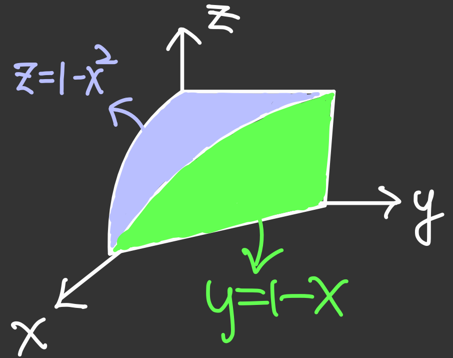
2021.6.4 助教課內容

The figure shows that the region for the integral

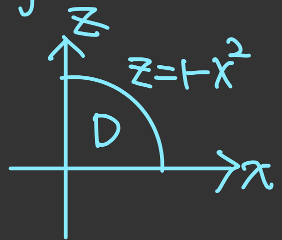
$\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,dz\,dx$.

Rewrite this integral as an equivalent iterated integral in five orders



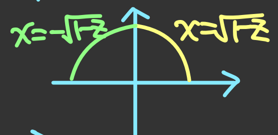

$$\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,\underbrace{dz\,dx}_{\text{bases at xz plane}}$$

Projection on xz plane




$E=\{(x,y,z)\mid (x,z)\in D,\ 0\le y\le 1-x$

$D=\{(x,z)\mid 0\le x\le\sqrt{1-z},\ 0\le z\le 1\}$ or $\{(x,z)\mid 0\le x\le 1,\ 0\le z\le 1-x^2\}$




Therefore, $I := \int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,dz\,dx$ ← original order

$$= \int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\,dy\,dx\,dz$$

Evaluate $\iiint_T xz\,dV$, where $T$ is the solid tetrahedron with vertices $(0,0,0), (1,0,1), (0,1,1), (0,0,1)$

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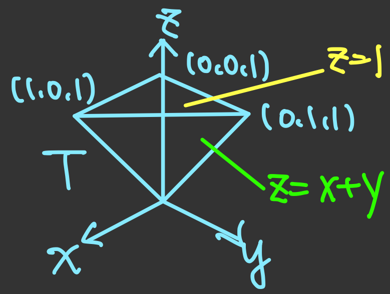
The plane passes $(0,0,0), (1,0,1), (0,1,1)$




$$\iiint_T xz\,dV = \int_0^1 \int_x^1 \int_0^{z-x} xz\,dy\,dz\,dx$$

$$= \int_0^1 \int_x^1 xz(z-x)\,dz\,dx$$

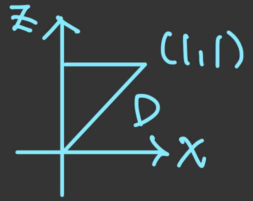
$$= \int_0^1 \left(\frac{1}{3}xz^3 - \frac{1}{2}x^2z^2\right)\Big|_{z=x}^{z=1} dx$$

$$= \int_0^1 \left(\frac{1}{3}x - \frac{1}{2}x^2 - \frac{1}{3}x^4 + \frac{1}{2}x^4\right) dx$$

$$= \left(\frac{1}{6}x^2 - \frac{1}{6}x^3 + \frac{1}{30}x^5\right)\Big|_0^1$$

$$= \frac{1}{6} - \frac{1}{6} + \frac{1}{30} = \frac{1}{30}$$

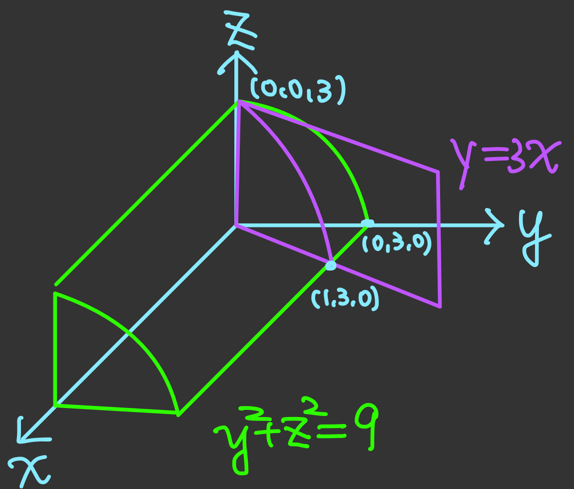
Projection on $xz$ plane



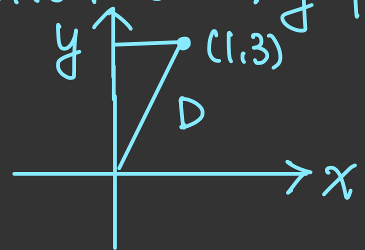

$D = \{(x,z) \mid 0 \le x \le 1,\ x \le z \le 1\}$

and $0 \le y \le z - x$,

Evaluate $\iiint_E z\,dV$, where E is bounded by the cylinder $y^2+z^2=9$ and the plane $x=0$, $y=3x$, and $z=0$ is first octant.




Projection on xy plane




$D=\{(x,y)\mid 0\le x\le 1,\ 3x\le y\le 3\}$

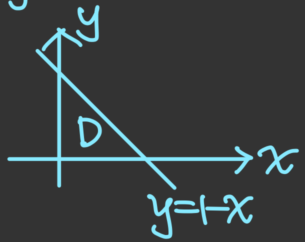
$$\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx = \int_0^1\int_{3x}^3 \frac{1}{2}(9-y^2)\,dy\,dx$$

$$= \int_0^1 \left(\frac{9}{2}y-\frac{1}{6}y^3\right)\Big|_{3x}^{3} dx$$

$$= \int_0^1 \left(9-\frac{27}{2}x+\frac{9}{2}x^3\right)dx$$

$$= \left(9x-\frac{27}{4}x^2+\frac{9}{8}x^4\right)\Big|_0^1 = \frac{27}{8}$$

Projection on xy plane




$E=\{(x,y,z)\mid (x,y)\in D,\ 0\le z\le 1-x^2\}$

$D=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le 1-x\}$

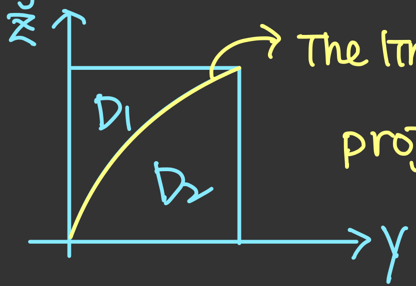
$\overset{\text{or}}{=}\{(x,y)\mid 0\le x\le 1-y,\ 0\le x\le 1\}$

Therefore,

$I=\int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx$

$=\int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy$

Projection on yz plane. E splits into two pieces.




The line intersects by $z=1-x^2$, $y=1-x$ project onto xz plane.

$z=1-x^2=1-(1-y)^2=2y-y^2$

or $y=1-\sqrt{1-z}$

$E_1=\{(x,y,z)\mid (y,z)\in D_1,\ 0\le x\le\sqrt{1-z}\}$

$D_1=\{(y,z)\mid 0\le y\le 1,\ 2y-y^2\le z\le 1\}$

$=\{(y,z)\mid 0\le y\le 1-\sqrt{1-z},\ 0\le z\le 1\}$

$E_2=\{(x,y,z)\mid (y,z)\in D_2,\ 0\le x\le 1-y\}$

$D_2=\{(y,z)\mid 0\le x\le 1,\ 0\le z\le 2y-y^2\}$

$\overset{\text{or}}{=}\{(y,z)\mid 1-\sqrt{1-z}\le y\le 1,\ 0\le z\le 1\}$

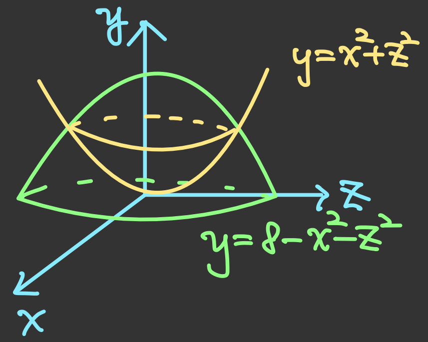
$I=\int_0^1\int_{2y-y^2}^1\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy$

$+\int_0^1\int_0^{2y-y^2}\int_0^{1-y} f(x,y,z)\,dx\,dz\,dy$

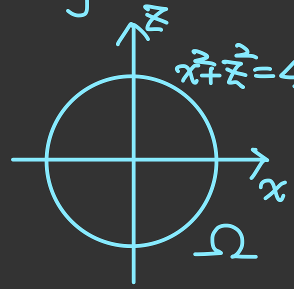
$=\int_0^1\int_0^{1-\sqrt{1-z}}\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz$

$+\int_0^1\int_{1-\sqrt{1-z}}^1\int_0^{1-y} f(x,y,z)\,dx\,dy\,dz$

Find the volume of solid enclosed by the paraboids $y=x^2+z^2$ and $y=8-x^2-z^2$




Project the solid on xz plane




$x^2+z^2=4$ (intersect by $y=x^2+z^2$ and $8-x^2-z^2$)

$$x^2+z^2 \le y \le 8-x^2-z^2$$

$$x = r\cos\theta,\quad z = r\sin\theta$$

$$\frac{\partial(x,z)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\tilde{\Omega} = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

$$f(x,z) = 8-2(x^2+z^2)$$

$$g(r,\theta) = 8-2r^2$$

$$\text{Volume} = \iiint_E 1\, dV$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{8-x^2-z^2} 1\, dy\, dz\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (8-2(x^2+z^2))\, dz\, dx$$

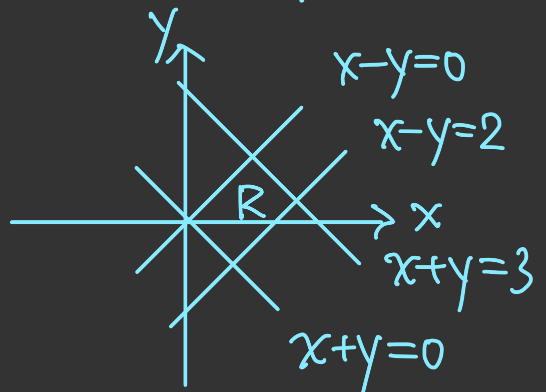
$$= \int_0^{2\pi}\int_0^2 (8-2r^2)\cdot r\, dr\, d\theta$$

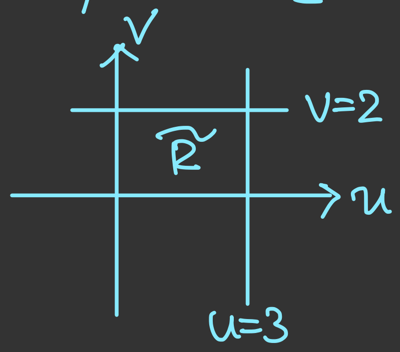
$$= 2\pi \cdot \left(4r^2 - \tfrac{1}{2}r^4\right)\Big|_0^2 = 16\pi$$

Evaluate $\iint_R (x+y)e^{x^2-y^2}\,dA$, where $R$ is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$

Consider $f: R\to \mathbb{R}$ by $f(x,y)=(x+y)e^{x^2-y^2}$ is a continuous function

Let $u=x+y,\ v=x-y \Rightarrow \begin{array}{l} x(u,v)=\frac{1}{2}(u+v) \\ y(u,v)=\frac{1}{2}(u-v)\end{array}$ which is an one-to-one map



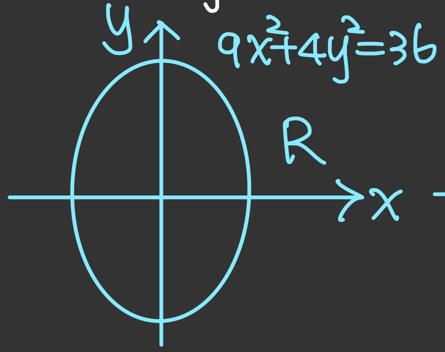



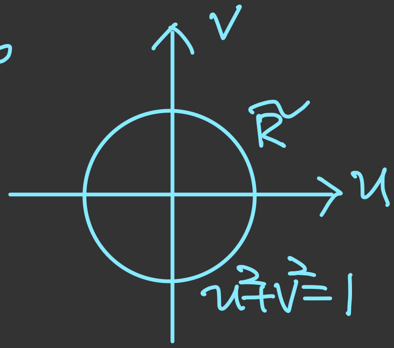

$$g(u,v)=f(x(u,v),y(u,v)) = ue^{uv}$$

$$J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix} = -\frac{1}{2}$$

$$\iint_R f(x,y)\,dA = \iint_{\tilde{R}} g(u,v)\,|J(u,v)|\,du\,dv$$
$$= \int_0^3\int_0^2 ue^{uv}\cdot\left|-\tfrac{1}{2}\right|\,dv\,du$$
$$= \int_0^3 \tfrac{1}{2}e^{uv}\Big|_0^2\,du$$
$$= \int_0^3 \left(\tfrac{1}{2}e^{2u}-\tfrac{1}{2}\right)du$$
$$= \left(\tfrac{1}{4}e^{2u}-\tfrac{1}{2}u\right)\Big|_0^3 = \left(\tfrac{1}{4}e^{6}-\tfrac{3}{2}\right)-\tfrac{1}{4} = \tfrac{1}{4}e^{6}-\tfrac{7}{4}$$

Evaluate $\iint_R x^2\,dA$, where $R$ is the region bounded by the ellipse $9x^2+4y^2=36$. Use the transformation $x=2u,\ y=3v$




$$f(x,y)=x^2$$

$$g(u,v)=f(2u,3v)=4u^2$$

$$J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

$$\iint_R x^2\,dA = \iint_{\tilde{R}} 4u^2 \cdot |6|\,du\,dv$$

$$= \int_0^{2\pi}\int_0^1 24r^2\cos^2\theta \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} \underbrace{2\cos^2\theta}_{=\,1+\cos 2\theta}\,d\theta \cdot \int_0^1 12r^3\,dr$$

$$= \left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} \cdot (3r^4)\Big|_0^1$$

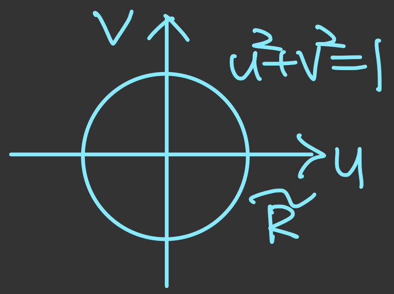
$$= 6\pi$$

Evaluate $\iint_R (x^2-xy+y^2)\,dA$, where $R$ is the region bounded by the ellipse $x^2-xy+y^2=2$. Use the transformation $x=\sqrt{2}u-\sqrt{\frac{2}{3}}v$, $y=\sqrt{2}u+\sqrt{\frac{2}{3}}v$

Note that $2=x^2-xy+y^2=(\sqrt{2}u-\frac{\sqrt{2}}{\sqrt{3}}v)^2-(\sqrt{2}u-\frac{\sqrt{2}}{\sqrt{3}}v)(\sqrt{2}u+\frac{\sqrt{2}}{\sqrt{3}}v)+(\sqrt{2}u+\sqrt{\frac{2}{3}}v)^2$

$$=2u^2-\frac{1}{\sqrt{3}}uv+\frac{2}{3}v^2-2u^2+\frac{2}{3}v^2+2u^2+\frac{1}{\sqrt{3}}uv+\frac{2}{3}v^2$$

$$=2u^2+2v^2$$




$$J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\sqrt{2} & \frac{\sqrt{2}}{\sqrt{3}}\\ \sqrt{2} & -\frac{\sqrt{2}}{\sqrt{3}}\end{vmatrix}=-\frac{4}{\sqrt{3}}$$

$f(x,y)=x^2-xy+y^2$

$g(u,v)=2u^2+2v^2$

$$\iint_R (x^2-xy+y^2)\,dA$$

$$=\iint_{\tilde{R}} 2(u^2+v^2)\left|\frac{-4}{\sqrt{3}}\right|du\,dv$$

$$=\int_0^{2\pi}\int_0^1 \frac{8}{\sqrt{3}}r^2\cdot r\,dr\,d\theta$$

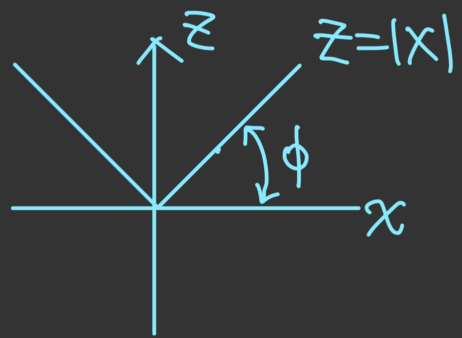
$$=2\pi\cdot\frac{2}{\sqrt{3}}r^4\Big|_0^1=\frac{4}{\sqrt{3}}\pi$$

Find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy plane, and below the cone $z=\sqrt{x^2+y^2}$


Let $x=\rho\cos\theta\sin\phi$

$y=\rho\sin\theta\sin\phi$ $\quad J(\rho,\theta,\phi)=\rho^2\sin\phi$

$z=\rho\cos\phi$




$\tilde{E}=\{(\rho,\phi,\theta)\mid 0\le\rho\le 2,\ \frac{\pi}{4}\le\phi\le\frac{\pi}{2},\ 0\le\theta\le 2\pi\}$

$$\text{Volume} = \iiint_E 1\,dV$$

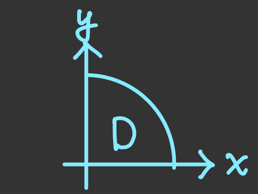
$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

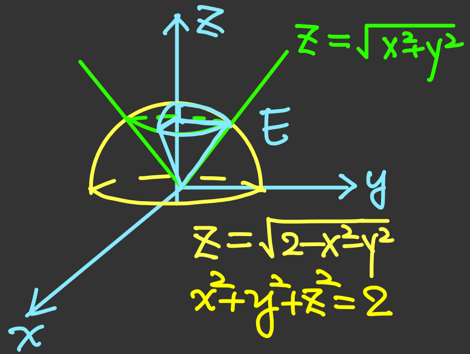
$$= 2\pi\cdot\int_{\pi/4}^{\pi/2}\sin\phi\,d\phi\cdot\int_0^2\rho^2\,d\rho$$

$$= 2\pi\cdot -\cos\phi\Big|_{\pi/4}^{\pi/2}\cdot\frac{1}{3}\rho^3\Big|_0^2$$

$$= \frac{16}{3}\pi\cdot\frac{1}{\sqrt{2}} = \frac{16}{3\sqrt{2}}\pi$$

Evaluate $\underbrace{\int_0^1\int_0^{\sqrt{1-x^2}}}_{D}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$




Let $x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi$

$z=\sqrt{x^2+y^2} \Rightarrow \cos\phi=\sin\phi \Rightarrow \phi=\frac{\pi}{4}\ (0\le\phi\le\pi)$

$\tilde{E}=\{(\rho,\phi,\theta)\mid 0\le\rho\le\sqrt{2},\ 0\le\phi\le\frac{\pi}{4},\ 0\le\theta\le\frac{\pi}{2}\}$

$f(x,y,z)=xy,\ g(\rho,\phi,\theta)=\rho^2\sin^2\phi\cos\theta\sin\theta$

$$\iiint_E xy\,dz\,dy\,dx=\iiint_{\tilde{E}}\rho^2\sin^2\phi\cos\theta\sin\theta\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}}\rho^4\sin^3\phi\cos\theta\sin\theta\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\,d\theta\cdot\int_0^{\frac{\pi}{4}}\underbrace{\sin^3\phi}_{(1-\cos^2\phi)\sin\phi}\,d\phi\cdot\int_0^{\sqrt{2}}\rho^4\,d\rho$$

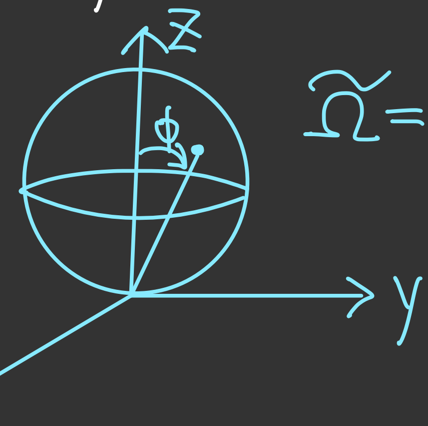
$$=\frac{1}{2}\sin^2\theta\Big|_0^{\frac{\pi}{2}}\cdot\int_1^{\frac{1}{\sqrt{2}}}(u^2-1)\,du\cdot\frac{1}{5}\rho^5\Big|_0^{\sqrt{2}}$$

$$=\frac{1}{\cancel{2}}\cdot\left(\frac{u^3}{3}-u\right)\Big|_1^{\frac{1}{\sqrt{2}}}\cdot\frac{\overset{2}{\cancel{4}}\sqrt{2}}{5}$$

$$=\frac{2\sqrt{2}}{5}\left(\frac{1}{6\sqrt{2}}-\frac{1}{\sqrt{2}}-\frac{1}{3}+1\right)=\frac{2\sqrt{2}}{5}\left(\frac{2}{3}-\frac{5}{6\sqrt{2}}\right)=\frac{4\sqrt{2}-5}{15}$$

Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{\frac{3}{2}}\,dz\,dy\,dx$

$$2-\underset{\substack{\uparrow\\\text{upper half}}}{\sqrt{4-x^2-y^2}} \le z \le 2+\underset{\substack{\uparrow\\\text{lower half}}}{\sqrt{4-x^2-y^2}}$$




$$\tilde{\Omega}=\{(\rho,\phi,\theta)\mid 0\le\rho\le 4\cos\phi,\ 0\le\phi\le\frac{\pi}{2},\ 0\le\theta\le 2\pi\}$$

$$x^2+y^2+(z-2)^2=4$$

$$\Omega=\{(x,y,z)\mid x^2+y^2+(z-2)^2\le 4\}$$

$$\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+(\rho\cos\phi-2)^2=4 \Rightarrow \rho=4\cos\phi$$

$$f(x,y,z)=(x^2+y^2+z^2)^{\frac{3}{2}}$$

$$g(\rho,\phi,\theta)=\rho^3$$

$$\iiint_\Omega (x^2+y^2+z^2)^{\frac{3}{2}}\,dV=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{4\cos\phi}\rho^3\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\frac{1}{6}\rho^6\sin\phi\Big|_0^{4\cos\phi}\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\frac{2048}{3}\sin\phi\cos^6\phi\,d\phi\,d\theta$$

$$=2\pi\cdot\frac{2048}{3}\cdot\left(\frac{-\cos^7\phi}{7}\right)\Big|_0^{\pi/2}=\frac{4096\pi}{21}$$

Find the region $E$ for which the triple integral $\iiint_E (1-x^2-2y^2-3z^2)\,dV$ is maximum and evaluate the value.

E must be the region such that $1-x^2-2y^2-3z^2$ is positive on $E$ and negative out of $E$

That is $E=\{(x,y,z) \mid x^2+2y^2+3z^2 \le 1\}$

Consider $x=u,\ y=\frac{v}{\sqrt{2}},\ z=\frac{s}{\sqrt{3}}$ is one-to-one map, $\tilde{E}=\{(u,v,s) \mid u^2+v^2+s^2 \le 1\}$

$$\frac{\partial(x,y,z)}{\partial(u,v,s)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial s} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{vmatrix} = \frac{1}{\sqrt{6}}, \quad g(u,v,s) = 1-u^2-v^2-s^2$$

$$\iiint_E (1-x^2-2y^2-3z^2)\,dV = \iiint_{\tilde{E}} (1-u^2-v^2-s^2)\cdot\frac{1}{\sqrt{6}}\,du\,dv\,ds$$

Consider $u=\rho\sin\phi\cos\theta,\ v=\rho\sin\phi\sin\theta,\ s=\rho\cos\phi,\ \frac{\partial(u,v,s)}{\partial(\rho,\phi,\theta)}=\rho^2\sin\phi$

$\tilde{\tilde{E}}=\{(\rho,\phi,\theta) \mid 0\le\rho\le1,\ 0\le\phi\le\pi,\ 0\le\theta\le2\pi\},\ h(\rho,\phi,\theta)=1-\rho^2$

$$\iiint_{\tilde{E}} (1-u^2-v^2-s^2)\left|\frac{1}{\sqrt{6}}\right|du\,dv\,ds = \frac{1}{\sqrt{6}}\int_0^{2\pi}\int_0^{\pi}\int_0^1 (1-\rho^2)\cdot\rho^2\cdot\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \frac{1}{\sqrt{6}}\cdot 2\pi\cdot\int_0^{\pi}\sin\phi\,d\phi\cdot\int_0^1(\rho^2-\rho^4)\,d\rho$$

$$= \frac{1}{\sqrt{6}}\cdot 2\pi\cdot 2\cdot\frac{2}{15} = \frac{8}{15\sqrt{6}}\pi$$