CALCULUS I LECTURE 1: FUNCTIONS

1. Introduction

Calculus is a study of functions. It includes two major parts: **differentiation** and **Integration**, where the former defines the instantaneous rate of change for functions and the later defines the area or volume bounded by functions. These two complete different concepts connected by the fundamental theorem of calculus, which tells us that they are inverse of each other. To start the first lecture, we need to review the concept of functions.

2. Definition of functions

Here we start from one of the most basic concepts in mathematics. When we say f is a function, it means that f associates each element in a set X to exactly one element in another set Y. We usually call X the domain of f and Y the co-domain of f. To make it more precise, we have:

Definition 2.1. Suppose X and Y are two sets. A **function** f mapping from X to Y (denoted by $f : X \to Y$) is a collection of ordered pairs $(x, y) \in X \times Y$(denoted by $\Gamma(f)$) such that:

1. For any $x \in X$, there is $(x, y) \in \Gamma(f)$.
2. The choice of y in 1. is unique, i.e., for any $(x, y), (x, y') \in X \times Y$, $y = y'$.

If $(x, y) \in \Gamma(f)$, we will denote y by $f(x)$. According to the first condition in Definition 1.1, there exists $f(x)$ for any $x \in X$. According to the second condition in Definition 1.1, the choice of $f(x)$ is unique. We usually call $f(x)$ the value of f at x.

To describe a function, one should be able to write down the value $f(x)$ specifically. For example, when we say a function $f : \mathbb{R}^1 \to \mathbb{R}^1$ is the function sending x to x^2, we can write down specifically the value of $f(x) = x^2$ because we have the rule. In fact, if we can write $f(x)$ as one polynomial over the other, then the value of f will be computable (by hand) when the denominator is not zero. In this circumstance, we call f a **rational function**. Otherwise, f is an **irrational function**.

There are many irrational functions we have faced before. For examples, $f(x) = \sin(x)$, $f(x) = \sqrt{x}$ and $f(x) = e^x$ are all irrational. These functions are **not** computable by hand. To find the value, one has to derive a way to approximate the value (that is also the way computers find the value for these functions).

Example 2.2. Let us consider $f(x) = \sin(x)$. One should notice that

$$\sin(2x) = 2\sin(x)\cos(x) = 2\sin(x)\Big(1 - 2\sin^2\left(\frac{x}{2}\right)\Big). \tag{2.1}$$

Here one should notice that the right hand side of (2.1) only involves x and $\frac{x}{2}$. So presumably the iteration will give us the approximation of $\sin(x)$ for large x when we have a good approximation for the value of sin with small input. We claim here that $\sin(x) \sim x$ is a good approximation when x is small.

By the first picture in Pic. 1, we can see that $\sin(x) \leq x$ (let us suppose $x > 0$, since the other side can be obtained by the same trick).

Pic.1

By the second picture in Pic. 2. One can easy obtain that $r = \frac{\sin x}{\sqrt{1-\sin^2(x)}}$ and

$$\frac{x}{2} \leq \frac{\sin(x)}{2\sqrt{1-\sin^2(x)}} \leq \frac{\sin(x)}{2\sqrt{1-x^2}} \leq \frac{\sin(x)}{2(1-x^2)} \tag{2.2}$$

when $x \leq 1$. So we have

$$x - x^3 \leq \sin(x) \leq x \tag{2.3}$$

when $x < 1$. Because $x^3 \ll x$ when x is small, so $\sin(x) \sim x$ when x is small.

Now we have the following cubic approximation for $\sin(x)$: By plugin $\sin(x) \sim x$ on the right hand side in (2.1), we have

$$\sin(2x) \sim 2x - x^3.$$

So

$$\sin(x) \sim x - \frac{x^3}{8}. \tag{2.4}$$

Exercise 2.3.
a. Find a degree 5 polynomial to approximate $\sin(x)$.
b. (Extra credit) Suppose that we can write

$$\sin(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}.$$

Prove inductively that $a_{2n+1} = (-1)^n \frac{1}{(2n+1)!}$

3. Properties of functions

In this course, we will mainly focus on those functions with $X, Y \subset \mathbb{R}$. So unless we make additional assumptions, otherwise X, Y are subsets of $\mathbb{R}$.

We call a function $f : X \to Y$ **injective** or **one to one** if and only if $f(x) \neq f(y)$ for any $x \neq y$; we call a function **surjective** or **onto** if and only if for any $y \in Y$ there exists $x \in X$ such that $f(x) = y$. When an injective function which is also surjective, we call it a **bijective** function.

For any two functions $f : X \to Y$ and $g : Y \to Z$, we can define the composition of these two functions $g \circ f : X \to Z$. This is a new function sending any $x \in X$ to $g(f(x))$.

For any bijective function f, there exists an **inverse function** $f^{-1} : Y \to X$ such that $f^{-1} \circ f(x) = x$ for any $x \in X$ and $f \circ f^{-1}(y) = y$ for any $y \in Y$. If the function is not bijective, then there is no inverse function. However, one can modified its domain and co-domain to make the function bijective on that small portion. Then there will be an inverse function for it. For examples, $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ are in this case.

4. Exponential functions

Let $a > 0$. For any rational number $q \in \mathbb{Q}$, we can define the value a^q: suppose $q = \frac{m}{n}$, then a^q is the solution of the algebraic equation $x^n = a^m$. The fundamental theorem of algebra tells us that such solution exists. Moreover, the solution we want is a positive real number, which is unique.

Therefore, we have a function $f : \mathbb{Q} \to \mathbb{R}$, which sends x to a^x. Then we want to extend the domain of this function to the whole $\mathbb{R}$. We will see in the future that this extension, if it is continuous, is unique.

5. Axiom of choice

Here we introduce the axiom which is intuitively correct (at least for many mathematicians), but it is impossible to be proved by other elementary axioms of set theory. This axiom states as the following:

Axiom of choice:
For any set I and class $\{S_i\}_{i\in I}$ with $S_i \neq \emptyset$, there exists a function $f : I \to \cup_{i\in I} S_i$ such that $f(i) \in S_i$.

When I is a finite set, this statement is obviously true. One can imagine that there are many non-empty boxes S_i labelled by $i \in I$. Axiom of choice tells us that we can find a way to "select" exactly one element form each box. When I is finite set, this kind of selection exhausted within finitely many times. So it can be proved inductively. However, when I is not finite, this statement cannot be proved directly and the existence of such a function f is unclear. By assuming this axiom, we wouldn't need to worry about this problem anymore.
We will use this axiom later when we discuss the limit of functions. Readers can always assume this is true in this course.