

Linear Algebra: Midterm Exam 2B

This is a 120-minutes exam.

4 pages in total

1. (30 pts) Determine whether the following statements are true (T) or false (F)? (A reasoning is required.)

- (1) If a 4×3 matrix A has a rank 3, then the dimension of $N(A)$ is 3.
- (2) Let A be a $n \times n$ matrix. If $N(A) = \{0\}$, then the system $A\mathbf{x} = \mathbf{b}$ has infinite solution for a given vector $\mathbf{b} \in R^n$.
- (3) The set $\{x, x + 3, x^2 - 1\}$ is spanning the vector space P_3 .
- (4) The following vectors $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}$ are linearly independent in $M_{2 \times 2}$.
- (5) Let V have two bases by $B = \{\sin(x) - \cos(x), 2\sin(x)\}$ and $B' = \{\sin(x), \cos(x)\}$, then the transition matrix from B to B' is $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$.
- (6) If A is a 4×5 matrix, then rank of A is at most 5
- (7) Let A be a 4×5 matrix. If $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and \mathbf{a}_4 are linearly independent and $\mathbf{a}_5 = 2\mathbf{a}_1 - \mathbf{a}_2 + 3\mathbf{a}_4$, the reduced row echelon form of A is
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$
- (8) The following transformation $L: R^3 \rightarrow R^3$ defined by $L(\mathbf{x}) = \begin{bmatrix} 1 + x_1 \\ x_1 + x_2 \\ x_1 - 2x_3 \end{bmatrix}$ is a linear transformation.
- (9) Let A be a 2×2 matrix, and let L_A be the linear operator defined by $L_A(\mathbf{x}) = A\mathbf{x}$. We can conclude that L_A maps the vector space R^2 onto the row space of A .
- (10) Let $L_1: R^2 \rightarrow R^2$ be the orthogonal projection onto the x -axis, and $L_2: R^2 \rightarrow R^2$ be the rotation counterclockwise with the angle θ , then $L_2 \circ L_1 = L_1 \circ L_2$.

(11) The matrix $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is a rotation matrix.

(12) If A is a nonsingular matrix, and B is similar to A , then A^{-1} is also similar to B^{-1} .

(13) Let $x = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then the angle between x and y is $\theta = 30^\circ$.

(14) Is it possible for a matrix A to have the vector $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ in row space of A and

$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ in the null space of A ?

(15) If the subspace S of R^3 is spanned by $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, then $\begin{bmatrix} 10 \\ -1 \\ -1 \end{bmatrix}$ is in the orthogonal complement of S .

2. (8 pts) Determine the kernel and range of each of the following linear operators on R^3

(a) $L(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 \\ x_2 - 2x_3 \\ x_1 + x_3 \end{bmatrix}$ (b) $L(\mathbf{x}) = \begin{bmatrix} x_1 \\ 2x_1 \\ x_1 \end{bmatrix}$

3. (5 pts) Given the two bases $V = \left\{ v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$, and $U = \left\{ u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$, please find the transition matrix from V to U .

4. (12 pts) Consider a set of matrices with size 2×2 : $S = \left\{ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \right\}$.

(1) Find the coordinate vector of the matrix $B_1 = \begin{bmatrix} 4 & -18 \\ -18 & 10 \end{bmatrix}$ with respect to the basis S .

(2) Find the coordinate vector of the matrix $B_2 = \begin{bmatrix} 4 & -9 \\ -9 & 10 \end{bmatrix}$ with respect to the basis S .

(3) Does the matrix $B_3 = \begin{bmatrix} 4 & 9 \\ -9 & 10 \end{bmatrix}$ be in the vector space $M = \text{Span}\{A_1, A_2, A_3\}$.

5. (12 pts) Let $L_1, L_2, L_3, L_4: R^2 \rightarrow R^2$ be the following linear transformations:

L_1 : Reflection with respect to the x -axis;

L_2 : Reflection with respect to the line l of angle 30° ;

L_3 : Rotation counterclockwise by the angle 60° ;

L_4 : Deformation: x_1 direction with factor $k = 2$.

(1) Find the representing matrices A_1, A_2, A_3 and A_4 of $L_1(x) = A_1x, L_2(x) = A_2x, L_3(x) = A_3x$, and $L_4(x) = A_4x$, respectively.

(2) Find the representing matrix C of the composition transformation $L_2 \circ L_1$

(3) Find the vector of $L_1 \circ L_3 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$.

(4) Plot the composite transformation $L_4 \circ L_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ in geometric diagram.

6. (9 pts) Let $L: R^3 \rightarrow R^4$ be a linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix}, L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}, L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ -9 \\ 2 \end{bmatrix}.$$

(1) Find the representing matrix A for the linear transformation L .

(2) If $L(x) = \begin{bmatrix} 3 \\ 3 \\ -26 \\ 5 \end{bmatrix}$, then find x .

(3) For the matrix A found in (1), if we are given a vector $b \in R^4$, then please find the solution of the system $Ax = b$.

7. (12 pts) Let the operator $L: P_3 \rightarrow P_3$ be defined by $L(f(x)) = x^2 f(1) + f'(x)$.

(1) Find the matrix A representing L with respect to the basis $\{x^2, x, 1\}$.

(2) Find the matrix B representing L with respect to the basis $\{x^2 + 1, x + 1, 1\}$.

(3) Find the matrix S such that $B = S^{-1}AS$.

8. (12 pts) Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 3 \\ 3 & 0 & 3 & 4 & 5 \\ 0 & 4 & 0 & 4 & 4 \end{bmatrix}$.

(1) Find a basis for $N(A)$ (the null space of A) and the nullity of A .

(2) Find a basis for the row space of A and find a basis for the column space of A .

(3) Find the $N(A^T)$ and the nullity of A^T .

(4) Find the range space of A^T and prove that $N(A) \perp R(A^T)$.