Chapter 1: Fundamental Principles of Counting

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Outline

- ***The Rules of Sum and Product***
- Permutations
- Combinations: The Binomial Theorem
- Combinations with Repetition

The Rules of Sum (1/2)

- The Rule of Sum: If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m + n$ ways.

The Rules of Sum (2/2)

- **Example 1.1:** A college library has 40 textbooks on sociology and 50 textbooks dealing with anthropology. By the rule of sum, a student at this college can select among $40 + 50 = 90$ textbooks in order to learn more about one or the other of these two subjects.

The Rules of Product (1/2)

- The Rule of Product: If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.

The Rules of Product (2/2)

- **Example 1.6:** Considering the manufacture of license plates consisting of 2 letters followed by 4 digits.
  a) If no letter or digit can be repeated, there are $26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3,276,000$ different possible plates.
  b) With repetitions of letters and digits allowed, $26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6,760,000$ different license plates are possible.
  c) If repetitions are allowed, as in part (b), how many of the plates have only vowels (A, E, I, O, U) and even digits? (0 is an even integer.)

The Rules of Sum and Product

- **Example 1.8:** At the AWL corporation Mrs. Foster operates the Quick Snake Coffee Shop. The menu at her shop is limited: 6 kinds of muffins, 8 kinds of sandwiches, and 5 beverages (hot coffee, hot tea, iced tea, cola, and orange juice). Ms. Dodd, an editor at AWL, sends her assistant Carl to the shop to get her lunch – either a muffin and a hot beverage or a sandwich and a cold beverage. How many ways in which Carl can purchase Ms. Dodd's lunch?

  $6 \times 2 + 8 \times 3 = 12 + 24 = 36$

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Permutations (1/2)

- **Example 1.9:** In a class of 10 students, 5 are to be chosen and seated in a row for a picture. How many such linear arrangements are possible?

  $10 \times 9 \times 8 \times 7 \times 6 = 30{,}240$

- **Definition 1.1:** For an integer $n \geq 0$, n factorial (denoted $n!$) is defined by

$$0! = 1,$$

$$n! = (n)(n-1)(n-2)\ldots(3)(2)(1), \quad \text{for } n \geq 1.$$

Permutations (2/2)

- **Definition 1.2:** Given a collection of n distinct objects, any (linear) arrangement of these objects is called a permutation of the collection.

If there are n distinct objects and r is an integer, with $1 \le r \le n$, then by the rule of product, the number of permutations of size r for the n objects is

$$P(n,r) = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) = \frac{n!}{(n-r)!}$$

Permutations with Indistinguishable Objects (1/2)

- **Example 1.12:** Consider the arrangements of all 9 letters in DATABASES.

  (2!)(3!)(Number of arrangements of the letters in DATABASES) = (Number of permutations of the symbols D, A_1, T, A_2, B, A_3, S_1, E, S_2)

  9!/(2!3!) = 30,240

Permutations with Indistinguishable Objects (2/2)

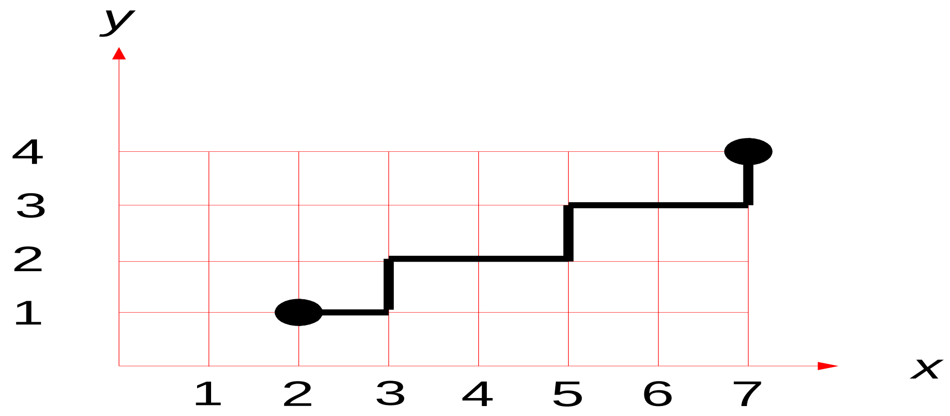
If there are n objects with n_1 indistinguishable objects of a first type, n_2 indistinguishable objects of a second type, ..., and n_r indistinguishable objects of an rth type, where $n_1 + n_2 + ... + n_r = n$,

then there are $\dfrac{n!}{n_1!n_2!...n_r!}$ (linear) arrangements of the given n objects.

Examples (1/2)

- **Example 1.14:** Determine the number of (staircase) paths in the *xy*-plane from (2, 1) to (7, 4), where each such path is made up of individual steps going one unit to the right (R) or one unit upward (U).
- (Figure Here!)

  $8!/(5!3!) = 56$



Examples (2/2)

- **Example 1.15:** (combinatorial proof) Prove that if n and k are positive integers with $n = 2k$, then $n!/2^k$ is an integer.
  (Consider the n symbols $x_1, x_1, x_2, x_2, \ldots, x_k, x_k$.)

The number of ways in which we can arrange all of these n=2k symbols is an integer that equals

$$\frac{n!}{2!2!\cdots 2!} = \frac{n!}{2^k}$$

Nonlinear Arrangement (1/2)

- **Example 1.16:** If 6 people, designated as A, B, …, F, are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation?

  There are $6!/6 = 5! = 120$ arrangements.

Nonlinear Arrangement (2/2)

- **Example 1.17:** Suppose now the 6 people of Example 1.16 are 3 married couples and that A, B, and C are the females. We want to arrange the 6 people around the table so that the sexes alternate. (Once again, arrangements are considered identical if one can be obtained from the other by rotation.)

  $3 \times 2 \times 2 \times 1 \times 1 = 12$ ways

Outline

- The Rules of Sum and Product
- Permutations
- ***<u>Combinations: The Binomial Theorem</u>***
- Combinations with Repetition

Combinations: The Binomial Theorem

- If we start with n distinct objects, each selection, or combination, of r of these objects, with no reference to order, corresponds to $r!$ permutations of size r from the n objects. Thus the number of combinations of size r from a collection of size n is

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} \qquad 0 \le r \le n$$

Combination Examples (1/2)

- **Example 1.19:** Lynn and Patti decide to buy a PowerBall ticket. To win the grand prize for PowerBall, one must match five numbers selected from 1 to 49 inclusive and then must also match the powerball, an integer from 1 to 42 inclusive. How many ways can Lynn and Patti select for their ticket?

  $C(49, 5)C(42, 1) = 80{,}089{,}128$ ways

Combination Examples (2/2)

- **Example 1.20:**
  a) A student taking a history examination is directed to answer any 7 of 10 essay questions.
  b) The student must answer 3 questions from the first 5 and 4 questions from the last 5.
  c) The student must answer 7 of 10 questions where at least 3 are selected from the first 5.

  a) $C(10, 7) = 120$ ways

  b) $C(5, 3)C(5,4) = 10 \times 5 = 50$ ways

  c) $C(5, 3)C(5, 4) + C(5, 4)C(5, 3) + C(5, 5)C(5, 2)$
  $= 110$ ways

Arrangements and Combinations (1/2)

- **Example 1.23:** The number of arrangements of the letters in TALLAHASSEE is

$$\frac{11!}{3!2!2!2!1!1!} = 831{,}600.$$

How many of these arrangement have no adjacent A's?

$$\begin{array}{ccccccccccccccccc} & \mathrm{E} & & \mathrm{E} & & \mathrm{S} & & \mathrm{T} & & \mathrm{L} & & \mathrm{L} & & \mathrm{S} & & \mathrm{H} & \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \end{array}$$

$$\frac{8!}{2!2!2!1!1!} \times C(9,3) = 5040 \times 84$$

$$= 423{,}360 \text{ arrangements}$$

Arrangements and Combinations (2/2)

- **Example 1.24:** Consider strings made up from symbols 0, 1, and 2. Suppose $x = x_1 x_2 x_3 \ldots x_n$ is one of strings of length n. We define the weight of x, denoted $\mathrm{wt}(x)$, by $\mathrm{wt}(x) = x_1 + x_2 + x_3 + \ldots + x_n$. Among the 3^{10} strings of length 10, we wish to determine how many have even weight. Such a string has even weight precisely when the number of 1's in the string is even.

$$2^{10} + C(10,2)2^8 + C(10,4)2^6 + C(10,6)2^4 + C(10,8)2^2$$

$$+ C(10,10) = \sum_{n=0}^{5} C(10,2n)2^{10-2n}$$

Overcounting (1/2)

- **Example 1.25:**
  a) Suppose that Ellen draws 5 cards from a standard deck of 52 cards. In how many ways can her selection result in a hand with no clubs?

  C(39, 5)

  b) Now suppose we want to count the number of Ellen's 5-card selections that contain at least one club.

  C(52, 5) – C(39, 5) = 2,023,303 vs.
  $\boxed{\text{C(13, 1)C(51, 4) = 3,248,700}}$

  Overcounting!

Overcounting (2/2)

- **Example 1.25 (cont.):**
  Another way to arrive at the answer:

$$C(13,1)C(39,4)+C(13,2)C(39,3)+C(13,3)C(39,2)$$

$$+C(13,4)C(39,1)+C(13,5)C(39,0)$$

$$=\sum_{i=1}^{5}C(13,i)C(39,5-i)$$

$$=(13)(82,251)+(78)(9139)+(286)(741)+(715)(39)$$

$$+(1287)(1)=2,023,203.$$

The Binomial Theorem (1/3)

- **Theorem 1.1** The Binomial Theorem. If x and y are variables and n is a positive integer, then

$$(x+y)^n = C(n,0)x^0y^n + C(n,1)x^1y^{n-1} + \ldots + C(n,n)x^ny^0$$
$$= \sum_{k=0}^{n} C(n,k)x^ky^{n-k}.$$

  - There are $C(n, k)$ different ways to select k x's and $n - k$ y's from the n available factors.
  - $C(n, k)$ is often referred to as a binomial coefficient

The Binomial Theorem (2/3)

- **Example 1.26:**
  a) What is the coefficient of x^5y^2 in the expansion of $(x + y)^7$?
  b) What is the coefficient of a^5b^2 in the expansion of $(2a - 3b)^7$?

  a) C(7, 5) = 21

  b) $C(7, 5)(2)^5(-3)^2 = 6048$

The Binomial Theorem (3/3)

- **Corollary 1.1** For each integer $n > 0$,
  a) $C(n, 0) + C(n, 1) + \ldots + C(n, n) = 2^n$, and
  b) $C(n, 0) - C(n, 1) + \ldots + (-1)^n C(n, n) = 0$

- **Proof.**
  a) Set $x = y = 1$
  b) Set $x = -1$, $y = 1$

The Multinomial Theorem (1/3)

- **Theorem 1.2** The Multinomial Theorem. For positive integers n, t, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots x_t^{n_t}$ in the expansions of $(x_1 + x_2 + \ldots + x_t)^n$ is

$$\frac{n!}{n_1! n_2! n_3! \ldots n_t!}$$

where each n_i is an integer with $0 \leq n_i \leq n$, for all $1 \leq i \leq t$, and $n_1 + n_2 + n_3 + \ldots + n_t = n$.

The Multinomial Theorem (2/3)

- **Proof of Theorem 1.2:**
  - The coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots x_t^{n_t}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, …

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\ldots-n_{t-1}}{n_t} = \frac{n!}{n_1!n_2!\cdots n_t!} = \boxed{\binom{n}{n_1, n_2, \cdots, n_t}}$$

a multinomial coefficient

The Multinomial Theorem (3/3)

- **Example 1.27:**

  a) What is the coefficient of x^3z^4 in the expansion of $(x + y + z)^7$?
  b) What is the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$?

a) $\binom{7}{3,\, 0,\, 4} = \frac{7!}{3!\, 0!\, 4!} = 35$

b) $\binom{16}{2,\, 3,\, 2,\, 5,\, 4}(1)^2(2)^3(-3)^2(2)^5(5)^4$

$= 435{,}891{,}456{,}000{,}000$

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- ***Combinations with Repetition***

Combinations with Repetition (1/)

- **Example 1.28:** 7 high school freshmen stop at a restaurant, where each of them has one of the following: a cheeseburger, a hot dog, a taco, or a fish sandwich. How many different purchases are possible (from the viewpoint of the restaurant)?

(a)		(b)	
1.	c, c, h, h, t, t, f	1.	x x | x x | x x | x
2.	c, c, c, c, h, t, f	2.	x x x x | x | x | x
3.	c, c, c, c, c, c, f	3.	x x x x x x | | | x
4.	h, t, t, f, f, f, f	4.	| x | x x | x x x x
5.	t, t, t, t, t, f, f	5.	| | x x x x x | x x
6.	t, t, t, t, t, t, t	6.	| | x x x x x x x |
7.	f, f, f, f, f, f, f	7.	| | | x x x x x x x

$$\frac{10!}{7!\,3!} = \binom{10}{7}.$$

Combinations with Repetition (2/)

When we wish to select, with repetition, r of n distinct objects, we find that we are considering all arrangements of r x's and $n - 1$ |'s and that their number is

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}.$$

Consequently, the number of combinations of n objects taken r at a time, with repetition, is $C(n + r - 1, r)$.