CALCULUS I LECTURE 4: DERIVATIVES OF FUNCTIONS

1. Derivatives of Algebraic functions

Let us consider a polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. By Proposition 2.2 in the previous lecture, the derivative of f will be

$$\frac{df}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1}. \tag{1.1}$$

Now, suppose we have $q = \frac{m}{n} \in \mathbb{Q}^+$, $m, n \in \mathbb{N}$. We can compute

$$\frac{dx^q}{dx} = \frac{d}{dx}(x^{\frac{1}{n}})^m \tag{1.2}$$

by the chain rule. Then the right hand side of (1.2) will be

$$mx^{\frac{m-1}{n}} \frac{d}{dx}(x^{\frac{1}{n}}).$$

To obtain $\frac{d}{dx}(x^{\frac{1}{n}})$, one can use the product rule

$$1 = \frac{dx}{dx} = nx^{\frac{n-1}{n}} \frac{d}{dx}(x^{\frac{1}{n}}).$$

So

$$\frac{d}{dx}(x^{\frac{1}{n}}) = \frac{1}{n} x^{\frac{1}{n}-1}. \tag{1.3}$$

Combine (1.2) and (1.3), we will have

Proposition 1.1. Let $f(x) = x^q$ for some $q \in \mathbb{Q}^+$. Then

$$\frac{df}{dx} = qx^{q-1} \tag{1.4}$$

for all $x > 0$.

Now, we use the product rule on $1 = \frac{1}{x} x$,

$$0 = \frac{d1}{dx} = \frac{1}{x} + x\frac{d(\frac{1}{x})}{dx}$$

So we have

$$\frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2}. \tag{1.5}$$

Combining (1.5) and Proposition 1.1, we have

$$\frac{df}{dx} = qx^{q-1}$$

for $f(x) = x^q$ and $q \in \mathbb{Q}$.

Notice that, if we have $r_1 < r_2 \in \mathbb{R}^+$, then

$$x^{r_2} - a^{r_2} < \max\{x^{r_2-r_1}, a^{r_2-r_1}\}(x^{r_1} - a^{r_1})$$

So by the comparison of limits, we have

$$\frac{dx^{r_2}}{dx} \leq \frac{dx^{r_1}}{dx} \tag{1.6}$$

for any $r_1 < r_2$. Therefore, we have

Proposition 1.2. Let $f(x) = x^r$ for some $r \in \mathbb{R}$. Then

$$\frac{df}{dx} = rx^{r-1} \tag{1.7}$$

for all $x > 0$.

2. Derivative of Exponential functions

Let $f(x) = p^x$ for some $p \geq 1$. Here we compute the derivative of f. One should notice that, for any $a \in \mathbb{R}$, the derivative of f at a is

$$\frac{dp^x}{dx}(a) = \lim_{x \to 0} \frac{p^x - p^a}{x - a} = \lim_{h \to 0} \frac{p^{a+h} - p^a}{h} = p^a \lim_{h \to 0} \frac{p^h - p^0}{h} = p^a \frac{dp^x}{dx}(0), \tag{2.1}$$

or we can simply write $f'(a) = p^a f'(0)$ for any $a \in \mathbb{R}$. Therefore, one can obtain the derivative of p^x by finding the value $\frac{dp^x}{dx}(0)$.

Meanwhile, suppose we have $p_1 > p_2 > 1$, then by the chain rule

$$\begin{aligned} \frac{dp_2^x}{dx}(a) &= \frac{d(p_1)^{(\log_{p_1} p_2)x}}{dx}(a) \\ &= \log_{p_1} p_2 \frac{dp_1^x}{dx}((\log_{p_1} p_2)a). \end{aligned}$$

So we have

$$\frac{dp_2^x}{dx}(0) = \log_{p_1} p_2 \frac{dp_1^x}{dx}(0).$$

This implies that, for example,

$$\frac{dp^x}{dx}(0) = \log_2 p \frac{d2^x}{dx}(0),$$

if we take $p_2 = p$, $p_1 = 2$. The value $\log_2 p$ goes to 0 as p goes to 1 and goes to ∞ as p goes to ∞. So $\frac{dp^x}{dx}(0)$ runs all non-negative real numbers as p runs all values which are greater than or equal to 1.

Therefore, there exists a unique number e, which is call the **Eular number**, such that

$$\frac{de^x}{dx}(0) = 1. \tag{2.2}$$

So

$$\frac{de^x}{dx} = e^x.$$

Exercise 2.1. Write down the derivative of p^x in terms of e^x.

3. Derivatives of Trigonometric functions

Recall that, in the first Lecture, we prove that

$$x - x^3 \le \sin x \le x$$

when x is small. By the comparison of limits, we have

$$1 = \lim_{x\to 0} \frac{x - x^3}{x} \le \lim_{x\to 0} \frac{\sin(x)}{x} \le \lim_{x\to 0} \frac{x}{x} = 1.$$

So $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. One can also obtain the following by the formula $\cos(x) = 1 - 2\sin^2(\frac{x}{2})$:

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x} = \lim_{x\to 0} \frac{-2\sin(\frac{x}{2})^2}{x} = 0.$$

We obtain

$$\frac{d\sin}{dx}(0) = 1; \tag{3.1}$$

$$\frac{d\cos}{dx}(0) = 0. \tag{3.2}$$

Now, the derivative of $\sin(x)$ equals

$$\begin{aligned}\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} &= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{x} \\ &= \sin(x) \lim_{h\to 0} \left(\frac{\cos(h) - 1}{h}\right) + \cos(x) \lim_{h\to 0} \left(\frac{\sin(h)}{h}\right) \\ &= \cos(x).\end{aligned}$$

Therefore we have

$$\frac{d\sin(x)}{dx} = \cos(x). \tag{3.3}$$

To obtain the derivative of cos, we can use the formula again

$$\cos(x) = 1 - 2\sin^2(\frac{x}{2}).$$

By product rule, we have

$$\frac{d\cos(x)}{dx} = -2\sin(\frac{x}{2})\cos(\frac{x}{2}) = -\sin(x). \tag{3.4}$$

One can obtain the derivatives of all trigonometric functions.

Exercise 3.1. Show that
a. $\frac{d\tan(x)}{dx} = \sec^2(x)$;
b. $\frac{d\cot(x)}{dx} = -\csc^2(x)$;
c. $\frac{d\sec(x)}{dx} = \sec(x)\tan(x)$;
d. $\frac{d\csc(x)}{dx} = -\csc(x)\cot(x)$.

4. Derivatives of Logarithmic functions

To obtain the derivative of $\log_a(x)$ (we will use "ln" or "log" to denote $\log_e$) for some $a > 1$, we should firstly notice that

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

So we have

$$\frac{d\log_a(x)}{dx} = \frac{1}{\ln(a)}\frac{d\log_e(x)}{dx}. \tag{4.1}$$

Therefore, to obtain the derivative of $\log_a(x)$, we only need to know the derivative of $\ln(x)$.

Now, notice that $\ln(x)$ is the inverse function of e^x. That is to say,

$$e^{\ln(x)} = x. \tag{4.2}$$

By chain rule, we have

$$\begin{aligned} 1 = \frac{dx}{dx} = \frac{de^{\ln(x)}}{dx} &= \frac{de^y}{dy}(\ln(x))\frac{d\ln(x)}{dx} \\ &= x\frac{d\ln(x)}{dx}. \end{aligned}$$

So we have

$$\frac{d\ln(x)}{dx} = \frac{1}{x} \tag{4.3}$$

when $x > 0$. One can also have $\frac{d\ln(|x|)}{dx} = \frac{1}{x}$ by extending the domain of $\ln(x)$ to $\mathbb{R} - \{0\}$.