Chapter 5
Steady-State Sinusoidal Analysis

5.1 Sinusoidal Currents and Voltages

A sinusoidal voltage

$$v(t) = V_m \cos(\omega t + \theta)$$

Peak value

Phase angle(相位)

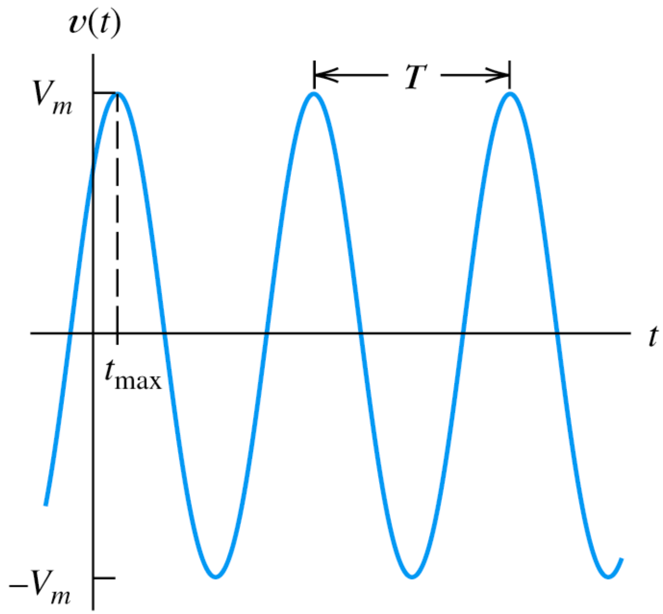


Figure 5.1 A sinusoidal voltage waveform given by $v(t) = V_m \cos(\omega t + \theta)$. Note: Assuming that θ is in degrees, we have $t_{max} = \frac{-\theta}{360} \times T$. For the waveform shown, θ is $-45°$.

ω Angular frequency (角頻率, rad/sec)

T Period (週期)

$$\omega T = 2\pi$$

$$f = \frac{1}{T}$$ frequency (頻率, Hz)

$$\omega = 2\pi f = \frac{2\pi}{T}$$

V_m is the **peak value**

ω is the **angular frequency** in radians per second

θ is the **phase angle (0^o~360^o)**

T is the **period**

Frequency $$f = \frac{1}{T}$$

Angular frequency $$\omega = \frac{2\pi}{T} = 2\pi f$$

$$\omega t + \theta$$

Radians Degree

($0 \sim 2\pi$) ($0^o \sim 360^o$)

For uniformity, we express sinusoidal functions by cosine function.

$$\longrightarrow \quad \sin(z) = \cos(z - 90^\circ)$$

Ex

$$v_x(t) = 10\sin(200t + 30^\circ) = 10\cos(200t + 30^\circ - 90^\circ)$$

$$= 10\cos(200t - 60^\circ)$$

Root-Mean-Square (均方根)Values

$$V_{\mathrm{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$$

$$I_{\mathrm{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}$$

Root-Mean-Square (均方根)Values

假設電壓源($v(t)$)週期為T, 與電阻連接，計算其平均功率 P_{avg}

$$p(t)=\frac{v^2(t)}{R} \qquad E_T=\int_0^T p(t)dt$$

$$P_{avg}=\frac{E_T}{T}=\frac{1}{T}\int_0^T p(t)dt=\frac{1}{T}\int_0^T\frac{v^2(t)}{R}dt=\frac{\left[\sqrt{\frac{1}{T}\int_0^T v^2(t)dt}\right]^2}{R}$$

$$\Longrightarrow \quad P_{avg}=\frac{V_{rms}^2}{R}=I_{rms}^2R$$

RMS Value of a Sinusoid

$$v(t) = V_m \cos(\omega t + \theta)$$

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \cos^2(\omega t + \theta)dt}$$

$$\because \qquad \cos^2(z) = \frac{1}{2} + \frac{1}{2}\cos(2z)$$

$$V_{rms} = \sqrt{\frac{V_m^2}{2T}\int_0^T \left[1 + \cos(2\omega t + 2\theta)\right]dt}$$

RMS Value of a Sinusoid

$$V_{rms} = \sqrt{\frac{V_m^2}{2T}\left[t + \frac{1}{2\omega}\sin(2\omega t + 2\theta)\right]_0^T}$$

$$V_{rms} = \sqrt{\frac{V_m^2}{2T}\left[T + \frac{1}{2\omega}\sin(2\omega T + 2\theta) - \frac{1}{2\omega}\sin(2\theta)\right]}$$

$$\omega = \frac{2\pi}{T}$$

$$\because \quad \frac{1}{2\omega}\sin(2\omega T + 2\theta) - \frac{1}{2\omega}\sin(2\theta) = \frac{1}{2\omega}\sin(4\pi + 2\theta) - \frac{1}{2\omega}\sin(2\theta)$$

$$= \frac{1}{2\omega}\sin(2\theta) - \frac{1}{2\omega}\sin(2\theta)$$

$$= 0$$

$$\Rightarrow \quad V_{rms} = \frac{V_m}{\sqrt{2}}$$

RMS Value of a Sinusoid

$$V_{\mathrm{rms}} = \frac{V_m}{\sqrt{2}}$$

The rms value for a sinusoid is the peak value divided by the square root of two. This is not true for other periodic waveforms such as square waves or triangular waves.

Example 5.1 Power delivered to a resistance by a sinusoidal source

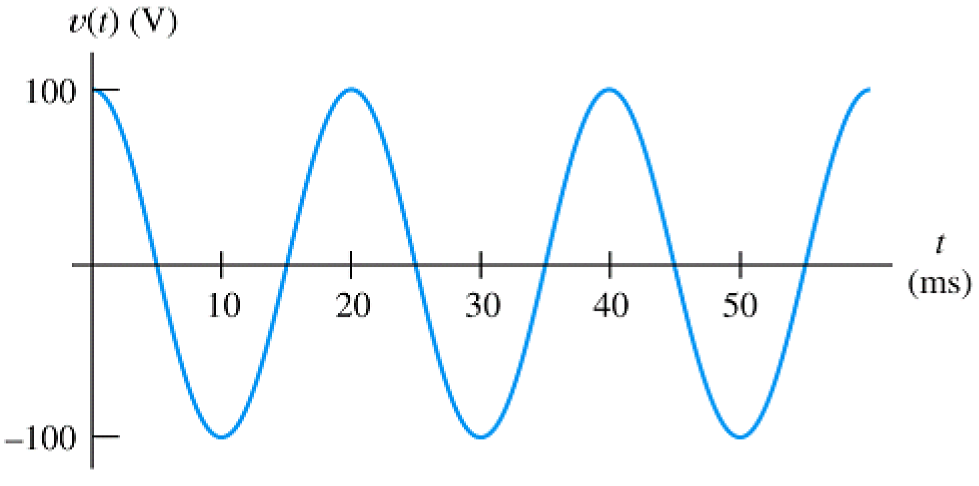
•A voltage $v(t) = 100\cos(100\pi t)$ is applied to a 50-Ω resistance. Find

•the rms value of the voltage.

•Find the average power delivered to the resistance.

•Find the power as a function of time and sketch to scale.

1. $\omega = 100\pi = \dfrac{2\pi}{T} \quad \Rightarrow \quad T = \dfrac{2\pi}{100\pi} = 20ms$



Example 5.1 Power delivered to a resistance by a sinusoidal source

2. The rms value of the voltage

$$\because \quad V_m = 100V \quad \Rightarrow \quad V_{rms} = \frac{V_m}{\sqrt{2}} = 70.71\,\text{V}$$

3. The average power

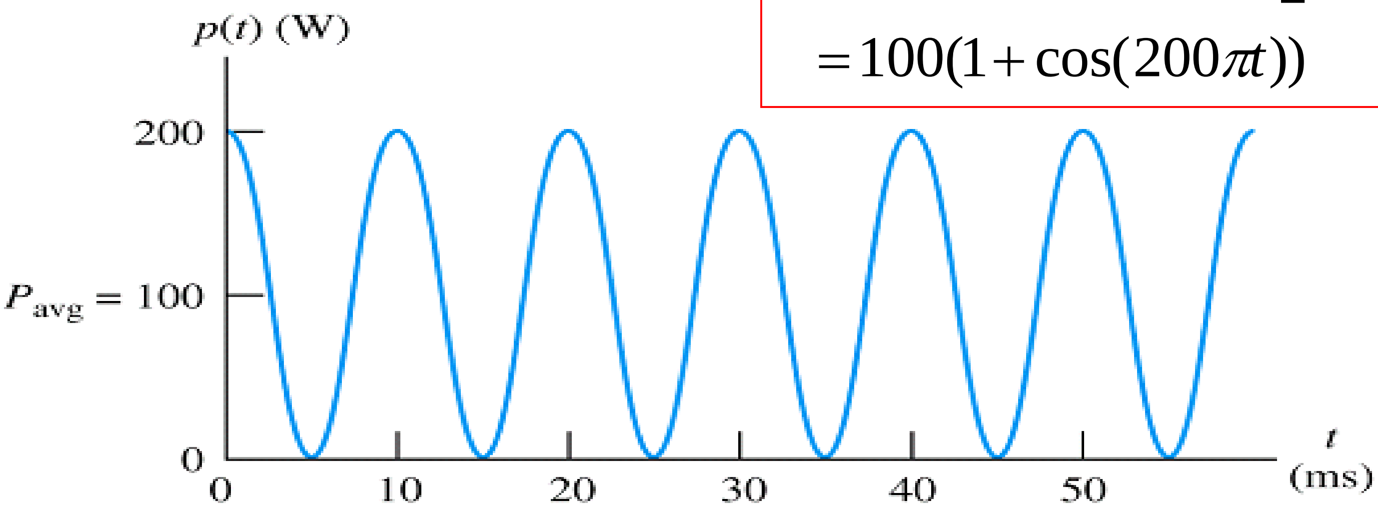
$$P_{avg} = \frac{V_{rms}^2}{R} = \frac{(70.71)^2}{50} = 100W$$

Example 5.1 Power delivered to a resistance by a sinusoidal source

4. The power as a function of time is

$$p(t) = \frac{v^2(t)}{R} = \frac{100^2\cos^2(100\pi t)}{50} = 200\cos^2(100\pi t)W$$

$$200\cos^2(100\pi t) = \frac{200}{2}(1+\cos(200\pi t))$$
$$= 100(1+\cos(200\pi t))$$



5.2 Phasors (相量)

Refer to Appendix A Complex Numbers

Phasor Definition

- Phasor 以複數(complex numbers)來表示 sinusoidal voltages or currents. Ex.

$$\text{Phasor}: \quad \mathbf{V} = V_m \angle \theta$$

- Magnitude (V_1) 代表最大振福(peak value).
- Angle (θ_1)代表相位(phase).

A sinusoidal cosine voltage

$$v_1(t) = V_1 \cos(\omega t + \theta_1)$$ (Time function, 時間函數)

其 phasor 為

$$\text{Phasor:} \quad \mathbf{V}_1 = V_1 \angle \theta_1$$

A sinusoidal sine voltage

$$v_2(t) = V_2 \sin(\omega t + \theta_2)$$

$\because$ $$\sin(z) = \cos(z - 90^\circ)$$

$\Longrightarrow$ $$v_2(t) = V_2 \cos(\omega t + \theta_2 - 90^\circ)$$

$\Longrightarrow$ $$\mathbf{V}_2 = V_2 \angle \theta_2 - 90^\circ$$

For sinusoidal currents

$$i_1 = I_1 \cos(\omega t + \theta_1)$$

phasor $$\mathbf{I}_1 = I_1 \angle \theta_1$$

$$i_2 = I_2 \sin(\omega t + \theta_2)$$

phasor $$\mathbf{I}_2 = I_2 \angle \theta_2 - 90^\circ$$

Adding Sinusoids Using Phasors

Step 1: Determine the phasor for each term.

Step 2: Add the phasors using complex arithmetic.

Step 3: Convert the sum to polar form.

Step 4: Write the result as a time function.

Using Phasors to Add Sinusoids

$$v_1(t) = 20\cos\left(\omega t - 45^\circ\right)$$
$$v_2(t) = 10\sin\left(\omega t + 60^\circ\right)$$

To find $v_s(t)=v_1(t)+v_2(t)$.

1. The phasors

$$\mathbf{V}_1 = 20\angle -45^\circ \qquad \mathbf{V}_2 = 10\angle -30^\circ$$

2. Complex arithmetic

$$\mathbf{V}_1 = x_1 + jy_1 = 20\cos(-45^o) + j20\sin(-45^o) = 14.14 - j14.14$$

$$\mathbf{V}_2 = x_2 + jy_2 = 10\cos(-30^o) + j10\sin(-30^o) = 8.66 - j5$$

$$\mathrm{V}_s = 14.14 - j14.14 + 8.660 - j5 = 22.80 - j19.14$$

Using Phasors to Add Sinusoids

3. Convert the sum to polar form

$$\mathrm{V}_s = 22.80 - j19.14$$

$$V_m = \sqrt{(22.80)^2 + (19.14)^2} = 29.77$$

$$\theta = \arctan\left(\frac{-19.14}{29.77}\right) = -40.01^\circ$$

$$V_s = 29.77\angle -40.01^\circ$$

$$v_s(t) = 29.77\cos(\omega t - 40.01^\circ)$$

Using Euler's formula to Add Sinusoids

$$v_1(t) = 20\cos\left(\omega t - 45^\circ\right) = 20\,\mathrm{Re}(e^{j(wt-45^\circ)})$$

$$v_2(t) = 10\sin\left(\omega t + 60^\circ\right) = 10\cos(wt - 30^\circ) = 10\,\mathrm{Re}(e^{j(wt-30^\circ)})$$

$$v_s(t) = v_1(t) + v_2(t) = \mathrm{Re}\left[(20e^{-j45^\circ} + 10e^{-j30^\circ})e^{jwt}\right]$$

$$20e^{-j45^\circ} + 10e^{-j30^\circ} = 29.77\angle -40.01^\circ = 29.77e^{-j40.01}$$

polar form

$$v_s(t) = \mathrm{Re}\left[29.77e^{-j40.01}e^{jwt}\right] = 29.77\cos(wt - 40.11^\circ)$$

Visualization of Sinusoids

A sinusoidal voltage $v(t) = V_m \cos(\omega t + \theta)$

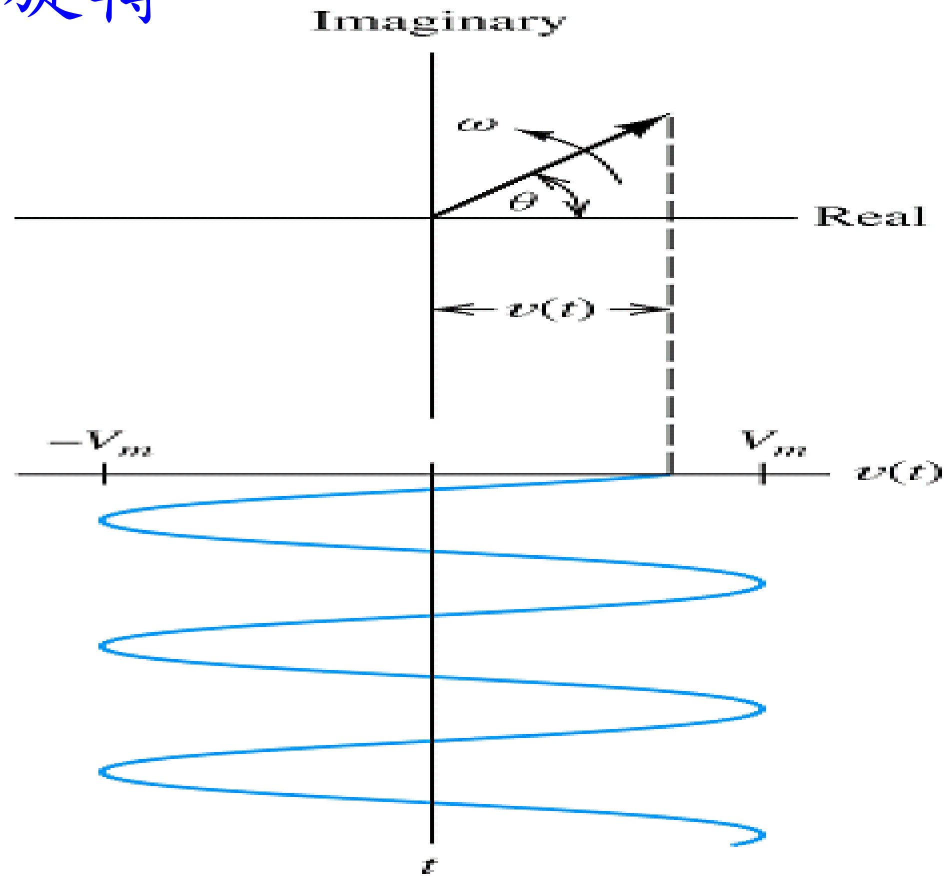
Exponential form $v(t) = \mathrm{Re}[V_m e^{j(\omega t + \theta)}]$

Complex Polar form $V_m e^{j(\omega t + \theta)} = V_m \angle \omega t + \theta$

- Sinusoids 可視為複數向量(complex vectors) 隨時間改變在實數軸的投影量。

$$V_m e^{j(\omega t+\theta)} = V_m \angle \omega t + \theta$$

可視為Vector with length V_m以 ω rad/s 角速度 (angular velocity)在複數平面以逆時針 (counterclockwise) 方向旋轉。



•$v(t)$為$V_m e^{j(\omega t+\theta)}$在實數軸的投影量。

•$v(t)$ 的Phasor $V_m\angle\theta$ 為 $V_m e^{j(\omega t+\theta)}$ 在$t=0$時的向量。

Phasors as Rotating Vectors

- Sinusoids can be visualized as the real-axis projection of vectors rotating in the complex plane.

- The phasor for a sinusoid is a snapshot of the corresponding rotating vector at $t = 0$.

Phase Relationships from Phasors

1. To determine phase relationships from a phasor diagram, consider the phasors to rotate counterclockwise.

2. Then when standing at a fixed point, if V_1 arrives first followed by V_2 after a rotation of θ , we say that V_1 leads V_2 by θ .

3. Alternatively, we could say that V_2 lags V_1 by θ . (Usually, we take θ as the smaller angle between the two phasors.)

Phase Relationships from Phasors

V_1 leads V_2 by 60º .

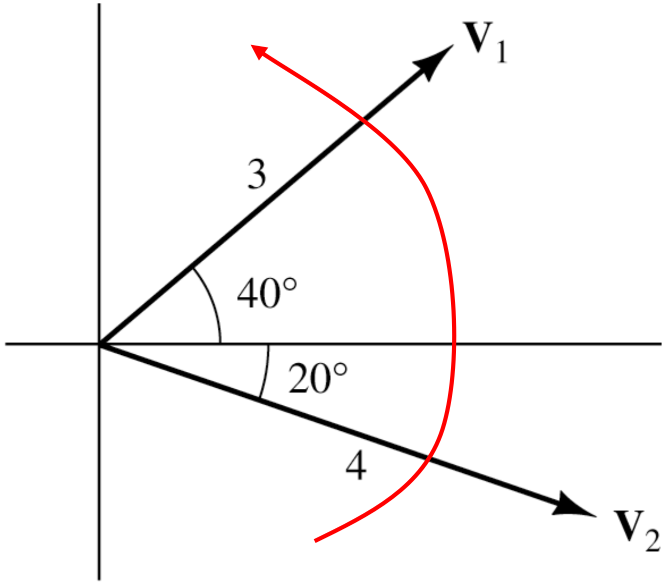


Figure 5.5 Because the vectors rotate counterclockwise, $\mathbf{v}_1$ leads $\mathbf{v}_2$ by 60° (or, equivalently, $\mathbf{v}_2$ lags $\mathbf{v}_1$ by 60°.)

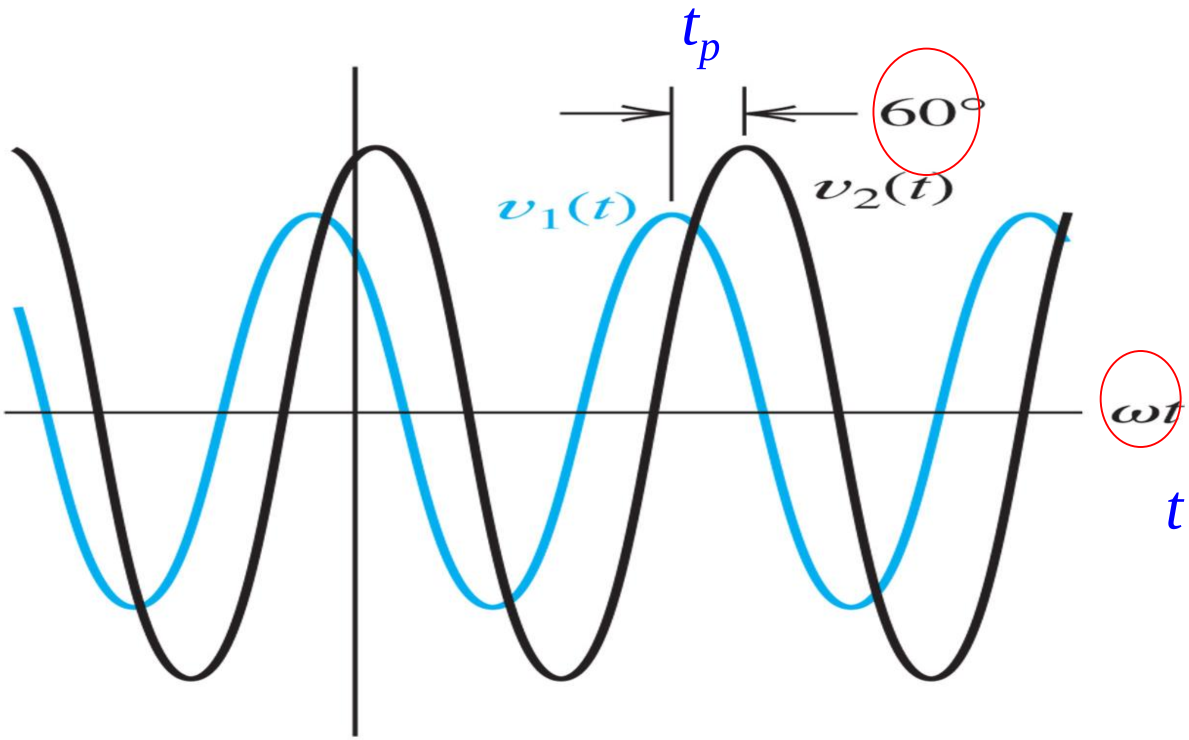
Phase Relationships from Time Functions

To determine phase relationships between sinusoids from their plots versus time,

1. find the shortest time interval t_p between positive peaks of the two waveforms.
2. Then, the phase angle is
   $\theta = (t_p/T) \times 360^\circ$.
3. If the peak of $v_1(t)$ occurs first, we say that $v_1(t)$ leads $v_2(t)$ or that $v_2(t)$ lags $v_1(t)$.

$v_1(t)$ leads $v_2(t)$ by θ (60°).

$\theta = (t_p / T) \times 360^\circ.$

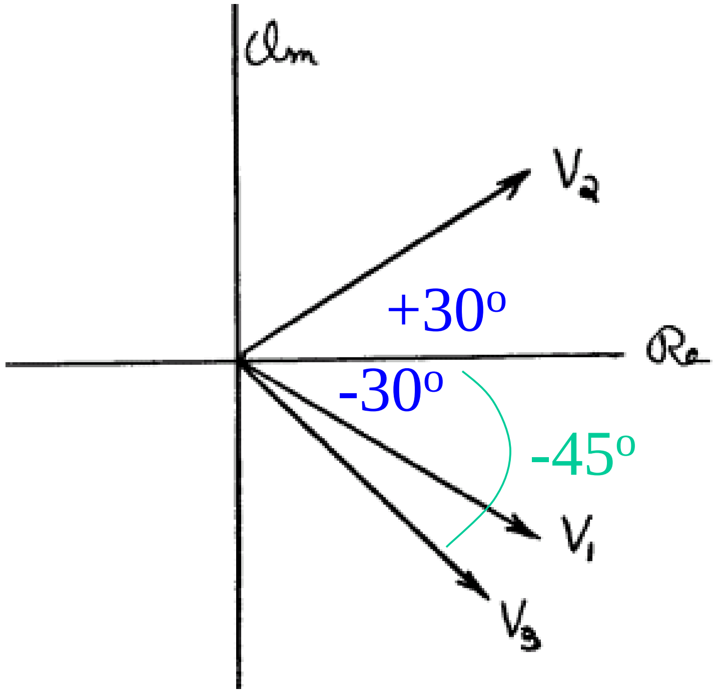


Exercise 5.5

$$v_1(t) = 10\cos(\omega t - 30^\circ)$$
$$v_2(t) = 10\cos(\omega t + 30^\circ)$$
$$v_3(t) = 10\sin(\omega t + 45^\circ)$$

State the phase relationship between each pair of voltages.

The phasors are $\mathbf{V}_1 = 10\angle -30^\circ$ $\mathbf{V}_2 = 10\angle +30^\circ$ and $\mathbf{V}_3 = 10\angle -45^\circ$



v_1 lags v_2 by 60° (or we could say v_2 leads v_1 by 60°)

v_1 leads v_3 by 15° (or we could say v_3 lags v_1 by 15°)

v_2 leads v_3 by 75° (or we could say v_3 lags v_2 by 75°)

5.3 COMPLEX IMPEDANCES (複數阻抗)

Inductance

假設通過電感L的 sinusoidal current 為

$$i_L(t) = I_m \sin(\omega t + \theta)$$

則電感L兩端的voltage為

$$v_L(t) = L\frac{di_L(t)}{dt} = \omega L I_m \cos(\omega t + \theta) \quad \text{(亦為sinusoid)}$$

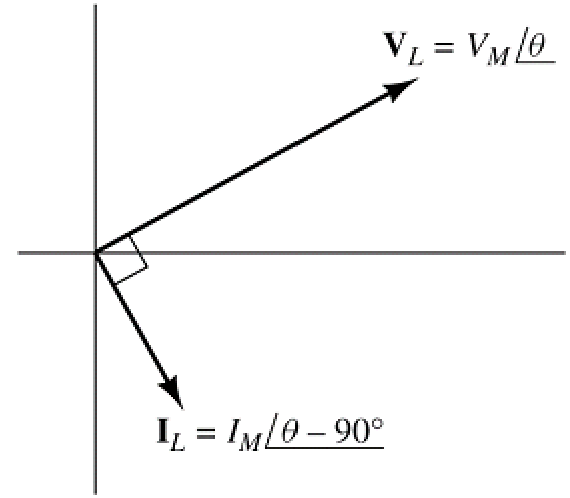
Inductance

Voltage & current 的 phasors 為

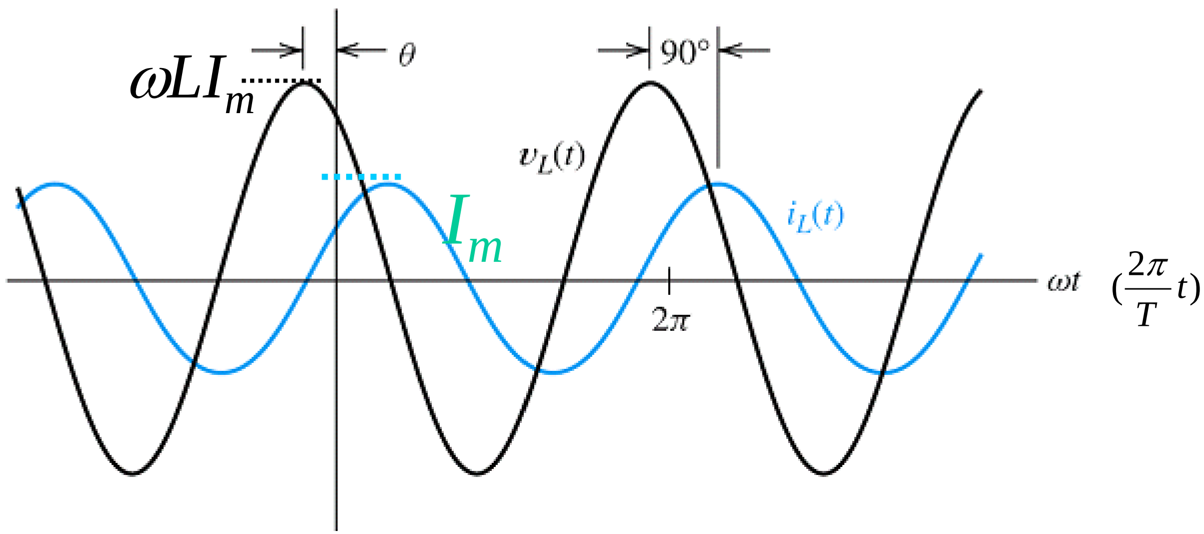
$$I_L = I_m \angle \theta - 90^\circ$$

$$V_L = \omega L I_m \angle \theta = V_m \angle \theta$$

Current lags the voltage by 90°



(a) Phasor diagram



(b) Current and voltage versus time

Ohm's Law

$$V = IR$$

What is the relationship between phaor vltage and phasor current? Does it be similar to Ohm's law?

$$I_L = I_m \angle \theta - 90^\circ$$

$$V_L = \omega L I_m \angle \theta = (\omega L \angle 90^\circ) \times I_m \angle \theta - 90^\circ$$

$$= (\omega L \angle 90^\circ) \times I_L = j\omega L \times I_L$$

$$\left(\because \quad \omega L \angle 90^\circ = \omega L(\cos 90^\circ + j \sin 90^\circ) \right)$$

$\Rightarrow$ $Z_L = j\omega L = \omega L \angle 90^\circ$ 稱為L的阻抗(impedance)

$$V_L = Z_L \times I_L$$

Phasor voltage = impedance ×phasor current (similar to Ohm's law)

Note the impedance of an inductance is an imaginary number 虛數 (called reactance). Resistance is a real number.

Capacitance

假設通過電容C的 sinusoidal voltage為

$$v_c(t) = V_m \cos(\omega t + \theta)$$

則電容C兩端的current為

$$i_c(t) = C\frac{dv_c(t)}{dt} = -\omega C V_m \sin(\omega t + \theta)$$

$\Longrightarrow$ $$V_C = V_m \angle \theta$$

$$I_c = -\omega C V_m \angle \theta - 90^\circ = \omega C V_m \angle \theta - 90^\circ + 180^\circ$$
$$= \omega C V_m \angle \theta + 90^\circ = I_m \angle \theta + 90^\circ$$

$$\because \quad -1 = \cos(180^\circ)$$

Capacitance

$$Z_c = \frac{V_c}{I_c} = \frac{V_m \angle \theta}{\omega C V_m \angle \theta + 90^\circ} = \frac{1}{\omega C} \angle -90^\circ$$

$$\left(= -j \frac{1}{\omega C} = \frac{1}{j\omega C} \right)$$

$\because$

$$V_C = V_m \angle \theta$$

$$I_c = \omega C V_m \angle \theta + 90^\circ$$

→ Current leads the voltage by 90°

Capacitance

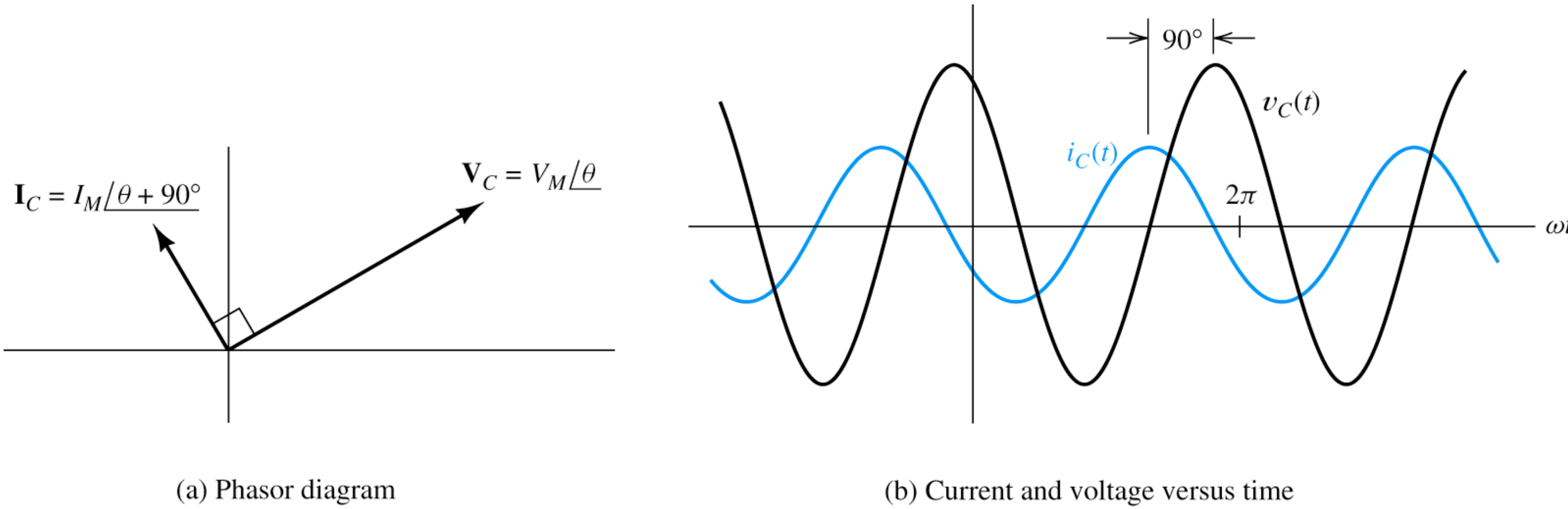


Figure 5.8 Current leads voltage by 90° in a pure capacitance.

Resistance

$$V_R = RI_R$$

The current and voltage are in phase. (同相位)

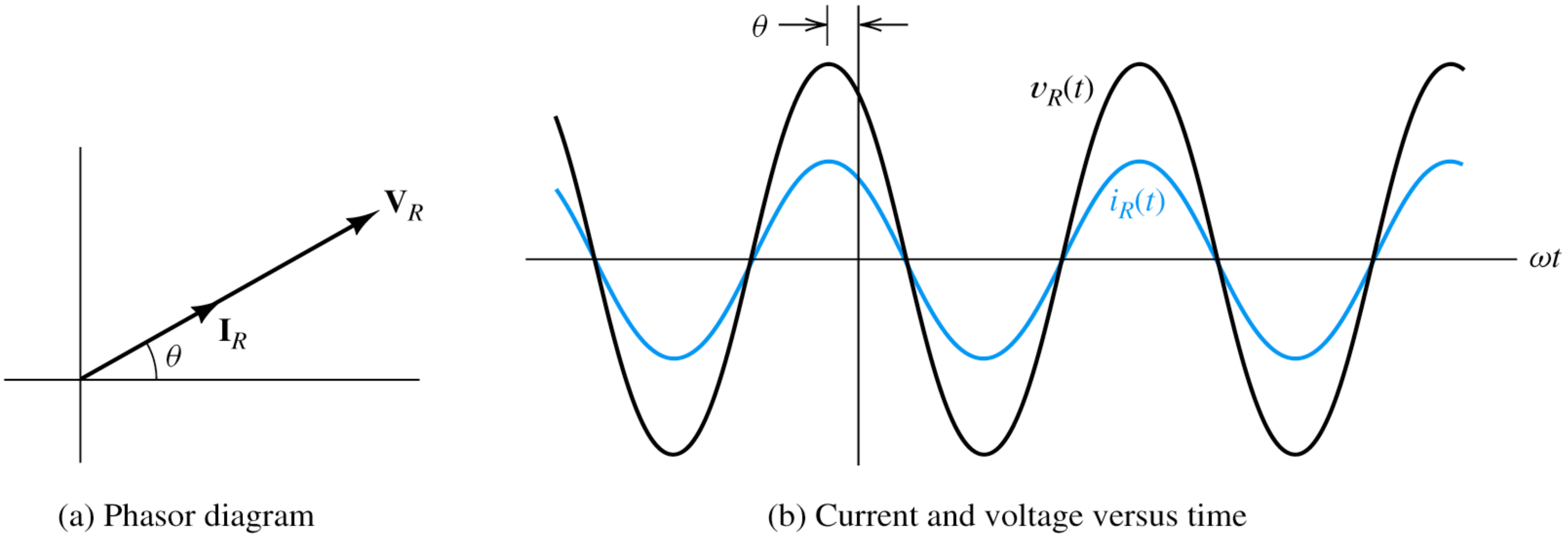
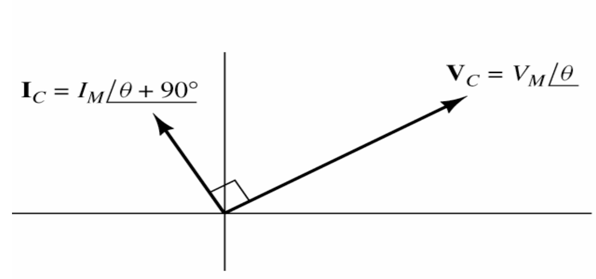
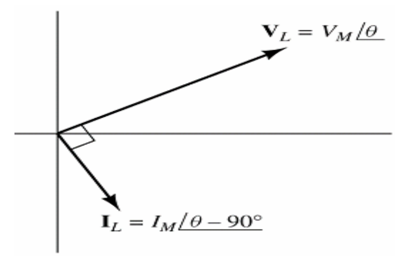
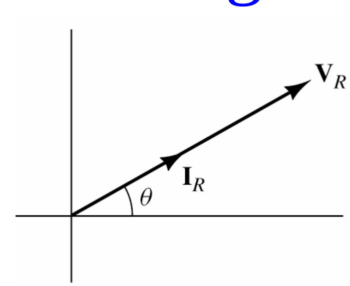


Figure 5.9 For a pure resistance, current and voltage are in phase.

電容阻抗	$Z_c = -j\dfrac{1}{\omega C}$	Current leads voltage by 90º 
電感阻抗	$Z_L = j\omega L$	Current lags voltage by 90º 
電阻	$Z_R = R$	Current and voltage are in phase 

5.4 Circuit Analysis with Phasors and Complex Impedances

Kirchhoff's Laws in Phasor Form

We can apply KVL directly to phasors. The sum of the phasor voltages equals zero for any closed path.

KCL-The sum of the phasor currents entering a node must equal the sum of the phasor currents leaving.

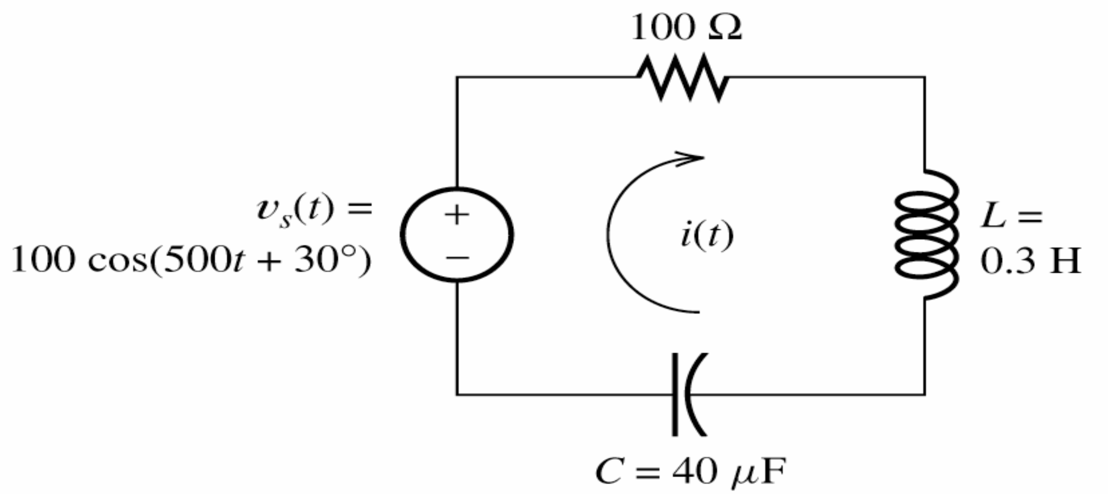
Circuit Analysis Using Phasors and Impedances

1. Replace the time descriptions of the voltage and current sources with the corresponding phasors. (All of the sources must have the same frequency.)

2. Replace inductances by their complex impedances $Z_L = j\omega L$. Replace capacitances by their complex impedances $Z_C = 1/(j\omega C)$ or $-j(1/\omega C)$. Resistances have impedances equal to their resistances.

3. Analyze the circuit using any of the techniques studied earlier in Chapter 2, performing the calculations with complex arithmetic.

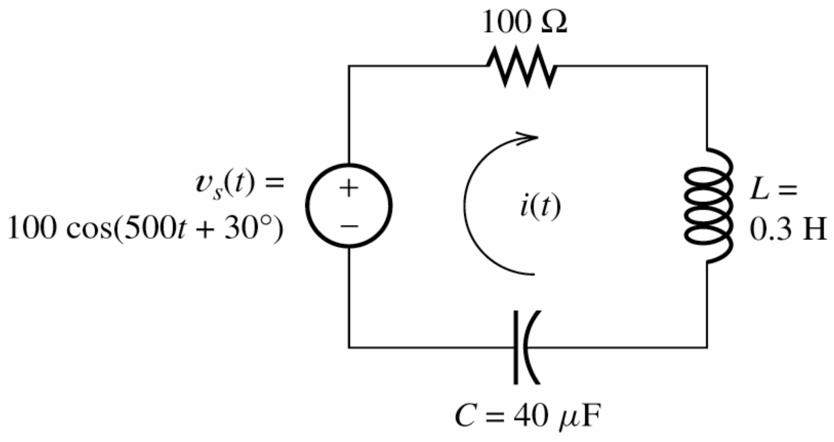
Example 5.4 Steady-state AC Analysis of a Series Circuit

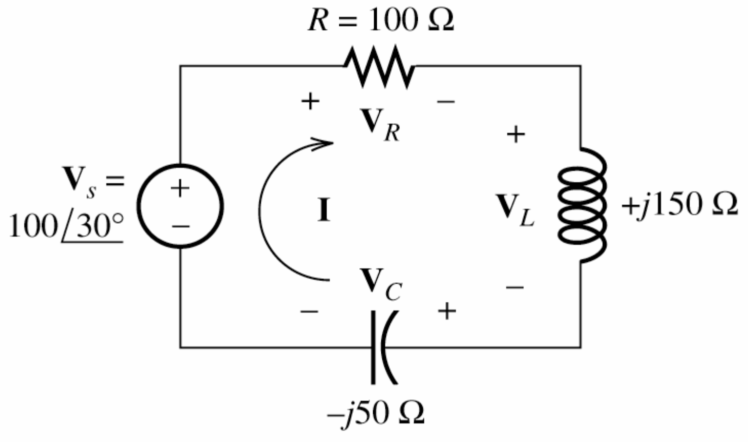


I?
V_R?
V_L?
V_C?

1. Phasors

$$V_s = 100\angle 30^\circ$$





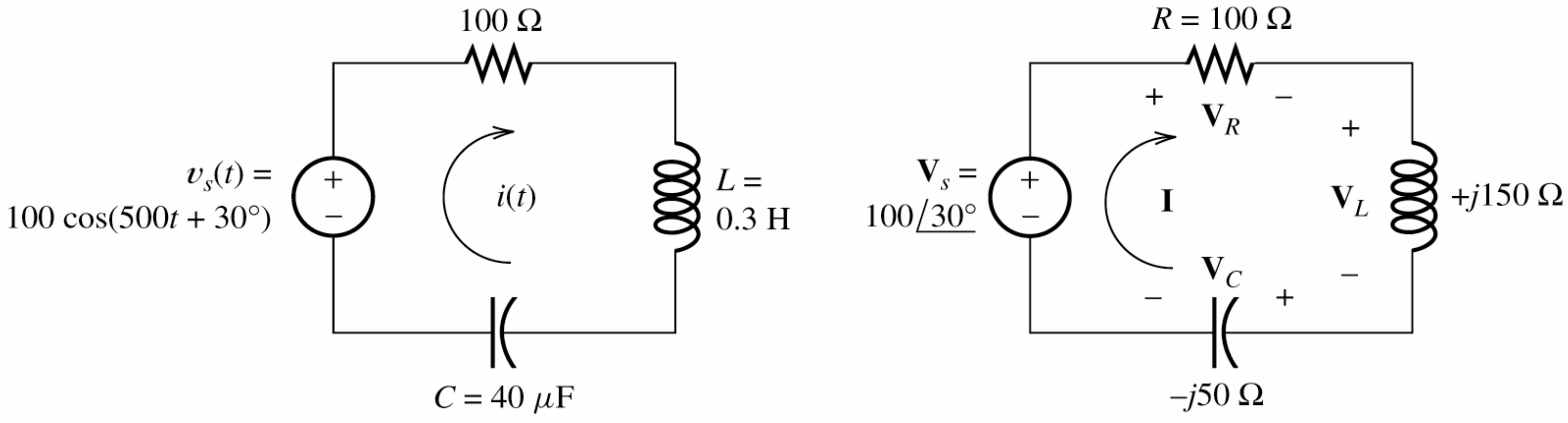
2. Complex impedances

$$Z_L = j\omega L = j500 \times 0.3 = j150\Omega$$

$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{500 \times 40 \times 10^{-6}} = -j50\Omega$$

$$Z_{eq} = R + Z_L + Z_C = 100 + j150 - j50 = 100 + j100$$

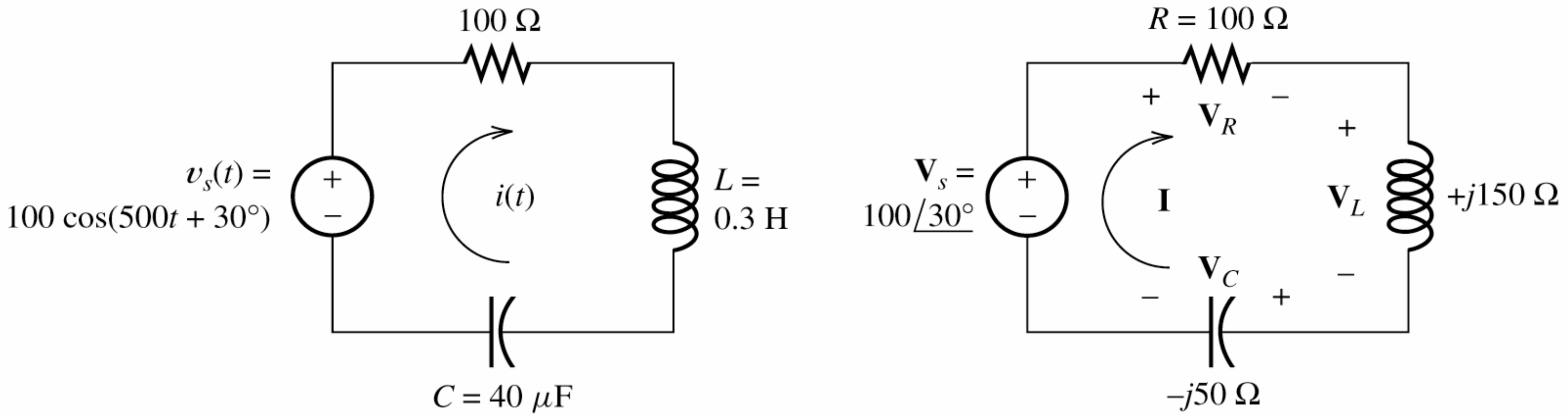
$$= 141.4\angle 45^\circ \qquad \left(\sqrt{100^2 + 100^2}\angle \arctan\frac{100}{100}\right)$$



3. Circuit Analysis

$$I = \frac{V_S}{Z_{eq}} = \frac{100\angle 30^\circ}{141.4\angle 45^\circ} = \frac{100}{141.4}\angle(30^\circ - 45^\circ) = 0.707\angle -15^\circ$$

$$\Rightarrow \quad i(t) = 0.707\cos(500t - 15^\circ)$$



3. Circuit Analysis

$$V_R = R \times I = 100 \times 0.707\angle -15^\circ = 70.7\angle -15^\circ$$

$$V_L = j\omega L \times I = \omega L\angle 90^\circ \times I = 150\angle 90^\circ \times 0.707\angle -15^\circ$$

$$= 106.1\angle 75^\circ$$

$$V_C = -j\frac{1}{\omega C} \times I = \frac{1}{\omega C}\angle -90^\circ \times I = 50\angle -90^\circ \times 0.707\angle -15^\circ$$

$$= 35.4\angle -105^\circ$$

Note $j = \angle 90^\circ, -j = \angle -90^\circ$

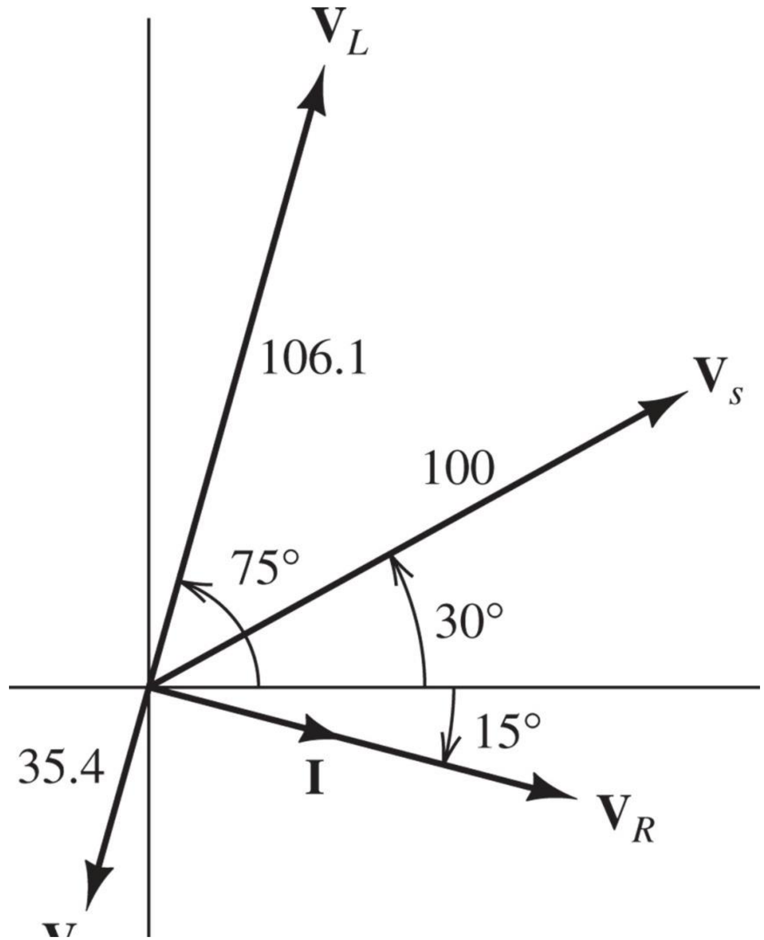
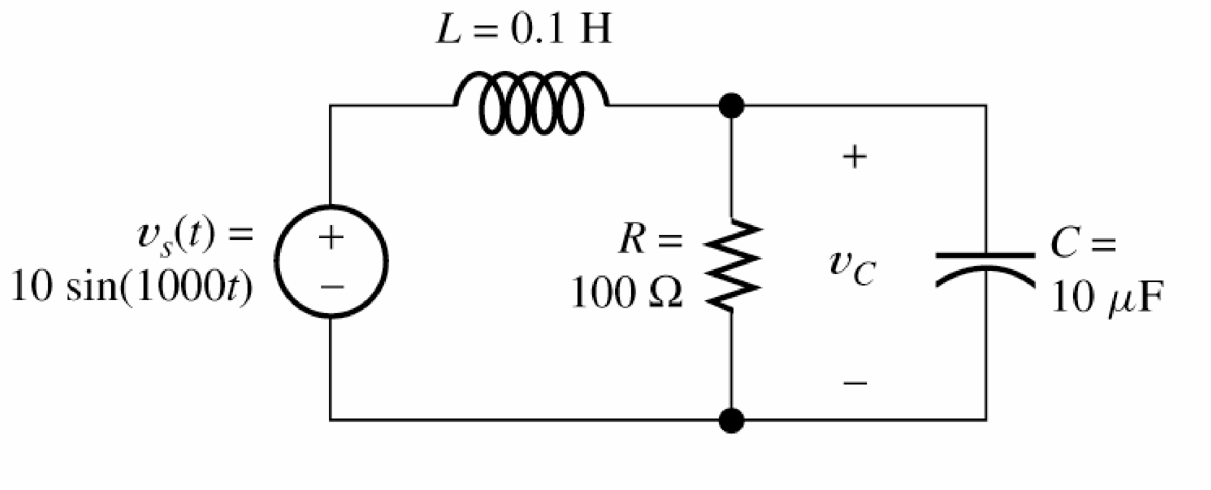


Figure 5.13 Phasor diagram for Example 5.4.

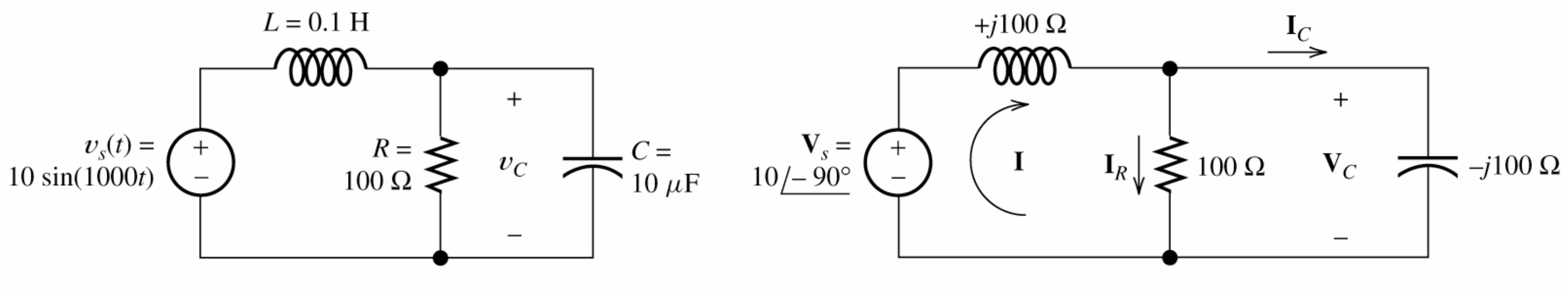
Example 5.5 Series and Parallel Combinations of Complex Impedances.



V_C?
I_L?
I_R?
I_C?

1. Phasors

$$V_s = 10\angle -90^{\circ}$$



2. Complex impedances

$$Z_L = j\omega L = j1000 \times 0.1 = j100\Omega$$

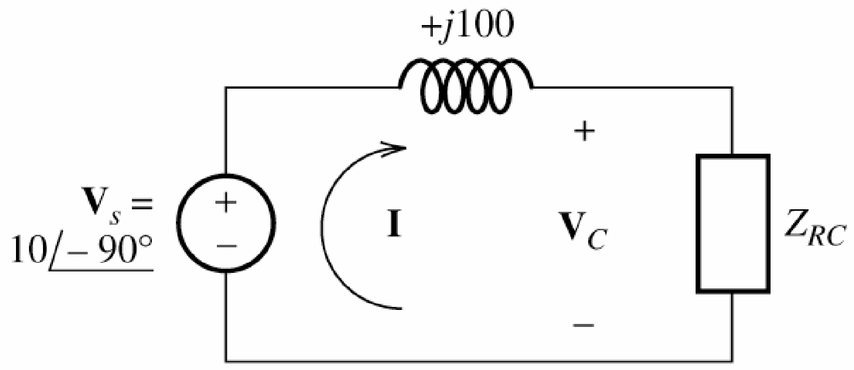
$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{1000 \times 10 \times 10^{-6}} = -j100\Omega$$

$$Z_{RC} = \frac{1}{1/R + 1/Z_C} = \frac{1}{1/100 + 1/(-j100)} = \frac{1}{0.01 + j0.01}$$

$$= \frac{1\angle 0^\circ}{0.01414\angle 45^\circ} = \boxed{70.71\angle -45^\circ} = 70.71(\cos -45^\circ + j\sin -45^\circ) = \boxed{50 - j50}$$

Or $$\frac{1}{0.01 + j0.01} = \frac{1}{0.01 + j0.01} \cdot \boxed{\frac{0.01 - j0.01}{0.01 - j0.01}} = \boxed{50 - j50}$$

$$= \sqrt{50^2 + 50^2}\angle \arctan\frac{-50}{50} = \boxed{70.71\angle -45^o}$$



$$50 - j50$$
$$= 70.71\angle -45^o$$

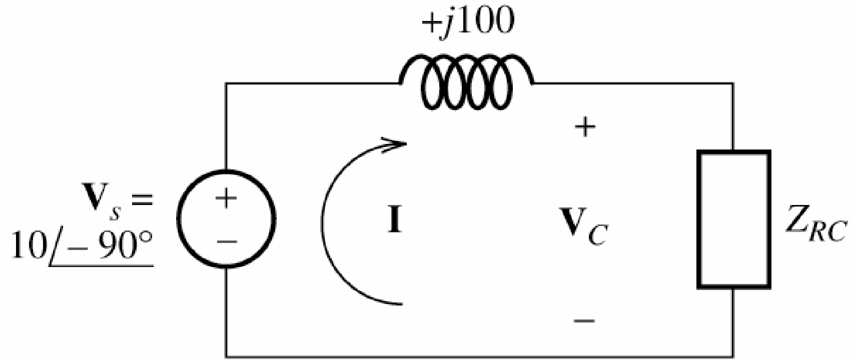
3. Circuit Analysis

$$V_C = V_s \frac{Z_{RC}}{Z_L + Z_{RC}} = 10\angle -90^\circ \frac{70.71\angle -45^\circ}{j100 + 50 - j50}$$

$$= 10\angle -90^\circ \frac{70.71\angle -45^\circ}{50 + j50} = 10\angle -90^\circ \frac{70.71\angle -45^\circ}{70.71\angle 45^\circ}$$

$$= 10\angle -180^\circ$$

$$v_C(t) = 10\cos(1000t - 180^\circ) = -10\cos(1000t)$$

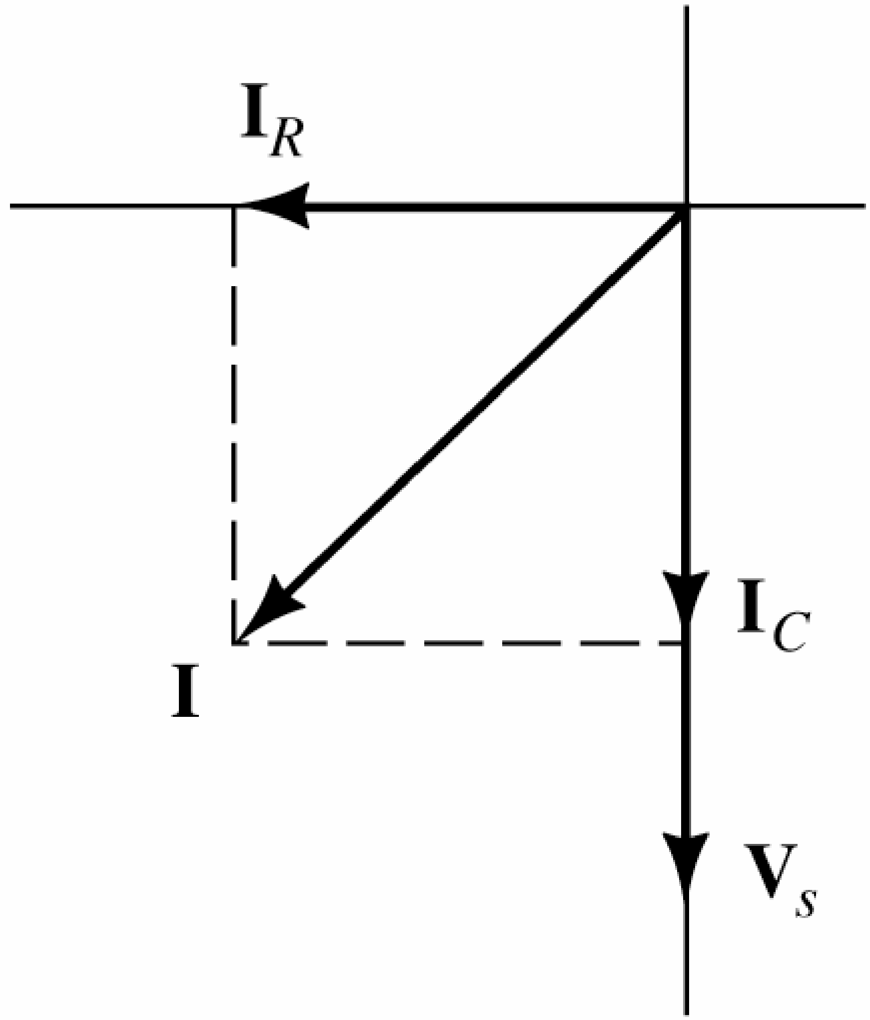


$$50 - j50$$
$$= 70.71\angle -45^{o}$$

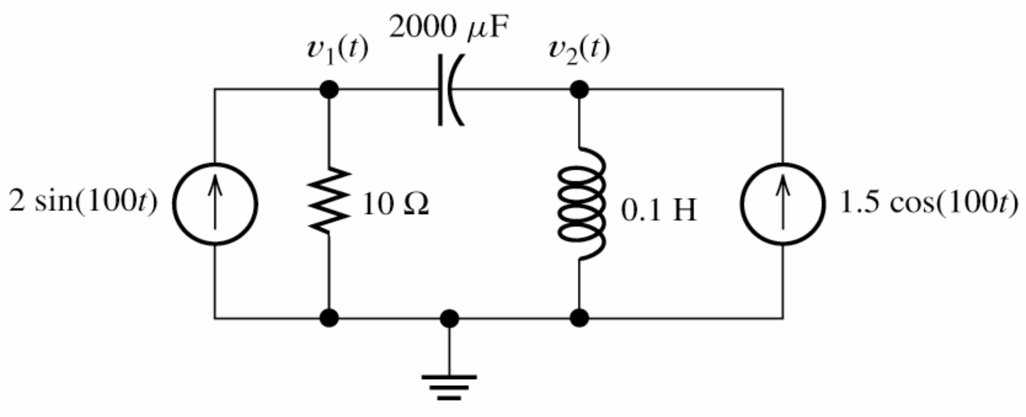
$$I = \frac{V_s}{Z_L + Z_{RC}} = \frac{10\angle -90^\circ}{j100 + 50 - j50} = \frac{10\angle -90^\circ}{50 + j50} = \frac{10\angle -90^\circ}{70.71\angle 45^\circ} = 0.1414\angle -135^\circ$$

$$I_R = \frac{V_C}{R} = \frac{10\angle -180^\circ}{100} = 0.1\angle -180^\circ$$

$$I_C = \frac{V_C}{Z_C} = \frac{10\angle -180^\circ}{-j100} = \frac{10\angle -180^\circ}{100\angle -90^\circ} = 0.1\angle -90^\circ$$


$\mathbf{I}_R$
$\mathbf{I}_C$
$\mathbf{I}$
$\mathbf{V}_s$

Example 5.6 Steady-State AC Node-Voltage Analysis

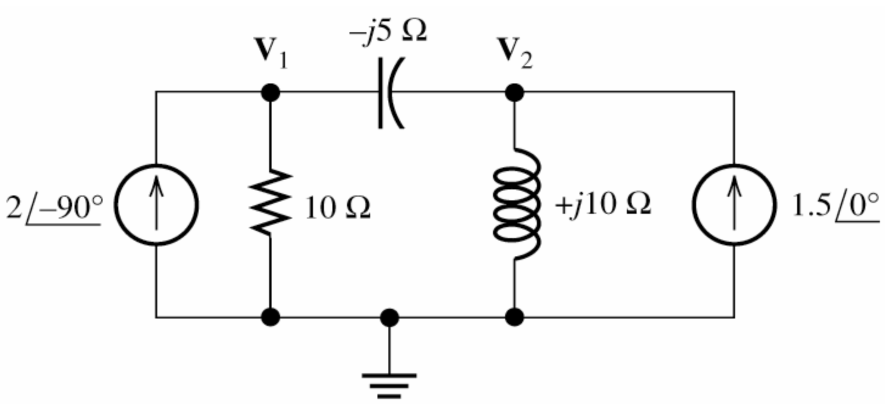


$v_1(t)$?

1. Phasors

$$2\sin(100t) = 2\angle -90^\circ$$

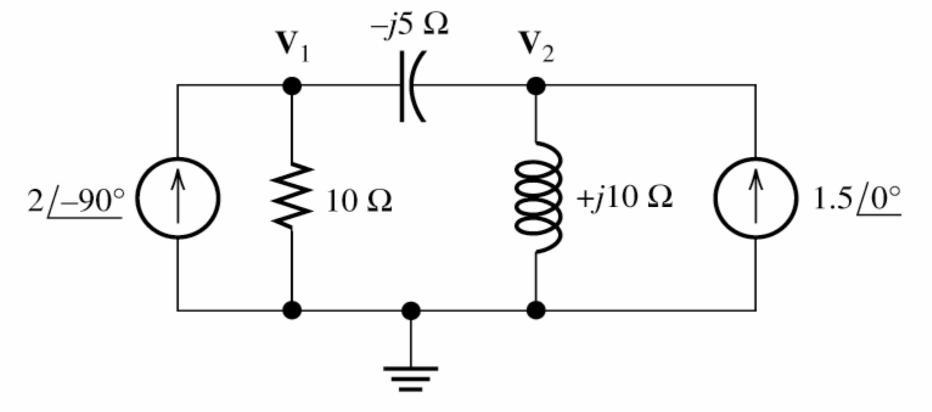
$$1.5\cos(100t) = 1.5\angle 0^\circ$$



2. Complex impedances

$$jwL = j100 \times 0.1 = j10$$

$$-j\frac{1}{wc} = -j\frac{1}{100 \times 2000 \times 10^{-6}} = -j5$$



3. Circuit Analysis

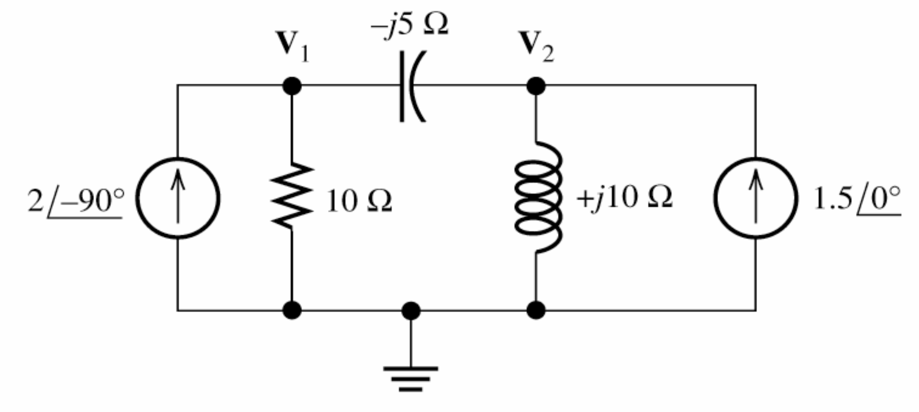
KCL Node 1
$$\frac{V_1}{10}+\frac{V_1-V_2}{-j5}=2\angle-90^\circ$$

KCL Node 2
$$\frac{V_2}{j10}+\frac{V_2-V_1}{-j5}=1.5\angle 0^\circ$$

$$(0.1+j0.2)V_1-j0.2V_2=-j2 \quad \because \boxed{\frac{1}{-j5}=\frac{1}{-j5}\cdot\frac{j}{j}=j0.2}$$

$$-j0.2V_1+j0.1V_2=1.5$$

$$\Rightarrow \quad (0.1-j0.2)V_1=3-j2$$

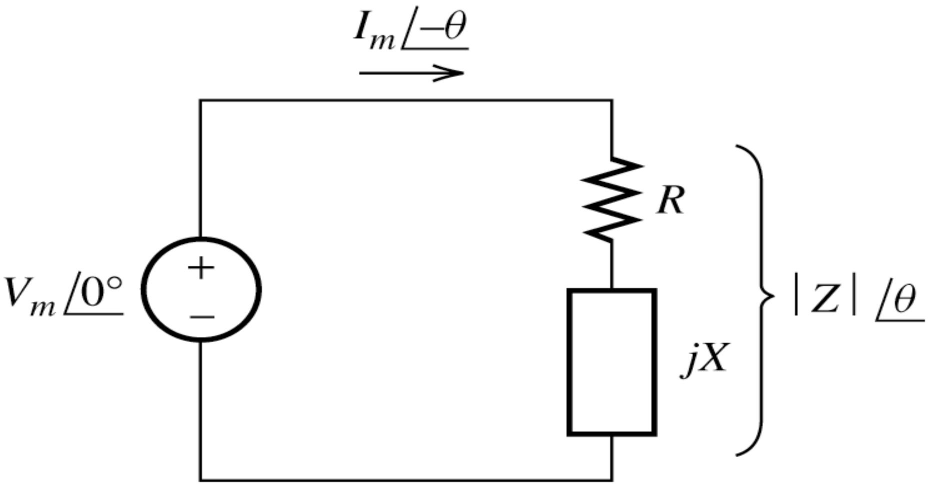


$$V_1 = \frac{3-j2}{0.1-j0.2} = \frac{3-j2}{0.1-j0.2}\cdot\frac{0.1+j0.2}{0.1+j0.2}$$

$$= \frac{0.7+j0.4}{0.05} = 14+j8 = 16.1\angle 29.7^\circ$$

$$\Longrightarrow\ v_1(t) = 16.1\cos(100t + 29.7^\circ)$$

5.5 Power in AC Circuits



A voltage source delivering power to a load impedance $Z = R + jX$.

阻抗有實數(電阻)或虛數 (電感, 電容),
功率有無實虛?

$$Z = R + jX = |Z|\angle\theta$$

$$I = \frac{V}{Z} = \frac{V_m}{|Z|}\angle -\theta = I_m\angle -\theta$$

$$\boxed{I_m = \frac{V_m}{|Z|}}$$

1. If X=0 → 實阻抗(純電阻)

$$\boxed{\begin{aligned} Z &= R, \\ \theta &= 0 \end{aligned}}$$

$$v(t) = V_m \cos(\omega t)$$

$$i(t) = I_m \cos(\omega t)$$

$$p(t) = v(t)i(t) = V_m I_m \cos^2(\omega t)$$

(參考Ch5.1)

$$p_{avg} > 0$$

→ 實功率 (real power)

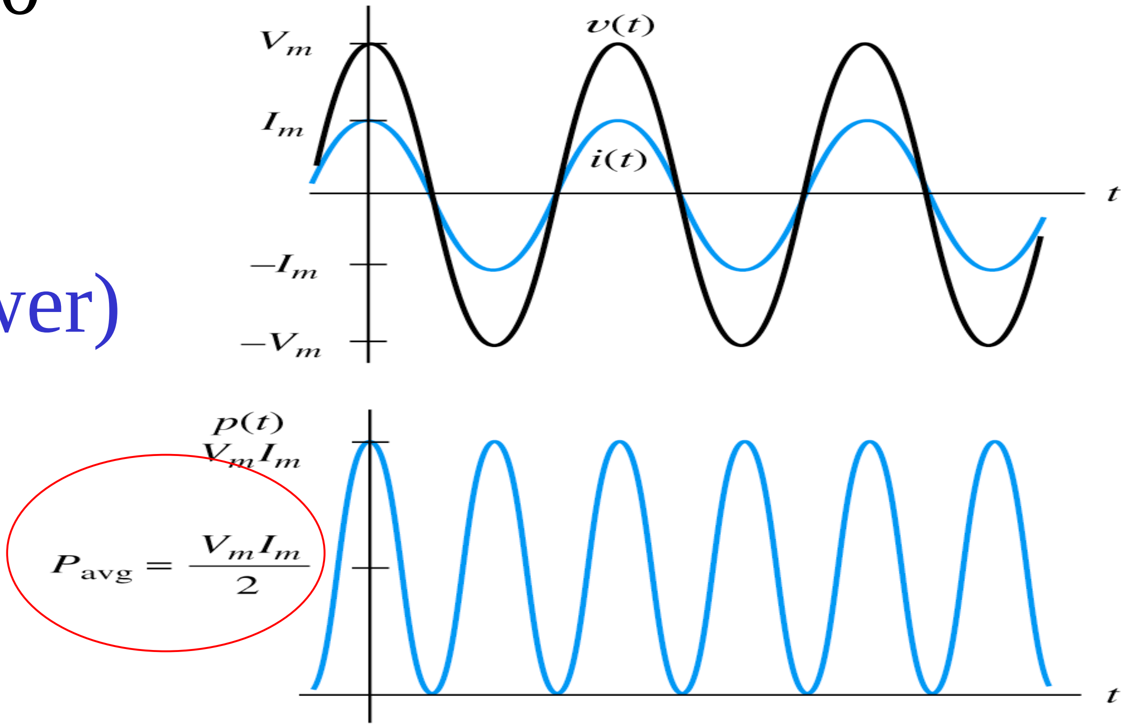


Figure 5.20 Current, voltage, and power versus time for a purely resistive load.

1. If X=0 → 實阻抗(純電阻)

$$P_{avg}=\frac{V_{rms}^2}{R}=I_{rms}^2 R$$

$$P_{avg}^2=\frac{V_{rms}^2}{R}\cdot I_{rms}^2 R=V_{rms}^2\cdot I_{rms}^2$$

$$P_{avg}=V_{rms}\cdot I_{rms}=\frac{V_m}{\sqrt{2}}\cdot\frac{I_m}{\sqrt{2}}=\frac{V_m I_m}{2}$$

$$\boxed{V_{rms}=\frac{V_m}{\sqrt{2}}}$$

2. If R=0, X>0 → 虛阻抗 (電感性)

$$\boxed{Z = jX, \quad \theta = 90^\circ}$$

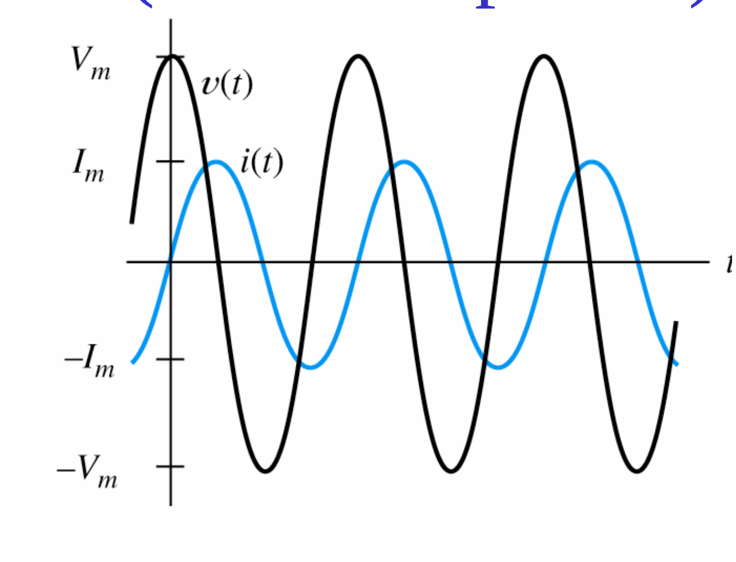
$$v(t) = V_m \cos(\omega t)$$

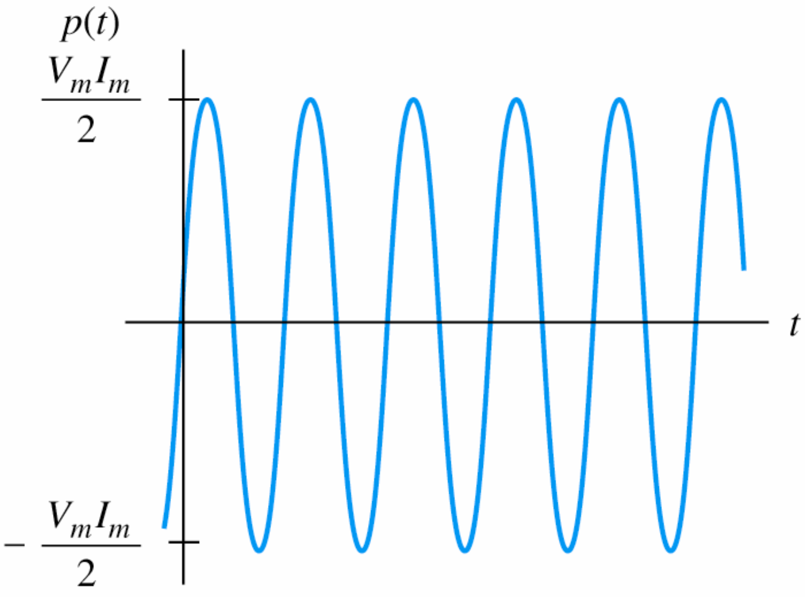
$$i(t) = I_m \cos(\omega t - 90^\circ) = I_m \sin(\omega t)$$

$$p(t) = v(t) i(t) = V_m I_m \cos(\omega t) \sin(\omega t) = \frac{V_m I_m}{2} \sin(2\omega t)$$

$$\boxed{p_{avg} = 0}$$

→ 虛功率 or 無效功率 (Reactive power)





3. If R=0, X<0 → 虛阻抗 (電容性)

$$v(t) = V_m \cos(\omega t)$$

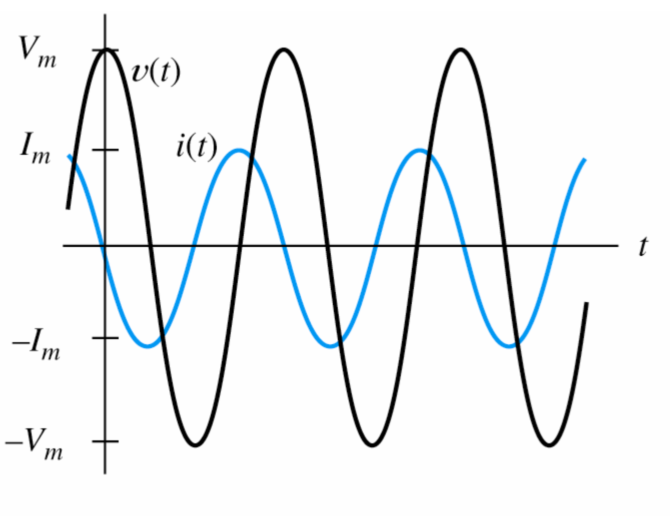
$$i(t) = I_m \cos(\omega t + 90^\circ) = -I_m \sin(\omega t)$$

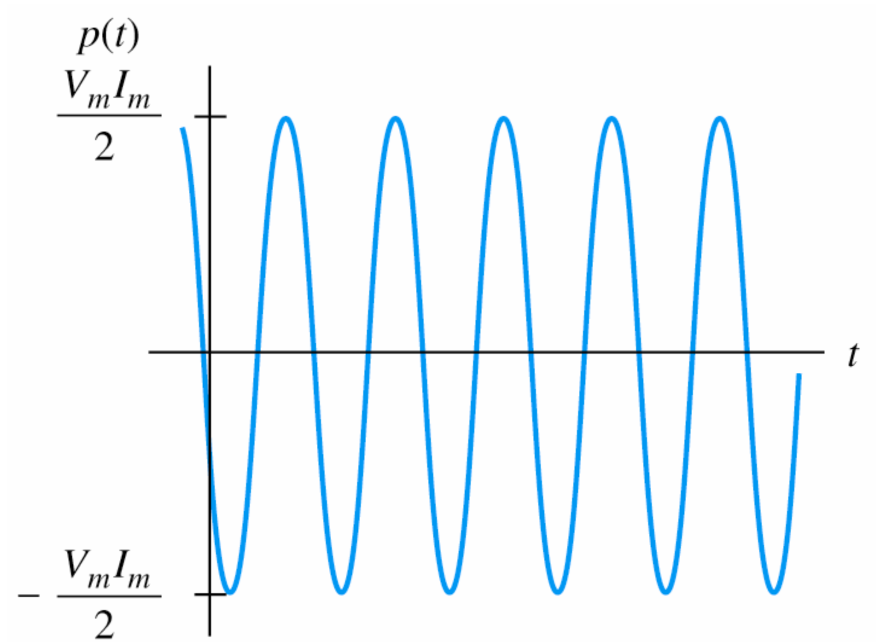
$$\boxed{Z = jX, \quad \theta = -90^\circ}$$

$$p(t) = v(t)i(t) = -V_m I_m \cos(\omega t)\sin(\omega t) = -\frac{V_m I_m}{2}\sin(2\omega t)$$

$$\boxed{p_{avg} = 0}$$

→ 虛功率 or 無效功率 (Reactive power)





Power for a General Load

一般狀況 R≠0且X≠0, RLC 電路 (有實與虛阻抗)

$$v(t) = V_m \cos(\omega t)$$

$$\boxed{Z = R + jX, \quad -90^0 < \theta < 90^\circ}$$

$$i(t) = I_m \cos(\omega t - \theta)$$

$$p(t) = V_m I_m \cos(\omega t)\cos(\omega t - \theta)$$

$$= V_m I_m \cos(\theta)\cos^2(\omega t) + V_m I_m \sin(\theta)\cos(\omega t)\sin(\omega t)$$

$$(\because \cos(\omega t - \theta) = \cos(\theta)\cos(\omega t) + \sin(\theta)\sin(\omega t))$$

$$= \frac{V_m I_m}{2}\cos(\theta)[1 + \cos(2\omega t)] + \frac{V_m I_m}{2}\sin(\theta)\sin(2\omega t)$$

$$(\because \cos^2(\omega t) = \frac{1}{2}(1 + \cos(2\omega t)), \quad \cos(\omega t)\sin(\omega t) = \frac{1}{2}\sin(2\omega t))$$

$$P_{avg} = \frac{1}{T}\int_0^T p(t)dt$$

$$= \frac{1}{T}\int_0^T \frac{V_m I_m}{2}\cos(\theta)[1+\underbrace{\cos(2wt)}_{\text{積分為0}}] + \frac{V_m I_m}{2}\sin(\theta)\underbrace{\sin(2wt)}_{\text{積分為0}}dt$$

$$= \frac{V_m I_m}{2}\cos(\theta)$$

$$\because \quad V_{\mathrm{rms}} = \frac{V_m}{\sqrt{2}}, I_{\mathrm{rms}} = \frac{I_m}{\sqrt{2}}$$

$$\Rightarrow \quad P_{avg} = V_{rms} I_{rms} \cos(\theta)$$

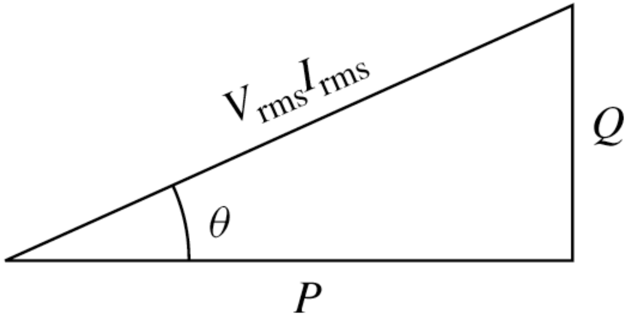
$$P_{avg} = P = V_{rms} I_{rms} \cos(\theta)$$ 為real power (有效功律or 實功率)
單位為W

$\cos(\theta)$ 為power factor, PF (功率因子)

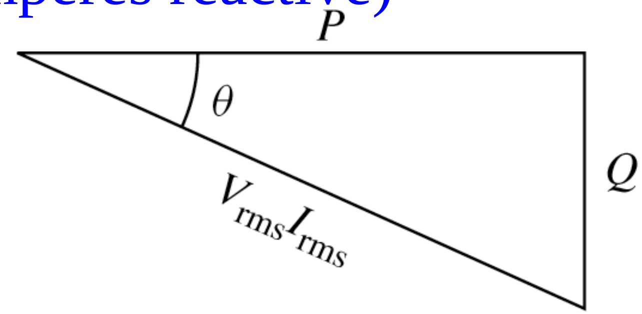
$$\theta = \theta_v - \theta_i$$ 為power angle (功率角)代表電流lags電壓的角度

$V_{rms} I_{rms}$ 為apparent power (視出功律) 單位為VA(volt-amperes)

$$Q = V_{rm} I_{rms} \sin(\theta)$$ 為reactive power (無效功律or 虛功率), 單位為 VAR (volt amperes reactive)



(a) Inductive load (θ positive)



(b) Capacitive load (θ negative)

Power triangles for inductive and capacitive loads.

AC Power Calculations

$$P = V_{\mathrm{rms}} I_{\mathrm{rms}} \cos(\theta) \quad \text{(W)}$$

$$\mathrm{PF} = \cos(\theta)$$

$$\theta = \theta_v - \theta_i$$

$$Q = V_{\mathrm{rms}} I_{\mathrm{rms}} \sin(\theta) \quad \text{(VAR)}$$

$$V_{rms} I_{rms} = \sqrt{P^2 + Q^2} \quad \text{(VA)}$$

$$\sqrt{P^2 + Q^2} = V_{\mathrm{rms}} I_{\mathrm{rms}}$$ (VA)

$$P = I_{\mathrm{rms}}^2 R$$ (W)

$$Q = I_{\mathrm{rms}}^2 X$$ (VAR)

$$P = I_{rms}^{2} R$$

$$\cos(\theta) = \frac{R}{|Z|}$$

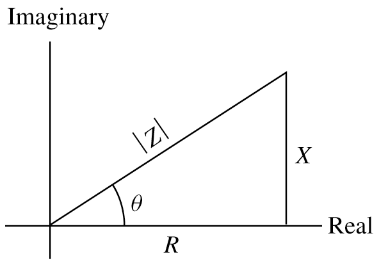


Figure 5.23 The load impedance in the complex plane.

Proof:

$$P = I_{rms} V_{rms} \cos\theta = I_{rms} \frac{V_m}{\sqrt{2}} \frac{R}{|Z|} = I_{rms} \frac{I_m}{\sqrt{2}} R = I_{rms}^{2} R$$

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

$$I = \frac{V}{Z}, I_m = \frac{V_m}{|Z|}$$

$$Q = I_{rms}^{2} X$$

Proof:

$$Q = I_{rms} V_{rms} \sin\theta = I_{rms} \frac{V_m}{\sqrt{2}} \frac{X}{|Z|} = I_{rms} \frac{I_m}{\sqrt{2}} X = I_{rms}^{2} X$$

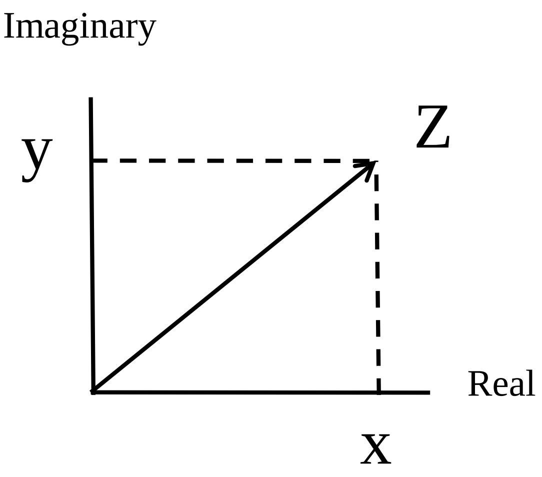
Appendix A
Complex Numbers

Basic Complex-Number Concepts

- Complex numbers involve the imaginary number(虛數) $j = \sqrt{-1}$

- $Z=x+jy$ has a **real part (實部)** x and an **imaginary part (虛部)** y , we can represent complex numbers by points in the **complex plane (複數平面)**

- The complex numbers of the form $x+jy$ are in **rectangular form (直角座標)**



- The **complex conjugate (共軛複數)** of a number in rectangular form is obtained by changing the sign of the imaginary part.

- For example if $Z_2 = 3 - j4$ then the complex conjugate of Z_2 is

$$Z_2^* = 3 + j4$$

- $j^2 = -1$

Example A.1 Complex Arithmetic in Rectangular Form

Given that $Z_1 = 5 + j5$ and $Z_2 = 3 - j4$, reduce $Z_1 - Z_2$, $Z_1 Z_2$, Z_1 / Z_2 to rectangular form

- Solution:

For the sum, we have

$$Z_1 + Z_2 = (5 + j5) + (3 - j4) = 8 + j1$$

The difference is

$$Z_1 - Z_2 = (5 + j5) - (3 - j4) = 2 + j9$$

Example A.1 Complex Arithmetic in Rectangular Form

For the product , we get

$$Z_1 Z_2 = (5 + j5)(3 - j4)$$

$$= 15 - j20 + j15 - j^2 20$$

$$= 15 - j20 + j15 + 20$$

$$= 35 - j5$$

To divide the numbers, we obtain

$$\frac{Z_1}{Z_2} = \frac{5+j5}{3-j4}$$

We can reduce this expression to rectangular form by multiplying the numberator and denominator by the complex conjugate of the denominator

$$\frac{Z_1}{Z_2} = \frac{5+j5}{3-j4} \times \frac{Z_2^*}{Z_2^*}$$

$$= \frac{5+j5}{3-j4} \times \frac{3+j4}{3+j4}$$

$$= \frac{15+j20+j15+j^2 20}{9+j12-j12-j^2 16}$$

$$= \frac{15+j20+j15-20}{9+j12-j12+16}$$

$$= \frac{-5+j35}{25}$$

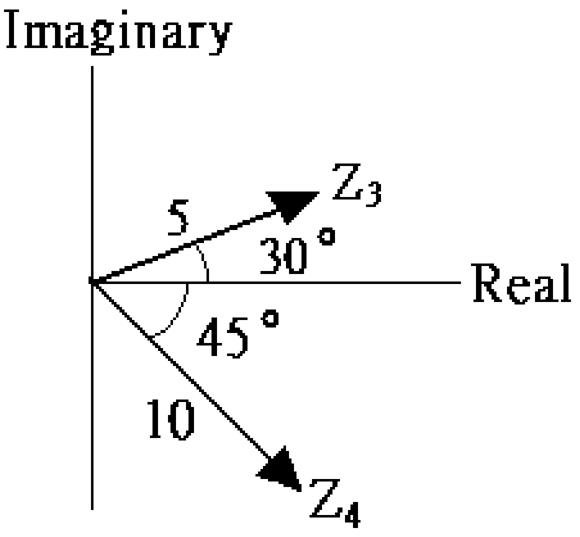
$$= -0.2+j1.4$$

Complex Numbers in Polar Form(極座標)

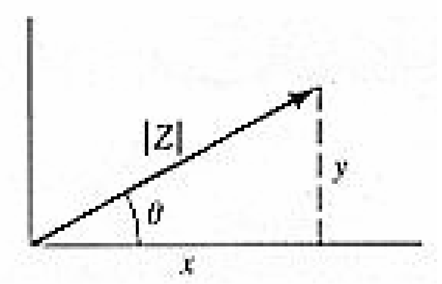
- Complex numbers can be expressed in **polar form (極座標).** Examples of complex numbers in polar form are :

$$Z_3 = 5\angle 30^\circ \quad \text{and} \quad Z_4 = 10\angle -45^\circ$$

The length of the arrow that represents a complex number Z is denoted as |Z| and is called the **magnitude** (幅值 or 大小) of the complex number.



- Using the magnitude |Z| , the real part *x*, and the imaginary part *y* form a right triangle (直角三角形).



$$|Z|^2 = x^2 + y^2 \qquad \text{(A.1)}$$

- Using trigonometry, we can write the following relationships:

$$\tan(\theta) = \frac{y}{x} \qquad \text{(A.2)}$$

$$x = |Z| \cos(\theta) \qquad \text{(A.3)}$$

$$y = |Z| \sin(\theta) \qquad \text{(A.4)}$$

These equations can be used to convert numbers from polar to rectangular form.

Example A.2 Polar-to Rectangular Conversion

$$\text{Convert } Z_3 = 5\angle 30^\circ \text{ to rectangular form}$$

Solution :

Using Equation A.3 and A.4 (pre. page) =>

$$x = |Z|\cos(\theta) = 5\cos(30^\circ) = 4.33$$

$$y = |Z|\sin(\theta) = 5\sin(30^\circ) = 2.5$$

$$\therefore Z_3 = 5\angle 30^\circ = x + jy = 4.33 + j2.5$$

Example A.3 Rectangular-to-Polar Conversion

Convert $Z_5 = 10 + j5$ and $Z_6 = -10 + j5$ to polar form.

Solution :

First, using Equation A.1 to find the magnitudes of each of the numbers

$$|Z_5| = \sqrt{x_5^2 + y_5^2} = \sqrt{10^2 + 5^2} = 11.18$$

$$|Z_6| = \sqrt{x_6^2 + y_6^2} = \sqrt{(-10)^2 + 5^2} = 11.18$$

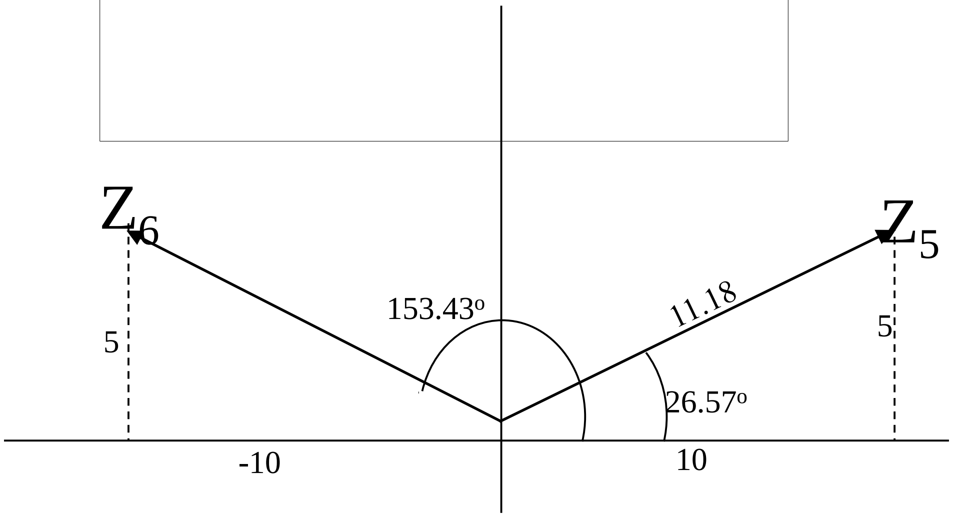


Figure A.4

To find the angles, we use Equation A.2.

For Z_5

$$\tan(\theta_5) = \frac{y_5}{x_5} = \frac{5}{10} = 0.5$$

$$\theta_5 = \arctan(0.5) = 26.57^\circ$$

$$\therefore Z_5 = 10 + j5$$

$$= 11.18\angle 26.57^\circ$$

For Z_6

$$\tan(\theta_6) = \frac{y_6}{x_6} = \frac{5}{-10} = -0.5$$

$$\theta_6 = 180 + \arctan(\frac{y_6}{x_6})$$

$$= 180 - 26.57 = 153.43^\circ$$

$$\therefore Z_6 = 10 + j5$$

$$= 11.18\angle 153.43^\circ$$

- The procedures that we have illustrated in Examples A.2 and A.3 can be carried out with a relatively simple calculator. However, if we find the angle by taking the arctangent of *y/x*, we must consider the fact that the principal value of the arctangent is the true angle only if the real part x is positive. If x is negative,

  we have:

$$\theta = \arctan\left(\frac{y}{x}\right) \pm 180^\circ$$

Euler's Identities

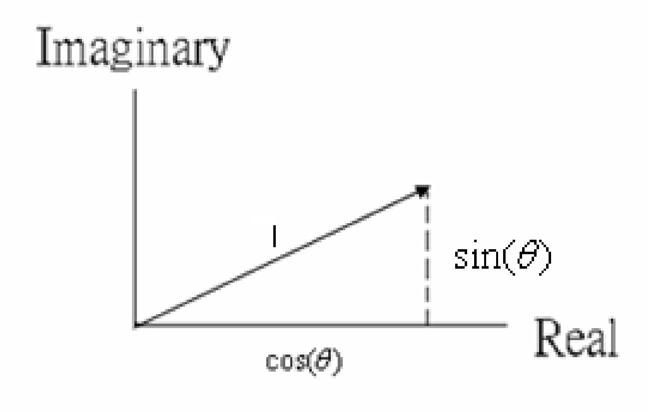
- The connection between sinusoidal signals and complex number is through Euler's identities, which state that

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \text{ and } \qquad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- Another form of these identities is

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \text{ and } \qquad e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

- $e^{j\theta}$ is a complex number having a real part of $\cos(\theta)$ and an imaginary part of $\sin(\theta)$



- The magnitude is

$$\left|e^{j\theta}\right| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

- The angle of $e^{j\theta}$ is θ

$$e^{j\theta} = 1\angle\theta = \cos(\theta) + j\sin(\theta)$$

$$e^{-j\theta} = 1\angle -\theta = \cos(\theta) - j\sin(\theta)$$

- A complex number such as $A\angle\theta$ can be written as

$$A\angle\theta = A\times(1\angle\theta) = Ae^{j\theta} = A\cos(\theta) + jA\sin(\theta)$$

- We call $Ae^{j\theta}$ the exponential form (指數形式)of a complex number.

- Given complex number can be written in three forms:
  - The rectangular form
  - The polar form
  - Exponential form

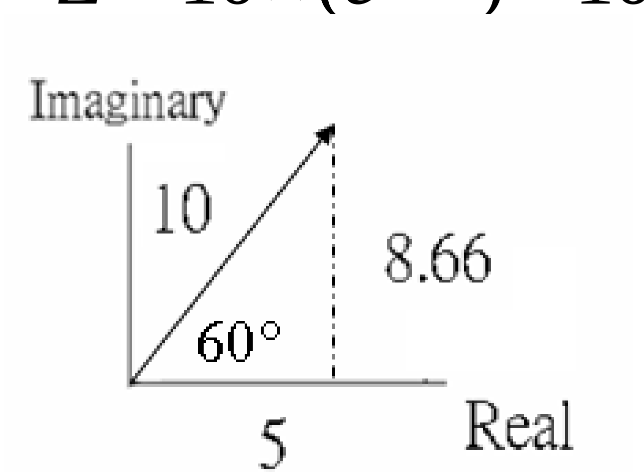
Example A.4 Exponential Form of a Complex Number

Express the complex number $Z = 10\angle 60^{\circ}$ in exponential and rectangular forms. Sketch the number in the complex plane

- Solution:

$$Z = 10\angle 60° = 10e^{j60°}$$

$$Z = 10 \times (e^{j60°}) = 10 \times [\cos(60°) + j\sin(60°)] = 5 + j8.66$$



Arithmetic Operations in Polar and Exponential Form

- To add (or subtract) complex numbers, we must first convert them to rectangular form. Then, we add (or subtract) real part to real part and imaginary to imaginary.

- Two complex numbers in exponential form:

$$Z_1 = |Z_1| e^{j\theta_1} \text{ and } Z_2 = |Z_2| e^{j\theta 2}$$

- The polar forms of these numbers are

$$Z_1 = |Z_1| \angle \theta_1 \text{ and } Z_2 = |Z_2| \angle \theta_2$$

- For multiplication of numbers in exponential form, we have $Z_1 \times Z_2 = |Z_1| e^{j\theta_1} \times |Z_2| e^{j\theta_2} = |Z_1||Z_2| e^{j(\theta_1+\theta_2)}$

- In polar form, this is $Z_1 \times Z_2 = |Z_1| \angle \theta_1 \times |Z_2| \angle \theta_2 = |Z_1||Z_2| \angle \theta_1 + \theta_2$

Proof:

$$Z_1 \times Z_2 = |Z_1|(\cos\theta_1 + j\sin\theta_1) \times |Z_2|(\cos\theta_2 + j\sin\theta_2)$$

$$= |Z_1||Z_2|(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + j(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1))$$

$$= |Z_1||Z_2|(\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2))$$

$$\left(\begin{aligned} sin(\alpha \pm \beta) &= sin\alpha cos\beta \pm \cos\alpha sin\beta \\ cos(\alpha \pm \beta) &= cos\alpha cos\beta \mp \sin\alpha sin\beta \end{aligned} \right)$$

- Now consider division: $$\frac{Z_1}{Z_2} = \frac{|Z_1| e^{j\theta_1}}{|Z_2| e^{j\theta_2}} = \frac{|Z_1|}{|Z_2|} e^{j(\theta_1 - \theta_2)}$$
- In polar form, this is $$\frac{Z_1}{Z_2} = \frac{|Z_1| \angle \theta_1}{|Z_2| \angle \theta_2} = \frac{|Z_1|}{|Z_2|} \angle \theta_1 - \theta_2$$

Example A.5
Complex Arithmetic in Polar Form

Given $Z_1 = 10\angle 60^\circ$ and $Z_2 = 5\angle 45^\circ$, find $Z_1 Z_2$, Z_1 / Z_2, and $Z_1 + Z_2$ in polar form.

- Solution:

$$Z_1 \times Z_2 = 10\angle 60^\circ \times 5\angle 45^\circ = 50\angle 105^\circ$$

$$\frac{Z_1}{Z_2} = \frac{10\angle 60^\circ}{5\angle 45^\circ} = 2\angle 15^\circ$$

Before we can add (or subtract) the numbers, we must convert them to rectangular form.

$$Z_1 = 10\angle 60^\circ = 10\cos(60^\circ) + j10\sin(60^\circ) = 5 + j8.66$$

$$Z_2 = 5\angle 45^\circ = 5\cos(45^\circ) + j5\sin(45^\circ) = 3.54 + j3.54$$

The sum as Zs:

$$Z_s = Z_1 + Z_2 = 5 + j8.66 + 3.54 + j3.54 = 8.54 + j12.2$$

Convert the sum to polar form:

$$|Z_s| = \sqrt{(8.54)^2 + (12.2)^2} = 14.9$$

$$\tan\theta_s = \frac{12.2}{8.54} = 1.43 \qquad \theta_s = \arctan(1.43) = 55^\circ$$

Because the real part of Z_s is positive, the correct angle is the principal value of the arctangent.

$$Z_s = Z_1 + Z_2 = 14.9\angle 55^\circ$$