Solutions to Final Exam

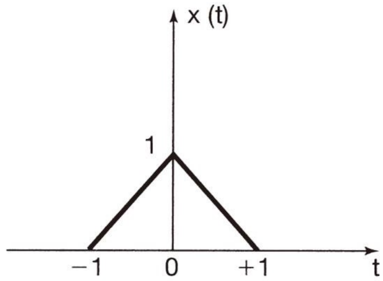
1. (a) True. Periodic $\xleftrightarrow{\mathcal{F}}$ Discrete.
   (b) True.
   (c) False. $\because \omega_s < 2 * 3$.
   (d) False. $\because X(e^{j(\omega+2\pi)}) \neq X(e^{j\omega})$.
   (e) False. $x(t)$ should be real.
   (f) False. "real and odd" $\xleftrightarrow{\mathcal{F}}$ "purely imaginary and odd".
   (g) True. $\because x[0] = 0$.
   (h) True. $\because \sum_{n=-\infty}^{\infty} x[n] = 0$.
   (i) True.
   (j) False. "real and even" $\xleftrightarrow{\mathcal{F}}$ "real and odd".

2. Let $Y(j\omega) = \text{sinc}\left(\frac{\omega}{2\pi}\right)$. Since

$$X(j\omega) = \text{sinc}^2\left(\frac{\omega}{2\pi}\right) = Y(j\omega) \cdot Y(j\omega)$$

we have

$$x(t) = y(t) * y(t) = \text{rect}(t) * \text{rect}(t) = \begin{cases} 1+t, & -1 \le t \le 0 \\ 1-t, & 0 \le t \le 1 \\ 0, & \text{otherwise.} \end{cases}$$



3. Since

$$\text{sgn}(t) \xleftrightarrow{\mathcal{F}} \frac{2}{j\omega}$$

we have

$$\frac{2}{jt} \xleftrightarrow{\mathcal{F}} 2\pi\text{sgn}(-\omega) = -2\pi\text{sgn}(\omega)$$

according to the duality. Hence,

$$x(t) = \frac{1}{\pi t} \xleftrightarrow{\mathcal{F}} -j\text{sgn}(\omega).$$

4. (a) We know

$$\mathrm{sgn}[n] \stackrel{\mathcal{F}}{\longleftrightarrow} 2/(1-e^{-j\omega}).$$

Since $u[n] = (1+\mathrm{sgn}[n])/2$, we have

$$U(e^{j\omega}) = \frac{1}{2}\left(\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-2\pi k) + \frac{2}{1-e^{-j\omega}}\right) = \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega-2\pi k).$$

(b) Let

$$y[n] = \sum_{m=-\infty}^{n} x[m] = x[n] * u[n].$$

We have the Fourier transform of $y[n]$ given by

$$\begin{aligned} X(e^{j\omega})U(e^{j\omega}) &= \frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi \sum_{k=-\infty}^{\infty} X(e^{j\omega})\delta(\omega-2\pi k) \\ &= \frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi \sum_{k=-\infty}^{\infty} X(e^{j2\pi k})\delta(\omega-2\pi k) \\ &= \frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega-2\pi k). \end{aligned}$$

5. From (iii),

$$\begin{aligned} x_o[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j\Im\{X(e^{j\omega})\} &= j\sin(\omega) - j\sin(2\omega) \\ &= \frac{1}{2}e^{j\omega} - \frac{1}{2}e^{-j\omega} - \frac{1}{2}e^{2j\omega} + \frac{1}{2}e^{-2j\omega}. \end{aligned}$$

Hence,

$$x_o[n] = \frac{1}{2}\delta[n+1] - \frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n+2] + \frac{1}{2}\delta[n-2].$$

Since $x_o[n] = (x[n]+x[-n])/2$ and (i), we have, for $n<0$,

$$x[n] = 2x_o[n] = \delta[n+1] - \delta[n-1] - \delta[n+2] + \delta[n-2] = \delta[n+1] - \delta[n+2].$$

Furthermore, according to Parseval's relation and (iv), we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2 = |x[0]|^2 + 2 = 3$$

implying

$$x[0] = \pm 1.$$

From (ii), we have $x[0] = 1$. Therefore,

$$x[n] = \delta[n] + \delta[n+1] - \delta[n+2].$$

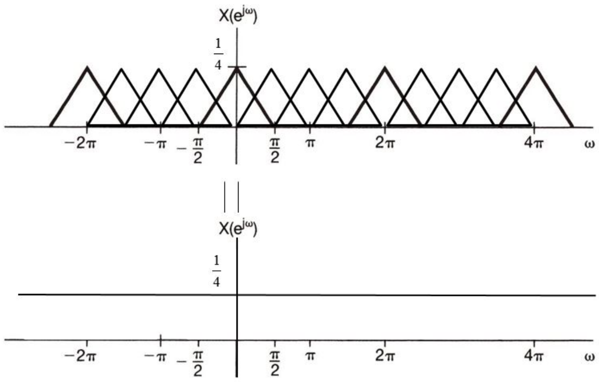
6. (a) The Fourier transform of the impulse train $p[n]$ is

$$P(e^{j\omega}) = \frac{2\pi}{4} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2k\pi}{4}) = \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{k\pi}{2}).$$

(b) The Fourier transform of $w[n]$ is

$$\begin{aligned} W(e^{j\omega}) &= \frac{1}{2\pi} \int_{2\pi} P(e^{j\theta}) X(e^{j(\omega-\theta)}) d\,\theta \\ &= \frac{1}{2\pi} \int_{-\frac{\pi}{4}}^{\frac{7\pi}{4}} \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \delta\left(\theta - \frac{k\pi}{2}\right) X(e^{j(\omega-\theta)}) d\,\theta \\ &= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{7\pi}{4}} \sum_{k=0}^{3} \delta\left(\theta - \frac{k\pi}{2}\right) X(e^{j(\omega-\theta)}) d\,\theta \\ &= \frac{1}{4} \sum_{k=0}^{3} X\left(e^{j(\omega - \frac{k\pi}{2})}\right). \end{aligned}$$

The spectrum is depicted below.



7. Let

$$x_1(t) = A \cdot \text{rect}\left(\frac{t}{T}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} X_1(j\omega) = AT\text{sinc}\left(\frac{\omega T}{2\pi}\right)$$

and

$$x_2(t) = \cos(\omega_c t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_2(j\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c).$$

The Fourier transform of $x(t)$ is

$$\begin{aligned} X(j\omega) &= \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega) \\ &= \frac{AT}{2}\left[\text{sinc}\left(\frac{\omega T}{2\pi}\right) * \delta(\omega - \omega_c) + \text{sinc}\left(\frac{\omega T}{2\pi}\right) * \delta(\omega + \omega_c)\right] \\ &= \frac{AT}{2}\text{sinc}\left(\frac{(\omega - \omega_c)T}{2\pi}\right) + \frac{AT}{2}\text{sinc}\left(\frac{(\omega + \omega_c)T}{2\pi}\right). \end{aligned}$$