Relativity

Einstein: The physical laws are the same in all inertial frames. (所有 inertial frams 有相同地位.)

speed of E.M. wave $= \sqrt{\frac{1}{\mu_0 \epsilon_0}} = 3 \times 10^8$ m/sec in all inertial frames.

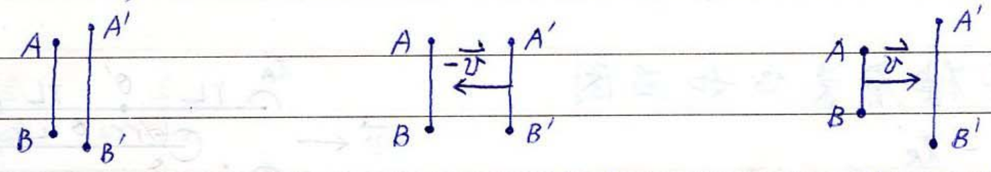
experiment $\pi^0 \rightarrow 2\gamma$

when velocity of $\pi^0 = 0.99975C$, vel. of $\gamma = C$

" " " " $= 0$, " " " = "

① 若A相对於B以$\vec{v}$運動,則B相对於A以$-\vec{v}$運動
(若$|\vec{v}_{AB}| \neq |\vec{v}_{BA}|$,則A&B有不同的"地位".)

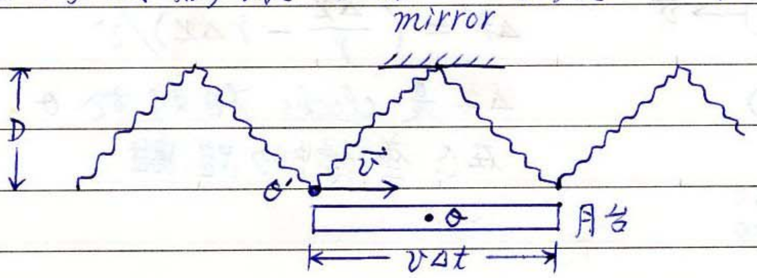
② 尺運動時,其與運動方向垂直(横向)的長度不變。
若有横向長度改變.假設運動時横向長度变短



靜止時　　A&A'(B&B')会相碰　　A&A'(B&B')不会相碰

矛盾,故長度不能變.

③ 時間膨脹與縱向長度收縮



在 O 的座標

E1(放光) E2(收光) two events 間

$\Delta t' = 2D/C$ (同地点)

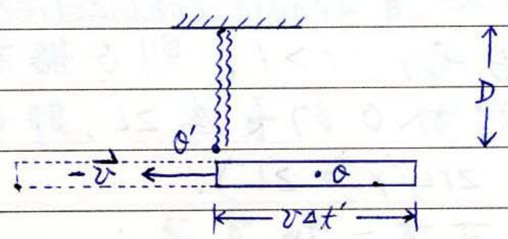
$\Delta t = 2\sqrt{D^2 + (\frac{1}{2} v \Delta t)^2} / C$ (不同地)

$c^2 \Delta t^2 = 4D^2 + v^2 \Delta t^2$

$(c^2 - v^2)\Delta t^2 = c^2 \Delta t'^2$

$\therefore \Delta t = \Delta t' / \sqrt{1 - v^2/c^2}$

∴ O'的鐘因運動而變慢了!



在 O' 的座標

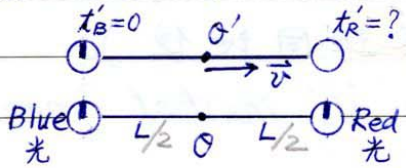
月台長度:相对於 $O = v\Delta t \equiv L$

" " " $O' = v\Delta t' \equiv L'$

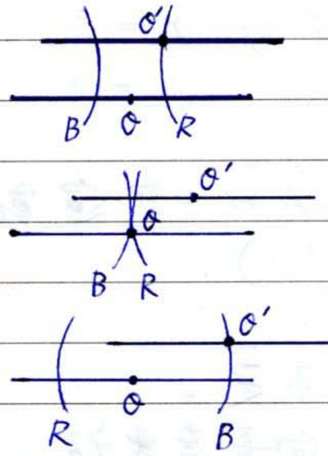
$\therefore \frac{L'}{L} = \frac{\Delta t'}{\Delta t} = \sqrt{1 - v^2/c^2}$

∴ O的尺(月台),因O運動而变短了!

④"同時"的相对性



二太空船端点对齊的瞬間，Red 與 Blue 光被發出，× 表發光瞬間.



後列附註

"facts"：

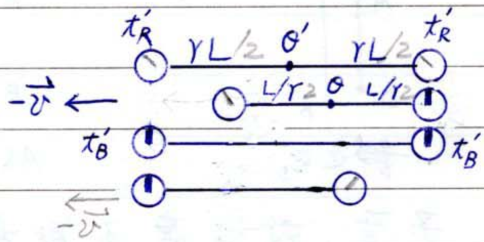
ⓐ O' 先碰上 Red，再碰到 Blue

ⓑ Red 與 Blue 的發光地点对 O'(或 O) 有相同距離，因都在船的端点上。

所以对 O' 而言，Red 與 Blue 的先後到達是因發光時間不同時 (因等距而光速又都是 c)，∴ $t'_B > t'_R$.

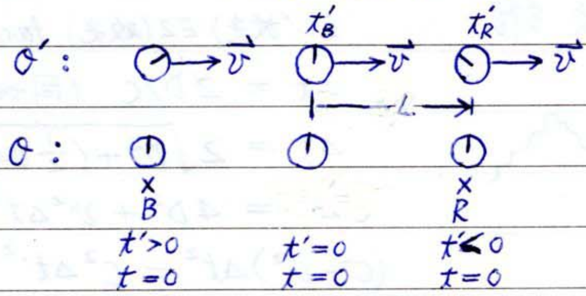
但对 O 而言，$t_B = t_R$，∴"同時"是相对的

~~故~~在 O' 的座標看来必如右圖



$$\therefore \Delta t'_{BR} \equiv t'_B - t'_R = \left(\gamma 2L - \frac{2L}{\gamma}\right)/v$$

故一串被 synchronized 的 clocks 相对於一觀察者運動時



O' 鐘面讀数 ($t'_B=0$)

$$t' = \left(\frac{L}{\gamma} - \gamma L\right)/v$$

L 是 clocks 相对於 O 在 O 中的距離.

附註：亦可推導 time dilation & length contraction 如下

i) 由"facts"知物体運動必有 length contraction. 假設靜止時長 l 的物体運動時縮為 l/γ, $\gamma>1$。則 O 船相对於 O' 的長度為 $2L/\gamma$。而 O' 船相对於 O 的長為 $2L$，那 O' 的靜止長度應為 $2\gamma L$ (故運動後長 $2\gamma L/\gamma = 2L$).

發光時間差 $\Delta t' \equiv t'_B - t'_R$ 可有二种算法：

ⓐ 由图知 $\Delta t' = 2(\gamma L - L/\gamma)/v$

ⓑ Red 與 Blue 同時碰到 O

但 R 發出後追上 O 需時 $= \frac{L}{\gamma}/(c-v)$

B 〃 〃 〃 碰上 O 〃 〃 $= \frac{L}{\gamma}/(c+v)$

故相对於 O' 的發光時間差 $\Delta t' = \frac{L}{\gamma}\left(\frac{1}{c-v} - \frac{1}{c+v}\right)$

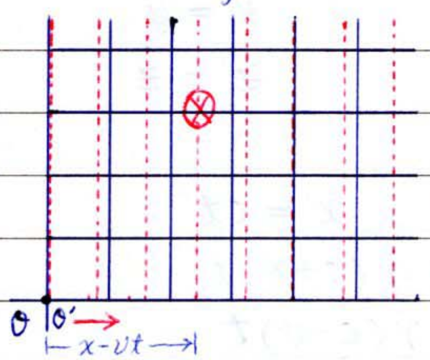
由 ⓐⓑ 相等得 $2(\gamma L - \frac{L}{\gamma})/v = \frac{L}{\gamma}\frac{2v}{c^2-v^2}$

$\Rightarrow \gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$ length contraction factor

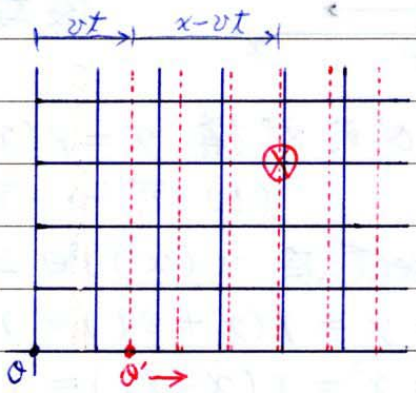
ii) 由 fact ⓑ 知, O' 被光碰到的時間差即等於發光時間差 $\Delta t' = \frac{L}{\gamma}\left(\frac{1}{c-v} - \frac{1}{c+v}\right)$。O' 被光碰到的兩事件是發生在同地點但不同時間。但对 O 而言,這是不同地點的兩事件, $\Delta t = \frac{L}{c-v} - \frac{L}{c+v}$

$\therefore \Delta t = \gamma \Delta t'$ time dilation.

Lorentz transformation



$t=0$ 時



$t>0$

O 與 O' 各帶一座標網, 在網的每一交叉點放一 clock, 且在 O 或 O' 座標內的所有 clocks 都被調為同時。

假設有一 O' 的 clock$^{c'}$ ⊗ 在 (x, y, z, t) 碰上一個 event,

$t=0$ 時該 clock$^{c'}$ 所在位置為 $(x-vt, y, z)$

$\therefore$ $t=0$ 時該 clock$^{c'}$ 的讀數為

$$\left(\frac{L}{\gamma} - \gamma L\right)/v = \left(\frac{x-vt}{\gamma} - \gamma(x-vt)\right)/v = (x-vt)\left(-\gamma + \frac{1}{\gamma}\right)/v$$

在 t 時, 該 clock$^{c'}$ 的讀數已增加了 t/γ (因 time dilation)

$$t' = +(x-vt)\left(\frac{1}{\gamma} - \gamma\right)/v + t/\gamma$$

$\therefore t' = (x-vt)\left(\sqrt{1-\frac{v^2}{c^2}} - \frac{1}{\sqrt{1-v^2/c^2}}\right)/v + t\sqrt{1-v^2/c^2} = \left(t - \frac{v}{c^2}x\right)/\sqrt{1-v^2/c^2}$, $x' = \gamma L = \frac{x-vt}{\sqrt{1-v^2/c^2}}$

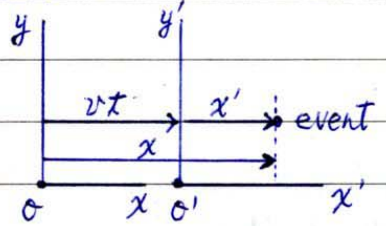
$$t' = (x - vt)\left(\sqrt{1-v^2/c^2} - \frac{1}{\sqrt{1-v^2/c^2}}\right)/v + t\sqrt{1-v^2/c^2}$$
$$= \left(t - \frac{v}{c^2}x\right)/\sqrt{1-v^2/c^2}$$

而該 clock ⊗ 的空間座標（即該 event 的空間座標）因 O' 座標網變密而為

$$x' = \gamma L = (x - vt)/\sqrt{1-\frac{v^2}{c^2}}$$

$$\begin{cases} x' = (x - vt)/\sqrt{1-\frac{v^2}{c^2}} \\ y' = y \\ z' = z \\ t' = \left(t - \frac{v}{c^2}x\right)/\sqrt{1-\frac{v^2}{c^2}} \end{cases} \qquad \begin{cases} x = (x' + vt')/\sqrt{1-\frac{v^2}{c^2}} \\ y = y' \\ z = z' \\ t = \left(t' + \frac{v}{c^2}x'\right)/\sqrt{1-\frac{v^2}{c^2}} \end{cases}$$

代數法：



$$x^2 + y^2 + z^2 = c^2t^2$$
$$x'^2 + y'^2 + z'^2 = c^2t'^2$$

假設 coord. transf. 是 $x' = \gamma(x - vt)$
$y' = y$,
$z' = z$

由 O & O' 的对稱 $x = \gamma(x' + vt')$
（即 $\vec{v} \to -\vec{v}$）

考慮 ~~event~~ 在 $x(x')$ 軸上 的光, $x = ct$, $x' = ct'$

$$ct = x = \gamma(x' + vt') = \gamma(ct' + vt') = \gamma(c+v)t'$$
$$ct' = x' = \gamma(x - vt) = \gamma(ct - vt) = \gamma(c-v)t$$
$$\therefore ct = \gamma(c+v)t' = \gamma(c+v)[\gamma(c-v)t/c] = \gamma^2(c^2 - v^2)t/c$$
$$c^2 = \gamma^2(c^2 - v^2), \quad \gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$$

$t' = ?$

$$x = \gamma(x' + vt') = \gamma(\gamma(x - vt) + vt')$$
$$(1 - \gamma^2)x + \gamma^2 vt = \gamma v t'$$
$$t' = \frac{1}{v}\sqrt{1-\frac{v^2}{c^2}}\left(1 - \frac{1}{1-v^2/c^2}\right)x + t/\sqrt{1-\frac{v^2}{c^2}}$$
$$= \left(t - \frac{v}{c^2}x\right)/\sqrt{1-v^2/c^2}$$

很容易証明 $c^2t^2 - x^2 - y^2 - z^2$ 與座標無関.

由 Lorentz transf. 導 length contraction & time dilation

time dilation:

考慮一 clock 靜止於 O' frame，clock 指到時間 t_1' 與指到 t_2' 為發生於同地點但不同時間的 two events，即 $\Delta x'=0$，$\Delta t'\neq 0$.

相対於 O，$\Delta t = \gamma(\Delta t' + \frac{v}{c^2}\Delta x')$

$= \gamma \Delta t'$，即 $\Delta t > \Delta t'$ time dilation

length contraction:

考慮一尺靜止於 O' frame. O frame 的 observer 要量尺的長度必需同時取下尺兩端的座標，即 $\Delta t = 0$，$\Delta x =$ 尺長

$\Delta x' = \gamma(\Delta x - v\Delta t)$

$= \gamma \Delta x$ ， $\Delta x' > \Delta x$

$\Delta x'$ 為尺靜止於 O' 的長度，Δx 為 O 量得的長度，

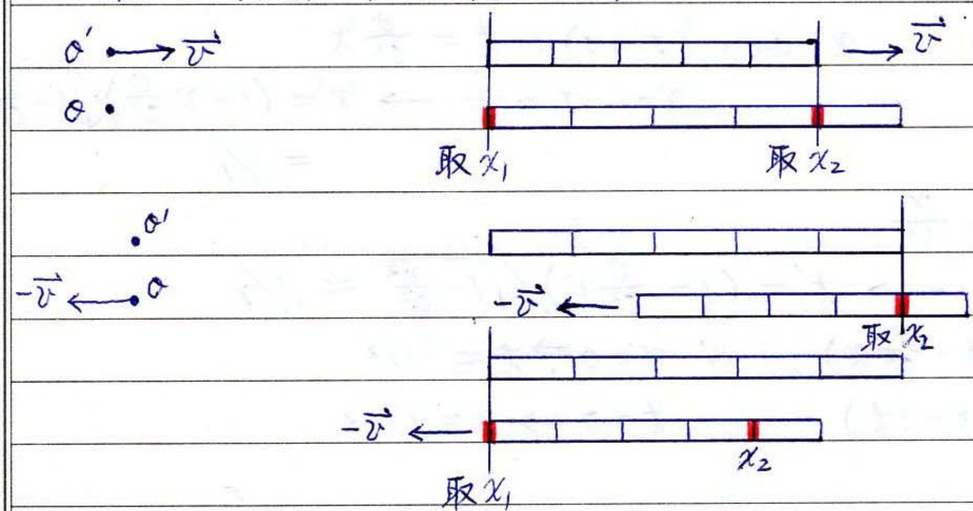
∴ length contraction.

$t=0$ 時 O' clocks 的讀數：

$t' = (t - \frac{v}{c^2}x)/\sqrt{1-\frac{v^2}{c^2}} = -\frac{v}{c^2}x/\sqrt{1-\frac{v^2}{c^2}}$

$= (\frac{x}{\gamma} - \gamma x)/v$ 如以前所得.

O 說 O' 的尺短，O' 說 O 的尺短，矛盾？



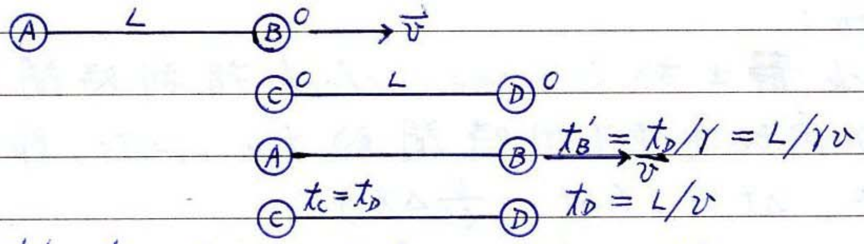
在 O 中同時取.

∴ 不矛盾，在 O' 中取 x_1 與取 x_2 不同時.

O 說 O' 的 clock 慢, O' 說 O 的慢, 矛盾?

O'-frame:

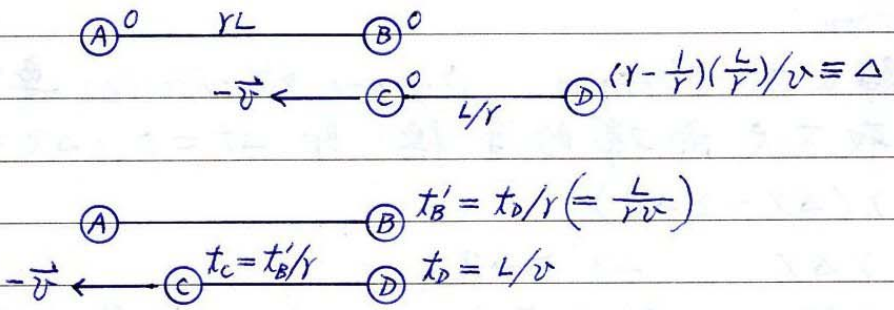
$\begin{pmatrix} O' \longrightarrow \vec{v} \\ O\ \cdot \end{pmatrix}$



$t_B' < t_D$ (B & D clocks are compared)

O'-frame:

$\begin{pmatrix} \cdot\ O' \\ \longleftarrow O \end{pmatrix}$



$\because\ t_D - t_C = \Delta$

$$\therefore\ t_C = t_D - \Delta = \frac{L}{v} - \left(\gamma - \frac{1}{\gamma}\right)\left(\frac{L}{\gamma}\right)/v$$

$$= \frac{1}{\gamma}\frac{L}{\gamma v} = t_B'/\gamma$$

$t_C < t_B'$ (C & B clocks are compared)

$\therefore$ 不矛盾, 被比較的 clocks 不一樣 (B & D 及 B & C).

Transformation of velocity

$$\Delta x = \gamma(\Delta x' + v\Delta t')$$

$$\Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)$$

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v\,dt')}{\gamma(dt' + \frac{v}{c^2}dx')} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u_x' + v}{1 + u_x' v/c^2}$$

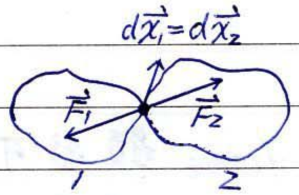
$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + \frac{v}{c^2}dx')} = \frac{u_y'\sqrt{1 - v^2/c^2}}{1 + u_x' v/c^2} = \frac{u_y'/\gamma}{1 + u_x' v/c^2}$$

$$u_z = \frac{u_z'\sqrt{1 - v^2/c^2}}{1 + u_x' v/c^2} = \frac{u_z'/\gamma}{1 + u_x' v/c^2}$$

例: 若光相对於 O' 以 $u_x' = c$ 前進, 則相对於 O

$$u_x = \frac{c + v}{1 + cv/c^2} = c$$

~~假設~~作用力-反作用力定律在 relativity 中仍成立 (for contact force, not for force at distance.)



$$\vec{F}_2 = -\vec{F}_1$$

作用點 $dt_1 = dt_2$, $d\vec{x}_1 = d\vec{x}_2$

$$d\vec{p} \equiv \vec{F} dt, \quad dE \equiv \vec{F} \cdot d\vec{x}$$

則 $d(\vec{P}_1 + \vec{P}_2) = \vec{F}_1 dt_1 + \vec{F}_2 dt_2 = \vec{F}_1 dt_1 + (-\vec{F}_1) dt_1 = 0$

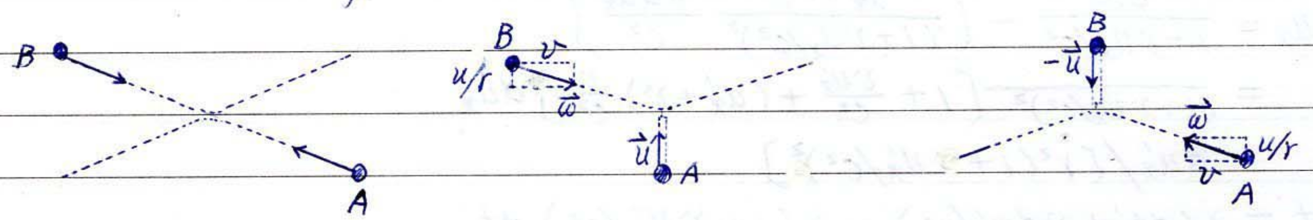
$$d(E_1 + E_2) = \vec{F}_1 \cdot d\vec{x}_1 + \vec{F}_2 \cdot d\vec{x}_2 = 0$$

$\therefore$ $\vec{p}$ & $\vec{E}$ are conserved for an isolated system.

$\because$ space is isotropic, $\therefore$ $\vec{p} \propto \vec{v}$

assume $\vec{p} = m(v)\vec{v}$, $m(v)$ is some unknown function.

Elastic collision of two identical masses



C.M. frame　　　O(A) frame　　　O′(B) frame

$u'_x = 0$, $u'_y = -u$ (for B) $\Longrightarrow$ $u_y = -u/\gamma$ for B in O frame

momentum conservation in y-direction

$$m(u)u - m(w)u/\gamma = -m(u)u + m(w)u/\gamma$$

$$2m(u)u = 2m(w)u/\gamma$$

$$\therefore m(w) = m(u)\gamma$$

let $u \to 0$, $m(v) = m(0)\gamma = m_0 / \sqrt{1 - \frac{v^2}{c^2}}$

momentum $\vec{p} = m\vec{v} = \frac{m_0}{\sqrt{1 - v^2/c^2}} \vec{v}$

$$\vec{F} = d\vec{p}/dt = \frac{d}{dt}\left(\frac{m_0}{\sqrt{1 - v^2/c^2}}\vec{v}\right) \neq m_0 \vec{a}$$
$$\neq m\vec{a}$$

transverse acc. ($v = $ const.) : $\vec{F} = m(v)\, d\vec{v}/dt = m\vec{a}$

longitudinal acc.:

$$F = \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1-v^2/c^2}}\right) = m_0\left[\frac{\sqrt{1-v^2/c^2} - v\left(1-\frac{v^2}{c^2}\right)^{-1/2}\frac{1}{2}\left(-\frac{2v}{c^2}\right)}{1-v^2/c^2}\right]\frac{dv}{dt}$$

$$= m_0\left[\frac{1-\frac{v^2}{c^2}+\frac{v^2}{c^2}}{(1-v^2/c^2)^{3/2}}\right]a = \frac{m_0}{(1-v^2/c^2)^{3/2}}a$$

$m_0/\left(1-\frac{v^2}{c^2}\right)^{3/2}$: longitudinal mass

$m_0/\left(1-\frac{v^2}{c^2}\right)^{1/2}$: transverse mass.

Mass & Energy

$$W = \int F\,dx = \int \frac{dp}{dt}dx = \int \frac{dp}{dt}v\,dt = \int v\,dp = \int d(pv) - \int p\,dv$$

$$= \int_0^{v_f} d(mv^2) - \int_0^{v_f} mv\,dv$$

$$= m_f v_f^2 - \int_0^{v_f}\frac{m_0 v}{\sqrt{1-v^2/c^2}}dv = m_f v_f^2 + \int_0^{v_f} m_0 c^2\, d\left(\sqrt{1-v^2/c^2}\right)$$

$$\therefore K.E. = W = mv^2 + m_0c^2\sqrt{1-v^2/c^2}\,\Big|_0^{v}$$

$$= mv^2 + m_0c^2\sqrt{1-v^2/c^2} - m_0c^2$$

$$= mv^2 + mc^2\left(1-\frac{v^2}{c^2}\right) - m_0c^2$$

$$= (m-m_0)c^2 = \cancel{\Delta m c^2}$$

$$\therefore mc^2 = m_0c^2 + K.E. = E \quad \text{total energy}$$

$$K.E. = \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)m_0c^2$$

if $v \ll c$

由 $(1+x)^n = 1 + nx + n(n-1)x^2/2 + \cdots$

$$\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots$$

$$\therefore K.E. = m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2} + \cdots\right)$$

$$= \frac{1}{2}m_0v^2 + \cdots$$

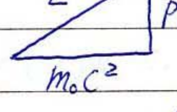
Energy & Momentum

$$E = \gamma m_0 c^2, \quad p = \gamma m_0 v$$

$$E^2 - p^2c^2 = \gamma^2 m_0^2 c^4 - \gamma^2 m_0^2 v^2 c^2 = \gamma^2 m_0^2 c^2 (c^2 - v^2)$$

but $\gamma^2 = 1/\left(1-\frac{v^2}{c^2}\right) = c^2/(c^2-v^2)$

$$\therefore E^2 - p^2c^2 = m_0^2c^4 \quad \text{or} \quad E = \sqrt{m_0^2c^4 + p^2c^2}$$



$$\beta = \frac{v}{c} = \frac{cp}{E}$$