COMPLEX VARIABLES

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CHAPTER 1

COMPLEX NUMBERS

1.1 Complex Numbers

A complex number is a number of the form

$$z = a + ib \tag{1.1}$$

where the imaginary unit is defined as [1]

$$i = \sqrt{-1} \tag{1.2}$$

and a is the real part of z, written $a = Re(z)$, and b is the imaginary part of z, written $b = Im(z)$. a and b are real numbers.

Complex numbers have real and imaginary parts. However, if a number has no real part, then it is called 'pure imaginary'.

It may be easily verified from Eq. (1.2) that $i^2 = -1$, and that

$$\frac{1}{i} = -i \tag{1.3}$$

[1]In electrical engineering, a lower case i represents a time-dependent current, so it is their convention to use the symbol j in place of i.

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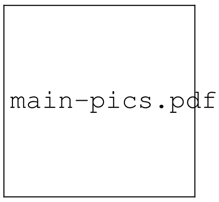

Figure 1.1 A complex number is easily visualized as a "phasor" in the complex plane.

1.2 The Complex Plane

A complex number can be visualized in a two-dimensional number line, known as an Argand diagram, or the complex plane as shown in Fig. 1.1. The complex plane replaces the number line as a visualization tool for real numbers. However, rather than plot points in the complex plane, it is conventional to represent a complex number as a vector in the complex plane. Instead of calling them complexified vectors, they are referred to as “phasors.” Note that because the visualization is 2-dimensional, a polar form for complex numbers is suggested. This is discussed below.

As an addition side note about the complex plane, complex analysis can be used to study improper real integrals that couldn’t otherwise be solved. Basically, for an integral that extends from $-\infty$ to ∞, instead of integrating over the real number line, one may perform a closed path line integral that includes the real number line and a semicircle of radius R, which is allowed to approach ∞. The residue theorem relates the closed loop integral to a real value. However, it can be shown that the integral over the infinite semicircle approaches zero, and our integral of interest is obtained.

1.3 Complex Magnitude

From Fig. 1.1, it can be easily seen (using the Pythagorean theorem) that the magnitude, or length, of the vector representing a complex number is

$$|z| = \sqrt{a^2 + b^2} \tag{1.4}$$

Thus, the complex magnitude is the square root of the sum of the squares of the real and imaginary parts of the complex number. This definition generalizes the absolute value function of a real number.

1.4 Complex Conjugate

The complex conjugate of a complex number z, is denoted $\bar{z}$, and is defined

$$\bar{z} = a - ib$$

We may think of the conjugation process as “replacing i with $-i$.”

Where would the vector $\bar{z}$ fit on the complex plane in Fig 1.1?

EXAMPLE 1.1 Complex Conjugate

Prove that $\bar{z}z = |z|^2$.

Proof:

$$\begin{aligned}
\bar{z}z &= (a-ib)(a+ib) \\
&= a^2+b^2 \\
&= (a^2+b^2)^{1/2}(a^2+b^2)^{1/2} \\
&= |z|^2
\end{aligned}$$

■

Note that $\bar{z}z$ is real and positive. A quotient of complex numbers can be written separated into real and imaginary parts using the above conjugate relation as shown in the next example.

EXAMPLE 1.2

Complex Fractions
Separate the complex number $z = \frac{3-i4}{5+i12}$ into real and imaginary parts.

Proof: The solution involves the well known rationalizing the denominator technique of multiplying the top and bottom of the quotient by the conjugate of the denominator.

$$\begin{aligned}
z &= \frac{3-i4}{5+i12} \\
&= \frac{3-i4}{5+i12}\frac{5-i12}{5-i12} \\
&= \frac{(15-48)+i(-20-36)}{13} \\
&= -\frac{33+i56}{13}
\end{aligned}$$

■

It may be noted that $|\bar{z}| = |z|$. Also, the conjugate of a product is a product of conjugates so that $\overline{(uv)} = \bar{u}\bar{v}$. Similarly, the conjugate of a sum is the sum of conjugates so that $\overline{(u+v)} = \bar{u}+\bar{v}$. Finally, the conjugate of a conjugate is the function itself, i.e. $\overline{(\bar{z})} = z$. Put another way, complex conjugation "toggles."

EXAMPLE 1.3

Consider the function $f(z) = 3z^2 + (2+i7)z + i6$ where z is a complex variable.
a) What is the conjugate of the function, $f^*(z)$?
b) What is $f(\bar{z})$?
c) What is $f^*(\bar{z})$?

Proof: Following the simple steps:

$$\begin{aligned}
a) f^*(z) &= [3z^2 + (2+i7)z + i6]^* \\
&= (3z^2)^* + [(2+i7)z]^* + (i6)^* \\
&= 3z^{*2} + [(2+i7)^* \bar{z}] + i^* 6^* \\
&= 3z^{*2} + (2-i7)\bar{z} - i6 \\
b) f(\bar{z}) &= 3z^{*2} + (2+i7)\bar{z} + i6 \\
c) f^*(\bar{z}) &= 3z^2 + (2-i7)z - i6
\end{aligned}$$

■

1.5 Polar Form of a Complex Number

Let r be the magnitude of the complex number z, and let θ be the angle that the line from origin to the complex number z makes with the positive x-axis. Here, note that θ is not defined if $z = 0$. The complex number z can be expressed in terms of a magnitude, r, and the angle, θ, as

$$z = r(cos\theta + isin\theta) \tag{1.5}$$

Due to Euler, we have a well known result

$$e^{i\theta} = cos(\theta) + isin(\theta) \tag{1.6}$$

This is one of the most powerful results in all of mathematics. Basically, all of the trignometric identities can be derived from it. Substituting the Euler Relation (Eq 1.6) into (Eq 1.5) yields

$$z = re^{i\theta} \tag{1.7}$$

Thus, a complex number can be thought of as having two forms: a rectangular form (Eq. 1.1) and a polar form (Eq. 1.7).

The complex conjugate of Eq. (1.6) is

$$e^{-i\theta} = cos(\theta) - isin(\theta) \tag{1.8}$$

Adding Eqs. (1.6) and (1.8) and dividing by 2 yields the important result

$$cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{1.9}$$

Similarly,

$$sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{1.10}$$

While the Euler relation is the most important result here, these two closely follow. While extremely useful, it is also generally interesting that the sum of complex functions yields a real function.

In the polar form the conjugate of z is

$$\bar{z} = re^{-i\theta} \tag{1.11}$$

Using the polar form, the result from Example A.1 becomes transparent.

EXAMPLE 1.4

The Euler Relation
Use Euler's identity to derive the formula for the cos of the sum of two angles.

Proof: If $Re\{\}$ represents an operator which takes the real part of a complex number or function, then from Eq. (1.9)

$$\begin{aligned} cos(a+b) &= Re\{e^{i(a+b)}\} \\ &= Re\{e^{ia}e^{ib}\} \\ &= Re\{[cos(a)+isin(a)][cos(b)+isin(b)]\} \\ &= Re\{[cos(a)cos(b)-sin(a)sin(b)]+i[sin(a)cos(b)+cos(a)sin(b)]\} \\ &= cos(a)cos(b)-sin(a)sin(b) \end{aligned}$$

Clearly the $sin(a+b)$ is also readily obtained. ■

1.6 Hyperbolic Sin and Cos

It is clear that the sin and cos of a real number is a real number, but what about the sin and cos of a number that is pure imaginary? From Eqs. (1.9) and (1.10) it follows that the sin and cos of a pure imaginary number is ... drumroll, please ... real! This was the inspiration for defining hyberbolic cos and sin. They are defined by simply erasing the "i's" in Eqs. (1.9) and (1.10):

$$cosh(\theta) = \frac{e^{\theta} + e^{-\theta}}{2} \tag{1.12}$$

$$sinh(\theta) = \frac{e^{\theta} - e^{-\theta}}{2} \tag{1.13}$$

Of course, once you have $sinh$ and $cosh$, you can define $tanh$, $coth$, $arcsinh$, ... In addition, there are hyperbolic trigonmetric identities. For example, by looking at the above equations, you should be able to confirm (without pencil and paper!) that

$$cosh^2(x) - sinh^2(x) = 1 \tag{1.14}$$

The switching from the variable θ to x was intentional. The argument of a sinusoid is an angle (in radians), while the argument of a hyberbolic sinusoid is not (it's dimensionless).

1.7 Converting Between Polar and Rectangular Forms

A complex number written in polar form may be converted to rectangular form by the relations

$$a = Acos(\theta) \tag{1.15}$$

$$b = Asin(\theta) \tag{1.16}$$

These are immediately obtained by substituting the Euler relation into the polar form of a complex number. Conversely, these equations may be inverted, and a complex number written in rectangular form may be converted to polar form by the relations

$$A = \sqrt{a^2 + b^2} \tag{1.17}$$

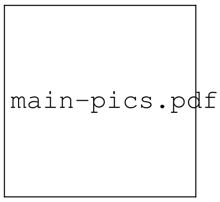

Figure 1.2 The nth root of a complex number is an angle which $1/n$th the original number.

$$\theta = tan^{-1}(b/a) \tag{1.18}$$

These four formulas are identical to normal polar-to-Cartesian and vice-versa conversions. Quite often in complex number calculations, one switches between the two forms.

1.8 Powers and Roots of Complex Numbers

A most logical way to continuing our study of complex numbers would be to look at the sin and cos of a complex number, the exponential function of a complex number, powers and roots of complex numbers... Basically, separate any elementary function[2] of a complex number into real and imaginary parts. The sin, cos, and exponential functions are easy, and left to the reader as an exercise. Here we consider powers and roots of complex numbers.

As a first step in this method is to write your complex number in polar form. With this done, the power of a complex number is easily calculated:

$$z^n = A^n e^{in\theta} \tag{1.19}$$

If desired (or required), one would then convert this back to the rectangular form. Solving problems involving complex numbers and functions often involves switching back and forth between rectangular and polar form.

Roots of complex numbers may be obtained in a nearly identical manner:

$$z^{1/n} = A^{1/n} e^{i\theta/n} \tag{1.20}$$

It is interested and inciteful to interpret these results graphically. The angle is reduced by a factor of $1/n$ and the magnitude is affected in the same way as the square root of a real number is. For a complex number of unit magnitude, a plot of a complex number with three of its roots are shown. At times it is useful to have the formula for a root of a complex number in rectangular form. While this can't be done in general, the square root is tractable:

$$\sqrt{a+ib} = \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + isgn(b)\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}} \tag{1.21}$$

where $sgn(b)$ is the signum function, which is also known as the sign of b. The signum function is generally defined as $b/|b|$. We define $sgn(0)$ to be unity.

[2]and special functions such as Bessel functions too!

1.9 The Problem that "Can't Be Done"

In pre-calculus and even in calculus, you may have been told that calculating the $arccos$ of a number greater than 1 can't be done. Since the cos function oscillates between -1 and 1, calculating $arccos(3)$ would be "difficult." When I ask my calculator to calculate $arccos(3)$, it says "Error 0."

Of course the calculator and the early math courses are restricting themselves to the real number system.[3] The $arccos(3)$, for example, is just a complex number. In fact, it is pure imaginary, as we will now show.

EXAMPLE 1.5

Inverse Trig Functions
Calculate $arccos(3)$.

Proof: let

$$yuivarccos(3) \tag{1.22}$$

, then $cos(y) = 3$. But from the Euler Relation, the cos function can be written as a sum of complex exponentials:

$$A\frac{e^{iy} + e^{-iy}}{2} = 3 \tag{1.23}$$

A Multiplying both sides by $2e^{iy}$, and moving all terms to the left hand side of the equation results in

$$A\left(e^{iy}\right)^2 - 6e^{iy} + 1 = 0 \tag{1.24}$$

A which is a quadratic equation in e^{iy}. The solutions are

$$Ae^{iy} = 3 \pm 2\sqrt{2} \tag{1.25}$$

A Taking the natural log of both sides and multiplying by $-i$ results in

$$Ay = -iln(3 \pm 2\sqrt{2}) \tag{1.26}$$

A ∎

1.10 PROBLEMS

PROBLEMS Separate the following into real and imaginary parts:
a) $\frac{3+i4}{5+i7}$
b) $(3+i4) + i(4+i5) + (2+i3)(4+i5)^2$
c) $tan(3+i4)$
d) e^{3+i4}
e) $\sqrt{1+i2}$
f) $ln(3+i4)$
g) $sin^{-1}(3)$

[3] Your calculator may not have this restriction.

h) i^i
Hint: i can be thought of as a complex number in rectangular form.
i) There are an infinite number of values for i^i, what are they?
Hint: $e^{i\theta} = e^{i(\theta+2n\pi)}$

PROBLEMS Consider a series AC electrical circuit with two resistors and a capacitor. The output complex voltage is related to the input complex voltage by the voltage divider

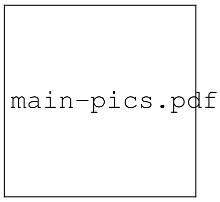

Figure 1.3 A simple AC circuit.

law

$$\hat{V}_{out} = \frac{R_2}{R_1 + R_2 - i/(\omega C)} \hat{V}_{in}$$

If $R_1 = 100\Omega$, $R_2 = 200\Omega$, $C = 50\mu F$, and $\omega = 2\pi(60)$ cycles/s, and $\hat{V}_{in} = 100V$, then what is the

a) magnitude of the output voltage,
b) phase of the output voltage.
c) plot $|\hat{V}_{out}/\hat{V}_{in}|$ as a function of differents ω's.

PROBLEMS If Ψ is the wave function from the Schrödinger equation, then the probability density of finding a particle at a particular place is given by $P(x) = \Psi^*\Psi$. Suppose that you have solved the Schrödinger equation for a given potential functions, and you find that

$$\Psi(x) = \frac{p}{1 + qx^2}$$

where $p = p_r + ip_i$, $q = q_r + iq_i$ are complex constants, and $P = p^*p$, and $Q = q^*q$. Compute the probability density distribution in this case. (Note: x is a real variable).

PROBLEMS One problem of interest is the writing of the n^{th} power of cos into a Fourier series. Using trigonometric identities it is generally difficult to prove that

$$cos^m(\omega t) = \frac{1}{2^m} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} cos[(m-2k)\omega t]$$

Use the Euler relation to derive the above result[4].

[4] As the student may be aware, the binomial theorem is $(a+b)^m = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} a^k b^{m-k}$

PROBLEMS Plot the following phasors tail-to-tip on a piece of graph paper:

$$2 - \sqrt{29}e^{-itan^{-1}(5/2)} + (\sqrt{34}/5)e^{+itan^{-1}(3/5)} - 2 + (\sqrt{136}/5)e^{-itan^{-1}(3/5)}$$

$$-\sqrt{5}e^{-itan^{-1}(2)} - \sqrt{5}e^{+itan^{-1}(2)} + (\sqrt{34}/5)e^{+itan^{-1}(3/5)} - (\sqrt{29}/5)e^{+itan^{-1}(5/2)} + 2$$

Hints: 1) Start near the bottom middle of your graph paper. 2) The sum of the complex numbers is zero, so the last phasor should end at the same place your first complex number started.

PROBLEMS The complex propagation constant for an electromagnetic wave propagating in a conductive medium can be obtained from the formula

$$\begin{aligned} {k_0}^2 &= \omega^2\mu\epsilon - i\mu\sigma\omega \\ &\equiv (\beta_0 - i\alpha_0)^2 \end{aligned}$$

where β_0 is the propagation constant and α_0 is the loss per length.

a) If the skin depth $\delta = 1/\alpha_0$, then obtain an analytical expression for δ in terms of ω, μ, ϵ, and σ.

b) Simply you expression for $\frac{\sigma}{\omega\epsilon} << 1$.

PROBLEMS Use the Euler Relation to derive the trigonometric identity for the sin of the sum of two different angles: $sin(a + b)$.

PROBLEMS Often times books show a proof of Euler's Identity by looking at the Taylor series expansion for $sin(x)$ and $cos(x)$, comparing it to the expansion for e^x and saying "Tada, it works!" Here the goal is to $derive$ the Euler relation, assuming that we know a little bit about differential equations. Put another way, we don't want to just show that it happens to work, but that it $must$ work. First, we consider the following differential equation which represents simple harmonic motion such as from a pendulum (small angles) or a spring:

$$\frac{d^2y}{dt^2} + \omega^2 y(t) = 0$$

Complete the following steps of the derivation:

a) Show that $y(t) = asin(\omega t) + bcos(\omega t)$ are solutions to the differential equation.

b) Apply the boundary conditions $y(0) = y_0$ and $y'(0) = y'_0$ to this solution.

c) Show that $y(t) = Ae^{i\omega t} + Be^{-i\omega t}$ are solutions to the differential equation.

d) Apply the boundary conditions $y(0) = y_0$ and $y'(0) = y'_0$ to this solution.

e) Since the differential equation is second order, it has only two independent solutions. Thus coefficients of the y_0 terms in the two equations must be equal to each other. Set them equal to each other, and solve for $e^{i\omega t}$.

PROBLEMS Make separate plots of the following roots as phasors in the complex plane:
a) $z^2 = 1$
b) $z^3 = 1$
c) $z^4 = 1$
d) $z^5 = 1$
e) $z^3 = \frac{1+i}{\sqrt{2}}$

CHAPTER 2

ANALYTIC FUNCTIONS

The theory of functions of complex variable is utmost important in solving a large number of problems in the field of engineering. Many complicated integrals of real functions are solved with the help of functions of a complex variable.

2.1 Introduction

Let Z and W be two non-empty set of complex numbers. A rule f assigns to each element $z \in Z$, a unique $w \in W$, is called as complex function or single valued function. i.e.,

$$f : Z \to W$$

We may also write,

$$w = f(z)$$

Here z and w are complex variables. As $z = x+iy$, x and y are independent real variables. Let $w = u + iv$, a function of z, which implies that u and v are functions x and y as z is function of x and y. i.e,

$$u \equiv u(x, y)$$
$$v \equiv v(x, y)$$

Thus

$$w = f(z) = u(x, y) + iv(x, y)$$

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EXAMPLE 2.1

Write the function $w = z^2 + 2z$ in the form $w = u(x, y) + \imath v(x, y)$.

By setting $z = x + \imath y$ we obtain

$$w = (x + \imath y)^2 + 2(x + \imath y) = x^2 - y^2 + \imath 2xy + 2x + \imath 2y$$

Which then can be rewritten as

$$w = (x^2 - y^2 + 2x) + \imath(2xy + 2y).$$

■

2.2 Limits and Continuity of Complex Functions

The concepts of limits and continuity for complex functions are very similar to those for real functions. Let's first examine the concept of the limit of a complex-valued function.

Definition 2.1 (Limit) *Let f be a function defined in some neighborhood of z_0, with the possible exception of the point z_0 itself. We say that the limit of $f(z)$ as z approaches z_0 is the number w_0 and write*

$$\lim_{z \to z_0} f(z) = w_0,$$

or equivalently,

$$f(z) \to w_0 \text{ as } z \to z_0,$$

if for any $\epsilon > 0$ there exists a positive number δ such that

$$|f(z) - w_0| < \epsilon$$

whenever $0 < |z - z_0| < \delta$ (deleted neighborhood)[1]

If $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Let $z_0 = x_0 + iy_0$, then

$$\lim_{z \to z_0} f(z) = w_0 = \alpha + i\beta \quad \text{or} \quad \lim_{x+iy \to x_0+iy_0} f(z) = \alpha + i\beta$$

$$\Leftrightarrow \quad \lim_{x \to x_0, y \to y_0} u(x, y) = \alpha \quad \text{and} \quad \lim_{x \to x_0, y \to y_0} v(x, y) = \beta$$

Definition 2.2 (Continuous) *Let $f(z)$ be a complex valued function defined in a neighborhood of z_0. Then , we say f is continuous at z_0 if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

That is, for f to be continuous at z_0, it must have a limiting value at z_0, and this limiting value must be $f(z_0)$.

In other words, The function $f(z)$ of a complex variable z is said to continuous at the point z_0, if for any given positive number ϵ, we can find a number δ such that

$$|f(z) - f(z_0)| < \epsilon \ ,$$

[1]A subset C_N of the complex plane containing z_0 is said to be neighborhood of z_0, if for some real number $\delta > 0$, the set $\{z \in C : |z - z_0| < \delta\} \subseteq C_N$. Further the set $C_N - \{z_0\}$ is called deleted neighborhood of z_0.

for all points z of the domain satisfying $|z - z_0| < \delta$.

Also, if $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is continuous at $z_0 = x_0 + iy_0$, then u and v are separately continuous at the point $z_0 = x_0 + iy_0$.

Definition 2.3 *$f(z)$ is said to be continuous in domain if continuous at each point of that domain.*

In fact, the properties of limits and continuous functions for real functions are also hold for complex-valued functions.

Theorem 2.1 *If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then*

(i) $\lim_{z \to z_0}(f(z) \pm g(z)) = A \pm B$,

(ii) $\lim_{z \to z_0} f(z)g(z) = AB$,

(iii) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$, if $B \neq 0$.

Theorem 2.2 *If $f(z)$ and $g(z)$ are continuous at z_0, then so are $f(z) \pm g(z)$ and $f(z)g(z)$. The quotient $\frac{f(z)}{g(z)}$ is also continuous at z_0 provided $g(z_0) \neq 0$.*

Here are some simple examples using these concepts of limits and continuity.

EXAMPLE 2.2

Find the limit as $z \to 2\imath$ of the function $f(z) = z^2 - 2z + 1$.

Solution: Since $f(z)$ is continuous at $z = 2\imath$, we simply evaluate it there,

$$\lim_{z \to 2\imath} f(z) = f(2\imath) = 2(2\imath)^2 - 2(2\imath) + 1 = -3 - 4\imath.$$

■

EXAMPLE 2.3

Find the limit as $z \to 2\imath$ of the function $f(z) = \frac{z^2+4}{z(z-2\imath)}$.

Solution: The function $f(z)$ is not continuous at $z = 2\imath$ because it is not defined there. However, for $z \neq 2\imath$ and $z \neq 0$ we have

$$\lim_{z \to 2\imath} f(z) = \frac{(z + 2\imath)(z - 2\imath)}{z(z - 2\imath)} = \frac{z + 2\imath}{z} = \frac{2\imath + 2\imath}{2\imath} = \frac{4\imath}{2\imath} = 2.$$

■

EXAMPLE 2.4

The function $z, Re(z), Im(z)$ and $|\overline{z}|$ are continuous in the entire plane.

2.3 Complex Differentiation

In general, a complex function of a complex variable, $f(z)$, is an arbitrary mapping from the *xy*-plane to the *uv*-plane. A complex function is split into real and imaginary parts,

u and v, and any pair $u(x, y)$ and $v(x, y)$ of two-variable functions gives us a complex function $u + \imath v$.

Consider the following example,

$$u_1(x, y) = x^2 - y^2, \quad v_1(x, y) = 2xy$$

as opposed to

$$u_2(x, y) = x^2 - y^2, \quad v_2(x, y) = 3xy$$

The difference is that the first complex function $u_1 + \imath v_1$ can be written as function of $z = x + \imath y$, z as a single "unit", because $x^2 - y^2 + \imath 2xy = (x + \imath y)^2$. These are the types of functions that are complex differentiable.

Definition 2.4 *Let the complex function $f(z)$ be defined in neighbourhood of z_0, the complex derivative of $f(z)$ at z_0, is defined as,*

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

provided this limit exists, and is denoted by $f'(z_0)$.

This means that the value of the limit is independent of the manner in which $\Delta z \to 0$. Here Δz is a complex number, so it can approach zero in many different ways. Which shows study becomes seem slightly more difficult.

If the complex derivative exists at a point, then we say that the function is *complex differentiable* at z_0. Although, $\Delta z \to 0$ in many different ways, the rules for differentiating real functions apply in the same way for complex-valued functions (as long as the complex-valued function is in a form where $z = x + \imath y$ is treated as a single unit).

Theorem 2.3 *If f and g are differentiable at z, then*

1. $(f \pm g)'(z) = f'(z) \pm g'(z)$,
2. $(cf)'(z) = cf'(z)$ *for any constant c,*
3. $(fg)'(z) = f(z)g'(z) + f'(z)g(z)$,
4. $\left(\frac{f}{g}\right)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2}$ *if* $g(z) \neq 0$
5. *if g is differentiable at z and f is differentiable at $g(z)$, then the chain rule holds:*

$$\frac{d}{dz} f(g(z)) = f'(g(z))g'(z).$$

EXAMPLE 2.5

Show that, for any positive integer n,

$$\frac{d}{dz} z^n = nz^{n-1}.$$

Solution: Using Definition 2.4 we have

$$\frac{(z + \Delta z)^n - z^n}{\Delta z} = \frac{nz^{n-1}\Delta z + \frac{n(n-1)}{2} z^{n-2}(\Delta z)^2 + \cdots + (\Delta z)^n}{\Delta z}.$$

Thus

$$\frac{d}{dz}z^n = \lim_{\Delta z \to 0}\left[nz^{n-1} + \frac{n(n-1)}{2}z^{n-2}\Delta z + \cdots + (\Delta z)^{n-1}\right] = nz^{n-1}.$$

■

EXAMPLE 2.6

Show that $f(z) = \overline{z}$ is not differentiable.

Solution: Here,

$$\lim_{\Delta z \to 0}\frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0}\frac{\overline{z+\Delta z} - \overline{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0}\frac{\overline{\Delta z}}{\Delta z}$$

First we take $\Delta z = \Delta x$ (i.e., $\Delta z \to 0$ along x-direction only) and evaluate the limit.

$$\lim_{\Delta x \to 0}\frac{\Delta x}{\Delta x} = 1$$

Now, $\Delta z = \imath \Delta y$ (i.e., $\Delta z \to 0$ along y-direction only).

$$\lim_{\Delta y \to 0}\frac{-\imath \Delta y}{\imath \Delta y} = -1$$

Since the limit depends on the way that $\Delta z \to 0$, the function is nowhere differentiable. ■

EXAMPLE 2.7

Prove that the function $f(z) = |z|^2$ is continuous everywhere but nowhere differentiable except at the origin.

Solution: Since $f(z) = |z|^2 = x^2 + y^2$, the continuity of the function $f(z)$ is evident because of the continuity of $x^2 + y^2$.

Let us consider its differentiability,

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta z \to 0}\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \\
&= \lim_{\Delta z \to 0}\frac{|z_0+\Delta z|^2 - |z_0|^2}{\Delta z} \\
&= \lim_{\Delta z \to 0}\frac{(z_0+\Delta z)\overline{(z_0+\Delta z)} - z_0\overline{z_0})}{\Delta z} \\
&= \lim_{\Delta z \to 0}\overline{z_0} + \Delta\overline{z} + z_0\frac{\Delta\overline{z}}{\Delta z} \\
&= \lim_{\Delta z \to 0}\overline{z_0} + z_0\frac{\Delta\overline{z}}{\Delta z} \qquad \text{(Since } \Delta z \to 0 \Rightarrow \Delta\overline{z} \to 0\text{)}
\end{aligned}$$

Now at $z_0 = 0$, the above lim,it is clearly zero, so that $f'(0) = 0$. Let us now choose $z_0 \neq 0$, let

$$\Delta z = re^{i\theta}$$
$$\Rightarrow \frac{\Delta \overline{z}}{\Delta z} = e^{-2i\theta} = \cos 2\theta - i \sin 2\theta$$

Above does not not tend to a unique limit as this limit depends upon θ. Therefore, the given function is not differentiable at any other non-zero value of z. ■

EXAMPLE 2.8

If

$$f(z) = \begin{cases} \dfrac{x^3 y(y - ix)}{x^6 + y^2}, & z \neq 0 \\ = 0, & z = 0, \end{cases}$$

prove that $\dfrac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector, but not $z \to 0$ in any manner.

Solution: Here, $y - ix = -i(x + iy) = -iz$. Now,

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{\frac{x^3 y(y-ix)}{x^6+y^2} - 0}{z}$$
$$= \lim_{z\to 0} \frac{x^3 yi}{x^6 + y^2}$$

Now if $z \to 0$ along any radius vector, say $y = mx$, then

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{x^3 (mx) i}{x^6 + (mx)^2}$$
$$= \lim_{z\to 0} \frac{x^4 im}{x^6 + m^2 x^2}$$
$$= \lim_{z\to 0} \frac{x^2 im}{x^4 + m^2} = 0 \quad \text{Hence.}$$

Now let us suppose that $z \to 0$ along the curve $y = x^3$, then

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{x^3 (x^3) i}{x^6 + (x^3)^2}$$
$$= \lim_{z\to 0} \frac{x^6 i}{x^6 + x^6} = -\frac{i}{2}$$

along different paths, the value of $f'(z)$ is not unique (that is 0 along $y = mx$ and $-\dfrac{i}{2}$ along $y = x^3$). Therefore the function is not differentiable at $z = 0$. ■

2.4 Cauchy-Riemann Equations

Theorem 2.4 *The necessary and sufficient condition for the derivative of the function $f(z) = u + iv$, where u and v are real-valued functions of x and y, exist for all values of z in domain D, are*

1. $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ *and* $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.
2. $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ *are continuous functions of* x, y *in* D.

provided these four partial derivatives involved here should exist. The relations given in (1) is referred as Cauchy-Riemann Equations (some times CR Equations).

Proof:
Necessary Condition Let derivative of $f(z)$ exists, then

$$\begin{aligned} f'(z) &= \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \\ &= \lim_{\delta x, \delta y \to 0,0} \frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta x + i\delta y} \end{aligned}$$

since $f'(z)$ exists, the limit of above equation should be finite as $(\delta x, \delta y) \to (0,0)$ in any manner that we may choose. To begin with, we assume that δz is wholly real, i.e. $\delta y = 0$ and $\delta z = \delta x$. This gives

$$\begin{aligned} f'(z) &= \lim_{\delta x \to 0} \frac{[u(x+\delta x, y) + iv(x+\delta x, y)] - [u(x,y) + iv(x,y)]}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{[u(x+\delta x, y) - u(x,y)] + i[v(x+\delta x, y) - v(x,y)]}{\delta x} \qquad = u_x + iv_x \end{aligned}$$

Similarly, if we assume δz wholly imaginary number, then

$$\begin{aligned} f'(z) &= \lim_{\delta y \to 0} \frac{[u(x, y+\delta y) + iv(x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta y} \\ &= \lim_{\delta x \to 0} \frac{[u(x, y+\delta y) - u(x,y)] + i[v(x, y+\delta y) - v(x,y)]}{i\delta y} \qquad = \frac{1}{i}u_y + v_y = v_y - iu_y \end{aligned}$$

Since $f'(z)$ exists, it is unique, therefore

$$u_x + iv_x = v_y - iu_y$$

Equating then the real and imaginary parts, we obtain

$$u_x = v_y \text{ and } u_y = -v_x$$

Thus the necessary conditions for the existence of the derivative of $f(z)$ is that the CR equations should be satisfied.
Sufficient Condition
Suppose $f(z)$ possessing partial derivatives u_x, u_y, v_x, v_y at each point in D and the CR equations are satisfied.

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) \\ f(z+\delta z) &= u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y) \\ &= [u(x,y) + (u_x\delta x + u_y\delta y) + ...] + i[v(x,y) + (v_x\delta x + v_y\delta y) + ...] \\ &\qquad \text{(Using Taylor's Theorem for two variables)} \\ &= [u(x,y) + iv(x,y)] + (u_x + iv_x)\delta x + (u_y + iv_y)\delta y + ... \\ &= f(z) + (u_x + iv_x)\delta x + (u_y + iv_y)\delta y \\ &= \qquad \text{Leaving the higher order terms} \end{aligned}$$

$$\Rightarrow \quad f(z+\delta z) - f(z) = (u_x + iv_x)\delta x + (u_y + iv_y)\delta y$$

On using Cauchy Riemann equations

$$u_x = v_y;\ u_y = -v_x$$

we get,

$$\begin{aligned} f(z+\delta z) - f(z) &= (u_x + iv_x)\delta x + (-v_x + iu_x)\delta y \\ &= (u_x + iv_x)\delta x + i(iv_x + u_x)\delta y \\ &= (u_x + iv_x)(\delta x + i\delta y) \\ &= (u_x + iv_x)\delta z \\ \Rightarrow \quad \frac{f(z+\delta z) - f(z)}{\delta z} &= u_x + iv_x \\ \Rightarrow \quad \lim_{z\to 0} \frac{f(z+\delta z) - f(z)}{\delta z} &= u_x + iv_x \\ \Rightarrow \quad f'(z) &= u_x + iv_x \end{aligned}$$

Since u_x, v_x exist and are unique, therefore we conclude that $f'(z)$ exists. Hence $f(z)$ is analytic.

■

Remark : $\dfrac{dw}{dz} = u_x + iv_x = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial}{\partial x}(u+iv) = \dfrac{\partial w}{\partial x}$. Also, $\dfrac{dw}{dz} = -i\dfrac{\partial w}{\partial y}$(why?).

EXAMPLE 2.9

Discuss the differentiability of exponential function.

$$w = e^z = \phi(x,y) = e^x(\cos y + \imath \sin y)$$

Solution: We use the Cauchy-Riemann equations to show that the function is entire. Let

$$f(z) = u + iv = e^z = e^x(\cos y + \imath \sin y)$$

Hence

$$u = e^x \cos y \text{ and } v = e^x \sin y$$

Then,

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = e^x \sin y, \quad v_y = e^x \cos y$$

It follows that Cauchy-Riemann equations are satisfied. Since the function satisfies the Cauchy-Riemann equations and the first partial derivatives are continuous everywhere in the finite complex plane. Hence $f'(z)$ exists and

Now we find the value of the complex derivative.

$$\begin{aligned} f'(z) &= \frac{dw}{dz} = \frac{\partial w}{\partial x} \\ &= \frac{\partial}{\partial x}[e^x(\cos y + i \sin y)] \\ &= e^x(\cos y + i\sin y) \\ &= e^z \end{aligned}$$

■

Remark: The differentiability of the exponential function implies the differentiability of the trigonometric functions, as they can be written in terms of the exponential.

EXAMPLE 2.10

A function $f(z)$ is defined as follows:

$$f(z) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} + i\frac{x^3+y^3}{x^2+y^2}, z \neq 0 \\ 0, \qquad\qquad\qquad z = 0 \end{cases}$$

Show that $f(z)$ is continuous and that Cauchy-Riemann equations are satisfied at the origin. Also show that $f'(0)$ does not exist.

Solution: We have

$$u = \frac{x^3 - y^3}{x^2 + y^2} \quad \text{and} \quad v = \frac{x^3 + y^3}{x^2 + y^2}$$

For non zero values of z, $f(z)$ is continuous since u and v are rational functions of x and y with non-zero denominators. To prove its continuity at $z = 0$, we use polar coordinates. Then we have $u = r(\cos^3\theta - \sin^3\theta)$ and $v = r(\cos^3\theta + \sin^3\theta)$. It is seen that u and v tends to zero as $r \to 0$ irrespective of the values of θ. Since $u(0,0) = v(0,0) = 0$ (Given $f(z) = 0$ at $z = 0$), it follows that $f(z)$ is continuous at $(0,0)$. Thus $f(z)$ is continuous for all values of z.

Further,

$$(u_x)_{(0,0)} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{x^3/x^2}{x} = 1$$
$$(u_y)_{(0,0)} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{-y^3/y^2}{y} = -1$$
$$(v_x)_{(0,0)} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{x^3/x^2}{x} = 1$$
$$(v_y)_{(0,0)} = \lim_{y\to 0} \frac{v(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{y^3/y^2}{y} = 1$$

which show that the Cauchy-Riemann equations are satisfied at the origin. Finally, Now, let $z \to 0$ along $y = mx$, then

$$\begin{aligned} f'(0) &= \lim_{z\to 0} \frac{f(z) - f(0)}{z} \\ &= \lim_{x,y\to 0} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)} \\ &= \lim_{x\to 0} \frac{x^3(1 - m^3) + ix^3(1 + m^3)}{x^2(1 + m^2)x(1 + im)} \\ &= \lim_{x\to 0} \frac{(1 - m^3) + i(1 + m^3)}{(1 + m^2)(1 + im)} \end{aligned}$$

Since this limit depends on m therefore $f'(0)$ is not unique, it follows that $f'(z)$ does not exist at $z = 0$. ∎

2.5 Cauchy-Riemann Equations in Polar Form

We know that $x = r\cos\theta, y = r\sin\theta$ and u is a function x and y. Thus, we have

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$$

$$\Rightarrow f(z) = u + iv = f(re^{i\theta}) \tag{2.1}$$

Differentiating Equation (2.1) partially with respect to r, we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial v} = f'(re^{i\theta})e^{i\theta} \tag{2.2}$$

Again, differentiating Equation (1.10) partially with respect to θ, we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = if'(re^{i\theta})re^{i\theta} \tag{2.3}$$

Substituting the value of $f'(re^{i\theta})e^{i\theta}$ from Equation (2.2) into Equation (2.3), we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) ir$$

or

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Comparing real and imaginary parts of the above equation, we get

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \tag{2.4}$$

and

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{2.5}$$

2.6 Derivative of a complex function in Polar Form

We have

$$w = u + iv$$

Therefore

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

But

$$\begin{aligned}
\frac{dw}{dz} = \frac{\partial w}{\partial x} &= \frac{\partial w \partial r}{\partial r \partial x} + \frac{\partial w \partial \theta}{\partial \theta \partial x} \\
&= \frac{\partial w}{\partial r}\cos\theta - \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\frac{\sin\theta}{r} \quad (w = u + iv) \\
&= \frac{\partial u}{\partial r}\cos\theta - \left(-r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}\right)\frac{\sin\theta}{r}
\end{aligned}$$

Since $\dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r}$ and $\dfrac{\partial v}{\partial \theta} = r\dfrac{\partial u}{\partial r}$. Therefore

$$\frac{dw}{dz} = \frac{\partial w}{\partial r}\cos\theta - i\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\sin\theta = \frac{\partial w}{\partial r}\cos\theta - i\frac{\partial}{\partial r}(u + iv)\sin\theta$$

$$\therefore \quad \frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r} \tag{2.6}$$

Again, we have

$$\frac{dw}{dz} = \frac{\partial w \partial r}{\partial r \partial x} + \frac{\partial w \partial \theta}{\partial \theta \partial x} = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

Using the Cauchy-Riemann equations in Polar form, we get

$$= (\frac{1}{r}\frac{\partial v}{\partial \theta} - i\frac{1}{r}\frac{\partial u}{\partial \theta})\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r} = -\frac{1}{r}(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta})\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

$$\therefore \quad \frac{dw}{dz} = -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta} \tag{2.7}$$

2.7 Analytic Functions

We will consider a single valued function throughout the section.

2.7.1 Analyticity at a point

The function $f(z)$ is said to be analytic at a point $z = z_0$ in the domain D if its derivative $f'(z)$ exists at $z = z_0$ and at every point in some neighborhood of z_0.

2.7.2 Analyticity in a domain

A function $f(z)$ is said to be analytic in a domain D, if $f(z)$ is defined and differentiable at all points of the domain.

Note that complex differentiable has a different meaning than analytic. Analyticity refers to the behavior of a function on an open set. A function can be complex differentiable at isolated points, but the function would not be analytic at those points. Analytic functions are also called *regular* or *holomorphic*. If a function is analytic everywhere in the finite complex plane, it is called *entire*.

EXAMPLE 2.11

Consider z^n, $n \in \mathbb{Z}^+$, Is the function differentiable? Is it analytic? What is the value of the derivative?

Solution: We determine differentiability by trying to differentiate the function. We use the limit definition of differentiation. We will use Newton's binomial formula to expand $(z + \Delta z)^n$.

$$\begin{aligned}
\frac{d}{dz}z^n &= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\left(z^n + nz^{n-1}\Delta z + \frac{n(n-1)}{2}z^{n-2}\Delta z^2 + \cdots + \Delta z^n\right) - z^n}{\Delta z} \\
&= \lim_{\Delta z \to 0} \left(nz^{n-1} + \frac{n(n-1)}{2}z^{n-2}\Delta z + \cdots + \Delta z^{n-1}\right) \\
&= nz^{n-1}
\end{aligned}$$

The derivative exists everywhere. The function is analytic in the whole complex plane so it is entire. The value of the derivative is $\frac{d}{dz}z^n = nz^{n-1}$. ■

Remark: The definition of the derivative of a function of complex variable is identical in form of that the derivative of the function of real variable. Hence the rule of differentiation for complex functions are the same as those of real functions. Thus if a complex function is once known to be analytic, it can be differentiated just like ordinary way.

EXAMPLE 2.12

If $w = \log z$, find $\dfrac{dw}{dz}$ and determine the value of z at which function ceases to be analytic.

Solution: We have

$$w = \log z = \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$$

i.e.,

$$u = \frac{1}{2}\log(x^2+y^2)$$

$$v = \tan^{-1}\frac{y}{x}$$

$$\therefore \quad u_x = \frac{x}{x^2+y^2},\, u_y = \frac{y}{x^2+y^2}$$

$$\text{and} \quad v_x = \frac{-y}{x^2+y^2},\, v_y = \frac{x}{x^2+y^2}$$

Since, the CR equations are satisfied and the partial derivatives are continuous except at (0,0). Hence w is analytic everywhere except at $z = 0$.

$$\therefore \quad \frac{dw}{dz} = u_x + iv_x = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{1}{x+iy} = \frac{1}{z}$$

where $z \neq 0$.
(Note direct differentiations of $\log z$ also gives $1/z$). ■

EXAMPLE 2.13

Show that for the analytic function $f(z) = u+iv$, the two families of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are orthogonal[2].

Solution: Families of curves

$$u(x,y) = c_1 \tag{2.8}$$

$$v(x,y) = c_2 \tag{2.9}$$

On differentiating equation 2.8,

$$u_x dx + u_y dy = 0 \quad \Rightarrow \quad m_1 = \frac{dy}{dx} = -\frac{u_x}{u_y} \tag{2.10}$$

On differentiating equation 2.8,

$$v_x dx + v_y dy = 0 \quad \Rightarrow \quad m_2 = \frac{dy}{dx} = -\frac{v_x}{v_y} \tag{2.11}$$

[2]Two curves are said to be orthogonal if they intersect at right angle at each point of intersection. Mathematically, if the curves have slopes m_1 and m_2, then the curves are orthogonal if $m_1 m_2 = -1$.

The product of two slopes

$$m_1 m_2 = \left(-\frac{u_x}{u_y}\right)\left(-\frac{v_x}{v_y}\right) \tag{2.12}$$

Since, $u + iv$ is analytic, Hence. Cauchy Riemann equations are

$$u_x = v_y \quad and \quad u_y = -v_x$$

Hence equation 2.12 reduces to

$$m_1 m_2 = \left(-\frac{u_x}{u_y}\right)\left(\frac{u_y}{u_x}\right) = -1$$

Hence the two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal. ■

2.7.3 Analytic Functions can be Written in Terms of z.

Consider an analytic function expressed in terms of x and y, $\phi(x, y)$. We can write ϕ as a function of $z = x + \imath y$ and $\overline{z} = x - \imath y$.

$$f\left(z, \overline{z}\right) = \phi\left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}\right)$$

We treat z and $\overline{z}$ as independent variables. We find the partial derivatives with respect to these variables.

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \imath\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \overline{z}} = \frac{\partial x}{\partial \overline{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \overline{z}}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath\frac{\partial}{\partial y}\right)$$

Since ϕ is analytic, the complex derivatives in the x and y directions are equal.

$$\frac{\partial \phi}{\partial x} = -\imath\frac{\partial \phi}{\partial y}$$

The partial derivative of $f\left(z, \overline{z}\right)$ with respect to $\overline{z}$ is zero.

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x} + \imath\frac{\partial \phi}{\partial y}\right) = 0$$

Thus $f\left(z, \overline{z}\right)$ has no functional dependence on $\overline{z}$, it can be written as a function of z alone.

If we were considering an analytic function expressed in polar coordinates $\phi(r, \theta)$, then we could write it in Cartesian coordinates with the substitutions:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(x, y) \text{ or } tan^{-1}(x, y).$$

Thus we could write $\phi(r, \theta)$ as a function of z alone.

EXAMPLE 2.14

If n is real, show that $f(re^{i\theta}) = r^n(\cos n\theta + i \sin n\theta)$ is analytic expect possibly when $r = 0$ and that its derivative is $nr^{n-1}[\cos(n-1)\theta + i\sin(n-1)\theta]$.

Solution: Let $w = f(z) = u + iv = r^n(\cos n\theta + i\sin n\theta)$. Therefore $u = r^n \cos n\theta, v = r^n \sin n\theta$

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta, \frac{\partial u}{\partial \theta} = -nr^n \sin n\theta$$

$$\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta, \frac{\partial v}{\partial \theta} = nr^n \cos n\theta$$

Thus, we have

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} = nr^{n-1}\cos n\theta$$

and

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} = -nr^{n-1}\sin n\theta$$

Hence, the Cauchy-Riemann equations are satisfied. Thus, the function $w = r^n(\cos n\theta + i\sin n\theta)$ is analytic for all finite values of z, if $\frac{dw}{dz}$ exits . we have

$$\frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r} = (\cos\theta - i\sin\theta)nr^{n-1}(\cos n\theta + i\sin n\theta)$$

$$= nr^{n-1}[\cos(n-1)\theta + i\sin(n-1)\theta]$$

Thus, $\dfrac{dw}{dz}$ exits for all values of r, including zero, except when $r = 0$ and $n \leq 1$. ■

EXAMPLE 2.15

Show that the function $f(z) = e^{-z^{-4}}$ $(z \neq 0)$ and $f(0) = 0$ is not analytic at $z = 0$. Although Cauchy-Riemann equations are satisfied at the point. How would you explain this?

Solution: Here

$$f(z) = e^{-z^{-4}}$$

$$= e^{-\frac{1}{(x+iy)^4}} = e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}} = e^{-\frac{(x^4+y^4-6x^2y^2) - i4xy(x^2-y^2)}{(x^2+y^2)^4}}$$

$$\Rightarrow u + iv = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} e^{-i\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}}$$

This gives,

$$u = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} \cos\left(\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right) \text{ and } v = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} \sin\left(\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right)$$

At $z = 0$

$$\frac{\partial u}{\partial x} = \lim_{h\to 0}\frac{u(0+h,0) - u(0,0)}{h} = \lim_{h\to 0}\frac{e^{-h^{-4}}}{h}$$

$$= \lim_{h\to 0}\frac{1}{he^{\frac{1}{h^4}}} = \lim_{h\to 0}\left[\frac{1}{h\left(1 + \frac{1}{h^4} + \frac{1}{2!h^8} + \frac{1}{3!h^{12}} + ...\right)}\right] = 0$$

$$(\because e^x = 1 + x + \frac{x^2}{2!} + ...)$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \lim_{k\to 0}\frac{u(0,0+k)-u(0,0)}{k} = \lim_{k\to 0}\frac{e^{-k^{-4}}}{k}\\ &= \lim_{k\to 0}\frac{1}{ke^{\frac{1}{k^4}}} = 0\end{aligned}$$

$$\begin{aligned}\frac{\partial v}{\partial x} &= \lim_{h\to 0}\frac{v(0+h,0)-v(0,0)}{h} = \lim_{h\to 0}\frac{e^{-h^{-4}}}{h}\\ &= \lim_{h\to 0}\frac{1}{h.e^{\frac{1}{h^4}}} = 0\end{aligned}$$

$$\begin{aligned}\frac{\partial v}{\partial y} &= \lim_{k\to 0}\frac{v(0,0+k)-v(0,0)}{k} = \lim_{k\to 0}\frac{e^{-k^{-4}}}{k}\\ &= \lim_{k\to 0}\frac{1}{k.e^{\frac{1}{k^4}}} = \lim_{k\to 0}\frac{1}{k.e^{\frac{1}{k^4}}} = 0\end{aligned}$$

Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (C-R equations are satisfied at $z = 0$)
But

$$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{e^{-z^{-4}}-0}{z}$$

Along $z = re^{i\frac{\pi}{4}}$

$$\begin{aligned}f'(0) &= \lim_{r\to 0}\frac{e^{-r^{-4}}e^{-\left(e^{i\frac{\pi}{4}}\right)^4}}{re^{i\frac{\pi}{4}}} = \lim_{r\to 0}\frac{e^{-r^{-4}}e}{re^{i\frac{\pi}{4}}}\\ &= \frac{e}{e^{i\frac{\pi}{4}}}\lim_{r\to 0}\frac{1}{re^{-r^{-4}}} = 0\end{aligned}$$

Showing that $f'(z)$ does not exist at $z = 0$. Hence $f(z)$ is not analytic at $z = 0$. ■

PROBLEMS

2.1 Determine which of the following functions are analytic:

a) $x^2 + iy^2$

b) $2xy + i(x^2 - y^2)$

c) $\sin x \cosh y + i \cos x \sinh y$

d) $\dfrac{1}{(z-1)(z+1)}$

e) $\dfrac{x-iy}{x-iy+a}$

f) $\dfrac{x-iy}{x^2+y^2}$

2.2 Consider the function $f(z) = (4x+y) + i(-x+4y)$ and discuss $\dfrac{df}{dz}$

2.3 For what values of z, the function w defined as

$$w = \rho(\cos\phi + i\sin\phi); \quad \text{where } z = \ln\rho + i\phi$$

cases to be analytic.

2.4 For what values of z the function $z = \sinh u \cos v + i \cosh u \sin v$, where $w = u + iv$ ceases to be analytic.

2.5 For what values of z the function $z = e^{-v}(\cos u + i \sin u)$, where $w = u + iv$ ceases to be analytic.

2.6 If

$$f(z) = \begin{cases} \dfrac{x^3 y(y - ix)}{x^6 + y^2}, & z \neq 0 \\ = 0, z = 0, \end{cases}$$

then discuss $\dfrac{df}{dz}$ at $z = 0$.

2.7 Show that the complex variable function $f(z) = |z|^2$ is differentiable only at the origin.

2.8 Using the Cauchy-Riemann equations, show that $f(z) = z^3$ is analytic in the entire z-plane.

2.9 Test the analyticity of the function $w = \sin z$ and hence derive that:

$$\frac{d}{dz}(\sin z) = \cos z$$

2.10 Find the point where the Cauchy-Riemann equations are satisfied for the function:

$$f(z) = xy^2 + ix^2 y$$

where does $f'(z)$ = exist? Where is $f(z)$ analytic?

2.11 Find the values of a and b such that the function

$$f(z) = x^2 + ay^2 - 2xy + i(bx^2 - y^2 + 2xy)$$

is analytic. Also find $f'(z)$.

2.12 Show that the function $z|z|$ is not analytic anywhere.

2.13 Discuss the analyticity of the function $f(z) = z\bar{(z)}$.

2.14 Show that the function $f(z) = u + iv$, where

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, & z \neq 0 \\ \text{=0, z=0} \end{cases}$$

satisfy the Cauchy-Riemann conditions at $z = 0$. Is the function analytic at $z = 0$? justify your answer.

2.15 Show that the function defined by $f(z) = \sqrt{|xy|}$ satisfy Cauchy Riemann equations at the origin but is not analytic at the point.

2.8 Harmonic Functions

Any real valued function of x and y satisfying Laplace equation[3] is called Harmonic function.
If $f(z) = u + iv$ is analytic function, then u and v are harmonic functions.
If $f(z)$ is analytic, we have CR Equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

On differentiating first equation partially with respect to x and second equation partially with respect to y, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

On adding both equations, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which shows that u is harmonic. Similarly, on differentiating first equation partially with respect to y and second equation partially with respect to x, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial x^2} \qquad \frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x^2}$$

On subtracting both equations, we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

which shows that v is harmonic. Hence, if $f(z) = u + iv$ is some analytic function then u and v are harmonic functions.

2.9 Determination of conjugate functions

If $f(z) = u + iv$ is an analytic function, $v(x, y)$ is called conjugate function of $u(x, y)$. In this section we have to device a method to compute $v(x, y)$ provided $u(x, y)$ is given. From partial differentiation, we have

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \tag{2.13}$$

But, $f(z)$ is analytic, which implies

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.14}$$

[3]The following equation is known as Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Using equations 2.14, Equation 2.13 reduces to

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \tag{2.15}$$

Here $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$, which gives

$$\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial y} = -\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right)$$

Since $f(z)$ is analytic, u is harmonic

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial y} = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$$

which shows that Eq 2.15 is an exact differential equation. It can be integrated to obtain v.

EXAMPLE 2.16

Show that $u = x^2 - y^2$ is harmonic and find the its conjugate

Solution: Here u is given, we may compute following

$$u_x = 2x \qquad u_y = -2y$$

$$u_{xx} = 2 \qquad u_{yy} = -2$$

This implies $u_{xx} + u_{yy} = 0$, i.e., Laplace equation holds. Therefore, the given function is harmonic. From partial differentiation and CR Equations, we have

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

which gives

$$dv = 2ydx + 2xdy$$

As this differential equation is exact, we may use method of solving an exact differential equation[4], which is as

$$\begin{aligned} v &= \int_{y \text{ as constant}} (2y)dx + c \\ &= 2xy + c \end{aligned}$$

which is required harmonic conjugate of u. ■

[4]See Appendix

EXAMPLE 2.17

If ϕ and ψ are function of x and y satisfying Laplace's equation, show that $s + it$ is analytic, where
$s = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}$ and $t = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}$.

Solution:

Since ϕ and ψ are function of x and y satisfying Laplace's equations.

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{2.16}$$

and

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial y^2} = 0. \tag{2.17}$$

For the function $s + it$ to be analytic,

$$\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} \tag{2.18}$$

$$\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x} \tag{2.19}$$

must satisfy.

Now,

$$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} \tag{2.20}$$

$$\frac{\partial t}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2} \tag{2.21}$$

$$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} \tag{2.22}$$

and

$$\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y} \tag{2.23}$$

From (2.18), (2.20) and (2.21), we have

$$\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Which is true by (2.17).

Again from (2.19), (2.22) and (2.23), we have

$$\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} = -\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial y}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

which is also true by (2.16).

Hence the function $s + it$ is analytic.

■

EXAMPLE 2.18

Show that an analytic function with constant modulus is constant.

Solution:

Solution : Let $f(z) = u + iv$ be an analytic function with constant modulus. Then,

$$|f(z)| = |u + iv| = \text{Constant}$$

$$\sqrt{u^2 + v^2} = \text{Constant} = c \text{ (say)}$$

Squaring both sides, we get

$$u^2 + v^2 = c^2 \tag{2.24}$$

Differentiating equation (2.24) partially w.r.t. x, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \tag{2.25}$$

Again, differentiating equation (2.24) partially w.r.t. y, we get

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

$$-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \tag{2.26}$$

$\because \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$. Squaring and adding equations (2.25) and (2.26), we get

$$(u^2 + v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \qquad \because u^2 + v^2 = c^2 \neq 0$$

$$|f'(z)|^2 = 0 \qquad [\because f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}]$$

$$|f'(z)| = 0$$

Hence $f(z)$ is constant.

■

2.10 Milne Thomson Method

Consider the problem to determine the function $f(z)$ of which u is given. One procedure may be as previous section, compute its harmonic conjugate v, and finally combine them to compute $f(z) = u + iv$. To overcome the length of the mechanism Milne's introduced another method which is a direct way to compute $f(z)$ for a given u. We have $z = x + iy$ which implies

$$x = \frac{z + \overline{z}}{2} \qquad y = \frac{z - \overline{z}}{2i}$$

$$\begin{aligned} w = f(z) &= u + iv = u(x,y) + iv(x,y) \\ &= u\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) + iv\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) \end{aligned}$$

On putting $z = \overline{z}$, we get

$$f(z) = u(z,0) + iv(z,0)$$

We have (CR Equation are used.)

$$f'(z) = \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

Here, $\frac{\partial u}{\partial x} = \phi_1(x,y)$ and $\frac{\partial u}{\partial y} = \phi_2(x,y)$ Thus

$$f'(z) = \phi_1(x,y) - i\phi_2(x,y)$$

or

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$

On integrating, we obtain the required function

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]dz + K$$

where K is complex constant.

Remark: In case, v is given, $\phi_1(x,y) = u_x = v_y$ and $\phi_2(x,y) = u_y = -v_x$

EXAMPLE 2.19

Find the regular[5] function for the given $u = x^2 - y^2$.

Solution: Here u is given, we may compute following

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = 2x \qquad \phi_2(x,y) = \frac{\partial u}{\partial y} = -2y$$

$$\begin{aligned} f'(z) &= \phi_1(z,0) - i\phi_2(z,0) \\ &= 2z - i(0) \\ &= 2z \end{aligned}$$

On integrating,

$$f(z) = 2\int z dz = z^2 + K$$

which is required function. ■

EXAMPLE 2.20

If $u - v = (x-y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find the $f(z)$ in terms of z.

Solution: We have,

$$u + iv = f(z)$$

[5] Analytic function is also called as Regular function.

This gives

$$iu - v = if(z)$$

On adding these both

$$(u - v) + i(u + v) = (1 + i)f(z)$$

Let $U = u - v$ and $V = u + v$ then

$$U + iV = (1 + i)f(z) = F(z) \ \ (say)$$

Here $U = (u - v)$ gives,

$$U = (x - y)(x^2 + 4xy + y^2)$$
$$U = x^3 + 3x^2y - 3xy^2 - y^3$$

Now, we can use Milne's Method to find $F(z)$.

$$\phi_1(x, y) = \frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2 \qquad \phi_2(x, y) = \frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2$$

$$F'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= 3z^2 - i3z^2 = (1 - i)3z^2$$

Hence

$$F(z) = (1 - i)z^3 + K$$

where K is complex constant.
Since we have $F(z) = (1 + i)f(z)$,

$$f(z) = \frac{F(z)}{(1 + i)} = \frac{(1 - i)z^3 + K}{1 + i}$$

$$f(z) = -iz^3 + K_1$$

where K_1 is complex constant. ■

PROBLEMS

2.1 Show that the following functions are harmonic and determine the conjugate functions.

a) $u = 2x(1 - y)$
b) $u = 2x - x^3 + 3xy$
c) $u = \frac{1}{2}log(x^2 + y^2)$
d) $x^3 - 3xy^2 + 3x^2 - 3y^2$

2.2 Determine the analytic function, whose imaginary part is

a) $x^2 - y^2 + 5x + y - \frac{y}{x^2+y^2}$
b) $\cos x \cosh y$
c) $3x^2y + 2x^2 - y^3 - 2y^2$
d) $e^{-x}(x \sin y - y \cos y)$
e) $e^{2x}(x \cos 2y - y \sin 2y)$
f) $v = \log(x^2 + y^2) + x - 2y$
g) $v = \sinh x cosy$
h) $v = \frac{x-y}{x^2+y^2}$
i) $v = \left(r - \frac{1}{r}\right) sin\theta$

2.3 If $f(z) = u+iv$ is an analytic function of $z = x+iy$ and $u-v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find $f(z)$ subject to the condition that $f\left(\frac{\pi}{2}\right) = \frac{(3-i)}{2}$

2.4 Find an analytic function $f(z) = u(r,\theta) + iv(r,\theta)$ such that $v(r,\theta) = r^2 \cos 2\theta - r\cos\theta + 2$

2.5 Show that the function $u = x^2 - y^2 - 2xy - 2x - y - 1$ is harmonic. Find the conjugate harmonic function v and express $u + iv$ as a function of z where $z = x + iy$

2.6 Construct an analytic function of the form $f(z) = u + iv$, where v is $tan^{-1}(\frac{y}{x})$, $x \neq 0, y \neq 0$

2.7 If $f(z)$ is a regular function of z, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2$

CHAPTER 3

COMPLEX INTEGRATION

3.1 Integration in complex plain

In case of real variable, the path of the integration of $\int_a^b f(x)dx$ is always along the x-axis from $x = a$ to $x = b$. But in case of a complex function $f(z)$ the path of a complex function $f(z)$ the path of the definite integral $\int_\alpha^\beta f(z)dz$ can be along any curve from $z = \alpha$ to $z = \beta$.

EXAMPLE 3.1

Evaluate $\int_0^{2+i} \bar{z}^2 dz$ along the real axis from $z = 0$ to $z = 2$ and then along parallel to y-axis from $z = 2$ to $z = 2 + i$.

Solution:

$$\begin{aligned}\int_0^{2+i} \bar{z}^2 dz &= \int_0^{2+i} (x - iy)^2 (dx + idy) \\ &= \int_0^{2+i} (x^2 - y^2 - 2ixy)(dx + idy)\end{aligned}$$

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Figure 3.1

Along real axis from $z = 0$ to $z = 2$ (y=0) :

$$y = 0 \Rightarrow dy = 0, dz = d(x + iy) = dx$$

$$z = 0, y = 0 \Rightarrow x = 0$$

and

$$z = 2, y = 0 \Rightarrow x = 2$$

$$\begin{aligned} \int_0^{2+i} \bar{z}^2 dz &= \int_0^2 (x^2)(dx) \\ &= \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} \end{aligned}$$

Along parallel to y-axis from $z = 2$ to $z = 2 + i$ (x=2)

$$x = 2 \Rightarrow dx = 0, dz = d(x + iy) = idy$$

$$z = 2, x = 2 \Rightarrow y = 0 \quad and \quad z = 2 + i, x = 2 \Rightarrow y = 1$$

$$\begin{aligned} \int_0^{2+i} \bar{z}^2 dz &= \int_0^1 (4 - y^2 - 4iy)(i.dy) \\ &= i\left[4y - \frac{y^3}{3} - 4i\frac{y^2}{2}\right]_0^1 = \left[\frac{11}{3}i + 2\right] \end{aligned}$$

$\int_0^{2+i} \bar{z}^2 dz$ along the real axis from $z = 0$ to $z = 2$ then along parallel to y-axis from $z = 2$ to $z = 2 + i$

$$= \frac{8}{3} + \frac{11}{3}i + 2 = \frac{1}{3}(14 + 11i)$$

■

PROBLEMS

3.1 Find the value of the integral

$$\int_0^{1+i} (x - y + ix^2) dz$$

a) Along the straight line from $z = 0$ to $z = 1 + i$.
b) along the real axis from $z = 0$ to $z = 1$ and then along parallel to y-axis from $z = 1$ to $z = 1 + i$.

3.2 Integrate $f(z) = x^2 + ixy$ from $A(1, 1)$ to $B(2, 8)$ along

a) the straight line AB
b) the curve $C, x = t,\ y = t^3$.

3.3 Evaluate the integral $\int_c (3y^2dx + 2ydy)$, where c is the circle $x^2 + y^2 = 1$, counter-clockwise from $(1, 0)$ to $(0, 1)$.

3.2 Cauchy's Integral Theorem

Theorem 3.1 *If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points within and on a simple closed curve c, then* $\int_c f(z)dz = 0$.

Proof: Let $f(z) = u + iv$ and $z = x + iy$ and region enclosed by the curve c be R, then

$$\int_c f(z)dz = \int_c (u + iv)(dx + idy) = \int_c (udx - vdy) + i(vdx + udy)$$
$$= \int\int_R (-v_x - u_y)dxdy + i\int\int_R (u_x - v_y)dxdy$$

By Cauchy-Riemann equations,

$$= \int\int_R (u_y - u_y)dxdy + i\int\int_R (u_x - u_x)dxdy = 0$$

■

EXAMPLE 3.2

Find the integral $\int_c \frac{3z^2 + 7z + 1}{z + 1}dz$, where c is the circle $|z| = \frac{1}{2}$.

Solution: Poles of integrand are given by

$$z + 1 = 0$$

That is, $z = -1$. Since given circle $|z| = \frac{1}{2}$, with centre $z = 0$ and radius $1/2$ does not enclose $z = -1$. Thus it is obvious that the integrand is analytic everywhere. Hence, by Cauchy's Theorem,

$$\int_c \frac{3z^2 + 7z + 1}{z + 1}dz = 0$$

■

Theorem 3.2 (Cauchy's integral theorem for multi-connected region) *If a function $f(z)$ is analytic in region R between two simple closed curves c_1 and c_2, then*

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$

Proof: Since $f(z)$ is analytic in region R, By Cauchy's Theorem

$$\int f(z)dz = 0$$

where path of integration is along AB, and curves C_2 in clockwise direction and along BA and along C_1 in anticlockwise direction.

We may write,

$$\int_{AB} f(z)dz - \int_{c_2} f(z)dz + \int_{BA} f(z)dz + \int_{c_1} f(z)dz = 0$$

or

$$-\int_{c_2} f(z)dz + \int_{c_1} f(z)dz = 0$$

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$

■

3.3 Cauchy Integral Formula

Theorem 3.3 *If a function $f(z)$ is analytic within and on a closed curve c, and if a is any point within c , then*

$$f(a) = \frac{1}{2\pi i} \int_z \frac{f(z)}{(z-a)} dz$$

Proof: Let $z = a$ be a point within a closed curve c. Describe a circle γ such that $|z - a| = \rho$ and it lies entirely within c. Now consider the function

Figure 3.2 Cauchy Integral Formula

$$\phi(z) = \frac{f(z)}{(z-a)}$$

Obviously, this function is analytic in region between γ and c. Hence by Cauchy's integral theorem for multiconnected region, we have

$$\int_c \phi(z)dz = \int_\gamma \phi(z)dz$$

or

$$\begin{aligned}
\int_c \frac{f(z)}{(z-a)} dz &= \int_\gamma \frac{f(z)}{(z-a)} dz \\
&= \int_\gamma \frac{f(z) - f(a) + f(a)}{(z-a)} dz \\
&= \int_\gamma \frac{f(z) - f(a)}{(z-a)} dz + \int_\gamma \frac{f(a)}{(z-a)} dz \\
&= I_1 + I_2
\end{aligned}$$

Now, since $|z-a|=\rho$, we have $z=a+\rho e^{i\theta}$ and $dz=i\rho e^{i\theta}d\theta$. Hence

$$\begin{aligned} I_1 &= \int_\gamma \frac{f(z)-f(a)}{(z-a)}dz \\ &= \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})-f(a)}{[(a+\rho e^{i\theta})-a]} i\rho e^{i\theta} d\theta \\ &= \int_0^{2\pi} [f(a+\rho e^{i\theta})-f(a)] i d\theta \\ &= 0 \qquad \text{as } \rho \text{ tends to } 0 \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_\gamma \frac{f(a)}{(z-a)}dz \\ &= \int_0^{2\pi} \frac{f(a)}{[(a+\rho e^{i\theta})-a]} i\rho e^{i\theta} d\theta \\ &= f(a)\int_0^{2\pi} i d\theta \\ &= 2\pi i f(a) \end{aligned}$$

Hence,

$$\int_c \frac{f(z)}{(z-a)}dz = I_1 + I_2$$

That is

$$\int_c \frac{f(z)}{(z-a)}dz = 0 + 2\pi i f(a)$$

or

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)}dz$$

■

EXAMPLE 3.3

Evaluate (i) $\displaystyle\int_c \frac{e^z}{z+2}dz$ and (ii) $\displaystyle\int_c \frac{e^z}{z}dz$, where c is circle $|z|=1$.

Solution: (i) The function $\dfrac{e^z}{z+2}$ is analytic everywhere except at $z=-2$. This point lies outside the circle $|z|=1$. Thus function is analytic within and on c, by Cauchy's Theorem, we have

$$\int_{|z|=1} \frac{e^z}{z+2}dz = 0$$

(ii) The function $\dfrac{e^z}{z}$ is analytic everywhere except at $z=0$. The point $z=0$ strictly inside $|z|=1$. Hence by Cauchy's Integral formula, we have

$$\int_{|z|=1} \frac{e^z}{z}dz = 2\pi i (e^z)_{z=0} = 2\pi i$$

■

3.4 Cauchy Integral Formula For The Derivatives of An Analytic Function

Theorem 3.4 *If a function $f(z)$ is analytic within and on a closed curve c, and if a is any point within c , then its derivative is also analytic within and on closed curve c, and is given as*

$$f'(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^2}dz$$

Proof: We know Cauchy's Integral formula

$$f(a) = \frac{1}{2\pi}\int_c \frac{f(z)}{(z-a)}dz$$

Differentiating, wrt a , we get

$$\begin{aligned} f'(a) &= \frac{1}{2\pi}\frac{d}{da}\left[\int_c \frac{f(z)}{(z-a)}dz\right] \\ &= \frac{1}{2\pi}\int_c f(z)\frac{\partial}{\partial a}\left[\frac{1}{(z-a)}\right]dz \\ &= \frac{1}{2\pi}\int_c \frac{f(z)}{(z-a)^2}dz \end{aligned}$$

We may generalize it,

$$f^n(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}dz$$

■

EXAMPLE 3.4

Evaluate the following integral $\int_c \frac{1}{z}\cos z dz$, where c is the ellipse $9x^2 + 4y^2 = 1$.

Solution: Here function $\frac{1}{z}\cos z$ has a simple pole at $z = 0$. The given ellipse $9x^2 + 4y^2 = 1$. encloses pole $z = 0$.
By Cauchy Integral formula

$$\int_c \frac{cosz}{z}dz = 2\pi i(cosz)_{z=0} = 2\pi i$$

■

EXAMPLE 3.5

Evaluate the complex integral $\int_c \tan z dz$, where c is $|z| = 2$.

Solution: We have

$$\int_c tanz.dz = \int_c \frac{\sin z}{\cos z}dz$$

$|z| = 2$, is a circle with centre at origin and radius = 2. Poles are given by putting the denominator equal to zero. i.e.,

$$\cos z = 0 \quad \Rightarrow \quad z = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

The integrand has two poles at $z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$ inside the given circle $|z| = 2$.
On applying Cauchy integral formula

$$\begin{aligned}\int_c \frac{\sin z}{\cos z} dz &= \int_{c_1} \frac{\sin z}{\cos z} dz + \int_{c_2} \frac{\sin z}{\cos z} dz \\ &= 2\pi i [\sin z]_{z=\frac{\pi}{2}} + 2\pi i [\sin z]_{z=-\frac{\pi}{2}} \\ &= 2\pi i(1) + 2\pi i(-1) = 0\end{aligned}$$

■

EXAMPLE 3.6

Evaluate $\int_c \frac{e^z}{z^2+1} dz$ over the circular path $|z| = 2$.

Solution: Here,

$$z^2 + 1 = 0, \;\; z^2 = -1, \;\; z = \pm i$$

Figure 3.3

Both points are inside the given circle with at origin and radius 2.

$$\begin{aligned}\int_c \frac{1}{2i}\{\frac{e^z}{z-i} - \frac{e^z}{z+i}\} dz &= \int_c \frac{1}{2i}\frac{e^z}{z-i} dz - \frac{1}{2i}\int_c \frac{e}{z+i} dz \\ &= \frac{1}{2i}[2\pi i (e^z)_{z=i} - 2\pi i (e^z)_{z=-i}] \\ &= \frac{2\pi i}{2i}[e^i - e^{-i}] = 2\pi i \sin(1)\end{aligned}$$

Second Method

$$\begin{aligned}\int_c \frac{e^z}{z^2+1} dz &= \int_c \frac{e^z d_z}{(z+i)(z-i)} \\ &= \int_{c1} \frac{\frac{e^z}{z-i}}{z+i} dz + \int_c \frac{\frac{e^z}{z+i}}{z-i} dz \\ &= 2\pi i (\frac{e^z}{z-i})_{z=-i} + 2\pi i (\frac{e^z}{z+i})_{z=i} \\ &= [2\pi i \frac{e^{-i}}{-i-i} + 2\pi i \frac{e^i}{i+i}] = \pi[-e^{-i} + e^i] \\ &= \pi(2isin1) = 2\pi i \sin 1\end{aligned}$$

■

EXAMPLE 3.7

Evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)} dz$ where c is $|z - i| = 2$.

Solution: The centre of the circle is at $z = i$ and its radius is 2. Poles are obtained by putting the denominator equal to zero.

$$(z+1)^2(z-2) = 0 \quad \Rightarrow \quad z = -1, -1, 2$$

The integral has two Poles at $z = -1$ (second order) and $z = 2$ (simple pole) of which $z = -1$ is inside the given circle. We can rewrite

$$\int_c \frac{(z-1)dz}{(z+1)^2(z-2)} = \int_{c1} \frac{\frac{z-1}{z-2}}{(z+1)^2} dz$$

By Cauchy Integral formula

$$\int \frac{f(z)}{(z+1)^2} dz = 2\pi i f'(-1)$$

Here

$$f(z) = \frac{z-1}{z-2}$$
$$f'(z) = \frac{(z-2).1 - (z-1).1}{(z-2)^2} = \frac{-1}{(z-2)^2} = \frac{-1}{(z-2)^2}$$
$$\Rightarrow \quad f'(-1) = \frac{-1}{(-1-2)^2} = \frac{-1}{9}$$
$$\therefore \int \frac{(z-1)}{(z+1)^2(z-2)} dz == -\frac{2\pi i}{9}$$

■

EXAMPLE 3.8

Use Cauchy integral formula to evaluate .

$$\int_c \frac{sin\pi z^2 + cos\pi z^2}{(z-1)(z-2)} dz$$

where c is the circle $|z| = 3$.

Solution: Poles of the integrand are given by putting the denominator equal to zero.

$$(z-1)(z-2) = 0, \quad z = 1, 2$$

The integrand has two poles at $z = 1, 2$. The given circle $|z| = 3$ with centre at $z = 0$ and radius 3 encloses both the poles $z = 1$, and $z = 2$.

$$\begin{aligned}
\int_c \frac{sin\pi z^2 + cos\pi z^2}{(z-1)(z-2)} dz &= \int_{c_1} \frac{\frac{sin\pi z^2 + cos\pi z^2}{(z-2)}}{(z-1)} dz + \int_{c_2} \frac{\frac{sin\pi z^2 + cos\pi z^2}{(z-1)}}{(z-2)} dx \\
&= 2\pi i \left[\frac{sin\pi z^2 + cos\pi z^2}{z-2}\right]_{z=1} + 2\pi i \left[\frac{sin\pi z^2 + cos\pi z^2}{z-1}\right]_{z=2} \\
&= 2\pi i \left[\frac{sin\pi + cos\pi}{1-2}\right] + 2\pi i \left[\frac{sin4\pi + cos4\pi}{2-1}\right] \\
&= 2\pi i \left(\frac{-1}{-1}\right) + 2\pi i \left(\frac{1}{1}\right) = 4\pi i
\end{aligned}$$

■

PROBLEMS

3.1 Evaluate the following by using Cauchy integral formula:

a) $\int_c \frac{1}{z-a}dz$, where c is a simple closed curve and the point $z = a$ is (i) outside c; (ii) inside c.

b) $\int_c \frac{e^z}{z-1}dz$, where c is the circle $|z| = 2$.

c) $\int_c \frac{cos\pi z}{z-1}dz$, where c is the circle $|z| = 3$.

d) $\int_c \frac{cos\pi z^2}{(z-1)(z-2)}dz$, where c is the circle $|z| = 3$.

e) $\int_c \frac{e^{-z}}{(z+2)^5}dz$, where c is the circle $|z| = 3$.

f) $\int_c \frac{e^{2z}}{(z+1)^4}dz$, where c is the circle $|z| = 2$.

g) $\int_c \frac{3z^2+z}{z^2-1}dz$, where c is the circle $|z-1| = 1$.

3.2 Evaluate the following integral using Cauchy integral formula

$$\int_c \frac{4-3z}{z(z-1)(z-2)}dz$$

where c is the circle $|z| = \frac{3}{2}$.

3.3 Integrate $\frac{1}{(z^3-1)^2}$ the counter clockwise sense around the circle $|z-1| = 1$.

3.4 Find the value of $\int_c \frac{2z^2+z}{z^2-1}dz$, if c is circle of unit radius with centre at $z = 1$.

In this chapter, we discuss the power series expansion of a complex function, viz, the Taylor's and Laurent's series

3.5 Taylor's Theorem

Theorem 3.5 *If function $f(z)$ is analytic at all points inside a circle c, with its centre at the point a and radius R, then at each point z inside c,*

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + ... + \frac{(z-a)^n}{n!}f^n(a) + ...$$

Proof: Let $f(z)$ be analytic function within and on the circle circle c of radius r centered at a so that

$$c : |z-a| = r$$

Draw another circle $\gamma : |z-a| = \rho$, where $\rho < r$ Hence by Cauchy's integral formula, we

Figure 3.4 Taylor's Theorem

have

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$$

But,

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{(w-a)-(z-a)} \\ &= \frac{1}{(w-a)}\frac{1}{\left[1-\frac{z-a}{w-a}\right]} \\ &= \frac{1}{(w-a)}\left[1-\frac{z-a}{w-a}\right]^{-1} \\ &= \frac{1}{(w-a)}+\frac{(z-a)}{(w-a)^2}+\frac{(z-a)^2}{(w-a)^3}+... \\ &\qquad +\frac{(z-a)^{n-1}}{(w-a)^n}+\frac{(z-a)^n}{(w-a)^{n+1}}\left[1-\frac{z-a}{w-a}\right]^{-1}\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)}dw+\frac{(z-a)}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^2}dw+\frac{(z-a)^2}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^3}dw \\ &\quad +...+\frac{(z-a)^{n-1}}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^n}dw+R_n \\ &\quad \text{where } R_n=\frac{(z-a)^n}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)(w-a)^n}dw\end{aligned}$$

But, we have

$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}dw = \frac{1}{n!}f^{(n)}(z)$Hence,

$$f(z)=f(a)+\frac{(z-a)}{1!}f'(a)+...+\frac{(z-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+R_n$$

which is called Taylor's formula, R_n being the *remainder*.

Now, let M denotes the maximum value of $f(w)$ on γ. Since $|z-a|=\rho$, $|w-a|=r$ and $|w-z|>r-\rho$, hence

$$\begin{aligned}|R_n| &\leq \frac{|(z-a)^n|}{|2\pi i|}\int_\gamma \frac{|f(w)|}{|(w-z)||(w-a)^n|}|dw| \\ &\leq \frac{|\rho^n|}{2\pi}\frac{M}{(r-\rho)r^n}\int_\gamma |dw| \\ &= \frac{M}{1-\frac{\rho}{r}}\left(\frac{\rho}{r}\right)^n\end{aligned}$$

which tends to zero as n tends to infinity since $\frac{\rho}{r}<1$. Thus

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+...+\frac{(z-a)^n}{n!}f^n(a)+...$$

Remark: If $a=0$, Taylor's series is

$$f(z)=f(0)+(z)f'(0)+\frac{(z)^2}{2!}f''(0)+...+\frac{(z)^n}{n!}f^n(0)+...$$

which is called the *Maclaurin's series* of $f(z)$. ■

EXAMPLE 3.9

Find the first four terms of the Taylor's series expansion of the complex variable function

$$f(z) = \frac{z+1}{(z-3)(z-4)}$$

about $z = 2$. Find the region of convergence.

Solution: Given $f(z) = \dfrac{z+1}{(z-3)(z-4)}$

If centre of a circle is at $z = 2$, then the distance of the singularities $z = 3$ and $z = 4$ from the centre are 1 and 2. Hence if a circle ($|z-2| = 1$) of radius 1 and with centre $z = 2$ is drawn, the given function function $f(z)$ is analytic, hence it can be expanded in a Taylor's series within the circle $|z-2| = 1$, which is therefore the circle of convergence (or region of convergence).

$$\begin{aligned}
f(z) &= \frac{z+1}{(z-3)(z-4)} = \frac{-4}{z-3} + \frac{5}{z-4} \quad \text{Using partial fraction} \\
&= \frac{-4}{(z-2)-1} + \frac{5}{(z-2)-2} \\
&= 4[1-(z-2)]^{-1} - \frac{5}{2}\left[1 - \frac{z-2}{2}\right]^{-1} \\
&= 4[1+(z-2)+(z-2)^2+(z-2)^3+...] - \frac{5}{2}\left[1 + \frac{z-2}{2} + \frac{(z-2)^2}{4} + \frac{(z-2)^3}{8} + ...\right] \\
&= \frac{3}{2} + \frac{11}{4}(z-2) + \frac{27}{8}(z-2)^2 + \frac{59}{16}(z-2)^3
\end{aligned}$$

■

EXAMPLE 3.10

Find the first three terms of the Taylor series expansion of $f(z) = \frac{1}{z^2+4}$ about $z = -i$. Find the region of convergence.

Solution: Here

$$f(z) = \frac{1}{z^2+4}$$

Poles are given by

$$z^2 + 4 = 0 \quad \Rightarrow \quad z = \pm 2i$$

If the centre of a circle is $z = -i$, then the distance of the singularities $z = 2i$ and $z = -2i$ from the centre are 3 and 1. Hence if a circle of radius 1 is drawn at the centre $z = -i$, then within the circle $|z+i|$=1, the given function $f(z)$ is analytic. Thus the function can be expanded in Taylor Series within the circle $|z+i| = 1$, which is therefore the region of convergence ($|z+i| < 1$).
This problem could be solved as previous example. Here we use alternative approach.
We have Taylor Series about $z = -i$,

$$f(z) = f(-i) + (z+i)\frac{f'(-i)}{1!} + (z+i)^2\frac{f''(-i)}{2!} + ...$$

Now since $f(z) = \dfrac{1}{z^2+4}$,

$$f(-i) = \frac{1}{3}$$

$$f'(z) = \frac{-2z}{(z^2+4)^2} \quad \Rightarrow \quad f'(-i) = \frac{2i}{9}$$

$$f''(z) = \frac{2(z^2+4) - 8z^2}{(z^2+4)^3} \quad \Rightarrow \quad f''(-i) = -\frac{14}{27}$$

Hence Taylor series,

$$f(z) = \frac{1}{3} + \frac{2i}{9}(z+i) + \frac{7}{27}(z+i)^2 + ...$$

■

3.6 Laurent's Theorem

In expanding a function $f(z)$ by Taylor's series at a point $z = a$, we require that function $f(z)$ is analytic at $z = a$. Laurent's series gives an expansion of $f(z)$ at a point $z = a$ even if $f(z)$ is not analytic there.

Theorem 3.6 *If function $f(z)$ is analytic between and on two circles c_1 and c_2 having common centre at $z = a$ and radii r_1 and r_2, then for all points z in this region, $f(z)$ can be expanded by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where

$$a_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(w)}{(w-a)^{n+1}} dw$$
$$b_n = \frac{1}{2\pi i}\int_{c_2} \frac{f(w)}{(w-a)^{-n+1}} dw$$

the integrals being taken in counter clockwise sense.

Proof: Let $f(z)$ be analytic function between and on the circles c_1 and c_2 of radii r_1 and r_2 centered at a so that

$$c_1 : |z-a| = r_1 \qquad c_2 : |z-a| = r_2$$

Draw another circle $\gamma : |z-a| = \rho$, where $r_2 < \rho < r_1$. Let w be a point on circle γ, then by Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_{c_1} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i}\int_{c_2} \frac{f(w)}{w-z} dw \tag{3.1}$$

But,

$$\begin{aligned}
\frac{1}{w-z} &= \frac{1}{(w-a)-(z-a)} \\
&= \frac{1}{(w-a)} \frac{1}{\left[1-\frac{z-a}{w-a}\right]} \\
&= \frac{1}{(w-a)} \left[1-\frac{z-a}{w-a}\right]^{-1} \\
&= \frac{1}{(w-a)} + \frac{(z-a)}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + ... \\
&\qquad + \frac{(z-a)^{n-1}}{(w-a)^n} + \frac{(z-a)^n}{(w-a)^{n+1}} \left[1-\frac{z-a}{w-a}\right]^{-1}
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c_1} \frac{f(w)}{w-z}dw &= \frac{1}{2\pi i}\int_{c_1} \frac{f(w)}{(w-a)}dw + \frac{(z-a)}{2\pi i}\int_{c_1} \frac{f(w)}{(w-a)^2}dw + \frac{(z-a)^2}{2\pi i}\int_{c_1} \frac{f(w)}{(w-a)^3}dw \\
&+ ... + \frac{(z-a)^{n-1}}{2\pi i}\int_{c_1} \frac{f(w)}{(w-a)^n}dw + R_n \\
&\text{where } R_n = \frac{(z-a)^n}{2\pi i}\int_{c_1} \frac{f(w)}{(w-z)(w-a)^n}dw
\end{aligned}$$

R_n being the *remainder*.

Now, let M denotes the maximum value of $f(w)$ on c_1. Since $|z-a| = \rho$, $|w-a| = r_1$ and $|w-z| > r_1 - \rho$, hence

$$\begin{aligned}
|R_n| &\leq \frac{|(z-a)^n|}{|2\pi i|}\int_{c_1} \frac{|f(w)|}{|(w-z)||(w-a)^n|}|dw| \\
&\leq \frac{|\rho^n|}{2\pi}\frac{M}{(r_1-\rho){r_1}^n}\int_{c_1}|dw| \\
&= \frac{M}{1-\frac{\rho}{r_1}}\left(\frac{\rho}{r_1}\right)^n
\end{aligned}$$

which tends to zero as n tends to infinity since $\dfrac{\rho}{r_1} < 1$. Thus

$$\frac{1}{2\pi i}\int_{c_1} \frac{f(w)}{w-z}dw = a_0 + a_1(z-a) + a_2(z-a)^2 + ... + a_n(z-a)^n + ... \qquad (3.2)$$

where,

$$a_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(w)}{(w-a)^{n+1}}dw$$

Similarly, since

$$
\begin{aligned}
\frac{1}{w-z} &= \frac{1}{(w-a)-(z-a)} \\
&= -\frac{1}{(z-a)}\frac{1}{\left[1-\frac{w-a}{z-a}\right]} \\
&= -\frac{1}{(z-a)}\left[1-\frac{w-a}{z-a}\right]^{-1} \\
&= -\left[\frac{1}{(z-a)}+\frac{(w-a)}{(z-a)^2}+\frac{(w-a)^2}{(z-a)^3}+...\right]
\end{aligned}
$$

Hence,

$$
\begin{aligned}
-\frac{1}{2\pi i}\int_{c_2}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{c_2}\frac{f(w)}{(z-a)}dw + \frac{1}{2\pi i}\int_{c_2}\frac{(w-a)f(w)}{(z-a)^2}dw + \frac{1}{2\pi i}\int_{c_2}\frac{(w-a)^2f(w)}{(z-a)^3}dw \\
+ ... + \frac{1}{2\pi i}\int_{c_2}\frac{(w-a)^{n-1}f(w)}{(z-a)^n}dw + R_n' \\
\text{where } R_n' = \frac{1}{2\pi i}\int_{c_2}\frac{(w-a)^n f(w)}{(w-z)(z-a)^n}dw
\end{aligned}
$$

Now, let M denotes the maximum value of $f(w)$ on c_2. Since $|z-a| = \rho$, $|w-a| = r_2$ and $|w-z| > r_2 - \rho$, hence

$$
\begin{aligned}
|R_n'| &\leq \frac{1}{|2\pi i|}\int_{c_1}\frac{|(w-a)^n||f(w)|}{|(w-z)||(z-a)^n|}|dw| \\
&\leq \frac{|r_2^n|}{2\pi}\frac{M}{(r_2-\rho)\rho^n}\int_{c_1}|dw| \\
&= \frac{M}{1-\frac{\rho}{r_2}}\left(\frac{r}{\rho}\right)^n
\end{aligned}
$$

which tends to zero as n tends to infinity since $\dfrac{r_2}{\rho} < 1$. Thus

$$
-\frac{1}{2\pi i}\int_{c_1}\frac{f(w)}{w-z}dw = \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + ... + \frac{b_n}{(z-a)^n} + ... \tag{3.3}
$$

where

$$
b_n = \frac{1}{2\pi i}\int_{c_1}\frac{f(w)}{(w-a)^{-n+1}}dw
$$

Hence from equation 3.1, 3.2 and 3.3, we get

$$
f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ... + a_n(z-a)^n + ... + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + ... + \frac{b_n}{(z-a)^n} + ...
$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{c_1}\frac{f(w)}{(w-a)^{n+1}}dw$ and $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{c_2}\frac{f(w)}{(w-a)^{-n+1}}dw.$ ■

EXAMPLE 3.11

Show that, if $f(z)$ has a pole at $z = a$ then $|f(z)| \to \infty$ as $z \to a$.

Solution: Suppose the pole is of order m, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{m} b_n (z-a)^{-n}$$

Its principal part is $\sum_{n=1}^{m} b_n (z-a)^{-n}$

$$\begin{aligned}
\sum_{n=1}^{m} b_n (z-a)^{-n} &= \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + ... + \frac{b_n}{(z-a)^m} \\
&= \frac{1}{(z-a)^m}[b_m + b_{m-1}(z-a) + ...b_1(z-a)^{m-1}] \\
&= \frac{1}{(z-a)^m}[b_m + \sum_{n=1}^{m-1} b_n (z-a)^{m-n}] \\
|\sum_{n=1}^{m} b_n (z-a)^{-n}| &= |\frac{z}{(z-a)^m}[b_m + \sum_{n=1}^{m-1} b_n (z-a)^{m-n}]| \\
&\geq |\frac{1}{(z-a)^m}| \left[|b_m| - \sum_{n-1}^{m-1} |b_n||z-a|^{m-n} \right]
\end{aligned}$$

This tends to $b_m |a_1 + a_2| \geq |a_1| - |a_2|$. As $z \to -a$ R.H.S. =∞ ■

EXAMPLE 3.12

Write all possible Laurent series for the function

$$f(z) = \frac{1}{z(z+2)^3}$$

about the pole $z = -2$, using appropriate Laurent series.

Solution: To expand $\frac{1}{z(z+2)^3}$ about $z = -2$, i.e., in powers of $(z+2)$, we put $z + 2 = t$. Then

$$\begin{aligned}
f(z) &= \frac{1}{z(z+2)^3} = \frac{1}{(t-2)t^3} = \frac{1}{t^3}.\frac{1}{t-2} \\
&= \frac{1}{t^3}.\frac{1}{-2}.\frac{1}{1-\frac{t}{2}} = -\frac{1}{2t^3}(1-\frac{t}{2})^{-1}
\end{aligned}$$

$0 < |z+2| < 1$ or $0 < |t| < 1$

$$\begin{aligned}
f(z) &= -\frac{1}{2t^3}[1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} + \frac{t^4}{16} + \frac{t^5}{32} + ...] \\
&= -\frac{1}{2t^3} - \frac{1}{4t^2} - \frac{1}{8t} - \frac{1}{16} - \frac{t}{32} - \frac{t^2}{64}... \\
&= -\frac{1}{2(z+2)^3} - \frac{1}{4(z+2)^2} - \frac{1}{8(z+2)} - \frac{1}{16} - \frac{z+2}{32} - \frac{(z+2)^2}{64}...
\end{aligned}$$

■

EXAMPLE 3.13

Expand $f(z) = \cosh\left(z + \dfrac{1}{z}\right)$

Solution: $f(z)$ is analytic except $z = 0$. $f(z)$ can be expanded by Laurent's theorem.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

where

$$a_n = \frac{1}{2\pi i}\int_c \frac{f(z)dz}{(z-0)^{n+1}}, \text{ and } b_n = \frac{1}{2\pi i}\int_c f(z)z^{n-1}dz$$

$$\begin{aligned}
a_n &= \frac{1}{2\pi i}\int_c \frac{\cosh(z+\frac{1}{z})dz}{z^{n+1}} \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\cosh(2\cos\theta)dz}{z^{n+1}} \\
&\qquad\qquad\qquad\qquad z = e^{i\theta} \\
&\qquad\qquad\qquad\qquad dz = ie^{i\theta}d\theta \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\cosh(2\cos\theta)e^{i\theta}d\theta}{e^{i(n+1)\theta}} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos)e^{-ni\theta}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta d\theta - \frac{i}{2\pi}\int_c \cosh(2\theta)\sin n\theta d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta d\theta + 0
\end{aligned}$$

Since,

$$\begin{aligned}
b_n &= a_{-n} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos(-n\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos(n\theta)d\theta \\
&= a_n
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n} = a_0 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{a_n}{z^n} \\
&= a_0 + \sum_{n=1}^{\infty} a_n(z^n + z^{-n})
\end{aligned}$$

■

EXAMPLE 3.14

$$\text{Show that } f(z) = e^{\frac{c}{2}\left(z-\frac{1}{z}\right)} = \sum_{-\infty}^{\infty} a_n z^n., \text{ where } a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)d\theta.$$

Solution: $f(z)$ is the analytic function except at $z = 0$ so $f(z)$ can be expanded by Laurent's series.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_c \frac{f(z)dz}{z^{n+1}}$ and $b_n = \dfrac{1}{2\pi i}\displaystyle\int_c \frac{f(z)dz}{z^{-n+1}}$

Since $f(z)$ remains unaltered if $\frac{-1}{z}$ is written for z, hence

$$b_n = (-1)^n a^n$$

$$\begin{aligned}
\therefore \quad f(z) &= \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \\
&= \sum_{n=0}^{\infty} a_n z^n + (-1)\sum_{n=0}^{\infty} \frac{a_n}{z^n} \\
&= \sum_{-\infty}^{\infty} a_n z^n
\end{aligned}$$

Now,

$$\begin{aligned}
a_n &= \frac{1}{2\pi i}\int_c \frac{f(z)dz}{z^{n+1}} \\
&= \frac{1}{2\pi i}\int_c \frac{e^{\frac{c}{2}\left(z-\frac{1}{z}\right)}dx}{z^{n+1}} \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{\frac{c}{2}(2isin\theta)}ie^{i\theta}d\theta}{e^{(n+1)i\theta}} \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{cisin\theta}e^{-sini\theta}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i(csin\theta)-ni\theta}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} [cos(csin\theta - n\theta) - isin(csin\theta - n\theta)]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} cos(csin\theta - n\theta)d\theta
\end{aligned}$$

$$\because \text{ if, } f(2a-x) = -f(x), then \int_a^{2a} f(x)dx = 0$$

■

PROBLEMS

3.1 Expand $f(z) = \frac{1}{(z-1)(z-2)}$ for $1 < |z| < 2$.

3.2 Obtain the Taylor's or Laurent's series which represents the function $f(z) = \dfrac{1}{(1+z^2)(z+2)}$ when (i) $1 < |z| < 2$ (ii) $|z| > 2$.

3.3 Expand $\dfrac{e^z}{(z-1)^2}$ about $z = 1$

3.4 Expand $f(z) = \sin\left\{c\left(z + \frac{1}{z}\right)\right\}$

3.5 Expand $f(z) = e^{\frac{c}{2}\left(z - \frac{1}{z}\right)}$

3.6 $\dfrac{z-1}{z+1}$ (a) about $z = 0$ (b) about $z = 1$

3.7 Expand $\frac{1}{(z+1)(z+3)}$ in Laurent's series, if (a) $1 < |z| < 3$ (b) $|z| < 3$ (c) $-3 < z < 3$ (d) $|z| < 1$.

3.8 Find the Taylor's and Laurent's series which represents the function $\frac{z^2-1}{(z+2)(z+3)}$ when (i) $|z| < 2$ (ii) $2|z| < 3$.

3.9 Expand $\frac{z}{(z^2+1)(z^2+4)}$ in $1 < |z| < 2$.

3.10 Represent the function f(z) = $\frac{4z+3}{z(z-3)(z+2)}$ in Laurent's series (i) within $|z| = 1$ (ii) in the annular region between $|z| = 2$ and $|z| = 3$.

3.11 Write all possible Laurent Series for the function f(z) = $\frac{z^2}{(z-1)^2(z+3)}$ about the singularity z=1, stating the region of convergence in each case.

3.12 Obtain the expansion

$$f(z) = f(a) + 2\left\{\frac{z-a}{2} f'\left(\frac{z+a}{2}\right) + \frac{(z-a)^3}{2^3.3} f'''\left(\frac{z+a}{2}\right) + \frac{(z-a)^5}{2^5.5} f^v\left(\frac{z+a}{2}\right) + \ldots.\right\}$$

3.13 Expand $\frac{z^2-6z-1}{(z-1)(z+2)(z-3)}$ in $3 < |z+2| < 5$.

CHAPTER 4

ZEROS AND SINGULARITIES

4.1 Definitions

4.1.1 Zeros

Definition 4.1 *The value of z for which analytic function $f(z) = 0$ is called zero of the function $f(z)$.*

Let z_0 be a zero of an analytic function $f(z)$. Since $f(z)$ is analytic at z_0, there exists a neighbourhood of z_0 at which $f(z)$ can be expanded in a Taylor's series. i.e.,

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...a_n(z - z_0)^n + ...$$

where $|z - z_0| < \rho$, where $a_0 = f(z_0)$ and $a_n = \dfrac{f^{(n)}(z_0)}{n!}$.

Since z_0 is zero of $f(z)$, $f(z_0) = 0$ which gives $a_0 = 0$ and $a_1 \neq 0$, then such z_0 is said to be a *simple zero*. If $a_0 = 0$ and $a_1 = 0$ but $a_2 \neq 0$ then z_0 is called double zero. In general, if $a_0 = a_1 = a_2 = ... = a_{m-1} = 0$ but $a_m \neq 0$ then z_0 is called zero of order m. Thus zero of order m may be defined as the condition

$$f(z_0) = f'(z_0) = f''(z_0) = ... = f^{(m-1)}(z_0) = 0 \text{ and } f^{(m)}(z_0) \neq 0 \tag{4.1}$$

In this case function $f(z)$ may be rewritten as

$$f(z) = (z - z_0)^m g(z) \tag{4.2}$$

where $g(z)$ is analytic and $g(z_0) = a_m$, which is a non-zero quantity.

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EXAMPLE 4.1

The function $f(z) = \dfrac{1}{(z-1)}$ has a simple zero at infinity.

EXAMPLE 4.2

The function $f(z) = (z-1)^3$ has a zero of order 3 at $z = 1$.

EXAMPLE 4.3

Find the zero of the function defined by

$$f(z) = \frac{(z-3)}{z^3} \sin \frac{1}{(z-2)}$$

Solution: To find zero, we have $f(z) = 0$

$$\frac{(z-3)}{z^3} \sin \frac{1}{(z-2)} = 0$$

Hence $(z-3) = 0$ or $\sin \dfrac{1}{(z-2)} = 0 \Rightarrow \dfrac{1}{(z-2)} = n\pi$, where $n = 0, \pm 1, \pm 2, ...$ It follows that $z = 3$ or $z = 2 + \dfrac{1}{n\pi}$, where $n = 0, \pm 1, \pm 2, ...$ ■

4.1.2 Singularity

Definition 4.2 *If a function $f(z)$ is analytic at every point in the neighbourhood of a point z_0 except at z_0 itself, then z_0 is called a singularity or a singular point of the function.*

A singularity of an analytic function is the point z at which function ceases to be analytic.

4.1.3 Types of Singularities

Isolated and Non-isolated Singularities Let $z = a$ be a singularity of $f(z)$ and if there is no other singularity in neighbourhood of the point $z = a$, then this point $z = a$ is said to be an *isolated singularity* and otherwise it is termed as *non-isolated singularity*. (Note when a sequence of singularities is obtained the limit point of the sequence is isolated singularity.)

EXAMPLE 4.4

The function $f(z) = \frac{1}{\sin \frac{\pi}{z}}$ is analytic everywhere except those points at which

$$\sin \frac{\pi}{z} = 0$$

$$\Rightarrow \qquad \frac{\pi}{z} = n\pi \qquad \Rightarrow \qquad z = \frac{1}{n} \qquad (n = 1, 2, 3, ...)$$

Thus point $z = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., z = 0$ are the singularity of $f(z)$. Since no other singularity of $f(z)$ in neighbourhood of these points (except $z = 0$), these are isolated singularities. But at $z = 0$, there are infinite number of other singularities, where n is very large. Therefore $z = 0$ is non isolated singularity.

EXAMPLE 4.5

The function $f(z) = \dfrac{1}{(z-a)(z-b)}$ is analytic everywhere except $z = a$ and $z = b$. Thus point $z = a$ and $z = b$ are singularity of $f(z)$. Also there is no other singularity of $f(z)$ in neighbourhood of these points, these are isolated singularities.

Principal Part of $f(z)$ at isolated singularity Let $f(z)$ be analytic function within domain D except at the point $z = a$ which is an isolated singularity. Now draw a circle C centered at $z = a$ and of radius as small as we please. draw another concentric circle of any radius say R lying wholly within the domain D. The function $f(z)$ is analytic in ring shaped region between these two circles. Hence by Laurent's Theorem, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

The second term $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ in this expansion is called *principal part* of $f(z)$ at the singularity $z = a$.

- If there is no of term in principal part. Then singularity $z = a$ is called *removal singularity*.

$$f(z) = \frac{\sin(z-a)}{(z-a)} = \frac{1}{(z-a)}\left[(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - ...\infty\right]$$
$$= 1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} - ...$$

- If there are infinite number of terms in principal part. Then singularity $z = a$ is called *essential singularity*.

$$f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + ...$$

- If there are finite number of term in principal part. (Say m terms). Then singularity $z = a$ is called *pole*. and m is called *Order of pole*.

$$f(z) = \frac{\sin(z-a)}{(z-a)^3} = \frac{1}{(z-a)^3}\left[(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - ...\right]$$
$$= (z-a)^{-2} - \frac{1}{3!} + \frac{(z-a)^2}{5!} - ... + \infty$$

Definition 4.3 *A functions is said to be meromorphic function if it has poles as its only type of singularity.*

Definition 4.4 *A functions is said to be entire function if it has no singularity.*

Definition 4.5 *Limit point of zeros is called isolated essential singularity.*

Definition 4.6 *Limit point of poles is called non-isolated essential singularity.*

EXAMPLE 4.6

Show that the function e^z has an isolated essential singularity at $z = \infty$

Solution: **Note:** The behaviour of function $f(z)$ at ∞ is same as the behaviour $f\left(\frac{1}{z}\right)$ at $z = 0$.

Here

$$f\left(\frac{1}{z}\right) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$

Thus, $f\left(\frac{1}{z}\right)$ contains an infinite number of terms in the negative power of z. Therefore, by definition, $z = 0$ is an essential singularity of function $f\left(\frac{1}{z}\right)$. Consequently $f(z)$ has essential singularity at $z = \infty$. ■

PROBLEMS

4.1 Find out the zeros and discuss the nature of singularities of $f(z) = \frac{z-2}{z^2} \sin \frac{1}{z-1}$

4.2 What kind of singularities, the following functions have:

a) $f(z) = \frac{1}{1-e^z}$ at $z = 2\pi i$

b) $f(z) = \frac{1}{\sin z - \cos z}$ at $z = \frac{\pi}{4}$

c) $f(z) = \cos z - \sin z$ at $z = \infty$

d) $f(z) = \frac{\cot \pi z}{(z-a)^2}$ at $z = 0$ and $z = \infty$

e) $f(z) = \sin \frac{1}{1-z}$ at $z = 1$

f) $f(z) = \tan \frac{1}{z}$ at $z = 0$

g) $f(z) = \frac{1}{\cos \frac{1}{z}}$ at $z = 0$

h) $f(z) = \frac{1}{\sin \frac{1}{z}}$ at $z = 0$

i) $f(z) = \frac{1-e^z}{1+e^z}$ at $z = \infty$

j) $f(z) = z$ cosec$z$ at $z = \infty$

4.2 The Residue at Poles

The coefficient of $\frac{1}{(z-a)}$ in the principal part of Laurent's Expansion is called the RESIDUE of function $f(z)$. The coefficient b_1 which is given as

$$b_1 = \frac{1}{2\pi i} \int_C f(z)dz \quad = RES[f(z)]_{z=a}$$

4.2.1 Methods of finding residue at poles

4.2.1.1 The residue at a simple pole *(Pole of order one.)*

$$R = \lim_{z \to a} [(z-a).f(z)]$$

If function of form $f(z) = \frac{\phi(z)}{\psi(z)}$ Then

$$R = \left[\frac{\phi(z)}{\psi'(z)}\right]_{z=a}$$

4.2.1.2 The residue at multiple pole (Pole of order m.)

$$R = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}(z-a)^m.f(z)\right]_{z=a}$$

4.2.1.3 The residue at pole of any order

Pole is $z = a$

Put $z - a = t$. Expand function. Now the coefficient of $1/t$ is residue.

EXAMPLE 4.7

Determine the poles and the residue at each pole of the function

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$

Solution: The poles of the function $f(z)$ are given by putting the denominator equal to zero. i.e.,

$$(z-1)^2(z+2) = 0$$

$$z = 1, 1, -2$$

The function $f(z)$ has a simple pole[1] at $z = -2$ and pole of order 2 at $z = 1$.

$$\begin{aligned}
\text{Residue of } f(z) \text{ at } (z=-2) &= \lim_{z\to -2}(z+2).f(z) \\
&= \lim_{z\to -2}(z+2)\frac{z^2}{(z-1)^2(z+2)} \\
&= \lim_{z\to -2}\frac{z^2}{(z-1)^2} = \frac{4}{9}
\end{aligned}$$

$$\begin{aligned}
\text{Residue of } f(z) \text{ at double pole (order 2) } (z=1) &= \frac{1}{(2-1)!}\left[\frac{d^{2-1}}{dz^{2-1}}(z-1)^2.f(z)\right]_{z=1} \\
&= \left[\frac{d}{dz}(z-1)^2\frac{z^2}{(z-1)^2(z+2)}\right]_{z=1} \\
&= \left[\frac{d}{dz}\frac{z^2}{(z+2)}\right]_{z=1} \\
&= \left[\frac{z^2+4z}{(z+2)^2}\right]_{z=1} = \frac{5}{9}
\end{aligned}$$

■

EXAMPLE 4.8

Determine the poles and residue at each pole of the function $f(z) = \cot z$

Solution: Here $\cot z = \dfrac{\cos z}{\sin z}$. i.e. $\phi(z) = \cos z$ and $\psi(z) = \sin z$. The poles of the function $f(z)$ are given by

$$\sin z = 0 \Rightarrow z = n\pi, \text{ where } n = 0, \pm 1, \pm 2, ...$$

[1] Pole of order one is called simple pole.

$$\text{Residue of } f(z) \text{ at } (z = n\pi) = \left[\frac{\phi(z)}{\psi'(z)}\right]_{z=n\pi}$$
$$= \left[\frac{\cos z}{\frac{d}{dz}\sin z}\right]_{z=n\pi}$$
$$= \frac{\cos z}{\cos z} = 1$$

■

EXAMPLE 4.9

Find the residue of $\dfrac{ze^z}{(z-a)^3}$ at its poles.

Solution: The pole of $f(z)$ is given by $(z-a)^3 = 0$, i.e., $z = a$ (pole of order 3)
Putting $z = t + a$

$$f(z) = \frac{ze^z}{(z-a)^3}$$
$$\Rightarrow f(z) = \frac{(t+a)e^{t+a}}{t^3}$$
$$= \left(\frac{a}{t^3} + \frac{1}{t^2}\right) e^{t+a}$$
$$= e^a \left(\frac{a}{t^3} + \frac{1}{t^2}\right) e^t$$
$$= e^a \left(\frac{a}{t^3} + \frac{1}{t^2}\right) \left(1 + \frac{t}{1!} + \frac{t^2}{2!} + ...\right)$$
$$= e^a \left(\frac{a}{t^3} + \frac{a}{t^2} + \frac{a}{2t} + \frac{1}{t^2} + \frac{1}{t} + \frac{1}{2} + ...\right)$$

Hence the residue at $(z = a)$ =Coefficient of $\frac{1}{t} = e^a\left(\frac{a}{2} + 1\right)$. ■

4.2.2 Residue at Infinity

Residue of $f(z)$ at $z = \infty$ is defined as $-\frac{1}{2\pi i}\int_C f(z)dz$, where integration is taken round in anti-clockwise direction. Where C is a large circle containing all finite singularities of $f(z)$. We also have, Residue at Infinity = $\lim_z \to \infty(-zf(z))$

Residue at Infinity = - Residue at zero of $f(1/z)$

EXAMPLE 4.10

Find the residue of $f(z) = \dfrac{z^3}{z^2 - 1}$ at $z = \infty$.

Solution:

Here,

$$f(z) = \frac{z^3}{z^2 - 1}$$

Therefore

$$f(1/z) = \frac{1}{z(z^2 - 1)}$$

Residue at $z = 0$

$$\lim_{z \to 0} z.\frac{1}{z(z^2 - 1)} = -1$$

Hence Residue at Infinity of $f(z)$ = - Residue at zero of $f(1/z)$ = 1 ■

PROBLEMS

4.1 Determine poles of the following functions. Also find the residue at its poles.

a) $\frac{z-3}{(z-2)^2(z+1)}$

b) $\frac{ze^{iz}}{z^2+a^2}$

c) $\frac{e^z}{(z^2+a^2)}$

d) $\frac{z^2}{(z+2)(z^2+1)}$

e) $z^2 e^{1/z}$

f) $z^2 \sin \frac{1}{z}$

g) $\frac{e^{2z}}{(1+e^z)}$

h) $\frac{1+e^z}{\sin z + z \cos z}$ at $z = 0$

i) $\frac{1}{z(e^z-1)}$

j) $\frac{\cot \pi z}{(z-a)^2}$

4.2 Find the residue at $z = 1$ of $\frac{z^3}{(z-1)^4(z-2)(z-3)}$

4.3 Locate the poles of $\frac{e^{az}}{\cosh \pi z}$ and evaluate the residue at the pole of the smallest positive value of z.

4.4 Find the residue of $\frac{z^3}{z^2-1}$ at $z = \infty$

4.5 Find the residue of $\frac{z^2}{(z-a)(z-b)(z-c)}$ at $z = \infty$

4.6 The function $f(z)$ has a double pole at $z = 0$ with residue 2, a simple pole at $z = 1$ with residue 2, is analytic at all finite points of the plane and is bounded as $|z| \to \infty$ if $f(2) = 5$ and $f(-1) = 2$ find $f(z)$

4.7 Let $\dfrac{P(z)}{Q(z)}$, where both $P(z)$ and $Q(x)$ are complex polynomial of degree 2. If $f(0) = f(-1) = 0$. and only singularly of $f(z)$ is of order 2 at $z = 1$ with residue -1, then find $f(z)$.

CHAPTER 5

RESIDUE

5.1 Residue Theorem

5.1.1 Residue

By Laurent series expansion of an analytic function $f(z)$, we have

$$f(x) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt$$

As we have from definition, the coefficient of $\frac{1}{(z-a)}$ in Laurent expansion about $z = a$ is called as *Residue*, Which may be obtained by putting $n = -1$ in above equation. Thus,

$$RES(a) = \frac{1}{2\pi i} \int_C f(t) dt$$

which implies

$$\int_C f(t) dt = 2\pi i . RES(a)$$

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5.1.2 Residue Theorem

Theorem 5.1 *If $f(z)$ be analytic within and on a simple closed curve C except at number of poles. (Say $z_1, z_2, z_3, ..., z_n$ are poles.) then the integral*

$$\int_C f(t)dt = 2\pi i (\text{Sum of Residue of } f(z) \text{ at each poles})$$

Proof: Let $f(z)$ be analytic within and on simple closed curve C except at number of poles $z_1, z_2, z_3, ..., z_n$.

Let $C_1, C_2, ..., C_n$ be small circles with centres at $z_1, z_2, z_3, ..., z_n$ respectively. Then by Cauchy extension theorem, we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ... + \int_{C_n} f(z)dz$$

Now, we have,

$$\int_{C_1} f(z)dt = 2\pi i.RES(z_1)$$

$$\int_{C_2} f(z)dt = 2\pi i.RES(z_2)$$

$$... \quad ... \quad ...$$

$$\int_{C_n} f(z)dt = 2\pi i.RES(z_n)$$

Therefore,

$$\int_C f(z)dz = 2\pi i.RES(z_1) + 2\pi i.RES(z_2) + ... + 2\pi i.RES(z_n)$$

$$\int_C f(z)dz = 2\pi i.[RES(z_1) + RES(z_2) + ... + RES(z_n)]$$

$$\int_C f(z)dz = 2\pi i.[\text{Sum of all Residues at each poles.}]$$

$$\int_C f(z)dz = 2\pi i.Res$$

We use Res for ‘Sum of Residues’ through out the text. ■

5.2 Evaluation of Real Integrals by Contour Integration

Type $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$

For such questions consider a unit radius circle with center at origin, as contour

$$|z| = 1 \qquad \Rightarrow z = e^{i\theta} \qquad d\theta = \frac{dz}{iz}$$

Now use following relations

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}(z + \frac{1}{z})$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}(z - \frac{1}{z})$$

We get, the whole function is converted into a function of $f(z)$, Now integral become

$$I = \int_C f(z)dz$$

where C is unit circle. The value of this integral may be obtained by using Residue Theorem, which is $2\pi i$.(Sum of Residue inside C)

5.2.1 Form I

Form $\int_0^{2\pi} \frac{1}{a+b\cos\theta} d\theta$ or $\int_0^{2\pi} \frac{1}{a+b\sin\theta} d\theta$

EXAMPLE 5.1

Use residue calculus to evaluate the following integral $\int_0^{2\pi} \frac{1}{5-4sin\theta} d\theta$

Solution: Put $z = e^{i\theta}$ so that $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$. Then

$$\int_0^{2\pi} \frac{1}{5 - 4\sin\theta} d\theta = \oint_c \frac{1}{5 - 4\left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]} \frac{dz}{iz}$$
$$= \oint_c \frac{1}{5iz - 2z^2 + 2} dz$$

Poles of integrand are given by

$$-2z^2 + 5iz + 2 = 0$$

or

$$z = \frac{-5i \pm \sqrt{-25 + 16}}{-4} = \frac{-5 \pm 3i}{-4} = 2i, \frac{i}{2}$$

Only $z = \frac{i}{2}$ lies inside c. Residue at the simple pole at $z = \frac{i}{2}$ is

$$\lim_{z \to \frac{i}{2}} \left(z - \frac{i}{2}\right) \times \left(\frac{1}{(2z - i)(-z + 2i)}\right) = \frac{1}{3i}$$

Hence by Cauchy's residue theorem

$$\int_0^{2\pi} \frac{1}{5 - 4\sin\theta} d\theta = 2\pi i \times Res$$
$$= 2\pi \times \frac{1}{3i} = \frac{2\pi}{3}$$

■

EXAMPLE 5.2

Using complex variables, evaluate the real integral

$$\int_0^{2\pi} \frac{d\theta}{1 - 2p\sin\theta + p^2}$$

where $p^2 < 1$

Solution:

We have,

$$I=\int_0^{2\pi}\frac{d\theta}{1-2p\sin\theta+p^2}=\int_0^{2\pi}\frac{d\theta}{1-2p\frac{(e^{i\theta}-e^{-i\theta})}{2i}+p^2}$$

Put $z=e^{i\theta}, dz=ie^{i\theta}d\theta=\frac{dz}{zi}$

$$\begin{aligned}I&=\int_c\frac{1}{1+ip(z-\frac{1}{z})+p^2}\frac{dz}{zi}\\&=\int_c\frac{dz}{zi-pz^2+p+p^2zi}=\int_c\frac{dz}{(iz+p)(izp+1)}\end{aligned}$$

Pole are given by

$$(iz+p)(ipz+1)=0$$
$$z=-\frac{p}{i},-\frac{1}{ip}\text{ or }z=ip,\frac{i}{p}$$

ip is the only poles inside the unit circle. Residue at$(z=ip)$

$$=\lim_{z\to ip}\frac{(z-pi)}{(iz+p)(izp+1)}=\lim_{z\to ip}\frac{1}{i(izp+1)}=\frac{1}{i}\left(\frac{1}{-p^2+1}\right)$$

Hence by Cauchy's residue theorem

$$\int_0^{2\pi}\frac{d\theta}{1-2psin+p^2}=2\pi i\times Res=2\pi i\times\frac{1}{i}\left(\frac{1}{1-p^2}\right)=\frac{2\pi}{1-p^2}$$

■

5.2.2 Form II

Form $\int_0^{2\pi}\frac{\cos m\theta}{a+b\cos\theta}d\theta$, $\int_0^{2\pi}\frac{\cos m\theta}{a+b\sin\theta}d\theta$, $\int_0^{2\pi}\frac{\sin m\theta}{a+b\cos\theta}d\theta$ or $\int_0^{2\pi}\frac{\sin m\theta}{a+b\sin\theta}d\theta$

EXAMPLE 5.3

Using complex variable techniques evaluate the real integral $\int_0^{2\pi}\frac{\sin^2\theta d\theta}{5-4\cos\theta}$

Solution: We may write,

$$\int_0^{2\pi}\frac{\sin^2\theta d\theta}{5-4\cos\theta}=\frac{1}{2}\int_0^{2\pi}\frac{1-\cos2\theta}{5-4\cos\theta}d\theta$$

Put $z=e^{i\theta}$so that $\sin\theta=\frac{1}{2i}\left(z-\frac{1}{z}\right)$and $d\theta=\frac{dz}{iz}$. Then

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi}\frac{1-\cos2\theta}{5-4\cos\theta}d\theta&=\text{Real part of }\frac{1}{2}\int_0^{2\pi}\frac{1-e^{2i\theta}}{5-4cos\theta}d\theta\\&=\text{Real part of }\frac{1}{2}\oint_c\frac{1-z^2}{5-2(z+\frac{1}{z})}(\frac{dz}{iz})\\&=\text{Real part of }\frac{1}{2i}\oint_c\frac{z^2-1}{2z^2-5z+2}dz\end{aligned}$$

Poles of integrand are given by

$$2z^2-5z+2=0$$

$$(2z-1)(z-2)=0$$
$$z=\frac{1}{2},2$$

So poles inside the contour c there is a simple pole at $z=\frac{1}{2}$. Residue at the simple pole $(z=\frac{1}{2})$ is

$$\lim_{z\to\frac{1}{2}}(z-\frac{1}{2})\frac{z^2-1}{(2z-1)(z-2)}$$
$$=\lim_{z\to\frac{1}{2}}\frac{z^2-1}{2(z-2)}=\frac{\frac{1}{4}-1}{2(\frac{1}{2}-2)}=\frac{1}{4}$$
$$\oint_c\frac{z^2-1}{2z^2-5z+2}dz=2\pi i\times Res=\frac{2\pi i}{4}$$
$$\therefore\int_0^{2\pi}\frac{sin^2\theta}{5-4cos\theta}d\theta=\frac{1}{2i}\oint_c\frac{z^2-1}{2z^2-5z+2}dz=\frac{1}{2i}\times\frac{2\pi i}{4}=\frac{\pi}{4}$$

■

5.2.3 Form III

$\int_0^{2\pi}\frac{1}{(a+b\cos\theta)^2}d\theta$ or $\int_0^{2\pi}\frac{1}{(a+b\sin\theta)^2}d\theta$

EXAMPLE 5.4

Use the residue theorem to show that

$$\int_0^{2\pi}\frac{d\theta}{(a+bcos\theta)^2}=\frac{2\pi a}{(a^2-b^2)^{3/2}}$$

Solution: Put $e^{i\theta}=z$,so that $e^{i\theta}(id\theta)=dz$or $izd\theta=dz=d\theta=\frac{dz}{iz}$

$$\int_0^{2\pi}\frac{d\theta}{(a+bcos\theta)^2}=\int_c\frac{1}{[a+\frac{b}{2}(z+\frac{1}{z})]^2}\frac{dz}{iz}$$

where c is the unit circle $z=1$

$$=\int_0\frac{-4izdz}{(bz^2+2az+b)^2}=-\frac{4i}{b^2}\int_c\frac{zdz}{(z^2+\frac{2az}{b}+1)^2}$$

The pole are given by putting the denominator

$$(z^2+\frac{2a}{b}z+1)^2=0$$

$$(z-\alpha)^2(z-\beta)^2=0$$

where

$$[\alpha+\beta=-\frac{2a}{b}],\alpha\beta=1$$

$$\alpha=\frac{-\frac{2a}{b}+\sqrt{\frac{4a^2}{b^2}-4}}{2}=\frac{-a+\sqrt{a^2-b^2}}{b}$$

$$\beta=\frac{-\frac{2a}{b}-\sqrt{\frac{4a^2}{b^2}-4}}{2}=\frac{-a-\sqrt{a^2-b^2}}{b}$$

There are two poles at $z = \alpha$ and $z = \beta$, each of order 2.
Now,
Let

$$f(Z) = \frac{-4iz}{b^2(z-\alpha)^2(z-\beta)^2} = \frac{\phi(z)}{(z-\alpha)^2}$$

where $\phi(z) = \dfrac{-4iz}{b^2(z-\beta)^2}$. Here $z = \alpha$ is only point inside circle c. Residue at the double pole $z = \alpha$

$$= \lim_{z\to\alpha}\left(\frac{d}{dz}(z-\alpha)^2\frac{\phi(z)}{(z-\alpha)^2}\right) = \lim_{z\to\alpha}\frac{d}{dz}\phi(z)$$

$$= -\lim_{z\to\alpha}\frac{4i}{b^2}\left[\frac{(z-\beta)^2.1 - z.2(z-\beta)}{(z-\beta)^4}\right] = -\frac{4i}{b^2}\lim_{z\to\alpha}\left[\frac{(z-\beta)-2z}{(z-\beta)^3}\right]$$

$$= \frac{4i}{b^2}\frac{\alpha+\beta}{(\alpha-\beta)^3} = \frac{4i}{b^2}\frac{-\frac{2a}{b}}{\left(\frac{2\sqrt{a^2-b^2}}{b}\right)^3} = \frac{-ai}{(a^2-b^2)^{\frac{3}{2}}}$$

Hence

$$\int_0^{2\pi}\frac{d\theta}{(a+bcos\theta)^2} = 2\pi i \times \frac{-ai}{(a^2-b^2)^{\frac{3}{2}}} = \frac{2\pi a}{(a^2-b^2)^{\frac{3}{2}}}$$

■

5.2.4 Form IV

$\int_0^{2\pi}\frac{1}{a^2+\cos^2\theta}d\theta$, $\int_0^{2\pi}\frac{1}{a^2+\sin^2\theta}d\theta$

EXAMPLE 5.5

Show by the method of residue, that

$$\int_0^{\pi}\frac{ad\theta}{a^2+sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}}$$

$$I = \int_0^{\pi}\frac{ad\theta}{a^2+sin^2\theta} = \int_0^{\pi}\frac{2ad\theta}{2a^2+2sin^2\theta}$$

$$\because (cos2\theta = 1 - 2sin^2)$$

$$= \int_0^{\pi}\frac{2ad\theta}{2a^2+1-cos2\theta} = \int_0^{2\pi}\frac{ad\phi}{2a^2+1-cos\phi}$$

Putting $2\theta = \phi$

$$= \int_0^{2\pi}\frac{ad\phi}{2a^2+1-\frac{1}{2}(e^{i\phi}+e^{-i\phi})} = \int_0^{2\pi}\frac{2ad\phi}{4a^2+2-(e^{i\phi}+e^{-i\phi})}$$

Now let $z = e^{i\phi} \Rightarrow d\phi\frac{dz}{iz}$

$$I = \int_0^{2\pi}\frac{2a}{4a^2+2-(z+\frac{1}{z})}\frac{dz}{iz}$$

$$= 2ai\int_0^{2\pi}\frac{dz}{[z^2-(4a^2+2)z-1]}$$

The pole are given by putting the denominator

$$z^2 - (4a^2 + 2)z - 1 = 0$$

$$z = \frac{(4a^2 + 2) \pm \sqrt{(4a^2 + 2)^2 - 4}}{2} = \frac{(4a^2 + 2) \pm \sqrt{16a^4 + 16a^2}}{2}$$

$$= (2a^2 + 1) \pm 2a\sqrt{a^2 + 1}$$

Let

$$\alpha = (2a^2 + 1) + 2a\sqrt{a^2 + 1}$$

$$\beta = (2a^2 + 1) - 2a\sqrt{a^2 + 1}$$

Let $z^2 - (4a^2 + 2)z + 1 = (z - \alpha)(z - \beta)$. Therefore product of the roots $= \alpha\beta = 1$ or $|\alpha\beta| = 1$. But $|\alpha| > 1$ therefore $|\beta| < 1$.

i.e., Only β lies inside the circle c.

Residue (at $z = \beta$) is

$$= \lim_{z \to \beta} (z - \beta) \frac{2ai}{(z - \alpha)(z - \beta)} = \frac{-2ai}{(\beta - \alpha)}$$
$$= \frac{2ai}{\left[(2a^2 + 1) - 2a\sqrt{a^2 + 1}\right] - \left[(2a^2 + 1) + 2a\sqrt{a^2 + 1}\right]}$$
$$= \frac{2ai}{-4a\sqrt{a^2 + 1}} = \frac{-i}{2\sqrt{a^2 + 1}}$$

Hence by Cauchy's residue theorem

$$I = 2\pi i \times Res$$

$$= 2\pi i \frac{-i}{2\sqrt{a^2 + 1}} = \frac{\pi}{\sqrt{a^2 + 1}}$$

PROBLEMS

5.1 Use Residue of calculus to evaluate the following integrals:

a) $\displaystyle\int_0^{\pi} \frac{1}{3 + 2\cos\theta} d\theta$

b) $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{3 + \sin\theta} d\theta$

c) $\displaystyle\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\sin\theta} d\theta$

d) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta - 2\cos\theta}{2 + \cos\theta} d\theta$

e) $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$

f) $\displaystyle\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta - n\theta)] d\theta$

g) $\displaystyle\int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta}{3+2\cos\theta} d\theta$

h) $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{1-2p\sin\theta+p^2} d\theta$ where $p^2 < 1$.

5.2.5 Some Important Results

Jordan's Inequality

Consider the relation $y = \cos\theta$. As θ increases, $\cos\theta$ decreases and therefore y decreases. The mean ordinate between 0 and θ is $\displaystyle\frac{1}{\theta}\int_0^{\theta} \cos\theta d\theta = \frac{\sin\theta}{\theta}$ which implies when

$$0 < \theta < \frac{\pi}{2} \text{ then } \frac{2}{\pi} < \frac{\sin\theta}{\theta} < 1$$

Theorem 5.2 *Let AB be the arc $\alpha < \theta < \beta$ of the circle $|z-a| = r$. If* $\lim_{z\to a}(z-a)f(z) = k$ *then*

$$\lim_{r\to 0}\int_{AB} f(z)dz = i(\beta-\alpha)k$$

Theorem 5.3 *Let AB be the arc $\alpha < \theta < \beta$ of the circle $|z| = R$. If* $\lim_{z\to\infty} z.f(z) = k$ *then*

$$\lim_{R\to\infty}\int_{AB} f(z)dz = i(\beta-\alpha)k$$

Theorem 5.4 *If $f(z) \to 0$ uniformly as $|z| \to \infty$, then* $\lim_{R\to\infty}\int_{C_R} e^{imz} f(z)dz = 0$, *where C_R denotes the semicircle $|z| = R, \quad m > 0$.*

5.2.6 Type $\int_{-\infty}^{\infty} f(x)dy$

Consider the integral $\int_{-\infty}^{\infty} f(x)dy$, such that $f(x) = \frac{\phi(x)}{\psi(x)}$, where $\psi(x)$ has no real roots and the degree of $\psi(x)$ is greater than $\phi(x)$.

Procedure Consider a function $f(z)$ corresponding to function $f(x)$. Again consider the integral $\int_C f(z)dz$, where C is a curve, consisting of upper half of the circle $|z| = R$, and part of real axis from $-R$ to R. Here R is on our choice and can be taken as such that there is no singularity on its circumference C_R.

Now by Cauchy's theorem,we have

$$\int_C f(z)dz = 2\pi i.\sum R_k$$

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i.\sum R_k$$

Now as,

$$\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$$

We get,

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i.\sum R_k$$

Which is required integral.

EXAMPLE 5.6

Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$.

Solution: Consider the integral $\int_C f(z)dz$ where $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$ and C is the contour consisting of the semi circle C_R which is upper half of a large circle $|z| = R$ of radius R together with the part of the real axis from$-R$ to $+R$.

For the poles

$$(z^2 + 1)(z^2 + 4) = 0 \quad \Rightarrow \quad z = \pm i, z = \pm 2i$$

So $z = i, 2i$ are the only poles inside C.

The residue at $z = i$

$$\begin{aligned} &= \lim_{z \to i}(z - i)\frac{z^2}{(z+i)(z-i)(z^2+4)} \\ &= \lim_{z \to i} \frac{z^2}{(z+i)(z^2+4)} = \frac{-1}{2i(-1+4)} = -\frac{1}{6i} \end{aligned}$$

The residue at $z = 2i$

$$\begin{aligned} &= \lim_{z \to 2i}(z - 2i).\frac{z^2}{(z^2+1)(z+2i)(z-2i)} \\ &= \lim_{z \to 2i} \frac{z^2}{(z^2+1)(z+2i)} = \frac{(2i)^2}{(-4+1)(2i+2i)} = \frac{1}{3i} \end{aligned}$$

By theorem of residue;

$$\begin{aligned} \int_C f(z)dz =& 2\pi i[Res(i) + Res(2i)] \\ =& 2\pi i(-\frac{1}{6i} + \frac{1}{3i}) = \frac{\pi}{3} \end{aligned}$$

i.e.

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{\pi}{3}$$

Hence by making $R \to \infty$, relation (1) becomes

$$\int_{-\infty}^{\infty} f(x)dx + \lim_{R \to \infty} \int_{C_R} f(z)dz = \frac{\pi}{3}$$

Now

$$
\begin{aligned}
\left|\int_{C_R} f(z)dz\right| &= \left|\int_{C_R} \frac{z^2 dz}{(z^2+1)(z^2+4)}\right| \\
&\le \int_{C_R} \frac{\left|z^2 dz\right|}{|(z^2+1)|\,|(z^2+4)|} \\
&\le \int_{C_R} \frac{\left|z^2 dz\right|}{|(|z|^2-1)|\,|(|z|^2-4)|} \\
&\qquad\qquad \because |z| = R \\
&\qquad\qquad z = Re^{i\theta}, 0 < \theta < \pi \\
&\qquad\qquad |dz| = Rd\theta, \\
&\le \int_{C_R} \frac{R^2 R d\theta}{(R^2-1)(R^2-4)} \\
&= \frac{\pi R^3}{(R^2-1)(R^2-4)} \to 0 \text{ as } R \to \infty
\end{aligned}
$$

Thus

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{3}$$

i.e.,

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}$$

■

EXAMPLE 5.7

Evaluate by the model of complex variables , the integral $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3} dx$

Solution: Consider the integral $\int_C f(z)dz$ where $f(z) = \frac{z^2}{(1+z^2)^3}$ and C is the contour consisting of the semi circle C_R which is upper half of a large circle $|z| = R$ of radius R together with the part of the real axis from$-R$ to $+R$. For the poles

$$(1+z^2)^3 = 0$$

$$\Rightarrow \; z = \pm i$$

$\therefore z = i$ and $z = -i$ are the two poles each of order 3. But only $z = i$ lies within the C. Residue at $z = i$

To get residue at $z = i$, put $z = i + t$, then

$$
\begin{aligned}
\frac{z^2}{(1+z^2)^3} &= \frac{(i+t)^2}{[1+(i+t)^2]^3} = \frac{-1+2it+t^2}{[1-1+2it+t^2]^3} \\
&= \frac{(-1+2it+t^2)}{(2it)^3(1+\frac{1}{2i}t)^3} = \frac{(-1+2it+t^2)}{-8it^3}(1+\frac{t}{2i})^{-3} \\
&= -\frac{1}{8i}\left(-\frac{1}{t^3}+\frac{2i}{t^2}+\frac{1}{t}\right)\left(1-\frac{3t}{2i}+\frac{(-3)(-4)}{2}\frac{t^2}{-4}+...\right) \\
&= -\frac{1}{8i}\left[-\frac{1}{t^3}+\frac{2i}{t^2}+\frac{1}{t}\right]\left[1-\frac{3}{2i}t-\frac{3}{2}t^2+...\right]
\end{aligned}
$$

Here coefficient of $\frac{1}{t}$ is $\frac{-1}{8i}(\frac{3}{2}-3+1)$ = $\frac{-i}{16}$, which is therefore the residue at $z=i$.

Using Cauchy's theorem of residues we have

$$\int_c f(z)dz = 2\pi i \times Res$$

where Res = Sum of the residues of $f(z)$ at the poles within c.

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i(-\frac{i}{16})$$

$$\int_{-R}^{R} \frac{x^2}{(1+x^2)^3}dx + \int_{C_R} \frac{z^2}{(1+z^2)^3}dz = \frac{\pi}{8}$$

Now

$$\left|\int_{C_R} \frac{z^2}{(1+z^2)^3}dz\right| \leq \int_{C_R} \frac{|z|^2\,|dz|}{|1+z^2|^3} \leq \int_{C_R} \frac{|z|^2|dz|}{(|z^2|-1)^3}$$

Since $z=Re^{i\theta}, |dz|=Rd\theta$

$$\leq \frac{R^2}{(R^2-1)^3}\int_0^{\pi} Rd\theta$$

$$= \frac{\pi R^3}{(R^2-1)^3} \to 0 \text{ as } R \to \infty$$

Hence

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3}dx = \frac{\pi}{8}$$

■

EXAMPLE 5.8

Using the complex variables techniques, evaluate the integral

$$\int_0^{\infty} \frac{dx}{x^4+16}$$

Solution: Consider the integral $\int_C f(z)dz$ where $f(z)=\frac{1}{z^4+16}$ and C is the contour consisting of the semi circle C_R which is upper half of a large circle $|z|=R$ of radius R together with the part of the real axis from$-R$ to $+R$.

For the poles

$$z^4+16=0 \;\Rightarrow\; z^4=-16 \;\Rightarrow\Rightarrow\; z^4=16e^{i\pi}=16e^{i(2n+1)\pi} \;\Rightarrow\; z=2e^{(2n+1)i\pi/4}$$

Now for $n=0,1,2,3$

$$z_1 = 2e^{i\pi/4}2(cos\frac{\pi}{4}+isin\frac{\pi}{4}) = 2(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}) = \sqrt{2}(1+i)$$
$$z_2 = 2e^{3i\pi/4}2(cos\frac{3\pi}{4}+isin\frac{3\pi}{4}) = 2(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}) = \sqrt{2}(-1+i)$$
$$z_3 = 2e^{5i\pi/4}2(cos\frac{5\pi}{4}+isin\frac{5\pi}{4}) = 2(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}) = -\sqrt{2}(1+i)$$
$$z_4 = 2e^{7i\pi/4}2(cos\frac{7\pi}{4}+isin\frac{7\pi}{4}) = 2(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}) = \sqrt{2}(1-i)$$

There are four poles , but only two poles at z_1 and z_2 lie within C.

Residue (at $z = 2e^{\frac{i\pi}{4}}$)

$$= \left[\frac{1}{\frac{d}{dz}(z^4+16)}\right]_{z=2e^{\frac{i\pi}{4}}} = \left[\frac{1}{4z^3}\right]_{z=2e^{\frac{i\pi}{4}}} = \frac{1}{4(2e^{i\frac{\pi}{4}})^3} = \frac{1}{32e^{i\frac{3\pi}{4}}} = -\frac{e^{i\frac{\pi}{4}}}{32}$$

Residues (at $z = 2e^{\frac{3i\pi}{4}}$)

$$= \left[\frac{1}{\frac{d}{dz}(z^4+16)}\right]_{z=2e^{\frac{3i\pi}{4}}} = \left[\frac{1}{4z^3}\right]_{z=2e^{\frac{3i\pi}{4}}} = \frac{1}{4(2e^{3i\frac{\pi}{4}})^3} = \frac{1}{32e^{i\frac{9\pi}{4}}} = \frac{e^{-i\frac{\pi}{4}}}{32} \text{ Note Here.}$$

$$\int_c f(z)dz = 2\pi i \times \left(\frac{-e^{i\pi/4}+e^{-i\pi/4}}{32}\right)$$

$$\int_c f(z)dz = -\left(2\pi i\frac{i\sin\frac{\pi}{4}}{16}\right) = \frac{\sqrt{2}\pi}{16}$$

where Res= Sum of residues at poles within C

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz == \frac{\sqrt{2}\pi}{16}$$

$$\int_{-R}^{R} \frac{1}{x^4+16}dx + \int_{C_R} \frac{1}{z^4+16}dz = \frac{\sqrt{2}\pi}{16}$$

Now

$$\begin{aligned}\left|\int_{C_R} \frac{1}{z^4+16}dz\right| &\le \int_{C_R} \frac{|dz|}{|z^4+16|}\\ &\le \int_{C_R} \frac{|dz|}{(|z|^4-16)}\\ &\quad \text{Since } z = Re^{i\theta}, |dz| = Rd\theta\\ &\le \int_0^{\pi} \frac{Rd\theta}{R^4-16} = \frac{R\pi}{R^4-16} \to 0 \text{ as } R\to\infty\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} \frac{1}{x^4+16}dx == \frac{\sqrt{2}\pi}{16}$$

$$2\int_0^{\infty} \frac{1}{x^4+16}dx = \frac{\sqrt{2}\pi}{16}$$

$$\int_0^{\infty} \frac{1}{x^4+16}dx = \frac{\sqrt{2}\pi}{32}$$

■

EXAMPLE 5.9

Using complex variables, evaluate the real integral

$$\int_0^{\infty} \frac{cos(3x)dx}{(x^2+1)(x^2+4)}$$

Solution: Consider the integral $\int_C f(z)dz$ where $f(z) = \frac{e^{3iz}}{(z^2+1)(z^2+4)}$ and C is the contour consisting of the semi circle C_R which is upper half of a large circle $|z| = R$ of radius R together with the part of the real axis from $-R$ to $+R$.

For the poles

$$(z^2+1)(z^2+4) = 0 \Rightarrow z^2+1 = 0 \text{ or } z = \pm i \Rightarrow z^2+4 = 0 \text{ or } z = \pm 2i$$

The Poles at $z = i$ and $z = 2i$ lie within the contour.

Residue (at $z = i$)

$$= \lim_{z\to i} \frac{(z-i)e^{3iz}}{(z^2+1)(z^2+4)} = \lim_{z\to i} \frac{e^{3iz}}{(z+i)(z^2+4)} = \frac{e^{-3}}{6i}$$

Residue (at $z = 2i$)

$$= \lim_{z\to 2i} \frac{(z-2i)e^{3iz}}{(z^2+1)(z^2+4)} = \lim_{z\to i} \frac{e^{3iz}}{(z^2+1)(z+2i)} = -\frac{e^{-6}}{12i}$$

By theorem of Residue

$$\int_C f(z)dz = 2\pi i \times Res$$

$$\int_{-R}^{R} \frac{e^{3iz}dz}{(z^2+1)(z^2+4)} + \int_{C_R} \frac{e^{3iz}dz}{(z^2+1)(z^2+4)} = 2\pi i \left[\frac{e^{-3}}{6i} + \frac{e^{-6}}{-12i}\right]$$

$$\lim_{R\to\infty} \int \frac{e^{3izdz}}{(z^2+1)(z^2+4)} = 0, \text{ By Jordan's Lemma}$$

$$\int_{-\infty}^{\infty} \frac{e^{3ix}}{(x^2+1)(x^2+4)}dx = \pi \left[\frac{e^{-3}}{3} - \frac{e^{-6}}{6}\right]$$

$$\int_{-\infty}^{\infty} \frac{cos(3x)dx}{(x^2+1)(x^2+4)} = \text{ Real part of} \int_{-\infty}^{\infty} \frac{e^{3ix}dx}{(x^2+1)(x^2+4)}$$

$$\int_{-\infty}^{\infty} \frac{cos(3x)dx}{(x^2+1)(x^2+4)} = \pi \left[\frac{e^{-3}}{3} - \frac{e^{-6}}{6}\right]$$

$$\int_{0}^{\infty} \frac{cos(3x)dx}{(x^2+1)(x^2+4)} = \frac{\pi}{2} \left[\frac{e^{-3}}{3} - \frac{e^{-6}}{6}\right]$$

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PROBLEMS

5.1 Use Residue of calculus to evaluate the following integrals:

a) $\displaystyle\int_0^\infty \frac{\cos mx}{(x^2+1)}dx$

b) $\displaystyle\int_0^\infty \frac{\log(1+x^2)}{(x^2+1)}dx$

c) $\displaystyle\int_0^\infty \frac{1}{(x^2+1)^3}dx$

d) $\displaystyle\int_0^\infty \frac{1}{(x^6+1)}dx$

e) $\displaystyle\int_0^\infty \frac{\cos x^2 + \sin x^2 - 1}{(x^4+16)}dx$

5.2 By contour integration, prove that $\displaystyle\int_0^\infty \frac{\sin mx}{x}dx = \frac{\pi}{2}$

5.3 Show that, if $a \geq b \geq 0$, then $\displaystyle\int_0^\infty \frac{\cos 2ax - \cos 2bx}{x^2} dx = \pi(b-a)$

5.4 Show that $\displaystyle\int_0^\infty \frac{x^3 \sin x}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{2(a^2-b^2)}[a^2e^{-a} - b^2e^{-b}]$ where $a > b > 0$

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