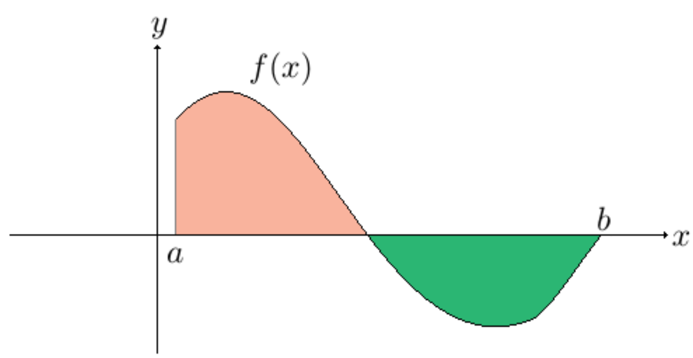
CALCULUS I LECTURE 12: INTEGRALS I

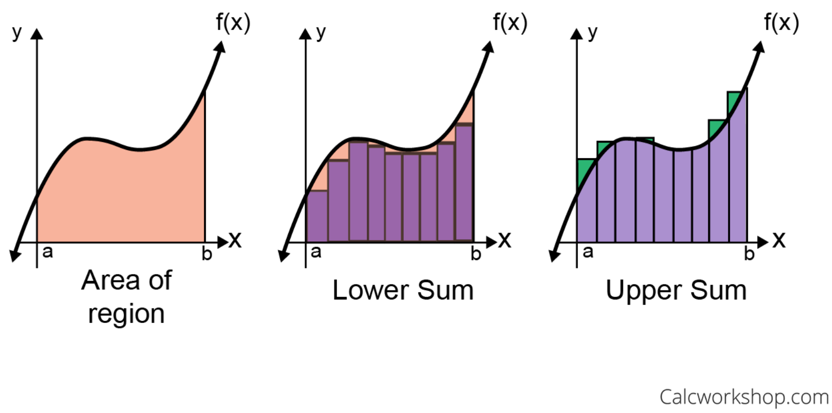
1. Riemann Sum

Suppose that $f : [a, b] \to \mathbb{R}$ is a differentiable function with $|f'(x)| < R$ for all $x \in [a, b]$. We define the **signed area** to be the area bounded by $x-$axis and the portion of graph $\{(x, f(x)|f(x) > 0)\}$ minus the area bounded by $x-$axis and the portion of graph $\{(x, f(x)|f(x) < 0)\}$. For example, in the following graph, the signed area bounded by f is the area of sand color part minus the area of green part.



Pic.1

Suppose the function f is positive now. Here we can approximate the area by n stripes, each of them have width $\frac{b-a}{n}$. We call this number Δ_n. Now, there are two way to approximate the area bounded by f, which can be demonstrated below.



Pic.2 (Cited from: https://calcworkshop.com/integrals/riemann-sum/)

To be more precise, let $M_k = \max\{f(x)|x \in [a+(k-1)\Delta_n, a+k\Delta_n]\}$. Then we define an **upper Riemann sum** to be

$$S_n = \sum_{k=1}^{n} M_k \Delta_n. \tag{1.1}$$

Similarly, by taking $m_k = \{f(x)|x \in [a+(k-1)\Delta_n, a+k\Delta_n]\}$, we have an **lower Riemann sum**

$$s_n = \sum_{k=1}^{n} m_k \Delta_n. \tag{1.2}$$

Here and in the following, we call the set $\{a, a+\Delta_n, a+2\Delta_n, \cdots, a+n\Delta_n = b\}$ a **partition of** $[a,b]$. One should notice that S_n, s_n are **not** infinite series, because each of them sums up only finitely many terms. Moreover, we always have $s_n \leq S_n$ for all $n \in \mathbb{N}$.

It is not always true that $S_m \leq S_n$ when $m > n$. However, if we consider the sequence S_{2^k} instead, then we can see that $S_{2^m} < S_{2^n}$ for any $m > n$. This is because the partition provided by S_{2^n} is a subset of partition provided by S_{2^m}. According to this observation, we conclude that

$$s_{2^n} \leq s_{2^m} \leq S_{2^m} \leq S_{2^n} \tag{1.3}$$

for all $m > n$. That is to say, $\{S_{2^k}\}_{k\in\mathbb{N}}$ is monotonic decreasing with a lower bound whereas $\{s_{2^k}\}_{k\in\mathbb{N}}$ is monotonic increasing with a upper bound. So both of them have limit. We will prove in a minute that the limit will be the same if f is differentiable. We call this limit the **definite integral** of f on $[a,b]$. Denote by

$$\int_a^b f(x)dx := \lim_{k\to\infty} S_{2^k} = \lim_{k\to\infty} s_{2^k}. \tag{1.4}$$

Proposition 1.1. Suppose f' is bounded on (a,b), then the following limits exist and have same value.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n. \tag{1.5}$$

Proof. Let $|f'(x)| < R$ for some $R > 0$. We firstly prove that $\{S_n\}$, $\{s_n\}$ are Cauchy sequences. Notice that for any $m, n \in \mathbb{N}$, we have

$$S_m = \sum_{k=1}^{m} M_k \Delta_m \tag{1.6}$$

where $M_k = \max\{f(x)|x \in [a+(k-1)\Delta_m, a+k\Delta_m]\}$, which can be denoted by $f(x_k)$ for some $x_k \in [a+(k-1)\Delta_m, a+k\Delta_m]$. Meanwhile, we can write $\Delta_m = n\Delta_{mn}$ and write

$$S_{mn} = \sum_{k=1}^{m} \sum_{j=1}^{n} f(x^*_{k,j})\Delta_{mn} \tag{1.7}$$

with $x^*_{k,j} \in [a+(k-1)\Delta_m+(j-1)\Delta_{mn}, a+(k-1)\Delta_m+j\Delta_{mn}]$. So by (1.6), (1.7) and the mean value theorem, we have

$$|S_m - S_{mn}| = \Big|\sum_{k=1}^{m}\sum_{j=1}^{n}\Big(f(x_k) - f(x^*_{k,j})\Big)\Delta_{mn}\Big| \tag{1.8}$$

$$\leq \sum_{k=1}^{m}\sum_{j=1}^{n}\Big|f(x_k) - f(x^*_{k,j})\Big|\Delta_{mn} \tag{1.9}$$

$$\leq \sum_{k=1}^{m}\sum_{j=1}^{n} \leq R\Delta_m\Delta_{mn} \leq R\Delta_m(b-a). \tag{1.10}$$

((1.10) is obtained from (1.9) and the mean value theorem). Notice that $R\Delta_m(b-a)$ converges to 0 as m goes to ∞. So we have

$$|S_m - S_n| \leq |S_m - S_{mn}| + |S_n - S_{mn}| \leq R(\Delta_m + \Delta_n)(b-a) \tag{1.11}$$

with the right hand side goes to 0 as m, n goes to ∞. Therefore, $\{S_n\}$ is a Cauchy sequence. By the same token, one can prove that $\{s_n\}$ is also a Cauchy sequence. Finally, since

$$|S_n - s_n| = |\sum_{k=1}^{n}(M_k - m_k)\Delta_n| \leq nR\Delta_n^2 = \frac{R(b-a)^2}{n}, \tag{1.12}$$

we have $\lim_{n\to\infty}(S_n - s_n) = 0$. This implies (1.5). □

Example 1.2. For any $a < b$, we have

$$\int_a^b x^2 dx = \frac{1}{3}(b^3 - a^3). \tag{1.13}$$

Here we consider the upper Riemann sum S_n. One should notice that

$$M_k = \Big(a + k\frac{(b-a)}{n}\Big)^2.$$

So

$$\begin{aligned} S_n = \sum_{k=1}^{n}(a + k\frac{b-a}{n})^2\Delta_n &= \sum_{k=1}^{n}(a^2 + 2ak\Delta_n + k^2\Delta_n^2)\Delta_n \\ &= a^2 n\Delta_n + n(n+1)a\Delta_n^2 + \frac{n(n+1)(2n+1)}{6}\Delta_n^3 \\ &= ba^2 - a^3 + \frac{n(n+1)}{n^2}(ab^2 - 2a^2b + a^3) \\ &\qquad + \frac{n(n+1)(n+\frac{1}{2})}{n^3}\frac{1}{3}(b-a)^3. \end{aligned}$$

By taking the limit, we have

$$\begin{aligned} \lim_{n\to\infty} S_n &= ba^2 + ab^2 - 2a^2b + a^3 + \frac{1}{3}(b^3 - b^2a + ba^2 - a^3) \\ &= \frac{1}{3}(b^3 - a^3). \end{aligned}$$

Remark 1.3. There are also several different type of Riemann sums. For example, the **right Riemann sum** and the **left Riemann sum**:

$$S_n^l = \sum_{k=1}^{n} f(a + (k-1)\Delta_n)\Delta_n; \tag{1.14}$$

$$S_n^r = \sum_{k=1}^{n} f(a + k\Delta_n)\Delta_n. \tag{1.15}$$

By using Proposition 1.1 and comparison theorem of sequences, we have all these Riemann sums have the same limit, which equals the integral.