Optimal Receiver in AWGN using Complex Baseband Representation

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Passband Signals in Passband Noise

Consider M-ary passband signaling over a channel with passband Gaussian noise

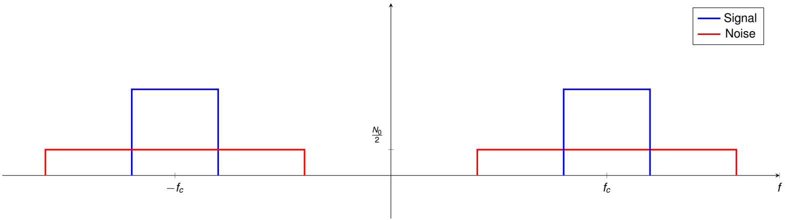
$$H_i : y_p(t) = s_{i,p}(t) + n_p(t), \quad i = 1, \ldots, M$$

where

$y_p(t)$ Real passband received signal

$s_{i,p}(t)$ Real passband signals

$n_p(t)$ Real passband GN with PSD $\frac{N_0}{2}$



Note: A WSS random process is passband if its autocorrelation function is a passband signal

Passband Signals in Passband Noise

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where

$y_p(t)$	Real passband received signal
$s_{i,p}(t)$	Real passband signals
$n_p(t)$	Real passband GN with PSD $\frac{N_0}{2}$

The equivalent problem in complex baseband is

$$H_i : y(t) = s_i(t) + n(t), \quad i = 1, \dots, M$$

where

$y(t)$	Complex envelope of $y_p(t)$
$s_i(t)$	Complex envelope of $s_{i,p}(t)$
$n(t)$	Complex envelope of $n_p(t)$

What is the optimal receiver in terms of the complex baseband signals?

Optimal Receiver in AWGN using Complex Envelopes

- Optimal receiver using passband representations

$$\delta_{MPE}(y_p) \quad = \quad \underset{1 \le i \le M}{\operatorname{argmax}} \; \langle y_p, s_{i,p} \rangle - \frac{\|s_{i,p}\|^2}{2} + \sigma^2 \log \pi_i$$

- Recall that $\langle u_p, v_p \rangle = \operatorname{Re}\left(\langle u, v \rangle\right)$ and $\|u_p\|^2 = \|u\|^2$
- Optimal receiver using complex baseband representations

$$\delta_{MPE}(y) \quad = \quad \underset{1 \le i \le M}{\operatorname{argmax}} \; \langle y, s_i \rangle - \frac{\|s_i\|^2}{2} + \sigma^2 \log \pi_i$$

  where $y(t), s_i(t)$ are the complex envelopes of $y_p(t), s_{i,p}(t)$ respectively
- But what about the performance analysis?
- We need to understand the statistics of $n(t)$, the complex envelope of the passband Gaussian noise process $n_p(t)$

Complex Envelope of Passband Gaussian Noise

- The complex baseband representation of $n_p(t)$ is given by

$$n(t) = n_c(t) + jn_s(t) = \frac{1}{\sqrt{2}} \left[n_p(t) + j\hat{n}_p(t) \right] e^{-j2\pi f_c t}$$

where $\hat{n}_p(t)$ is the Hilbert transform of $n_p(t)$

- The in-phase and quadrature components of $n(t)$ are given by

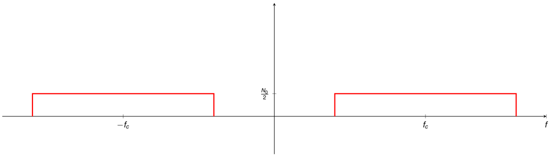
$$\begin{aligned} n_c(t) &= \frac{1}{\sqrt{2}} \left[n_p(t) \cos 2\pi f_c t + \hat{n}_p(t) \sin 2\pi f_c t \right] \\ n_s(t) &= \frac{1}{\sqrt{2}} \left[\hat{n}_p(t) \cos 2\pi f_c t - n_p(t) \sin 2\pi f_c t \right] \end{aligned}$$

- $n_c(t)$ and $n_s(t)$ are jointly Gaussian and i.i.d. random processes (Proof in Proakis Section 2.9)
- Random processes $X(t)$ and $Y(t)$ are jointly Gaussian if any $n, m \in \mathbb{Z}^+$ and $t_1, t_2, \ldots, t_n, t_1', t_2', \ldots, t_m' \in \mathbb{R}$, the random variables $X(t_1), X(t_2), \ldots, X(t_n), Y(t_1'), Y(t_2'), \ldots, Y(t_m')$ are jointly Gaussian random variables.

Complex Envelope PSD

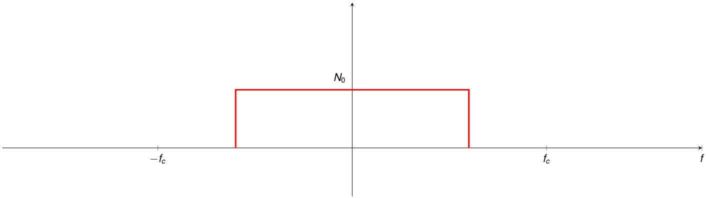
$$S_{n_p}(f) = \begin{cases} \frac{N_0}{2} & |f - f_c| < W \\ 0 & \text{otherwise} \end{cases}$$

Passband Gaussian Noise PSD



Recall that $S_n(f) = 2S_{n_p}(f + f_c)u(f + f_c) \implies S_n(f) = \begin{cases} N_0 & |f| < W \\ 0 & \text{otherwise} \end{cases}$

Complex Envelope PSD



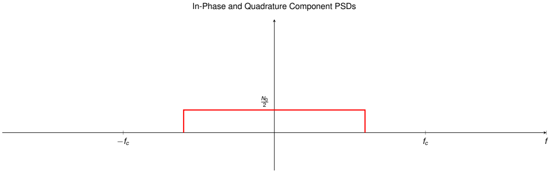
Complex Envelope PSD

- By the independence of $n_c(t)$ and $n_s(t)$, we have

$$R_n(\tau) = E\left[n(t+\tau)n^*(t)\right] = R_{n_c}(\tau) + R_{n_s}(\tau) \implies S_n(f) = S_{n_c}(f) + S_{n_s}(f)$$

- As $n_c(t)$ and $n_s(t)$ are identically distributed, we get

$$S_{n_c}(f) = S_{n_s}(f) = \begin{cases} \frac{N_0}{2} & |f| < W \\ 0 & \text{otherwise} \end{cases}$$



- If $n_c(t)$ and $n_s(t)$ are approximated by white Gaussian noise, $n(t)$ is said to be complex white Gaussian noise

Complex White Gaussian Noise

Definition

Real random processes $X(t)$ and $Y(t)$ are jointly Gaussian if any $n, m \in \mathbb{Z}^+$ and $t_1, t_2, \dots, t_n, t'_1, t'_2, \dots, t'_m \in \mathbb{R}$, the random variables $X(t_1), X(t_2), \dots, X(t_n), Y(t'_1), Y(t'_2), \dots, Y(t'_m)$ are jointly Gaussian random variables.

Definition (Complex Gaussian Random Process)

A complex random process $Z(t) = X(t) + jY(t)$ is a complex Gaussian random process if $X(t)$ and $Y(t)$ are jointly Gaussian random processes.

Definition (Complex White Gaussian Noise)

A complex Gaussian random process $Z(t) = X(t) + jY(t)$ is complex white Gaussian noise with PSD N_0 if $X(t)$ and $Y(t)$ are independent white Gaussian noise processes with PSD $\frac{N_0}{2}$.

Optimal Receiver using Signal Space Representation

- The continuous time hypothesis testing problem in complex baseband

$$H_i : y(t) = s_i(t) + n(t), \quad i = 1, \dots, M$$

where

$y(t)$	Complex envelope of $y_p(t)$
$s_i(t)$	Complex envelope of $s_{i,p}(t)$
$n(t)$	Complex white Gaussian noise with PSD $N_0 = 2\sigma^2$

- The equivalent problem in terms of complex random vectors

$$H_i : \mathbf{Y} = \mathbf{s}_i + \mathbf{N}, \quad i = 1, \dots, M$$

where $\mathbf{Y}$, $\mathbf{s}_i$ and $\mathbf{N}$ are the projections of $y(t)$, $s_i(t)$ and $n(t)$ respectively onto the signal space spanned by $\{s_i(t)\}$.

- $\mathbf{N}$ is a vector of complex Gaussian random variables

$$\mathbf{N} = \begin{bmatrix} N_{c,1} + jN_{s,1} \\ N_{c,2} + jN_{s,2} \\ \vdots \\ N_{c,K} + jN_{s,K} \end{bmatrix}$$

Optimal Receiver using Signal Space Representation

- Each component of $\mathbf{N}$ has independent real and imaginary parts
- Different components are also independent of each other
- The $K \times 1$ complex vectors in $\mathbf{Y} = \mathbf{s}_i + \mathbf{N}$, that is

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_K \end{bmatrix} = \begin{bmatrix} s_{i,1} \\ \vdots \\ s_{i,K} \end{bmatrix} + \begin{bmatrix} N_1 \\ \vdots \\ N_K \end{bmatrix}$$

can be written as $2K \times 1$ real vectors

$$\begin{bmatrix} Y_{1,c} \\ Y_{1,s} \\ Y_{2,c} \\ Y_{2,s} \\ \vdots \\ Y_{K,c} \\ Y_{K,s} \end{bmatrix} = \begin{bmatrix} s_{i,1,c} \\ s_{i,1,s} \\ s_{i,2,c} \\ s_{i,2,s} \\ \vdots \\ s_{i,K,c} \\ s_{i,K,s} \end{bmatrix} + \begin{bmatrix} N_{1,c} \\ N_{1,s} \\ N_{2,c} \\ N_{2,s} \\ \vdots \\ N_{K,c} \\ N_{K,s} \end{bmatrix}$$

where $Y_{j,c} = \mathrm{Re}\left(Y_j\right), Y_{j,s} = \mathrm{Im}\left(Y_j\right), s_{i,j,c} = \mathrm{Re}\left(s_{i,j}\right), s_{i,j,s} = \mathrm{Im}\left(s_{i,j}\right), N_{j,c} = \mathrm{Re}\left(N_j\right), N_{j,s} = \mathrm{Im}\left(N_j\right)$

- The joint pdf of the real Gaussian random vectors can be used for performance analysis

ML Receiver for QPSK

- QPSK signals where $q(t)$ is a real baseband pulse, A is a real number and $1 \leq i \leq 4$

$$
\begin{aligned}
s_{i,p}(t) &= \sqrt{2}Aq(t)\cos\left(2\pi f_c t + \frac{\pi(2i-1)}{4}\right) \\
&= \mathrm{Re}\left[\sqrt{2}Aq(t)e^{j\left(2\pi f_c t + \frac{\pi(2i-1)}{4}\right)}\right] \\
&= \mathrm{Re}\left[\sqrt{2}Aq(t)e^{j\frac{\pi(2i-1)}{4}}e^{j(2\pi f_c t)}\right]
\end{aligned}
$$

- Complex Envelope of QPSK Signals

$$
s_i(t) = Aq(t)e^{j\frac{\pi(2i-1)}{4}}, \quad 1 \leq i \leq 4
$$

- Orthonormal basis for the complex envelope consists of only

$$
\phi(t) = \frac{q(t)}{\sqrt{E_q}}
$$

where $E_q = \|q\|^2$

ML Receiver for QPSK

Let $\sqrt{E_b} = \frac{A\sqrt{E_q}}{\sqrt{2}}$. The vector representation of the QPSK signals is

$$\begin{aligned} s_1 &= \sqrt{E_b} + j\sqrt{E_b} \\ s_2 &= -\sqrt{E_b} + j\sqrt{E_b} \\ s_3 &= -\sqrt{E_b} - j\sqrt{E_b} \\ s_4 &= \sqrt{E_b} - j\sqrt{E_b} \end{aligned}$$

The hypothesis testing problem in terms of vectors is

$$H_i : \begin{bmatrix} Y_c \\ Y_s \end{bmatrix} = \begin{bmatrix} s_{i,c} \\ s_{i,s} \end{bmatrix} + \begin{bmatrix} N_c \\ N_s \end{bmatrix}, \quad i = 1, \dots, 4$$

where $s_{i,c} = \text{Re}(s_i)$, $s_{i,s} = \text{Im}(s_i)$, $N_c \sim \mathcal{N}(0, \sigma^2)$, $N_s \sim \mathcal{N}(0, \sigma^2)$, $N_c \perp N_s$

The ML rule is given by

$$\delta_{ML}(\mathbf{y}) = \underset{1 \le i \le 4}{\text{argmin}} \, (y_c - s_{i,c})^2 + (y_s - s_{i,s})^2 = \underset{1 \le i \le 4}{\text{argmin}} \, \|\mathbf{y} - \mathbf{s}_i\|^2$$

References

- Sections 3.4, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008