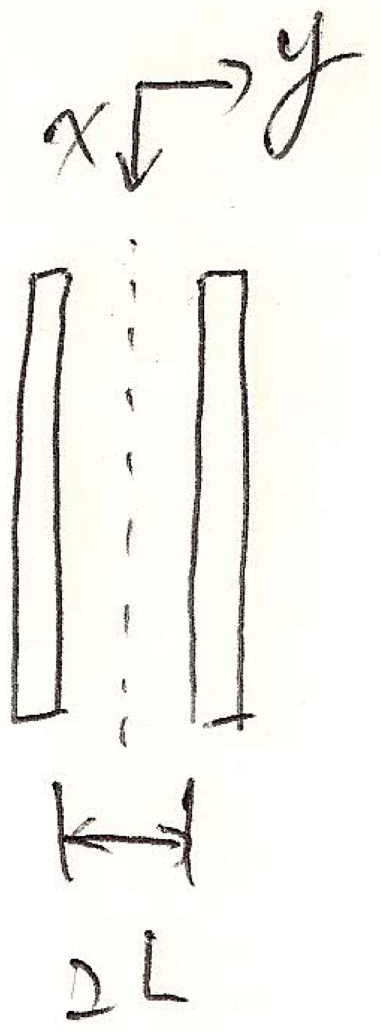
$$= -4\pi\mu R V_\infty$$

The z-component of shear force exerting on the sphere

$F_z = 4\pi\mu R V_\infty$ in the positive direction

方向寫錯扣 2分

8. (a)



$\boxed{\text{Solution I}}$

if $x=0$ is in the center.

~~For~~ Because the flow is laminar,

$v_x = v_x(x, y)$ $\quad v_x$ is not function of z

$v_y = 0$

$v_z = 0$

Because ρ is const. and the flow is steady, the continuity equation becomes

$$\nabla \cdot \vec{v} = 0$$

$$\Rightarrow \frac{\partial v_x}{\partial x} = 0 \quad v_x \text{ is not function of } x$$

(2分)

From Navier-Stokes equation

$$x\text{-direction} \quad 0 = \rho g + \mu \frac{d^2 v_x}{dy^2}$$

B.C.s

① $y = L$ $\quad v_x = 0$

② $y = -L$ $\quad v_x = 0$

③ $y = 0$ $\quad v_x = V_{max}, \ \tau_{yx} = 0$

(4分)

8 (a) (continuous)

$$\frac{d^2 \hat{V}_x}{dy^2} = -\frac{\rho g}{\mu}$$

$$\int d\left(\frac{d\hat{V}_x}{dy}\right) = -\frac{\rho g}{\mu}\int dy$$

$$\frac{d\hat{V}_x}{dy} = -\frac{\rho g}{\mu}y + C_1$$

$$\int d\hat{V}_x = \int\left[-\frac{\rho g}{\mu}y + C_1\right]dy$$

$$V_x = -\frac{\rho g}{2\mu}y^2 + C_1 y + C_2$$

From B.C. 1 and 2

$$0 = -\frac{\rho g}{2\mu}L^2 + C_1 L + C_2 \quad ①$$

$$0 = -\frac{\rho g}{2\mu}L^2 - C_1 L + C_2 \quad ②$$

① − ② :

$C_1 = 0$ substitute into eq ①

$$\Rightarrow C_2 = \frac{\rho g}{2\mu}L^2$$

$$\therefore \hat{V}_x = \frac{\rho g}{2\mu}\left[L^2 - y^2\right] \#$$

(b) $V_{max} = \hat{V}_x\big|_{y=0} = \frac{\rho g L^2}{2\mu}$ (2分)

f (b) (cont's)

$$V_{avg} = \frac{\int_0^w \int_{-L}^{L} V_x \, dy \, dz}{\int_0^w \int_{-L}^{L} dy \, dz} \qquad (2分)$$

$$= \frac{1}{2L} \int_{-L}^{L} \frac{\rho g}{2\mu} [L^2 - y^2] \, dy$$

$$= \frac{\rho g}{4\mu L} \left[L^2 y - \frac{1}{3} y^3 \right]_{-L}^{L}$$

$$= \frac{\rho g}{4\mu L} \left(2L^3 - \frac{1}{3} \cdot 2L^3 \right)$$

$$= \frac{\rho g}{\cancel{3}4\mu L} \cdot \frac{4}{3} L^3$$

$$= \frac{\rho g L^2}{3\mu}$$

$$\frac{V_{avg}}{V_{max}} = \frac{\frac{\rho g L^2}{3\mu}}{\frac{\rho g L^2}{2\mu}} = \frac{2}{3}$$

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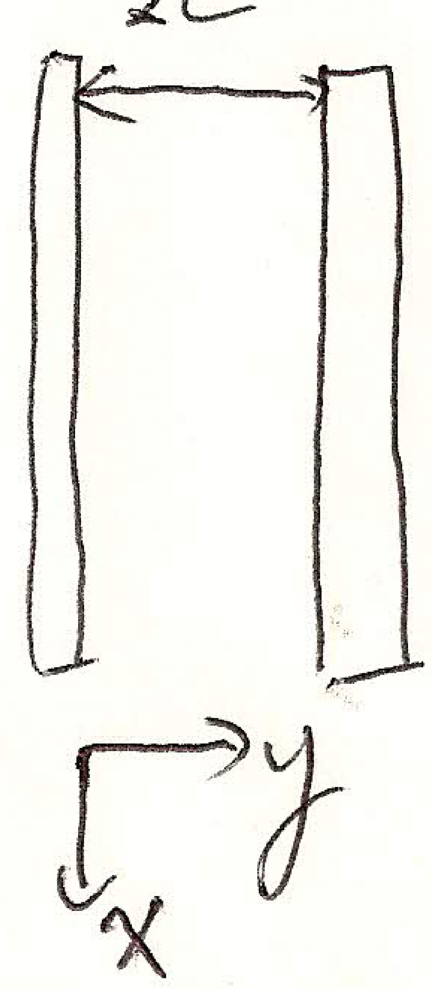
f (a) $\boxed{\text{Solution II}}$



if $x = 0$ is on the wall

f.

~~From~~ The only difference

B.C.s

1. $y=0$, $v_x=0$

2. $y=2L$, $v_x=0$

3. $y=L$, $v_x=V_{max}$, $\tau_{yx}=0$

$$v_x=-\frac{\rho g}{2\mu}y^2+C_1y+C_2$$

From B.C. 1 and 2.

$$C_2=0$$

$$0=-\frac{2\rho gL^2}{\mu}+2LC_1+C_2$$

$$\Rightarrow C_1=\frac{\rho gL}{\mu}$$

$$\therefore v_x=-\frac{\rho g}{2\mu}y^2+\frac{\rho gL}{\mu}y$$ #

& (b) You can derive

$$\frac{V_{avg}}{V_{max}}=\frac{2}{3}$$

9. (a)

$$\begin{cases} \dfrac{\partial \phi}{\partial r} = \left(A - \dfrac{2B}{r^3}\right)\cos\theta \\[2ex] \dfrac{\partial \phi}{\partial \theta} = -\left(Ar + \dfrac{B}{r^2}\right)\sin\theta \end{cases}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = \frac{\partial}{\partial r}\left[\left(Ar^2 - \frac{2B}{r}\right)\cos\theta\right]$$

$$= \left(2Ar + \frac{2B}{r^2}\right)\cos\theta$$

$$\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left[-\left(Ar + \frac{B}{r^2}\right)\sin^2\theta\right]$$

$$= -\left(Ar + \frac{B}{r^2}\right)\cdot 2\sin\theta\cos\theta \quad (2分)$$

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right)$$

$$= \frac{1}{r^2}\cdot\left(2Ar + \frac{2B}{r^2}\right)\cos\theta + \frac{1}{r^2 \cancel{\sin\theta}}\left[-\left(Ar + \frac{B}{r^2}\right)\cdot 2\cancel{\sin\theta}\cos\theta\right]$$

$$= 2\left(\frac{A}{r} + \frac{B}{r^4}\right)\cos\theta - 2\left(\frac{A}{r} + \frac{B}{r^4}\right)\cos\theta$$

$$= 0$$

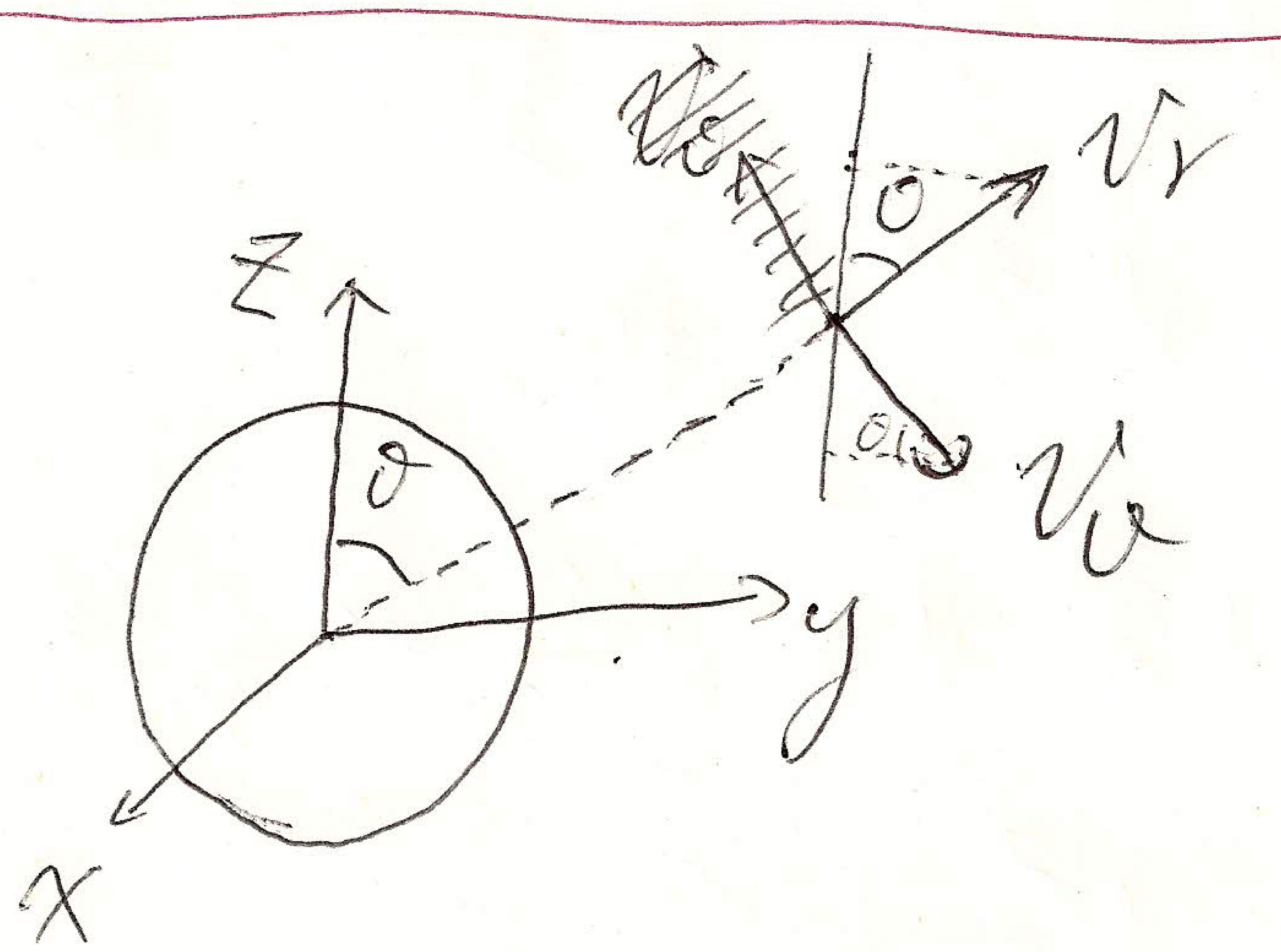
$\therefore$ The solution satisfies ~~the~~ $\nabla^2 \phi = 0$

4 (b) B.C.s'

1. $\underline{r = a, \quad v_r = 0, \quad v_\theta = ?}$

(3分)

2. $\underline{r \to \infty, \quad v_z \to v_\infty}$ $v_r \to v_\infty$



$$\underline{v_z = v_r \cos\theta - v_\theta \sin\theta}$$

$$v_r = -\frac{\partial \phi}{\partial r} = -\left[A - \frac{2B}{r^3}\right]\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{1}{r}\left[Ar + \frac{B}{r^2}\right](-\sin\theta)$$

$$= \left[A + \frac{B}{r^3}\right]\sin\theta$$

$$\Rightarrow \boxed{v_z = v_r \cos\theta - v_\theta \sin\theta}$$

$$= \left[-A + \frac{2B}{r^3}\right]\cos^2\theta - \left[A + \frac{B}{r^3}\right]\sin^2\theta$$

$$= -A(\cos^2\theta + \sin^2\theta) + \frac{B}{r^3}(2\cos^2\theta - \sin^2\theta)$$

From B.C. 1 and 2

$$0 = \left(A - \frac{2B}{a^3}\right)\cos\theta$$

$$-A = V_\infty$$

$$\therefore A = -V_\infty$$

$$B = -\frac{a^3 V_\infty}{2}$$

$$\Rightarrow \begin{cases} v_r = \left[V_\infty - V_\infty a^3 \cdot \frac{1}{r^3}\right]\cos\theta = V_\infty\left[1 - \left(\frac{a}{r}\right)^3\right]\cos\theta \\ v_\theta = \left[-V_\infty - \frac{a^3 V_\infty}{2}\frac{1}{r^3}\right]\sin\theta = -V_\infty\left[1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right]\sin\theta \end{cases}$$

※

(c) $r = a$

$$v_r|_{r=a} = 0$$

$$v_\theta|_{r=a} = -V_\infty \cdot \frac{3}{2}\sin\theta = -\frac{3}{2}V_\infty \sin\theta \quad (2分)$$

From Bernoulli's eq.

$$P_\infty + \frac{1}{2}\rho V_\infty^2 = P|_{r=a} + \frac{1}{2}\rho \cdot \frac{9}{4}V_\infty^2 \sin^2\theta$$

$$P|_{r=a} = P_\infty + \frac{1}{2}\rho V_\infty^2\left[1 - \frac{9}{4}\sin^2\theta\right]$$

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