Calculus 2 Final Exam version 1

Version 1: For who has the last digit of your student ID being even.

Lecturer: Ryosuke Takahashi; Date: 2021/6/23

Let ab be the final two digits of your student's ID. Define A and B as the following:

$$A = \begin{cases} \frac{1}{2} & \text{when } a \text{ is even} \\ 2 & \text{when } a \text{ is odd} \end{cases}.$$

and

$$B = \begin{cases} -2 & \text{when } b \in \{0,3,6,9\} \\ 1 & \text{when } b \in \{1,4,7\} \\ 3 & \text{when } b \in \{2,5,8\} \end{cases}$$

1. (12 points) Compute the following double integral:

$$\iint_\Omega y^2 dxdy$$

with $\Omega := \{(x,y) \in \mathbb{R}^2 | \frac{x^2}{A^2} + \frac{y^2}{B^2} \le 1\}$.
Hint. Change the variables to make Ω a disc of radius 1.

2. (12 points) Let $F = (9 + 6xy, 3x^2 - 9y^3)$ be a vector field defined on $\mathbb{R}^2$. Show that F is conservative and find the function f such that $F = \nabla f$.

3. (12 points) Let $R = \{(x,y) \in \mathbb{R}^2 | 1 \le x^2 + y^2 \le (B-2)^2 + 1, x \le 0\}$. Find

$$\iint_R (2Ax + By^2) dxdy.$$

4. (16 points) Let $S = \{(x,y,z) | z = x^2 + y^2 + 2, z \le 11\}$ be a surface. We can parametrize the surface by variables $(u,v) = (x,y)$.
a. Find the unit normal vector $\mathbf{n}$ of this surface.
b. Find the area of this surface.

5. (12 points) Let $\Omega = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \le |B|\}$. Find

$$\iiint_\Omega (x^2 + 2y^2)\sqrt{x^2 + y^2 + z^2} dxdydz.$$

6. (12 points) Let C be the boundary of the region enclosed by $y = x^2$, $x = y^2$. Find

$$\int_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy.$$

7. (12 points) Let $\Omega = \{(x, y, z) \in \mathbb{R}^3 | z^2 \geq x^2 + y^2, 0 \leq z \leq 2\}$ and $V = (2x^3, 2y^3, 2z^3)$. Find

$$\iint_{\partial\Omega} V \cdot \vec{n} dS.$$

where $\vec{n}$ is the outer unit normal of $\partial\Omega$.

8. (12 points) Let $S = \{(x, y, z) \in \mathbb{R}^3 | x^2 + \frac{y^2}{B^2} + \frac{z^2}{A^2} = 1\}$. Explain why

$$\iint_S \operatorname{curl}(V) \cdot \vec{n} dS = 0$$

for any vector field V defined on $\mathbb{R}^3$.