# CALCULUS I LECTURE 10: SEQUENCES, SERIES AND THEIR LIMITS II

## 1. Series with positive terms and Alternating series

Here we introduce two type of series. First one is the series $s_n = \sum_{i=1}^{n} a_i$ with all $a_i \geq 0$. In this case, the sequence of partial sums $\{s_n\}$ is a monotonic increasing sequence. It converges whenever there exists an upper bound.

Now for any two sequences $\{a_i\}$, $\{b_i\}$ with the $a_i, b_i \geq 0$ and $a_i \leq b_i$. Suppose we already know that $\sum_{i=1}^{\infty} b_i$ is convergent. Then $\sum_{i=1}^{\infty} a_i$ is also convergent. Conversely, if $\sum_{i=1}^{\infty} a_i$ diverges, so does $\sum_{i=1}^{\infty} b_i$.

**Proposition 1.1.** Suppose $0 \leq a_i \leq b_i$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} b_i$ convergent. Then we have $\sum_{i=1}^{\infty} a_i$ convergent.

*Proof.* Let $M = \sum_{i=1}^{\infty} b_i$, So we have

$$s_n := \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i \leq M. \tag{1.1}$$

Namely, $s_n$ is a monotone, increasing sequence with an upper bound. So it is convergent. □

We usually call this Proposition the **comparison test** of two sequences.

**Exercise 1.2.** Show that $\sum_{i=1}^{\infty} \frac{1}{i}$ divergent.

**Proposition 1.3.** Let $\{a_n\}$, $\{b_n\}$ be two positive sequences (i.e., $a_n, b_n > 0$ for all $n$). Suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} \tag{1.2}$$

converges to some $c > 0$. Then either both of $\sum_{i=1}^{\infty} a_i$, $\sum_{i=1}^{\infty} b_i$ converges or both of them diverges.

*Proof.* Here we just sketch the proof. Since $c > 0$, we have $a_n \leq (c+1)b_n$ and $b_n \leq (\frac{1}{c}+1)a_n$ when $n$ is sufficiently large. Therefore, by Proposition 1.1, we can prove the result immediately. □

We call this Proposition the **limit comparison test**.

The second type of series we will discuss here is called **alternating series**. It can be written as

$$\sum_{i=1}^{\infty} (-1)^{i+1} a_i \tag{1.3}$$

or

$$a_1 - a_2 + a_3 - a_4 + \cdots \tag{1.4}$$

with all $a_i > 0$

Recall the last Proposition in the last lecture. A necessary condition for the alternating series convergent is $\lim_{n\to} a_n = 0$. We will prove in the following proposition, if $a_n$ is monotonic decreasing, then the alternating series will convergent.

**Proposition 1.4.** Let $a_n$ monotonic decreasing and $\lim_{n\to\infty} a_n = 0$. Then $\sum_{i=1}^{\infty}(-1)^{(i+1)}a_i$ converges.

*Proof.* Firstly, we can observe that by taking $b_n = s_{2n}$, $c_n = s_{2n+1}$ for all $n \in \mathbb{N}$, $\{b_n\}$ is monotonic increasing and $\{c_n\}$ is monotonic decreasing. This is because

$$b_n = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}); \tag{1.5}$$

$$c_n = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n} - a_{2n+1}). \tag{1.6}$$

Moreover, since $c_n = b_n + a_{2n+1}$, we have $b_n < c_n$. So we have $c_1 > c_n > b_n > b_1$. Therefore both $b_n$ and $c_n$ converge because $\{b_n\}$ has an upper bound $c_1$ and $\{c_n\}$ has a lower bound $b_1$.

Therefore, we have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n - \lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} c_n. \tag{1.7}$$

One can check, by Definition, this implies $\sum_{i=1}^{\infty} a_i$ converges. □

## 2. Absolute converges, Ratio test and Root test

For a general series, there are three most common ways to check whether it is convergent. The first one is called **absolutely convergent**.

**Definition 2.1.** For any sequence $\{a_n\}$, we call the series $\sum_{i=1}^{\infty} a_i$ absolutely convergent if and only if $\sum_{i=1}^{\infty} |a_i|$ converges.

**Proposition 2.2.** If $\sum_{i=1}^{\infty} a_i$ is absolutely convergent, then it is convergent.

*Proof.* Let $s_n$ be the partial sum of $\{a_n\}$ and $s'_n$ be the partial sum of $\{|a_n|\}$. Now, since the original series is absolutely convergent, so $\{s'_n\}$ is a Cauchy sequence. Meanwhile, we have

$$|s_m - s_n| = |\sum_{i=n+1}^{m} a_i| \leq \sum_{i=n+1}^{m} |a_i| \leq |s'_m - s'_n|. \tag{2.1}$$

So $\{s_n\}$ is also a Cauchy sequence, which is equivalently to say, $\lim_{n\to\infty} s_n$ converges. □

*Remark* 2.3. This Proposition doesn't imply conversely. Namely, there are some series which are not absolutely convergent, but still convergent. In these cases, we call those series **conditionally convergent**.

**Exercise 2.4.** Show that $\sum_{i=1}^{\infty}(-1)^{(i+1)}\frac{1}{i}$ is conditionally convergent.

There are two other methods to check whether a series is convergent. These two methods, however, are all based on the observation of the behavior of geometric series.

**Proposition 2.5. (Ratio test).** Suppose the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ exists. Then we have the series $\sum_{i=1}^{\infty} a_i$ converges when $L < 1$; the series $\sum_{i=1}^{\infty} a_i$ diverges when $L > 1$. When $L = 1$, no conclusion can be made.

*Proof.* Suppose $L < 1$, then there exists $N = N(\frac{1}{2}(1-L))$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \frac{1}{2} + \frac{L}{2} < 1. \tag{2.2}$$

for all $n > N$. So we can take $\frac{1}{2} + \frac{L}{2} = r < 1$ and write

$$|a_{n+1}| < r|a_n| < r^{n-N}|a_N| \tag{2.3}$$

for all $n > N$. So $\{|a_n|\}$ is bounded by a geometric sequence with ratio $r < 1$. This implies the original series is absolutely convergent. By Proposition 2.2, it is convergent.

We leave the proof of $L > 1$ for readers. One can show that, by the same argument, $a_n$ will not converge to 0. So the series will diverge. □

The final test I should mention is the root test. This one is better than ratio test (which means root test can show us some series convergent that ratio test can't).

**Proposition 2.6. (Root test)**. Suppose the limit $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ exists. Then we have $\sum_{i=1}^{\infty} a_i$ converges when $L < 1$; $\sum_{i=1}^{\infty} a_i$ diverges when $L > 1$. When $L = 1$, no conclusion can be made.

*Proof.* Here we only prove the case $L < 1$. We can take $N = N(\frac{1}{2}(1-L))$ again, $r = \frac{1}{2} + \frac{L}{2}$. So

$$\sqrt[n]{|a_n|} < r^n \tag{2.4}$$

when $n > N$. So again, the sequence $\{|a_n|\}$ will be bounded by a geometric sequence. So the original series converges. □

**Exercise 2.7.** Show that $\sum_{n=1}^{\infty} 3^{-n+(-1)^n}$ convergent. Does ratio test work for this series?

## 3. Some examples of series

One example readers need to know is the series

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)}. \tag{3.1}$$

Notice that

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}. \tag{3.2}$$

So the partial sum

$$s_n = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{n-1} - \frac{1}{n}) \tag{3.3}$$

$$= 1 - \frac{1}{n}, \tag{3.4}$$

which has limit 1. This implies

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = 1 \tag{3.5}$$

By this result and Proposition 2.2, we have

$$\sum_{i=1}^{\infty} \frac{1}{i^2} \tag{3.6}$$

is also convergent. By Proposition 1.1, we have the following conclusion.

**Proposition 3.1.** $\sum_{i=1}^{\infty} \frac{1}{i^s}$ converges when $s \geq 2$, diverges when $s \leq 1$. In fact, this series is converges when $s > 1$.