1. Convert the following decimal numbers to a 32-bit binary number. (6%)

(a) -3125.3125

(b) 10^{-100}

(a) -3125.3125

$$
\begin{aligned}
3125&=2\times1562+1\\
1562&=2\times 781+0\\
781&=2\times 390+1\\
390&=2\times 195+0\\
195&=2\times 97+1\\
97&=2\times 48+1\\
48&=2\times 24+0\\
24&=2\times 12+0\\
12&=2\times 6+0\\
6&=2\times 3+0\\
3&=2\times 1+1\\
1&=2\times 0+1
\end{aligned}
$$

$$
\begin{aligned}
&2\times0.3125=0.625 \quad ,int(0.625)=0 \quad ,0.625-0=0.625\\
&2\times0.6250=1.250 \quad ,int(1.250)=1 \quad ,1.250-1=0.25\\
&2\times0.2500=0.500 \quad ,int(0.500)=0 \quad ,0.500-0=0.5\\
&2\times0.5000=1.000 \quad ,int(1.000)=1 \quad ,1.000-1=0
\end{aligned}
$$

故$(-3125.3125)_{10}=(-1100\ 0011\ 0101.0101)_2=(-1)^1(1.100\ 0011\ 0101\ 0101)_2\times2^{11}$

由$(-1)^s=2^{c-127}\times(1.f)_2 \Rightarrow s=1$, $f=100\ 0011\ 0101\ 0101$, $c-127=11 \Rightarrow c=138$

$$
\begin{aligned}
138&=2\times69+0\\
69&=2\times34+1\\
34&=2\times17+0\\
17&=2\times 8+1\\
8&=2\times 4+0\\
4&=2\times 2+0\\
2&=2\times 1+0\\
1&=2\times 0+1
\end{aligned}
$$

$(138)_{10}=(1000\ 1010)_2$

$$
\boxed{\begin{aligned}&故(-3125.3125)_{10}\\&=(1\ 1000\ 1010\ 1000\ 0110\ 1010\ 1010\ 0000\ 000)_2\end{aligned}}
$$

(b) 10^{-100}

無法計算,因32-bit最小只能計算到 2^{-126} ,而$10^{-100} << 2^{-126}$

2. Derive the following finite difference formula. (10%)

$$\frac{\partial f_i}{\partial x} = \frac{2f_{i+3} - 9f_{i+2} + 18f_{i+1} - 11f_i}{6\Delta x} + O(\Delta x^3)$$

Let $\Delta x = h$

$$f_i = f_i$$

$$f_{i+1} = f_i + hf_i' + \frac{h^2}{2!}f_i'' + \frac{h^3}{3!}f_i''' + \frac{h^4}{4!}f_i'''' + \cdots$$

$$f_{i+2} = f_i + (2h)f_i' + \frac{(2h)^2}{2!}f_i'' + \frac{(2h)^3}{3!}f_i''' + \frac{(2h)^4}{4!}f_i'''' + \cdots$$

$$f_{i+3} = f_i + (3h)f_i' + \frac{(3h)^2}{2!}f_i'' + \frac{(3h)^3}{3!}f_i''' + \frac{(3h)^4}{4!}f_i'''' + \cdots$$

$$\rightarrow f_i' = \frac{1}{h}\left(f_{i+1} - f_i - \frac{h^2}{2!}f_i'' - \frac{h^3}{3!}f_i''' + O(h^4)\right) - ①$$

$$f_i' = \frac{1}{2h}\left(f_{i+2} - f_i - \frac{(2h)^2}{2!}f_i'' - \frac{(2h)^3}{3!}f_i''' + O(h^4)\right) - ②$$

$$f_i' = \frac{1}{3h}\left(f_{i+3} - f_i - \frac{(3h)^2}{2!}f_i'' - \frac{(3h)^3}{3!}f_i''' + O(h^4)\right) - ③$$

$$f_i' = \frac{1}{h}\left(f_{i+1} - f_i - \frac{h^2}{2!}f_i'' - \frac{h^3}{3!}f_i''' + O(h^4)\right) \quad - \quad ①$$

$$f_i' = \frac{1}{2h}\left(f_{i+2} - f_i - \frac{(2h)^2}{2!}f_i'' - \frac{(2h)^3}{3!}f_i''' + O(h^4)\right) \quad - \quad ②$$

$$f_i' = \frac{1}{3h}\left(f_{i+3} - f_i - \frac{(3h)^2}{2!}f_i'' - \frac{(3h)^3}{3!}f_i''' + O(h^4)\right) \quad - \quad ③$$

①$\times 3$ −② $\times 3 +$ ③ $\times 1$

$$\rightarrow f_i' = \frac{3}{h}f_{i+1} - \frac{3}{2h}f_{i+2} + \frac{1}{3h}f_{i+3} + \left(-\frac{3}{h} + \frac{3}{2h} - \frac{1}{3h}\right)f_i + \frac{O(h^4)}{h}$$

$$\rightarrow \frac{\partial f_i}{\partial x} = \frac{2f_{i+3} - 9f_{i+2} + 18f_{i+1} - 11f_i}{6\Delta x} + O(\Delta x^3)$$

3. Given the data table.

x	1.0	1.3	1.6	1.9
f(x)	0.76	0.62	0.46	0.28

(a) Use the Newton's divided-difference formula. Find f(1.5). (8%)

(b) Use the third-degree Lagrange polynomial. Find f(5.5). (7%)

i	0	1	2	3
x_i	1	1.3	1.6	1.9
$f(x_i)$	0.76	0.62	0.46	0.28

(a)

Let $p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$

$\rightarrow p(x) = a_0 + (x - x_0)\{a_1 + (x - x_1)[a_2 + a_3(x - x_2)]\}$

$a_0 = f[x_0]$
$a_1 = f[x_0, x_1]$
$a_2 = f[x_0, x_1, x_2]$
$a_3 = f[x_0, x_1, x_2, x_3]$

x	$f[\,]$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
x_0	$f[x_0]$			
		$f[x_0, x_1]$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

x	$f[\ ]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$
1	0.76			
		$f[1, 1.3]$		
1.3	0.62		$f[1, 1.3, 1.6]$	
		$f[1.3, 1.6]$		$f[1, 1.3, 1.6, 1.9]$
1.6	0.46		$f[1.3, 1.6, 1.9]$	
		$f[1.6, 1.9]$		
1.9	0.28			

$$a_0 = f[1] = 0.76$$

$$a_1 = f[1, 1.3] = \frac{0.62 - 0.76}{1.3 - 1} = -\frac{7}{15}$$

$$f[1.3, 1.6] = -\frac{8}{15}$$

$$f[1.6, 1.9] = -\frac{9}{15}$$

$$a_2 = f[1, 1.3, 1.6] = \frac{-\frac{8}{15} - \left(-\frac{7}{15}\right)}{1.6 - 1} = -\frac{1}{9}$$

$$f[1.3, 1.6, 1.9] = -\frac{1}{9}$$

$$a_3 = f[1, 1.3, 1.6, 1.9] = \frac{-\frac{1}{9} - \left(-\frac{1}{9}\right)}{1.9 - 1} = 0$$

$a_0 = 0.76$

$a_1 = -\frac{7}{15}$

$a_2 = -\frac{1}{9}$

$a_3 = 0$

代入 $p(x)$

$\rightarrow p(x) = 0.76 + (x-1)\left\{-\frac{7}{15} + (x-1.3)\left[-\frac{1}{9} + 0(\,x-1.6)\right]\right\} \approx f(x)$

$\rightarrow f(1.5) = 0.51\bar{5}$

(b)Use the third-degree Lagrange polynomial. Find f(5.5). (7%)

i	0	1	2	3
x_i	1	1.3	1.6	1.9
$f(x_i)$	0.76	0.62	0.46	0.28

Let $p(x) = \sum_{i=0}^{3} l_i(x) f(x_i) = l_0 f(x_0) + l_1 f(x_1) + l_2 f(x_2) + l_3 f(x_3)$

$$l_0(x) = \left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)\left(\frac{x-x_3}{x_0-x_3}\right)$$

$$l_1(x) = \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)$$

$$l_2(x) = \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right)$$

$$l_3(x) = \left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)$$

$$\rightarrow f(5.5) \approx p(5.5)$$

$$= \left(\frac{5.5-1.3}{1-1.3}\right)\left(\frac{5.5-1.6}{1-1.6}\right)\left(\frac{5.5-1.9}{1-1.9}\right) 0.76 + \left(\frac{5.5-1}{1.3-1}\right)\left(\frac{5.5-1.6}{1.3-1.6}\right)\left(\frac{5.5-1.9}{1.3-1.9}\right) 0.62$$

$$+ \left(\frac{5.5-1}{1.6-1}\right)\left(\frac{5.5-1.3}{1.6-1.3}\right)\left(\frac{5.5-1.9}{1.6-1.9}\right) 0.46 + \left(\frac{5.5-1}{1.9-1}\right)\left(\frac{5.5-1.3}{1.9-1.3}\right)\left(\frac{5.5-1.6}{1.9-1.6}\right) 0.28$$

$$= -276.64 + 725.4 - 579.6 + 127.4$$

$$= -3.44$$

4. The Pade’ scheme are derived by writing approximations of the form:

$$\beta f'_{i-2} + \alpha f'_{i-1} + f'_i + \alpha f'_{i+1} + \beta f'_{i+2} = c\frac{f_{i+3} - f_{i-3}}{6h} + b\frac{f_{i+2} - f_{i-2}}{4h} + a\frac{f_{i+1} - f_{i-1}}{2h}$$

list the Taylor table and find α, β, a, b, and c.(10%)

$$\beta f'_{i-2} + \alpha f'_{i-1} + f'_i + \alpha f'_{i+1} + \beta f'_{i+2} = c\frac{f_{i+3} - f_{i-3}}{6h} + b\frac{f_{i+2} - f_{i-2}}{4h} + a\frac{f_{i+1} - f_{i-1}}{2h}$$

$$a\frac{f_{i+1} - f_{i-1}}{2h} + b\frac{f_{i+2} - f_{i-2}}{4h} + c\frac{f_{i+3} - f_{i-3}}{6h} - f'_i - \alpha(f'_{i+1} + f'_{i-1}) - \beta(f'_{i+2} + f'_{i-2}) = 0$$

	$\boldsymbol{f_i}$	$\boldsymbol{f'_i}$	$\boldsymbol{f''_i}$	$\boldsymbol{f'''_i}$	$\boldsymbol{f_i^{(4)}}$	$\boldsymbol{f_i^{(5)}}$
$\mathbf{f_{i+1} - f_{i-1}}$	0	$2 \times \frac{h}{1!}$	0	$2 \times \frac{h^3}{3!}$	0	$2 \times \frac{h^5}{5!}$
$\mathbf{f_{i+2} - f_{i-2}}$	0	$2 \times \frac{2h}{1!}$	0	$2 \times \frac{(2h)^3}{3!}$	0	$2 \times \frac{(2h)^5}{5!}$
$\mathbf{f_{i+3} - f_{i-3}}$	0	$2 \times \frac{3h}{1!}$	0	$2 \times \frac{(3h)^3}{3!}$	0	$2 \times \frac{(3h)^5}{5!}$
$\mathbf{f'_i}$	0	1	0	0	0	0
$\mathbf{f'_{i+1} + f'_{i-1}}$	0	2	0	$2 \times \frac{h^2}{2!}$	0	$2 \times \frac{h^4}{4!}$
$\mathbf{f'_{i+2} + f'_{i-2}}$	0	2	0	$2 \times \frac{(2h)^2}{2!}$	0	$2 \times \frac{(2h)^4}{4!}$

$$a\frac{f_{i+1}-f_{i-1}}{2h}+b\frac{f_{i+2}-f_{i-2}}{4h}+c\frac{f_{i+3}-f_{i-3}}{6h}-f_i'$$
$$-\alpha(f_{i+1}'+f_{i-1}')-\beta(f_{i+2}'+f_{i-2}')=0$$

	f_i	f_i'	f_i''	f_i'''	$f_i^{(4)}$	$f_i^{(5)}$
$\mathbf{f_{i+1}-f_{i-1}}$	0	$2\times\frac{h}{1!}$	0	$2\times\frac{h^3}{3!}$	0	$2\times\frac{h^5}{5!}$
$\mathbf{f_{i+2}-f_{i-2}}$	0	$2\times\frac{2h}{1!}$	0	$2\times\frac{(2h)^3}{3!}$	0	$2\times\frac{(2h)^5}{5!}$
$\mathbf{f_{i+3}-f_{i-3}}$	0	$2\times\frac{3h}{1!}$	0	$2\times\frac{(3h)^3}{3!}$	0	$2\times\frac{(3h)^5}{5!}$
$\mathbf{f_i'}$	0	1	0	0	0	0
$\mathbf{f_{i+1}'+f_{i-1}'}$	0	2	0	$2\times\frac{h^2}{2!}$	0	$2\times\frac{h^4}{4!}$
$\mathbf{f_{i+2}'+f_{i-2}'}$	0	2	0	$2\times\frac{(2h)^2}{2!}$	0	$2\times\frac{(2h)^4}{4!}$

$$\boldsymbol{f_i'}\rightarrow\frac{a}{2h}\left(2\times\frac{h}{1!}\right)+\frac{b}{4h}\left(2\times\frac{2h}{1!}\right)+\frac{c}{6h}\left(2\times\frac{3h}{1!}\right)-1-2\alpha-2\beta=a+b+c-1-2\alpha-2\beta=0$$

$$\boldsymbol{f_i'''}\rightarrow\frac{a}{2h}\left(2\times\frac{h^3}{3!}\right)+\frac{b}{4h}\left(2\times\frac{(2h)^3}{3!}\right)+\frac{c}{6h}\left(2\times\frac{(3h)^3}{3!}\right)-\alpha\left(2\times\frac{h^2}{2!}\right)-\beta\left(2\times\frac{(2h)^2}{2!}\right)$$
$$=h^2\left[\frac{a}{3!}+\frac{2^2\times b}{3!}+\frac{3^2\times c}{3!}-2\left(\frac{\alpha}{2!}+\frac{2^2\times\beta}{2!}\right)\right]=0$$

$$\boldsymbol{f_i^{(5)}}\rightarrow\frac{a}{2h}\left(2\times\frac{h^5}{5!}\right)+\frac{b}{4h}\left(2\times\frac{(2h)^5}{5!}\right)+\frac{c}{6h}\left(2\times\frac{(3h)^5}{5!}\right)-\alpha\left(2\times\frac{h^4}{4!}\right)-\beta\left(2\times\frac{(2h)^4}{4!}\right)$$
$$=h^4\left[\frac{a}{5!}+\frac{2^4\times b}{5!}+\frac{3^4\times c}{5!}-2\left(\frac{\alpha}{4!}+\frac{2^4\times\beta}{4!}\right)\right]=0$$

$$f_i' \rightarrow a + b + c - 1 - 2\alpha - 2\beta = 0 \quad \longrightarrow -a - b - c + 2\alpha + 2\beta = -1$$

$$f_i''' \rightarrow h^2\left[\frac{a}{3!} + \frac{2^2 \times b}{3!} + \frac{3^2 \times c}{3!} - 2\left(\frac{\alpha}{2!} + \frac{2^2 \times \beta}{2!}\right)\right] = 0 \longrightarrow -a - 2^2 \times b - 3^2 \times c + 6(\alpha + 2^2 \times \beta) = 0$$

$$f_i^{(5)} \rightarrow h^4\left[\frac{a}{5!} + \frac{2^4 \times b}{5!} + \frac{3^4 \times c}{5!} - 2\left(\frac{\alpha}{4!} + \frac{2^4 \times \beta}{4!}\right)\right] = 0 \longrightarrow -a - 2^4 \times b - 3^4 \times c + 10(\alpha + 2^4 \times \beta) = 0$$

$$f_i^{(7)} \rightarrow h^6\left[\frac{a}{7!} + \frac{2^6 \times b}{7!} + \frac{3^6 \times c}{7!} - 2\left(\frac{\alpha}{6!} + \frac{2^6 \times \beta}{6!}\right)\right] = 0 \longrightarrow -a - 2^6 \times b - 3^6 \times c + 14(\alpha + 2^6 \times \beta) = 0$$

$$f_i^{(9)} \rightarrow h^8\left[\frac{a}{9!} + \frac{2^8 \times b}{9!} + \frac{3^8 \times c}{9!} - 2\left(\frac{\alpha}{8!} + \frac{2^8 \times \beta}{8!}\right)\right] = 0 \longrightarrow -a - 2^8 \times b - 3^8 \times c + 18(\alpha + 2^8 \times \beta) = 0$$

$$-a-b-c+2\alpha+2\beta=-1$$

$$-a-2^2\times b-3^2\times c+6(\alpha+2^2\times\beta)=0$$

$$-a-2^4\times b-3^4\times c+10(\alpha+2^4\times\beta)=0$$

$$-a-2^6\times b-3^6\times c+14(\alpha+2^6\times\beta)=0$$

$$-a-2^8\times b-3^8\times c+18(\alpha+2^8\times\beta)=0$$

→使用高斯消去法計算

$$\begin{array}{cl}\left[\begin{array}{ccccc|c}-1 & -1 & -1 & 2 & 2 & -1\\ -1 & -2^2 & -3^2 & 6 & 24 & 0\\ -1 & -2^4 & -3^4 & 10 & 160 & 0\\ -1 & -2^6 & -3^6 & 14 & 896 & 0\\ -1 & -2^8 & -3^8 & 18 & 4608 & 0\end{array}\right] & \begin{array}{l} \\ \times(-1)\\ \times(-1)\\ \times(-1)\\ \times(-1)\end{array}\end{array}$$

$$\begin{array}{cl}\left[\begin{array}{ccccc|c}-1 & -1 & -1 & 2 & 2 & -1\\ 0 & -3 & -8 & 4 & 22 & 1\\ 0 & -15 & -80 & 8 & 158 & 1\\ 0 & -63 & -728 & 12 & 894 & 1\\ 0 & -255 & -6560 & 16 & 4606 & 1\end{array}\right] & \begin{array}{l} \\ \\ \times(-5)\\ \times(-21)\\ \times(-85)\end{array}\end{array}$$

$$\left[\begin{array}{ccccc|c} -1 & -1 & -1 & 2 & 2 & -1 \\ 0 & -3 & -8 & 4 & 22 & 1 \\ 0 & 0 & -40 & -12 & 48 & -4 \\ 0 & 0 & -560 & -72 & 432 & -20 \\ 0 & 0 & -5880 & -324 & 2736 & -84 \end{array}\right] \quad \begin{array}{l} \times(-14) \\ \times(-147) \end{array}$$

$$\left[\begin{array}{ccccc|c} -1 & -1 & -1 & 2 & 2 & -1 \\ 0 & -3 & -8 & 4 & 22 & 1 \\ 0 & 0 & -40 & -12 & 48 & -4 \\ 0 & 0 & 0 & 96 & -240 & 36 \\ 0 & 0 & 0 & 1440 & -7320 & 504 \end{array}\right] \quad \times(-15)$$

$$\left[\begin{array}{ccccc|c} -1 & -1 & -1 & 2 & 2 & -1 \\ 0 & -3 & -8 & 4 & 22 & 1 \\ 0 & 0 & -40 & -12 & 48 & -4 \\ 0 & 0 & 0 & 96 & -240 & 36 \\ 0 & 0 & 0 & 0 & -720 & -36 \end{array}\right]$$

$$-720\beta = -36$$

$$96\alpha - 240\beta = 36$$

$$-40c - 12\alpha + 48\beta = -4$$

$$-3b - 8c + 4\alpha + 22\beta = 1$$

$$-a - b - c + 2\alpha + 2\beta = -1$$

a=1.41667
b=0.67333
c=0.01000
α=0.50000
β=0.05000

5. A general Padé type boundary scheme for the first derivative can be written as

$$f_0' + \alpha f_1' = \frac{1}{h}(af_0 + bf_1 + cf_2 + df_3)$$

show that requiring this scheme to be at least third-order accurate would constrain the coefficients to

$$\mathrm{a} = -\frac{11+2\alpha}{6}, \qquad \mathrm{b} = \frac{6-\alpha}{2}, \qquad c = \frac{2\alpha-3}{2}, \qquad d = \frac{2-\alpha}{6}$$

Which value of α would you choose and why? (10%)

$$f_0' + \alpha f_1' = \frac{1}{h}(af_0 + bf_1 + cf_2 + df_3)$$

$$f_0' + \alpha f_1' - \frac{1}{h}(af_0 + bf_1 + cf_2 + df_3) = 0$$

$$a = -\frac{11+2\alpha}{6}, b = \frac{6-\alpha}{2}, c = \frac{2\alpha-3}{2}, d = \frac{2-\alpha}{6}$$

	f_0	f_0'	f_0''	f_0'''	f_0''''	f_0'''''
f_0	1	0	0	0	0	0
f_1	1	$\frac{h}{1!}$	$\frac{h^2}{2!}$	$\frac{h^3}{3!}$	$\frac{h^4}{4!}$	$\frac{h^5}{5!}$
f_2	1	$\frac{2h}{1!}$	$\frac{(2h)^2}{2!}$	$\frac{(2h)^3}{3!}$	$\frac{(2h)^4}{4!}$	$\frac{(2h)^5}{5!}$
f_3	1	$\frac{3h}{1!}$	$\frac{(3h)^2}{2!}$	$\frac{(3h)^3}{3!}$	$\frac{(3h)^4}{4!}$	$\frac{(3h)^5}{5!}$
f_0'	0	1	0	0	0	0
f_1'	0	1	$\frac{h}{1!}$	$\frac{h^2}{2!}$	$\frac{h^3}{3!}$	$\frac{h^4}{4!}$

$$f_0 \rightarrow (a+b+c+d) = -\frac{11+2\alpha}{6} + \frac{6-\alpha}{2} + \frac{2\alpha-3}{2} + \frac{2-\alpha}{6} = 0$$

$$f_0' \rightarrow 1+\alpha - \frac{1}{h}(hb + 2hc + 3hd) = 1 + \alpha + \frac{-6+\alpha}{2} - 2\alpha + 3 + \frac{-2+\alpha}{2} = 0$$

$$f_0'' \rightarrow \frac{h}{1!}\alpha - \frac{1}{h}\left(\frac{h^2}{2!}\,b + \frac{(2h)^2}{2!}c + \frac{(3h)^2}{2!}d\right) = \frac{h}{2!}[2\alpha - b - 4c - 9d\,]$$

$$= \frac{h}{2!}\left[2\alpha - \frac{6-\alpha}{2} - 4\frac{2\alpha-3}{2} - 9\frac{2-\alpha}{6}\right] = 0$$

$$f_0''' \rightarrow \frac{h^2}{2!}\alpha - \frac{1}{h}\left(\frac{h^3}{3!}\,b + \frac{(2h)^3}{3!}c + \frac{(3h)^3}{3!}d\right) = \frac{h^2}{3!}[3\alpha - b - 8c - 27d\,]$$

$$= \frac{h^2}{3!}\left[3\alpha - \frac{6-\alpha}{2} - 8\frac{2\alpha-3}{2} - 27\frac{2-\alpha}{6}\right] = 0$$

$$f_0'''' \to \frac{h^3}{3!}\alpha - \frac{1}{h}\left(\frac{h^4}{4!}\ b + \frac{(2h)^4}{4!}c + \frac{(3h)^4}{4!}d\right) = \frac{h^3}{4!}[4\alpha - b - 16c - 81d\ ]$$

$$= \frac{h^3}{4!}\left[4\alpha - \frac{6-\alpha}{2} - 16\frac{2\alpha-3}{2} - 81\frac{2-\alpha}{6}\right] = \frac{h^3}{4!}[2\alpha - 6\ ]$$

$$f_0''''' \to \frac{h^4}{4!}\alpha - \frac{1}{h}\left(\frac{h^5}{5!}\ b + \frac{(2h)^5}{5!}c + \frac{(3h)^5}{5!}d\right) = \frac{h^4}{5!}[5\alpha - b - 32c - 243d\ ]$$

$$= \frac{h^4}{4!}\left[5\alpha - \frac{6-\alpha}{2} - 32\frac{2\alpha-3}{2} - 243\frac{2-\alpha}{6}\right] = \frac{h^4}{5!}[14\alpha - 36\ ]$$

α 等於任意實數(除α=3)時，上述coefficients將滿足3階精準度。

當α (除α=3)等於任意實數，我們已經證明 $f_0 = f_0' = f_0'' = f_0''' = 0$，誤差項將由

$f_0'''' \to \frac{h^3}{4!}[2\alpha - 6\ ]$產出。

6. Let $f(x) = (x+2)(x+1)x(x-1)^3(x-2)$.

(a) Which root of $f(x)$ can be found by the Bisection method for the interval [-3.0, 2.5], why? (10%)

(b) Comments on the disadvantage of the Bisection method. (5%)

(a)（方程式的正解為-2, -1, 0, 1, 1, 1, 2）

$$f(-3) = -1920, \quad f(2.5) = 66.4453, \quad f(2.5) \times f(-3) \leq 0$$

$$c_1 = \frac{1}{2} \times \left(2.5 + (-3)\right) = \mathbf{-0.25}, \quad f(-0.25) = -1.442$$

$$c_2 = \frac{1}{2} \times \left(2.5 + (-0.25)\right) = \mathbf{1.125}, \quad f(1.125) = -0.0128$$

$$c_3 = \frac{1}{2} \times (2.5 + 1.125) = \mathbf{1.8125}, \quad f(1.8125) = -1.9546$$

（到這裡會發現Bisection method的解會在1.125及2.5之間，即用Bisection method可找出的解為2。）

$$c_4 = \frac{1}{2} \times (2.5 + 1.8125) = \mathbf{2.1562}, \quad f(2.1562) = 6.832$$

$$c_5 = \frac{1}{2} \times (2.15625 + 1.8125) = \mathbf{1.9843}, \quad f(1.9843) = -0.3517$$

$$c_6 = \frac{1}{2} \times (2.15625 + 1.984375) = \mathbf{2.0703}, \quad f(2.0703) = 2.2305$$

$$c_7 = \frac{1}{2} \times (2.0703125 + 1.984375) = \mathbf{2.0273}, \quad f(2.0273) = 0.7328$$

$$c_8 = \frac{1}{2} \times (2.02734375 + 1.984375) = \mathbf{2.0058} \quad , f(2.005859375) = 0.144$$ （解慢慢逼近2。）

(b) Comments on the disadvantage of the Bisection method. (5%)

1. 如果要逼近到精確值，相對比其他的方法需要更多次的運算才能得到答案。
2. 需設定函數值為一正一負的區間
3. 設定的區間內不一定會有根。
4. 設定的區間若有多個根，不一定能全部找出。

7. Use the finite difference method and the TDMA method. Find the simultaneous equations in matrix form. The length of the rod is 5 and number of grid points is 6.

$$\frac{\partial^2 T}{\partial x^2} = 10$$

i	1	2	3	4	5	6
T_i	T_1	T_2	T_3	T_4	T_5	T_6

Boundary conditions : $T_1 = 100$, $T_5 = T_6$ (10%)

$$f_{j+1} = f_j + hf'_j + \frac{h^2}{2}f''_j + \frac{h^3}{6}f'''_j + \cdots\cdots$$

$$f_{j-1} = f_j - hf'_j + \frac{h^2}{2}f''_j - \frac{h^3}{6}f'''_j + \cdots\cdots$$

$$\rightarrow \frac{h^2}{2}f''_j = f_{j+1} - f_j - hf'_j - \frac{h^3}{6}f'''_j + \cdots\cdots$$

$$\frac{h^2}{2}f''_j = f_{j-1} - f_j + hf'_j + \frac{h^3}{6}f'''_j + \cdots\cdots$$

$$\rightarrow f''_j \approx \frac{f_{j+1} - 2f_j + f_{j-1}}{h^2}$$

$$\frac{d^2T}{dx^2} = 10 \rightarrow \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} = 10 \rightarrow T_{i+1} - 2T_i + T_{i-1} = 10\Delta x^2$$

Boundary condition: $T_1 = 100$; $T_5 = T_6 \rightarrow T_5 - T_6 = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 100 \\ 10\Delta x^2 \\ 10\Delta x^2 \\ 10\Delta x^2 \\ 10\Delta x^2 \\ 0 \end{bmatrix}$$

8. Using the Taylor series in two variables (x, y) of the form

$$f(x+h, y+k) = f(x, y) + hf_x(x, y) + kf_y(x, y) + \cdots$$

where $f_x = \partial f / \partial x$ and $f_y = \partial f / \partial y$, establish that Newton's method for solving the two simultaneous nonlinear equations

$$\begin{cases} f(x, y) = 0 \\ g(x, y) = 0 \end{cases}$$

can be described with the formulas

$$\begin{cases} x_{n+1} = x_n - \dfrac{f g_y - g f_y}{f_x g_y - g_x f_y} \\ y_{n+1} = y_n - \dfrac{f_x g - g_x f}{f_x g_y - g_x f_y} \end{cases}$$

Here the functions f, f_x, and so on are evaluated at (x_n, y_n). (10%)

Let $F\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_{(x,y)} \\ g_{(x,y)} \end{bmatrix}$; $F' = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$

Newton's method:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}^{-1} F\begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

$$= \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}^{-1} \begin{bmatrix} f_{(x_n,y_n)} \\ g_{(x_n,y_n)} \end{bmatrix}$$

$$= \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \left[\frac{1}{f_x g_y - f_y g_x}\right] \begin{bmatrix} g_y & -f_y \\ -g_x & f_x \end{bmatrix} \begin{bmatrix} f_{(x_n,y_n)} \\ g_{(x_n,y_n)} \end{bmatrix}$$

$$\rightarrow x_{n+1} = x_n - \frac{g_y f - f_y g}{f_x g_y - f_y g_x} \quad ; \quad y_{n+1} = y_n - \frac{-g_x f + f_x g}{f_x g_y - f_y g_x}$$

9. Solve the parameter x in this pair of simultaneous nonlinear equations by

(a) Newton's method (7%). Start with the initial value $x_0 = 2.0$ and iterate 3 times.

(b) Bisection method (7%). With the initial interval [2, 3] and iterate 3 times.

$$\begin{cases} x^3 - 2xy + y^7 - 4x^3y = 5 \\ y \sin x + 3x^2y + \tan x = 4 \end{cases}$$

$$x^3 - 2xy + y^7 - 4x^3y = 5 \qquad -①$$

$$y \sin x + 3x^2y + \tan x = 4 \qquad -②$$

(a)

$$y \sin x + 3x^2y + \tan x = 4 \rightarrow y\,(\sin x + 3x^2) = 4 - \tan x$$

$$\rightarrow y = \frac{4-\tan x}{\sin x+3x^2} \qquad -③$$

③代入①

$$\rightarrow x^3-2x\left(\frac{4-\tan x}{\sin x+3x^2}\right)-\left(\frac{4-\tan x}{\sin x+3x^2}\right)^7-4x^3\left(\frac{4-\tan x}{\sin x+3x^2}\right)=5$$

$$\rightarrow f=x^3-2x\left(\frac{4-\tan x}{\sin x+3x^2}\right)-\left(\frac{4-\tan x}{\sin x+3x^2}\right)^7-4x^3\left(\frac{4-\tan x}{\sin x+3x^2}\right)-5$$

$$f'=3x^2-2\left(\frac{4-\tan x}{\sin x+3x^2}\right)-2x\left(\frac{-\sec^2 x(\sin x+3x^2)-(4-\tan x)(\cos x+6x)}{(\sin x+3x^2)^2}\right)$$
$$-7\left(\frac{4-\tan x}{\sin x+3x^2}\right)^6\left(\frac{-\sec^2 x(\sin x+3x^2)-(4-\tan x)(\cos x+6x)}{(\sin x+3x^2)^2}\right)-12x^2\left(\frac{4-\tan x}{\sin x+3x^2}\right)$$
$$-4x^3\left(\frac{-\sec^2 x(\sin x+3x^2)-(4-\tan x)(\cos x+6x)}{(\sin x+3x^2)^2}\right)$$

Newton's method: $x_{i+1} = x_i - \frac{f_{(x_i)}}{f'_{(x_i)}}$

$$x_1 = 2$$

$$x_2 = x_1 - \frac{f_{(x_1)}}{f'_{(x_1)}} = 2 - \frac{(-14.2539)}{19.6989} = 2.736$$

$$x_3 = x_2 - \frac{f_{(x_2)}}{f'_{(x_2)}} = 2.736 - \frac{(-1.7145)}{20.4798} = 2.8073$$

$$x_4 = x_3 - \frac{f_{(x_3)}}{f'_{(x_3)}} = 2.8073 - \frac{0.0565}{21.8493} = \boxed{2.8047}$$

(b) Bisection method (7%). With the initial interval [2, 3] and iterate 3 times.

a=2;b=3

$f_{(a)}$ = -14.2539 ; $f_{(b)}$ =4.6002 ; $f_{(a)} \cdot f_{(b)} < 0$

$\rightarrow c = \frac{a+b}{2} = 2.5$

$f_{(c)}$ = -5.9357

$f_{(a)} \cdot f_{(c)} \geqq 0 \rightarrow a = 2.5 ; b = 3; c = \frac{a+b}{2} = 2.75$

$f_{(a)}$ = -5.9357 ; $f_{(b)}$ = 4.6002 ; $f_{(c)}$ = -1.1682

$f_{(a)} \cdot f_{(c)} \geqq 0 \rightarrow a = 2.75 ; b = 3 ; x = c = \frac{a+b}{2} = \boxed{2.875}$