

Gaussian Random Variables

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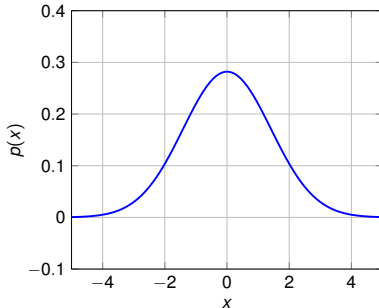
Gaussian Random Variable

Definition

A continuous random variable with pdf of the form

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty,$$

where μ is the mean and σ^2 is the variance.



Notation

- $\mathcal{N}(\mu, \sigma^2)$ denotes a Gaussian distribution with mean μ and variance σ^2
- $X \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow X$ is a Gaussian RV with mean μ and variance σ^2
- If $X \sim \mathcal{N}(0, 1)$, then X is a standard Gaussian RV

Affine Transformations Preserve Gaussianity

Theorem

If X is Gaussian, then $aX + b$ is Gaussian for $a, b \in \mathbb{R}$.

Remarks

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.
- If $X \sim \mathcal{N}(\mu, \sigma^2)$ and $\sigma \neq 0$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$.

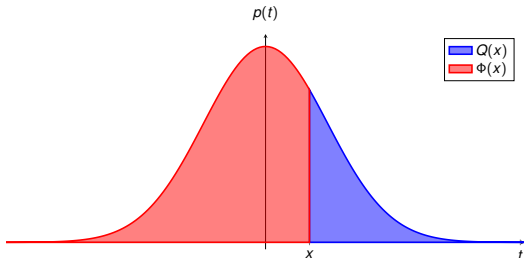
CDF and CCDF of Standard Gaussian

- Cumulative distribution function of $X \sim \mathcal{N}(0, 1)$

$$\Phi(x) = P[X \leq x] = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$$

- Complementary cumulative distribution function of $X \sim \mathcal{N}(0, 1)$

$$Q(x) = P[X > x] = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$$



Properties of $Q(x)$

- $\Phi(x) + Q(x) = 1$
- $Q(-x) = \Phi(x) = 1 - Q(x)$
- $Q(0) = \frac{1}{2}$
- $Q(\infty) = 0$
- $Q(-\infty) = 1$
- $X \sim \mathcal{N}(\mu, \sigma^2)$

$$P[X > \alpha] = Q\left(\frac{\alpha - \mu}{\sigma}\right)$$

$$P[X \leq \alpha] = Q\left(\frac{\mu - \alpha}{\sigma}\right)$$

Jointly Gaussian Random Variables

Jointly Gaussian Random Variables

Definition (Jointly Gaussian RVs)

Random variables X_1, X_2, \dots, X_n are jointly Gaussian if any linear combination is a Gaussian random variable.

$a_1 X_1 + \dots + a_n X_n$ is Gaussian for all $(a_1, \dots, a_n) \in \mathbb{R}^n$.

Example (Not Jointly Gaussian)

$X \sim \mathcal{N}(0, 1)$

$$Y = \begin{cases} X, & \text{if } |X| > 1 \\ -X, & \text{if } |X| \leq 1 \end{cases}$$

$Y \sim \mathcal{N}(0, 1)$ and $X + Y$ is not Gaussian.

Covariance

- For real random variables X and Y , the covariance is defined as

$$\text{cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

where $\mu_X = E[X]$ and $\mu_Y = E[Y]$

- Properties

- $\text{var}(X) = \text{cov}(X, X)$
- If X and Y are independent, then $\text{cov}(X, Y) = 0$
- If $\text{cov}(X, Y) = 0$, then they are said to be uncorrelated
- $\text{cov}(X + a, Y + b) = \text{cov}(X, Y)$ for any $a, b \in \mathbb{R}$
- Covariance is a bilinear function

$$\begin{aligned}\text{cov}(a_1 X_1 + a_2 X_2, a_3 X_3 + a_4 X_4) &= a_1 a_3 \text{cov}(X_1, X_3) \\ &\quad + a_1 a_4 \text{cov}(X_1, X_4) \\ &\quad + a_2 a_3 \text{cov}(X_2, X_3) \\ &\quad + a_2 a_4 \text{cov}(X_2, X_4)\end{aligned}$$

- Correlation coefficient of X and Y is defined as

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X) \text{var}(Y)}}.$$

- $|\rho(X, Y)| \leq 1$ with equality $\iff \Pr[Y = aX + b] = 1$ for some constants a, b

Mean Vector and Covariance Matrix

- Let $\mathbf{X} = (X_1, \dots, X_n)^T$ be a $n \times 1$ random vector
- The mean vector of \mathbf{X} is given by $\mathbf{m}_X = E[\mathbf{X}] = (E[X_1], \dots, E[X_n])^T$
- The covariance matrix \mathbf{C}_X of \mathbf{X} is an $n \times n$ matrix with (i, j) th entry given by

$$\begin{aligned}\mathbf{C}_X(i, j) &= E[(X_i - E[X_i])(X_j - E[X_j])] \\ &= E[X_i X_j] - E[X_i]E[X_j]\end{aligned}$$

- A compact notation for \mathbf{C}_X is

$$\mathbf{C}_X = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T] = E[\mathbf{X}\mathbf{X}^T] - E[\mathbf{X}](E[\mathbf{X}])^T$$

- If $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ where \mathbf{A} is $m \times n$ constant matrix and \mathbf{b} is an $m \times 1$ constant vector, then

$$\begin{aligned}\mathbf{m}_Y &= \mathbf{A}\mathbf{m}_X + \mathbf{b} \\ \mathbf{C}_Y &= \mathbf{A}\mathbf{C}_X\mathbf{A}^T\end{aligned}$$

Gaussian Random Vector

Definition (Gaussian Random Vector)

A random vector $\mathbf{X} = (X_1, \dots, X_n)^T$ whose components are jointly Gaussian.

Notation

$\mathbf{X} \sim \mathcal{N}(\mathbf{m}, \mathbf{C})$ where

$$\mathbf{m} = E[\mathbf{X}], \quad \mathbf{C} = E[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^T]$$

Definition (Joint Gaussian Density)

If \mathbf{C} is invertible, the joint density is given by

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x} - \mathbf{m})\right)$$

Example (\mathbf{C} is not invertible)

$\mathbf{X} = (X_1, X_2)^T$ where $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 = 2X_1 + 3$

Affine Transformations Preserve Joint Gaussianity

- If \mathbf{X} is a Gaussian vector, then $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ is also a Gaussian vector
 - Here \mathbf{X} is an $n \times 1$ vector, \mathbf{A} is an $m \times n$ constant matrix, and \mathbf{b} is an $m \times 1$ constant vector
 - Any linear combination of Y_1, \dots, Y_m is a constant plus a linear combination of X_1, \dots, X_n , which is a Gaussian random variable
- Since \mathbf{Y} is a Gaussian random vector, its distribution is completely characterized by its mean vector and covariance matrix

$$\mathbf{X} \sim \mathcal{N}(\mathbf{m}, \mathbf{C}) \implies \mathbf{Y} \sim \mathcal{N}(\mathbf{A}\mathbf{m} + \mathbf{b}, \mathbf{A}\mathbf{C}\mathbf{A}^T)$$

Uncorrelated Random Variables and Independence

- Recall that X_1 and X_2 are said to be uncorrelated if $\text{cov}(X_1, X_2) = 0$
- If X_1 and X_2 are independent,

$$\text{cov}(X_1, X_2) = 0.$$

- If X_1, \dots, X_n are jointly Gaussian and pairwise uncorrelated, then they are independent. Consider the case when $\text{var}(X_i) \neq 0$ for each i .

$$\begin{aligned} p(\mathbf{x}) &= \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x} - \mathbf{m})\right) \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(x_i - m_i)^2}{2\sigma_i^2}\right) \end{aligned}$$

where $m_i = E[X_i]$ and $\sigma_i^2 = \text{var}(X_i)$.

Uncorrelated Gaussian RVs may not be Independent

Example

- $X \sim \mathcal{N}(0, 1)$
- W is equally likely to be +1 or -1
- W is independent of X
- $Y = WX$
- $Y \sim \mathcal{N}(0, 1)$
- X and Y are uncorrelated
- X and Y are not independent

References

- Section 3.1, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008